2019

JEE MAIN

COMPLETE MATHEMATICS

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McGraw Hill Education

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To Our Readers...

How to Crack the JEE

This new edition of our book, ***Complete Mathematics JEE Main,*** is primarily meant for the candidates due to appear in CBSE's JEE (Main). It thoroughly covers all the topics prescribed in the Mathematics syllabus of this examination. The book is divided into 28 chapters. Each chapter is organised as follows:

1. It begins with a review of **important definitions, formulae, theorems, tips, tricks and techniques** to solve problems. Difficult concepts have been explained by giving appropriate **illustrations** in sufficient numbers.
2. This theory is followed by examples. These **examples (solved in detail)** are divided into three categories:
   (a) **Concept Based** examples to further strengthen and enrich your conceptual understanding
   (b) **Level 1** examples carefully selected to groom you for the JEE Main examination. This section contains both **Straight Objective Type Questions** and **Assertion-Reason Type Questions**.
   (c) **Level 2** examples added to further boost your confidence by initiating you into tougher questions
3. The Solved Examples are followed by Exercises. These have been divided into five parts.
   (i) Concept Based Questions
   (ii) Level 1 Questions
   (iii) Level 2 Questions
   (iv) Previous Years' AIEEE/JEE (Main) Questions
   (v) Previous Years' B. Architecture Entrance Examination Questions.
   The philosophy behind (i), (ii) and (iii) is the same as that in Solved Examples. We have added (iv) and (v) to give you glimpse of the questions that have been asked in the previous years (till 2016). Questions from online JEE (Main) papers from 2013 and 2016 have also been added. As the Mathematics syllabus for B. Architecture Entrance Examination Questions is the same as that of JEE (Main) examination, questions from these papers have also been added for further practice.
4. Each exercise has been provided with **answers**.
5. **Hints and Solutions** to all the questions in (i), (ii), (iii), (iv) and (v) have been given at the end of the chapters.

How to Use this Book

Step 1

Read the theory given in the chapter carefully. Take meaningful notes in a notebook.

Step 2

Go to solved examples. First attempt the questions given in this section (Concept-based). Try to solve each example yourself. If you fail to do it within a reasonable time limit, go through the solution carefully.

The key to improvement is NOT merely understanding the explanation given in the solution, but to see exactly what clues in the question lead us to the correct answer.

Step 3

Repeat Step 2 with Concept-based questions in the exercise.

Step 4

Repeat Step 2 and Step 3 with Solved Examples and questions in Exercises at Level-1 and Level-2.

Important: Leave Previous Years' Questions for the future.

Facing the Examination

The analysis of previous fourteen years' papers shows that it is not possible to leave any topic. You get questions from each and every topic of mathematics. Thus, it is essential that you study each and every topic thoroughly.

Beginning the Preparation

The best time to begin preparation for JEE (Main) is when you have studied almost fifty percent of your Class XI course, that is, almost one and a half year before the day you wish to appear in the JEE (Main).

Effective Studying

You need to set priorities so that you can get the best return for time invested. Write down the time you wish to devote to Physics, Chemistry and Mathematics. As you study, you will develop a list of your problem areas, which will direct you further. This may require you to rearrange your priorities. Do not overestimate the time you have, extra preparation always helps. The key is to set up a workable plan and stick to it.

Some Tips for Preparation

- Identify your strengths and weaknesses.

  It is important to identify your problem areas early on. Your study plan would need to evolve as you identify these areas.

  Equally important is to know your strength areas. You should ensure that these areas are further strengthened and that you are not a victim of overconfidence.

  Your study plan should strike a balance between the time kept for strength areas and for problem areas. Though you need to concentrate more on your problem areas, mastering your strength areas in between would ensure that you do not get dejected.

- Make notes and continually review each section.

  Take meaningful notes in a notebook which will be of a great help at a later date. Good notes are short and crisp, but cover the key concepts. Very long notes are redundant as you never have time to go through them and too short notes are useless as they do not trigger any recall.

  Rapidly review the section you have just studied. Recall all the major ideas of this portion. If you have any difficulties, check your notes.

- Study for short durations.

  Your brain needs time to assimilate your thoughts and develop a deep understanding of each concept that you are learning. Try to study in short durations with rest breaks in between. During these rests, first blank out your mind so as to relax and then go over the key concepts that you have learnt in the just completed leg of your study.

- **Do each exercise keeping time in mind and look for clues.**

  This is very important. Before proceeding to another section, rework all the questions about which you were not sure or which you could not solve correctly. And make sure that you complete them in the time allocated for each section.

- **Always have a positive attitude.**

  Be motivated. Don't be discouraged if you are unable to solve a problem. When you cannot understand a problem you tend to think in different directions. This helps you to develop your mental faculties.

- **Always keep your goal in mind.**
- **Get your body clock in rhythm with the examination timings a few days before the examination.**

  Some students are in the habit of studying at night and sleeping during the day time. This habit should not be continued till the day of the exam. You should develop a habit of working during the same clock hours as the actual exam at least 15-20 days before the exam. This will help you to work effectively during the exam.

Test Taking Strategies

- Read the instructions in the question paper carefully and scrupulously follow these instructions.
- Read the entire question before attempting to answer it, and recognise the key concepts in the same. Each question typically contains certain key concepts; certain pieces of information that, if you recognise accurately, your ability to come to the right answer increases significantly. The correct answer will typically incorporate all the key concepts contained in the question.
- Read each question carefully. Answer the questions asked, not the one you may have expected.
- Budget your time. Work swiftly. Don't devote too much time to any difficult problem. There may be some simple ones waiting for you. Concentrate on those questions which you are absolutely or reasonably certain you know. Leave those questions for later where you have some doubts. Go through the exam answering questions that you can answer and skip questions you can't answer. Then go through the paper again, deliberating and answering the questions you had left unattended earlier.

Best wishes for your endeavours!

RAVI PRAKASH
AJAY KUMAR
USHA GUPTA

Please log on to www.mhhe.com/jeemaincompletemathematics for Seven Sectional Test Papers, Five Practice Test Papers and Miscellaneous Questions from the entire book.

Trend Analysis

Chap. No.	Chapter Name	JEE (Main) (Offline) 2017	Online 8/4/17	Online 9/4/17	Architecture 2017	JEE (Main) (Offline) 2016	Online 9/4/16	Online 10/4/16	2016 (Architecture)
1.	Sets, Relations and Functions	8	4	4	4	4	4	4	8
2.	Complex Numbers		4	4	4	4	4	4	4
3.	Quadratic Equations	4	4	4	4	4	8	4	4
4.	Determinants	4	8		4	4	4		8
5.	Matrices	8	4	4	4	4	4	8	4
6.	Permutations and Combinations	4	4	4	4	4	8	8	4
7.	Mathematical Induction and Binomial Theorem	4	4	4	4	4	4	4	8
8.	Progressions	4	8	8	8	8	4	4	4
9.	Limits and Continuity	4	4	4	4	4		8	4
10.	Differentiability and Differentiation	4		8	4	8	4		4
11.	Applications of Derivatives	8	8	4	4	4	12	8	8
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13.	Definite Integrals	4	4	8	4	4	4	8	4
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15.	Differential Equations	4	4	4	4	4	4	4	4
16.	Cartesian System of Rectangular Coordinates and Straight Lines	4		4	8	4	4	8	8
17.	Circles and Systems of Circles		8	4	4	8	8	4	
18.	Parabola	4		4		4		4	4
19.	Ellipse	4	4	4	4		4	4	
20.	Hyperbola	4	4		4	4	4		4
21.	Three Dimensional Geometry	8	8	12	8	8	8	8	12
22.	Vectors	4	4	4	4	4	4	4	4
23.	Statistics		4	4		4	4	4	
24.	Probability	12	8	8	12	4	4	4	4
25.	Trigonometrical Ratios, Identities and Equations	4	4	4	4	4	4	4	
26.	Inverse Trigonometric Functions			4					4
27.	Heights and Distances	4	4		4	4		4	
28.	Mathematical Reasoning	4	4	4	4	4	4	4	4
	Total	**120**	**120**	**120**	**120**	**120**	**120**	**120**	**120**

About JEE Main

1. Introduction and Scheme of Examination

The Joint Entrance Examination from the year 2013 for admission to the undergraduate programmes in Engineering is being held in two parts, JEE-Main and JEE-Advanced. Only the top 2,00,000 candidates (including all categories) based on performance in JEE Main will qualify to appear in the JEE Advanced examination. Admissions to IITs will be based only on category-wise All India Rank (AIR) in JEE Advanced, subject to condition that such candidates are in the top 20 percentile categories. Admission to NITs will be based on 40% weightage for performance in Class XII board marks (normalized) and the remainder 60% weightage would be given to performance in JEE Main and a combined All India Rank (AIR) would be decided accordingly.

In case any State opts to admit students in the engineering Colleges affiliated to state Universities where States require separate merit list to be provided based on relative weightages adopted by the states, then the merit list shall be prepared with such relative weightages as may be indicated by States.

2. Eligibility Criteria and List of Qualifying Examinations for JEE(Main) Exam

The minimum academic qualification for appearing in JEE(Main) is that the candidate must have passed in final examination of 10+2 (Class XII) or its equivalent referred to as the qualifying examination (see below).

However, admission criteria in the concerned institution/university will be followed as prescribed by concerned university/institution and as per the guidelines & criteria prescribed by AICTE.

Qualifying Examinations

List of Qualifying Examinations

(i) The +2 level examination in the 10+2 pattern of examination of any recognized Central/State Board of Secondary Examination, such as Central Board of Secondary Education, New Delhi, and Council for Indian School Certificate Examination, New Delhi

(ii) Intermediate or two-year Pre-University Examination conducted by a recognized Board/University.

(iii) Final Examination of the two-year course of the Joint Services Wing of the National Defence Academy.

(iv) Any Public School/Board/University Examination in India or in foreign countries recognized by the Association of Indian Universities as equivalent to 10+2 system.

(v) H.S.C. Vocational Examination.

(vi) A pass grade in the Senior Secondary School Examination conducted by the National Open School with a minimum of five subjects.

(vii) 3 or 4-year diploma recognized by AICTE or a State Board of Technical Education.

3. Pattern of Examination

Subject combination for each paper and type of questions in each paper are given below:

	Subjects	Type of Questions	Duration
Paper 1	Physics, Chemistry & Mathematics	Objective type questions with equal weightage to Physics, Chemistry & Mathematics	3 Hours
Paper 2	Mathematics – Part I	Objective type questions	3 Hours
	Aptitude Test – Part II &	Objective type questions	
	Drawing Test – Part III	Questions to test Drawing Aptitude	

Requirement of papers for different courses is given in the table below:

Course	Papers
B.E/B.TECH	Paper – 1
B.ARCH/B. PLANNING	Paper – 2

Scoring and Negative Marking

There will be objective type questions with four options having single correct answer. For each incorrect response, one fourth (1/4) of the total marks allotted to the question would be deducted. No deduction from the total score will, however, be made if no response is indicated for an item in the answer sheet.

Syllabus

Unit 1: Sets, Relations and Functions

Sets and their representation; Union, intersection and complement of sets and their algebraic properties; Power set; Relation, Types of relations, equivalence relations, functions;. one-one, into and onto functions, composition of functions.

Unit 2: Complex Numbers and Quadratic Equations

Complex numbers as ordered pairs of reals, Representation of complex numbers in the form $a + ib$ and their representation in a plane, Argand diagram, algebra of complex numbers, modulus and argument (or amplitude) of a complex number, square root of a complex number, triangle inequality, Quadratic equations in real and complex number system and their solutions. Relation between roots and co-efficients, nature of roots, formation of quadratic equations with given roots.

Unit 3: Matrices and Determinants

Matrices, algebra of matrices, types of matrices, determinants and matrices of order two and three. Properties of determinants, evaluation of determinants, area of triangles using determinants. Adjoint and evaluation of inverse of a square matrix using determinants and elementary transformations, Test of consistency and solution of simultaneous linear equations in two or three variables using determinants and matrices.

Unit 4: Permutations and Combinations

Fundamental principle of counting, permutation as an arrangement and combination as selection, Meaning of $P(n, r)$ and $C(n, r)$, simple applications.

Unit 5: Mathematical Induction

Principle of Mathematical Induction and its simple applications.

Unit 6: Binomial Theorem and its Simple Applications

Binomial theorem for a positive integral index, general term and middle term, properties of Binomial coefficients and simple applications.

Unit 7: Sequences and Series

Arithmetic and Geometric progressions, insertion of arithmetic, geometric means between two given numbers. Relation between A.M. and G.M. Sum upto n terms of special series: $\sum n$, $\sum n^2$, $\sum n^3$. Arithmetico - Geometric progression.

Unit 8: Limit, Continuity and Differentiability

Real—valued functions, algebra of functions, polynomials, rational, trigonometric, logarithmic and exponential functions, inverse functions. Graphs of simple functions. Limits, continuity and differentiability. Differentiation of the sum, difference, product and quotient of two functions. Differentiation of trigonometric, inverse trigonometric, logarithmic, exponential, composite and implicit functions; derivatives of order upto two. Rolle's and Lagrange's Mean Value Theorems. Applications of derivatives: Rate of change of quantities, monotonic—increasing and decreasing functions, Maxima and minima of functions of one variable, tangents and normals.

Unit 9: Integral Calculus

Integral as an anti-derivative. Fundamental integrals involving algebraic, trigonometric, exponential and logarithmic functions. Integration by substitution, by parts and by partial fractions. Integration using trigonometric identities.

Evaluation of simple integrals of the type

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{\sqrt{x^2 \pm a^2}}, \int \frac{dx}{a^2 - x^2}, \int \frac{dx}{\sqrt{a^2 - x^2}}, \int \frac{dx}{ax^2 + bx + c}$$

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}}, \int \frac{(px + q)\,dx}{ax^2 + bx + c}, \int \frac{(px + q)\,dx}{\sqrt{ax^2 + bx + c}}$$

$$\int \sqrt{a^2 \pm x^2}\, dx \text{ and } \int \sqrt{x^2 - a^2}\, dx$$

Integral as limit of a sum. Fundamental Theorem of Calculus. Properties of definite integrals. Evaluation of definite integrals, determining areas of the regions bounded by simple curves in standard form.

Unit 10: Differential Equations

Ordinary differential equations, their order and degree. Formation of differential equations. Solution of differential equations by the method of separation of variables, solution of homogeneous and linear differential equations of the type:

$$\frac{dy}{dx} + p(x)y = q(x)$$

Unit 11: Co-ordinate Geometry

Cartesian system of rectangular co-ordinates in a plane, distance formula, section formula, locus and its equation, translation of axes, slope of a line, parallel and perpendicular lines, intercepts of a line on the coordinate axes.

Straight lines

Various forms of equations of a line, intersection of lines, angles between two lines, conditions for concurrence of three lines, distance of a point from a line, equations of internal and external bisectors of angles between two lines, coordinates of centroid, orthocentre and circumcentre of a triangle, equation of family of lines passing through the point of intersection of two lines.

Circles, conic sections

Standard form of equation of a circle, general form of the equation of a circle, its radius and centre, equation of a circle when the end points of a diameter are given, points of intersection of a line and a circle with the centre at the origin and condition for a line to be tangent to a circle, equation of the tangent. Sections of cones, equations of conic sections (parabola, ellipse and hyperbola) in standard forms, condition for $y = mx + c$ to be a tangent and point (s) of tangency.

Unit 12: Three Dimensional Geometry

Coordinates of a point in space, distance between two points, section formula, direction ratios and direction cosines, angle between two intersecting lines. Skew lines, the shortest distance between them and its equation. Equations of a line and a plane in different forms, intersection of a line and a plane, coplanar lines.

Unit 13: Vector Algebra

Vectors and scalars, addition of vectors, components of a vector in two dimensions and three dimensional space, scalar and vector products, scalar and vector triple product.

Unit 14: Statistics and Probability

Measures of Dispersion: Calculation of mean, median, mode of grouped and ungrouped data. Calculation of standard deviation, variance and mean deviation for grouped and ungrouped data.

Probability: Probability of an event, addition and multiplication theorems of probability, Baye's theorem, probability distribution of a random variate, Bernoulli trials and Binomial distribution.

Unit 15: Trigonometry

Trigonometrical identities and equations. Trigonometrical functions. Inverse trigonometrical functions and their properties. Heights and Distances.

Unit 16: Mathematical Reasoning

Statements, logical operations And, or, implies, implied by, if and only if. Understanding of tautology, contradiction, converse and contrapositive.

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Format of Questions in this Book

STRAIGHT OBJECTIVE TYPE QUESTIONS

Each Question has four choices—(a), (b), (c) and (d) out of which **ONLY ONE** is correct

ASSERTION-REASON TYPE QUESTIONS

Each Question has four choices—(a), (b), (c) and (d) out of which **ONLY ONE** is correct

Write (a), (b), (c) or (d) according to the following rules.

(a) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 **is** a correct explanation for STATEMENT-1
(b) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 **is NOT** a correct explanation for STATEMENT-1
(c) STATEMENT-1 is True, STATEMENT-2 is False
(d) STATEMENT-1 is False, STATEMENT-2 is True

CHAPTER ONE

Sets, Relations and Functions

By a *set* we mean any collection of objects. For example, we may speak of the set of all living Indians, the set of all letters of the English alphabet or the set of real numbers less than 5. The objects constituting a set are *elements* or *members* of the set.

If X is a set and an element x is a member of, or belongs to the set X, then this is expressed, as $x \in X$.

There are two methods of representing a set.

(a) Roster method (or tabular method) In this method, a set is represented by listing all its elements in curly brackets and separating them by commas.

For example, set having elements 1, 2, 3, 5 only is written as $\{1, 2, 3, 5\}$.

(b) Property method In this method a set is represented by stating all the properties which are satisfied by the elements of the set and not by other elements outside the set.

If X contains all values of 'x' for which the condition $P(x)$ is true, then we write $X = \{x : P(x)\}$.

Illustration 1

The set of all real roots of the equation $x^4 - 2x^2 - 3 = 0$ is denoted by

$\{x : x \in \mathbf{R}, x^4 - 2x^2 - 3 = 0\}$ or equivalently by $\left\{\sqrt{3}, -\sqrt{3}\right\}$.

Illustration 2

The set of integers strictly between 2 and 8 is represented by

$\{x : x \in \mathbf{I}, 2 < x < 8\}$ or equivalently by $\{3, 4, 5, 6, 7\}$.

Illustration 3

The set of Presidents of India can be represented by

$\{x : x$ is/was President of India$\}$ or equivalently

$A = \{$Rajendra Prasad, Sarvepalli Radhakrishnan,
Zakir Hussain, V.V. Giri, ...$\}$

The number of elements of A or the cardinality of A is denoted by $n(A)$. In Illustration 3, $n(A) = 13$.

FINITE SET AND INFINITE SET

A set is called a *finite set* if it contains only finite number of elements. A set which does not contain finite number of elements is called *infinite set*.

Null set A set containing no element is called a *null set* or *empty set* or *void set* and is denoted by 'ϕ'. Equivalently $P(x)$ is a property satisfied by no object at all.

Singleton set A set containing exactly one element is called a *singleton set*.

If a set X has 'r' elements then we write $n(X)$ (number of elements in X) = r.

Subset A set 'A' is said to be subset of the set X if every element of A is an element of X and we write $A \subseteq X$. A is said to be a proper subset of X if $A \subseteq X$. and $A \neq X$, this will be written as $A \underset{\neq}{\subset} X$. The denial of $A \subset B$ is written as $A \not\subset B$.

For two sets A and B, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

The set of all subsets of X is called *Power set* of X denoted by $P(X)$ i.e. $P(X) = \{A : A \subseteq X\}$.

Some Basic Properties:

(i) $A \subseteq A$

(i) $A \subseteq B$, $B \subseteq C$ then $A \subseteq C$

(iii) The only subset of ϕ is ϕ itself.

(iv) The subsets of $\{x\}$ are ϕ and $\{x\}$.

(v) The subsets of $\{x, y\}$ are ϕ, $\{x\}$, $\{y\}$, $\{x, y\}$

(vi) If $n(X) = r$ then number of all subsets of X will be 2^r i.e. $n(P(X)) = 2^r$.

ALGEBRA OF SETS

Union of two Sets

Union of two sets A and B is denoted by $A \cup B$

and, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Clearly $A \subseteq A \cup B$, $B \subseteq A \cup B$

and $A \cup B = B \cup A$, $A \cup A = A$,

$A \cup B = B$ if and only if $A \subseteq B$,

$(A \cup B) \cup C = A \cup (B \cup C)$ (Associativity).

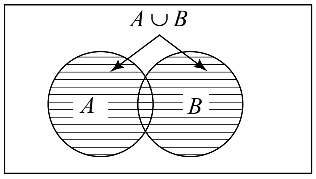


Fig. 1.1

Intersection of Two Sets

Intersection of two sets A and B is denoted by $A \cap B$ and, $A \cap B = \{x : x \in A \text{ and } x \in B\}$
Trivially, $A \cap B \subseteq A$, $A \cap B \subseteq B$ and $A \cap B = B \cap A$, $(A \cap B) \cap C = A \cap (B \cap C)$ (Associativity).

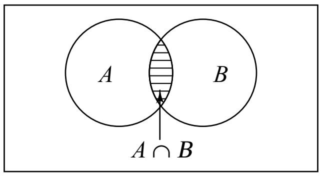


Fig. 1.2

Complement of a Set

Complement of a set B in a set A is written as

$$A \sim B = \{x : x \in A, x \notin B\}$$

e.g., $\mathbf{I} \sim \mathbf{N} = \ \dots -3, -2, -1, 0\}$. The set of irrational numbers = $\mathbf{R} \sim \mathbf{Q}$

Where **I** is the set of integers
N is the set of natural numbers
R is the set of real number and
Q is the set of rational numbers

In any discussion involving sets and their operations, we presume that all these sets are subsets of a parent set called *Universal set*, U. The complement of A in U is denoted by A^c or $A' = \{x : x \notin A\}$.

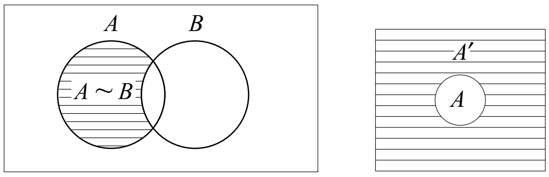


Fig. 1.3

SOME PROPERTIES OF OPERATIONS ON SETS

1. $A \cap (B_1 \cup B_2 \cup \dots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_n)$ (Intersection is distributive over union)
2. $A \cup (B_1 \cap B_2 \cap \dots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \dots \cap (A \cup B_n)$ (Union is distributive over intersection)
3. $(A \cap B)^c = A^c \cup B^c$ and $(A \cup B)^c = A^c \cap B^c$. (De Morgan's Laws)
4. $A \sim \left(\bigcap_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} (A \sim A_i)$.
5. $A \sim \left(\bigcup_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n} (A \sim A_i)$.
6. $A \cap A^c = \phi$
7. $A \cup A^c = X$
8. $\phi^c = X$
9. $X^c = \phi$
10. $(A^c)^c = A$
11. $A \sim B = A \cap B^c$
12. $A \subseteq B$ if and only if $B^c \subseteq A^c$.
13. $A \sim A = \phi$.
14. $(A \sim B) \sim C = (A \sim C) \sim B$
15. $A \sim (B \cup C) = (A \sim B) \cap (A \sim C)$
16. If A and B are finite sets, then
$n(A \cup B) = n(A) + n(B) - n(A \cap \mathrm{B})$
17. If A, B and C are finite sets then
$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Symmetric Difference of Two Sets

$$A \,\Delta\, B = (A \sim B) \cup (B \sim A)$$

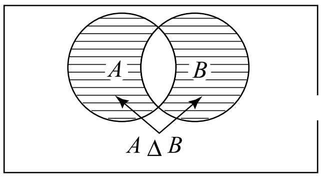


Fig. 1.4

$A \,\Delta\, A = \phi$, $A \,\Delta\, B = B \,\Delta\, A$

Cartesian Product

The cartesian product of sets A and B is denoted by $A \times B$ and, $A \times B = \{(a, b): a \in A, b \in B\}$

In general $A \times B \neq B \times A$, moreover,

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

and $$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Illustration 4

If $A = \{0, 1, 2, 3, 5\}$, $B = \{0, 1, 2, 3)$, $C = \{0, 1, 4, 5\}$ then $B \subseteq A$, $C \not\subseteq A$, $A \cap C = \{0, 1, 5\}$, $B \cap C = \{0, 1\}$, $A \sim B = \{5\}$, $A \sim C = \{2, 3\}$, $B \sim C = \{2, 3\}$, $C \sim B = \{4, 5\}$, $A \,\Delta\, B = \{5\}$, $A \,\Delta\, C = \{2, 3, 4\}$ and $B \,\Delta\, C = \{2, 3, 4, 5\}$

Illustration 5

Simplify $(A \cup B \cup C) \cap (A \cap B^c \cap C^c)^c \cap C^c$

$$
\begin{aligned}
&(A \cup B \cup C) \cap (A \cap B^c \cap C^c)^c \cap C^c \\
&\quad = [(A \cup B \cup C) \cap (A^c \cup B \cup C)] \cap C^c \\
&\quad = [(A \cap A^c) \cup (B \cup C] \cap C^c \\
&\quad = [\phi \cup (B \cup C)] \cap C^c \\
&\quad = (B \cup C) \cap C^c = (B \cap C^c) \cup (C \cap C^c). \\
&\quad = (B \cap C^c) \cup \phi = B \cap C^c = B \sim C.
\end{aligned}
$$

Relations

A relation from a set of X to a set Y is a subset R of $X \times Y$. If $(a, b) \in R$, we say a is related to b and often write it as $a\,R\,b$. If $X = Y$, we say that R is relation on X.

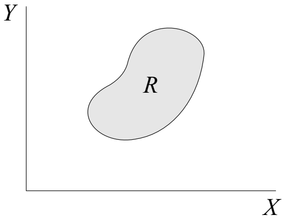


Fig. 1.5

Let R be a relation from X to Y. For each subset A of X, we write

$$R(A) = \{y \in Y\text{: } x\,R\,y \text{ for some } x \in A\}$$

and call it the (direct) image of A under R, and for each subset B of Y, we write $R^{-1}(B) = \{x \in X : x\,R\,y \text{ for some } y \in B\}$ and call it the *inverse image* of B under R.

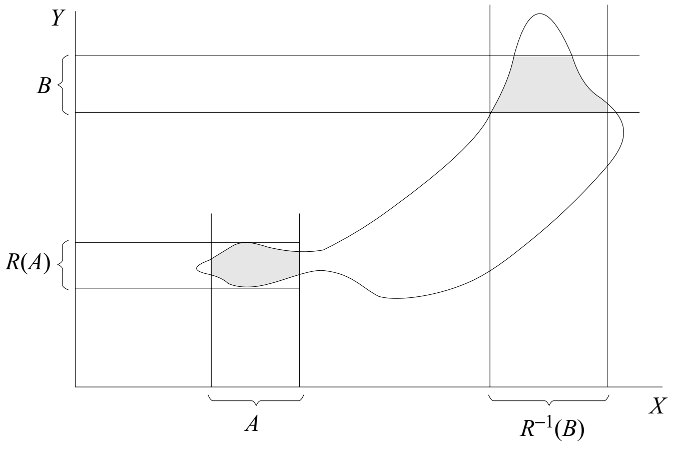


Fig. 1.6

One can think of R^{-1} as a relation from Y to X, where

$$yR^{-1}\,x \Leftrightarrow xRy$$

that is, $\quad R^{-1} = \{(y, x) : (x, y) \in R\}$

R^{-1} is called the *reverse* of the relation R.

Domain of a relation R from a set X to a set Y is the set of all first components of the elements of R, i.e., dom $R = \{a \in X\text{: } (a, b) \in R \text{ for some } b \in R\}$

Let R be a relation from X to Y and S from Y to Z, then $SoR = \{(x, z)|\ x \in X, z \in Z \text{ and } \exists\ y \in Y$ such that and $(x, y) \in R$ and $(y, z) \in S$.

If $n(A), n(B) < \infty$, then the number of relation from A to B will be $2^{n(A)\,n(B)}$

Equivalence Relation

A relation R on a set X (i.e. $R \subset X \times X$) is said to be

(i) *reflexive* if $x\,R\,x$ for all $x \in X$.
(ii) *symmetric* if $x\,R\,y \Rightarrow y\,R\,x$, where $x, y \in X$
(iii) *transitive* if $x\,R\,y$, $y\,R\,z \Rightarrow x\,R\,z$, where $x, y, z \in X$

Further, a relation R in a set X is said to be an equivalence relation if it is reflexive, symmetric and transitive. **R** is called anti-symmetric if $x\,R\,y$ and $y\,R\,x \Rightarrow x = y$.

Illustration 6

For $x, y \in$ **I**, write $x\,R\,y$ if $x - y$ divisible by 6; this is an equivalence relation and is usually written as $x \equiv y$ (mod. 6).

Illustration 7

The relation R on **R** defined as $R = \{(a, b)\text{: } a \le b\}$ is not symmetric as $(1, 3) \in R$ but $(3, 1) \notin R$ but is transitive and reflexive.

Illustration 8

If $A = \{1, 2, 3\}$; let $R = \{1, 1), (2, 2)\ (3, 3), (1, 2)\ (2, 1)\}$ then R is reflexive, symmetric and transitive.

The number of equivalence relations that can be defined on a set containing k elements is given

$$B_k = \sum_{n=0}^{k-1} \binom{k-1}{n} B_n$$

E.g. $\quad B_0 = 1, B_1 = 1, B_2 = 2, B_3 = \binom{2}{0}B_0 + \binom{2}{1}B_1 + \binom{2}{2}B_2$
$\qquad = 1 + 2 + 2 = 5$

Functions

A relation F from A to B is said to be a function if for each $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in F$. b is called image of 'a' under F and is denoted by $F(a)$. Thus every function is a relation but the converse may not be true.

If $n(A) = n$ and $n(B) = m$ then total number of functions from A to B is n^m.

SOME DEFINITIONS

Let A and B be two non-empty sets. A function f from A to B can also be defined as a rule that assigns to each element in the set A, one and only one element of the set B. In general, the sets A and B need not be sets of real numbers. However, we consider only those functions for which A and B are both subsets of the real numbers. We shall denote by **R**, the set of all real numbers.

The set A in the above definition is called the **domain** of the function f. We usually denote it by dom f. If x is an element in the domain of a function f, then the element that

f associates with x is denoted by the symbol $f(x)$, and is called the *image* of x under f, or the *value* of f at x. The set of all possible values of $f(x)$ as x varies over the domain is called the **range of f**. If $f: A \to B$, then the range of f is a subset of B and the set B is called **co-domain of f**.

Remark

- ☐ If x is an element in the domain of a function f, the definition of a function requires that f assigns one and only one value to x. This means that a function cannot be *multiple-valued.* For example, the expression $\pm\sqrt{x}$ does not define a function of x, since it assigns two values to each positive x.

ALGEBRAIC OPERATIONS ON FUNCTIONS

1. If f and g are two functions, then *sum of the functions* $f + g$, is defined for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$ by
$$(f+g)(x) = f(x) + g(x).$$
2. If k is any real number and f is a function, then kf is defined for all $x \in \operatorname{dom} f$ by $(kf)(x) = kf(x)$.
3. If f and g are two functions, then the *pointwise product* fg is defined for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$ by
$$(fg)(x) = f(x)\, g(x).$$
4. If f and g are functions, then f/g is defined for all $x \in \operatorname{dom}(f) \cap \operatorname{dom}(g) \cap \{x: g(x) \neq 0\}$ by $(f/g)(x) = f(x)/g(x)$.
5. *Composition of functions* Let $f: A \to B$ and $g: B \to C$ be functions, then $gof: A \to C$ defined by $(gof)(x) = g(f(x))$.

We have the following formulae for domains of functions.

1. $\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g$
2. $\operatorname{dom}(fg) = \operatorname{dom} f \cap \operatorname{dom} g$
3. $\operatorname{dom}(f/g) = \operatorname{dom} f \cap \operatorname{dom} g \cap \{x : g(x) \neq 0\}$
4. $\operatorname{dom}\sqrt{f} = \operatorname{dom} f \cap \{x : f(x) \geq 0\}$

Note

Note that if gof is defined, then fog may not be defined. For example if f : **A** $\to$ **B** and g : **B** $\to$ **C** then gof is defined but fog is not. Let f: **R** $\to$ **R** and g: **R** $\to$ **R** be defined by $f(x) = \cos x$ and $g(x) = x^3$. Then $gof(x) = g(f(x)) = g(\cos x) = \cos^3 x$ and $fog(x) = f(x^3) = \cos x^3$. Thus even if fog and gof are both defined it is not necessary that $fog = gof$.

DOMAINS AND RANGES OF SOME FUNCTIONS

1. *Constant functions* A function that assigns the same value to every member of its domain is called a constant function. The domain of the constant function $f(x) = c$ is **R** and its range is $\{c\}$.

2. *Polynomial functions* A function of the form cx^n, where c is a constant and n is a non-negative integer, is called a *monomial* in x. Examples are $2x^3$, $5x^4$, $-6x$ and x^8. The function $4x^{1/2}$ and x^{-3} and not monomials because the powers of x are not non-negative integers. A function that is expressible as the sum of finitely many monomials in x is called a *polynomial* in x. Thus

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is a polynomial. The domain of a polynomial is **R** and its range is a subset of **R**.

3. The domain of $f(x) = \log_e x$ (= ln x) is $(0, \infty) = \{x \in \mathbf{R}, x > 0\} = \mathbf{R}^+$, and its range is $(-\infty, \infty)$ (i.e., the whole of **R**).

4. The domain of $f(x) = e^x$ is **R** and its range is $\mathbf{R}^+$.

TYPES OF FUNCTIONS

1. *Rational function.* This function is defined as the ratio of two polynomials

$$y = \frac{a_0 x^n + a_1 x^{n-1} + \cdots + a_n}{b_0 x^m + b_1 x^{m-1} + \cdots + b_m}$$

For example, $y = (x^2 + 3)/(x^3 + 4)$ is rational function.

2. *Irrational function.* If in the function $y = f(x)$, the operations of addition, subtraction, multiplication, division and raising to a power with rational non-integral exponents are performed on the right-hand side, the function $y = f(x)$ is said to be irrational. Examples are $y = (3x^2 + \sqrt{x})/\sqrt{2+4x}$ and $y = \sqrt{x}$.

3. *Even function.* A function $y = f(x)$ is said to be an even function if $f(-x) = f(x)\ \forall\ x \in \operatorname{dom}(f)$. Examples are $y = |x|$, $y = \cos x$ and $y = x^{2n}$. The graph of an even function will be symmetric about y-axis

4. *Odd function.* A function $y = f(x)$ is said to be an odd function if $f(-x) = -f(x)\ \forall\ x \in \operatorname{dom}(f)$. Examples are $y = \sin x$ and $y = x^{2n+1}$.

Clearly, $y = f(x) + f(-x)$ is always an even function, and $y = f(x) - f(-x)$ is always odd. Any function $y = f(x)$ can be expressed uniquely as the sum of an even and an odd function as follows:

$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$$

5. *Periodic function.* A function $y = f(x)$ is said to be periodic if there exists a number $T > 0$ such that $f(x + T) = f(x)$ for all x in the domain of f. The least such T is called the period of f. For example, the period of $\sin x$ and $\cos x$ is 2π, and that of $\tan x$ is π.

If $f(x)$ is a periodic function with period T, then the function $f(ax + b)$, $a > 0$, is periodic with period T/a. For example, $\sin 2x$ has a period π, $\cos 3x$ has a period $2\pi/3$ and $\tan(2x+4)$ has a period $\pi/2$. The function

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbf{Q} \\ -1 \text{ if } x \in \mathbf{R} \sim \mathbf{Q} \end{cases}$$

is a periodic function without any period. The sum of two periodic functions may not be periodic e.g., $f(x) = \{x\}$, the fractional part of x and $g(x) = \sin x$.

6. *Onto function (or Surjective Function).* If a function $f: A \to B$ is such that each element in B is the f-image of at least one element in A, then we say that f is a function of A 'onto' B. Equivalently a function f is an onto function if co-domain of f = Range of f.

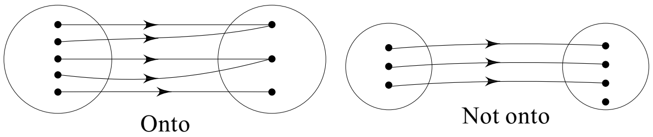


Fig. 1.7

For example, $f: \mathbf{R} \to [-1, 1]$ defined by $f(x) = \sin x$ is an onto function but $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \sin x$ is not onto since Range of $f = [-1, 1]$ and co-domain of $f = \mathbf{R}$.

In order to show that a function $f: A \to B$ is onto we start with any $y \in B$ and try to find $x \in A$ such that $f(x) = y$.

7. *One-to-one function (or injective function).* A function f is said to be *one-to-one* if it does not take the same value at two distinct points in its domain. For example, $f(x) = x^3$ is one-to-one, whereas $f(x) = x^2$ is not, as $f(1) = 1$ and $f(-1) = 1$. Note that a periodic function $f: R \to R$ cannot be one-to-one as $f(x + T) = f(x)$ for some $T > 0$.

In order to show that a function $f: A \to B$ is one-to-one, we may take any $x, y \in A$ such that $f(x) = f(y)$ and try to show that $x = y$.

If $n(A) = m$ and $n(B) = n (m \le n)$, then the number of injections (or one-one functions) from A to B is

$$^nP_m = \frac{n!}{(n-m)!}.$$

8. *Bijective function (One-to-One and Onto).* If a function f is both one-to-one and onto, then f is said to be a *bijective function*. For example an identity function $i_A : A \to A$ defined by $i_A(x) = x$ is trivially a bijective function. The function $f: [-\pi/2, \pi/2] \to [-1, 1]$ defined by $f(x) = \sin x$ is a bijective function. The function $f: (-\pi/2, \pi/2) \to \mathbf{R}$ defined by $f(x) = \tan x$ is also a bijective function.

If A and B are finite sets and $f: A \to B$ is a bijection then $n(A) = n(B)$. If $n(A) = n$, then the number of bijections from A to B is the total number of arrangements of n items taken all at a time which is $n!$

9. *Bounded and unbounded functions* A function f defined on an interval I is said bounded on I if there is $k > 0$ such that $|f(x)| \le k$ for all $x \in I$. Equivalently there is m and M such that $m \le f(x) \le M$ for all $x \in I$. E.g. $f(x) = \frac{1}{x}$ is not bounded on (0, 1) and $f(x) = x^2$ is bounded on [0, 1]. $f(x) = \sin x$ or $\cos x$ are bounded functions on $\mathbf{R}$ whereas $f(x) = \tan x$ is unbounded on $(-\pi/2, \pi/2)$.

GRAPHS OF SOME FUNCTIONS

1. *Constant function* $f(x) = c$ represents a constant function (Fig. 1.8).

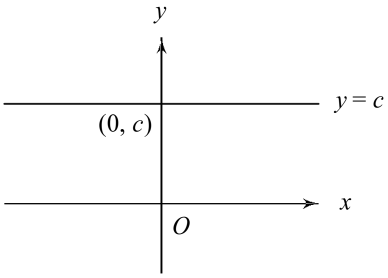


Fig. 1.8

2. *Proportional values*. If variables y and x are direct proportional, then the functional dependence between them is represented by the equation:

$$y = kx,$$

where k is a constant a *factor of proportionality*.

A graph of a direct proportionality is a straight line, going through an origin of coordinates and forming with an x-axis an angle α, a tangent of which is equal to k: $\tan \alpha = k$. Therefore, a factor of proportionality is called also a *slope*. These are shown in three graphs with $k = 1/3$, $k = 1$ and $k = -3$ on Fig.1.9

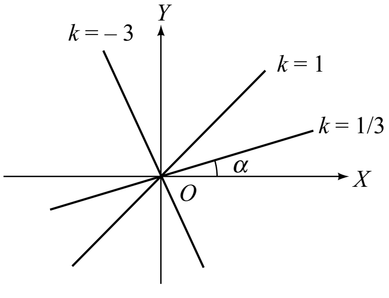


Fig. 1.9

3. *Linear function*. If variables y and x are related by the 1-st degree equation:

$$Ax + By = C,$$

(at least one of numbers A or B is non-zero), then a graph of the functional dependence is a straight line. If $C = 0$, then it goes through an origin of coordinates, otherwise – not. Graphs of linear functions for different combinations of A, B, C are represented on Fig.1.10

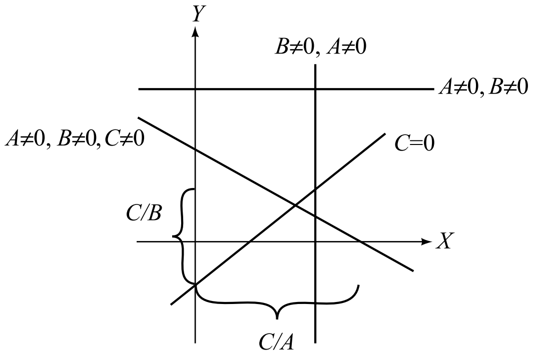


Fig. 1.10

4. *Inverse proportionality*. If variables y and x are *inverse proportional*, then the functional dependence between them is represented by the equation:

$$y = k/x,$$

where k is a constant.

A graph of an inverse proportionality is a curve, having two branches (Fig. 1.11). This curve is called a *hyperbola*. These curves are received at crossing a circular cone by a plane. As shown on Fig. 1.11, a product of coordinates of a hyperbola points is a constant value, equal in this case to 1. In general case this value is k, as it follows from a hyperbola equation:

$$xy = k$$

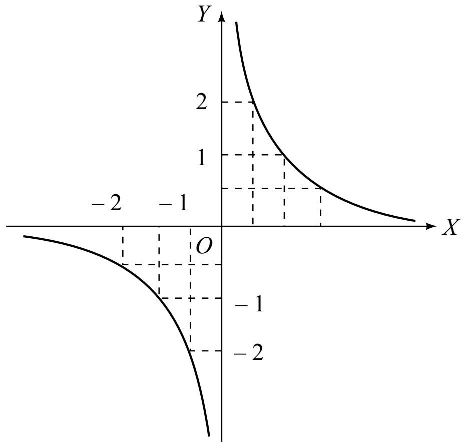


Fig. 1.11

The main characteristics and properties of hyperbola:

- the function domain: $x \neq 0$, and codomain: $y \neq 0$;
- the function is monotone (decreasing) at $x < 0$ and at $x > 0$, but it is not monotone on the whole, because of a point of discontinuity $x = 0$
- the function is unbounded, discontinuous at a point $x = 0$, odd, non-periodic;
- there are no zeros of the function.

5. *Quadratic function*. This is the function: $y = ax^2 + bx + c$, where a, b, c – constants, $a \neq 0$. In the simplest case we have $b = c = 0$ and $y = ax^2$. A graph of this function is a *quadratic parabola* – a curve going through an origin of coordinates. Every parabola has an axis of symmetry OY, which is called an *axis of parabola*. The point O of intersection of a parabola with its axis is a *vertex of parabola*.

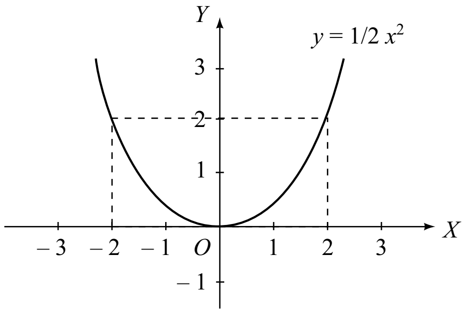


Fig. 1.12

A graph of the function $y = ax^2 + bx + c$ is also a quadratic parabola of the same shape, that $y = ax^2$, but its vertex is not an origin of coordinates, this is a point whose coordinates:

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$$

The form and location of a quadratic parabola in a coordinate system depends completely on two parameters: the coefficient a of x^2 and *discriminant* D : $D = b^2 - 4ac$. These properties follow from analysis of the quadratic equation roots. All possible different cases for a quadratic parabola are shown on Fig. 1.13

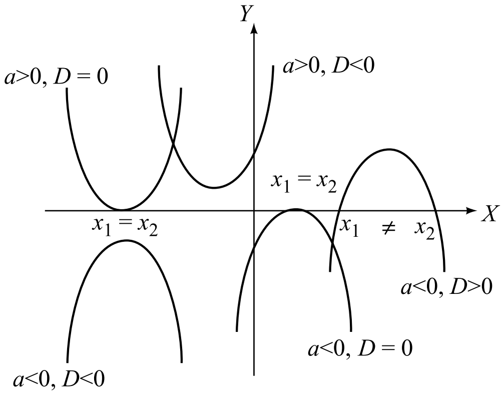


Fig. 1.13

The main characteristics and properties of a quadratic parabola:

- the function domain: $-\infty < x < +\infty$ (i.e. x is any real number)
- the function is not monotone on the whole, but to the right or to the left of the vertex it behaves as a monotone function;
- the function is unbounded, continuous in everywhere, even at $b = c = 0$, and non-periodic;
- the function has no zeros at $D < 0$.

6. *Power function*. This is the function: $y = ax^n$ where a, n are constants. At $n = 1$ we obtain the function, called a *direct proportionality*: $y = ax$; at $n = 2$ – a *quadratic parabola*; at $n = -1$ – an *inverse proportionality* or *hyperbola*. So, these functions are particular cases of a power function. We know, that a zero power of every non-zero number is 1, thus at $n = 0$ the power function becomes a constant: $y = a$, i.e. its graph is a straight line, parallel to an x-axis, except an origin of coordinates. All these cases (at $a = 1$) are shown on Fig. 1.14 ($n \geq 0$) and Fig. 1.15 ($n < 0$).

Negative values of x are not considered here, because then some of functions:

$$y = x^{1/2},\ \ y = x^{1/4}$$

lose a meaning.

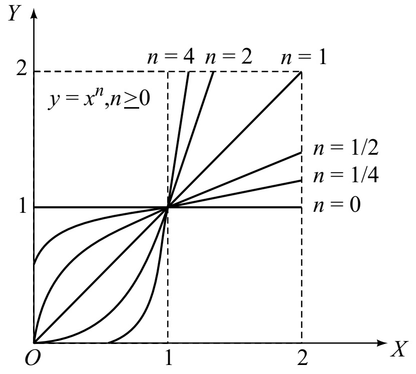


Fig. 1.14

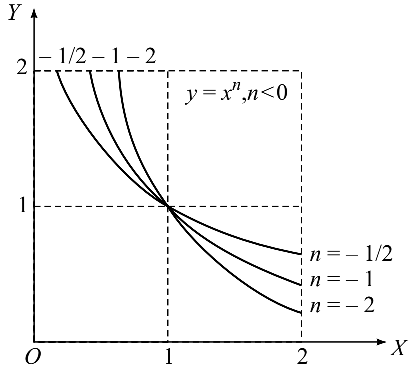


Fig. 1.15

If n – integer, power functions have a meaning also at $x < 0$, but their graphs have different forms depending on that is n an even or an odd number. In Fig. 1.16 two such power functions are shown: for $n = 2$ and $n = 3$.

At $n = 2$ the function is even and its graph is symmetric relatively an axis Y; at $n = 3$ the function is odd and its graph is symmetric relatively an origin of coordinates. The function $y = x^3$ is called a *cubic parabola*.

On Fig. 1.17 the function $y = \pm\sqrt{x}$ is represented. This function is inverse to the quadratic parabola $y = x^2$, its graph is received by rotating the quadratic parabola graph around a bisector of the 1-st coordinate angle. We see by the graph, that this is the two-valued function (the sign ± before the square root symbol says about this). Such functions are not studied in an elementary mathematics, therefore we consider usually as a function one of its branches: either an upper or a lower branch.

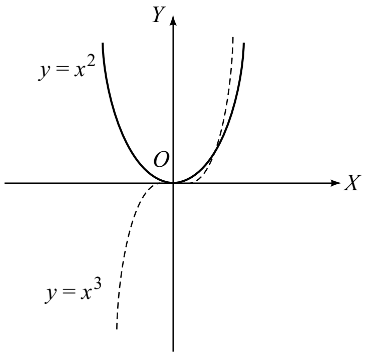


Fig. 1.16

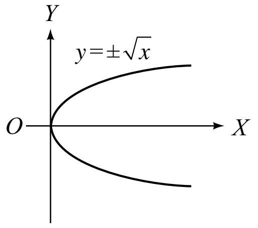


Fig. 1.17

7. *Exponential function*. The function $y = a^x$, where a is a positive constant number, is called an *exponential function*. The argument x adopts *any real* values; as the function values *only positive numbers* are considered, because otherwise we will have a multi-valued function. So, the function $y = 81^x$ has at $x = 1/4$ four different values: $y = 3$, $y = -3$, $y = 3i$ and $y = -3i$. But we consider as the function value only $y = 3$. Graphs of an exponential function for $a = 2$ and $a = 1/2$ are shown on Fig. 1.18. All they are going through the point (0, 1). At $a = 1$ we have as a graph a straight line, parallel to x-axis, i.e. the function becomes a constant value, equal to 1. At $a > 1$ an exponential function increases, and at $0 < a < 1$ – decreases.

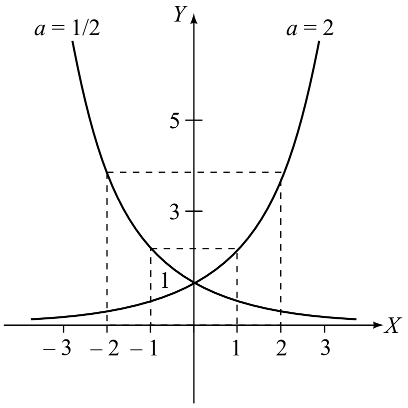


Fig. 1.18

The main characteristics and properties of an exponential function:

- the function domain: — $\infty < x < +\infty$ (i.e. x is any real number) and its codomain: $y > 0$;
- this is a monotone function: it increases at $a > 1$ and decreases at $0 < a < 1$;
- the function is unbounded, continuous in everywhere, non-periodic;
- the function has no zeros.

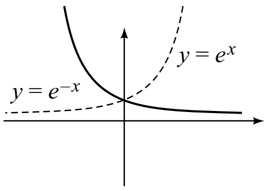


Fig. 1.19

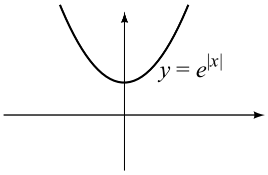


Fig. 1.20

8. *Logarithmic function*. The function $y = \log_a x$, where a is a positive constant number, not equal to 1 is called a *logarithmic function*. This is an inverse function relatively to an exponential function; its graph (Fig.1.19) can be obtained by rotating a graph of an exponential function around of a bisector of the 1-st coordinate angle.

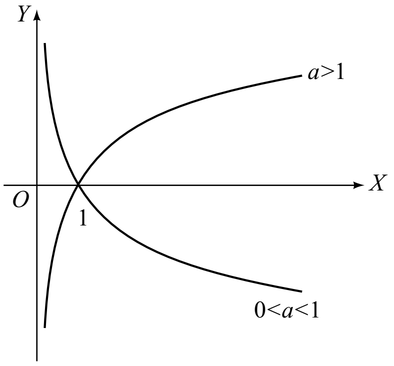


Fig. 1.21

The main characteristics and properties of a logarithmic function:

- the function domain: $x > 0$ and its codomain: $-\infty < y < +\infty$ (i.e. y is any real number);
- this is a monotone function: it increases for $a > 1$ and decreases for $0 < a < 1$;
- the function is unbounded, continuous in everywhere, non-periodic;
- the function has one zero: $x = 1$.

9. *Absolute-value function* is given by

$$y = |x| = \max\{x, -x\} = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

It is depicted in Fig. 1.22.

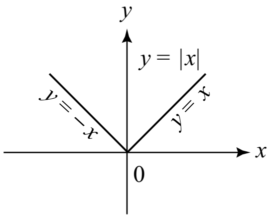


Fig. 1.22

Greatest Integer Function and **Fractional part function** $f(x) = [x]$ will denote the integral part of x or greatest integer less than or equal to x. E.g. $[5] = 5$, $[5.2]$, $= 5$, $[-5.1] = -6$, $[\pi] = 3$, $[e] = 2$. The domain of this function is **R** and the range is **I**. For positive x, this function might represent, for example, the legal age of a person is a function of his chronological age x.

The *fractional part* of x is the fractional part $\{x\}$ of x so $\{x\} = x - [x]$ e.g. $\{5.04\} = 0.04$, $\{-4\} = 0$, $\{-1.7\} = 0.3$, $\{\pi\} = \pi - 3$. The domain of this function is **R** whereas the range is $[0, 1)$.

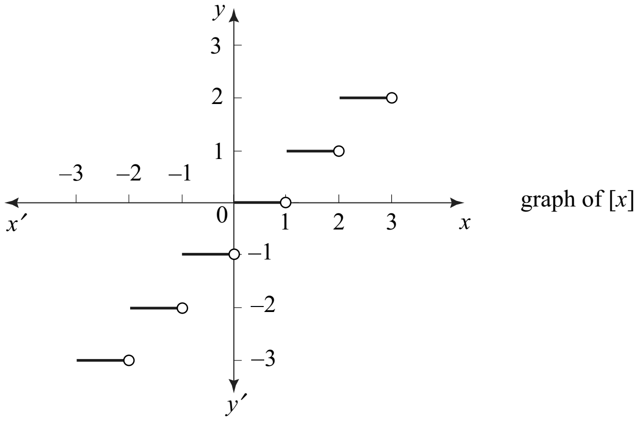


Fig. 1.23

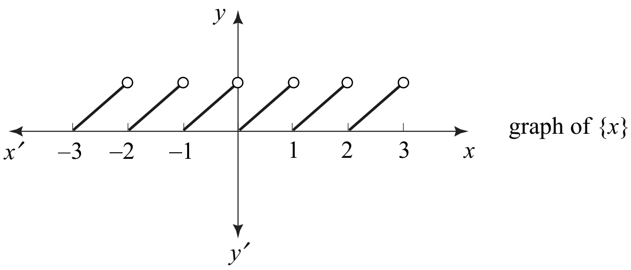


Fig. 1.24

Some Properties

1. $x - 1 < [x] \leq x$
2. $[\{x\}] = 0 = \{[x]\}$
3. If $n \leq x < n + 1$, $n \in$ **I**, then $[x] = n$ and conversely.
4. If $x \in$ I, then $[x] = x$ otherwise $[x] < x$.
5. $[-x] = \begin{cases} -[x], & \text{if } x \in \mathbf{I} \\ -[x]-1, & \text{if } x \notin \mathbf{I} \end{cases}$

$$\text{in other word, } [x] + [-x] = \begin{cases} 0, & \text{if } x \in \mathbf{I} \\ -1, & \text{if } x \notin \mathbf{I} \end{cases}$$

6. $[x + y] = [x] + [y]$, if x or $y \in \mathbf{I}$.

7. $[x + y] = \begin{cases} [x]+[y], & \text{if } \{x\}+\{y\}<1 \\ [x]+[y]+1, & \text{if } \{x\}+\{y\}\geq 1 \end{cases}$

   Moreover, $[x + y] \geq [x] + [y]$.

8. $\{x\} = \begin{cases} x, & \text{if } 0<x<1 \\ 0, & x\in \mathbf{I} \end{cases}$

9. $\{x\} + \{-x\} = \begin{cases} 0, & x\in \mathbf{I} \\ 1, & \text{if } x\notin \mathbf{I} \end{cases}$

10. $[x]+\left[x+\frac{1}{n}\right]+\cdots+\left[x+\frac{n-1}{n}\right]=[n\,x]$.

11. *Signum function* is given by

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and it is depicted in Fig. 1.25

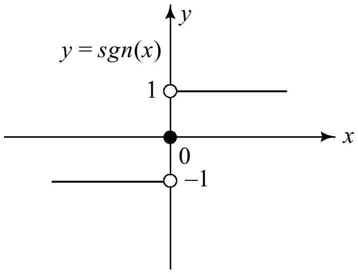


Fig. 1.25

12. The *least integer function*: The function whose value at any number x is the smallest integer greater than or equal to x is called *the least integer function*, e.g. if $f(x) = [x]'$ = smallest integer, greater than or equal to x.

   then $[1.5]' = 2$ $[1.99]' = 1$ $[1.4]' = 2$.

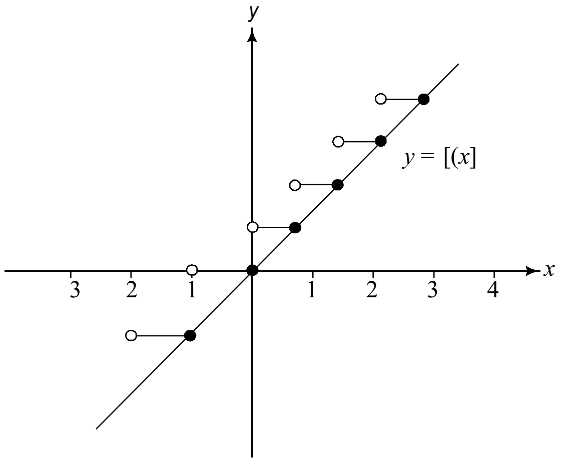


Fig. 1.26

Figure 1.26 shows the graph. For positive value of x, this function might represent for example, the cost of parking x hours in a parking lot which charges Rs 10 for each hour or part of an hour in this case $f(x) = 10\,[x]'$.

INVERSE OF A FUNCTION

The function $f(x) = x^{1/3}$ and $g(x) = x^3$ have the following property:

$$f(g(x)) = f(x^3) = (x^3)^{1/3} = x$$
$$g(f(x)) = g(x^{1/3}) = (x^{1/3})^3 = x$$

Similarly, the functions $f(x) = \log_e x$ and $g(x) = e^x$ cancel the effect of each other. If two functions f and g satisfy $f(g(x)) = x$ for every x in the domain of g, and $g(f(x)) = x$ for every x in the domain of f, we say that *f is the inverse of g and g is the inverse of f*, and we write $f = g^{-1}$, or $g = f^{-1}$. (**The symbol f^{-1} does not mean $1/f$.**) To find the inverse of f, write down the equation $y = f(x)$ and then solve x as a function of y. The resulting equation is $x = f^{-1}(y)$.

If f is one-to-one, then f has an inverse defined on its range and, conversely, if f has an inverse, then f is one-to-one. f^{-1} is defined on the range of f. If f is one-to-one from A to B, and g is one-to-one from B to C, then $f\,o\,g$ is one-to-one from A to C, and $(f\,o\,g)^{-1} = g^{-1}\,o\,f^{-1}$.

If $f : X \to Y$ is bijective then $f^{-1} : Y \to X$ is bijective and $\left(f^{-1}\right)^{-1} = f$. Moreover, $f^{-1}(\{y\}) = \{f^{-1}(y)\}$ for all $y \in Y$.

Illustration 9

Let $f(x) = \dfrac{ax+b}{cx+d}$. The domain of f is $\mathbf{R} \sim \left\{-\dfrac{d}{c}\right\}$.

Let $y = \dfrac{ax+b}{cx+d}$ so $cxy + dy = ax + b \Rightarrow x = \dfrac{b-dy}{cy-a}$.

Hence $f^{-1}(x) = \dfrac{b-dx}{cx-a}$.

DIRECT AND INVERSE IMAGES

Let f be a function with domain X and co-domain Y. If $A \subseteq X$, then the direct image of A under f is the subset of Y (denoted by $f(A)$) is defined to be $\{f(x) : x \in A\}$.

For example, let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = x^2$. If $A = \{-3, -1, 0, 1, 3\}$, then $f(A) = \{0, 1, 9\}$.

Note that if $f(X) = Y$, then the function $f : X \to Y$ is onto. The following relations can be verified easily.

(i) $A \subset B \Rightarrow f(A) \subset f(B)$

(ii) $f(A \cup B) = f(A) \cup f(B)$

(iii) $f(A \cap B) \subset f(A) \cap f(B)$. The inclusion may be strict.

Let f be a function with domain X and range Y and let B be a subset of Y. The inverse image of B under f is a subset of X (denoted by $f^{-1}(B)$) is defined to be $\{x : f(x) \in B\}$.

Illustration 10

If f: $\mathbf{R} \to \mathbf{R}$ is a function defined by $f(x) = |x|$ and $A = (-\infty, 0)$, then $f^{-1}(A) = \{x: f(x) \in A\} = \{x: |x| \in (-\infty, 0)\} = \phi$ (the empty set). If $B = (-3, 3)$, then $f^{-1}(B) = (-3, 3)$.

Note that if f is one-to-one, then $f^{-1}(\{x\})$ is a singleton set. The following relations can be verified easily for $A, B \subset Y$.

(i) $A \subset B = f^{-1}(A) \subset f^{-1}(B)$
(ii) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
(iii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: Let **I** be set of integers, **N** = the set of non-negative integers, N_p = the set of non-positive integers. Then the sets A and B satisfying $A \cap B = \phi$ are

(a) $A = I \sim N_p$, $B = N \sim N_p$
(b) $A = I \sim N$, $B = I \sim N_p$
(c) $A = N \Delta N_p$, $B = I \sim N_p$
(d) $A = N \Delta N_p$, $B = (I \sim N) \cup \{0\}$

Ans. (b)

Solution: $I \sim \mathbf{N} = \{\ldots -3, -2, -1\}$, $I \sim N_p = \{1, 2, 3\ldots\}$, $\mathbf{N} \sim N_p = \{1, 2, \ldots\}$, $N \Delta N_p = (\mathbf{N} \sim N_p) \cup (N_p \sim \mathbf{N})$

$$= \{0, 1, 2, \ldots\}$$

$$(I \sim \mathbf{N}) \cup \{0\} = \{\ldots, -3, -2, -1, 0\}$$

The disjoint sets are $\mathbf{I} \sim \mathbf{N}$ and $\mathbf{I} \sim N_p$.

Example 2: Which of the following equality is not true.

(a) $A \cap (B \sim C) = A \cap B \sim (A \cap C)$
(b) $A \sim (A \cap B) = A \sim B$
(c) $A \sim (B \sim C) = (A \sim B) \cup (A \cap C)$
(d) $A \sim (B \Delta C) = (A \sim B) \Delta (A \sim C)$

Ans. (d)

Solution: For equality (a),

$$\begin{aligned} A \cap (B \sim C) &= A \cap (B \cap C') \\ &= \phi \cup (A \cap B \cap C') \\ &= (A \cap B \cap A') \cup (A \cap B \cap C') \\ &= A \cap B \cap (A' \cup C') \\ &= A \cap B \sim (A \cap C) \end{aligned}$$

For equality (b),

$$\begin{aligned} A \sim (A \cap B) &= A \cap (A' \cup B') \\ &= (A \cap A') \cup (A \cap B') \\ &= \phi \cup (A \cap B') = A \cap B' = A \sim B \end{aligned}$$

For equality (c)

$$\begin{aligned} A \sim (B \sim C) &= A \sim (B \cap C') = A \cap (B' \cup C) \\ &= (A \cap B') \cup (A \cap C) \\ &= (A \sim B) \cup (A \cap C) \end{aligned}$$

For (d) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5\}$, $C = \{1, 2, 3\}$

So, $B \Delta C = \{4, 5\} \cup \{1, 2\} = \{1, 2, 4, 5\}$

Thus $A \sim (B \Delta C) = \{3\}$

$$A \sim B = \{1, 2\}; A \sim C = \{4, 5\}$$

Therefore $(A \sim B) \Delta (A \sim C) = (\{1, 2\} \sim \{4, 5\}) \cup (\{4, 5\} \sim \{1, 2\})$

$$= \{1, 2\} \cup \{4, 5\} = \{1, 2, 4, 5\}.$$

Hence $A \sim (B \Delta C) \neq (A \sim B) \Delta (A \sim C)$

Example 3: A boating club consists of 82 members, each member is either a sailboat owner or a powerboat owner. If 53 members owned sailboats and 38 members owned powerboats, the number of members owned both sailboat and powerboat is

(a) 6 (b) 7
(c) 9 (d) 4

Ans. (c)

Solution: Let S = the set of all members owning sailboats and P = the set of all members owning powerboats

$$n(S \cup P) = n(S) + n(P) - n(S \cap P)$$

$$82 = 53 + 38 - n(S \cap P)$$

$$\Rightarrow \quad n(S \cap P) = 91 - 82 = 9.$$

Example 4: If, $B \underset{\neq}{\subset} A'$, then which of the following sets is equal to A'.

(a) $(A \cap B) \cup B$ (b) $(A \cap B) \cup A'$
(c) $(A \cup B) \cap A'$ (d) $(A \cup B) \cap B$

Ans. (b)

Solution: For (a), $(A \cap B) \cup B = (A \cup B) \cap (B \cup B)$

$$= (A \cup B) \cap B = B.$$

For (b), $(A \cap B) \cup A' = (A \cup A') \cap (B \cup A')$

$$= X \cap A' = A'$$

For (c), $(A \cup B) \cap A' = (A \cap A') \cup (B \cap A')$

$$= \phi \cup (B \cap A')$$

$$= B \cap A' = B.$$

For (d) $(A \cup B) \cap B = B$.

Example 5: If $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] = \frac{5x}{6}$, then x is any term of the following

(a) 3, 6, 9, 12,… (b) 9, 18, 27, 36, …
(c) 6, 12, 18, 24,… (d) $\frac{6}{5}, \frac{12}{5}, \frac{18}{5}, \ldots$

Ans. (c)

◎ **Solution:** Since $\left[\frac{x}{2}\right], \left[\frac{x}{3}\right] \in \boldsymbol{I}$, so $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] \in \boldsymbol{I}$

Thus $\frac{5x}{6} \in \boldsymbol{I} \Rightarrow x = \frac{6}{5}n, n \in \boldsymbol{I}$.

Substituting this value in $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] = \frac{5x}{6}$, we have

$$\left[\frac{3}{5}n\right] + \left[\frac{2}{5}n\right] = n$$

$$\Rightarrow \quad \frac{3}{5}n - \left\{\frac{3}{5}n\right\} + \frac{2}{5}n - \left\{\frac{2}{5}n\right\} = n$$

$$\Rightarrow \quad \left\{\frac{3}{5}n\right\} + \left\{\frac{2}{5}n\right\} = 0$$

$$\Rightarrow \quad \left\{\frac{3}{5}n\right\} = 0 = \left\{\frac{2}{5}n\right\} \qquad (\text{Since } 0 \le \{x\} < 1)$$

Thus $3n = 5m_1$, $2n = 5m_2$, Therefore $xn = \frac{2n \cdot 3n}{5}$

$$= \frac{5m_1 \times 5m_2}{5} = 5m_1\, m_2 \Rightarrow \frac{x \cdot 5m_1}{3} = 5m_1\, m_2$$

$$\Rightarrow \quad x = 3m_2$$

Similarly $\frac{x \cdot 5m_2}{2} = 5m_1\, m_2 \Rightarrow x = 2m_1$

Hence x is multiple of 2 and 3 so of 6 and $x \in \mathbf{I}$

◉ **Example 6:** The relation R defined by '>' on the set **N** is

(a) reflexive (b) symmetric
(c) transitive (d) equivalence relation

Ans. (c)

◎ **Solution:** $2 \not> 2$ so > is not reflexive, $3 > 2$ but $2 \not< 3$ so $(3, 2) \in R$ but $(2, 3) \notin R$. Thus R is not symmetric. If $(a, b) \in R$ and $(b, c) \in R$ then $a > b$, $b > c \Rightarrow a > c$ so $(a, c) \in R$. R is not an equivalence relation.

◉ **Example 7:** The relation $a\, R\, b$ defined by a is factor of b on $\boldsymbol{N}$ is not

(a) reflexive (b) transitive
(c) anti symmetric (d) symmetric

Ans. (d)

◎ **Solution:** For $a \in \boldsymbol{N}$, a is a factor of a so R is reflexive. If a is factor b and b is factor of c then a is factor of c so R is transitive. If a is factor of b and b is factor of a then $a = b$ so R is anti symmetric, 2 is factor of 4 but 4 is not a factor of 2.

◉ **Example 8:** The domain of the function $f(x) = \log_2 \sin x$ is

(a) **R** (b) $\mathbf{R} \sim \{n\,\pi : n \in \mathbf{I}\}$
(c) $R \sim \{n\pi : n \in \mathbf{N}\}$ (d) $\bigcup\limits_{n \in \mathbf{I}} (2n\pi, (2n+1)\pi)$

Ans. (d)

◎ **Solution:** $f(x) = \log_2 \sin x$ is defined for all x for which $\sin x > 0$. But $\sin x > 0$ if $x \in (0, \pi) \cup (2\pi, 3\pi) \cup \ldots = \bigcup\limits_{n \in \mathbf{I}} (2n\pi, (2n+1)\pi)$.

◉ **Example 9:** The domain of $y = \cos^{-1} \frac{1-2x}{4}$ is

(a) $\left[-\frac{3}{2}, \frac{5}{2}\right]$ (b) [–1, 1]
(c) [0, 2] (d) $\left[-\frac{1}{2}, \frac{3}{2}\right]$

Ans. (a)

◎ **Solution:** The given function is defined if

$$-1 \le \frac{1-2x}{4} \le 1 \quad \text{i.e. if } -4 \le 1 - 2x \le 4,$$

$$\Rightarrow -5 \le -2x \le 3 \Rightarrow \frac{-3}{2} \le x \ge \frac{5}{2}$$

◉ **Example 10:** Which of the following functions is bounded

(a) $y = 1 - \log_{10}x$ (b) $y = e^{-2x}$
(c) $y = \sin^{-1}(2x + 1)$ (d) $y = \tan(4x + 1)$

Ans. (c)

◎ **Solution:** The range $\log_{10}x$ is $(-\infty, \infty)$ so $y = 1 - \log_{10}x$ is unbounded. $y = e^{-2x}$ is unbounded from below as $x \to -\infty$, $y \to \infty$. The range of $\sin^{-1}x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ so $y = \sin^{-1}(2x + 1)$ is a bounded function. The range of $\tan x$ is **R** so $y = \tan(4x + 1)$ is unbounded

◉ **Example 11:** A function out of the following whose period is not π is

(a) $\sin^2 x$ (b) $\cos^2 x$
(c) $\tan(2x + 3)$ (d) $y = |\sin x|$

Ans. (c)

◎ **Solution:** $y = \sin^2 x = \frac{1}{2}[1 - \cos 2x]$. Since period of $\cos x$ is 2π so period of $\cos 2x$ is π.

$y = 1 + \cos^2 x = 1 + \frac{1}{2}(1 + \cos 2x)$. The period of this is again π. The period of $\tan x$ is π so period of $\tan(2\pi + 3)$ is $\pi/2$. If $f(x) = |\sin x|$ then $f(x + \pi) = |\sin(x + \pi)| = |-\sin x| = |\sin x| = f(x)$. Thus the period of f is π.

◉ **Example 12:** Which of the following functions is an odd function

(a) $y = x^4 - 2x^2$ (b) $y = x - x^2$
(c) $y = \cos x$ (d) $y = x - \frac{x^3}{6} + \frac{x^5}{40}$

Ans. (d)

◎ **Solution:** If $f(x) = x^4 - 2x^2$ then $f(-x) = (-x)^4 - 2(-x)^2 = x^4 - 2x^2 = f(x)$. Hence f is an even function. If $u(x) = x - x^2$ then $u(-x) = x - x^2$ so u is neither even nor an odd function. $p(x) = \cos x$, $p(-x) = \cos(-x) = \cos x = p(x)$ so p is even function. If $s(x) = x - \frac{x^3}{6} + \frac{x^5}{40}$ then $s(-x) = -x - \frac{x^3}{6} - \frac{x^5}{40} = -\left(x - \frac{x^3}{6} + \frac{x^5}{40}\right) = -s(x)$, so s is an odd function.

LEVEL 1

Straight Objective Type Questions

Example 13: Let $X = \{x : x = n^3 + 2n + 1, n \in \mathbf{N}\}$ and $Y = \{x : x = 3n^2 + 7, n \in \mathbf{N}\}$ then

(a) $X \cap Y$ is a subset of $\{x : x = 3n + 5, n \in \mathbf{N}\}$
(b) $X \cap Y \subseteq \{x : x = n^2 + n + 1, n \in \mathbf{N}\}$
(c) $34 \in X \cap Y$
(d) none of these

Ans. (c)

Solution: If $n^3 + 2n + 1 = 3n^2 + 7$

$$\Rightarrow \quad n^3 - 3n^2 + 2n - 6 = 0$$
$$\Rightarrow \quad (n - 3)(n^2 + 2) = 0$$
$$\Rightarrow \quad n = 3 \text{ as } n \in \mathbf{N}$$
$$\text{So,} \quad x = 3 \times 3^2 + 7 = 34 \in X \cap Y.$$

In (a) and (b) $x \neq 34$, for any $n \in \mathbf{N}$.

Example 14: A, B, C are the sets of letters needed to spell the words STUDENT, PROGRESS and CONGRUENT, respectively. Then $n(A \cup (B \cap C))$ is equal to

(a) 8
(b) 9
(c) 10
(d) 11

Ans. (b)

Solution:
$$A = \{D, E, N, S, T, U]$$
$$B = \{E, G, O, P, R, S\}$$
$$C = \{C, E, G, N, O, R, T, U\}$$

So $A \cup (B \cap C)$
$$= \{D, E, N, S, T, U\} \cup \{E, G, O, R\}$$
$$= \{D, E, G, O, N, R, S, T, U\}$$

and $n[(A \cup (B \cap C)] = 9$.

Example 15: Let $A = \{x : x$ is a prime factor of $240\}$
$B = \{x : x$ is the sum of any two prime factors of $240\}$.

Then

(a) $5 \notin A \cap B$
(b) $7 \in A \cap B$
(c) $8 \in A \cap B$
(d) $8 \in A \cup B$

Ans. (d)

Solution: $240 = 2 \times 3 \times 5 \times 8$

So $A = \{2, 3, 5\}$, $B = \{5, 7, 8\}$.

Clearly $8 \in A \cup B$.

Example 16: A, B, C are three sets such that $n(A) = 25$, $n(B) = 20$, $n(c) = 27$, $n(A \cap B) = 5$, $n(B \cap C) = 7$ and $A \cap C = \phi$ then $n(A \cup B \cup C)$ is equal to

(a) 60
(b) 65
(c) 67
(d) 72.

Ans. (a)

Solution: $A \cap C = \phi \quad \Rightarrow \quad A \cap B \cap C = \phi$.

$$\therefore \; n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$
$$= 25 + 20 + 27 - 5 - 7 - 0 + 0 = 60.$$

Example 17: Let $X = \{(x,y,z) \mid x,y,z \in \mathbf{N}.\ x + y + z = 10,\ x < y < z\}$ and $Y = \{(x,y,z) \mid x,y,z \in \mathbf{N},\ y = |x - z|\}$ then $X \cap Y$ is equal to

(a) $\{(2,3,5)\}$
(b) $\{1,4,5\}$
(c) $\{5,1,4\}$
(d) $\{(2,3,5), (1,4,5)\}$

Ans. (d)

Solution: $X = \{(1,2,7), (1,3,6), (1,4,5), (2,3,5)\}$.
Elements of X which belong to Y are $(1,4,5)$ and $(2,3,5)$ both so they belong to $X \cap Y$.

Example 18: If A,B,C are three non-empty sets such that $A \cap B = \phi$, $B \cap C = \phi$, then

(a) $A = C$
(b) $A \subset C$
(c) $C \subset A$
(d) none of these

Ans. (d)

Solution: Let $A = \{1,2,3,4,5\}$, $B = \{6,7,8,9\}$ and $C = \{11,12,13\}$ which satisfy the given conditions but none of (a), (b) or (c).

Example 19: Two finite sets have m and n elements respectively. The total number of subsets of first set is 56 more than the total number of subsets of the second set. The values of m and n respectively are

(a) 7, 6
(b) 6, 3
(c) 5, 1
(d) 8, 7

Ans. (b)

Solution: According to the given condition, we have $2^m = 2^n + 56$
$\Rightarrow 2^{m-3} - 2^{n-3} = 7 \Rightarrow 2^{n-3}(2^{m-n} - 1) = 7$. Since 7 is a prime number so we must have $n - 3 = 0$ (clearly $m \neq n$). Thus $n = 3$. Therefore, $2^m = 2^3 + 56 = 64 = 2^6 \Rightarrow m = 6$.

Example 20: Among employee of a company taking vacations last years, 90% took vacations in the summer, 65% in the winter, 10% in the spring, 7% in the autumn, 55% in winter and summer, 8% in the spring and summer, 6% in the autumn and summer, 4% in the winter and spring, 4% in winter and autumn, 3% in the spring and autumn, 3% in the summer, winter and spring, 3% in the summer, winter and autumn, 2% in the summer, autumn and spring, and 2% in the winter, spring and autumn. Percentage of employee that took vacations during every season:

(a) 4 (b) 3
(c) 2 (d) 8

Ans. (c)

◎ **Solution:** Suppose that number of employee taking vacations is 100.

Su – set of employee taking leave in Summer
W – set of employee taking leave in Winter
Sp – set of employee taking leave in Spring
A – set of employee taking leave in Autumn

$$n(Su) = 90,\ n(W) = 65,\ n(Sp) = 10,\ n(A) = 7$$
$$n(W \cap Su) = 55,\ n(Sp \cap Su) = 8,\ n(A \cap Su) = 6$$
$$n(W \cap Sp) = 4,\ n(W \cap Au) = 4,\ n(Sp \cap A) = 3$$
$$n(Su \cap A) = 3,\ n(Su \cap W \cap A) = 3$$
$$n\ (Su \cap W \cap Sp) = 3,\ n(Su \cap A \cap Sp) = 2$$
$$n(W \cap Sp \cap A) = 2$$

$$\begin{aligned} n(Su \cap Sp \cap W \cap A) &= n(Su) + n(Sp) + n(W) + n(A) - n(Su \cap Sp) \\ &\quad - n(Sp \cap W) - n(W \cap A) - n(Su \cap A) - n(Su \cap W) \\ &\quad - n(Sp \cap A) + n(Su \cap Sp \cap W) + n(Su \cap W \cap A) \\ &\quad + n(W \cap A \cap Su) + n(Su \cap Sp \cap A) \\ &\qquad - n(Sp \cup Su \cup A \cup W) \\ &= 90 + 65 + 10 + 7 - 55 - 8 - 6 - 4 - 4 - 3 \\ &\qquad + 3 + 3 + 2 + 2 - 100 = 2 \end{aligned}$$

◉ **Example 21:** If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$ then the number of elements in $(A \cup B) \times (A \cap B) \times (A \,\Delta\, B)$ is

(a) 5 (b) 30
(c) 10 (d) 4

Ans. (b)

◎ **Solution:** $A \cup B = \{1, 2, 3, 4, 5\}$. $n(A \cup B) = 5$

$$A \cap B = \{3, 4\},\ n(A \cap B) = 2$$
$$A \,\Delta\, B = (A \sim B) \cup (B \sim A) = \{1, 2\} \cup \{5\} = \{1, 2, 5\}$$
$$n(A \,\Delta\, B) = 3. \text{ Hence } n((A \cup B) \times (A \cap B) \times (A \,\Delta\, B)\} = 5 \times 2 \times 3 = 30.$$

◉ **Example 22:** Let **I** be the set of integers. For $a, b \in \mathbf{I}$, $a\,R\,b$ if and only if $|a - b| < 1$, then

(a) R is not reflexive
(b) R is not symmetric
(c) $R = \{(a,a);\ a \in \mathbf{I}\}$
(d) R is not an equivalence relation.

Ans. (c)

◎ **Solution:** For any integers a, b, $|a - b| < 1$ if and only if $|a - b| = 0$ so $a = b$. Hence $R = \{(a,a);\ a \in \mathbf{I}\}$.Thus R is reflexive, symmetric and transitive.

◉ **Example 23:** Let W denote the words in the English Dictionary. Define the relation **R** by $\mathbf{R} = \{(x, y) \in W \times W$: the words x and y have at least one letter common$\}$, then **R** is

(a) reflexive, not symmetric and transitive
(b) not reflexive, symmetric and transitive
(c) reflexive, symmetric and not transitive
(d) reflexive, symmetric and transitive

Ans. (c)

◎ **Solution:** $(x, x) \in \mathbf{R}\quad \forall\ x \in W$ as all letters in both are common. If $(x, y) \in \mathbf{R}$ then x and y have a letter in common $\Rightarrow (y, x) \in \mathbf{R}$.

Next, let x = fix, y = six and z = son then$(x, y) \in \mathbf{R}$, $(y, z) \in \mathbf{R}$ but $(x, z) \notin \mathbf{R}$

So **R** is reflexive, symmetric but not transitive

◉ **Example 24:** If the relation $R : A \to B$, where $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$ is defined by $R = \{(x, y);\ x < y,\ x \in A,\ y \in B\}$ then RoR^{-1} is

(a) $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
(b) $\{(3, 1), (5, 1), (5, 2), (5, 3), (5, 4)\}$
(c) $\{(3, 3), (3, 5), (5, 3), (5, 5)$
(d) none of these

Ans. (c)

◎ **Solution:** $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$

and $\quad R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$.

Thus $\quad RoR^{-1} = \{(3, 3), (3, 5), (5, 3), (5, 5)\}$.

◉ **Example 25:** Let $A = \{x \in \mathbf{R} : [x + 3] + [x + 4] \le 3\}$ and $B = \left\{x \in \mathbf{R} : 3^x \left(\sum_{r=1}^{\infty} \frac{3}{10^r}\right)^{x-3} < 3^{-3x}\right\}$ then

(a) $A = B$ (b) $A \underset{\neq}{\subset} B$
(c) $B \underset{\neq}{\subset} A$ (d) $A \cap B = \phi$

Ans. (a)

◎ **Solution:** Let $x \in A$, $[x + 3] + [x + 4] \le 3$

$$\Rightarrow \quad [x] + 3 + [x] + 4 \le 3$$
$$\Rightarrow \quad 2[x] \le -4 \ \Rightarrow\ [x] \le -2$$
$$\Rightarrow \quad x \in (-\infty, -1)$$
$$A = (-\infty, -1)$$

If $x \in B$ then $3^x\, 3^{x-3}\left(\sum_{r=1}^{\infty} \frac{1}{10^r}\right)^{x-3} < 3^{-3x}$

$$\Rightarrow \quad 3^{2x-3}\left(\frac{1/10}{1-1/10}\right)^{x-3} < 3^{-3x}$$
$$\Rightarrow \quad 3^{2x-3}\left(3^{-2}\right)^{x-3} < 3^{-3x}$$
$$\Rightarrow \quad 3^3 < 3^{-3x} \ \Rightarrow\ 3 < -3x$$

so $x \in (-\infty, -1)$

Hence $B = (-\infty, -1)$. Thus $A = B$.

◉ **Example 26:** The range of the function $f(x) = {}^{7-x}P_{x-3}$ is

(a) $\{1, 2, 3, 4\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
(c) $\{1, 2, 3\}$ (d) $\{1, 2, 3, 4, 5\}$

Ans. (c)

Solution: $7 - x \geq 1, x - 3 \geq 0$

and $\quad 7 - x \geq x - 3$

$\Rightarrow \quad x \leq 6, x \geq 3, x \leq 5.$

Thus $\quad 3 \leq x \leq 5$

$$\therefore \quad \text{Range} = \{4P_0, 3P_1, 2P_2\} = \{1, 3, 2\}$$

Example 27: The domain of the function

$f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

(a) [1, 2] (b) [2, 3)

(c) [1, 3) (d) [1, 2]

Ans. (b)

Solution: $-1 \leq x - 3 \leq 1$ and $9 - x^2 > 0$

$\Rightarrow \quad 2 \leq x \leq 4$ and $-3 < x < 3$.

So domain of f is [2, 3).

Example 28: The solution of $8x \equiv 6 \pmod{14}$ is

(a) [8], [6] (b) [6], [14]

(c) [6], [13] (d) [8], [14], [16]

where $[a] = \{a + 14k : k \in \mathbf{I}\}$

Ans. (c)

Solution: We need to solve $14|(8x - 6)$ i.e., $8x - 6 = 14k$, for $k \in I$. The values 6 and 13 satisfy this equation, while 8, 14 and 16 do not.

Example 29: Let $R = \{(x, y): x, y \in A, x + y = 5\}$, where $A = \{1, 2, 3, 4, 5\}$ then

(a) R is not reflexive, symmetric and not transitive

(b) R is an equivalence relation

(c) R is reflexive, symmetric but not transitive

(d) R is not reflexive, not symmetric but transitive

Ans. (a)

Solution: $R = \{(1, 4), (4, 1), (2, 3), (3, 2)\}$, so R is not reflexive as $(1, 1) \notin R$. R is symmetric by definition and R is not transitive as $(1, 4) \in R$, $(4, 1) \in R$ but $(1, 1) \notin R$.

Example 30: Let R be a relation on a set A such that $R = R^{-1}$ then R is

(a) reflexive (b) symmetric

(c) transitive (d) an equivalence relation

Ans. (b)

Solution: If $(a, b) \in R$ then $(b, a) \in R^{-1} = R$ so R is symmetric. The relation in Example 29 satisfy $R = R^{-1}$ but is neither reflexive nor transitive.

Example 31: For $x, y \in \mathbf{R}$, define a relation R by $x\,R\,y$ if and only if $x - y + \sqrt{2}$ is an irrational number. Then R is

(a) reflexive (b) symmetric

(c) transitive (d) none of these

Ans. (a)

Solution: Since $x - x + \sqrt{2} = \sqrt{2}$ which is an irrational number so $x\,R\,x$ for all $x \in \mathbf{R}$. Hence R is reflexive. R is not symmetric as $(\sqrt{2}, 1) \in R$ but $(1, \sqrt{2}) \notin R$. Again R is not transitive since $(\sqrt{2}, 1) \in R$ and $(1, 2\sqrt{2}) \in R$ but $(\sqrt{2}, 2\sqrt{2}) \notin R$.

Example 32: If $n|q$ and $A = \{z \in C : z^n = 1\}$, $B = \{z : z^q = 1\}$ then

(a) $A = B$ (b) $A \cap B = \{1\}$

(c) $B \subseteq A$ (d) $A \subsetneq B$

Ans. (d)

Solution: $q = p\,n$ for some $p \in \mathbf{N}$

$$z^q - 1 = (z^n)^p - 1 = (z^n - 1)\,(z^{(p-1)n} + \cdots + z^n + 1)$$

Every root of $z^n - 1$ is a root of $z^q - 1$ and every root of $z^{(p-1)n} + \cdots + z^n + 1 = 0$ is also a root of $z^q - 1$. Hence $A \subsetneq B$ and $A \cap B = A$.

Example 33: If $A = \{z : (1 + 2i)\,\bar{z} + (1 - 2i)\,z + 2 = 0\}$ and $B = \{z : (3 + 2i)\,\bar{z} + (3 - 2i)\,z + 3 = 0\}$, then

(a) $A \cap B$ is a singleton set

(b) $A \subseteq B$

(c) $B \subseteq A$

(d) $A \cap B = \phi$.

Ans. (a)

Solution: Equations in sets A and B represent straight line with $\bar{\alpha}_1 = 1 + 2i$ and $\bar{\alpha}_2 = 3 + 2i$. Since $\dfrac{\bar{\alpha}_1}{\alpha_1} \neq \dfrac{\bar{\alpha}_2}{\alpha_2}$ so the lines are intersecting, hence $A \cap B$ is a singleton set.

Example 34: Let $x, y \in \mathbf{I}$ and suppose that a relation R on $\mathbf{I}$ is defined by $x\,R\,y$ if and only if $x \leq y$ then

(a) R is reflexive but not symmetric

(b) R is an equivalence relation

(c) R is neither reflexive nor symmetric

(d) R is symmetric but not transitive

Ans. (a)

Solution: Since $x \leq x$ for all $x \in \mathbf{I}$ so R is reflexive but is not symmetric as $(1, 2) \in R$ and $(2, 1) \notin R$. Also R is transitive as $x \leq y, y \leq z \Rightarrow x \leq z$.

Example 35: If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 + 1$, then value of $f^{-1}(17)$ and $f^{-1}(-3)$ are, respectively,

(a) ϕ, {4, –4} (b) ϕ, {3, –3}

(c) {3, –3}, ϕ (d) {4, –4}, ϕ

Ans. (d)

Solution: For any $A \subseteq \mathbf{R}$, we have $f^{-1}(A) = \{x \in \mathbf{R} : f(x) \in A\}$. Thus,

$f^{-1}(17) = \{x : f(x) \in \{17\}\} = \{x : f(x) = 17\}$

$= \{x : x^2 + 1 = 17\} = \{4, -4\}$,

and similarly, $f^{-1}(-3) = \{x \in \mathbf{R} : x^2 + 1 = -3\} = \phi$.

Example 36: The functions f and g are given by $f(x) = \{x\}$, the fractional part of x and $g(x) = \frac{1}{2}\sin [x]\pi$, where $[x]$ denotes the integral part of x. Then range of gof is

(a) $[-1, 1]$ (b) $\{0\}$
(c) $\{-1, 1\}$ (d) $[0, 1]$

Ans. (b)

Solution: $(gof)(x) = g(f(x)) = 1/2 \sin [\{x\}]\pi = 0$, for all $x \in \mathbf{R}$. Hence the range of gof is $\{0\}$.

Example 37: The period of the function $f(x) = \cos^2 3x + \tan 4x$ is

(a) $\pi/3$ (b) $\pi/4$
(c) $\pi/6$ (d) π

Ans. (d)

Solution: $f(x) = (1/2)(1 + \cos 6x) + \tan 4x$. The period of $\cos 6x$ is $2\pi/6 = \pi/3$ and the period of $\tan 4x$ is $\pi/4$. Hence the period of f is l.c.m. of $\pi/3$ and $\pi/4 = \pi$.

Example 38: The domain of the function

$$f(x) = \sin^{-1}\left(\log_3 \frac{x}{3}\right) \text{ is}$$

(a) $[-1, 9]$ (b) $[1, 9]$
(c) $[-9, 1]$ (d) $[3, 9]$

Ans. (b)

Solution: The function f is defined only if $-1 \le \log_3 (x/3) \le 1$. This inequality is possible only if $1/3 \le x/3 \le 3$ i.e., $1 \le x \le 9$.

Example 39: The domain of the function

$$f(x) = \frac{\sqrt{-\log_{0.3}(x-1)}}{\sqrt{-x^2 + 2x + 8}} \text{ is}$$

(a) $(1, 4)$ (b) $(-2, 4)$
(c) $(2, 4)$ (d) none of these

Ans. (c)

Solution: Since for, $0 < a < 1$, $\log_a x < 0$ for $x > 1$ so $\log_{0.3}(x - 1) < 0$ for $x > 2$. Also $-x^2 + 2x + 8 > 0$ if and only if $x \in (-2, 4)$. Hence the domain of the given function is $(2, 4)$.

Example 40: The function f: $[-1/2, 1/2] \to [-\pi/2, \pi/2]$ defined by $f(x) = \sin^{-1}(3x - 4x^3)$ is

(a) both one-one and onto
(b) neither one-one nor onto
(c) onto but not one-one
(d) one-one but not onto

Ans. (a)

Solution: Since $\sin^{-1}(3x - 4x^3) = 3\sin^{-1} x \in [-\pi/2, \pi/2]$ i.e., $\sin^{-1} x \in [-\pi/6, \pi/6]$ or $x \in [-1/2, 1/2]$ so f is onto. Also $f'(x) = \frac{3}{\sqrt{1-x^2}} > 0$ for $-1/2 < x < 1/2$. Therefore, f increases on $[-1/2, 1/2]$ and hence f is one-one.

Example 41: Given $f(x) = \frac{1}{\sqrt{|x| - x}}$ and $g(x) = \frac{1}{\sqrt{x - |x|}}$ then

(a) dom $f \ne \phi$ and dom $g = \phi$
(b) dom $f = \phi$ and dom $g \ne \phi$
(c) f and g have the same domain
(d) dom $f = \phi$ and dom $g = \phi$

Ans. (a)

Solution: dom $f = \{x : |x| > x\}$ and dom $g = \{x : x > |x|\} = \phi$. Thus dom $f = \mathbf{R}^-$ (the set of all negative real numbers) and dom $g = \phi$.

Example 42: Which of the following functions is not onto

(a) f: $\mathbf{R} \to \mathbf{R}$, $f(x) = 3x + 4$
(b) f: $\mathbf{R} \to \mathbf{R}^+$, $f(x) = x^2 + 2$
(c) f: $R^+ \to R^+$, $f(x) = \sqrt{x}$
(d) none of these

Ans. (b)

Solution: The function $f(x) = 3x + 4$ is onto as for $y \in \mathbf{R}$, $f\left(\frac{y-4}{3}\right) = y$. The function $f : \mathbf{R}^+ \to \mathbf{R}^+$, $f(x) = \sqrt{x}$ is onto as for $y \in \mathbf{R}^+$, $f(y^2) = y$. $f : \mathbf{R} \to \mathbf{R}^+$, $f(x) = x^2 + 2$ is not onto as $1 \in \mathbf{R}^+$ has no pre-image.

Example 43: Which of the following functions is not one-one

(a) f: $\mathbf{R} \to \mathbf{R}$, $f(x) = 2x + 5$
(b) f: $[0, \pi] \to [-1, 1]$, $f(x) = \cos x$
(c) f: $[-\pi/2, \pi/2] \to [1, 7]$ $f(x) = 3 \sin x + 4$
(d) f : $\boldsymbol{R} \to [-1, 1]$, $f(x) = \sin x$

Ans. (d)

Solution: The function in (*a*) is one-one as $2x_1 + 5 = 2x_2 + 5$ is possible only if $x_1 = x_2$. The function in (*b*) is one-one as $\cos x_1 = \cos x_2$ if and only if $\sin \frac{x_1 - x_2}{2} = 0$ i.e., $x_1 = x_2$. Similarly the function in (*c*) is also one-one. The function in (*d*) is not one-one as $f(\pi) = f(2\pi) = 0$.

Example 44: Which of the following functions is non-periodic

(a) $f(x) = x - [x]$

(b) $f(x) = \begin{cases} 1 \text{ if } x \text{ is a rational number} \\ 0 \text{ if } x \text{ is an irrational number} \end{cases}$

(c) $f(x) = \sqrt{\frac{8}{1 + \cos x} + \frac{8}{1 - \cos x}}$

(d) none of these

Ans. (d)

◎ **Solution:** The function in (a) is periodic with period 1 and the function in (b) is also periodic since $f(x + r) = f(x)$ for every rational r. The function in (c) is equal to $\dfrac{4}{|\sin x|}$ and thus has period π.

◉ **Example 45:** Let $f(x) = \dfrac{\alpha x}{x+1}$, $x \neq -1$. Then, for what value of α is $f(f(x)) = x$?

(a) $\sqrt{2}$ (b) $-\sqrt{2}$
(c) 1 (d) –1

Ans. (d)

◎ **Solution:** $f(f(x)) = \dfrac{\alpha f(x)}{f(x)+1} = \dfrac{\alpha^2 x}{(\alpha+1)x+1}$

$$\text{Thus } f(f(x)) = x \Leftrightarrow \alpha^2 x = (\alpha+1)x^2 + x$$
$$\Leftrightarrow (\alpha^2 - 1)x = (\alpha + 1)\, x^2$$
$$\Leftrightarrow (\alpha + 1)\,((\alpha - 1) - (\alpha + 1)x) = 0$$

Since $(\alpha - 1) - (\alpha + 1)\, x \neq 0$ for all x so $\alpha = -1$.

◉ **Example 46:** Let $R = \{(x, y): x, y \in \mathbf{R}, x^2 + y^2 \leq 25\}$

$$R' = \left\{(x, y): x, y \in \mathbf{R}, y \geq \frac{4}{9}x^2\right\} \text{ then}$$

(a) dom $R \cap R' = [-4, 4]$
(b) range $R \cap R' = [0, 4]$
(c) range $R \cap R' = [0, 5]$
(d) $R \cap R'$ defines a function.

Ans. (c)

◎ **Solution:** The equation $x^2 + y^2 = 25$ represents a circle with centre (0, 0) and radius 5 and the equation $y = \dfrac{4}{9}x^2$ represents a parabola with vertex (0, 0) and focus (0, 1/9). Hence $R \cap R'$ is the set of points indicated in the Fig.1.27

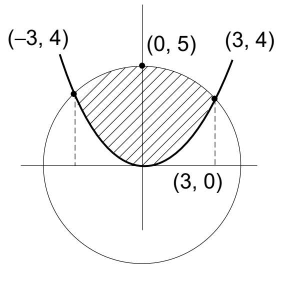


Fig. 1.27

$= \{(x, y): -3 \leq x \leq 3, 0 \leq y \leq 3]\}$. Thus
dom $R \cap R' = [-3, 3]$ and range $R \cap R' = [0, 5] \supset [0, 4]$
Since $(0, 0) \in R \cap R'$ and $(0, 5) \in R \cap R'$
$\therefore$ 0 is related to 0 as well as 5.
Hence $R \cap R'$ doesn't define a function.

◉ **Example 47:** In a factory 70% of the workers like oranges and 64% like apples. If x% like both oranges and apples, then

(a) $x \leq 34$ (b) $x \geq 64$
(c) $34 \leq x \leq 64$ (d) none of these.

Ans. (c)

◎ **Solution:** Let the total number of workers be 100. A, the set of workers who like oranges and B, the set of workers who like apples.

So $n(A) = 70, n(B) = 64, n(A \cap B) = x.$
Also $n(A \cup B) \leq 100.$
$\Rightarrow$ $n(A) + n(B) - n(A \cap B) \leq 100$
$\Rightarrow$ $70 + 64 - x \leq 100 \Rightarrow x \geq 34$
Since $n(A \cap B) \leq n(B) \Rightarrow x \leq 64$
Hence $34 \leq x \leq 64.$

◉ **Example 48:** The Cartesian product of $A \times A$ has 16 elements. $S = \{(a, b) \in A \times A | a < b\}$. (– 1, 2) and (0, 1) are two elements belonging to S. The remaining elements of S are given by.

(a) {(–1, 0), (– 1, 1), (0, 2), (1, 2)}
(b) {(–1, 0), (1, 1), (2, –1), (1, 2)}
(c) {(0, –1), (1, – 1), (0, 2), (1, 2)}
(d) none of these

Ans. (a)

◎ **Solution:** $(-1, 2) \in A \times A$
$\Rightarrow$ $-1 \in A,\ 2 \in A$ and $(0, 1) \in A \times A \Rightarrow 0 \in A, 1 \in A$
So, $A = \{-1, 0, 1, 2\}$ as A has four elements
and $S = \{(-1, 0), (-1, 1), (-1, 2), (0, 1), (0, 2), (1, 2)\}.$
Hence the required element of S are given by (a)

◉ **Example 49:** If R and R' are two symmetric relations (not disjoint) on a set A, then the relation $R \cap R'$ is

(a) reflexive (b) symmetric
(c) transitive (d) none of these.

Ans. (b)

◎ **Solution:**
Let $(a, b) \in R \cap R'$ for some $a, b \in A$.
$\Rightarrow$ $(a, b) \in R$ and $(a, b) \in R'$
$\Rightarrow$ $(b, a) \in R$ and $(b, a) \in R'$ as R and R' are symmetric
$\Rightarrow$ $(b, a) \in R \cap R'$ showing that $R \cap R'$ is symmetric.

◉ **Example 50:** α, β, γ denote respectively the sets containing the letters in the names Apoorv, Mannan and Manvi of three children playing together. Which of the following is not correct.

(a) $n(\alpha \cap \gamma) | n(\alpha \cup \beta \cup \gamma)$
(b) $n(\beta \cap \gamma) | n(\alpha \cup \beta \cup \gamma)$
(c) $n(\alpha \cup \beta \cup \gamma) = 8$
(d) $n(\alpha \cup \beta \cup \gamma) = n(\alpha \cup \gamma)$

Ans. (b)

◎ **Solution:** $\alpha = \{a, o, p, r, v\}, \beta = \{a, m, n\}$
$\gamma = \{a, i, m, n, v\}$
$\alpha \cap \beta = \{a\}, \alpha \cap \gamma = \{a, v\}, \beta \cap \gamma = \{a, m, n\}$
$\alpha \cup \beta \cup \gamma = \{a, i, m, n, o, p, r, v\} = \alpha \cup \gamma.$

Example 51: If $f: \mathbf{R} \rightarrow \mathbf{R}$, defined by $f(x) = x^3 + 7$, then the value of $f^{-1}(71)$ and $f^{-1}(-1)$ respectively are

(a) $\{4\}, \phi$ (b) $\phi, \{-2\}$

(c) $\{4\}, \{-2\}$ (d) $\{2\}, \{-4\}$

Ans (c)

Solution: $f(x) = x^3 + 7 = 71 \Rightarrow x^3 = 64 \Rightarrow x = 4$

$$\Rightarrow \qquad f^{-1}(71) = \{4\}$$

and $f(x) = x^3 + 7 = -1 \Rightarrow x^3 = -8 \Rightarrow x = -2 \Rightarrow f^{-1}(-1) = \{-2\}$

Example 52: If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$, $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$ and **N**, the set of natural numbers is the universal set, then $A' \cup [(A \cup B) \cap B']$ is

(a) A (b) A'

(c) B (d) **N**.

Ans. (d)

Solution: $(A \cup B) \cap B' = A$ as $A \cap B = \phi$

$$\Rightarrow \qquad A' \cup [(A \cup B) \cap B'] = A' \cup A = \mathbf{N}.$$

Example 53: If $X = \{1, 2, 3, 4,\}$, then a one-one onto mapping $f: X \rightarrow X$ such that $f(1) = 1$, $f(2) \neq 2$ and $f(4) \neq 4$ is given by

(a) $\{(1, 1), (2, 3), (3, 4), (4, 2)\}$

(b) $\{(1, 1), (2, 4), (3, 1), (4, 2)\}$

(c) $\{(1, 2), (2, 4), (3, 2), (4, 3)\}$

(d) none of these

Ans. (a)

Solution: f in (a) is clearly one-one and onto also satisfies $f(1) = 1$, $f(2) = 3 \neq 2$, $f(4) = 2 \neq 4$.

Example 54: Let $f(x) = (x + 1)^2 - 1$, $x \geq -1$ then the set $\{x : f(x) = f^{-1}(x)\}$ is equal to

(a) $\{0, -1\}$ (b) $\{0, 1\}$

(c) $\{-1, 1\}$ (d) $\{0\}$

Ans. (a)

Solution: Let $y = (x + 1)^2 - 1$, $x \geq -1$

$$\Rightarrow \qquad (x + 1)^2 = y + 1$$

$$\Rightarrow \qquad x + 1 = \sqrt{y+1} \text{ as } x \geq -1$$

$$\Rightarrow \qquad x = -1 + \sqrt{y+1}, \; y \geq -1$$

$$\text{Thus} \qquad f^{-1}(x) = -1 + \sqrt{x+1}$$

$$\text{So} \qquad f(x) = f^{-1}(x)$$

$$\Rightarrow \qquad (x + 1)^2 - 1 = -1 + \sqrt{x+1}$$

$$\Rightarrow \qquad \sqrt{x+1} = 0 \text{ or } (x + 1)^{3/2} = 1$$

$$\Rightarrow \qquad x = -1 \text{ or } x = 0$$

Example 55: Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets, then

(a) $P \subset Q$ and $Q \sim P \neq \phi$ (b) $Q \not\subset P$

(c) $P \not\subset Q$ (d) $P = Q$.

Ans. (d)

Solution: $\sin\theta - \cos\theta = \sqrt{2}\cos\theta$

$$\Leftrightarrow \qquad \sin\theta = (\sqrt{2} + 1)\cos\theta$$

$$\Leftrightarrow \qquad \cos\theta = (\sqrt{2} - 1)\sin\theta$$

$$\Leftrightarrow \qquad \sin\theta + \cos\theta = \sqrt{2}\sin\theta$$

$$\Rightarrow \qquad P = Q.$$

Example 56: If A, B, C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then

(a) $B = C$ (b) $A \cap B = \phi$

(c) $A = B$ (d) $A = C$

Ans. (a)

Solution: Let $x \in B \Rightarrow x \in A \cup B$

$\Rightarrow \qquad x \in A \cup C \Rightarrow x \in A$ or $x \in C$

If $x \in A$, then $x \in A \cap B = A \cap C \Rightarrow x \in C$

So $\qquad x \in B \Rightarrow x \in C$.

Similarly $\quad x \in C \Rightarrow x \in B$, Hence $B = C$.

Example 57: The domain of the function

$$f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}} \text{ is}$$

(a) $[1, 2]$ (b) $[2, 3]$

(c) $[1, 3]$ (d) $[1, 2]$

Ans. (b)

Solution: $x^2 < 9 \Rightarrow -3 < x < 3$

and $-1 \leq x - 3 \leq 1 \Rightarrow 2 \leq x < 3$

Example 58: Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can be formed such that $Y \subseteq X$, $Z \subseteq X$, and $Y \cap Z$ is empty is

(a) 2^5 (b) 5^3

(c) 5^2 (d) 3^5

Ans. (d)

Solution: For each $x \in X$, we have three choices

$x \in Y,\ x \notin Z; \quad x \notin Y,\ x \in Z; \quad x \notin Y,\ x \notin Z$

So the required number of ordered pairs is 3^5.

Example 59: Let $X = \{1, 2, 3, 4\}$. The number of equivalence relations that can be defined on X is

(a) 10 (b) 15

(c) 16 (d) 8

Ans. (b)

Solution: The number of equivalence relations

$$B_k = \sum_{n=0}^{k-1} \binom{k-1}{n} B_n; \quad B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5$$

$$B_4 = \binom{3}{0}B_0 + \binom{3}{1}B_1 + \binom{3}{2}B_2 + \binom{3}{3}B_3$$

$$= 1 + 3 + 3 \times 2 + 1 \times 5 = 15$$

Example 60: The function $f(x) = \sin\dfrac{\pi x}{n!} - \cos\dfrac{\pi x}{(n+1)!}$ is

(a) non periodic
(b) periodic with period $2(n!)$
(c) periodic with period $2\,(n+1)!$
(d) periodic with period $n!$

Ans. (c)

Solution: Since the period of sin x is 2π so the period of $\sin\dfrac{\pi x}{n!}$ is $\dfrac{2\pi n!}{\pi}$ = $2(n!)$. The period of $\cos\dfrac{\pi x}{(n+1)!}$ is $2(n+1)!$ The period of $f(x)$ = l.c.m. $(2(n!), 2(n+1)!)$ = $2(n+1)!$

Example 61: If $f(x)^2\ f\left(\dfrac{1-x}{1+x}\right)=x^3,\ x \neq -1, 1$ and $f(x) \neq 0$, then $\{f(-2)\}$ (the fractional part of $f(-2)$) is equal to

(a) 2/3 (b) 1/3
(c) 1/2 (d) 0

Ans. (a)

Solution: Replacing x by $\dfrac{1-x}{1+x}$ in the equation

$$(f(x))^2\, f\left(\frac{1-x}{1+x}\right)=x^3 \qquad \text{...(i)}$$

We have

$$\left(f\left(\frac{1-x}{1+x}\right)\right)^2 f(x) = \left(\frac{1-x}{1+x}\right)^3 \qquad \text{... (ii)}$$

Solving (i) and (ii), we have

$$\left(\frac{x^3}{(f(x))^2}\right)^2 f(x) = \left(\frac{1-x}{1+x}\right)^3$$

$$\Rightarrow \qquad \frac{(f(x))^3}{x^6} = \left(\frac{1+x}{1-x}\right)^3$$

$$\Rightarrow \qquad f(x) = x^2\left(\frac{1+x}{1-x}\right)$$

$$f(-2) = 4\left(\frac{-1}{3}\right) = -\frac{4}{3}$$

$$\{f(-2)\} = -\frac{4}{3}-\left[-\frac{4}{3}\right] = -\frac{4}{3}+2=\frac{2}{3}.$$

Example 62: Let $f(x)$ be a function such that $f(x-1) + f(x+1) = \sqrt{2}f(x)$ for all $x \in \mathbf{R}$. If $f(3) = 5$ then $\sum_{r=0}^{10} f(3+8r)$ is equal to

(a) 50 (b) 55
(c) 0 (d) 10

Ans. (b)

Solution: The given equation is

$$f(x-1) + f(x+1) = \sqrt{2}f(x) \qquad \text{... (i)}$$

Replace x by $x-1$ in (i)

$$f(x-2) + f(x) = \sqrt{2}f(x-1) \qquad \text{... (ii)}$$

Replace x by $x+1$ in (ii)

$$f(x) + f(x+2) = \sqrt{2}f(x+1) \qquad \text{... (iii)}$$

Adding (ii) and (iii), we have

$$\begin{aligned} f(x-2) + f(x+2) + 2f(x) &= \sqrt{2}\,(f(x-1) + f(x+1)) \\ &= \sqrt{2}\,\sqrt{2}f(x) = 2f(x) \end{aligned}$$

$$\Rightarrow \qquad f(x-2) = -f(x+2)$$

Replacing x by $x+2$, we have

$$f(x) = -f(x+4) = -(f(x+8)) = f(x+8)$$

$$\sum_{r=0}^{10} f(3+8r) = 11 \times f(3) = 11 \times 5 = 55.$$

Example 63:

$$\text{Let } f(\theta) = \begin{vmatrix} \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \end{vmatrix}$$

then f is

(a) a non periodic function
(b) periodic with period π
(c) periodic with period $\pi/2$
(d) odd function

Ans. (c)

Solution: $f(\theta) = \begin{vmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 1+\sin^2\theta & \cos^2\theta & 4\sin 2\theta \end{vmatrix}$

$$\begin{pmatrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{pmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ \begin{matrix} 1+\sin^2\theta \\ +4\sin 4\theta \end{matrix} & \cos^2\theta & 4\sin 4\theta \end{vmatrix} \qquad (C_1 \to C_1 + C_3)$$

$$= -(\cos^2\theta + 1 + \sin^2\theta + 4\sin 4\theta)$$

$$= -2\,(1 + 2\sin 4\theta)$$

which is periodic function with period $\dfrac{2\pi}{4}=\dfrac{\pi}{2}$.

Assertion-Reason Type Questions

Example 64: Let **R** be the real line. Consider the following subsets of the plane $\mathbf{R} \times \mathbf{R}$:

$$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$$
$$T = \{(x, y) : x - y \text{ is an integer}\}$$

Statement-1 : T is an equivalence relation on R but S is not an equivalence relation on R.
Statement-2 : S is neither reflexive nor symmetric but T is reflexive, symmetric and transitive.
Ans. (a)

Solution: Since $x \neq x + 1$, $(x, x) \notin S$, so S is not reflexive. Next $x, y \in S$, $\Rightarrow y = x + 1 \Rightarrow x = y - 1 \Rightarrow (y, x) \notin S$, so is not symmetric.
Since $x - x = 0$ is an integer $(x, x) \in T \; \forall \; x \in T$
$\Rightarrow$ T is reflexive.
Again $(x, y) \in T \Rightarrow x - y$ is an integer
$\Rightarrow$ $y - x$ is also an integer $\Rightarrow (y, x) \in T$
So T is symmetric.
Also $(x, y) \in T$, $(y, z) \in T$.
$\Rightarrow$ $x - y$ and $y - z$ are integers
$\Rightarrow$ $x - z = (x - y) - (y - z)$ is also an integer
$\Rightarrow$ $(x, z) \in T$
So T is Transitive.
Which shows that statement-2 is true and hence statement-1 is also true.

Example 65: Consider the following relations.
$R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$
$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right)\right\}$ m, n, p, q, are integer such that $n \cdot q \neq 0$ and $qm = pn\}$

Statement-1: S is an equivalence relation but R is not an equivalence relation.
Statement-2: R and S both are symmetric.
Ans. (c)

Solution: Since $(0, 1) \in R$ but $(1, 0) \notin R$, R is not symmetric and hence is not an equivalence relation so statement-2 is false.
Next, For the relation S, $qm = pn \Rightarrow \frac{m}{n} = \frac{p}{q}$

Thus $\left(\frac{m}{n}, \frac{p}{q}\right) \in S \Rightarrow \frac{m}{n} = \frac{p}{q}$ which shows that S is reflexive and symmetric

Again, $\left(\frac{m}{n}, \frac{p}{q}\right) \in S$ and $\left(\frac{p}{q}, \frac{r}{s}\right) \in S$

$$\Rightarrow \quad \frac{m}{n} = \frac{p}{q} = \frac{r}{s} \quad \Rightarrow \left(\frac{m}{n}, \frac{r}{s} \in S\right)$$

Thus S is transitive and hence S is an equivalence relation. So, statement 1 is true.

Example 66: Let R be a relation on the set N of natural numbers defined by $n \, Rm \Leftrightarrow n$ is a factor of m (*i.e.* $n \mid m$).
Statement-1: R is not an equivalence relation
Statement-2: R is not symmetric
Ans. (a)

Solution: Statement-2 is true as $2 \mid 6 \Rightarrow 2\mathbf{R}6$ but 6 does not divide 2 so R is not symmetric $\Rightarrow R$ is not an equivalence relation and the statement-1 is also true.

Example 67: Let $A = \{1, 2, 3\}$ and $B = \{3, 8\}$
Statement-1: $(A \cup B) \times (A \cap B) = \{(1, 3), (2, 3), (3, 3), (8, 3)\}$
Statement-2: $(A \times B) \cap (B \times A) = \{(3, 3)\}$
Ans. (b)

Solution: $A \cup B = \{1, 2, 3, 8\}$, $A \cap B = \{3\}$
$\Rightarrow$ $(A \cup B) \times (A \cap B) = \{(1, 3), (2, 3), (3, 3), (8, 3)\}$
$\Rightarrow$ Statement-1 is True.
$(x, y) \in (A \times B) \cap (B \times A)$
$\Rightarrow$ $(x, y) \in A \times B$ and $(x, y) \in B \times A$
$\Rightarrow$ $x \in A \cap B$, $y \in A \cap B$
$\Rightarrow$ $\{(3, 3)\} = (A \times B) \cap (B \times A) \Rightarrow$ Statement-2 is also true but is not a correct explanation for statement-1.

Example 68: Statement-1: The number of bijective functions from the set A containing 100 elements to itself is 2^{100}.
Statement-2: The total number of bijections from a set containing n elements to itself is $n!$
Ans. (d)

Solution: Statement-2 is true and so, statement-1 is False.

Example 69: Statement-1: $f : R \to R$ is a function defined by $f(x) = 5x + 3$. If $g = f^{-1}$, then $g(x) = \frac{x - 3}{5}$.
Statement-2: If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then fog is the identity function on A.
Ans. (c)

Solution: Let $y = 5x + 3 \Rightarrow x = \frac{y - 3}{5}$.

$\Rightarrow$ $g(x) = \frac{x - 3}{5}$ is the inverse of f, so statement-1 is True.

Statement-2 is false because $g : B \to A$ and $f : A \to B$

⇒ $fog : B \to B$ and $g = f^{-1} \Rightarrow fog$ is an identity function on B.

◉ **Example 70:** Let X and Y be two sets.

Statement-1: $X \cap (Y \cup X)' = \phi$

Statement-2: If $X \cup Y$ has m elements and $X \cap Y$ has n elements then symmetric difference $X \Delta Y$ has $m - n$ elements

Ans. (b)

◎ **Solution:** $X \cap (Y \cup X)' = X \cap (Y' \cap X') = X \cap X' \cap Y' = \phi$.

⇒ Statement-1 is True.

$X \Delta Y = (X \sim Y) \cup (Y \sim X) = (X \cup Y) \sim (X \cap Y)$

⇒ number of elements in $X \Delta Y = m - n$.

⇒ Statement-2 is True but does explain statement-1.

◉ **Example 71:** Let f be a function defined by

$f(x) = (x - 1)^2 + 1$, $(x \ge 1)$

Statement-1: The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$

Statement-2: f is a bijection and $f^{-1}(x) = 1 + \sqrt{x-1}$, $x \ge 1$

Ans. (a)

◎ **Solution:** Let $y = f(x) = (x - 1)^2 + 1$

⇒ $y - 1 = (x - 1)^2 \Rightarrow x = 1 + \sqrt{y-1}$, $y \ge 1$

Thus $f^{-1}(x) = 1 + \sqrt{x-1}$, $x \ge 1$. So statement-2 is true.

Now $f(x) = f^{-1}(x)$

⇒ $(x - 1)^2 = \sqrt{x-1}$

⇒ $\sqrt{x-1}\,[(x-1)^{3/2} - 1] = 0$

⇒ $x = 1, 2$.

So statement-1 is true and statement-2 is a correct explanation for statement-1.

◉ **Example 72:** Let R be the set of real numbers

Statement-1: $A = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y + x$ is an integer$\}$ is an equivalence relation on $\mathbf{R}$.

Statment-2: $B = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y = \alpha x$ for some rational number $\alpha\}$ is not equivalence relation on $\mathbf{R}$.

Ans. (d)

◎ **Solution:** A is neither reflexive nor transitive as $x + x$ may not be integer $\forall\, x \in \mathbf{R}$ and if $x + y$ and $y + z$ are integers, $x + z$ may not be an integer for $x, y, z \in \mathbf{R}$.

So statement-1 is false.

Statememt-2 is true as B is not symmetric, because $(\sqrt{3}, 0) \in B$ as $0 = \sqrt{3} \times 0$ for $\alpha = 0$, but $(0, \sqrt{3}) \notin B$.

◉ **Example 73:** Consider the following relation R on the set of real square matrices of order 3.

$R = \{ (A, B) : A = P^{-1}BP$ for some invertible matrix $P\}$

Statement-1: R is an equivalence relation.

Statement-2: For any two invertible 3×3 matrices M and N, $(MN)^{-1} = N^{-1}M^{-1}$.

Ans. (b)

◎ **Solution:** Statement-2 in true (See Text.)

In statement-1, $A = I^{-1} A I$

for all real square matrices A of order 3.

⇒ $(A, A) \in R \Rightarrow R$ is reflexive,

Next, let $(A, B) \in R$

⇒ $\exists$ a invertible matrix P of order 3.

such that $A = P^{-1} B P$

⇒ $B = P A P^{-1} = (P^{-1})^{-1} A (P^{-1})$

⇒ R in symmetric

If Now $(A, B) \in R$ and $(B, C) \in R$

Then $\exists$ invertible matrices P and Q

of order 3 such that

$A = P^{-1} B P$ and $B = Q^{-1} C Q$.

⇒ $A = P^{-1} Q^{-1} C Q P = (QP)^{-1} C QP$ (From statement-2)

⇒ $(A, C) \in R$ and thus R in transitive. Hence R is an equivalence relation and the statement-1 in also true but statement-2 is not a correct explanation for it.

LEVEL 2

Straight Objective Type Questions

◉ **Example 74:** From 50 students taking examinations in Mathematics, Physics and Chemistry, 37 passed Mathematics, 24 Physics and 43 Chemistry. At most 19 passed Mathematics and Physics, at most 29 Mathematics and Chemistry and at most 20 Physics and Chemistry. The largest possible number that could have passed all three exams is

(a) 10 (b) 12

(c) 9 (d) none of these.

Ans. (d)

◎ **Solution:** The given conditions can be expressed as

$n(M \cup P \cup C) = 50$, $n(M) = 37$, $n(P) = 24$, $n(C) = 43$, $n(M \cap P) \le 19$,

$n(M \cap C) \le 29$ and $n(P \cap C) \le 20$,

$$n(M \cup P \cup C) = n(M) + n(P) + n(C) - n(M \cap P) - n(M \cap C) - n(P \cap C) + n(M \cap P \cap C)$$

⇒ $50 = 37 + 24 + 43 - n(M \cap P) - n(P \cap C)$

$$- n(M \cap C) + n(M \cap P \cap C)$$

$$\Rightarrow \quad n(M \cap P \cap C) \le n(M \cap P) + n(M \cap C) + n(P \cap C) - 54.$$

Therefore, the number of students that could have passed all three exams is at most
19 + 29 + 20 − 54 = 14.

◉ **Example 75:** Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 5 elements and $B_1, B_2 \ldots, B_n$ are n sets each with 3 elements, let $\bigcup\limits_{i=1}^{30} A_i = \bigcup\limits_{j=1}^{n} B_j = S$ and each element of S belongs to exactly 10 of the A_i's and exactly 9 of the B_j's. Then n is equal to

(a) 15 (b) 3
(c) 45 (d) none of these

Ans. (c)

◎ **Solution:** $S = \bigcup\limits_{i=1}^{30} A_i$, so $n(S) = \dfrac{1}{10}(5 \times 30) = 15$ (since element in the union S belongs to exactly 10 of the A_i's).

Again $S = \bigcup\limits_{i=1}^{n} B_i$ so

$$n(S) = 1/9(3 \times n) = n/3 = 15 \Rightarrow n = 45$$

◉ **Example 76:** Let $R = \{(3, 3), (6, 6), (9, 9) (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set A = {3, 6, 9, 12}.

The relation is

(a) an equivalence relation.
(b) reflexive and symmetric only.
(c) reflexive and transitive only
(d) reflexive only.

Ans. (c)

◎ **Solution:** R is reflexive as
$(3, 3), (6, 6), (9, 9), (12, 12) \in R$.
R is not symmetric as $(6, 12) \in R$ but $(12, 6) \notin R$.
R is transitive as the only pair which needs verification is $(3, 6)$ and $(6, 12) \in R \Rightarrow (3, 12) \in R$.

◉ **Example 77:** Let $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$ be a relation on the set A = {1, 2, 3, 4}, the relation R is

(a) not symmetric (b) transitive
(c) a function (d) reflexive

Ans. (a)

◎ **Solution:** R is not symmetric as $(2, 3) \in R$ but $(3, 2) \notin R$. It is not transitive as $(1, 3), (3, 1) \in R$. but $(1, 1) \notin$ R so is not reflexive also.
Again as $(2, 4)$ and $(2, 3) \in R$, it is not a function.

◉ **Example 78:** Let **I** be the set of integers, **N** the set of non-negative integers; Np the set of non-positive integers ; E is the set of even integers and P is set of prime numbers. Then

(a) $\mathbf{N} \cap Np = \phi$ (b) $\mathbf{I} - \mathbf{N} = Np$
(c) $\mathbf{N} \,\Delta\, Np = I - \{0\}$ (d) $E \cap P = \phi$.

Ans. (c)

◎ **Solution:** $N \cap Np = \{0\}$,

$$\mathbf{I} - N = \{\ldots, -2, -1\}$$
$$N \,\Delta\, Np = \{N - Np) \cup (Np - N)$$
$$= \{1, 2, 3 \,-\,-\} \cup \{\ldots, -3, -2, -1\}$$
$$= \mathbf{I} \sim \{0\}$$

and $E \cap P = \{2\}$.

◉ **Example 79:** If $n(A) = n$ then $n\{(x, y, z);\ x, y, z \in A, x \ne y, y \ne z, z \ne x\} =$

(a) n^3 (b) $n(n-1)^2$
(c) $n^2(n-2)$ (d) $n^3 - 3n^2 + 2n$

Ans. (d)

◎ **Solution:** There are n choices for the first coordinate, $n - 1$ choices for second coordinate and $n - 2$ choices for the third coordinate, hence $n(\{(x, y, z);\ x, y, z \in A, x \ne y \ne z\}) = n(n-1)(n-2) = n^3 - 3n^2 + 2n$.

◉ **Example 80:** If $A = \{x : x \in \mathbf{I}, -2 \le x \le 2\}$, $B = \{x \in \mathbf{I}, 0 \le x \le 3\}$, $C = \{x: x \in \mathbf{N}, 1 \le x \le 2\}$ and $D = \{x, y) \in \mathbf{N} \times \mathbf{N};\ x + y = 8\}$. Then

(a) $n(A \cup (B \cup C) = 5$ (b) $n(D) = 6$
(c) $n(B \cup C) = 5$ (d) none of these

Ans. (d)

◎ **Solution:** $A = \{-2, -1, 0, 1, 2\}$, $B = \{0, 1, 2, 3\}$, $C = \{1, 2\}$

so $B \cup C = \{0, 1, 2, 3\}$,
$A \cup (B \cup C) = \{-2, -1, 0, 1, 2, 3\}$
so $n(A \cup (B \cup C) = 6$, $n(B \cup C) = 4$
and $D = \{(1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1)\}$ so
$n(D) = 7$.

◉ **Example 81:** If $A = \{4^n - 3n - 1 | n \in \mathbf{N}\}$ and $B = \{9n - 9: n \in \mathbf{N}\}$, then $A \cup B$ is equal to

(a) B (b) A
(c) **N** (d) none of these

Ans. (a)

◎ **Solution:** It can be shown by induction that $9 \mid 4^n - 3n - 1$ for every $n \in \mathbf{N}$. Thus $A \subseteq B$. Clearly $A \ne B$ as $27 \in B$ but $27 \notin A$. Thus $A \cup B = B$.

◉ **Example 82:** A and B are two sets having 3 and 4 elements respectively and having 2 elements in common. The number of relations which can be defined from A to B is

(a) 2^5 (b) $2^{10} - 1$
(c) 2^{12} (d) none of these.

Ans. (c)

◎ **Solution:** The number of elements in $A \times B$ is 12. Hence the number of subsets of $A \times B$ is 2^{12}, which includes the empty set

Example 83: If R and S are two symmetric relations then

(a) $R \; o \; S$ is a symmetric relation
(b) $S \; o \; R$ is a symmetric relation
(c) $R \; o \; S^{-1}$ is a symmetric relation
(d) $R \; o \; S$ is a symmetric relation if and only if $RoS = SoR$.

Ans. (d)

Solution: Since R and S are symmetric relations so $R^{-1} = R$ and $S^{-1} = S$. But $(R \; o \; S)^{-1} = S^{-1} \; o \; R^{-1} = S \; o \; R$. Thus RoS is symmetric if and only if $RoS = SoR$.

Example 84: Let A be the set of all determinants of order 3 with entries 0 or 1 only, B the subset of A consisting of all determinants with value 1, and C the subset consisting of all determinants with value –1. Then if $n(B)$ and $n(C)$ denote the number of elements in B and C, respectively, we have

(a) $C = \phi$ (b) $n(B) = n(C)$
(c) $A = B \cup C$ (d) $n(B) = 2n(C)$

Ans. (b)

Solution: C cannot be the empty set because, for instance,

$$-1 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \in C. \text{ We also have } \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2,$$

so $A \neq B \cup C$. In general, the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{22}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}, \end{matrix}$$

with the a's being 0 or 1, equals 1 only if $a_{11}a_{22}a_{33} = 1$ and the remaining terms are zero; if $a_{12}\,a_{23}\,a_{31} = 1$ and the remaining terms are zero; or if $a_{13}\,a_{21}\,a_{32} = 1$ and the remaining terms are zero. Since there are three similar relations for determinants that equal – 1, we must have $n\,(B) = n\,(C)$.

Example 85: The domain of the function

$$f(x) = \log_2\left(-\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) - 1\right) \text{ is}$$

(a) $0 < x < 1$ (b) $0 < x \leq 1$
(c) $x \geq 1$ (d) $x > 1$

Ans. (a)

Solution: For f to be defined we must have $\log_{1/2}\left(1 + \frac{1}{\sqrt[4]{x}}\right) < -1 \Leftrightarrow 1 + \frac{1}{\sqrt[4]{x}} > (2^{-1})^{-1} = 2$ which is possible if and only if $\frac{1}{\sqrt[4]{x}} > 1$ i.e., $0 < x < 1$.

Hence the domain of the given function is $\{x : 0 < x < 1\}$.

Example 86: The domain of definition of

$$f(x) = \sqrt{\log_{0.4}\left(\frac{x-1}{x+5}\right)} \times \frac{1}{x^2 - 36} \text{ is}$$

(a) $\{x : x < 0, x \neq -6\}$
(b) $\{x : x > 0, x \neq 1, x \neq 6\}$
(c) $\{x : x > 1, x \neq 6\}$
(d) $\{x : x \geq 1, x \neq 6\}$

Ans. (c)

Solution: For $\sqrt{\log_{0\cdot 4}\left(\frac{x-1}{x+5}\right)}$ to be defined, we must have $0 < \frac{x-1}{x+5} < 1$, which is true if $x > 1$. Morever, $\frac{1}{x^2 - 36}$ is defined for $x \neq \pm 6$. Hence the domain of f is $\{x : x > 1, x \neq 6\}$.

Example 87: The set of all x for which $f(x) = \log_{\frac{x-2}{x+3}} 2$ and $g(x) = \frac{1}{\sqrt{x^2 - 9}}$ are both not defined is

(a) $(-3, 2)$ (b) $[-3, 2)$
(c) $(-3, 2]$ (d) $[-3, 2]$

Ans. (d)

Solution: The function f is not defined for $-3 \leq x \leq 2$ and g is not defined for those x for which $x^2 - 9 \leq 0$ i.e., $x \in [-3, 3]$. Thus f and g are not defined on $[-3, 2]$.

Example 88: If $f(x)$ is a polynomial satisfying $f(x)\,f(1/x) = f(x) + f(1/x)$ and $f(3) = 28$, then $f(4)$ is given by

(a) 63 (b) 65
(c) 67 (d) 68

Ans. (b)

Solution: Any polynomial satisfying the functional equation $f(x) \cdot f(1/x) = f(x) + f(1/x)$ is of the form $x^n + 1$ or $-x^n + 1$. If $28 = f(3) = -3^n + 1$ then $3^n = -27$ which is not possible for any n. Hence $28 = f(3) = 3^n + 1 \Rightarrow 3^n = 27 \Rightarrow n = 3$. Thus $f(x) = x^3 + 1$, so $f(4) = 4^3 + 1 = 65$.

Example 89: Part of the domain of the function $f(x) = \sqrt{\frac{\cos x - 1/2}{6 + 35x - 6x^2}}$ lying in the interval $[-1, 6]$ is

(a) $[-1/6, \pi/3] \cup [5\pi/3, 6]$
(b) $(-1/6, \pi/3] \cup [5\pi/3, 6)$
(c) $(-1/6, 6)$
(d) none of these

Ans. (a)

Solution: The function f is meaningful only if $\cos x - 1/2 \geq 0$, $6 + 35x - 6x^2 > 0$ or $\cos x - 1/2 \leq 0$, $6 + 35x - 6x^2 < 0$ i.e., $\cos x \geq 1/2$, $(6 - x)(1 + 6x) > 0$ or $\cos x \leq 1/2$, $(6 - x)(1 + 6x) < 0$. These inequalities are satisfied if $x \in (-1/6, \pi/3] \cup [5\pi/3, 6)$.

Example 90: Let f: $\mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$. Then

(a) f is both one-one and onto
(b) f is one-one but not onto

(c) f is onto but not one-one
(d) f is neither one-one nor onto

Ans. (d)

◎ **Solution:** f is not one-one as $f(0) = 0$ and $f(-1) = 0$. f is also not onto as for $y = 1$ there is no $x \in \mathbf{R}$ such that $f(x) = 1$. If there is such a $x \in \mathbf{R}$ then $e^{|x|} - e^{-x} = e^x + e^{-x}$, clearly $x \neq 0$. For $x > 0$, this equation gives $-e^{-x} = e^{-x}$ which is not possible. For $x < 0$, the above equation gives $e^x = -e^{-x}$ which is also not possible.

◉ **Example 91:** Let $f(x) = x^2$ and $g(x) = 2^x$ then the solution set of $fog\ (x) = g\ of(x)$ is

(a) $\mathbf{R}$	(b) $\{0\}$
(c) $\{0, 2\}$	(d) none of these

Ans. (c)

◎ **Solution:** $fog\ (x) = f(g(x)) = f(2^x) = (2^x)^2 = 2^{2x}$ and $gof\ (x) = g\ (f(x)) = g\ (x^2) = 2^{x^2}$. Thus the solution of $2^{x^2} = 2^{2x}$ is given by $x^2 = 2x$ which is $x = 0, 2$.

◉ **Example 92:** A function $f: \mathbf{R} \to \mathbf{R}$ satisfies the equation $f(x) f(y) - f(xy) = x + y$ for all $x, y \in \mathbf{R}$ and $f(1) > 0$, then

(a) $f(x) = x + 1/2$	(b) $f(x) = (1/2)x + 1$
(c) $f(x) = (1/2)x - 1$	(d) $f(x) = x + 1$

Ans. (d)

◎ **Solution:** Taking $x = y = 1$, we get

$$f(1) f(1) - f(1) = 1 + 1 \Rightarrow f(1)^2 - f(1) - 2 = 0$$

$$\Rightarrow \quad (f(1) - 2)(f(1) + 1) = 0 \Rightarrow f(1) = 2 \ (\because f(1) > 0)$$

Taking $y = 1$, we get

$$f(x) f(1) - f(x) = x + 1 \Rightarrow 2f(x) - f(x) = x + 1$$

$$\Rightarrow \quad f(x) = x + 1.$$

◉ **Example 93:** If $f: [1, \infty) \to [2, \infty)$ is given by $f(x) = x + 1/x$ then $f^{-1}(x)$ equals

(a) $\dfrac{x + \sqrt{x^2 - 4}}{2}$	(b) $\dfrac{x}{1 + x^2}$
(c) $\dfrac{x - \sqrt{x^2 - 4}}{2}$	(d) $1 + \sqrt{x^2 - 4}$

Ans. (a)

◎ **Solution:** $y = x + 1/x \Rightarrow x^2 - xy + 1 = 0$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

Since $x \in [1, \infty)$ so $x = \dfrac{y + \sqrt{y^2 - 4}}{2}$.

Hence $f^{-1}(x) = \dfrac{x + \sqrt{x^2 - 4}}{2}$.

◉ **Example 94:** Consider the function $f = \{(x, \sin x) | -\infty < x < \infty\}$. Let $A = \left[0, \frac{\pi}{6}\right]$ and $B = \left[0, \frac{\pi}{2}\right]$ then

(a) $f(A \cap B) = f(A) \cap f(B)$	(b) $f(A \cap B) = [0, 1]$
(c) $f(A) \cap f(B) = [0, 1]$	(d) $f(A) \cup f(B) = \left[0, \frac{1}{2}\right]$

Ans. (a)

◎ **Solution:** $f(x) = \sin x$ is an increasing function on $[0, \pi/2]$ so $f(A) = \left[0, \sin\frac{\pi}{6}\right] = \left[0, \frac{1}{2}\right]$ and $f(B) = [0, 1]$. Thus $f(A) \cap f(B) = \left[0, \frac{1}{2}\right]$. Also $A \cap B = \left[0, \frac{\pi}{6}\right]$, so $f(A \cap B) = \left[0, \sin\frac{\pi}{6}\right] = \left[0, \frac{1}{2}\right]$. Thus $f(A) \cap f(B) = \left[0, \frac{1}{2}\right] = f(A \cap B)$, Also $f(A) \cup f(B) = [0, 1]$.

◉ **Example 95:** Let $f(x)$ be a polynomial of even degree satisfying $f(2x)\left(1 - f\left(\frac{1}{2x}\right)\right) + f\left(16x^2 y\right) = f(-2) - f(4xy)$ for all $x, y \in \mathbf{R} \sim \{0\}$ and $f(4) = -255$, $f(0) = 1$. Then the value of $\left|\dfrac{f(2) + 1}{2}\right|$ is

(a) 4	(b) 5
(c) 7	(d) 6

Ans. (c)

◎ **Solution:** Replacing y by $\dfrac{1}{8x^2}$ in the given functional equation, we obtain

$$f(2x)\left(1 - f\left(\frac{1}{2x}\right)\right) + f(2) = f(-2) - f\left(\frac{1}{2x}\right).$$

Since f is an even function so $f(2) = f(-2)$,

$$\text{so} \quad f(2x) - f(2x) f\left(\frac{1}{2x}\right) = -f\left(\frac{1}{2x}\right)$$

$$\Rightarrow \quad f(2x) + f\left(\frac{1}{2x}\right) = f(2x)\, f\left(\frac{1}{2x}\right)$$

Replacing x by $\dfrac{x}{2}$, we have

$$f(x) + f\left(\frac{1}{x}\right) = f(x)\, f\left(\frac{1}{x}\right)$$

Since f is a polynomial, so $f(x) = \pm x^n + 1$ (see Example 88) But $-255 = f(4) = \pm 4^n + 1$. Only negative sign is possible, thus $4^n = 256 \Rightarrow n = 4$. i.e. $f(x) = -x^4 + 1$. $f(2) = -2^4 + 1 = -15$

$$\left|\frac{f(2) + 1}{2}\right| = \left|-\frac{14}{2}\right| = 7.$$

◉ **Example 96:** The domain of the function $f(x) = \dfrac{1}{\sqrt{\sin x}} + \sqrt[3]{\sin x} + \log_{10} \dfrac{x - 5}{x^2 - 10x + 24}$ is

(a) $\{(2k\pi, (2k + 1)\pi) : k \in \mathbf{I}\}$
(b) $\{(2k\pi, (2k + 1)\pi) : k \in \mathbf{N}\}$
(c) $(6, \infty) \cup (4, 5)$
(d) $\{(2k\pi, (2k + 1)\pi) : k \in \mathbf{I}\} \cup (6, \infty)$

Ans. (b)

◎ **Solution:** The domain of $\sqrt[3]{\sin x}$ is **R**.

The domain of $\dfrac{1}{\sqrt{\sin x}} = \{x : \sin x > 0\}$

$= \{(2k\pi, (2k+1)\pi) : k \in \mathbf{I}\}$

The domain of $\log_{10} \dfrac{x-5}{x^2-10x+24} = \log_{10} \dfrac{x-5}{(x-6)(x-4)}$

$= \{x : x \neq 5, x > 5, x > 6, x > 4$ or $x \neq 5, x < 5, 4 < x < 6\}$

$= (4, 5) \cap (6, \infty)$

Thus the domain of $f(x)$

$= \{(2k\pi, (2k+1)\pi) : k \in \mathrm{I}\} \cap \{(4, 5) \cup (6, \infty)\}$

$= \{(2k\pi, (2k+1)\pi) : k \in \mathbf{N}\}$

(since (4, 5) does not intersect $\{(2k\pi, 2k+1)\pi : k \in \mathrm{I}\}$)

EXERCISE

Concept-based Straight Objective Type Questions

1. Let U be a Universal set and $n(U) = 12$. If $A, B \subseteq U$ are such that $n(B) = 6$ and $n(A \cap B) = 2$ then $n(A \cup B')$ is equal to
   (a) 6 (b) 10
   (c) 7 (d) 8

2. Let R be a relation on **R** defined as $a \, R \, b$ if $|a| \leq b$. Then, relation R is
   (a) reflexive (b) symmetric
   (c) transitive (d) not antisymetric

3. Let $f(x) = ax + b$, $x \in$ **R**, and $g(x) = x + d$, $x \in$ **R**, then $fog = gof$ if and only if
   (a) $f(a) = g(c)$ (b) $f(d) = g(b)$
   (c) $f(b) = g(d)$ (d) $f(c) = g(a)$

4. The domain of the function $f(x) = \dfrac{\sin x}{\sqrt{|x| - x}}$ is
   (a) **R** (b) **R** ~ {0}
   (c) $\mathbf{R}^+$ (d) $\mathbf{R}^-$

   ($\mathbf{R}^+$ is the set of positive real numbers and $\mathbf{R}^-$ is the set of negative real numbers)

5. The domain of $y = \cos^{-1}(1 - 2x)$ is
   (a) [– 1, 1] (b) [0, 1]
   (c) [–1, 0] (d) $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

6. If $f(x) = \sin x - \cos x$ is written as $f_1(x) + f_2(x)$ where $f_1(x)$ is even and $f_2(x)$ is odd then
   (a) $f_1(x) = \cos x$ (b) $f_1(x) = -\cos x$
   (c) $f_2(x) = -\sin x + \cos x$ (d) $f_2(x) = \sin(2\pi - x)$

7. The range of $y = 1 - \sin x$ is
   (a) [–1, 1] (b) [0, 1]
   (c) [–1, 2] (d) [0, 2]

8. The function $f(x) = x^2 \log \dfrac{1-x}{1+x}$ is
   (a) a periodic function (b) a bounded function
   (c) an odd function (d) an even function

LEVEL 1

Straight Objective Type Questions

9. If $A = \{2, 3, 4, 5, 7\}$, $B = \{1, 2, 4, 7, 9\}$ then $((A \sim B) \cup (B \sim A)) \cap A$ is equal to
   (a) {3, 5} (b) {2, 4}
   (c) {3, 7} (d) {2, 7}

10. If $A = \{2x : x \in N\}$, $B = \{3x : x \in N\}$ and $C = \{5x : x \in N)$ then $A \cap (B \cap C)$ is equal to
    (a) {15, 30, 45, ... }
    (b) {10, 20, 30 ... }
    (c) {30, 60, 90, ... }
    (d) {7, 14, 21 ... }.

11. If X and Y are two sets such that $X \cap Y = X \cup Y$, then
    (a) $X \subset Y, X \neq Y$ (b) $Y \subset X, Y \neq X$
    (c) $X = Y$ (d) none of these

12. If X, Y and A are three sets such that $A \cap X = A \cap Y$ and $A \cup X = A \cup Y$ then
    (a) $X \subset Y$ (b) $Y \subset X$
    (c) $X = Y$ (d) none of these

13. If $A = \{x : x \in R$ and satisfy $x^2 - 15x + 56 = 0\}$
    $B = \{x : x \in N$ and $x + 5 \leq 14\}$
    and $C = \{x : x \in N$ and $x/112\}$.
    Then $A \cup (B \cap C)$ is equal to
    (a) {1, 2, 3,....8} (b) {8, 7}
    (c) {1, 2, 8, 7} (d) {1, 2, 4, 7, 8}

14. If A is the set of letters needed to spell “MATHEMATICS” and B is the set of letters needed to spell STATISTICS, then

(a) $A \subset B$ (b) $A \sim B = \phi$
(c) $A \Delta B = A \sim B$ (d) none of these

15. If A and B are two sets such that $n(A \cup B) = 36$, $n(A \cap B) = 16$ and $n(A \sim B) = 15$, then $n(B)$ is equal to

(a) 21 (b) 31
(c) 20 (d) 52

16. The maximum number of sets obtainable from A and B by applying union and difference operations is

(a) 5 (b) 6
(c) 7 (d) 8

17. If A and B both contain same number of elements and are finite sets then

(a) $n(A \cup B) = n(A \cap B)$ (b) $n(A \sim B) = n(B \sim A)$
(c) $n(A \Delta B) = n(B)$ (d) $n(A \sim B) = n(A)$

18. If $A \Delta B = A \cup B$ then

(a) $A = B$ (b) $A \cap B = \phi$
(c) $A \Delta B = \phi$ (d) $A \Delta B = A \sim B$

19. In a class 60% passed their Physics examination and 58% passed in Mathematics. Atleast what percentage of students passed both their Physics and Mathematics examination?

(a) 18% (b) 17%
(c) 16% (d) 2%

20. If the relation $R: A \to B$, where $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ is defined by
$R = \{(x, y): x < y, x \in A, y \in B\}$, then

(a) $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5)\}$
(b) $R = \{(1, 1), (1, 5), (2, 3), (3, 5)\}$
(c) $R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 3)\}$
(d) $R^{-1} = \{(1, 1), (5, 1), (3, 2), (5, 3)\}$

21. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by

$$R = \{(x, y): |x^2 - y^2| < 16\}$$

is given by

(a) $R_1 = \{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$
(b) $R_2 = \{(2, 2), (3, 2), (4, 2), (2, 4)\}$
(c) $R_3 = \{(3, 3), (4, 3), (5, 4), (3, 4)$
(d) none of these

22. Let L be the set of all lines in a plane and R be a relation on L defined by $l_1 R\, l_2$ if and only if $l_1 \perp l_2$ then R is

(a) reflexive (b) symmetric
(c) transitive (d) an equivalence relation

23. For non-empty subsets A and B,

(a) Any subset of $A \times B$ defines a function from A to B.
(b) Any subset of $A \times B$ defines an equivalence relation
(c) Any subset of $A \times A$ defines a function on A
(d) none of these.

24. Let $f(x) = (x + 1)^2 - 1$ $(x \geq -1)$. Then the set $S = \{x : f(x) = f^{-1}(x)\}$ contains

(a) $\left\{0, -1, \dfrac{-3 + i\sqrt{3}}{2}, \dfrac{-3 - i\sqrt{3}}{2}\right\}$
(b) $\{0, 1, -1\}$
(c) $\{0, -1\}$
(d) none of these

25. If a set A has n elements then the number of all relations on A is

(a) 2^{n^2} (b) $2^{n^2} - 1$
(c) 2^n (d) none of these

26. The function f: $\mathbf{R} \to \mathbf{R}$ given by $f(x) = 3 - 2 \sin x$ is

(a) one-one (b) onto
(c) bijective (d) none of these

27. Which of the following are functions?

(a) $\{(x, y): y^2 = 4ax, x, y \in \mathbf{R}\}$
(b) $\{(x, y): y = |x|, x, y \in \mathbf{R}\}$
(c) $\{(x, y): x^2 + y^2 = 1, x, y \in \mathbf{R}\}$
(d) $\{(x, y): x^2 - y^2 = 1, x, y \in \mathbf{R}\}$

28. If f: $\mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^4 + 2$ then the value of f^{-1} (83) and $f^{-1}(-2)$ respectively are

(a) ϕ, $\{3, -3\}$ (b) $\{3, -3\}$, ϕ
(c) $\{4, -4\}$, ϕ (d) $\{4, -4\}$, $\{2, -2\}$.

29. The minimum number of elements that must be added to the relation $R = \{(1, 2), (2, 3)\}$ on the sub set $\{1,2,3\}$ of natural numbers so that it is an equivalence relation is

(a) 4 (b) 7
(c) 6 (d) 5

30. Let X be a non-empty set and $P(X)$ be the set of all subsets of X. For $A, B \in P(X)$, ARB if and only if $A \cap B = \phi$ then the relation

(a) R is reflexive
(b) R is symmetric
(c) R is transitive
(d) R is an equivalence relation

31. Let f be a function satisfying $2f(x) - 3f(1/x) = x^2$ for any $x \neq 0$. Then the value of $f(2)$ is

(a) – 2 (b) – 7/4
(c) – 7/8 (d) 4

32. If $f(x) = \begin{vmatrix} 1 & x & (x+1) \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & x(x-1)(x+1) \end{vmatrix}$

then

$f(50) + f(51) + \ldots\ldots + f(99)$ is equal to

(a) 0 (b) 1275
(c) 3725 (d) none of these

33. The domain of definition of the functions $y(x)$ given by the equation $a^x + a^y = a$ $(a > 1)$ is

(a) $0 < x \le 1$ (b) $0 \le x \le 1$
(c) $-\infty < x < 1$ (d) $-\infty < x \le 0$

34. Let $f(x) = \dfrac{x - [x]}{1 + x - [x]}$, where $[x]$ denotes the greatest integer less than or equal to x, then the range of f is

(a) $[0, 1/2]$ (b) $[0, 1)$
(c) $[0, 1/2)$ (d) $[0, 1]$

35. If $2f(x^2) + 3f(1/x^2) = x^2 - 1$, then $f(x^2)$ is

(a) $(1 - x^4)/5x^2$ (b) $(1 - x^2)/5x$
(c) $5x^2/(1 - x^4)$ (d) none of these

36. If $f(x + 3y, x - 3y) = 12xy$, then $f(x, y)$ is

(a) $2xy$ (b) $2(x^2 - y^2)$
(c) $x^2 - y^2$ (d) none of these

37. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$ then $f^{-1}(x)$ is

(a) $(1/2)^{x(x-1)}$ (b) $\dfrac{1}{2}\left(1 + \sqrt{1 + 4\log_2 x}\right)$
(c) $\dfrac{1}{2}\left(1 - \sqrt{1 + 4\log_2 x}\right)$ (d) not defined

38. If $n(A) = 3$ and $n(B) = 5$ then number of one-one functions that can be defined from A to B is

(a) 30 (b) 40
(c) 120 (d) 60

39. Let f: $\{x, y, z\} \to \{1, 2, 3\}$ be a one-one function. If it is given that exactly one of the following statements is true,

Statement-1: $f(x) = 1$, Statement-2: $f(y) \ne 1$, Statement-3: $f(z) \ne 2$.

then $f^{-1}(1)$ is

(a) x (b) y
(c) z (d) none of these

40. The value of $n \in \mathbf{z}$ for which the function

$f(x) = \dfrac{\sin nx}{\sin(x/n)}$ has 4π as its period is

(a) 2 (b) 3
(c) 4 (d) 5

41. Let R be a relation on $\mathbf{N}$ defined by

$$R = \{(m, n) : m, n \in N \text{ and } m = n^2\}.$$

Which of the following is true.

(a) $(n, n) \in R$. $\forall\, n \in \mathrm{N}$
(b) $(m, n) \in R \Rightarrow (n, m) \in R$
(c) $(m, n) \in R$, $(n, p) \in R \Rightarrow (m, p) \in R$
(d) none of these

42. Of the number of three athletic teams in a school, 21 are in the basketball team, 26 in hockey team and 29 in the football team, 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. The total number of members is

(a) 42 (b) 43
(c) 45 (d) none of these

43. E, I, R, O denote respectively the sets of all equilateral, isosceles, right angled and obtuse angled triangles in a plane, then which of the following is not true.

(a) $R \cap E = \phi, R \cap I \ne \phi$
(b) $E \cap O = \phi, O \cap I \ne \phi$
(c) $E \cap I = \phi, E \cap O \ne \phi$
(d) $E \cap I \ne \phi, E \subset I$

44. If $A = \{3^n : n \in \mathbf{N}, n \le 6\}$, $B = \{9^n : n \in \mathbf{N}, n \le 4\}$ then which of the following is false

(a) $A \,\Delta\, B = \{6561\}$
(b) $A \sim B = \{3, 27, 243\}$
(c) $A \cap B = \{9, 81, 729\}$
(d) $A \cup B = \{3, 9, 27, 81, 243, 729, 6561\}$

45. If $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = ax + \sin x + a$, then f is one-one and onto for all

(a) $a \in \mathbf{R}$ (b) $a \in \mathbf{R} \sim [-1, 1]$
(c) $a \in \mathbf{R} \sim \{0\}$ (d) $a \in \mathbf{R} \sim \{-1\}$

46. If f: $(0, \pi) \to \mathbf{R}$ is given by $f(x) = \sum_{k=1}^{n} [1 + \sin kx]$, $[x]$ denotes the greatest integer function, then the range of $f(x)$ is

(a) $\{n-1, n+1\}$ (b) $\{n\}$
(c) $\{n, n+1\}$ (d) $\{n-1, n\}$

47. If the number of elements in $(A \sim B) \sim C$, $(B \sim C) \sim A$, $(C \sim A) \sim B$ and $A \cap B \cap C$ is 10, 15, 20, and 5 respectively then the number of elements in $(A \Delta B) \Delta C$ is

(a) 35 (b) 50
(c) 40 (d) 45

48. Let $f(x) = \dfrac{x-3}{x+1}, x \ne -1$. Then $f^{2010}(2014)$ (where $f^n(x) = fof \ldots$ of (x) (n times)) is

(a) 2010 (b) 4020
(c) 4028 (d) 2014

Assertion-Reason Type Questions

49. Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ be a relation on set $A = \{1, 2, 3\}$

    Statement-1: R is not an equivalence relation on A.
    Statement-2: R is function from A to A.

50. **Statement-1:**

    If A is a set with 5 elements and B is a set containing 9 elements, then the number of injective mappings from A to B to $9! - 4!$.

    Statement-2: The total number of injective mappings from a set with m elements to a set with n elements, $m \le n$, is $\dfrac{n!}{(n-m)!}$.

51. Let $f(x) = \sin x + \cos x$, $g(x) = \dfrac{\sin x}{1-\cos x}$.

    Statement-1: f is neither an odd function nor an even function.

    Statement-2: g is an odd function.

52. **Statement-1:** A function $f : R \to R$ satisfies the equation $f(x) - f(y) = x - y \ \forall\ x, y \in R$ and $f(3) = 2$, then $f(xy) = xy - 1$

    Statement-2: $f(x) = f(1/x) \ \forall\ x \in R$, $x \ne 0$, and $f(2) = 7/3$ if $f(x) = \dfrac{x^2+x+1}{x^2-x+1}$.

53. **Statement-1:** Let $A = \{2, 3, 7, 9\}$ and $B = \{4, 9, 49, 81\}$ $f : A \to B$ is a function defined as $f(x) = x^2$. Then f is a bijection from A to B.

    Statement-2: A function f from a set A to a set B is a bijection if $f(A) = B$ and $f(x_1) \ne f(x_2)$ if $x_1 \ne x_2$ for all $x_1, x_2 \in A$ and $n(A) = n(B)$.

LEVEL 2

Straight Objective Type Questions

54. The range of the function $f(x) = \cos [x]$ for $-\pi/2 < x < \pi/2$ contains

    (a) $\{-1, 1, 0\}$
    (b) $\{\cos 1, 1, \cos 2\}$
    (c) $\{\cos 1, -\cos 1, 1\}$
    (d) $[-1, 1]$.

55. The domain of the function

    $f(x) = \log_{10} \dfrac{x-5}{x^2-10x+24} - 3\sqrt{x+5}$ is

    (a) $(-5, \infty)$
    (b) $(5, \infty)$
    (c) $(2, 5) \cup (5, \infty)$
    (d) $(4, 5) \cup (6, \infty)$

56. If $g(x) = 1 + \sqrt[3]{x}$ then a function f such that $f(g(x)) = 3 - 3(\sqrt[3]{x}) + x$ is

    (a) $f(x) = x^3 - 3x^2 + x + 5$
    (b) $f(x) = x^3 + 3x^2 - x - 5$
    (c) $f(x) = x^3 - 3x^2 + 5$
    (d) $f(x) = x^3 - 3x^2 + 3x + 3$

57. The function $f(x) = \dfrac{x}{e^x-1} + \dfrac{x}{2} + 1$ is

    (a) even
    (b) periodic
    (c) odd
    (d) neither even nor odd

58. Let f and g be two functions defined by $f(x) = \dfrac{x}{x+1}$, $g(x) = \dfrac{x}{1-x}$. Then $(f \ o\ g)^{-1}(x)$ is equal to

    (a) x
    (b) 1
    (c) $2x$
    (d) none of these

59. Let $f : (-1, 1) \to (0, \pi)$ be defined by

    $f(x) = \cot^{-1} \dfrac{2x}{1-x^2}$ Then

    (a) f is one-one but not onto
    (b) f is onto but not one-one
    (c) f is both one-one and onto
    (d) f is neither one-one nor onto

60. Let $f : X \to [1, 27]$ be a function by $f(x) = 5 \sin x + 12 \cos x + 14$. The set X so that f is one-one and onto is

    (a) $[-\pi/2, \pi/2]$
    (b) $[0, \pi]$
    (c) $[0, \pi/2]$
    (d) none of these

61. Let $f(x) = \dfrac{1-x}{1+x}$. Then $f \ o\ f(\cos x)$ is equal to

    (a) $\cos 2x$
    (b) $\cos x$
    (c) $\tan 2x$
    (d) $\tan x$

62. Let $f(x) = \sin\sqrt{x}$, then

    (a) $f(x)$ is periodic with period $\sqrt{2\pi}$
    (b) $f(x)$ is periodic with period $\sqrt{\pi}$
    (c) $f(x)$ is periodic with period $4\pi^2$
    (d) none of these

63. Let $X = Y = \mathbf{R} \sim \{1\}$. The function $f : X \to Y$ defined by $f(x) = \dfrac{x+2}{x-1}$ is
    (a) one-one but not onto
    (b) onto but not one-one
    (c) neither one-one nor onto
    (d) one-one and onto

64. If $f(x) = \dfrac{a^x + a^{-x}}{2}$ and $f(x+y) + f(x-y) = K f(x) f(y)$ then K is equal to
    (a) 2 (b) 4
    (c) -2 (d) none of these

65. If $f(x) = 2\sin^{-1}\sqrt{x-3}$, then the domain of f is
    (a) [3, 4] (b) [– 1, 1]
    (c) $[-\pi/2, \pi/2]$ (d) [6, 8]

66. Let $f(x) = x\,|x|$ and $g(x) = \sqrt{|x|}$ then the number of elements in the set $\{x \in \mathbf{R} : f(x) = g(x)\}$ is
    (a) 1 (b) 2
    (c) 3 (d) infinitely many

67. Which of the following functions are odd functions
    (a) $f(x) = x\left(\dfrac{a^x + a^{-x}}{a^x - a^{-x}}\right)$
    (b) $f(x) = \dfrac{a^x + x}{a^x - x}$
    (c) $f(x) = \dfrac{a^x - 1}{a^x + 1}$
    (d) $f(x) = x\log_2\left(x + \sqrt{x^2+1}\right)$

68. Let $g(x) = 1 + x - [x]$ and $f(x) = \text{sgn}\, x$. Then for all x, $f \circ g(x)$ is equal to
    (a) x (b) 1
    (c) $f(x)$ (d) $g(x)$

    Note $f(x) = \text{sgn}\, x = 1$ if $x > 0$
    $= 0$ if $x = 0$
    $= -1$ if $x < 0$

69. The domain of definition of the function $f(x) = \log_2(\log_{1/2}(x^2 + 4x + 3)) + \sin^{-1}(2[x]^2 - 3)$, $[x]$ denotes the greatest integer $\le x$ is
    (a) $0 < x \le 1$ (b) $0 \le x \le 1$
    (c) $-\infty < x \le 0$ (d) none of these

70. If $g(f(x)) = |\sin x|$ and $f(g(x)) = (\sin\sqrt{x})^2$, then f and g may be given by
    (a) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$
    (b) $f(x) = \sin x$, $g(x) = |x|$
    (c) $f(x) = x^2$, $g(x) = \sin x$
    (d) f and g cannot be determined

71. If $f(x) = 3x - 5$, then $f^{-1}(x)$
    (a) is given by $\dfrac{1}{3x-5}$
    (b) is given by $\dfrac{x+5}{3}$
    (c) does not exist because f is not one-one
    (d) does not exist because f is not onto

72. Let $f(x) = |x - a|$, $a \ne 0$ then
    (a) $f(x^2) = (f(x))^2$
    (b) $f(|x|) = |f(x)|$
    (c) $f(x+y) = f(x) + f(y)$
    (d) none of these

73. If $n(A) = n(B) = 4$ then number of bijections from A to B is
    (a) 6 (b) 24
    (c) 12 (d) 18

Previous Years' AIEEE/JEE Main Questions

1. Let R = {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)} be a relation on the set $A = \{1, 2, 3, 4\}$. The relation R is
    (a) not symmetric (b) transitive
    (c) a function (d) reflexive **[2004]**

2. The range of the function is $f(x) = {}^{7-x}P_{x-3}$ is
    (a) {1, 2, 3, 4} (b) {1, 2, 3, 4, 5, 6}
    (c) {1, 2, 3} (d) {1, 2, 3, 4, 5} **[2004]**

3. The domain of the function $f(x) = \dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is
    (a) [1, 2] (b) [2, 3]
    (c) [2, 3] (d) [1, 2] **[2004]**

4. If $f : R \to S$ defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$ is onto, then the interval of S is
    (a) [0, 1] (b) [–1, 1]
    (c) [0, 3] (d) [–1, 3] **[2004]**

5. The graph of the function $y = f(x)$ is symmetrical about $x = 2$ then

(a) $f(x) = f(-x)$ (b) $f(2 + x) = f(2 - x)$
(c) $f(x + 2) = f(x - 2)$ (d) $f(x) = -f(-x)$ **[2004]**

6. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$

The relation is

(a) an equivalence relation
(b) reflexive and symmetric only
(c) reflexive and transitive only
(d) reflexive only **[2005]**

7. A real valued function $f(x)$ satisfies the functional equation

$$f(x - y) = f(x)f(y) - f(a - x)f(a + y)$$

where a is a given constant and $f(0) = 1$, $f(2a - x)$ is equal to

(a) $f(a) + f(a - x)$ (b) $f(-x)$
(c) $-f(x)$ (d) $f(x)$ **[2005]**

8. Let W denote the words in the English Dictionary. Define the relation R by $R = \{(x, y) \in W \times W$: the words x and y have at least one letter in common$\}$, then R is

(a) reflexive, not symmetric and transitive.
(b) not reflexive, symmetric and transitive.
(c) reflexive, symmetric and not transition.
(d) reflexive, symmetric and transitive **[2006]**

9. The largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which the function $f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log \cos x$ is defined is

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(c) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) $\left[0, \frac{\pi}{2}\right)$ **[2007]**

10. Let **R** be the real line. Consider the following subsets of the plane **R** × **R**:

$S = \{(x, y): y = x + 1 \text{ and } 0 < x < 2\}$

$T = \{(x, y): x - y \text{ is an integer}\}$

which of the following is true :

(a) T is an equivalence relation on R but S is not
(b) Neither S nor T is an equivalence relation.
(c) Both S and T are equivalence relation on R.
(d) S is an equivalence relation but T is not. **[2008]**

11. Let $f : N \to Y$ be a function defined as

$$f(x) = 4x + 3 \text{ where } Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}.$$

Show that f is invertible and its inverse is

(a) $g(y) = \frac{3y + 4}{3}$ (b) $g(y) = 4 + \frac{y + 3}{4}$
(c) $g(y) = \frac{y + 3}{4}$ (d) $g(y) = \frac{y - 3}{4}$ **[2008]**

12. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then

(a) $B = C$ (b) $A \cap B = \phi$
(c) $A = B$ (d) $A = C$ **[2009]**

13. Consider the following relations

$R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$

$S = \left\{\left(\frac{m}{n}, \frac{p}{q}\right)\right\}$ $\{m, n, p, q$ are integers such that $n \cdot q \neq 0$ and $qm = pn\}$

Then

(a) S is an equivalence relation but R is not an equivalence relation
(b) R and S both are equivalence relations.
(c) R is an equivalence relation but S is not an equivalence relation.
(d) Neither R nor S is an equivalence relation. **[2010]**

14. Let R be the set of real numbers.

Statement-1:

$A = \{(x,y) \in R \times R : y - x$ is an integer$\}$ is an equivalence relation on R.

Statement-2:

$B = \{(x, y) \in R \times R : x = \alpha y$ for some rational numbers $\alpha\}$ in an equivalence relation on R. **[2011]**

15. Let f be a function defined by

$$f(x) = (x - 1)^2 + 1, (x \geq 1)$$

Statement-1:

The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$

Statement-2: f is a bijection and

$f^{-1}(x) = 1 + \sqrt{x - 1}, x \geq 1.$ **[2011]**

16. Consider the following relation R on the set of real square matrices of order 3.

$R = \{(A,B): A = P^{-1} B P$ for some invertible matrix $P\}$

Statement-1: R is an equivalence relation.

Statement-2: For any two invertible 3 × 3 matrices M and N, $(M N)^{-1} = N^{-1}M^{-1}$ **[2011]**

17. Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can be formed such that $Y \subseteq X$, $Z \subseteq X$ and $Y \cap Z$ is empty is

(a) 2^5 (b) 5^3
(c) 5^2 (d) 3^5. **[2012]**

18. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is
(a) 211 (b) 256
(c) 220 (d) 219 **[2013]**

19. Let P be the relation defined on the set of all real numbers such that
$P = \{(a, b) : \sec^2 a - \tan^2 b = 1\}$. Then P is
(a) reflexive and symmetric but not transitive.
(b) reflexive and transitive but not symmetric.
(c) symmetric and transitive but not reflexive.
(d) an equivalence relation. **[2014]**

20. Let $f(n) = \left[\frac{1}{3}+\frac{3n}{100}\right]n$, where $[n]$ denotes the greatest integer less than or equal to n. Then $\sum_{n=1}^{56} f(n)$ is equal to
(a) 689 (b) 1399
(c) 1287 (d) 56 **[2014]**

21. The function $f(x) = |\sin 4x| + |\cos 2x|$ is a periodic function with period
(a) $\pi/2$ (b) 2π
(c) π (c) $\pi/4$ **[2014]**

22. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \frac{|x|-1}{|x|+1}$ then f is
(a) onto but not one-one
(b) both one-one and onto
(c) one-one but not onto
(d) neither one-one nor onto **[2014]**

23. A relation on the set $A = \{x: |x| < 3, x \in \mathbf{Z}\}$, where $\mathbf{Z}$ is the set of integers is defined by $R = \{(x, y) : y = |x|, \ x \neq -1\}$. Then the number of elements in the power set of R is
(a) 32 (b) 16
(c) 8 (d) 64 **[2014]**

24. Let A and B be two sets containing four and two elements respectively. Then the number of subsets of the set $A \times B$, each having atleast three elements is
(a) 219 (b) 256
(c) 275 (d) 510 **[2015]**

25. In a certain town, 25% of the families own a phone and 15% own a car, 65% families own neither a phone nor a car and 2000 families own both a car and a phone. Consider the following three statements:
(a) 5% families own both a car and a phone
(b) 35% families own either a car or a phone
(c) 40,000 families live in town. Then,
(a) only (a) and (b) are correct
(b) only (a) and (c) are correct
(c) only (b) and (c) are correct
(d) all (a), (b) and (c) are correct **[2015, online]**

26. Let $A = \{x_1, x_2, \ldots x_7\}$ and $B = \{y_1, y_2, y_3\}$ be two sets containing seven and three distinct elements respectively. Then the total number of functions $f : A \to B$ that are onto, if there exist exactly three x in A such $f(x) = y_2$ is equal to
(a) $14.{}^7C_2$ (b) $16.{}^7C_3$
(c) $12.{}^7C_2$ (d) $14.{}^7C_3$ **[2015, online]**

27. If $f(x) + 2f\left(\frac{1}{x}\right) = 3x, x \neq 0$ and $S = \{x \in \mathbf{R} : f(x) = f(-x)\}$; then S
(a) is an empty set
(b) contains exactly one element
(c) contains exactly two elements
(d) contains more than two elements **[2016]**

28. If the function $f : [1, \infty[\to [1, \infty[$ is defined by $f(x) = 3^{x(x-1)}$, then $f^{-1}(x)$ is
(a) $\frac{1}{2}\left(1-\sqrt{1+4\log_3 x}\right)$ (b) $\frac{1}{2}\left(1+\sqrt{1+4\log_3 x}\right)$
(c) not defined 2(d) $\left(\frac{1}{3}\right)^{x(x-1)}$ **[2016]**

29. For $x \in \mathbf{R}$, $x \neq 0$, $x \neq 1$, let $f_0(x) = \frac{1}{1-x}$ and $f_{n+1}(x) = f_0(f_n(x))$, $n = 0, 1, 2, \ldots$. Then the $f_{100}(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right)$ is equal to
(a) $\frac{8}{3}$ (b) $\frac{4}{3}$
(c) $\frac{5}{3}$ (d) $\frac{1}{3}$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. The domain of the function
$f(x) = \sqrt{2x-3} + \sin x + \sqrt{x-1}$ is
(a) $(-\infty, 1]$ (b) $[0, 1]$
(c) $\left[\frac{3}{2}, \infty\right)$ (d) $[1, \infty]$ **[2006]**

2. Let $f : (1, \infty) \rightarrow (1, \infty)$ be defined by $f(x) = \dfrac{x+2}{x-1}$. Then
   (a) f is 1 – 1 and onto
   (b) f is 1 – 1 but not onto
   (c) f is not 1 – 1 but onto
   (d) f is neither 1 – 1nor onto **[2008]**

3. Let $A = \{(x, y) : x > 0, y > 0, x^2 + y^2 = 1\}$ and let $B = \{(x, y) : x > 0, y > 0, x^6 + y^6 = 1\}$ Then $A \cap B$
   (a) A (b) B
   (c) ϕ (d) $\{(0, 1), (1, 0)\}$ **[2008]**

4. A school awarded 38 medals in football, 15 in basketball and 20 in cricket. Suppose these medals went to a total of 58 students and only three students got medals in all three sports. If only 5 students got medals in football and basketball, then the number of medals received in exactly two of three sports is
   (a) 7 (b) 9
   (c) 11 (d) 13 **[2008]**

5. Let Q be the set of all rational numbers and R be the relation defined as

   $R = \{(x, y) : 1 + xy > 0, x, y \in Q\}$

   Then relation R is
   (a) symmetric and transitive
   (b) reflexive and transitive
   (c) an equivalence relation
   (d) reflexive and symmetric **[2009]**

6. The domain of the function $f(x) = \dfrac{1}{3 - \log_3 (x-3)}$ is
   (a) $(-\infty, 30)$ (b) $(-\infty, 30) \cup (30, \infty)$
   (c) $(3, 30) \cup (30, \infty)$ (d) $(4, \infty)$ **[2009]**

7. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a function defined by $f(x) = x^{2009} + 2009x + 2009$
   Then $f(x)$ is
   (a) one-one but not onto
   (b) not one-one but onto
   (c) neither one-one nor onto
   (d) one-one and onto **[2010]**

8. Let f be a function defined on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ by

   $f(x) = 3\cos^4 x - 6\cos^3 x - 6\cos^2 x - 3$

   Then the range of $f(x)$ is
   (a) $[-12, -3]$ (b) $[-6, -3]$
   (c) $[-6, 3)$ (d) $(-12, 3]$ **[2010]**

9. **Statement-1:** The function $f(x) = x^2 e^{-x^2} \sin|x|$ is even

   Statement-2: Product of two odd function is an even function **[2011]**

10. Consider the following relations
    $R_1 = \{(x, y) : x$ and y are integers and $x = ay$ or $y = ax$ for some integer $a\}$
    $R_2 = \{(x, y)$: x and y are integres and $ax + by = 1$ for some integers $a, b\}$
    Then
    (a) R_1, R_2 are not equivalence relations
    (b) R_1, R_2 are equivalence relation
    (c) R_1 is an equivalence relation but R_2 is not
    (d) R_2 is an equivalence relation but R_1 is not **[2012]**

11. Let f and g be functions defined by $f(x) = \dfrac{1}{x+1}$ $x \in \mathbf{R}$, $x \neq -1$ and $g(x) = x^2 + 1$, $x \in \mathbf{R}$. Then $g \circ f$ is
    (a) one-one but not onto
    (b) onto but not one-one
    (c) both one-one and onto
    (d) neither one-one nor onto **[2013]**

12. Let $\mathbf{N}$ be the set of natural number and for $a \in \mathbf{N}$, $a\mathbf{N}$ denotes the set $\{ax : x \in \mathbf{N}\}$.

    If $b\mathbf{N} \cap c\mathbf{N}, = d\mathbf{N}$, where b, c, d are natural numbers greater than 1 and the greatest common divisor of b and c is 1 then d equals
    (a) $\max\{b, c\}$ (b) $\min\{b, c\}$
    (c) bc (d) $b + c$ **[2014]**

13. Let $f(x) = (x+1)^2 - 1$, $x \geq -1$, then the set $\{x : f(x) = f^{-1}(x)\}$:
    (a) is an empty set
    (b) contains exactly one element
    (c) contains exactly two elements
    (d) contains more than two elements **[2014]**

14. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a function defined by

    $f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$, then f is
    (a) one-one and onto (b) one-one but not onto
    (c) onto but not one-one (d) neither onto nor one-one **[2015]**

15. If f is a function of real variable x satisfying $f(x+4) - f(x+2) + f(x) = 0$, then f is periodic with period
    (a) 8 (b) 10
    (c) 12 (d) 6 **[2016]**

Answers

Concept-based

1. (d) **2.** (c) **3.** (b) **4.** (d)
5. (b) **6.** (b) **7.** (d) **8.** (c)

Level 1

9. (a) **10.** (c) **11.** (c) **12.** (c)

13. (d)	**14.** (c)	**15.** (a)	**16.** (d)
17. (b)	**18.** (b)	**19.** (a)	**20.** (a)
21. (d)	**22.** (b)	**23.** (d)	**24.** (c)
25. (a)	**26.** (d)	**27.** (b)	**28.** (b)
29. (b)	**30.** (b)	**31.** (b)	**32.** (a)
33. (c)	**34.** (c)	**35.** (d)	**36.** (c)
37. (b)	**38.** (d)	**39.** (b)	**40.** (a)
41. (d)	**42.** (b)	**43.** (c)	**44.** (a)
45. (b)	**46.** (c)	**47.** (b)	**48.** (d)
49. (c)	**50.** (d)	**51.** (b)	**52.** (b)
53. (a)			

Level 2

54. (b)	**55.** (d)	**56.** (c)	**57.** (a)
58. (a)	**59.** (c)	**60.** (d)	**61.** (b)
62. (d)	**63.** (d)	**64.** (a)	**65.** (a)
66. (b)	**67.** (c)	**68.** (b)	**69.** (d)
70. (a)	**71.** (b)	**72.** (d)	**73.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (a)	**2.** (c)	**3.** (b)	**4.** (d)
5. (b)	**6.** (c)	**7.** (b)	**8.** (c)
9. (d)	**10.** (a)	**11.** (d)	**12.** (a)
13. (a)	**14.** (c)	**15.** (a)	**16.** (b)
17. (d)	**18.** (d)	**19.** (c)	**20.** (b)
21. (a)	**22.** (d)	**23.** (b)	**24.** (a)
25. (d)	**26.** (d)	**27.** (c)	**28.** (b)
29. (c)			

Previous Years' B-Architecture Entrance Examination Questions

1. (c)	**2.** (a)	**3.** (c)	**4.** (b)
5. (d)	**6.** (c)	**7.** (d)	**8.** (a)
9. (b)	**10.** (c)	**11.** (d)	**12.** (c)
13. (c)	**14.** (d)	**15.** (c)	

Hints and Solutions

Concept-based

1. $n(A \cup B') = n(A) + n(B') - n(A \cap B')$
$= n(A) + n(U) - n(B) - (n(A) - n(A \cap B))$
$= n(U) - n(B) + n(A \cap B)$
$= 12 - 6 + 2 = 8.$

2. If $a = -1$ than a is not related to a. So R is not reflexive. If $a = -1$, $b = 2$ then $|a| \le b$ but $|b| > a$, so R is not symmetric. If $|a| \le b$ and $|b| \le c$ then $|a| \le b \le |b| \le c$ so $|a| \le c$, hence a is related to c. Thus R is transitive. If $|a| \le b$, and $|b| \le a$ then $a \le |a| \le b \le |b| \le a$. So $a = b$, R is anti symmetric.

3. $f(g(x)) = g(f(x))$ for all $x \in \mathbf{R} \Leftrightarrow f(cx + d) = g(ax + b)$
$\Leftrightarrow a(cx + d) + b = c(ax + b) + d \Leftrightarrow ad + b = cb + d$
$\Leftrightarrow f(d) = g(b)$

4. For $f(x)$ to be defined, we have $|x| > x$. Since $|x| \ge x \;\forall\; x \in \mathbf{R}$ and $|x| = x$ for $x \in [0, \infty)$ so $|x| > x$ if $x \in \mathbf{R} = (-\infty, 0)$

5. The function is defined if $-1 \le 1 - 2x \le 1 \Leftrightarrow$
$-2 \le -2x \le 0$
$\Leftrightarrow 0 \le x \le 1.$

6. $f_1(x) = \dfrac{f(x)+f(-x)}{2} = \dfrac{1}{2}\,[\sin x - \cos x - \sin x - \cos x]$
$= -\cos x$
$f_2(x) = \dfrac{f(x)-f(-x)}{2} = \dfrac{1}{2}\,[\sin x - \cos x + \sin x + \cos x]$
$= \sin x$

7. $-1 \le \sin x \le 1 \Leftrightarrow -1 \le -\sin x \le 1 \Leftrightarrow 0 \le 1 - \sin x \le 2$

8. $f(-x) = (-x)^2 \log\dfrac{1+x}{1-x} = x^2 \log\left(\dfrac{1-x}{1+x}\right)^{-1} =$
$-x^2 \log\dfrac{1-x}{1+x} = -f(x).$

Level 1

9. $((A \sim B) \cup (B \sim A)) \cap A$
$= ((A \sim B) \cap A) \cup (B \sim A) \cap \text{A}$
$= (A \sim B) \cup \phi = (A \sim B) = \{3, 5\}$

10. $(A \cap B) \cap C = \{2 \times 3 \times 5\, x : x \in N\}$
$= \{30x : x \in \mathbf{N}\}$

11. $X \subset Y \Rightarrow X \cap Y = X, X \cup Y = Y$
$Y \subset X \Rightarrow X \cap Y = Y, X \cup Y = X.$
So $X \cap Y = X \cup Y \Rightarrow X = Y$

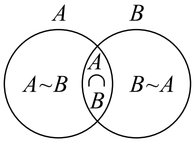
12. If $X \underset{\neq}{\subset} Y$, let $y \in Y$ and $y \notin X$ then either $y \notin A$. or $y \in A$
So if $y \in A$, then $y \in A \cap Y$ but $y \notin A \cap X$ and thus $A \cap X \ne A \cap Y$.
If $y \notin A$, then $y \in A \cup Y$ but $y \notin A \cup X$ and thus $A \cup X \ne A \cup Y$
So $X \subseteq Y$, similarly $Y \subseteq X$.
$\Rightarrow \qquad X = Y.$

13. $A = \{7, 8\}$, $B = \{1, 2, 3, ... 9\}$
$C = \{1, 2, 4, 7, 8, 14, 16, ...\}$
$B \cap C = \{1, 2, 4, 7, 8\}$
$A \cup (B \cap C) = \{1, 2, 4, 7, 8\}$

14. $A = \{\text{M, A, T, H, E, I, C, S}\}$
$B = \{\text{S, T, A, I, C}\}$
$B \sim A = \phi$, So $A \,\Delta\, B = (A \sim B) \cup (B \sim A)$
$= A \sim B.$
Verify (a) and (b) are not correct.

15. $n(A \sim B) = n(A) - n(A \cap B)$
$\Rightarrow \quad 15 = n(A) - 16 \Rightarrow n(A) = 31$
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
$\Rightarrow \quad 36 = 31 + n(B) - 16$
$\Rightarrow \quad n(B) = 21.$

16. $A \cup B, A \sim B, B \sim A$
$(A \cup B) \sim (A \sim B) = B$
$(A \cup B) \sim (B \sim A) = A$
$A \Delta B = (A \sim B) \cup (B \sim A)$
$(A \cup B) \sim (A \Delta B) = A \cap B$
and $(A \sim B) \sim A = \phi$
Thus, the required no is 8.



17. $n(A \sim B) = n(A) - n(A \cap B)$
$= n(B) - n(A \cap B) = n(B \sim A)$

18. $A \cup B = A \Delta B \cup (A \cap B)$
So $A \cup B = A \Delta B \Rightarrow A \cap B = \phi$.

19. $n(P) = 60$, $n(M) = 58$, $n(P \cup M) = 100$, $N(P \cap M)$
$= n(P) + M(M) - n(P \cup M)$

20. $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5)\}$

21. $(1, 2) \in R$ but $(1, 2) \notin R_1$ or R_2 or R_3 so $R \neq R_1$ or R_2 or R_3.

22. L_1 is not $\perp L_1$, so R is not reflexive
$L_1 \perp L_2 \Rightarrow L_2 \perp L_1 \Rightarrow R$ is symmetric.
$L_1 \perp L_2$ and $L_2 \perp L_3 \Rightarrow L_1 \parallel L_3 \Rightarrow R$ is not transitive.

23. See the definition of function

24. $f(x) = f^{-1}(x) \Rightarrow f(f(x)) = x.$
$\Rightarrow \quad [(x + 1)^2 - 1 + 1]^2 - 1 = x.$
$\Rightarrow \quad (x + 1)^2 = x + 1 \Rightarrow x = 0$ or $x = -1$
So $S = \{0, -1\}$.

25. $A \times A$ has $n \times n = n^2$ elements and any subset of $A \times A$ is a relation on A. The number of such subsets is 2^{n^2}.

26. $-1 \leq \sin x \leq 1 \Rightarrow 1 \leq f(x) \leq 5 \Rightarrow f(x)$ is not onto, $f(x) = f(x + 2\pi) \Rightarrow f$ is not one-one and f is not bijective.

27. In (a) $a \to 2a$ and $a \to -2a$, so it is not a function
In (b) for each $x \in R$ there is unique $|x| = y \in R$, so it is a function.

28. $f(x) = x^4 + 2 = y \Rightarrow x = (y - 2)^{1/4}$.
$\Rightarrow \quad f^{-1}(x) = (x - 2)^{1/4}$
$\Rightarrow \quad f^{-1}(83) = (81)^{1/4} = \pm 3$
and $f^{-1}(-2) = (-4)^{1/4} = \phi$.

29. R is reflexive if $(x, x). \in R$ for all $x \in \{1, 2, 3\}$
$\Rightarrow \quad (1, 1), (2, 2), (3, 3) \in R$
R is symmetric if $(1, 2), (2, 3) \in R$
$\Rightarrow \quad (2, 1), (3, 2) \in R.$
R is transitive if $(1, 2), (2, 3) \in R$
$\Rightarrow \quad (1, 3) \in R$ also $(3, 1) \in R$ as
R is symmetric.
So the total numbers of elements is 9.

30. $A \cap A = A \neq \phi$ if $A \neq \phi$, So R is not reflexive.
$A \cap B = \phi \quad \Rightarrow \quad B \cap A = \phi$
$ARB \quad \Rightarrow \quad BRA \quad \Rightarrow \quad R$ is symmetric
Note R is not transitive and not an equivalence relation.

31. $2f(2) - 3f(1/2) = 2^2 = 4$
$2f(1/2) - 3f(2) = (1/2)^2 = 1/4$
$\Rightarrow \quad 5(f(2) - f(1/2)) = 4 - 1/4 = 15/4$
$\Rightarrow \quad f(1/2) = f(2) - 3/4.$
So $\quad 2f(2) - 3[f(2) - 3/4] = 4$
$$\Rightarrow \quad f(2) = \frac{9}{4} - 4 = -\frac{7}{4}.$$

32. Applying $C_3 \to C_3 - (C_1 + C_2)$ we get $f(x) = 0$ for all x.

33. $a^{x-1} + a^{y-1} = 1$
$\Rightarrow \quad a^{y-1} = 1 - a^{x-1} \Rightarrow (y - 1)\log a = \log(1 - a^{x-1})$
R.H.S is defined if $1 - a^{x-1} > 0$
$\Rightarrow \quad a^{x-1} < 1 = a^{\circ} \Rightarrow x - 1 < 0$ as $a > 1$
$\Rightarrow \quad x < 1$ and the required domain is $-\infty < x < 1$.

34. If x is an integer, $f(x) = 0$
if x is not an integer $x - [x] = \dfrac{p}{q}$ where p and q are positive integers and $p < q$, so that $f(x) = \dfrac{p}{p+q} < \dfrac{1}{2}$
so $0 \leq f(x) < \dfrac{1}{2}$ and the required range is $[0, \dfrac{1}{2}]$

35. $2f(x^2) + 3f(1/x^2) = x^2 - 1$
$\Rightarrow \quad 2f(1/x^2) + 3f(x^2) = 1/x^2 - 1$
Subtracting, $f(x^2) - f(1/x^2) = 1/x^2 - x^2$
$\Rightarrow \quad f(x^2) = 1/x^2.$

36. $f(x + 3y, x - 3y) = 12xy = (x + 3y)^2 - (x - 3y)^2$
$\Rightarrow \quad f(x, y) = x^2 - y^2.$

37. Let $f(x) = y = 2^{x(x - 1)}$
$\Rightarrow \quad \log y = x(x - 1) \log^2$
$\Rightarrow \quad x^2 - x - \log_2 y = 0$
$$\Rightarrow \quad x = \frac{1 \pm \sqrt{1 + 4 \log_2 y}}{2}$$
$$\Rightarrow \quad x = f^{-1}(y) = \frac{1 \pm \sqrt{1 + 4 \log_2 y}}{2}$$
$$\Rightarrow \quad f^{-1}(x) = \frac{1 + \sqrt{1 + 4 \log_2 y}}{2} \text{ as } f^{-1}(x) \geq 1.$$

38. Number of one-one functions $= 5 \times 4 \times 3 = 60$.

39. If we take $f(x) = 1$, then $f(y) \neq 1$ as the function is one-one. So $f(x) \neq 1$
If $f(z) = 1$, then statements 2 and 3 both are true, so $f(z) \neq 1$ hence $f(y) = 1$ and $f^{-1}(1) = y$.
[**Note** $f(x) = 2, f(y) = 1, f(z) = 3$)]

40. $f(x + 4\pi) = f(x)$
$$\text{If } \frac{\sin n(x + 4\pi)}{\sin \frac{x + 4\pi}{n}} = \frac{\sin nx}{\sin \frac{x}{n}}$$
which holds if $n = 2$.

41. $(n, n) \in R$ only for $n = 1$

$m = n^2 \Rightarrow n = m^2$ unless $m = n = 1$

$m = n^2$ and $n = p^2 \;\not\Rightarrow\; m = p^2$ for $m, n, p \neq 1$

hence (a), (b), (c) none is true.

42. Let B, H, F be the sets of the three teams respectively

so $n(B) = 21, n(H) = 26, n(F) = 29,$

$n(H \cap B) = 14$ $n(H \cap F) = 15, n(F \cap B) = 12,$

$n(B \cap H \cap F) = 8$

and $n(B \cup H \cup F) = n(B) + n(H) + n(F) - n(H \cap B)$

$- n(H \cap F) - n(F \cap B) + n(B \cap H \cap F) = 21 + 26$

$+ 29 - 14 - 15 - 12 + 8 = 43$

43. An equilateral triangle cannot be obtuse angled or right angled but is an isosceles triangle.

So $E \subset I$ and $E \cap I = E$.

44. $A = \{3, 9, 27, 81, 243, 729\}$, $B = \{9, 81, 729, 6561\}$

$A \cup B = \{3, 9, 27, 81, 243, 729, 6561\}$

$A \cap B = \{9, 81, 729\}$

$A \sim B = \{3, 27, 243\}$

$B \sim A = \{6561\}$

$A \,\Delta\, B = (A \sim B) \cup (B \sim A) = \{3, 27, 243, 6561\}$

45. For $a \neq 0$, the range of f is $\mathbf{R}$. f is differentiable function so f is one-one and only if f is monotonic.

$$f'(x) = a + \cos x$$

If $a > 1$, $f'(x) > 0$ i.e. f is increasing

If $a < -1$, $f'(x) < 0$ i.e. f is decreasing

Thus f is monotonic if $a \in \mathbf{R} \sim [-1, 1]$.

46. $f(x) = \sum_{k=1}^{n} 1 + [\sin kx] = n + [\sin x] + [\sin 2x] + \cdots + [\sin nx]$. If $kx \neq \frac{\pi}{2}$ for any $k = 1, 2 \ldots n$ then $0 < kx < \pi$ and $kx \neq \pi/2$ so $0 < \sin kx < 1$. Hence $[\sin kx] = 0$, $k = 1, 2 \ldots n$. i.e. $f(x) = n$. If $kx = \frac{\pi}{2}$ for some k then $x = \frac{\pi}{2k}$, hence $\sin x, \sin 2x, \ldots, \sin(k-1)x$ will lie between 0 and 1 so $[\sin jx] = 0$ $1 \leq j \leq k-1$; $\sin kx = 1$ so $f(x)$ can be $n + 1$ or n.

47. $(A \Delta B) \Delta C$ is disjoint union of $(A \sim B) \sim C$, $(B \sim C) \sim A$, $(C \sim A) \sim B$ and $A \cap B \cap C$.

Threfore, number of elements is $(A \Delta B) \Delta C$ is $10 + 15 + 20 + 5 = 50$.

48. $f^2(x) = fof(x) = f(f(x)) = f\left(\frac{x-3}{x+1}\right)$

$$= \frac{\frac{x-3}{x+1} - 3}{\frac{x-3}{x+1} + 1} = -\frac{x+3}{x-1}$$

$$f^3(x) = f\left(-\frac{x+3}{x-1}\right) = \frac{\frac{-x-3}{x-1} - 3}{\frac{-x-3}{x-1} + 1} = x.$$

Hence $f^{3k}(x) = x$

$$f^{2010}(2014) = f^{3.670}(2014) = 2014.$$

49. $(1, 2) \in R$, But $(2, 1) \notin R \Rightarrow R$ is not symmetric and hence not an equivalence relation.

$\Rightarrow$ Statement-1 is True. Staement-2 is False as $1 \to 1, 2, 3$.

50. Statement-2 is True. Because if A has m elements, then first elements can be mapped to n elements. For the 2nd element the choice is $n - 1$ and so on. So the total number of injective mappings is $n(n - 1)(n - 2) \ldots\ldots (n - m + 1) = n! / (n - m)!$ Which shows that statement-1 is False.

51. $f(-x) = -\sin x + \cos x \neq f(x)$ or $-f(x)$, $\Rightarrow f$ is neither odd nor even. So statement-1 is True

$g(-x) = \frac{-\sin x}{1 - \cos x} = -(g(x)) \Rightarrow g$ is an odd function

$\Rightarrow$ Statement-2 is also True but does not lead to statement-1.

52. Taking $y = 3$, $f(x) - f(3) = x - 3$

$\Rightarrow f(x) = x - 3 + f(3) = x - 3 + 2 = x - 1$

$\Rightarrow f(xy) = xy - 1 \Rightarrow$ Statement –1 is True

$$f(x) = \frac{x^2 + x + 1}{x^2 - x + 1} \Rightarrow f(1/x) = \frac{1 + x + x^2}{1 - x + x^2} = f(x)$$

and $f^{(2)} = 7/3 \Rightarrow$ Statement-2 is also True, but does not lead to statement-1.

53 . Statement-2 is true by definition of a bijective mapping, using which statement-1 is also true.

Level 2

54. $-\pi/2 < x < \pi/2 \Rightarrow [x] = -2, -1, 0, 1$

$\Rightarrow f(x) = \cos[x]$

$= \cos(-2), \cos(-1), \cos(0), \cos 1$

$= \cos 2, \cos 1, 1 \cos 1$

55. $\frac{x-5}{x^2 - 10x + 24} > 0$ and $x + 5 > 0$

$\Rightarrow x > 5, x > 6 \Rightarrow x > 6 \Rightarrow x \in (6, \infty)$

or $x < 5, x > 4 \Rightarrow 4 < x < 5 \Rightarrow x \in (4, 5)$

56. Let $g(x) = 1 + \sqrt[3]{x} = y \Rightarrow x^{1/3} = y - 1$

so $f(g(x)) = 3 - 3x^{1/3} + x$

$\Rightarrow f(y) = 3 - 3(y - 1) + (y - 1)^3$

$= y^3 - 3y^2 + 5$

57. $f(-x) = \frac{-x}{e^x - 1} - \frac{x}{2} + 1$

$$= \frac{-xe^x}{1 - e^x} - \frac{x}{2} + 1$$

$$= \frac{x(e^x - 1 + 1)}{e^x - 1} - \frac{x}{2} + 1$$

$$= \frac{x}{e^x - 1} + \frac{x}{2} + 1 = f(x)$$

58. $(\text{fog})\,(x) = f(g(x)) = f\left(\dfrac{x}{1-x}\right)$

$$= \frac{\dfrac{x}{1-x}}{\dfrac{x}{1-x} + 1} = \frac{x}{x + 1 - x} = x$$

$\Rightarrow (\text{fog})^{-1}\,(x) = x$

59. $f(x) = \cot^{-1} \dfrac{2x}{1 - x^2}$ is clearly one-one as

$f(x_1) = f(x_2)$

$\Rightarrow x_1 = x_2$

Next, let $y = f(x) \Rightarrow \cot y = \dfrac{2x}{1 - x^2}$

$$= \frac{2\tan(\theta/2)}{1 - \tan^2(\theta/2)} = \tan\theta$$

$\Rightarrow y = \pi/2 - \theta = \pi/2 - 2\tan^{-1}x$. Taking $x = \tan(\theta/2)$

so $y \in (0, \pi) \Rightarrow 0 < y < \pi$

$\Rightarrow -\pi/4 < \tan^{-1} x < \pi/4$

$\Rightarrow -1 < x < 1 \Rightarrow x \in (-1, 1)$

and thus f is onto.

60. As $-\sqrt{5^2 + 12^2} \le 5 \sin x + 12 \cos x \le \sqrt{5^2 + 12^2}$ for all x

$\Rightarrow 14 - 13 \le 5 \sin x + 2 \cos x \le 14 + 13$

$\Rightarrow 1 \le f(x) \le 27$

f is not one-one as it is periodic.

61. $\text{fof}\,(\cos x) = f\left(\dfrac{1 - \cos x}{1 + \cos x}\right) = f(\tan^2(x/2))$

$$= \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} = \cos x.$$

62. Since $f(x + T) = \sin \sqrt{x + T} \ne \sin\sqrt{x}$ for any value T.

$f(x)$ is not periodic.

63. Let $y = f(x) = \dfrac{x + 2}{x - 1}$, $y \in Y = \mathbf{R} - \{1\}$

$\Rightarrow x = \dfrac{y + 2}{y - 1}$, $x \in X = \mathbf{R} \sim \{1\}$

$\Rightarrow f$ is onto

Also $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one.

64. $f(x + y) + f(x - y)$

$$= \frac{a^{x+y} + a^{-(x+y)}}{2} + \frac{a^{x-y} + a^{-(x-y)}}{2}$$

$$= \frac{a^x(a^y + a^{-y}) + a^{-x}(a^{-y} + a^y)}{2}$$

$$= \frac{\left(a^x + a^{-x}\right)\left(a^y + a^{-y}\right)}{2}$$

$= 2.\ f(x)\, f(y) \Rightarrow k = 2.$

65. $x - 3 \ge 0 \Rightarrow x \ge 3$ and $-1 \le \sqrt{x - 3} \le 1$

$\Rightarrow x \le 4. \Rightarrow x \in [3, 4].$

66. $x[x] = \sqrt{|x|} \Rightarrow x^2 = x$ if x is an integer

$\Rightarrow \ x = 0, 1$

67. $f(x) = x\dfrac{a^x + a^{-x}}{a^x - a^{-x}} \Rightarrow f(-x) = f(x) \Rightarrow f$ is not odd.

$f(x) = \dfrac{a^x + x}{a^x - x} \Rightarrow f(-\,x) = \dfrac{1 - xa^x}{1 + xa^x} \ne -f(x)$

$\Rightarrow f$ is not odd

$f(x) = \dfrac{a^x - 1}{a^x + 1} \Rightarrow f(-\,x) = \dfrac{1 - a^x}{1 + a^x} = -f(x) \Rightarrow f$ is odd.

$f(x) = x \log_2 \sqrt{\left(x + \sqrt{x^2 + 1}\right)} \Rightarrow f(-x)$

$= -\,x \log \left(-x + \sqrt{x^2 + 1}\right) \ne f(-x).$

68. $g(x) = 1 + x - [x] > 0$ for all x

$\Rightarrow (\text{fog})\,(x) = f(g(x)) = 1.$

69. The function $f(x) = \log_2 (\log_{1/2}(x^2 + 4x + 3) + \sin^{-1}[x]$ is defined if

$0 < x^2 + 4x + 3 < 1$ and $-1 \le x \le 2$ (I)

$x^2 + 4x + 3 > 0 \Rightarrow x > -1$ or $x < -3$ (II)

$x^2 + 4x + 3 < 1 \Rightarrow -2 - \sqrt{2} < x < -2 + \sqrt{2}$ (III)

So the domain of the function is $-1 < x < -2 + \sqrt{2}$.

70. $g\,(f(x)) = |\sin x| = \sqrt{\sin^2 x}$

which is satisfied if $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$

Also it satisfies $f(g(x)) = f\left(\sqrt{x}\right) = (\sin\sqrt{x}\,)^2$

71. $f(x) = 3x - 5 = y \Rightarrow x = \dfrac{y+5}{3} = f^{-1}(y)$

$\Rightarrow f^{-1}(x) = \dfrac{x+5}{3}$

72. $f(x^2) = |\,x^2 - a| \neq (f(x))^2$

$f(|x|) = |\,|x| - a| \neq |\,f(x)|$

$f(x + y) = |x + y - a| \neq |x - a| + |y - a|$.

73. Required number is 4! = 24.

Previous Years' AIEEE/JEE Main Questions

1. R is not symmetric as $(2, 3) \in R$ but $(3, 2) \notin R$

2. We must have

$7 - x \geq 1,\ x - 3 \geq 0$ and $7 - x \geq x - 3$

$\Rightarrow\ \ x \leq 6,\ x \geq 3$ and $x \leq 5$

Thus,

$3 \leq x \leq 5$

$\therefore$ Range of $f = \{{}^4P_0,\ {}^3P_1,\ {}^2P_2)$

$= \{1, 3, 2\}$

3. We have $-1 \leq x - 3 \leq 1$ and $9 - x^2 > 0$

$\Rightarrow\ \ 2 \leq x \leq 4$ and $-3 < x < 3$

$\therefore$ domain of f is [2, 3,)

4. We have

$-\sqrt{a^2+b^2} \leq a \sin x + b \cos x \leq \sqrt{a^2+b^2}$

Thus,

$-\sqrt{1+3} \leq \sin x - \sqrt{3}\cos x \leq \sqrt{1+3}$

$\therefore\ \ -2 + 1 \leq f(x) \leq 2 + 1\ \ \Rightarrow\ \ -1 \leq f(x) \leq 3$

5. A function $f(x)$ is symmetrical about the line $x = a$ if

$$f(a - x) = f(a + x).$$

6. Relation R is clearly reflexive. Since $(6, 12) \in R$ but $(12, 6) \notin R$, R is not symmetric.

Also, as $(3,3) \in R$, $(3, 6) \in R \Rightarrow (3, 6) \in R$;

$(3, 3) \in R$, $(3, 9) \in R \Rightarrow (3, 9) \in R$;

$(3, 3) \in R$, $(3, 12) \in R \Rightarrow (3, 12) \in R$

$(6, 6) \in R$, $(6, 12) \in R \Rightarrow (6, 12) \in R$

Thus, R is transitive.

$\therefore$ R is reflexive and transitive.

7. Putting $x = y = 0$, we get

$$f(0) = f(0)f(0) - f(a)f(a)$$

$\Rightarrow\qquad f(a)^2 = 1^2 - 1 = 0\ \ \Rightarrow f(a) = 0$

Putting $x = a$, $y = a$, we get

$$f(0) = [f(a)]^2 - f(0)f(2a)$$

$$1 = 0 - f(2a)\ \ \Rightarrow\ f(2a) = -1$$

Next, we put $x = 0$, $y = a$ to obtain

$$f(-a) = f(0)f(a) - f(a)f(2a) = 0$$

Now, putting $x = 2a$, $y = x$, we get

$$f(2a - x) = f(2a)\,f(x) - f(-a)\,f(a + x)$$

$$= (-1)\,f(x) - (0)\,f(a + x) = -f(x)$$

8. Let $\omega \in W$, then $(\omega, \omega) \in R$.

$\therefore$ R is reflexive.

Also, if $\omega_1, \omega_2, \in W$ and $(\omega_1, \omega_2) \in R$, then $(\omega_2, \omega_1) \in$ R.

$\therefore$ R is symmetric.

Next, let ω_1 = ink ω_2 = link and ω_3, = let, then $(\omega_1, \omega_2) \in R$,

$$(\omega_2, \omega_3) \in R \text{ but } (\omega_1, \omega_3) \notin R.$$

Thus, R is not transitive.

9. 4^{-x^2} is defined for all $x \in R$.

$\cos^{-1}\left(\dfrac{x}{2} - 1\right)$ is defined if

$-1 \leq \dfrac{x}{2} - 1 \leq 1$ or for $0 \leq x \leq 4$

and log cos x is defined if

$\cos x > 0$ or for $2n\pi - \dfrac{\pi}{2} < x < 2n\pi + \dfrac{\pi}{2}$

where $n \in N$.

Thus, domain of f is $0 \leq x < \dfrac{\pi}{2}$ or $\left[0, \dfrac{\pi}{2}\right)$

10. As $x \neq x + 1$, S is not reflexive

$\Rightarrow S$ is not an equivalence relation.

As $x - x = 0 \in \mathbf{I}$, T is reflexive

$xTy \Rightarrow x - y \in \mathbf{I}\ \ \Rightarrow y - x \in \mathbf{I} \Rightarrow yTx$

$\therefore$ T is symmetric

Suppose xTy and yTz

$\Rightarrow x - y \in \mathbf{I},\ \ y - z \in \mathbf{I}$

$\Rightarrow (x - y) + (y - z) \in \mathbf{I} \Rightarrow x - z \in \mathbf{I} \Rightarrow xTz$

$\therefore$ T is transitive

11. As f is one-one and onto, f is invertible. So

$y = 4x + 3 \Rightarrow x = \dfrac{1}{4}(y - 3)$

Thus, inverse of f is $g(y) = \dfrac{1}{4}(y - 3)$.

12. $B = ((A \cup B) - A) \cup (A \cap B)$

$= ((A \cup C) - A) \cup (A \cap C) = C.$

13. Note that R is not symmetric as $(0, 1) \in$ R but $(1, 0) \notin$ R.

Next, note that $qm = pm \Leftrightarrow \frac{m}{n} = \frac{p}{q}$

Thus, $\left(\frac{m}{n}, \frac{p}{q}\right) \in S \Rightarrow \frac{m}{n} = \frac{p}{q}$

Clearly, S is reflexive and symmetric

Next, note that

$$\left(\frac{m}{n}, \frac{p}{q}\right) \in S \text{ and } \left(\frac{p}{q}, \frac{r}{s}\right) \in S \Rightarrow \frac{m}{n} = \frac{p}{q} \text{ and } \frac{p}{q} = \frac{r}{s}$$

$$\Rightarrow \quad \frac{m}{n} = \frac{r}{s} \quad \Rightarrow \quad \left(\frac{m}{n}, \frac{r}{s}\right) \in \text{S}$$

Thus S is transitive so is an equivalence relation.

14. A is *reflexive*

Let $x \in \mathbf{R}$. $(x, x) \in A$ as $x - x = 0$ is an integer.
A is *symmetric*
Suppose $(x, y) \in A \Rightarrow x - y$ is an integer
$\Rightarrow$ $y - x$ is an integer
$\Rightarrow$ $(y, x) \in A$
A is *transitive*
Let $(x, y) \in A$ and $(y, z) \in A$
$\Rightarrow x - y$ and $y - z$ are integers
$\Rightarrow$ $(x - y) + (y - z)$ is an integer
$\Rightarrow x - z$ is an integer.
$\Rightarrow (x, z) \in A$
Thus, A is an equivalence relation.
B is not *symmetric*

Note that $(0, \sqrt{3}) \in B$ as $0 = (0)\sqrt{3}$ and 0 is a rational number, but $(\sqrt{3}, 0) \notin B$ as there is no rational number a such that $\sqrt{3} = (a)(0)$.

$\therefore$ Statement-1 is true but statement-2 is false. TIP As statement-2 is false, only possible answer is (*d*).

15. Let $y = f(x) = (x - 1)^2 + 1$

$\Rightarrow y - 1 = (x - 1)^2 \Rightarrow x = 1 + \sqrt{y-1}, y \geq 1.$
Thus, $f^{-1}(x) = 1 + \sqrt{x-1}, x \geq 1.$

Now, $f(x) = f^{-1}(x)$

$\Rightarrow (x - 1)^2 + 1 = \sqrt{x-1} + 1$

$\Rightarrow \sqrt{x-1}\,[(x - 1)^{3/2} - 1] = 0$

$\Rightarrow x = 1, 2$

16. As $A = I^{-1} AI$, we get
$(A, A) \in R$ for all A.
$\therefore$ R is reflexive.
Next, assume $(A, B) \in R$
$\Rightarrow$ there exists a non-singular matrix P such that $A = P^{-1}BP$
$\Rightarrow$ $B = PAP^{-1} = (P^{-1})^{-1}AP^{-1}$
Thus, $(B, A) \in R$
$\therefore$ R is symmetric.

Next, assume that $(A, B) \in R$ and $(B, C) \in R$

$\Rightarrow$ there exist non-singular matrices P and Q such that

$A = P^{-1}BP$ and $B = Q^{-1}CQ$

$\therefore \quad A = P^{-1}(Q^{-1}CQ)P$

$= (P^{-1}Q^{-1})\,C(QP)$

$= (QP)^{-1}\,C(QP)$ [using statement-2]

$\Rightarrow (A, C) \in R$

$\therefore$ Statement-1 is true and statement-2 is true and is correct explanation for statement-1.

17. For each $x \in X$, we have the following four choices:
(a) $x \in Y$ and $x \in Z$ (b) $x \notin Y$ and $x \in Z$
(c) $x \in Y$ and $x \notin$ Z (d) $x \notin Y$ and $x \notin Z$

As we do not want $x \in Y \cap Z$, we are left with three choices for each $x \in X$.

Thus, the total number of ways in which Y and Z can be chosen is 3^5.

18. A × B contain (2) (4) = 8 elements.
The number of subsets of A × B having 3 or more elements
= Number of subsets of A × B – [number of subsets with at most 2 elements]

$= 2^8 - ({}^8C_1 + {}^8C_1 + {}^8C_2) = 219$

19. $P = \{(a, b) : \sec^2 a - \tan^2 b = 1\}$
P is not reflexive as $(\pi/2, \pi/2) \notin P$.

P is symmetric.
Suppose $(a, b) \in P$, then
$\sec^2 a - \tan^2 b = 1$
$\Rightarrow \quad (1 + \tan^2 a) - (\sec^2 b - 1) = 1$
$\Rightarrow \quad \sec^2 a - \tan^2 b = 1$
$\Rightarrow \quad (b, a) \in P.$
Suppose $(a, b) \in P$, $(b, c) \in P$
$\Rightarrow \sec^2 a - \tan^2 b = 1$ and $\sec^2 b - \tan^2 c = 1$
Adding, we get
$\Rightarrow \quad \sec^2 a + \sec^2 b - \tan^2 b - \tan^2 c = 2$

$\Rightarrow \quad \sec^2 a + 1 - \tan^2 c = 2$

$\Rightarrow \quad \sec^2 a - \tan^2 c = 1$

$\therefore \ (a, c) \in P.$

20. $\frac{1}{3}+\frac{3n}{100}<1 \ \Rightarrow n \le 22;$

$1 \le \frac{1}{3}+\frac{3n}{100}<2 \quad \Rightarrow 300 \le 100 + 9n < 600$

$\Rightarrow \quad 23 \le n \le 55;$

Also, $\frac{1}{3}+\frac{3(56)}{100}=2\frac{1}{75}$

Thus, $\sum_{n=1}^{56} f(n) = \sum_{n=1}^{22} f(n) + \sum_{n=23}^{55} f(n) + f(56)$

$= 0 + \sum_{n=23}^{55} f(n) + 2(56)$

$= \frac{33}{2}(55+23)+112 = 1399$

21. Period $|\sin(4x)|$ is $\pi/4$ and period of $|\cos 2x|$ is $\pi/2$.

$\therefore$ period of $f(x) = |\sin 4x| + |\cos (2x)|$ is

$\left(l\text{cm}\left(\frac{1}{2},\frac{1}{4}\right)\right)\pi = \frac{\pi}{2}$

22. $f(-1) = f(1) = 0 \ \therefore f$ is not one-one.

Also $f(x) = \frac{|x|-1}{|x|+1} = 1 - \frac{2}{|x|+1} \le 1$ for all $x \in \mathbf{R}$

Thus f cannot be onto.

23. $A = \{-2, -1, 0, 1, 2\}$

$R = \{(-2, 2), (0, 0), (1, 1), (2, 2)\}$

As R has four elements, the power set of R contains 16 elements.

24. $n(A \times B) = n(A)n(B) = 4.2 = 8$

The number of subsets of $A \times B$ which contains at least three elements.

$= {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8$

$= 2^8 - ({}^8C_0 + {}^8C_1 + {}^8C_2) = 256 - (1 + 8 + 28)$

$= 219$

25. Let total number of families in the town be x. Then $n(P) = 0.25x$, $n(C) = 0.15x$, $n(P' \cap C') = 0.65x$,

$n(P \cup C) = x - n(P' \cap C') = 0.35x$

i.e. 35% of the families own either a car or a phone.

$n(P \cup C) = n(P) + n(C) - n(P \cap C)$

$\Rightarrow 0.35x = 0.25x + 0.15x - n(P \cap C)$

$\Rightarrow n(P \cap C) = 0.05x$

i.e. 5% of the families own a car and a phone

As $0.05x = 2000 \Rightarrow x = 40{,}000$

Thus, (a), (b), (c) are all true.

26. We can choose three elements out of seven in ${}^7C_3$ ways. These three elements are mapped to y_2. The remaining 4 elements are to be mapped to y_1 and y_3. This can be done is 2^4 ways. But these include two ways in which all the four elements are mapped to either y_1 or y_3. Therefore, there are $2^4 - 2 = 14$ ways to map containing four elements to y_1 and y_3. Thus the required number of onto mappings is $14.{}^7C_3$.

27. $f(x) + 2f\left(\frac{1}{x}\right) = 3x \ \forall \ x \ne 0 \qquad$ (i)

Replacing x by $\frac{1}{x}$, we have

$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x} \qquad$ (ii)

(i) – 2(ii) gives $f(x) - 4f(x) = 3x - \frac{6}{x}$

$\Rightarrow f(x) = \frac{2}{x} - x$

$f(x) = f(-x) \Rightarrow \frac{2}{x} - x = -\frac{2}{x} + x \Rightarrow \frac{4}{x} = 2x \Rightarrow x^2 = 2$

$x = \pm\sqrt{2}$. Thus S has exactly two elements.

28. Let $y = 3^{x(x-1)}$, so $y \ge 1$

$\Rightarrow \log_3 y = x(x - 1)$

$\Rightarrow x^2 - x - \log_3 y = 0$

$\Rightarrow x = \frac{1 \pm \sqrt{1 + 4\log_3 y}}{2}$. As $x \ge 1$, so

$x = \frac{1}{2}\left[1 + \sqrt{1 + 4\log_3 y}\right].$

Thus $f^{-1}(x) = \frac{1}{2}\left[1 + \sqrt{1 + 4\log_3 y}\right].$

29. $f_1(x) = f_0(f_0(x)) = \frac{1}{1 - f_0(x)} = \frac{1}{1 - \frac{1}{1-x}}$

$= \frac{1-x}{1-x-1} = 1 - \frac{1}{x}$

$f_2(x) = f_0(f_1(x)) = \frac{1}{1 - f_1(x)} = \frac{1}{1 - \left(1 - \frac{1}{x}\right)} = x$

$f_3(x) = f_0(f_2(x)) = f_0(x)$

Thus $f_{3k}(x) = f_0(x)$, $f_{3k+1}(x) = f_1(x)$,

$f_{3k+2}(x) = f_2(x)$ for all $x \in \mathbf{R} \sim \{0, 1\}$, $k \ge 0$

So, $f_{100}(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right)$

$= f_1(3) + f_1\left(\frac{2}{3}\right) + f_2\left(\frac{3}{2}\right)$

$= 1-\frac{1}{3}+1-\frac{3}{2}+\frac{3}{2} = \frac{5}{3}.$

Previous Years' B-Architecture Entrance Examination Questions

1. f is defined if $2x - 3 \ge 0$, $x \ge 1$

   So the domain of f is $\left[\frac{3}{2}, \infty\right)$.

2. If $\frac{x_1+2}{x_1-1} = \frac{x_2+2}{x_2-1} \Rightarrow x_1x_2 + 2x_2 - x_1 - 2$

   $= x_1x_2 + 2x_1 - x_2 - 2$

   $\Rightarrow 3x_2 = 3x_1 \Rightarrow x_1 = x_2$

   So f is one-one

   If $y = f(x) = \frac{x+2}{x-1} \Rightarrow yx - y = x + 2$

   $\Rightarrow (y - 1)x = y + 2$

   $\Rightarrow x = \frac{y+2}{y-1} \in (1, \infty)$

   Thus f is onto

3. $x^6 + y^6 = (x^2 + y^2)(x^4 - x^2y^2 + y^4)$, so if $(x, y) \in A \cap B$ then $1 = x^4 - x^2y^2 + y^4 = (x^2 + y^2)^2 - 3x^2y^2$

   $= 1 - 3x^2y^2$

   $\Rightarrow x^2y^2 = 0 \Rightarrow x = 0$ or $y = 0$ but $x > 0$, $y > 0$

   so $A \cap B = \phi$

4. $58 = n(F \cup B \cup C)$
   $= n(f) + n(B) + n(C) - n(F \cap B) - n(B \cap C)$
   $- n(F \cap C) + n(F \cap B \cap C)$

   $= 38 + 15 + 20 - n(F \cap B) - n(B \cap C)$
   $- n(F \cap C) + 3$

   $\Rightarrow n(F \cap B) + n(B \cap C) + n(F \cap C) = 18$

   Number of medals received in exactly two of three sports is

   $= n(F \cap B \cap C') + n(B \cap C \cap F')$
   $+ n(F \cap C \cap B')$

   $= n(F \cap B) + n(B \cap C) + n(F \cap C)$
   $- 3\,n(F \cap \mathrm{B} \cap C)$

   $= 18 - 9 = 9$

5. Since $1 + x^2 > 0$ for all $x \in Q$, so R is reflexive. If $(x, y) \in R \Rightarrow 1 + xy > 0$
   $\Rightarrow 1 + yx > 0 \Rightarrow (y, x) \in R$

   Thus R is symmetric. Take $x = 2$, $y = -\frac{1}{4}$, $z = -1$

   $1 + xy = 1 - \frac{1}{2} = \frac{1}{2} > 0$ so $(x, y) \in R$

   $1 + yz = 1 + \frac{1}{4} = \frac{5}{4} > 0$ so $(y, z) \in R$

   $1 + xz = 1 - 2 = -1 \Rightarrow (x, z) \notin R$. Hence R is not transitive.

6. f is defined if $x > 3$ and $3 \ne \log_3(x - 3)$

   $\Rightarrow x > 3$, $x - 3 \ne 27$ i.e. $x \in (3, \infty) \sim \{30\}$

   $= (3, 30) \cup (30, \infty\}$

7. f is continuous function and $f'(x) = 2009\,x^{2008} + 2009 > 0$. So f is one-one and onto.

8. For $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $0 \le \cos x \le 1$. So

   $0 \le \cos^4 x \le 1$ and $-5 \le -2\cos^2 x - 2\cos x - 1 \le -1$

   $\Rightarrow -4 \le \cos^4 x - 2\cos^2 x - 2\cos x - 1 \le -1$

   $\Rightarrow -12 \le f(x) \le -3$

9. $f(-x) = (-x)^2 e^{-(-x)^2} \sin|-x| = x^2e^{-x^2} \sin|x| = f(x)$. Product of two odd functions is an even function.

10. $x = 1\cdot x$ so $(x, x) \in R_1$. If $(x, y) \in R_1$ then $x = ay$ or $y = ax$ for some integer a

    $\Rightarrow y = ax$ or $x = ay \Rightarrow (y, x) \in R_1$

    If $x = ay$ or $y = ax$ and $z = by$ or $y = bz$

    for some integer a and b then $x = abz$ or $z = abx$, so $(x, z) \in R_1$. Thus R_1 is an equivalence relation.

    Since $(2, 2) \notin R_2$ so R_2 is not reflexive. Thus R_2 is not an equivalence relation.

11. $\text{gof}\,(x) = g\left(\frac{1}{x+1}\right) = \left(\frac{1}{x+1}\right)^2 + 1 = \frac{1+(x+1)^2}{(x+1)^2} \ge 1$
    so gof is not onto, gof is not one-one as gof (1) = gof (–3).

12. Since g: $cd\,(b, c) = 1$ so $bc = \text{l.c.m}\,(b, c)$. Hence $d = bc$

13. $y = f(x) = (x + 1)^2 - 1 \Rightarrow x = \sqrt{y+1} - 1$

    Hence $f^{-1}(x) = \sqrt{x+1} - 1$. Now $f(x) = f^{-1}(x)$

    $\Leftrightarrow (x + 1)^2 - 1 = \sqrt{x+1} - 1 \Rightarrow (x + 1)^4 = x + 1$

    $\Rightarrow (x + 1)\,[x^3 + 3x^2 + 3x] = 0$

$\Rightarrow (x + 1)\, x(x^2 + 3x + 3) = 0$

$\Rightarrow x = 0, -1$

14. For $x < 0$, $f(x) = \dfrac{e^{-x} - e^{-x}}{e^x + e^{-x}} = 0$

$\therefore$ f cannot be one-one

For $x \geq 0$, $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} \geq 0$

Thus $f(x)$ cannot take negative values so range of f cannot be **R**. Therefore f is not onto.

15. $f(x + 4) - f(x + 2) + f(x) = 0 \ \forall\ x \in \mathbf{R}$

Replace x by $x + 2$,

$f(x + 6) - f(x + 4) + f(x + 2) = 0$

Adding above two equations, we have

$f(x + 6) + f(x) = 0 \ \forall\ x \in \mathbf{R}$

Replace x by $x + 6$, we get

$f(x + 12) + f(x + 6) = 0$

$\Rightarrow f(x + 12) = -f(x + 6) = f(x)$

f is periodic with period 12.

CHAPTER TWO

Complex Numbers

DEFINITIONS

A number of the form $a + ib$ where $a, b \in \mathbf{R}$, the set of real numbers, and $i = \sqrt{-1}$, is called a *complex number*. A complex number can also be defined as an *ordered pair* of real numbers a and b, and may be written as (a, b), where the first number denotes the *real part* and the second number denotes the *imaginary part*. If $z = a + ib$, then the real part of z is denoted by $\text{Re}(z)$ and the imaginary part of z is denoted by $\text{Im}(z)$.

A complex number z is said to be *purely real* if $\text{Im}(z) = 0$ and is said to be *purely imaginary* if $\text{Re}(z) = 0$. Note that the complex number $0 = 0 + i0$ is both purely real and purely imaginary. It is the only complex number with this property.

We denote the set of all complex numbers by $\mathbf{C}$. That is, $\mathbf{C} = \{a + ib | \ a, b \in \mathbf{R}\}$. Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are said to be *equal* if $a_1 = a_2$ and $b_1 = b_2$.

Very Important

- ☐ The main advantage of complex number is to use **one** symbol z for an ordered pair (x, y) of real numbers. We lose this advantage whenever we replace z by $x + iy$.
- ☐ As far as possible do not write $z = x + iy$ while solving the problems. Putting $z = x + iy$ should be the last resort rather than the first option.

ALGEBRAIC OPERATIONS WITH COMPLEX NUMBERS

1. *Addition:* $(a + ib) + (c + id) = (a + c) + i(b + d)$
2. *Subtraction:* $(a + ib) - (c + id) = (a - c) + i(b - d)$
3. *Multiplication:* $(a + ib)(c + id)$
$$= (ac - bd) + i(ad + bc)$$
4. *Reciprocal:* If at least one of a, b is non-zero then the reciprocal of $a + ib$ is given by
$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$
5. *Quotient:* If at least one of c, d is non-zero, then quotient of $a + ib$ and $c + id$ is given by
$$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$
$$= \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2+d^2}$$

CONJUGATE OF COMPLEX NUMBER

Let $z = a + ib$ be a complex number. We define conjugate of z, denoted by $\bar{z}$ to be the complex number $a - ib$. That is, if $z = a + ib$, then $\bar{z} = a - ib$.

Properties of Conjugate of a Complex Number

(i) $z_1 = z_2 \Leftrightarrow \bar{z}_1 = \bar{z}_2$

(ii) $\overline{(\bar{z})} = z$

(iii) $z + \bar{z} = 2\,\text{Re}(z)$

(iv) $z - \bar{z} = 2i\,\text{Im}(z)$

(v) $z = \bar{z} \Leftrightarrow z$ is purely real

(vi) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary

(vii) $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$

(viii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(ix) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

(x) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(xi) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$ if $z_2 \neq 0$

(xii) If $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ where $a_0, a_1, \ldots a_n$ and z are complex number, then
$$\overline{P(z)} = \bar{a}_0 + \bar{a}_1(\bar{z}) + \bar{a}_2(\bar{z})^2 + \cdots + \bar{a}_n(\bar{z})^n$$
$$= \bar{P}(\bar{z})$$
where $\bar{P}(z) = \bar{a}_o + \bar{a}_1 z + \bar{a}_2 z^2 + \cdots + \bar{a}_n z^n$

(xiii) If $R(z) = \dfrac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are polynomials in z, and $Q(z) \neq 0$, then

$$\overline{R(z)} = \frac{\bar{P}(\bar{z})}{\bar{Q}(\bar{z})}$$

Illustration 1

$$\overline{\left(\frac{z+3z^2}{z-1}\right)} = \frac{\bar{z}+3\bar{z}^2}{\bar{z}-1}$$

(xiv) If $z = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, then $\bar{z} = \begin{vmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\ \bar{c}_1 & \bar{c}_2 & \bar{c}_3 \end{vmatrix}$

where a_i, b_i, c_i $(i = 1, 2, 3)$ are complex numbers.

MODULUS OF A COMPLEX NUMBER

Let $z = a + ib$ be a complex number. We define the *modulus* or the *absolute value* of z to be the real number $\sqrt{a^2+b^2}$ and denote it by $|z|$.
Note that $|z| \geq 0 \ \forall \ z \in \mathbf{C}$

Properties of Modulus

If z is a complex number, then

(i) $|z| = 0 \Leftrightarrow z = 0$

(ii) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(iii) $-|z| \leq \text{Re}(z) \leq |z|$

(iv) $-|z| \leq \text{Im}(z) \leq |z|$

(v) $z\bar{z} = |z|^2$

In particular, note that if $z \neq 0$, then $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$ so that $\text{Re}\left(\dfrac{1}{z}\right) = \dfrac{\text{Re}(z)}{|z|^2}$ and $\text{Im}\left(\dfrac{1}{z}\right) = -\dfrac{\text{Im}(\bar{z})}{|z|^2}$.

If z_1, z_2 are two complex numbers, then

(vi) $|z_1 z_2| = |z_1|\,|z_2|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $z_2 \neq 0$

(viii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1\bar{z}_2$

$= |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

(ix) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - \bar{z}_1 z_2 - z_1\bar{z}_2$

$= |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$

(x) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(xi) If a and b are real numbers and z_1, z_2 are complex numbers, then

$|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

(xii) If $z_1, z_2 \neq 0$, then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow z_1/z_2$ is purely imaginary.

(xiii) *Triangle Inequality*. If z_1 and z_2 are two complex numbers, then $|z_1 + z_2| \leq |z_1| + |z_2|$.

The equality holds if and only if $z_1\bar{z}_2 \geq 0$.

In general, $|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$ and the equality sign holds if and only if the ratio of any two non-zero terms is positive.

(xiv) $|z_1 - z_2| \leq |z_1| + |z_2|$

(xv) $\big||z_1| - |z_2|\big| \leq |z_1| + |z_2|$

(xvi) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

(xvii) If a_1, a_2, a_3 and a_4 are four complex numbers, then $|z - a_1| + |z - a_2| + |z - a_3| + |z - a_4| \geq \max\{|a_1 - a_l| + |a_m - a_n|$: l, m, n are distinct integers lying in $\{2, 3, 4\}$ and $m < n\}$.

GEOMETRICAL REPRESENTATION OF COMPLEX NUMBERS

A complex number $z = x + iy$ can be represented by a point P whose Cartesian co-ordinates are (x, y) referred to rectangular axes Ox and Oy, usually called the *real* and *imaginary* axes respectively. The plane is called the *Argand plane, complex plane* or *Gaussian plane*. The point $P(x, y)$ is called the *image* of the complex number z and z is said to be the *affix* or *complex co-ordinate* of point P.

Note

All purely real numbers lie on the real axis and all purely imaginary numbers lie on the imaginary axis. The complex number $0 = 0 + i0$ lies at the origin O.

We have $OP = \sqrt{x^2 + y^2} = |z|$. Thus, $|z|$ is the length of OP.

ARGUMENT OF A COMPLEX NUMBER

If z is a non-zero complex numbers represented by point P in the complex plane, then argument of z is the angle which OP makes with the positive direction of the real axis. See Fig. 2.1.

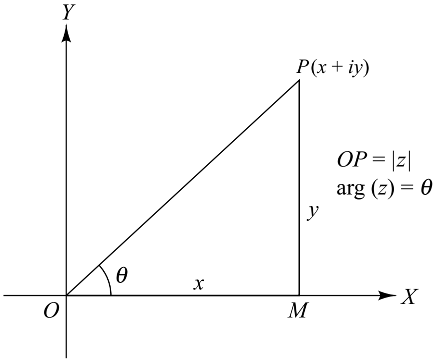


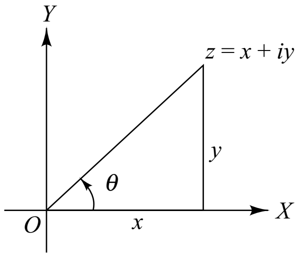
Fig. 2.1

Note

Argument of a non-zero complex number is not unique, since, if θ is a value of the argument, then $2n\pi + \theta$ where $n \in \mathbf{I}$, the set of integers, are also values of the argument of z. The value θ of the argument which satisfies the inequality $-\pi < \theta \leq \pi$ is called the ***principal value*** of the argument or ***principal argument***.

Principal Value of the Argument for Different Positions of *z* in the Complex Plane

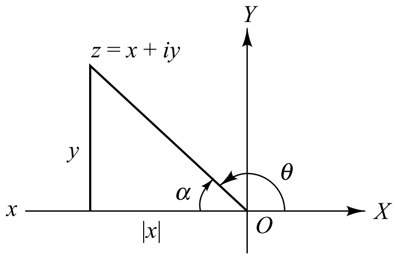
1. When z lies in the first quadrant



$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$$

Fig. 2.2 (i)

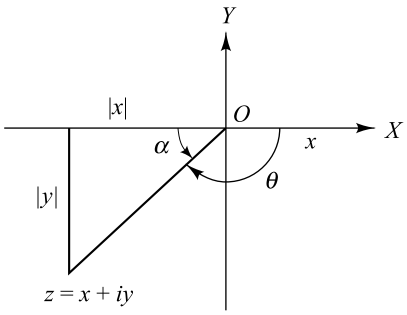
2. When z lies in the second quadrant



$$\arg(z) = \pi - \tan^{-1}\left(\frac{y}{|x|}\right)$$

Fig. 2.2 (ii)

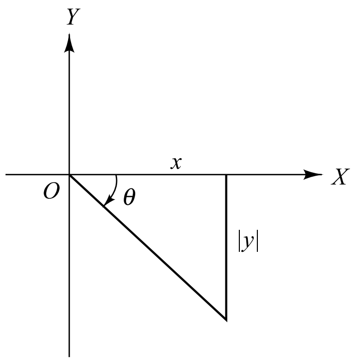
3. When z lies in the third quadrant



$$\arg(z) = -\pi + \tan^{-1}\left(\frac{|y|}{|x|}\right)$$

Fig. 2.2 (iii)

4. When z lies in the fourth quadrant



$$\arg(z) = -\tan^{-1}\left(\frac{|y|}{x}\right)$$

Fig. 2.2 (iv)

Thus, if $z = x + iy$, then

$$\arg(z) = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0, y \geq 0 \\ \tan^{-1}(y/x) - \pi & \text{if } x < 0, y < 0 \\ \pi/2 & \text{if } x = 0, y > 0 \\ -\pi/2 & \text{if } x = 0, y < 0 \end{cases}$$

Common Mistake

A usual mistake committed by the students is to take the argument of $z = x + iy$ as $\tan^{-1}(y/x)$ irrespective of the values of x and y. Kindly remember that $\tan^{-1}(y/x)$ lies in the interval $(-\pi/2, \pi/2)$ whereas the principal value of argument of z lies in the interval $(-\pi, \pi]$.

POLAR FORM OF A COMPLEX NUMBER

Let z be a non-zero complex number, then we can write

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \arg(z)$.

Illustration 2

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

and $$-1 - i = \sqrt{2}\left\{\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right\}$$

In fact, if $z = r(\cos\theta + i\sin\theta)$, then z is also given by

$$z = r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]$$

where k is any integer.

Euler's Formula

The complex number $\cos\theta + i\sin\theta$ is denoted by $e^{i\theta}$ or cis θ. That is

$$e^{i\theta} = \text{cis}\,\theta = \cos\theta + i\sin\theta$$

Some Important Results Involving Argument

If z, z_1 and z_2 are non-zero complex numbers, then

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) *$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

In fact,

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$

where

$$k = \begin{cases} 0 \text{ if } -\pi < \arg(z_1) + \arg(z_2) \le \pi \\ 1 \text{ if } -2\pi < \arg(z_1) + \arg(z_2) \le -\pi \\ -1 \text{ if } \pi < \arg(z_1) + \arg(z_2) \le 2\pi \end{cases}$$

(iii) *$\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$

(iv) *$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

In fact,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$$

where k is defined as in (ii) with + sign between $\arg(z_1)$ and $\arg(z_2)$ replaced by –

(v) $|z_1 + z_2| = |z_1 - z_2|$

$\Leftrightarrow \quad \arg(z_1) - \arg(z_2) = \pi/2$

(vi) $|z_1 + z_2| = |z_1| + |z_2|$

$\Leftrightarrow \quad \arg(z_1) = \arg(z_2)$

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

where $r_1 = |z_1|$, $r_2 = |z_2|$

and $\theta_1 = \arg(z_1)$, $\theta_2 = \arg(z_2)$, then

(vii) $$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$
$$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$$

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \cos(\theta_1 - \theta_2) = 0$$

$\Leftrightarrow Oz_1$ is at right angles to Oz_2.

(viii) $$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$
$$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$$

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2$$

$\Leftrightarrow Oz_1$ is perpendicular to Oz_2.

VECTOR REPRESENTATION OF COMPLEX NUMBERS

We can also represent the complex number $z = x + iy$ by using the vector $\overrightarrow{OP}$ joining the origin O of the complex plane to the point $P(x, y)$, instead of using point P itself. The length of the vector $\overrightarrow{OP}$, that is, $|\overrightarrow{OP}|$ is the modulus of z. The angle between the positive real axis and the vector $\overrightarrow{OP}$, more exactly, the angle through which the positive real axis must be rotated to cause it to have the same direction as $\overrightarrow{OP}$ (considered positive if the rotation is counter-clockwise and negative otherwise) is the argument of the complex number z.

GEOMETRICAL REPRESENTATION OF ALGEBRAIC OPERATIONS ON COMPLEX NUMBERS

Let z_1 and z_2 be two complex numbers represented by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ respectively.

Sum

By definition, $z_1 + z_2$ should be represented by the point $(x_1 + x_2, y_1 + y_2)$. This point is nothing but the vertex P which completes the parallelogram with the line segments joining the origin with z_1 and z_2 as the adjacent sides. See Fig. 2.3

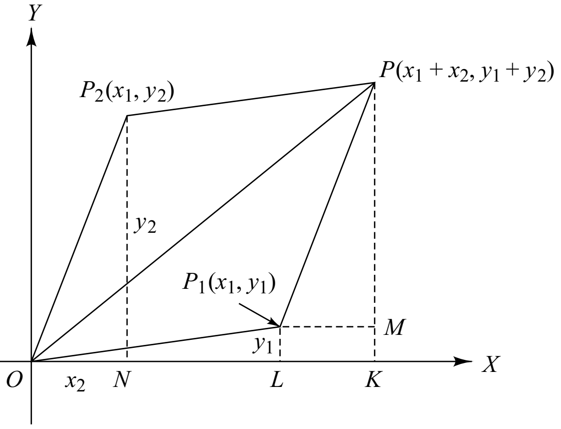


Fig. 2.3

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* L.H.S. and R.H.S. may differ by a multiple of 2π.

Note that the addition of two complex numbers z_1 and z_2 follows the same law of addition as that of vectors, represented both in magnitude and direction by the line segments joining the origin and the points representing z_1 and z_2, for

$$\mathbf{OP_1} + \mathbf{OP_2} = \mathbf{OP_1} + \mathbf{P_1P} = \mathbf{OP}$$

Difference

We first represent $-z_2$ by P'_2 so that $P_2P'_2$ is bisected at O. Complete the parallelogram $OP_1\ PP'_2$. Then, it can be easily seen that P represents the difference $z_1 - z_2$. See Fig. 2.4.

As $OP_1\ PP'_2$ is a parallelogram so $P_1P = OP'_2$. Using vector notation, we have

$$\begin{aligned} z_1 - z_2 &= \mathbf{OP_1} - \mathbf{OP_2} = \mathbf{OP_1} + \mathbf{P_2O} = \mathbf{OP_1} + \mathbf{OP'_2} \\ &= \mathbf{OP_1} + \mathbf{P_1P} = \mathbf{OP} = \mathbf{P_2P_1} \end{aligned}$$

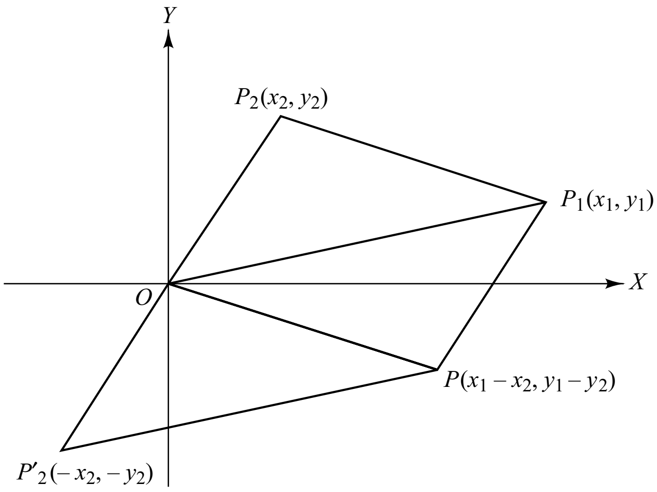


Fig. 2.4

Note the complex number $z_1 - z_2$ is represented by the vector $\mathbf{P_2P_1}$, where the points P_2 and P_1 represent the complex number z_2 and z_1 respectively.

Note that $\arg(z_1 - z_2)$ is the angle through which OX must be rotated in the anticlockwise direction so that it becomes parallel to P_2P_1.

Product (Multiplication)

Let $z_1 = r_1(\cos\ \theta_1 + i \sin\ \theta_1)$ and $z_2 = r_2\ (\cos\ \theta_2 + i\ \sin\ \theta_2)$. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i \sin\ \theta_1)\ (\cos\ \theta_2 + i\ \sin\ \theta_2) \\ &= r_1\ r_2\ \{\cos(\theta_1 + \theta_2) + i\ \sin\ (\theta_1 + \theta_2)\} \end{aligned}$$

Thus $|z_1 z_2| = r_1\ r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$

This shows that modulus of the product of two complex numbers is the product of their moduli, and the argument of the product of any two complex numbers is the sum of their arguments. Using this, we shall derive a geometrical interpretation of the product of two complex numbers.

Let E be a point on the x-axis such that $OE = 1$ unit. (See Fig 2.5.) Complete the triangle OP_1E. Now, taking OP_2 as the base, construct a triangle OPP_2 similar to OP_1E so that

$$OP : OP_1 = OP_2 : OE$$

i.e., $\quad OP = OP_1 \cdot OP_2 \qquad [\because\ OE = 1]$

Also $\quad \angle P_2OP = \angle EOP_1 = \angle XOP_1 = \theta_1$

Thus $\quad \angle XOP = \theta_1 + \theta_2$

Hence P represents the complex number for which the modulus is $r_1 r_2$ and the argument is $\theta_1 + \theta_2$. That is, it represents the complex number $z_1\, z_2$.

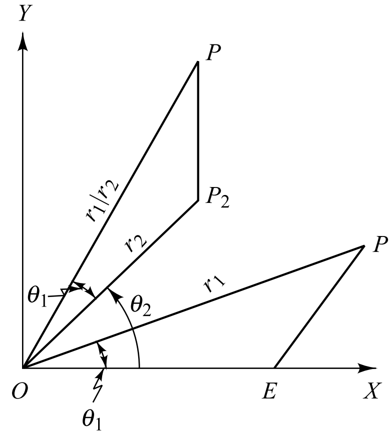


Fig. 2.5

Quotient

Let $z_1 = r_1\ (\cos\ \theta_1 + i\ \sin\ \theta_1)$ and $z_2 = r_2\ (\cos\ \theta_2 + i\ \sin\ \theta_2)$. We take $z_2 \neq 0$, so that $r_2 \neq 0$. Now

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1\ (\cos\theta_1 + i \sin\theta_1)}{r_2\ (\cos\theta_2 + i \sin\theta_2)} \\ &= \frac{r_1}{r_2}\ \{\cos\ (\theta_1 - \theta_2) + i\ \sin\ (\theta_1 - \theta_2)\} \end{aligned}$$

We shall use this to get a geometrical interpretation of the quotient of a complex number by a non-zero complex number.

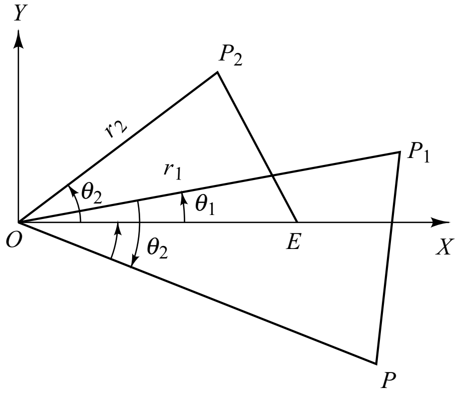


Fig. 2.6

Let P_1 and P_2 represent z_1 and z_2 respectively. On OP_1 construct the triangle OPP_1 similar to OEP_2, where E lies on the x-axis and $OE = 1$ unit. (See Fig. 2.6.)

Now, $\quad OP : OE = r_1 : r_2 \quad \Rightarrow \quad OP = \dfrac{r_1}{r_2}$

Also $\quad \angle XOP = \theta_1 - \theta_2$

The point P thus represents the quotient z_1/z_2, since its modulus is r_1/r_2 and its arguments is $\theta_1 - \theta_2$.

Remark

Note that if θ_1 and θ_2 are the principal values of arg z_1 and arg z_2 then $\theta_1 + \theta_2$ is not necessarily the principal value of $\arg(z_1\, z_2)$, nor is $\theta_1 - \theta_2$ necessarily the principal value of $\arg(z_1/z_2)$.

Interpretation of $\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)$

If z_1, z_2, z_3 are the vertices of a triangle ABC described in the counter-clockwise sense, then

(i) $\arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right) = \angle BAC = \alpha$ (say), and

(ii) $\dfrac{z_3 - z_1}{z_2 - z_1} = \dfrac{CA}{BA}(\cos\alpha + i\sin\alpha)$

See Fig. 2.7.

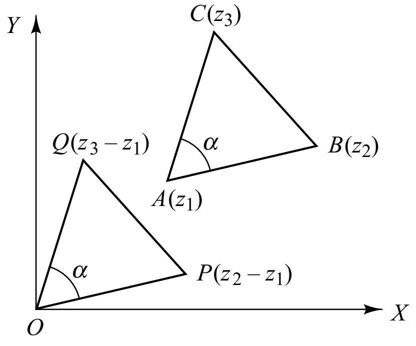


Fig. 2.7

Corollary The points z_1, z_2, z_3 will be collinear if and only if angle $\alpha = 0$ or π, i.e., if and only if $\dfrac{z_3 - z_1}{z_2 - z_1}$ is purely real.

Interpretation of arg $\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right)$

Let z_1, z_2, z_3 and z_4 be four complex numbers. Then the line joining z_4 and z_3 is inclined to the line joining z_2 and z_1 at the following angle:

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right)$$

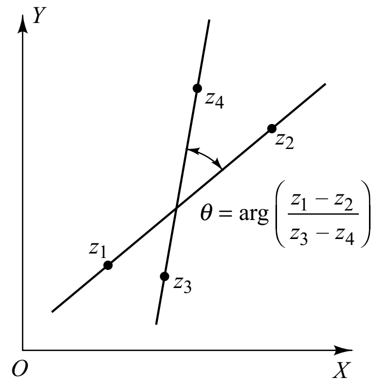


Fig. 2.8

Corollary The line joining z_4 and z_3 is inclined at 90° to the line joining z_2 and z_1 if

$$\arg\left(\frac{z_1 - z_2}{z_3 - z_4}\right) = \pm\frac{\pi}{2}$$

i.e., if $z_1 - z_2 = \pm\, ik(z_3 - z_4)$, where k is a non-zero real number. (Fig. 2.9).

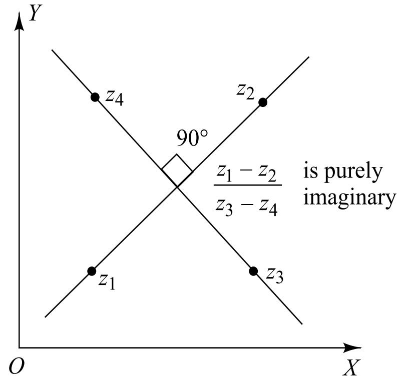


Fig. 2.9

SOME IMPORTANT GEOMETRICAL RESULTS AND EQUATIONS

1. **Distance Formula**

   Distance between $A(z_1)$ and $B(z_2)$ is given by

   $AB = |z_2 - z_1|$

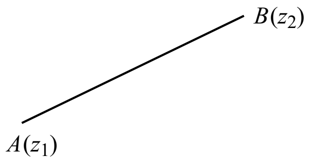


Fig. 2.10

2. **Section Formula**

   The point $P(z)$ which divides the join of the segment AB in the ratio $m : n$ is given by

$$z = \frac{mz_2 + nz_1}{m+n}$$

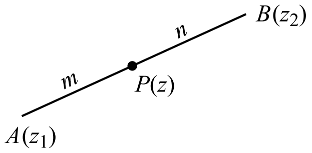

$B(z_2)$
n
m
$P(z)$
$A(z_1)$

Fig. 2.11

3. **Mid-point Formula**

Mid-point $M(z)$ of the segment AB is given by

$$z = \frac{1}{2}(z_1 + z_2)$$

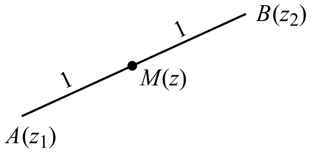


Fig. 2.12

4. **Condition(s) *for four non-collinear* $A(z_1)$, $B(z_2)$, $C(z_3)$ *and* $D(z_4)$ *to represent vertices of a***

(i) Parallelogram

The diagonals AC and BD must bisect each other

$$\Leftrightarrow \quad \frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4)$$

$$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$$

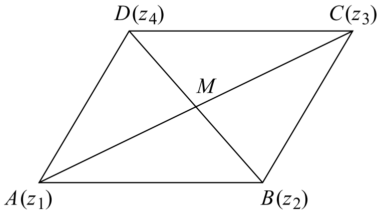

$D(z_4)$
$C(z_3)$
M
$A(z_1)$
$B(z_2)$

Fig. 2.13

(ii) Rhombus

(a) the diagonals AC and BD bisect each other

$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$, and

(b) a pair of two adjacent sides are equal, for instance, $AD = AB$

$\Leftrightarrow \quad |z_4 - z_1| = |z_2 - z_1|$

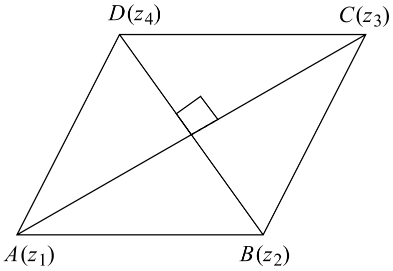

$D(z_4)$
$C(z_3)$
$A(z_1)$
$B(z_2)$

Fig. 2.14

(iii) Square

(a) the diagonals AC and BD bisect each other

$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$

(b) a pair of adjacent sides are equal; for instance, $AD = AB$

$\Leftrightarrow \quad |z_4 - z_1| = |z_2 - z_1|$

(c) the two diagonals are equal, that is,

$AC = BD \Leftrightarrow |z_3 - z_1| = |z_4 - z_2|$

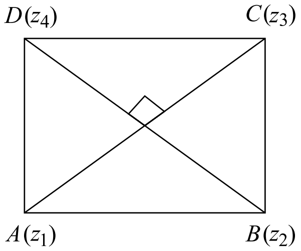

$D(z_4)$
$C(z_3)$
$A(z_1)$
$B(z_2)$

Fig. 2.15

(iv) Rectangle

(a) the diagonals AC and BD bisect each other

$\Leftrightarrow \quad z_1 + z_3 = z_2 + z_4$

(b) the diagonals AC and BD are equal

$\Leftrightarrow \quad |z_3 - z_1| = |z_4 - z_2|$.

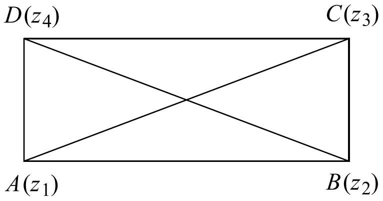
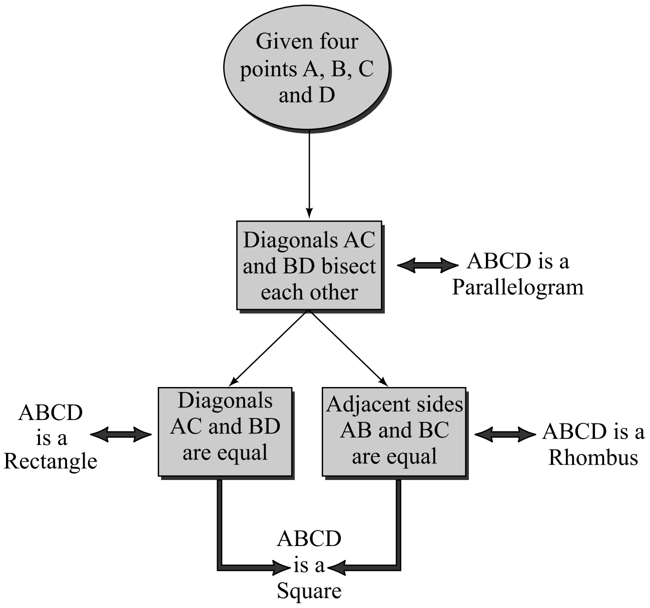

$D(z_4)$
$C(z_3)$
$A(z_1)$
$B(z_2)$

Fig. 2.16

Showing that four points, no three of which are collinear, form a Parallelogram/Rhombus/Square/Rectangle


Given four points A, B, C and D
Diagonals AC and BD bisect each other
ABCD is a Parallelogram
ABCD is a Rectangle
Diagonals AC and BD are equal
Adjacent sides AB and BC are equal
ABCD is a Rhombus
ABCD is a Square

5. Centroid, Incentre, Orthocentre and Circumcentre of a Triangle

Let ABC be a triangle with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$,

(i) **Centroid** $G(z)$ of the ΔABC is the point of concurrence of medians of ΔABC and is given by

$$z = \frac{1}{3}\,(z_1 + z_2 + z_3)$$

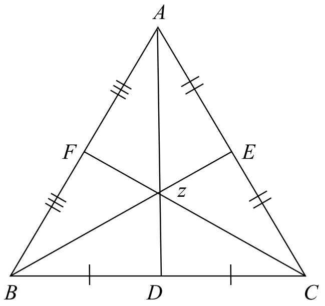


Fig. 2.17

(ii) Incentre $I(z)$ of the ΔABC is the point of concurrence of internal bisectors of angles of ΔABC and is given by

$$z = \frac{az_1 + bz_2 + cz_3}{a + b + c}$$

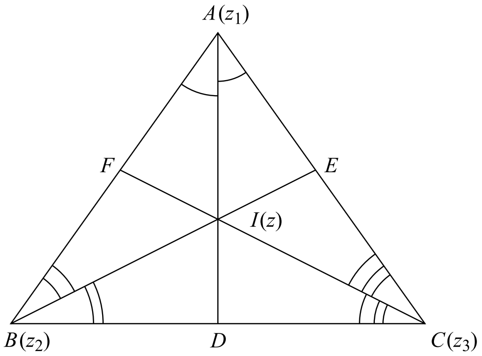


Fig. 2.18

(iii) Circumcentre $S(z)$ of the ΔABC is the point of concurrence of perpendicular bisectors of sides of ΔABC and is given by

$$z = \frac{(z_2 - z_3)|z_1|^2 + (z_3 - z_1)|z_2|^2 + (z_1 - z_2)|z_3|^2}{\bar{z}_1(z_2 - z_3) + \bar{z}_2(z_3 - z_1) + \bar{z}_3(z_1 - z_2)}$$

$$z = \frac{\begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{vmatrix}}$$

Also $$z = \frac{z_1(\sin 2A) + z_2(\sin 2B) + z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$$

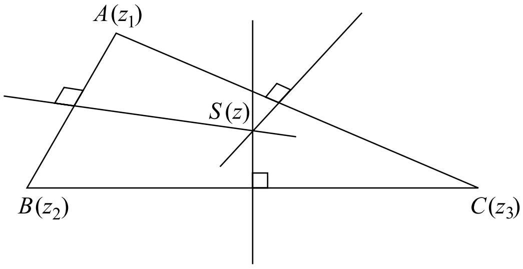


Fig. 2.19

(iv) **Orthocentre** $H(z)$ of the ΔABC is the point of concurrence of altitudes of ΔABC and is given by

$$z = \frac{\begin{vmatrix} z_1^2 & \bar{z}_1 & 1 \\ z_2^2 & \bar{z}_2 & 1 \\ z_3^2 & \bar{z}_3 & 1 \end{vmatrix} + \begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}}{\begin{vmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{vmatrix}}$$

or $$z = \frac{(\tan A)\,z_1 + (\tan B)\,z_2 + (\tan C)z_3}{\tan A + \tan B + \tan C}$$

or $$z = \frac{(a \sec A)\,z_1 + (b \sec B)z_2 + (c \sec C)z_3}{a \sec A + b \sec B + c \sec C}$$

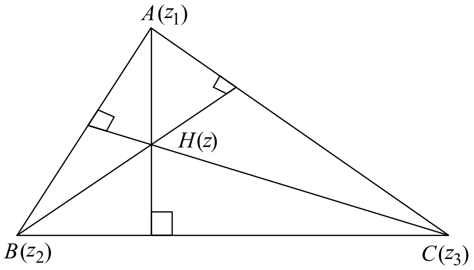


Fig. 2.20

Remark

In case circumcentre of ΔABC is at the origin, then orthocentre of triangle is given by $z_1 + z_2 + z_3$.

Important Note

It is not necessary to remember formulae for circumcentre and orthocentre of a triangle.

Euler's Line

The centroid G of a triangle lies on the segment joining the orthocentre H and the circumcentre S of the triangle. G divides the join of H and S in the ratio 2 : 1.

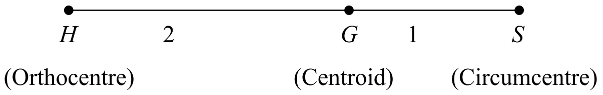


Fig. 2.21

Thus,

$$z_G = \frac{1}{3}(z_H + 2z_S)$$

6. Area of a Triangle

Area of ΔABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is given by

$$\Delta = \left|\frac{1}{4i}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right|$$

$$= \left|\frac{1}{2}\operatorname{Im}(\bar{z}_1 z_2 + \bar{z}_2 z_3 + \bar{z}_3 z_1)\right|$$

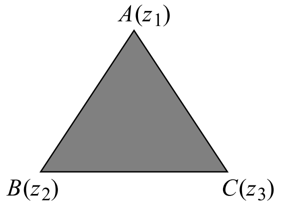


Fig. 2.22

7. Condition for Triangle to be Equilateral

Triangle ABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is equilateral if and only if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

$$\Leftrightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_2 z_3 + z_3 z_1 + z_1 z_2$$

$$\Leftrightarrow \quad z_1 \bar{z}_2 = z_2 \bar{z}_3 = z_3 \bar{z}_1$$

$$\Leftrightarrow \quad z_1^2 = z_2 z_3 \text{ and } z_2^2 = z_1 z_3$$

$$\Leftrightarrow \quad \begin{vmatrix} 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \\ 1 & z_1 & z_2 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad \frac{z_2 - z_1}{z_3 - z_2} = \frac{z_3 - z_2}{z_1 - z_3}$$

$$\Leftrightarrow \quad \frac{1}{z - z_1} + \frac{1}{z - z_2} + \frac{1}{z - z_3} = 0$$

where $$z = \frac{1}{3}(z_1 + z_2 + z_3)$$

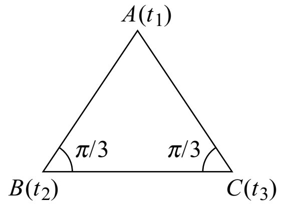


Fig. 2.23

8. Equation of a Straight Line

(i) Non-parametric form

An equation of a straight line joining the two points $A(z_1)$ and $B(z_2)$ is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

or $$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

or $$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1\bar{z}_2 - z_2\bar{z}_1 = 0$$

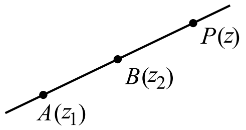


Fig. 2.24

(ii) Parametric form

An equation of the straight line joining the points $A(z_1)$ and $B(z_2)$ is

$$z = tz_1 + (1 - t)z_2$$

where t is a real parameter.

(iii) General Equation of a Straight Line

The general equation of a straight line is

$$\bar{a}z + a\bar{z} + b = 0$$

where a is a non-zero complex number and b is a real number.

9. Complex Slope of a Line

If $A(z_1)$ and $B(z_2)$ are two points in the complex plane, then complex slope of AB is defined to be

$$\mu = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$$

Two lines with complex slopes μ_1 and μ_2 are

(i) parallel, if $\mu_1 = \mu_2$

(ii) perpendicular, if $\mu_1 + \mu_2 = 0$

The complex slope of the line

$\bar{a}z + a\bar{z} + b = 0$ is given by $-(a/\bar{a})$

10. Length of Perpendicular from a Point to a Line

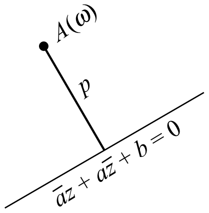


Fig. 2.25

Length of perpendicular of point $A(\omega)$ from the line $\bar{a}z + a\bar{z} + b = 0$ ($a \in \mathbf{C} - \{0\}, b \in \mathbf{R}$) is given by

$$p = \frac{|\bar{a}\omega + a\bar{\omega} + b|}{2|a|}$$

11. Some Results on Circle

(i) Equation of a circle

An equation of a circle with centre at z_0 and radius r is

$$|z - z_0| = r$$

or $\quad z = z_0 + re^{i\theta}, 0 \le \theta < 2\pi$ (parametric form)

or $\quad z\bar{z} - z_0\bar{z} - \bar{z}_0 z + z_0\bar{z}_0 - r^2 = 0$

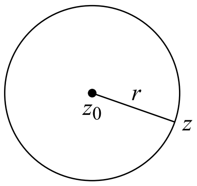


Fig. 2.26

(ii) General equation of a circle

General equation of a circle is

$$z\bar{z} + a\bar{z} + \bar{a}z + b = 0 \tag{1}$$

where a is a complex number and b is a real number such that $a\bar{a} - b \ge 0$

Centre of (1) is $-a$ and its radius is $\sqrt{a\bar{a} - b}$.

(iii) Diameter form of a circle

An equation of the circle one of whose diameter is the line segment joining $A(z_1)$ and $B(z_2)$ is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) = 0$$

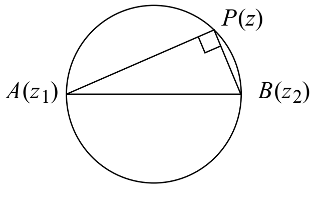


Fig. 2.27

(iv) An equation of circle through two points

An equation of the circle passing through two points $A(z_1)$ and $B(z_2)$ is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) + ik\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

where k is real a parameter.

(v) Equation of a circle passing through three non-collinear points.

Let three non-collinear points be $A(z_1)$, $B(z_2)$ and $C(z_3)$. Let $P(z)$ be any point on the circle. Then either

$\angle ACB = \angle APB$

[when angles are in the same segment]

$\angle ACB + \angle APB = \pi$

[when angles are in the opposite segment]

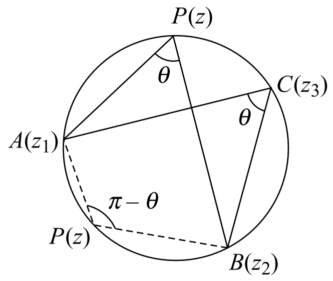


Fig. 2.28

$$\Rightarrow \quad \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) - \arg\left(\frac{z - z_2}{z - z_1}\right) = 0$$

or

$$\arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) + \arg\left(\frac{z - z_1}{z - z_2}\right) = \pi$$

$$\Rightarrow \quad \arg\left[\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\left(\frac{z - z_1}{z - z_2}\right)\right] = 0$$

$$\text{or} \quad \arg\left[\left(\frac{z - z_1}{z - z_2}\right)\left(\frac{z_3 - z_2}{z_3 - z_1}\right)\right] = \pi$$

[using $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$]

In any case, we get $\dfrac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)}$ is purely real.

$$\Leftrightarrow \quad \frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}$$

(vi) Condition for four points to be concyclic.

Four points z_1, z_2, z_3 and z_4 will lie on the same circle if and only if $\dfrac{(z_4 - z_1)(z_3 - z_2)}{(z_4 - z_2)(z_3 - z_1)}$ is purely real.

$$\Leftrightarrow \quad \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \text{ is purely real.}$$

RECOGNIZING SOME LOCI BY INSPECTION

(i) $\arg(z) = \alpha \ (-\pi < \alpha \le \pi)$

$\arg(z) = \alpha$ represents a ray starting at the origin (excluding the origin) and making an angle α with the positive direction of the real axis. See Fig. 2.29.

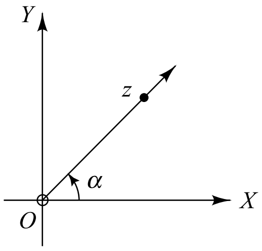


Fig. 2.29

(ii) $\arg(z - z_0) = \alpha \ (-\pi < \alpha \le \pi)$

$\arg(z - z_0) = \alpha$ represents a ray starting at the fixed point z_0 (excluding the point z_0) and making an angle α with the positive direction of the real axis.

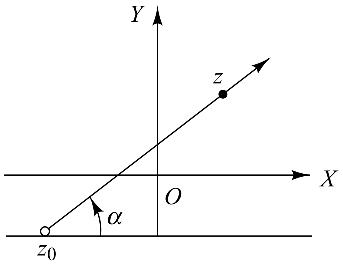


Fig. 2.30

(iii) If z_1 and z_2 are two fixed points, then

$$|z - z_1| = |z - z_2|$$

represents the perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.

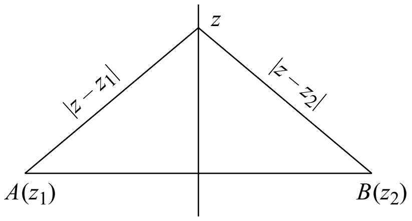


Fig. 2.31

(iv) If z_1 and z_2 are two fixed points, and $k > 0$, $k \ne 1$ is a real number, then

$$\frac{|z - z_1|}{|z - z_2|} = k$$

represents a circle.

For $k = 1$, it represents perpendicular bisector of the segment joining $A(z_1)$ and $B(z_2)$.

(v) $|z - z_1| + |z - z_2| = k$

Let z_1 and z_2 be two fixed points and k be a positive real number.

(a) If $k > |z_1 - z_2|$, then

$|z - z_1| + |z - z_2| = k$ represents an ellipse with foci at $A(z_1)$ and $B(z_2)$ and length of major axis $= k$. See Fig. 2.32

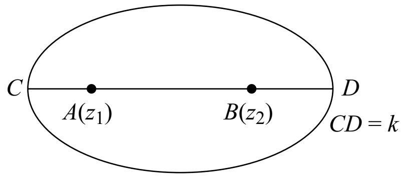


Fig. 2.32

(b) If $k = |z_1 - z_2|$, then

$|z - z_1| + |z - z_2| = k$

represents the segment joining z_1 and z_2.

(c) If $k < |z_1 - z_2|$, then

$|z - z_1| + |z - z_2| = k$

does not represent any curve in the *Argand* plane.

(vi) $|z - z_1| - |z - z_2| = k$

Let z_1 and z_2 be two fixed points, k be a positive real number.

(a) If $k < |z_1 - z_2|$, then

$||z - z_1| - |z - z_2|| = k$

represents a hyperbola with foci at $A(z_1)$ and $B(z_2)$. See Fig 2.33

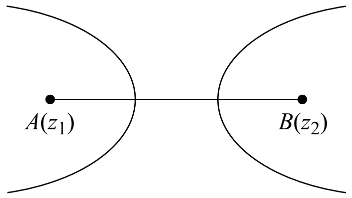


Fig. 2.33

(b) If $k = |z_1 - z_2|$, then $||z - z_1| - |z - z_2|| = k$ represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB. See Fig. 2.34.

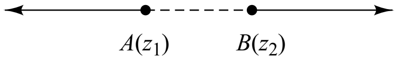


Fig. 2.34

Remark

If $k > |z_1 - z_2|$, then

$||z - z_1| - |z - z_2|| = k$

does not represent any curve in the *Argand* plane.

(vii) If z_1 and z_2 are two fixed points, then

$$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

represent a circle with z_1 and z_2 as the extremities of a diameter. See Fig. 2.35.

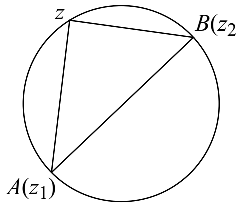


Fig. 2.35

(viii) $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$

Let z_1 and z_2 be two fixed points, and α be a real number such $0 \le \alpha \le \pi$.

(a) If $0 < \alpha < \pi$ and $\alpha \ne \pi/2$, then

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$$

represents a segment of the circle passing through $A(z_1)$ and $B(z_2)$. See Fig. 2.36.

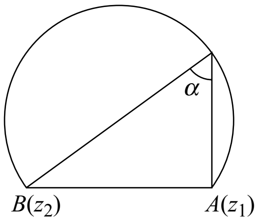


Fig. 2.36

(b) If $\alpha = \pi/2$, then

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha\left(= \frac{\pi}{2}\right)$$

represents a circle with diameter as the segment joining $A(z_1)$ and $B(z_2)$. See Fig. 2.37.

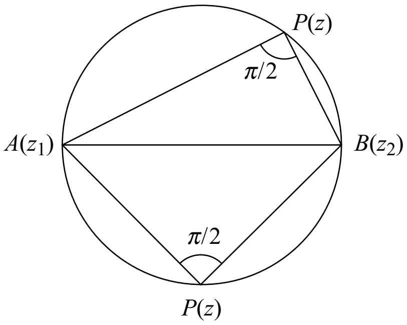


Fig. 2.37

(c) If $\alpha = \pi$, then

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha\ (= \pi)$$

represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB. See Fig. 2.38

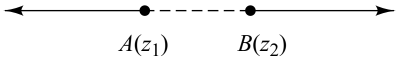


Fig. 2.38

(d) If $\alpha = 0$, then

$$\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha\ (= 0)$$

represents the segment joining $A(z_1)$ and $B(z_2)$ see Fig. 2.39.

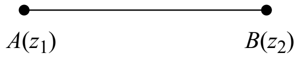


Fig. 2.39

GEOMETRIC INTERPRETATION OF MULTIPLYING A COMPLEX NUMBER BY $e^{i\alpha}$.

Let z be a non-zero complex number. We can write z in the polar form as follows:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

where $r = |z|$ and $\arg(z) = \theta$

We have

$$ze^{i\alpha} = re^{i\theta}e^{i\alpha} = re^{i(\theta + \alpha)}$$

Thus, $ze^{i\alpha}$ represents the complex number whose modulus is r and argument is $\theta + \alpha$. Geometrically, $ze^{i\alpha}$ can be obtained by rotating the segment joining O and $P(z)$ through an angle α in the anticlockwise direction. See Fig. 2.40

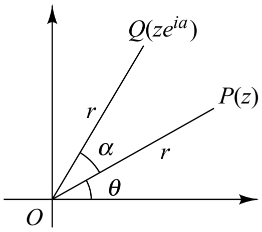


Fig. 2.40

In particular iz is obtained by rotating OP through an angle of $\pi/2$ in the anticlockwise direction.

Corollary If $A(z_1)$ and $B(z_2)$ are two complex number such that angle $\angle AOB = \theta$, then we can write

$$z_2 = \frac{|z_2|}{|z_1|} z_1 e^{i\theta}$$

Suppose $\quad z_1 = r_1 e^{i\alpha}$ and $z_2 = r_2 e^{i\beta}$,

where $\quad |z_1| = r_1, |z_2| = r_2$.

then $\quad \dfrac{z_2}{z_1} = \dfrac{r_2 e^{i\beta}}{r_1 e^{i\alpha}} = \dfrac{r_2}{r_1} e^{i(\beta-\alpha)}$

Note that

$$\beta - \alpha = \theta$$

Thus,

$$\frac{z_2}{z_1} = \frac{r_2}{r_1} e^{i\theta}$$

$$\Rightarrow \quad z_2 = \frac{|z_2|}{|z_1|} z_1 e^{i\theta}$$

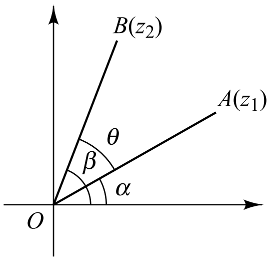


Fig. 2.41

Square Roots of a Complex Number

Let $z = x + iy$. If $a + ib$ is a square root of z, then

$$(a + ib)^2 = x + iy$$

$$\Rightarrow \quad a^2 - b^2 = x,\ 2ab = y$$

Also, $\quad a^2 + b^2 = \sqrt{x^2 + y^2}$

Thus, $\quad a = \pm \dfrac{1}{\sqrt{2}} \sqrt{\sqrt{x^2 + y^2} + x}$

$$b = \pm \frac{1}{\sqrt{2}} \sqrt{\sqrt{x^2 + y^2} - x}$$

If $y > 0$, then a and b both have the same signs.
If $y < 0$, then a and b have opposite signs.

Illustration 3

$$\sqrt{3 + 4i} = \pm \frac{1}{\sqrt{2}} \left[\sqrt{\sqrt{3^2 + 4^2} + 3} + \left(\sqrt{\sqrt{3^2 + 4^2} - 3}\right)i\right]$$

$$= \pm \frac{1}{\sqrt{2}} [2\sqrt{2} + \sqrt{2}i] = \pm (2 + i)$$

Alternatively we can use De Moivre's Theorem.

DE MOIVRE'S THEOREM AND ITS APPLICATIONS

(a) *De Moivre's Theorem for integral index.*
If n is an integer, then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

(b) *De Moivre's Theorem for rational index.*
If n is a rational number, then value of or one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos(n\theta) + i\sin(n\theta)$. In fact, if $n = p/q$ where $p, q \in \mathbf{I}, q > 0$ and p,q have no factors in common, then $(\cos\theta + i\sin\theta)^n$ has q distinct values, one of which is $\cos(n\theta) + i\sin(n\theta)$.

Note

The values of $(\cos\theta + i\sin\theta)^{p/q}$ where $p, q \in \mathbf{I}$, $q > 0$, hcf $(p,q) = 1$ are given by

$$\cos\left[\frac{p}{q}(2k\pi + \theta)\right] + i\sin\left[\frac{p}{q}(2k\pi + \theta)\right]$$

where $\quad k = 0, 1, 2, \ldots, q-1$.

nth Roots of Unity

By an nth root of unity we mean any complex number z which satisfies the equation

$$z^n = 1 \tag{1}$$

Since, an equation of degree n has n roots, there are n values of z which satisfy the equation (1). To obtain these n values of z, we write

$$1 = \cos(2k\pi) + i\sin(2k\pi)$$

where $k \in \mathbf{I}$ and

$$\Rightarrow \quad z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

[using the De Moivre's Theorem]

where $\quad k = 0, 1, 2, \ldots, n-1$.

Note

We may use any n consecutive integral values to k. For instance, in case of 3, we may take -1, 0 and 1 and in case of 4, we may take -1, 0, 1 and 2 or -2, -1, 0 and 1.

Notation $\quad$ Let $\omega = \cos\left(\dfrac{2\pi}{n}\right) + i\sin\left(\dfrac{2\pi}{n}\right)$

By using the De Moivre's theorem, we can write the nth roots of unity as

$$1, \omega, \omega^2, \ldots, \omega^{n-1}.$$

Sum of the Roots of Unity is Zero

We have

$$1 + \omega + \ldots + \omega^{n-1} = \frac{1 - \omega^n}{1 - \omega}$$

But $\quad \omega^n = 1$ as ω is a nth root of unity.

$$\therefore \quad 1 + \omega + \ldots + \omega^{n-1} = 0$$

Also, note that

$$\frac{1}{x-1} + \frac{1}{x-\omega} \ldots + \frac{1}{x-\omega^{n-1}} = \frac{nx^{n-1}}{x^n - 1}$$

Writing *n*th Roots of Unity When *n* is Odd

If $n = 2m + 1$, then nth roots of unity are also given by

$$z = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

where $k = -m, -(m-1), \ldots, -1, 0, 1, 2, \ldots m$.

Since $\qquad \cos\left(-\frac{2k\pi}{n}\right) = \cos\left(\frac{2k\pi}{n}\right)$

and $\qquad \sin\left(-\frac{2k\pi}{n}\right) = -\sin\left(\frac{2k\pi}{n}\right)$

we may take the roots as

$$1, \cos\left(\frac{2k\pi}{n}\right) \pm i\sin\left(\frac{2k\pi}{n}\right)$$

where $\quad k = 1, 2, \ldots, m$.

In terms, ω we may take nth roots of unity to be 1, $\omega^{\pm 1}$, $\omega^{\pm 2}, \ldots \omega^{\pm m}$.

Writing *n*th Roots of Unity When *n* is Even

If $n = 2m$, then nth roots of unity are given

$$z = \pm 1, \pm \omega, \pm \omega^2, \ldots \pm \omega^{m-1}$$

where $\omega = \cos\left(\frac{2\pi}{2m}\right) + i\sin\left(\frac{2\pi}{2m}\right) = \cos\left(\frac{\pi}{m}\right) + i\sin\left(\frac{\pi}{m}\right)$

Cube Roots of Unity

Cube roots of unity are given by 1, ω, ω^2, where

$$\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2} \text{ and } \omega^2 = \frac{-1-\sqrt{3}i}{2}$$

Some Results Involving Complex Cube Root of Unity (ω)

(i) $\omega^3 = 1$, $\bar{\omega} = \omega^2$ and $\frac{1}{\omega} = \omega^2$

(ii) $1 + \omega + \omega^2 = 0$

(iii) $x^3 - 1 = (x-1)(x-\omega)(x-\omega^2)$

(iv) ω and ω^2 are roots of $x^2 + x + 1 = 0$

(v) $a^3 - b^3 = (a-b)(a-b\omega)(a-b\omega^2)$

$\qquad = (a-b)(a\omega - b\omega^2)(a\omega^2 - b\omega)$

(vi) $a^2 + b^2 + c^2 - bc - ca - ab$

$\qquad = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(vii) $a^3 + b^3 + c^3 - 3abc$

$\qquad = (a+b+c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

(viii) $x^3 + 1 = (x+1)(x+\omega)(x+\omega^2)$

(ix) $a^3 + b^3 = (a+b)(a+b\omega)(a+b\omega^2)$

(x) Cube roots of real number a are $a^{1/3}$, $a^{1/3}\,\omega$, $a^{1/3}\,\omega^2$.

To obtain cube roots of a real number a, we write $x^3 = a$ as $y^3 = 1$ where $y = x/a^{1/3}$.

Solution of $\quad y^3 = 1$ are 1, ω, ω^2.

$\therefore \qquad x = a^{1/3}, a^{1/3}\,\omega, a^{1/3}\,\omega^2$.

Illustration 4

To obtain cube roots of –27, we write $\left(\frac{x}{-3}\right)^3 = 1$

$\Rightarrow \qquad x = -3, -3\omega, -3\omega^2$

*n*th Roots of a Complex Number

Let $z \neq 0$ be a complex number. We can write z in the polar form as follows:

$$z = r(\cos\theta + i\sin\theta)$$

where $\quad r = |z|$ and $\theta = \arg(z)$. Recall $-\pi < \theta \leq \pi$.

The nth root of z has n values one of which is equal to $z_0 = \sqrt[n]{|z|}\left[\cos\left(\frac{\arg z}{n}\right) + i\sin\left(\frac{\arg z}{n}\right)\right]$ and is called as the principal value of $\sqrt[n]{|z|}$. To obtain other values of $\sqrt[n]{|z|}$, we write z as

$$z = r\,[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$$

$$\Rightarrow \quad z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

$$= z_0\,\omega^k \text{ where } k = 0, 1, 2, \ldots, n-1.$$

and $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is a complex nth root of unity.

Thus, all the nth roots of z can be obtained by multiplying the principal value of $\sqrt[n]{|z|}$ by different roots of unity.

Rational Power of a Complex Number

If z is a complex number and m/n is a rational number such that m and n are relatively prime integers and $n > 0$. We define

$$z^{m/n} = \left(\sqrt[n]{z}\right)^m$$

Thus, $z^{m/n}$ has n distinct values which are given by

$$z^{m/n} = \left(\sqrt[n]{|z|}\right)^m\left[\cos\left(\frac{m}{n}(\theta + 2k\pi)\right) + i\sin\left(\frac{m}{n}(\theta + 2k\pi)\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$.

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: If $a + ib = \sum_{k=1}^{101} i^k$, then (a, b) equals

(a) (0, 1) (b) (0, 0)
(c) (0, –1) (d) (1, 1)

Ans. (a)

Solution: Write

$$a + ib = i + (i^2 + i^3 + i^4 + i^5) + (i^6 + i^7 + i^8 + i^9) + \dots (i^{98} + i^{99} + i^{100} + i^{101})$$

TIP Use

$$i^m + i^{m+1} + i^{m+2} + i^{m+3} = 0 \ \forall \ m \in \mathbf{I}$$

to obtain

$$a + ib = 0 + i \quad \Rightarrow \quad a = 0, b = 1$$

Example 2: If $\left(\frac{1+i}{1-i}\right)^n = -1$, n $\in \mathbf{N}$, then least value of n is

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (b)

Solution: Write $1 = -i^2$ in the numerator, to obtain

$$\frac{1+i}{1-i} = \frac{-i^2+i}{1-i} = \frac{i(1-i)}{1-i} = i$$

$$\therefore \qquad -1 = \left(\frac{1+i}{1-i}\right) = i^n$$

The least value of n is 2.

Example 3: The conjugate of a complex number z is $\frac{2}{1-i}$. Then Re (z) equals

(a) –1 (b) 0
(c) 1 (d) 2

Ans. (c)

Solution: $\bar{z} = \frac{2}{1-i} = \frac{1-i^2}{1-i} = 1 + i$

Now, Re (z)= Re $(\bar{z})$ = 1.

Example 4: The number of complex numbers z such that $|z - i| = |z + i| = |z + 1|$ is

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (b)

Solution:

TIP
Note z is circum-centre of the circle passing through three non-collinear points A $(0 + i)$, B $(0 - i)$ and $C(-1 + 0i)$

Thus, there is only one value of z.

Example 5: If $z + 2|z| = \pi + 4i$, then Im (z) equals

(a) π (b) 4
(c) $\sqrt{\pi^2 + 16}$ (d) none of these

Ans. (b)

Solution: The given equation can be written as

$$z = (\pi - 2|z|) + 4i$$

As $\pi - 2|z|$ is real, we get

$$\text{Im}(z) = 4$$

Example 6: If $|z| = z + 3 - 2i$, then z equals

(a) $7/6 + i$ (b) $-7/6 + 2i$
(c) $-5/6 + 2i$ (d) $5/6 + i$

Ans. (c)

Solution:

$$z = |z| - 3 + 2i$$

$$\Rightarrow \qquad |z|^2 = (|z| - 3)^2 + 4$$

$$= |z|^2 - 6|z| + 9 + 4$$

$$\Rightarrow \qquad |z| = 13/6$$

Thus, $\qquad z = -5/6 + 2i$

Example 7: If ω $(\neq 1)$ is a cube root of unity and $(1 + \omega^2)^{11} = a + b\omega + c\omega^2$, then (a, b, c) equals

(a) (1, 1, 0) (b) (0, 1, 1)
(c) (1, 0, 1) (d) (1, 1, 1)

Ans. (a)

Solution:

$$a + b\omega + c\omega^2 = (1 + \omega^2)^{11} = (-\omega)^{11}$$

$$= -(\omega^3)^3\omega^2 = 1 + \omega \qquad [\because 1 + \omega + \omega^2 = 0]$$

$$\Rightarrow \qquad a = 1, b = 1, c = 0$$

Example 8: If $x^2 + y^2 = 1$ and $x \neq -1$ then $\frac{1+y+ix}{1+y-ix}$ equals

(a) 1 (b) $x + iy$
(c) 2 (d) $y + ix$

Ans. (d)

Solution:

> **TIP**
> Sometimes, it is helpful to write $x^2 + y^2 = z\bar{z}$
> where $z = x + iy$ or $y + ix$

Put $z = y + ix$, then $1 = x^2 + y^2 = z\bar{z}$

Now,

$$\frac{1+y+ix}{1+y-ix} = \frac{z\bar{z}+z}{1+\bar{z}} = \frac{z(\bar{z}+1)}{1+\bar{z}}$$
$$= z = y + ix$$

Example 9: If z is a non-zero complex number, then $\arg(z) + \arg(\bar{z})$ equals

(a) 0 (b) π
(c) 2π (d) None of these

Ans. (d)

Solution: If $z \in \mathbf{R}$ and $z < 0$, then

$$\arg(z) = \arg(\bar{z}) = \pi$$
$$\Rightarrow \quad \arg(z) + \arg(\bar{z}) = 2\pi$$

Suppose $z \in \mathbf{C}$, $z \neq 0$ and z is not a negative real number.

Let $\arg(z) = \alpha$, where $-\pi < \alpha < \pi$

In this case $\arg(\bar{z}) = -\alpha$, so that

$\arg(z) + \arg(\bar{z}) = 0$

Example 10: If $z \in \mathbf{C}$ and $2z = |z| + i$, then z equals

(a) $\frac{\sqrt{3}}{6} + \frac{1}{2}i$ (b) $\frac{\sqrt{3}}{6} + \frac{1}{3}i$
(c) $\frac{\sqrt{3}}{6} + \frac{1}{4}i$ (d) $\frac{\sqrt{3}}{6} + \frac{1}{6}i$

Ans. (a)

Solution:

$$|2z|^2 = |z|^2 + 1$$
$$\Rightarrow \quad 3|z|^2 = 1 \Rightarrow |z| = 1/\sqrt{3}$$

Thus, $z = \frac{\sqrt{3}}{6} + \frac{1}{2}i$

Example 11: If $z = \left(\frac{1}{\sqrt{3}} + \frac{1}{2}i\right)^7 + \left(\frac{1}{\sqrt{3}} - \frac{1}{2}i\right)^7$, then

(a) $\mathrm{Re}(z) = 0$
(b) $\mathrm{Im}(z) = 0$
(c) $\mathrm{Re}(z) > 0$, $\mathrm{Im}(z) < 0$
(d) $\mathrm{Re}(z) < 0$, $\mathrm{Im}(z) > 0$

Ans. (b)

Solution:

$$\bar{z} = \left(\frac{1}{\sqrt{3}} - \frac{1}{2}i\right)^7 + \left(\frac{1}{\sqrt{3}} + \frac{1}{2}i\right)^7 = z$$

$\Rightarrow$ z is purely real.

$\therefore \quad \mathrm{Im}(z) = 0$

Example 12: If $\omega\,(\neq 1)$ is a complex cube root of unity and $(1 + \omega^4)^n = (1 + \omega^8)^n$, then the least positive integral value of n is

(a) 2 (b) 3
(c) 6 (d) 12

Ans. (b)

Solution:

As $\omega^4 = \omega$, $\omega^8 = \omega^2$, we get

$$(1 + \omega)^n = (1 + \omega^2)^n$$
$$\Rightarrow \quad (-\omega^2)^n = (-\omega)^n$$
$$\Rightarrow \quad \omega^n = 1$$
$$\therefore \quad n = 3$$

Example 13: If $z = \frac{1+\cos\theta + i\sin\theta}{\sin\theta + i(1+\cos\theta)}$ $(0 < \theta < \pi/2)$

then $|z|$ equals

(a) $2\,|\sin\theta|$ (b) $2|\cos\theta|$
(c) 1 (d) $|\cot(\theta/2)|$

Ans. (c)

Solution: Using $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$, we get

$$|z|^2 = \frac{(1+\cos\theta)^2 + \sin^2\theta}{\sin^2\theta + (1+\cos\theta)^2} = 1$$
$$\Rightarrow \quad |z| = 1$$

Example 14: All the roots of $(z + 1)^4 = z^4$ lie on

(a) a straight line parallel to x-axis
(b) a straight line parallel to y-axis
(c) a circle with centre at $-1 + 0i$
(d) a circle with centre at $1 + i$

Ans. (b)

Solution:

> **TIP**
> It is unnecessary to find roots of $(z + 1)^4 = z^4$.

If z is a root of $(z + 1)^4 = z^4$, then $|(z + 1)^4| = |z^4|$

$$\Rightarrow \quad |z + 1| = |z|$$
$$\Rightarrow \quad |z - (-1)| = |z - 0|$$

$\Rightarrow$ z lies on the perpendicular bisector of the segment joining $-1 + 0i$ and 0 i.e. z lies on the line $\mathrm{Re}(z) = -1/2$.

Example 15: If $\alpha\,(\neq 1)$ is a fifth root of unity and $\beta\,(\neq 1)$ is a fourth root of unity then

$$z = (1 + \alpha)(1 + \beta)(1 + \alpha^2)(1 + \beta^2)(1 + \alpha^3)(1 + \beta^3)$$

equals

(a) α (b) β
(c) $\alpha\beta$ (d) 0

Ans. (d)

Solution: As $\beta \neq 1$ is a fourth root of unity,

$$\beta^4 = 1 \Rightarrow (1 - \beta)(1 + \beta + \beta^2 + \beta^3) = 0$$

As $\beta \neq 1,\ 1 + \beta + \beta^2 (1 + \beta) = 0$

$\Rightarrow \quad (1 + \beta)(1 + \beta^2) = 0$

$\therefore \quad z = 0$

Example 16: Suppose z_1, z_2, z_3 are vertices of an equilateral triangle whose circumcentre $-3 + 4i$, then $|z_1 + z_2 + z_3|$ is equal to

(a) 5 (b) $10\sqrt{3}$

(c) 15 (d) $15\sqrt{3}$

Ans. (c)

Solution: As triangle is equilateral, circumcentre and centroid of the triangle coincides, therefore,

$$\frac{1}{3}(z_1 + z_2 + z_3) = -3 + 4i$$

$$\Rightarrow \quad |z_1 + z_2 + z_3| = 3\sqrt{(-3)^3 + 4^2} = 15$$

Example 17: If $z \neq 0$ lies on the circle $|z - 1| = 1$ and $\omega = 5/z$, then ω lies on

(a) a circle (b) an ellipse

(c) a straight line (d) a parabola

Ans. (c)

Solution: $z = 5/\omega$ and $|z - 1| = 1$

$$\Rightarrow \quad \left|\frac{5}{\omega} - 1\right| = 1 \Rightarrow |\omega - 5| = |\omega|$$

that is, ω lies on the perpendicular bisector of the segment joining 0 and $5 + 0i$.

Therefore ω lies on the straight line Re $(\omega) = 5/2$

Example 18: If $\bar{z} = 3i + \dfrac{25}{z + 3i}$, then $|z|$ cannot exceed

(a) 3 (b) 8

(c) 16 (d) 18

Ans. (b)

Solution: $(\bar{z} - 3i)(z + 3i) = 25$

$\Rightarrow \quad |z - 3i|^2 = 25$ or $|z - 3i| = 5$

Now, $\quad |z| = |(z - 3i) + 3i|$

$\quad \leq |z - 3i| + |3i| = 5 + 3 = 8$

Example 19: If $|z - 1| = |z + 1| = |z - 2i|$, then value of $|z|$ is

(a) 1 (b) 2

(c) 5/4 (d) 3/4

Ans. (d)

Solution: z is centre of the circle passing through $1 + 0i$, $-1 + 0i$ and $0 + 2i$. Clearly centre lies on the y-axis.

If $z = 0 + ai$ the centre, then

$$\sqrt{1 + a^2} = |a + 2|$$

$\Rightarrow \quad 1 + a^2 = a^2 + 4a + 4$

$\Rightarrow \quad a = 3/4$

$\therefore \quad |z| = 3/4$

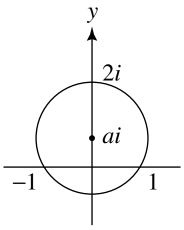


Fig. 2.42

Example 20: The number of complex numbers satisfying $\bar{z} = iz^2$ is

(a) 1 (b) 2

(c) 3 (d) 4

Ans. (d)

Solution: $|\bar{z}| = |iz^2| \quad \Rightarrow \quad |z| = |z|^2$

$\Rightarrow \quad |z| = 0 \quad$ or $\quad |z| = 1.$

If $|z| = 0$, then $z = 0$.

If $|z| = 1$, we get $\bar{z} = 1/z$, so that the equation becomes $1/z = iz^2$

or $\quad z^3 = -i = i^3 \Rightarrow z = i, i\omega, i\omega^2$

where ω ($\neq 1$) is a cube root of unity.

LEVEL 1

Straight Objective Type Questions

Example 21: If $z \in \mathbf{C}$, $z \notin \mathbf{R}$, and $a = z^2 + 3z + 5$, then a cannot take value

(a) –2/5 (b) 5/2

(c) 11/4 (d) –11/5

Ans. (c)

Solution: $a = \left(z + \dfrac{3}{2}\right)^2 + \dfrac{11}{4}$

As $z \notin \mathbf{R}$, $z \neq -\dfrac{3}{2}$, thus, $a \neq \dfrac{11}{4}$

Example 22: Suppose $a, b, c \in \mathbf{C}$, and $|a| = |b| = |c| = 1$ and $abc = a + b + c$, then $bc + ca + ab$ is equal to

(a) 0 (b) –1

(c) 1 (d) none of these

Ans. (c)

Solution: $|a|^2 = |b|^2 = |c|^2 = 1$

$\Rightarrow \quad a\bar{a} = b\bar{b} = c\bar{c} = 1$

Now, $\quad abc = a + b + c$

$\Rightarrow \quad \bar{a}\,\bar{b}\,\bar{c} = \bar{a} + \bar{b} + \bar{c}$

$$\Rightarrow \quad \frac{1}{abc} = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

$$\Rightarrow \quad bc + ca + ab = 1$$

Example 23: The number of complex numbers z which satisfy $z^2 + 2|z|^2 = 2$ is

(a) 0 (b) 2
(c) 3 (d) 4

Ans. (d)

Solution: As $z^2 = 2(1 - |z|^2)$ is real, z is either purely real or purely imaginary.

If z is purely real, then

$$z^2 = 2\,(1 - z^2) \quad \Rightarrow \quad z = \pm\sqrt{\frac{2}{3}}$$

TIP
If z is purely imaginary, then $|z|^2 = -z^2$.

In this case $z^2 = 2(1 + z^2) \quad \Rightarrow \quad z = \pm\sqrt{2}i$

Thus, there are four complex numbers satisfying

$$z^2 + 2|z|^2 = 2$$

Example 24: Suppose $a \in \mathbf{R}$ and the equation $z + a|z| + 2i = 0$ has no solution in $\mathbf{C}$, then a satisfies the relation.

(a) $|a| > 1$ (b) $|a| \ge 1$
(c) $|a| > \sqrt{2}$ (d) $|a| \ge \sqrt{2}$

Ans. (b)

Solution: $z = -a\,|z| - 2i$

$$\Rightarrow \quad |z|^2 = a^2\,|z|^2 + 4$$

$$\Rightarrow \quad |z|^2\,(1 - a^2) = 4$$

This equation has no solution if $1 - a^2 \le 0$ or $|a| \ge 1$

For $|a| < 1$, $|z| = \dfrac{2}{\sqrt{1-a^2}}$ and

$$z = \frac{2a}{\sqrt{1-a^2}} - 2i$$

Example 25: Suppose z is a complex number and $n \in \mathbf{N}$ be such that $z^n = (1 + z)^n = 1$, then the least value of n is

(a) 3 (b) 6
(c) 9 (d) 18

Ans. (b)

Solution: $|z|^n = |1 + z|^n = 1 \quad \Rightarrow \quad |z| = |z + 1| = 1$

$\Rightarrow$ $|z| = 1$ and z lies on the perpendicular bisector of the segment joining $0 + 0i$ and $-1 + 0i$, that z lies on the line $\text{Re}\,(z) = -1/2$.

Let $\quad z = -\dfrac{1}{2} + iy$, then $|z| = 1$

$$\frac{1}{4} + y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{\sqrt{3}}{2}$$

$$\therefore \quad z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \omega,\ \omega^2$$

where ω is complex cube root of unity.

If $z = \omega$, $\omega^n = (1 + \omega)^n = (-1)^n\,\omega^{2n}$

$\Rightarrow n$ is even and multiple of 3. Thus, least value of n is 6.

Similarly, for $z = \omega^2$.

Example 26: Let $z \ne \pm i$ be a complex number such that $\dfrac{z-i}{z+i}$ is purely imaginary number, then $z + \dfrac{1}{z}$ is

(a) a non-zero real number other than 1
(b) a purely imaginary number
(c) a non-zero real number
(d) 0

Ans. (c)

Solution: Let $\dfrac{z-i}{z+i} = ik$, where $k \in \mathbf{R}$.

$$\Rightarrow \quad z - i = ikz - k$$

$$\Rightarrow \quad z\,(1 - ik) = -k + i$$

$$\Rightarrow \quad z = \frac{-k+i}{1-ik}$$

Note that $|z|^2 = \dfrac{k^2+1}{1+k^2} = 1$

$$\Rightarrow \quad z\bar{z} = 1 \quad \Rightarrow \quad \bar{z} = 1/z$$

Thus, $\quad z + \dfrac{1}{z} = z + \bar{z}$, which is a real number.

Also, $\quad z + \bar{z} = 0$

$$\Rightarrow \quad 2\text{Re}\,(z) = 0 \quad \Rightarrow \quad \text{Re}\,(z) = 0$$

$\Rightarrow \quad z = ai$ for some $a \in \mathbf{R}$.

But in this case

$\dfrac{z-i}{z+i}$ is a real number

Therefore, $z + \bar{z} \ne 0$.

Example 27: The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if

(a) $z_1 + z_4 = z_2 + z_3$
(b) $z_1 + z_3 = z_2 + z_4$
(c) $z_1 + z_2 = z_3 + z_4$
(d) None of these

Ans. (b)

Solution: See theory.

Example 28: If the complex numbers z_1, z_2 and z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then

(a) $z_1 + z_2 + z_3 = 0$ (b) $z_1 + z_2 - z_3 = 0$
(c) $z_1 - z_2 + z_3 = 0$ (d) $z_1 + z_2 + z_3 \ne 0$

Ans. (a)

◎ **Solution:** Let $|z_1| = |z_2| = |z_3| = k$ (say),
⇒ z_1, z_2, z_3 lie on a circle with centre at the origin and radius k. As z_1, z_2, z_3 are vertices of an equilateral triangle, the circumcentre and the centroid of the triangle coincide. Therefore,

$$\frac{1}{3}(z_1 + z_2 + z_3) = 0 \quad \Rightarrow \quad z_1 + z_2 + z_3 = 0$$

● **Example 29:** The value of $S = \sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7} - i\cos\frac{2\pi k}{7}\right)$ is

(a) – 1 (b) 0
(c) – i (d) i

Ans. (d)

◎ **Solution:** $\sin\left(\frac{2\pi k}{7}\right) - i\cos\left(\frac{2\pi k}{7}\right)$

$$= -i\left[\cos\left(\frac{2\pi k}{7}\right) + i\sin\left(\frac{2\pi k}{7}\right)\right]$$

$$= -i\,\omega^k \text{ [De Moivre's Theorem]}$$

where $\omega = \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)$.

Note that $\omega^7 = 1$.

$$\therefore \qquad S = -i\sum_{k=1}^{6}\omega^k = \frac{-i\,\omega(1-\omega^6)}{1-\omega}$$

$$= \frac{-i\,(\omega - \omega^7)}{1-\omega} = \frac{-i\,(\omega - 1)}{1-\omega} = i$$

● **Example 30:** The complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other for

(a) $x = n\pi, n \in \mathbf{I}$ (b) $x = \left(n + \frac{1}{2}\right)\pi, n \in \mathbf{I}$
(c) $x = 0$ (d) no value of x.

Ans. (d)

◎ **Solution:** $\overline{\sin x + i\cos 2x} = \cos x - i\sin 2x$

⇒ $\sin x - i\cos 2x = \cos x - i\sin 2x$

⇒ $\sin x = \cos x$ and $\cos 2x = \sin 2x$

⇒ $\tan x = 1$ and $\tan 2x = 1$

These two equations cannot hold simultaneously.

● **Example 31:** If z_1 and z_2 are two complex numbers and a, b are two real numbers, then $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$ equals

(a) $(a^2 + b^2)|z_1 z_2|$
(b) $(a^2 + b^2)(z_1^2 + z_2^2)$
(c) $(a^2 + b^2)(|z_1|^2 + |z_2|^2$
(d) $2ab|z_1 z_2|$

Ans. (c)

◎ **Solution:** We have

$$|az_1 - bz_2|^2 + |bz_1 + az_2|^2$$
$$= a^2|z_1|^2 + b^2|z_2|^2 - abz_1\bar{z}_2 - ab\bar{z}_1 z_2$$
$$+ b^2|z_1|^2 + a^2|z_2|^2 + ba\,z\bar{z}_2 + ba\bar{z}_1 z_2$$
$$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$$

● **Example 32:** If a and b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle, then

(a) $a = b = 2 - \sqrt{3}$ should look like this
(b) $a = 2 - \sqrt{3}$, $b = \sqrt{3} - 1$
(c) $a = \sqrt{3} - 1$, $b = 2 - \sqrt{3}$
(d) none of these

Ans. (a)

◎ **Solution:** By the hypothesis $0 < a, b < 1$

and $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$

⇒ $|(a - 1) + i(1 - b)| = |1 + ib| = |a + i|$

⇒ $(a - 1)^2 + (1 - b)^2 = 1 + b^2 = a^2 + 1$

⇒ $a^2 - 2a + 1 - 2b = 0$ and $b^2 = a^2$

As $0 < a, b < 1$ and $a^2 = b^2$, we get $a = b$.

∴ $a^2 - 2a + 1 - 2a = 0 \Rightarrow a^2 - 4a + 1 = 0$

⇒ $a = 2 \pm \sqrt{3}$ As $0 < a < 1$, $a = 2 - \sqrt{3}$.

● **Example 33:** If $z \neq 0$ is a complex number such that $\arg(z) = \pi/4$, then

(a) $\text{Re}(z^2) = 0$ (b) $\text{Im}(z^2) = 0$
(c) $\text{Re}(z^2) = \text{Im}(z^2)$ (d) none of these

Ans. (a)

◎ **Solution:** As $\arg(z) = \pi/4$, we can write

$$z = r\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \text{where } r = |z|$$

$$\Rightarrow \qquad z^2 = r^2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

[Using De Moivre's Theorem]

$$= r^2(0 + i) = ir^2$$

⇒ $\text{Re}(z^2) = 0$

● **Example 34:** Let z and w be two non-zero complex numbers such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$. Then z equal

(a) w (b) $-w$
(c) $\bar{w}$ (d) $-\bar{w}$

Ans. (d)

◎ **Solution:** Let $|z| = |w| = r$ and $\arg(w) = \theta$, so that $\arg(z) = \pi - \theta$.

We have

$$z = r[\cos(\pi - \theta) + i\sin(\pi - \theta)]$$

$$= r(-\cos\theta + i\sin\theta)$$
$$= \overline{-r(\cos\theta + i\sin\theta)} = -\bar{w}$$

Example 35: If $|z| = 1$ and $w = \dfrac{z-1}{z+1}$ (where $z \neq -1$), then Re(w) equals

(a) 0 (b) $-\dfrac{1}{|z+1|^2}$

(c) $\left|\dfrac{z}{z+1}\right| \dfrac{1}{|z+1|^2}$ (d) $\dfrac{\sqrt{2}}{|z+|^2}$

Ans. (a)

Solution: $|z| = 1 \Rightarrow z\bar{z} = 1$.

$$2\text{Re}(w) = w + \bar{w} = \frac{z-1}{z+1} + \frac{\bar{z}-1}{\bar{z}+1}$$
$$= \frac{z-1}{z+1} + \frac{(1/z)-1}{(1/z)+1}$$
$$= \frac{z-1}{z+1} + \frac{1-z}{1+z} = 0$$

$\Rightarrow \quad \text{Re}(w) = 0$

Example 36: Let z and w be two complex numbers such that $|z| = |w| = 1$ and $|z + iw| = |z - i\bar{w}| = 2$. Then z equals

(a) 1 or i (b) i or $-i$
(c) 1 or -1 (d) i or -1

Ans. (c)

Solution: We have $|-iw| = |-i|\,|w| = 1$
and $|i\bar{w}| = |i|\,|\bar{w}| = 1$

$\Rightarrow -iw$ and $i\bar{w}$ lie on the circle $|z| = 1$.

As $|z - (-iw)| = |z - i\bar{w}| = 2$ we get z and $-iw$, as well as z and $i\bar{w}$ are the end points of the same diameter, with one end point at z.

$\therefore \quad -iw = i\bar{w} \quad \Rightarrow \quad w + \bar{w} = 0$

$\Rightarrow$ w is purely imaginary.

Let $w = ik$ where $k \in \mathbf{R}$.

As $|w| = 1$, we get $|ik| = 1$

$\Rightarrow \quad |k| = 1 \quad \Rightarrow \quad k = \pm 1$.

$\therefore \quad w = \pm i \Rightarrow -iw = i\bar{w} = \pm 1$

When $i\bar{w} = 1$, then $z = -1$ and
when $i\bar{w} = -1$ then $z = 1$

Example 37: The complex numbers $z = x + iy$ which satisfy the equation $\left|\dfrac{z-5i}{z+5i}\right| = 1$, lie on

(a) the x – axis
(b) the straight line $y = 5$
(c) a circle passing through origin
(d) none of these

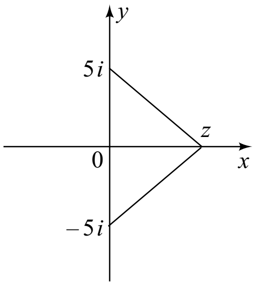


Fig. 2.43

Ans. (a)

Solution: $\left|\dfrac{z-5i}{z+5i}\right| = 1 \quad \Rightarrow \quad |z - 5i| = |z - (-5i)|$

$\Rightarrow$ z lies on the perpendicular bisector of the segment joining $5i$ and $-5i$, i.e., z lies on the x-axis.

Example 38: The inequality $|z - 4| < |z - 2|$ represents the region given by

(a) $\text{Re}(z) \geq 0$ (b) $\text{Re}(z) < 3$
(c) $\text{Re}(z) \leq 0$ (d) $\text{Re}\,z > 3$

Ans. (d)

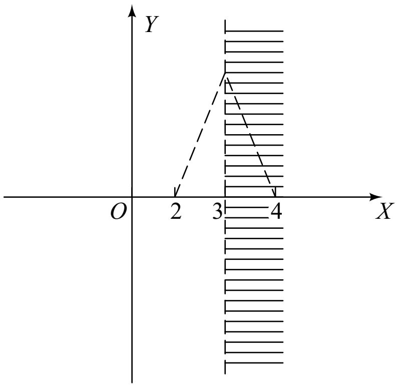


Fig. 2.44

Solution: If z satisfies $|z - 4| = |z - 2|$, then z lies on the perpendicular bisector of the segment joining $z = 2$ and $z = 4$.

i.e., $|z - 4| = |z - 2| \quad \Rightarrow \quad \text{Re}(z) = 3$.

As $z = 0$ does not satisfy $|z - 4| < |z - 2|$, we get $|z - 4| < |z - 2|$ represents the region $\text{Re}(z) > 3$.

Example 39: If z_1 and z_2 are two complex numbers such that $\left|\dfrac{z_1 - z_2}{z_1 + z_2}\right| = 1$, then

(a) $z_2 = kz_1, k \in \mathbf{R}$ (b) $z_2 = ikz_1, k \in \mathbf{R}$
(c) $z_1 = z_2$ (d) none of these

Ans. (b)

Solution: Note that at least one of z_1, z_2 is different from 0. Suppose $z_2 \neq 0$. We can write

$$\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = 1, \text{ as } \left|\frac{z_1/z_2 - 1}{z_1/z_2 + 1}\right| = 1 \text{ or as } \left|\frac{z_1}{z_2} - 1\right| = \left|\frac{z_1}{z_2} + 1\right|.$$

This shows that z_1/z_2 lies on the perpendicular bisector of the segment joining $A\,(-1 + i0)$ and $B(1 + i0)$ [See Theory]. Thus, z_1/z_2 lies on the imaginary axis.

$\therefore \quad \dfrac{z_1}{z_2} = ia$ for some $a \in \mathbf{R}$. $\Rightarrow \dfrac{z_2}{z_1} = \dfrac{1}{ia} = \dfrac{-i}{a}$

$\Rightarrow \quad z_2 = ikz_1$ for some $k \in \mathbf{R}$

Example 40: For any complex number z, the minimum value of $|z| + |z - 2i|$ is

(a) 0 (b) 1

(c) 2 (d) none of these

Ans. (c)

Solution: We have, for $z \in \mathbf{C}$

$$|2i| = |z + (2i - z)| \le |z| + |2i - z|$$

$$\Rightarrow \qquad 2 \le |z| + |z - 2i|$$

Thus, minimum value of $|z| + |z - 2i|$ is 2 and it is attained when $z = i$.

Example 41: If $x = 2 + 5i$, then the value of $x^3 - 5x^2 + 33x - 19$ is equal to

(a) – 5 (b) – 7

(c) 7 (d) 10

Ans. (d)

Solution: $x = 2 + 5i \;\Rightarrow\; x - 2 = 5i$

$\Rightarrow \quad (x - 2)^2 = (5i)^2 \;\Rightarrow\; x^2 - 4x + 4 = -25$

$\Rightarrow x^2 - 4x + 29 = 0$. We now divide $x^3 - 5x^2 + 33x - 19$ by $x^2 - 4x + 29$.

$$
\begin{array}{rl}
 & x-1 \\
x^2-4x+29\,\big) & x^3-5x^2+33x-19 \\
 & x^3-4x^2+29x \\
 & \ominus \quad \oplus \quad \ominus \\
\cline{2-2}
 & -x^2+4x-19 \\
 & -x^2+4x-29 \\
 & \oplus \quad \ominus \quad \oplus \\
\cline{2-2}
 & 10
\end{array}
$$

Thus, $x^3 - 5x^2 + 33x - 19 = (x - 1)(x^2 - 4x + 29) + 10$

$$= (x - 1)(0) + 10 = 10$$

Example 42: If $z = x + iy$ and $w = \dfrac{1 - iz}{z - i}$, then $|w| = 1$ implies, that, in the complex plane

(a) z lies on the imaginary axis

(b) z lies on the real axis

(c) z lies on the unit circle

(d) none of these

Ans. (b)

Solution: $|w| = 1 \quad \Rightarrow \quad \left|\dfrac{1 - iz}{z - i}\right| = 1$

$\Rightarrow \quad |-i^2 - iz| = |z - i| \quad \Rightarrow \quad |(-i)(z + i)| = |z - i|$

$\Rightarrow \quad |z + i| = |z - i|$

$\Rightarrow$ z lies on the perpendicular bisector of the segment joining i and $-i$.

$\Rightarrow$ z lies on the real axis.

Example 43: The real part of $z = \dfrac{1}{1 - \cos\theta + i\sin\theta}$ is

(a) $\dfrac{1}{1 - \cos\theta}$ (b) $\dfrac{1}{2}$

(c) $\dfrac{1}{2}\tan\theta$ (d) 2

Ans. (b)

Solution:

TIP

If $\quad z = \dfrac{1}{\omega}$, then

$$\text{Re}(z) = \frac{\text{Re}(\omega)}{|\omega|^2}$$

$$\therefore \quad \text{Re}(z) = \frac{1 - \cos\theta}{(1 - \cos\theta)^2 + \sin^2\theta}$$

$$= \frac{1 - \cos\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{1 - \cos\theta}{2\,(1 - \cos\theta)} = \frac{1}{2}$$

Example 44: If the imaginary part of $\dfrac{2z + 1}{iz + 1}$ is – 4, then the locus of the point representing z in the complex plane is

(a) a straight line (b) a parabola

(c) a circle (d) an ellipse

Ans. (c)

Solution: Let $z = x + iy$, then

$$\frac{2z + 1}{iz + 1} = \frac{2(x + iy) + 1}{i(x + iy) + 1} = \frac{(2x + 1) + 2iy}{(1 - y) + ix}$$

$$= \frac{[(2x + 1) + 2iy][(1 - y) - ix]}{(1 - y)^2 + x^2}$$

As $\text{Im}\left(\dfrac{2z + 1}{iz + 1}\right) = -4$, we get

$$\frac{2y(1 - y) - x(2x + 1)}{x^2 + (1 - y)^2} = -4$$

$$\Rightarrow \qquad 2x^2 + 2y^2 + x - 2y = 4x^2 + 4(y^2 - 2y + 1)$$

$$\Rightarrow \qquad 2x^2 + 2y^2 - x - 6y + 4 = 0$$

which represents a circle.

Example 45: The area of the triangle whose vertices are the points represented by the complex number z, iz and $z + iz$ is

(a) $\dfrac{1}{4}\,|z|^2$ (b) $\dfrac{1}{8}\,|z|^2$

(c) $\dfrac{1}{2}\,|z|^2$ (d) $\dfrac{1}{2}\,|z|$

Ans. (c)

Solution: Area of the triangle is given by

$$\Delta = \frac{1}{4}\left|\begin{vmatrix} z & \overline{z} & 1 \\ iz & -i\overline{z} & 1 \\ z+iz & \overline{z}-i\overline{z} & 1 \end{vmatrix}\right|$$

Applying $R_3 \to R_3 - R_1 - R_2$, we get

$$\Delta = \frac{1}{4}\left|\begin{vmatrix} z & \overline{z} & 1 \\ iz & -i\overline{z} & 1 \\ 0 & 0 & -1 \end{vmatrix}\right| = \frac{1}{4}\left|(-1)\begin{vmatrix} z & \overline{z} \\ iz & -i\overline{z} \end{vmatrix}\right|$$

$$= \frac{1}{4}\left|(-1)\,(i)z\,\overline{z}\right|\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \frac{1}{4}|z|^2\,(2) = \frac{1}{2}|z|^2$$

Example 46: If ω is a complex cube root of unity, then a root of the equation $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} = 0$ is

(a) $x = 1$ (b) $x = \omega$

(c) $x = \omega^2$ (d) $x = 0$

Ans. (d)

Solution: Let us denote the given determinant by Δ. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+1+\omega+\omega^2 & \omega & \omega^2 \\ x+1+\omega+\omega^2 & x+\omega^2 & 1 \\ x+1+\omega+\omega^2 & 1 & x+\omega \end{vmatrix} = \begin{vmatrix} x & \omega & \omega^2 \\ x & x+\omega^2 & 1 \\ x & 1 & x+\omega \end{vmatrix}$$

Clearly $\Delta = 0$ for $x = 0$.

Example 47: Let z_1 and z_2 be two non-zero complex numbers such that $\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1$, then the origin and points represented by z_1 and z_2

(a) lie on a straight line

(b) form a right triangle

(c) form an equilateral triangle

(d) none of these

Ans. (c)

Solution: Let $z = \frac{z_1}{z_2}$, then $z + \frac{1}{z} = 1 \Rightarrow z^2 - z + 1 = 0$

$$\Rightarrow \quad z = \frac{1 \pm \sqrt{3}\,i}{2} \quad \Rightarrow \quad \frac{z_1}{z_2} = \frac{1 \pm \sqrt{3}i}{2}$$

If z_1 and z_2 are represented by A and B respectively and O be the origin, then

$$\frac{OA}{OB} = \frac{|z_1|}{|z_2|} = \left|\frac{1 \pm \sqrt{3}i}{2}\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$\Rightarrow \quad OA = OB$

Also, $$\frac{AB}{OB} = \frac{|z_2 - z_1|}{|z_2|} = \left|1 - \frac{z_1}{z_2}\right|$$

$$= \left|1 - \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)\right| = \left|\frac{1}{2} \mp \frac{\sqrt{3}}{2}i\right| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$\Rightarrow \quad AB = OB$

Thus, $\quad OA = OB = AB$.

$\therefore$ ΔOAB is an equilateral triangle.

Example 48: If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}\,x^{2n}$, then value of $a_0 + a_3 + a_6 + \ldots$ is

(a) 1 (b) 2^n

(c) 2^{n-1} (d) 3^{n-1}

Ans. (d)

Solution: Putting $x = 1, \omega, \omega^2$ in

$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}\,x^{2n}$, we get

$(1 + 1 + 1)^n = a_0 + a_1 + a_2 + a_3 + \ldots + a_{2n}$,

$(1 + \omega + \omega^2)^n = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 + \ldots + a_{2n}\,\omega^{2n}$,

and $(1 + \omega^2 + \omega^4)^n = a_0 + a_1\omega^2 + a_2\omega^4 + a_3\omega^6 + \ldots + a_{2n}\,\omega^{4n}$

Adding the above three equations and using $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$ we get

$$3^n = 3(a_0 + a_3 + a_6 + \ldots)$$

$\Rightarrow a_0 + a_3 + a_6 + \ldots = 3^{n-1}$.

Example 49: Let

$$z = \begin{vmatrix} 1 & 1-2i & 3+5i \\ 1+2i & -5 & 10i \\ 3-5i & -10i & 11 \end{vmatrix}, \text{ then}$$

(a) z is purely imaginary

(b) z is purely real

(c) $z = 0$

(d) none of these

Ans. (b)

Solution: Conjugate of z equals the determinant obtained by taking conjugate of each of its element. Therefore,

$$\overline{z} = \begin{vmatrix} 1 & 1+2i & 3-5i \\ 1-2i & -5 & -10i \\ 3+5i & 10i & 11 \end{vmatrix} = \begin{vmatrix} 1 & 1-2i & 3+5i \\ 1+2i & -5 & 10i \\ 3-5i & -10i & 11 \end{vmatrix} = z$$

Thus, z is purely real.

Example 50: If $(x + iy)^{1/3} = a + ib$, then $\frac{x}{a} + \frac{y}{b}$ equals

(a) $4(a^2 - b^2)$ (b) $2(a^2 - b^2)$

(c) $2(a^2 + b^2)$ (d) none of these

Ans. (a)

Solution: $(x + iy)^{1/3} = a + ib$

$\Rightarrow \quad x + iy = (a + ib)^3 = a^3 + 3a^2(ib) + 3a(ib)^2 + (ib)^3$

$\Rightarrow \quad x + iy = (a^3 - 3ab^2) + (3a^2b - b^3)i$

Equating real and imaginary parts, we get

$$x = a^3 - 3ab^2 \text{ and } y = 3a^2b - b^3$$

$$\Rightarrow \quad \frac{x}{a} = a^2 - 3b^2 \text{ and } \frac{y}{b} = 3a^2 - b^2.$$

Thus, $\dfrac{x}{a} + \dfrac{y}{b} = 4a^2 - 4b^2 = 4(a^2 - b^2)$

Example 51: If $z \varepsilon \mathbf{C}$, the minimum value of $|z| + |z - i|$ is attained at

(a) exactly one point
(b) exactly two points
(c) infinite number of points
(d) none of these

Ans. (c)

Solution: We have

$$1 = |i| = |z + (i - z)| \le |z| + |i - z|$$

$$\Rightarrow \quad |z| + |z - i| \ge 1$$

The minimum value 1 is attained at all points $z = i\,t$ where $t \in [0, 1]$.

Example 52: For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is

(a) 0	(b) 2
(c) 7	(d) 17

Ans. (b)

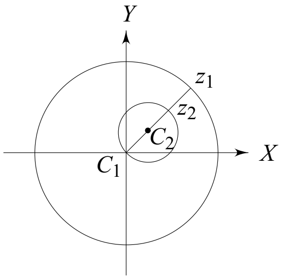


Fig. 2.45

Solution: $|z_1| = 12$ implies that z_1 lies on the circle with centre C_1 at the origin and radius 12 whereas $|z_2 - 3 - 4i| = 5$ implies z_2 lies on the circle with centre at C_2 $(3 + 4i)$ and radius 5. See Fig. 2.45. The quantity $|z_1 - z_2|$ will be least if z_1 and z_2 lie on the line joining C_1 and C_2 i.e. on the line $z = (3 + 4i)t$

In fact, when we take $z_2 = 6 + 8i$ and $z_1 = \dfrac{12}{5}$ $(3 + 4i)$, we obtain $|z_1 - z_2| = \left|\dfrac{12}{5}(3+4i) - (6+8i)\right| = \dfrac{2}{5}\,|3+4i| = 2.$

Example 53: If z lies on the circle $|z - 1| = 1$, then $\dfrac{z-2}{z}$ equals

(a) 0
(b) 2
(c) – 1
(d) none of these

Ans. (d)

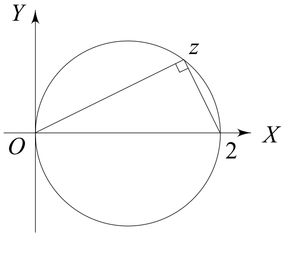


Fig. 2.46

Solution: Note that $|z - 1| = 1$ represents a circle with the segment joining $z = 0$ and $z = 2 + 0i$ as a diameter. See Fig. 2.46. If z lies on the circle, then $\dfrac{z-2}{z-0}$ is purely imaginary.

Example 54: If $1, \omega, \ldots, \omega^{n-1}$ are the nth roots of unity, then value of $\dfrac{1}{2-\omega} + \dfrac{1}{2-\omega^2} + \cdots + \dfrac{1}{2-\omega^{n-1}}$ equals

(a) $\dfrac{1}{2^n - 1}$	(b) $\dfrac{n(2^n - 1)}{2^n + 1}$
(c) $\dfrac{(n-2)\,2^{n-1}}{2^n - 1}$	(d) none of these

Ans. (d)

Solution: We know that

$$\frac{1}{x-1} + \frac{1}{x-\omega} + \frac{1}{x-\omega^2} + \cdots + \frac{1}{x-\omega^{n-1}} = \frac{n(x^{n-1})}{x^n - 1}$$

Putting $x = 2$, we get

$$\frac{1}{2-\omega} + \frac{1}{2-\omega^2} + \cdots + \frac{1}{2-\omega^{n-1}} = \frac{n(2^{n-1})}{2^n - 1}$$

Example 55: If $\omega = \cos\dfrac{\pi}{n} + i\,\sin\dfrac{\pi}{n}$, then value of $1 + \omega + \omega^2 + \ldots + \omega^{n-1}$ is

(a) $1 + i\cot\left(\dfrac{\pi}{2\pi}\right)$	(b) $1 + i\tan\left(\dfrac{\pi}{n}\right)$
(c) $1 + i$	(d) none of these

Ans. (a)

Solution: We have

$$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = \frac{1-\omega^n}{1-\omega} = \frac{2}{1-\omega}.$$

as $\omega^n = \cos\left(\dfrac{n\pi}{n}\right) + i\sin\left(\dfrac{n\pi}{n}\right) = \cos\pi + i\sin\pi = -1$

$$\text{Now,} \quad \frac{2}{1-\omega} = \frac{2(1-\bar{\omega})}{(1-\omega)(1-\bar{\omega})} = \frac{2(1-\bar{\omega})}{1-(\omega+\bar{\omega})+\omega\bar{\omega}}$$

$$= \frac{2(1-\bar{\omega})}{2-2\,\mathrm{Re}(\omega)} \qquad [\because\ \omega\bar{\omega} = |\omega|^2 = 1]$$

$$= \frac{1-\mathrm{Re}(\omega)+i\,\mathrm{Im}(\omega)}{1-\mathrm{Re}(\omega)} = 1 + i\frac{\mathrm{Im}(\omega)}{1-\mathrm{Re}(\omega)}$$

$$= 1 + i\frac{\sin\left(\dfrac{\pi}{n}\right)}{1-\cos\left(\dfrac{\pi}{n}\right)} = 1 + i\frac{2\sin\left(\dfrac{\pi}{2n}\right)\cos\left(\dfrac{\pi}{2n}\right)}{2\sin^2\left(\dfrac{\pi}{2n}\right)}$$

$$= 1 + i\cot(\pi/2n)$$

Example 56: If $|z| = 1$ and $z \ne \pm 1$, then all the values of $\dfrac{z}{1-z^2}$ lie on:

(a) a line not passing through the origin.
(b) $|z| = 2$
(c) the x – axis
(d) the y – axis

Ans. (d)

◎ **Solution:** As $|z| = 1$, we get $z\bar{z} = 1$.

Let $w = \dfrac{z}{1-z^2}$, then

$$w + \bar{w} = \frac{z}{1-z^2} + \frac{\bar{z}}{1-\bar{z}^2} = \frac{z}{1-z^2} + \frac{1/z}{1-(1/z)^2}$$

$$= \frac{z}{1-z^2} + \frac{z}{z^2-1} = 0$$

$\Rightarrow \quad 2\text{Re}(w) = 0 \quad \Rightarrow \quad \text{Re}(w) = 0$

Thus, w lies on the y-axis.

Alternate Solution

$$w = \frac{z}{1-z^2} = \frac{z}{z\bar{z}-z^2} = \frac{1}{\bar{z}-z}$$

$$= \frac{1}{-2i\,\text{Im}(z)}$$

$\Rightarrow w$ is purely imaginary, that is, w lies on the y-axis.

● **Example 57:** The locus of the centre of a circle which touches the circle $|z - z_1| = a$ and $|z - z_2| = b \neq a$ externally is

(a) an ellipse (b) a hyperbola
(c) a circle (d) a pair of straight lines.

Ans. (b)

◎ **Solution:** Suppose $|z - w| = r$ touches $|z - z_1| = a$ and $|z - z_2| = b$ externally.

Then $|w - z_1| = a + r$, $|w - z_2| = b + r$

$\Rightarrow \quad |w - z_1| - |w - z_2| = a - b$

$\Rightarrow \quad w$ lies on a hyperbola with focii at z_1 and z_2

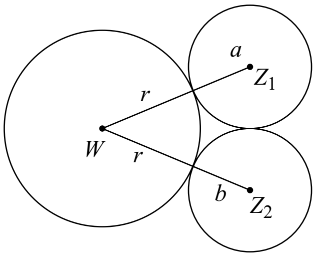


Fig. 2.47

● **Example 58:** If $|z^2 - 1| = |z|^2 + 1$, then z lies on

(a) a circle (b) the imaginary axis
(c) the real axis (d) an ellipse

Ans. (b)

◎ **Solution:** $|z^2 - 1| = |z|^2 + 1$ can be written as $|z^2 + (-1)| = |z^2| + |-1|$

$\Leftrightarrow \quad \dfrac{z^2}{-1}$ is a non-negative real number.

$\Leftrightarrow \quad z^2$ is a non-positive real number.

$\Leftrightarrow \quad z$ lies on the imaginary axis.

● **Example 59:** If $z^2 + z + 1 = 0$, where z is a complex number, then values of

$$S = \left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \text{---} + \left(z^6 + \frac{1}{z^6}\right)^2 \text{ is}$$

(a) 12 (b) 18
(c) 54 (d) 6

Ans. (a)

◎ **Solution:** $z^2 + z + 1 = 0 \quad \Rightarrow \quad z = \omega, \omega^2$.

Let $z = \omega$ then $(\omega + \omega^2)^2$

$= (\omega^2 + \omega^4)^2 = (\omega^4 + \omega^8)^2$

$= (\omega^5 + \omega^{10})^2 = 1$

and $(\omega^3 + \omega^6)^2 = (\omega^6 + \omega^{12})^2 = 4$.

Thus, $S = 12$.

● **Example 60:** If $|z + 4| \leq 3$, then maximum value of $|z + 1|$ is

(a) 4 (b) 10
(c) 6 (d) 0

Ans. (c)

◎ **Solution:** $|z+4| \leq 3$ represents the interior and boundary of the circle with center at – 4 and radius equal to 3. As – 1 is an end point of a diameter of the circle, maximum possible value of $|z + 1|$ is 6 which is attained when $z = -7$. See Fig. 2.48

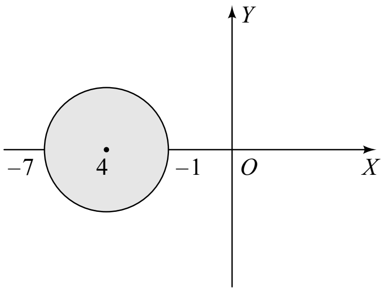


Fig. 2.48

● **Example 61:** Let z, w be two complex numbers such that $\bar{z} + i\bar{w} = 0$ and arg $(zw) = \pi$, then arg z equals

(a) $\dfrac{3\pi}{4}$ (b) $\dfrac{\pi}{2}$

(c) $\dfrac{\pi}{4}$ (d) $\dfrac{5\pi}{4}$

Ans. (a)

◎ **Solution:** $\bar{z} + i\bar{w} = 0 \quad \Rightarrow \quad z - iw = 0$

Now, $\quad \arg(zw) = \pi \quad \Rightarrow \quad \arg\left(\dfrac{z^2}{i}\right) = \pi$

$\Rightarrow \quad 2\arg(z) - \arg(i) = \pi$

$\Rightarrow \quad 2\arg(z) = \pi + \dfrac{\pi}{2} = \dfrac{3\pi}{2} \quad \Rightarrow \quad \arg(z) = \dfrac{3\pi}{4}$

Example 62: If $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then value of $z_1^2 + z_2^2 + z_3^2$ equals

(a) -1 (b) 0
(c) 1 (d) 3

Ans. (b)

Solution: $z_1^2 + z_2^2 + z_3^2$

$$= (z_1 + z_2 + z_3)^2 - 2(z_2 z_3 + z_3 z_1 + z_1 z_2)$$

$$= 0 - 2z_1 z_2 z_3 \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$$

$$= -2z_1 z_2 z_3 (\overline{z}_1 + \overline{z}_2 + \overline{z}_3)$$

$$[\because |z_1| = |z_2| = |z_3| = 1]$$

$$= -2z_1 z_2 z_3(0)$$

$$= 0$$

Example 63: If z satisfies the relation

$$|z - i|z|| = |z + i|z||, \qquad (1)$$

then

(a) $\text{Im}(z) = 0$ (b) $|z| = 1$
(c) $\text{Re}(z) = 0$ (d) none of these

Ans. (a)

Solution: $z = 0$ clearly satisfies (1).
For $z \neq 0$, (1) can be written as

$$\left|\frac{z}{|z|} - 1\right| = \left|\frac{z}{|z|} + i\right|$$

$\Rightarrow \dfrac{z}{|z|}$ is equidistant from i and $-i$

$\Rightarrow \dfrac{z}{|z|}$ lies on the real axis.

$\Rightarrow$ z is real.

Thus, $\text{Im}(z) = 0$.

Example 64: If α, β are distinct complex numbers with $|\beta| = 1$, then value of $\left|\dfrac{\beta - \alpha}{1 - \overline{\alpha}\beta}\right|$ equals

(a) 1 (b) $|\alpha|$
(c) 2 (d) none of these

Ans. (a)

Solution: $|\beta| = 1 \Rightarrow \beta\overline{\beta} = 1$

$$\therefore \quad \left|\frac{\beta - \alpha}{1 - \overline{\alpha}\beta}\right| = \left|\frac{\beta - \alpha}{1 - \overline{\alpha}/\overline{\beta}}\right| = \left|\frac{\overline{\beta}(\beta - \alpha)}{\overline{\beta} - \overline{\alpha}}\right| = |\beta| \frac{|\beta - \alpha|}{|\overline{\beta} - \overline{\alpha}|}$$

$$= |\beta|(1) = 1. \qquad [\because \ |z| = |\overline{z}|]$$

Example 65: Suppose $z \in \mathbf{C}$, and $z \notin \mathbf{R}$. If $w = \dfrac{1 - z + z^2}{1 + z + z^2}$ is a real number, then $|z|$ equals

(a) 1 (b) 2
(c) $\sqrt{3}$ (d) $2\sqrt{3}$

Ans. (a)

Solution: Note that $z \neq 0$ and $1 - w = \dfrac{2z}{1 + z + z^2} \in \mathbf{R}$.

$$\Rightarrow \frac{z}{1 + z + z^2} \in \mathbf{R} \Rightarrow \frac{1}{1 + z + 1/z} \in \mathbf{R} \Rightarrow z + \frac{1}{z} \in \mathbf{R}$$

$$\Rightarrow z + \frac{1}{z} = \overline{z} + \frac{1}{\overline{z}} \quad \Rightarrow \quad z - \overline{z} = \frac{1}{\overline{z}} - \frac{1}{z} = \frac{z - \overline{z}}{|z|^2}$$

As $z \notin \mathbf{R}$, $z \neq \overline{z}$, therefore $|z|^2 = 1$ or $|z| = 1$

Example 66: If $\left|z - \dfrac{4}{z}\right| = 2$, then the maximum value of $|z|$ is equal to

(a) 1 (b) $2 + \sqrt{2}$
(c) $\sqrt{3} + 1$ (d) $\sqrt{5} + 1$

Ans. (d)

Solution: $|z| - \dfrac{4}{|z|} \le \left||z| - \dfrac{4}{|z|}\right| \le \left|z - \dfrac{4}{z}\right| = 2$

$$\Rightarrow \quad |z|^2 - 2|z| - 4 \le 0$$

$$\Rightarrow \quad (|z| - 1)^2 \le 5 \Rightarrow |z| \le 1 + \sqrt{5}.$$

The maximum value $\sqrt{5} + 1$ is attained when $z = \sqrt{5} + 1$.

Example 67: If $|\omega| = 2$, then the set of points $x + iy = \omega - \dfrac{1}{\omega}$ lie on

(a) circle (b) ellipse
(c) parabola (d) hyperbola

Ans. (b)

Solution: $|\omega| = 2 \Rightarrow \omega = 2e^{i\theta} = 2(\cos\theta + i\sin\theta)$

$$\Rightarrow \quad x + iy = \omega - \frac{1}{\omega} = 2e^{i\theta} - \frac{1}{2}e^{-i\theta}$$

$$\Rightarrow \quad x = \frac{3}{2}\cos\theta,\ y = \frac{5}{2}\sin\theta \Rightarrow \frac{x^2}{9/4} + \frac{y^2}{25/4} = 1$$

which represents an ellipse.

Example 68: If $|z| = 1$, $z \neq 1$, then value of $\arg\left(\dfrac{1}{1 - z}\right)$ cannot exceed

(a) $\pi/2$ (b) π
(c) $3\pi/2$ (d) 2π

Ans (a)

Solution: As $|z| = 1$, $z \neq 1$, $z = \cos\theta + i\sin\theta$, $-\pi < \theta$, $\le \pi$, $\theta \neq 0$. Now

$$\omega = \frac{1}{1 - z} = \frac{1}{1 - \cos\theta - i\sin\theta}$$

$$= \frac{(1-\cos\theta)+i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta} = \frac{(1-\cos\theta)+i\sin\theta}{2(1-\cos\theta)}$$

$$= \frac{1}{2}+\frac{i}{2}\cot\left(\frac{\theta}{2}\right) = \frac{1}{2}+\frac{i}{2}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

This shows that ω lies on the line $x = 1/2$ and $-\pi/2 < \arg(\omega) < \pi/2$, $\text{Arg}(\omega) \neq 0$, The maximum value of Arg(ω) is never attained.

Example 69: If $z \neq 1$, $\dfrac{z^2}{z-1}$ is real, then point represented by the complex number z lies

- (a) on circle with centre at the origin.
- (b) either on the real axis or on a circle not passing through the origin.
- (c) on the imaginary axis.
- (d) either on the real axis or on a circle passing through the origin

Ans (b)

Solution: As $\dfrac{z^2}{z-1}$ is real, we get

$$\frac{z^2}{z-1} = \frac{\bar{z}^2}{\bar{z}-1}$$

$\Leftrightarrow \quad z^2(\bar{z} - 1) = \bar{z}^2 (z - 1)$

$\Leftrightarrow \quad z\bar{z}(z - \bar{z}) - (z - \bar{z})(z + \bar{z}) = 0$

$\Leftrightarrow \quad (z - \bar{z})(z\bar{z} - z - \bar{z}) = 0$

$\Rightarrow \quad z - \bar{z} = 0$ or $z\bar{z} - z - \bar{z} = 0$

$\Rightarrow \quad z$ lies on the real axis

or z lies on a circle through the origin.

Example 70: If $3^{49}(x + iy) = \left(\dfrac{3}{2}+\dfrac{i}{2}\sqrt{3}\right)^{100}$, $y \in \mathbf{N}$, and $x = ky$, then value of k is

(a) $\pm 1/3$	(b) $\pm 2\sqrt{2}$
(c) $\pm 1/\sqrt{3}$	(d) $\pm 1/\sqrt{3}$

Ans (b)

Solution: We have

$3^{49}\,|x + iy| = |3/2 + \sqrt{3}i/2|^{100}$

$\Rightarrow \quad 3^{49}\sqrt{x^2+y^2} = (9/4 + 3/4)^{50}$

$\Rightarrow \quad \sqrt{x^2+y^2} = 3 \Rightarrow (y)\sqrt{1+k^2} = 3$

$\Rightarrow \quad \sqrt{1+k^2} = 3,\ 3/2,\ 1,$ as $y \in \mathbf{N}$

$\Rightarrow \quad k = \pm\, 2\sqrt{2},\ \pm\sqrt{5}/2,\ 0$

Out of the given values, we have $k = \pm 2\sqrt{2}$.

Example 71: If $(4 + i)(z + \bar{z}) - (3 + i)(z - \bar{z}) + 26i = 0$, then the value of $|z|^2$ is

(a) 13	(b) 17
(c) 19	(d) 11

Ans (b)

Solution: Let $z = x + iy$, then

$2(4 + i)x - (3 + i)2iy + 26i = 0$

$\Rightarrow \quad 4x + y = 0,\ x - 3y + 13 = 0$

$\Rightarrow \quad x = -1,\ y = 4$

$\therefore \quad |z|^2 = 17$

Example 72: Let $z = a\left(\cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}\right)$, $a \in \mathbf{R}$, $|a| < 1$, then $S = z^{2015} + z^{2016} + z^{2017} + \ldots$ equals

(a) $\dfrac{a^{2015}}{z-1}$	(b) $\dfrac{a^{2015}}{1-z}$
(c) $\dfrac{z^{2015}}{1-a}$	(d) $\dfrac{z^{2015}}{a-1}$

Ans (a)

Solution: We have $|z| = |a| < 1$, thus

$$S = \frac{z^{2015}}{1-z}$$

But $\quad z^{2015} = a^{2015}[\cos(403\pi) + i\sin(403\pi)] = -a^{2015}$

$$\therefore \quad S = \frac{a^{2015}}{z-1}$$

Example 73: If $z = \sqrt{20i-21}+\sqrt{20i+21}$, than one of the possible value of $|\arg(z)|$ equals

(a) $\pi/4$	(b) $\pi/2$
(c) $3\pi/8$	(d) π

Ans (a)

Solution: $20i + 21 = (5 + 2i)^2$

and $\quad 20i - 21 = (2 - 5i)^2$

$\therefore \quad z = \pm(5 + 2i) \pm (2 + 5i)$

$\Rightarrow \quad z = 7(1 + i),\ 3(1 - i),\ 3(-1 + i),\ 7(-1 - i)$

$\Rightarrow \quad \arg(z) = \pi/4,\ -\pi/4,\ 3\pi/4,\ -3\pi/4$

Thus, one of the possible value of $|\arg(z)|$ is $\pi/4$.

Example 74: If $(a + bi)^{11} = x + iy$, where $a, b, x, y \in \mathbf{R}$, then $(b + ai)^{11}$ equals

(a) $y + ix$	(b) $-y - ix$
(c) $-x - iy$	(d) $x + iy$

Ans (b)

Solution: $\overline{(a+bi)^{11}} = \overline{x+iy}$

$\Rightarrow \quad (a - bi)^{11} = x - iy$

$\Rightarrow \quad [(-i)(b + ia)]^{11} = -i(y + ix)$

$\Rightarrow \quad (-i)^{11}(b + ia)^{11} = -i(y + ix)$

As $\quad (-i)^{11} = (-i)^8(-1)^3 i^3 = i$,

we get $\quad (b + ia)^{11} = -(y + ix) = -y - ix$

◉ **Example 75:** If $a, b, x, y \in \mathbf{R}$, $\omega \neq 1$, is a cube root of unity and $(a + b\omega)^7 = x + y\omega$, then $(b + a\omega)^7$ equals

(a) $y + x\omega$ (b) $-y - x\omega$

(c) $x + y\omega$ (d) $-x - y\omega$

Ans (a)

◎ **Solution:** Taking conjugate, we get

$$(a + b\omega^2)^7 = x + y\,\omega^2 \Rightarrow (a\omega^3 + b\omega^2)^7 = x\omega^3 + y\omega^2$$

$$\omega^{14}\,(a\omega + b)^7 = \omega^2\,(x\omega + y)$$

$$\Rightarrow \quad (a\omega + b)^7 = x\omega + y$$

Assertion-Reason Type Questions

◉ **Example 76:** Suppose z_1, z_2 are two distinct complex numbers and a, b are real numbers.

Statement-1: If $z_1 + z_2 = a$, $z_1 z_2 = b$, then $\arg(z_1 z_2) = 0$

Statement-2: If $z_1 + z_2 = a$, $z_1 z_2 = b$, then $\bar{z}_1 = z_2$

Ans. (c)

◎ **Solution:** Statement-2 is false. This can be seen by taking z_1 and z_2 to be distinct real numbers.

For statement-1, take $z_1 = \alpha_1 + i\beta_1$, $z_2 = \alpha_2 + i\beta_2$. We have

$$\beta_1 + \beta_2 = 0,\ \alpha_1\beta_2 + \alpha_2\beta_1 = 0$$

If $\beta_2 = 0$, then $\beta_1 = 0$, and statement-1 is true.

If $\beta_2 \neq 0$, then $\beta_1 = -\beta_2$ and $(\alpha_1 - \alpha_2)\beta_2 = 0$

$\Rightarrow \quad \alpha_1 - \alpha_2 = 0 \quad$ or $\quad \alpha_1 = \alpha_2$

Thus, $\quad \alpha_2 + i\beta_2 = \quad \alpha_1 - i\beta_1 \quad \Rightarrow \quad z_2 = \bar{z}_1$

In this case also $\arg(z_1 z_2) = \arg(z_1 \bar{z}_1)$

$= \arg(|z_1|^2) = 0$

◉ **Example 77:** Suppose z_1 and z_2 are two distinct non-zero complex numbers.

Statement-1: $|z_1 - z_2| = \left|\dfrac{1}{z_1} - \dfrac{1}{z_2}\right|$

$\Rightarrow \quad |z_1|\,|z_2| = 1$

Statement-2: $|z_1 - z_2| = \left|\dfrac{1}{z_1} - \dfrac{1}{z_2}\right|$

then both of z_1, z_2 have modulus 1

Ans (c)

◎ **Solution:** $|z_1 - z_2| = \left|\dfrac{1}{z_1} - \dfrac{1}{z_2}\right| = \dfrac{|z_2 - z_1|}{|z_2\, z_1|}$

As $z_1 \neq z_2$, we get $|z_1\, z_2| = 1$

$\Rightarrow \quad |z_1|\,|z_2| = 1$

$\therefore$ Statement-1 is true.

Statement-2 is false. For example take $z_1 = 3$, $z_2 = 1/3$.

◉ **Example 78:** Suppose $a, b, c \in \mathbf{R}$.

Statement-1: If $z = a + (b + ic)^{2017} + (b - ic)^{2017}$ then z is real.

Statement-2: If $z = \bar{z}$, then z is real

Ans (a)

◎ **Solution:** Statement-2 is true. See theory

We have

$$\bar{z} = a + (\overline{b + ic})^{2017} + (\overline{b - ic})^{2017}$$

$$= a + (b - ic)^{2017} + (b + ic)^{2017}$$

$$= z$$

$\Rightarrow \quad z$ is real

$\therefore$ Statement-1 is true and statement-2 is correct explanation for it.

◉ **Example 79:** Let $\omega \neq 1$, be a cube root of unity, and $a, b \in \mathbf{R}$.

Statement-1: $a^3 + b^3 = (a + b)\,(a\omega + b\omega^2)\,(a\omega^2 + b\omega)$

Statement-2: $x^3 - 1 = (x - 1)\,(x\omega^2 - \omega)\,(x\omega - \omega^2)$

for each $x \in \mathbf{R}$.

Ans (a)

◎ **Solution:** We have

$$(x\omega^2 - \omega)\,(x\omega - \omega^2)$$

$$= x^2\omega^3 - x\omega^2 - x\omega^4 + \omega^3$$

$$= x^2 - (\omega^2 + \omega)x + 1$$

$$= x^2 + x + 1 \qquad [\because\ \omega^2 + \omega = -1]$$

$\therefore \quad (x - 1)\,(x\omega^2 - \omega)\,(x\omega - \omega^2)$

$= (x - 1)\,(x^2 + x + 1) = x^3 - 1$

Thus, Statement-2 is true.

Replacing x by $-x$, we get

$$(-x)^3 - 1 = (-x - 1)\,(-x\,\omega^2 - \omega)\,(-x\omega - \omega^2)$$

$$\Rightarrow \quad x^3 + 1 = (x + 1)\,(x\,\omega^2 + \omega)\,(x\omega + \omega^2)$$

Taking conjugate of both the sides, we get

$$x^3 + 1 = (x + 1)\,(x\,\omega + \omega^2)\,(x\omega^2 + \omega)$$

$$[\because\ \omega = \omega^2]$$

If $b = 0$, statement-1 is clearly true. Suppose $b \neq 0$.

Replacing x by a/b we get

$$\left(\frac{a}{b}\right)^3 + 1 = \left(\frac{a}{b} + 1\right)\left(\frac{a}{b}w + w^2\right)\left(\frac{a}{b}w^2 + w\right)$$

$$\Rightarrow \quad a^3 + b^3 = (a + b)\,(aw + bw^2)\,(aw^2 + bw)$$

Thus, statement-1 is also true and statement-2 is a correct explanation for it.

Example 80: Let A, B, C be three set of complex numbers as defined below:

$$A = \{z : \text{Im } z \ge 1\}$$
$$B = \{z : |z - 2 - i| = 3\}$$
$$C = \{z : \text{Re }((1 - i)\, z)\} = \sqrt{2}\ \}$$

Statement-1: $|z + 1 - i|^2 + |z - 5 - i|^2 = 37$

$\forall\, z \in A \cap B \cap C$

Statement-2: $A \cap B \cap C$ consists of exactly one point.

Ans (d)

Solution: The set A consists of all points in the half plane $\text{Im }(z) \ge 1$, that is, all the points above and including the line through S and parallel to the real axis. The set B is the set of all points on the circle with centre at $2 + i$ and radius 3, and the set C consists of all the points on the line $x + y = \sqrt{2}$. The regions in A, B and C intersect in exactly one point *viz*. R. see Fig. 2.49.

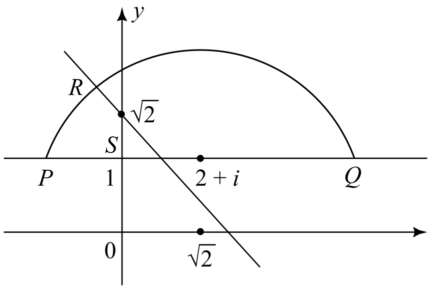


Fig. 2.49

$\therefore$ Statement-2 is true. Points $P\,(-1 + i)$ and $Q(5 + i)$ are the end points of diameter of the circle $|z - (2 + i)| = 3$

Now, $|z + 1 - i|^2 + |z - 5 - i|^2$

$$= PR^2 + QR^2 = PQ^2 = 36$$

$\therefore$ Statement-1 is false.

Example 81: Let z_1, z_2, z_3 be three distinct non-zero complex numbers such that $a = |z_1|$, $b = |z_2|$, $c = |z_3|$. and

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

Statement-1: If $z_1 + z_2 + z_3 = 0$, then triangle with vertices z_1, z_2 and z_3 is an equilateral triangle.

Statement-2: Area of triangle with vertices z_1, z_2, and z_3 is $\frac{\sqrt{3}}{4}|z_1 - z_2|^2$

Ans (c)

Solution: Applying $C_1 \to C_1 + C_2 + C_3$,

we get

$$\Delta = (a + b + c)\, \Delta_1 \text{ where}$$

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 0 & c - b & a - c \\ 0 & a - b & b - c \end{vmatrix}$$

$$= -(b - c)^2 - (a - b)\,(a - c)$$
$$= -[a^2 + b^2 + c^2 - bc - ca - ab]$$
$$= -\frac{1}{2}[(b - c)^2 + (c - a)^2 + (a - b)^2]$$

Now, $\Delta = 0$

$\Rightarrow\ (a + b + c)\,[(b - c)^2 + (c - a)^2 + (a - b)^2]$

As $a + b + c > 0$, we get

$$a = b = c$$

$$\Rightarrow\ |z_1| = |z_2| = |z_3|$$

$\Rightarrow$ z_1, z_2, z_3 lie on a circle with centre at the origin and radius equal to a.

If we take z_1, z_2 as opposite vertices of a diameter, then triangle is a right triangle with right angle at z_3, and its area is $\frac{1}{2}|z_3 - z_1|\,|z_3 - z_2|$

$\therefore$ Statement-2 is false.

If $z_1 + z_2 + z_3 = 0$, then $|z_1 - z_2|^2 + |z_3|^2$

$$= |z_1 - z_2|^2 + |-z_1 - z_2|^2$$
$$= 2|z_1|^2 + 2|z_2|^2 = 4a^2$$

$$\Rightarrow\ |z_1 - z_2| = \sqrt{3}a$$

Similarly, $|z_2 - z_3| = |z_3 - z_1| = \sqrt{3}a$

Thus, triangle is an equilateral triangle.

Example 82: **Statement-1:** If z_1, z_2, z_3 are such that $|z_1| = |z_2| = |z_3| = 1$, then maximum value of $|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$ is 9.

Statement-2: If z_1, z_2, z_3 are such that $|z_1| = |z_2| = |z_3| = 1$, then

$$\text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2) \ge -3/2$$

Ans. (a)

Solution: $0 \le |z_1 + z_2 + z_3|^2$

$$\Rightarrow\ 0 \le |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\,\text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2)$$

$$\Rightarrow\ \text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2) \ge -3/2$$

$$[\because |z_1| = |z_2| = |z_3| = 1]$$

Therefore, statement-2 is true.

Next, $|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$

$$= 2(|z_1|^2 + |z_2|^2 + |z_3|^2) - 2\,\text{Re}(z_2\bar{z}_3 + z_3\bar{z}_1 + z_1\bar{z}_2)$$
$$\leq 2(1 + 1 + 1) + 2(3/2) = 9$$

Maximum value is obtained when $z_1 = 1$, $z_2 = \omega$, $z_3 = \omega^2$, where ω is a cube root of unity.

Statement-1 is also true and statement-2 is correct explanation for it.

◉ **Example 83: Statement-1:** If $\omega \neq 1$ is a cube root of unity and z is a complex number such that $|z| = 1$, then

$$\left|\frac{2 + 3\omega + 4z\,\omega^2}{4\omega + 3\omega^2\, z + 2z}\right| = 1.$$

Statement-2: If z_1, z_2 are two complex numbers, then $|z_1| = |z_2| \Leftrightarrow z_1 = \bar{z}_2$

Ans. (c)

◎ **Solution:** $\left|\dfrac{2 + 3\omega + 4z\,\omega^2}{4\omega + 3\omega^2\, z + 2z}\right| = \left|\dfrac{1}{z}\,\dfrac{2 + 3\omega + 4\omega^2 z}{2 + 3\omega^2 + 4\omega z^{-1}}\right|$

$$= \left|\frac{1}{z}\right|\left|\frac{2 + 3\omega + 4\omega^2\, z}{2 + 3\bar{\omega} + 4\bar{\omega}^2\bar{z}}\right| = \frac{1}{|\,z\,|} = 1$$

$$[\because\ \bar{\omega} = \omega^2,\ \bar{z} = 1/z]$$

Thus, Statement-1 is true.

Statement-2 is false as $|z_1| = |z_2|$ does not imply $z_1 = \bar{z}_2$. However, $z_1 = \bar{z}_2 \Rightarrow |z_1| = |z_2|$

◉ **Example 84: Statement-1:** If z is a root of the equation $x^7 + 2x + 3 = 0$, then $1 \leq |z| < 3/2$.

Statement-2: If z lies in the annular region $1 < |z| \leq 3/2$, then z satisfies the

$$\frac{1}{z - 1} + \frac{1}{z - \omega} + \frac{1}{z - \omega^2} = 1$$

where $\omega \neq 1$ is a cube root of unity.

Ans. (c)

◎ **Solution:** Suppose $|z| < 1$ and $z^7 + 2z + 3 = 0$, then

$$3 = |-3| = |z^7 + 2z| \leq |z^7| + 2|z|$$

$$\Rightarrow \qquad 3 \leq |z|^7 + 2|z| < 1 + 2(1) = 3.$$

A contradiction.

Next, suppose that $|z| \geq 3/2$ and $z^7 + 2z + 3 = 0$, then $\omega = 1/z$ satisfies the equation $1 + 2\omega^6 + 3\omega^7 = 0$. Now, $1 = |-1| = |2\omega^6 + 3\omega^7| \leq 2|\omega|^6 + 3|\omega^7|$

$$\Rightarrow \qquad 1 \leq 2\left(\frac{2}{3}\right)^6 + 3\left(\frac{2}{3}\right)^7 = \left(\frac{2}{3}\right)^6[2 + 2] = \frac{2^8}{3^6} < 1$$

A contradiction.

Thus, if z satisfies the equation $z^7 + 2z + 3 = 0$, then $1 \leq |z| < 3/2$.

Thus, Statement-1 is true.

Statement-2 is false, as the given relation implies $z^3 - 3z^2 + 1 = 0$ which is satisfied by just 3 values of z where as $1 < |z| \leq 3/2$ contains infinite number of points.

◉ **Example 85: Statement-1** If $\omega \neq 1$, is a cube root of unity, then $A^2 = O$, where

$$A = \begin{pmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{pmatrix}$$

Statement-2 If $\omega \neq 1$, is a cube root of unity, then

$$\Delta = \begin{vmatrix} x + 1 & \omega & \omega^2 \\ \omega & x + \omega^2 & 1 \\ \omega^2 & 1 & x + \omega \end{vmatrix} = x^3$$

Ans. (b)

◎ **Solution:** It is easy to show

$$A^2 = \begin{pmatrix} y & y & y \\ y & y & y \\ y & y & y \end{pmatrix} = O$$

where $\qquad y = 1 + \omega + \omega^2 = 0.$

$\therefore$ Statement-1 is true.

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = x\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & x + \omega^2 & 1 \\ 1 & 1 & x + \omega \end{vmatrix} = x\begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & x + \omega^2 - \omega & 1 - \omega^2 \\ 0 & 1 - \omega & x + \omega - \omega^2 \end{vmatrix}$$

$$= x[(x + \omega^2 - \omega)(x + \omega - \omega^2) - (1 - \omega)(1 - \omega^2)]$$
$$= x[x^2 - (\omega - \omega^2)^2 - \{1 - \omega - \omega^2 + 1\}] = x^3$$

$\therefore$ Statement-2 is also true but is not the correct reason for the statement-1.

LEVEL 2

Straight Objective Type Questions

Example 86: Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}\left(z^{2m-1}\right)$ at $\theta = 2°$ is

(a) $\frac{1}{\sin 2°}$ (b) $\frac{1}{3\sin 2°}$

(c) $\frac{1}{\sin 2°}$ (d) $\frac{1}{4\sin 2°}$

Ans. (d)

Solution: $\sum_{m=1}^{15} \text{Im}\left(z^{2m-1}\right) = \text{Im}\sum_{m=1}^{15}\left(z^{2m-1}\right)$

$$= \text{Im}\left[\frac{z\left(1-\left(z^2\right)^{15}\right)}{1-z^2}\right] = \text{Im}\left(\frac{1-z^{30}}{\bar{z}-z}\right)$$

$$= \text{Im}\left[\frac{1}{-2i\sin\theta}\{1-\cos(30\theta)-i\sin(30\theta)\}\right]$$

$$= \frac{1}{2\sin\theta}[1-\cos(30\theta)] = \frac{1}{2\sin 2°}[1-\cos(60°)]$$

$$= \frac{1}{4\sin 2°}$$

Example 87: Let $z = x + iy$ be a complex number where x and y are integers. Then the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ is

(a) 48 (b) 32

(c) 40 (d) 80

Ans. (a)

Solution: $z\bar{z}^3 + \bar{z}z^3 = 350 \Rightarrow z\bar{z}\left(\bar{z}^2 + z^2\right) = 350$

$$\Rightarrow \left(x^2+y^2\right)\left[(x-iy)^2+(x+iy)^2\right]=350$$

$$\Rightarrow \left(x^2+y^2\right)\left(2x^2-2y^2\right) = 350$$

$$\Rightarrow x^4 - y^4 = 175$$

$$\Rightarrow x^4 \geq 175 \Rightarrow x^4 \geq 256$$

Let us try $x = \pm 4$

Thus, $y^4 = 256 - 175 = 81 \Rightarrow y = \pm 3$

$\therefore$ roots of $z\bar{z}^3 + \bar{z}z^3 = 350$ are $z = \pm 4 \pm 3i$

Area of rectangle whose vertices are $\pm 4 \pm 3i$ is (8) (6) = 48.

Example 88: Let z_1 and z_2 be two distinct complex numbers and let $z = (1 - t)\, z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg (w) denotes the principal argument of a nonzero complex number w, then which of the following is not true?

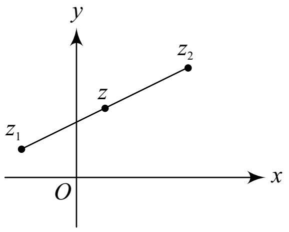


Fig. 2.50

(a) $|z-z_1|+|z-z_2|=|z_1-z_2|$

(b) Arg $(z - z_1)$ = Arg $(z - z_2)$

(c) $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix} = 0$

(d) Arg $(z - z_1)$ = Arg $(z_2 - z_1)$

Ans. (b)

Solution: $z - z_1 = t\,(z_2 - z_1)$

$\Rightarrow$ $|z - z_1| = t|z_2 - z_1| = t|z_1 - z_2|$ (1)

[$\because$ $t > 0$]

and $z - z_2 = (1 - t)\,(z_1 - z_2)$

$\Rightarrow$ $|z - z_2| = (1 - \text{t})\,|z_1 - z_2| = (1 - t)\,|z_2 - z_1|$ (2)

[$\because$ $1 - t > 0$]

From (1) and (2), we get

$$|z - z_1| + |z - z_2| = |z_2 - z_1|$$

Next,

$$\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix} = \begin{vmatrix} t\left(z_2-z_1\right) & t\left(\bar{z}_2-\bar{z}_1\right) \\ z_2-z_1 & \left(\bar{z}_2-\bar{z}_1\right) \end{vmatrix} = 0$$

[$\because$ R_1 and R_2 are identical]

Also, Arg $(z - z_1)$ = Arg $(z_2 - z_1)$ since z_1, z and z_2 lie on the same straight line and on the same side of z_1.

Example 89: For complex numbers $z_1 = x_1 + iy$, and $z_2 = x_2 + iy_2$, we write $z_1 \cap z_2$ if $x_1 \leq x_2$ and $y_1 \leq y_2$. Let z be a complex number such that $1 \cap z$, then

(a) $\frac{1-z}{1+z} \cap -i$ (b) $1 \cap \frac{1-z}{1+z}$

(c) $\frac{1-z}{1+z} \cap 0$ (d) $\frac{1+z}{1-z} \cap 0$

Ans. (c)

Solution: Let $z = x + iy$. As $1 \cap z$, we get $1 \le x$ and $0 \le y$

Now, $$\frac{1-z}{1+z} = \frac{(1-x)-iy}{(1+x)+iy} = \frac{[(1-x)-iy][(1+x)-iy]}{(1+x)^2+y^2}$$

$$= \frac{(1-x^2)-y^2-iy\,(1+x+1-x)}{(1+x)^2+y^2}$$

$$= \frac{1-(x^2+y^2)-2iy}{(1+x)^2+y^2}$$

As $x \ge 1$, and $y \ge 0$, we get

$$\frac{1-(x^2+y^2)}{(1+x)^2+y^2} \le 0 \text{ and } \frac{-2y}{(1+x)^2+y^2} \le 0$$

Thus, $\frac{1-z}{1+z} \cap 0$

Example 90: The complex number z_1, z_2, z_3 are the vertices of an equilateral triangle. If z_0 is the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2$ is equal to

(a) z_0^2 (b) $3z_0^2$

(c) z_0^3 (d) $3z_0^3$

Ans. (b)

Solution: As the triangle with vertices z_1, z_2 and z_3 is an equilateral triangle, the circumcentre and the centroid of the triangle coincides. Thus,

$$z_0 = \frac{1}{3}\,(z_1 + z_2 + z_3)$$

$$\Rightarrow \quad (3z_0)^2 = z_1^2 + z_2^2 + z_3^2 + 2(z_2 z_3 + z_3 z_1 + z_1 z_2) \qquad (1)$$

As the triangle is an equilateral triangle,

$$z_2 z_3 + z_2 z_1 + z_1 z_3 = z_1^2 + z_2^2 + z_3^2 \qquad (2)$$

From (1) and (2) we get

$$9\,z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2\,(z_1^2 + z_2^2 + z_3^3)$$

$$\Leftrightarrow \quad 3z_0^2 = z_1^2 + z_2^2 + z_3^2\,.$$

Example 91: If ω is an imaginary cube root of unity, then value of the expression

$1(2 - \omega)(2 - \omega^2) + 2(3 - \omega)(3 - \omega^2) + \ldots$
$+ (n - 1)(n - \omega)(n - \omega^2)$ is

(a) $\frac{1}{4}n^2(n+1)^2 - n$ (b) $\frac{1}{4}n^2(n+1)^2 + n$

(c) $\frac{1}{4}n^2(n+1) - n$ (d) $\frac{1}{4}n\,(n+1)^2 - n$

Ans. (a)

Solution: rth term of the given expression is $r\,(r + 1 - \omega)(r + 1 - \omega^2)$

$$= (r + 1 - 1)(r + 1 - \omega)(r + 1 - \omega^2)$$
$$= (r + 1)^3 - 1$$
$$[\therefore\ (x - 1)(x - \omega)(x - \omega^2) = x^3 - 1]$$

$$\therefore \quad S = \sum_{r=1}^{n-1}\left[(r+1)^3 - 1\right] = \sum_{r=0}^{n-1}\left[(r+1)^3 - 1\right]$$

$$= \sum_{r=0}^{n} r^3 - n = \frac{1}{4}n^2(n+1)^2 - n$$

Example 92: The number of complex numbers z such that $|z| < 1/3$, and $\sum_{r=1}^{n} a_r\, z^r = 1$ where $|a_r| < 2$, is

(a) 0 (b) 1

(c) 4 (d) infinite

Ans. (a)

Solution: We have

$$1 = \left|\sum_{r=1}^{n} a_r\, z^r\right| \le \sum_{r=1}^{n} |a_r|\,|z|^r$$

$$< \sum_{r=1}^{n} 2\left(\frac{1}{3}\right)^r < \sum_{r=1}^{\infty} 2\left(\frac{1}{3}\right)^r = \frac{2/3}{1-2/3} = 1$$

A contradiction.

Example 93: If a, b, c are integers, not all equal and $\omega(\neq 1)$ is a cube root of unity, then minimum value of $|a + b\omega + c\omega^2|$ is

(a) $\sqrt{3}$ (b) 1

(c) 2 (d) 3

Ans. (b)

Solution: $\left|a + b\omega + c\omega^2\right|^2$

$$= (a + b\omega + c\omega^2)(a + b\bar{\omega} + c\bar{\omega}^2)$$
$$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$
$$= a^2 + b^2 + c^2 + (bc + ca + ab)(\omega + \omega^2)$$
$$= a^2 + b^2 + c^2 - bc - ca - ab$$
$$= \frac{1}{2}\,[(b - c)^2 + (c - a)^2 + (a - b)^2]$$

As a, b, c are integers not all equal, at least two of $b - c$, $c - a$ and $a - b$ are of different from zero. Therefore minimum value of $(b - c)^2 + (c - a)^2 + (a - b^2)$ is 2.

Thus, $\left|a + b\omega + c\omega^2\right|^2 \ge 1 \quad \Rightarrow \quad |a + b\omega + c\omega^2| \ge 1$.

Example 94: The region of the argand plane defined by $|z - i| + |z + i| \le 4$ is

(a) interior of an ellipse
(b) exterior of a circle
(c) interior and boundary of an ellipse
(d) none of these

Ans. (c)

◎ **Solution:** Let A and B be the point $0 + i$ and $0 - i$ and $P(z)$ be any point satisfying $|z - i| + |z + i| \le 4$.

$\Rightarrow \quad PA + PB \le 4$

Thus, P lies in the interior or on the boundary on the ellipse with foci A and B and length of major axis = 4. See Fig. 2.51

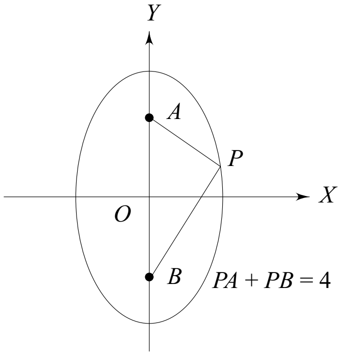


Fig. 2.51

◉ **Example 95:** If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to

(a) $-\pi$ (b) $-\pi/2$
(c) $\pi/2$ (d) 0

Ans. (d)

◎ **Solution:** Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ where $r_1 = |z_1|$, $r_2 = |z_2|$, $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.

We have

$$|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$$
$$= (r_1 + r_2)^2 + 2r_1r_2\{\cos(\theta_1 - \theta_2) - 1\}$$

Now, $|z_1 + z_2| = |z_1| + |z_2|$

$\Leftrightarrow \cos(\theta_1 - \theta_2) = 1 \Leftrightarrow \theta_1 - \theta_2 = 0 \Leftrightarrow \theta_1 = \theta_2$.

◉ **Example 96:** If $|z - 25i| \le 15$, then

|maximum arg (z) – minimum arg (z)| equals

(a) $2\cos^{-1}(3/5)$ (b) $2\cos^{-1}(4/5)$
(c) $\pi/2 + \cos^{-1}(3/5)$ (d) $\sin^{-1}(3/5) - \cos^{-1}(3/5)$

Ans. (b)

◎ **Solution:** If $|z - 25i| \le 15$, then z lies either in the interior and or on the boundary of the circle with centre at C $(25i)$ and radius equal to 15.

From Fig. 2.52 it is clear that least argument is for point A and the greatest argument is for point B.

From right ΔOAC, $\cos\left(\dfrac{\pi}{2} - \theta\right) = \dfrac{OA}{OC} = \dfrac{20}{25}$

$\Rightarrow \quad \pi/2 - \theta = \cos^{-1}(4/5)$

Now, for $|z - 25i| \le 15$

|maximum (arg z) – minimum (arg z)|

$= |\arg(B) - \arg(A)|$
$= \angle BOA = \angle BOX - \angle AOX$
$= \pi/2 + (\pi/2 - \theta) - \theta = \pi - 2\theta$
$= 2\cos^{-1}(4/5)$

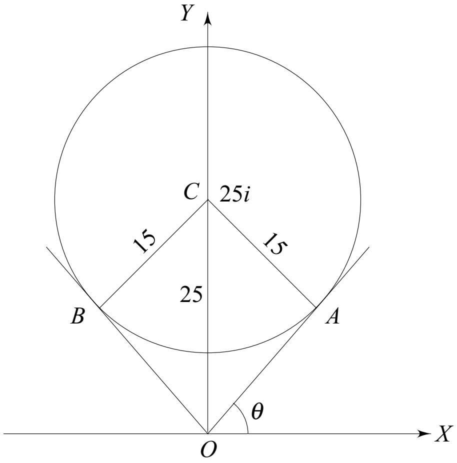


Fig. 2.52

◉ **Example 97:** If $|z| = 3$, the area of the triangle whose sides are z, ωz and $z + \omega z$ (where ω is a complex cube root of unity) is

(a) $9\sqrt{3}/4$ (b) $3\sqrt{3}/2$
(c) $5/2$ (d) $8\sqrt{3}/3$

Ans. (a)

◎ **Solution:** We have $|z| = 3$, $|\omega z| = |\omega|\,|z| = (1)(3) = 3$ and

$$|z + \omega z| = |(1 + \omega)z| = |(-\omega^2)z|$$
$$= |-\omega^2|\,|z| = (1)(3) = 3.$$

$\therefore$ The given triangle is equilateral and its area is

$$\frac{\sqrt{3}}{4}|z|^2 = \frac{9\sqrt{3}}{4}.$$

◉ **Example 98:** The greatest and the least value of $|z_1 + z_2|$ if $z_1 = 24 + 7i$ and $|z_2| = 6$ are respectively

(a) 31, 19 (b) 25, 19
(c) 31, 25 (d) none of these

Ans. (a)

◎ **Solution:** Note that $|z| = 6$ represents a circle. As $|z_2| = 6$, $|z_1 + z_2| = |z_2 - (-24 - 7i)|$ represent distance between a point on the circle $|z| = 6$ and the point $(-24 - 7i)$.

$|z_1 + z_2|$ will be greatest and least at points B and A which are the end points of the diameter of the circle through C. As $OC = 25$, $CA = OC - OA = 25 - 6 = 19$ and $CB = OC + OB = 25 + 6 = 31$. See Fig. 2.53

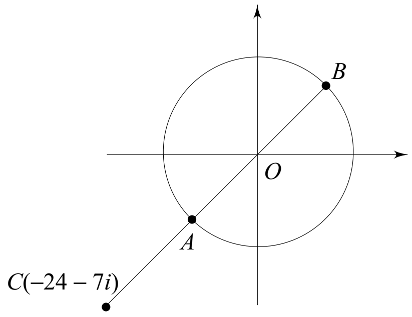


Fig. 2.53

Alternate Solution

$$|z_2| = 6 \Rightarrow z_2 = 6e^{i\theta} \text{ where } \theta \,\varepsilon\, \mathbf{R}.$$

$$\begin{aligned} \therefore \quad |z_1 + z_2|^2 &= |24 + 7i + 6(\cos\theta + i\sin\theta)|^2 \\ &= (24 + 6\cos\theta)^2 + (7 + 6\sin\theta)^2 \\ &= 576 + 36\cos^2\theta + 288\cos\theta + 49 + 36\sin^2\theta + 84\sin\theta \\ &= 625 + 36 + 12(24\cos\theta + 7\sin\theta) \\ &= 661 + 12(25)\sin(\theta + \alpha) \\ &\qquad [\text{put } 7 = r\cos\alpha \text{ and } 24 = r\sin\alpha] \\ &= 661 + 300\sin(\theta + \alpha) \end{aligned}$$

Thus, greatest possible value of $|z_1 + z_2|^2$ is 661 + 300 = 961 and the least possible value of $|z_1 + z_2|^2$ is 361.

$\therefore$ greatest and least possible values of $|z_1 + z_2|$ are 31 and 19 respectively.

Example 99: If α, β are the roots of $x^2 + px + q = 0$, and ω is a cube root of unity, then value of $(\omega\alpha + \omega^2\beta)(\omega^2\alpha + \omega\beta)$ is

(a) p^2 (b) $3q$
(c) $p^2 - 2q$ (d) $p^2 - 3q$

Ans. (d)

Solution: We have $\alpha + \beta = -p$, $\alpha\beta = q$

$$\begin{aligned} \text{Now} \quad & (\omega\alpha + \omega^2\beta)(\omega^2\alpha + \omega\beta) \\ &= \omega^3\alpha^2 + \omega^4\alpha\beta + \omega^2\alpha\beta + \omega^3\beta^2 \\ &= \alpha^2 + \beta^2 + (\omega + \omega^2)\alpha\beta = \alpha^2 + \beta^2 - \alpha\beta \\ &= (\alpha + \beta)^2 - 3\alpha\beta = p^2 - 3q \end{aligned}$$

Example 100: Maximum distance from the origin of the points z satisfying the relation $|z + 1/z| = 1$ is

(a) $(\sqrt{5} + 1)/2$ (b) $(\sqrt{5} - 1)/2$
(c) $3 - \sqrt{5}$ (d) $(3 + \sqrt{5})/2$

Ans. (a)

Solution: We may assume $|z| \geq \dfrac{1}{|z|}$ for otherwise, we may interchange z and $1/z$ in the given equation.

We have

$$|z| - \frac{1}{|z|} = \left||z| - \frac{1}{|z|}\right| = \left||z| - \left|-\frac{1}{z}\right|\right| \leq \left|z - \left(-\frac{1}{z}\right)\right| = \left|z + \frac{1}{z}\right| = 1$$

Thus, $|z| - \dfrac{1}{|z|} \leq 1 \Rightarrow |z|^2 - |z| - 1 \leq 0$

$\Rightarrow$ $|z|$ lies between the roots of $|z|^2 - |z| - 1 = 0$

$$\Rightarrow \quad \frac{1}{2}(1 - \sqrt{5}) \leq |z| \leq \frac{1}{2}(1 + \sqrt{5})$$

As $z \neq 0$, $|z| > 0$, therefore, $0 < |z| \leq \dfrac{1}{2}(\sqrt{5} + 1)$.

Taking $z = \dfrac{i}{2}(\sqrt{5} + 1)$, we get $\left|z + \dfrac{1}{z}\right| = 1$. Thus, maximum possible value of $|z|$ is $(\sqrt{5} + 1)/2$.

Example 101: If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$, then area of the triangle whose vertices are z_1, z_2, z_3 is

(a) $3\sqrt{3}/4$ (b) $\sqrt{3}/4$
(c) 1 (d) 2

Ans. (a)

Solution: $|z_2 - z_3|^2 + |z_2 + z_3|^2 = 2|z_2|^2 + 2|z_3|^2$

$$\Rightarrow \quad |z_2 - z_3|^2 + |-z_1|^2 = 2(1) + 2(1)$$

$$[\because \quad z_1 + z_2 + z_3 = 0]$$

$$\Rightarrow \quad |z_2 - z_3| = \sqrt{3}$$

Similarly, $|z_3 - z_1| = |z_1 - z_2| = \sqrt{3}$

Thus, area of triangle with vertices z_1, z_2, z_3 is

$$\frac{\sqrt{3}}{4}\left(\sqrt{3}\right)^2 = \frac{3\sqrt{3}}{4}$$

Example 102: An equation of straight line joining the complex numbers a and ib (where $a, b \,\varepsilon\, \boldsymbol{R}$ and $a, b \neq 0$) is

(a) $z\left(\dfrac{1}{a} - \dfrac{i}{b}\right) + \bar{z}\left(\dfrac{1}{a} + \dfrac{i}{b}\right) = 2$

(b) $z(a - ib) + \bar{z}(a + ib) = 2(a^2 + b^2)$

(c) $z(a + ib) + \bar{z}(a - ib) = 2ab$

(d) none of these

Ans. (a)

Solution: An equation of straight line joining a and ib is

$$\begin{vmatrix} z & \bar{z} & 1 \\ a & a & 1 \\ ib & -ib & 1 \end{vmatrix} = 0$$

$$\Rightarrow z(a + ib) - \bar{z}(a - ib) - 2iab = 0$$

$$\Rightarrow z\left(\frac{1}{a} - \frac{i}{b}\right) + \bar{z}\left(\frac{1}{a} + \frac{i}{b}\right) = 2$$

Example 103: Two non-parallel lines meet the circle $|z| = r$ in the points a, b and c, d respectively. The point of intersection of these lines is

(a) $\dfrac{a^{-1}+b^{-1}+c^{-1}+d^{-1}}{a^{-1}b^{-1}+c^{-1}d^{-1}}$

(b) $\dfrac{ab+cd}{a+b+c+d}$

(c) $\dfrac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$

(d) none of these

Ans. (c)

◎ **Solution:** An equation of straight line passing through $A(a)$ and $B(b)$ is

$$\begin{vmatrix} z & \bar{z} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \qquad z(\bar{a} - \bar{b}) - \bar{z}(a-b) + a\bar{b} - \bar{a}b = 0 \qquad (1)$$

As a, b lie on $|z| = r$, we get

$$|a| = |b| = r \Rightarrow a\bar{a} = b\bar{b} = r^2.$$

Equation (1), now can be written as

$$z\left(\frac{r^2}{a} - \frac{r^2}{b}\right) - \bar{z}(a-b) + \frac{ar^2}{b} - \frac{br^2}{a} = 0$$

$$\Rightarrow \qquad \frac{zr^2}{ba}(b-a) + \bar{z}\,(b-a) = (b^2-a^2)\,\frac{r^2}{ab}$$

$$\Rightarrow \qquad za^{-1}b^{-1} + \frac{\bar{z}}{r^2} = a^{-1} + b^{-1} \qquad (2)$$

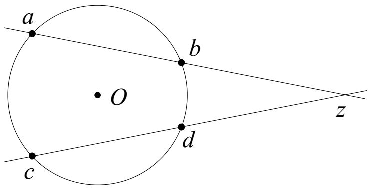


Fig. 2.54

Similarly, equation of straight line joining c and d is

$$zc^{-1}d^{-1} + \frac{\bar{z}}{r^2} = c^{-1} + d^{-1} \qquad (3)$$

Subtracting (3) from (2), we get

$$z(a^{-1}b^{-1} - c^{-1}d^{-1}) = a^{-1} + b^{-1} - c^{-1} - d^{-1}$$

$$\Rightarrow \qquad z = \frac{a^{-1}+b^{-1}-c^{-1}-d^{-1}}{a^{-1}b^{-1}-c^{-1}d^{-1}}$$

EXERCISE

Concept-based Straight Objective Type Questions

1. The number of complex numbers z such that $(1+i)z = i\,|z|$
   (a) 0 (b) 1
   (c) 2 (d) infinite

2. Suppose a, b, $c \in \mathbf{R}$ and $C < 0$. Let $z = a + (b + ic)^{2015} + (b - ic)^{2015}$, then
   (a) Re $(z) = 0$
   (b) Im $(z) = 0$
   (c) Re $(z) > 0$, Im $(z) < 0$
   (d) Re $(z) < 0$, Im $(z) > 0$

3. The number of solutions of $z^2 + |z| = 0$ is
   (a) 1 (b) 2
   (c) 3 (d) infinite

4. The equation $\left|\dfrac{(1+i)z-2}{(1+i)z+4}\right| = k$
   does not represent a circle when k is
   (a) 2 (b) π
   (c) e (d) 1

5. If $|z| \ge 5$, then least value of $\left|z - \dfrac{1}{z}\right|$ is
   (a) 5 (b) 24/5
   (c) 8 (d) 8/3

6. Principal argument of $z = \dfrac{i-1}{i\left(1-\cos\dfrac{2\pi}{7}\right)+\sin\dfrac{2\pi}{7}}$ is
   (a) $\dfrac{\pi}{28}$ (b) $\dfrac{3\pi}{28}$
   (c) $\dfrac{17\pi}{28}$ (d) $\dfrac{19\pi}{28}$

7. If $x + iy = \sqrt{\dfrac{a+ib}{c+id}}$, then $(x^2+y^2)^2\,(c^2+d^2)$ equals
   (a) a^2+b^2 (b) $\sqrt{a^2+b^2}$
   (c) $\dfrac{a^2+b^2}{c^2+d^2}$ (d) $\sqrt{\dfrac{a^2+b^2}{c^2+d^2}}$

8. Suppose z_1, z_2, z_3 are three complex numbers, and

$$\Delta = \frac{1}{4i}\begin{vmatrix} 1 & z_1 & \bar{z}_1 \\ 1 & z_2 & \bar{z}_2 \\ 1 & z_2 & \bar{z}_3 \end{vmatrix},$$

then

(a) $\text{Re}\,(\Delta) = 0$ (b) $\text{Im}\,(\Delta) = 0$

(c) $\text{Re}\,(\Delta) \geq 0$ (d) $\text{Im}\,(\Delta) \leq 0$

9. If $x, y, a, b \in \mathbf{R}$, $a \neq 0$ and
$(a + ib)(x + iy) = (a^2 + b^2)\, i$,
then (x, y) equals

(a) (a, b) (b) $(a, 0)$

(c) $(0, b)$ (d) (b, a)

10. If ω ($\neq 1$) is a cube root of unity, then the value of $\tan\,[(\omega^{2017} + \omega^{2225})\,\pi - \pi/3]$

(a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$

(c) $-\sqrt{3}$ (d) $\sqrt{3}$

11. If z is purely imaginary and $\text{Im}\,(z) < 0$, then $\arg\,(i\bar{z}) + \arg\,(z)$ is equal to

(a) π (b) 0

(c) $\pi/2$ (d) $-\pi/2$

12. The inequality $a + ib > c + id$ is true when

(a) $a > c, b > d > 0$ (b) $a > c, b = d = 0$

(c) $a > c, b = d > 0$ (d) none of these

13. Let $z \in \mathbf{C}$ be such that $\text{Re}(z^2) = 0$, then

(a) $|\text{Re}(z)| + \text{Im}\,(z) = 0$

(b) $|\text{Re}(z)| = |\text{Im}\,(z)|$

(c) $\text{Re}(z) + |\text{Im}\,(z)| = 0$

(d) $\text{Re}(z) = 0$ or $\text{Im}\,(z) = 0$

14. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$ equals

(a) $0, \pi$ (b) $\pi, -\pi$

(c) $\frac{\pi}{2}, \frac{3\pi}{2}$ (d) $0, 2\pi$

15. If $z = x + iy$ and $0 \leq \sin^{-1}\left(\frac{z-4}{2i}\right) \leq \frac{\pi}{2}$ then

(a) $x = 4, 0 \leq y \leq 2$ (b) $0 \leq x \leq 4, 0 \leq y \leq 2$

(c) $x = 0, 0 \leq y \leq 2$ (d) none of these

16. If $a > 0$ and $z|z| + az + 3i = 0$, then z is

(a) 0

(b) purely imaginary

(c) a positive real number

(d) a negative real number

17. If $z \neq 0$ is a complex number such that $\text{Re}(z) = 0$, then

(a) $\text{Re}\,(z^2) = 0$ (b) $\text{Im}\,(z^2) = 0$

(c) $\text{Re}\,(z^2) = \text{Im}\,(z^2)$ (d) $\text{Im}\,(z^2) < 0$

18. If $z_k = \cos\left(\frac{k\pi}{10}\right) + i\sin\left(\frac{k\pi}{10}\right)$, then $z_1 z_2 z_3 z_4$ is equal to

(a) -1 (b) 2

(c) -2 (d) 1

19. If $|z_1| = |z_2| = 1$, $z_1 z_2 \neq -1$ and $z = \frac{z_1 + z_2}{1 + z_1 z_2}$ then

(a) z is a purely real number

(b) z is a purely imaginary number

(c) $|z| = 1$

(d) none of these

20. If $z \in \mathbf{C}$, then $\text{Re}(\bar{z}^2) = k^2$, $k > 0$, represents

(a) an ellipse (b) a parabola

(c) a circle (d) a hyperbola

LEVEL 1

Straight Objective Type Questions

21. If $\omega \neq 1$ is a cube root of unity, then $1, \omega, \omega^2$

(a) are vertices of an equilateral triangle

(b) lie on a straight line

(c) lie on a circle of radius $\sqrt{3/2}$

(d) none of these

22. If α, β, γ are the cube roots of p, $p < 0$, then for any x, y and z which does not make denominator zero, the expression $\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha}$ equals

(a) $\omega, 1$ (b) ω, ω^2

(c) $\omega^2, 1$ (d) $1, \omega, \omega^2$

23. *ABCD* is a rhombus, its diagonal *AC* and *BD* intersect at the point *M* and satisfy $BD = 2AC$. If the points *D* and *M* are the complex numbers $1 + i$ and $2 - i$ respectively, then *A* represent the complex number.

(a) $3-\frac{1}{2}i, 1+\frac{3}{2}i$
(b) $3+\frac{1}{2}i, 1+\frac{3}{2}i$
(c) $3-\frac{1}{2}i, 1-\frac{3}{2}i$
(d) $3+\frac{1}{2}i, 1-\frac{3}{2}i$

24. Let α and β be the roots of the equation $x^2 + x + 1 = 0$. The equation whose roots are α^{19}, β^7 is
(a) $x^2 - x - 1 = 0$
(b) $x^2 - x + 1 = 0$
(c) $x^2 + x - 1 = 0$
(d) $x^2 + x + 1 = 0$

25. If ω is an imaginary cube root of unity, then the value of

$$\sin\left\{(\omega^{10} + \omega^{23})\pi - \frac{\pi}{4}\right\}$$

is
(a) $-\frac{\sqrt{3}}{2}$
(b) $-\frac{1}{\sqrt{2}}$
(c) $\frac{1}{\sqrt{2}}$
(d) $\frac{\sqrt{3}}{2}$

26. If $\omega(\neq 1)$ is a cube root of unity and $(1 + \omega)^{2017} = A + B\omega$. Then A and B are respectively the numbers
(a) 0,1
(b) 1, 1
(c) 1, 0
(d) – 1, 1

27. If $\omega(\neq 1)$ is a cube root of unity, then

$$\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$$

equals
(a) 0
(b) 1
(c) i
(d) ω

28. If ω $(\neq 1)$, is a cube root of unity, then value of $(1 + \omega - \omega^2)^7$ equals
(a) 128ω
(b) -128ω
(c) $128\omega^2$
(d) $-128\omega^2$

29. If $\Delta = \begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then
(a) $x = 3$, $y = 1$
(b) $x = 1$, $y = 3$
(c) $x = 0$, $y = 3$
(d) $x = 0$, $y = 0$

30. If $\arg(z) < 0$, then $\arg(-z) - \arg(z)$ equals
(a) π
(b) $-\pi$
(c) $-\frac{\pi}{2}$
(d) $\frac{\pi}{2}$

31. If z_1, z_2, z_3 are complex numbers such that

$$|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1,$$

then $|z_1 + z_2 + z_3|$ is
(a) equal to 1
(b) less than 1
(c) greater than 3
(d) equal to 3

32. Let z_1 and z_2 be nth roots of unity which subtend a right angle at the origin. Then n must be of the form
(a) $4k + 1$
(b) $4k + 2$
(c) $4k + 3$
(d) $4k$

33. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1}{2}(1 - \sqrt{3}i)$ are vertices of a triangle which is
(a) of area zero
(b) right-angled isosceles
(c) equilateral
(d) obtuse-angle isosceles

34. Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then the value of the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} \text{ is}$$

(a) 3ω
(b) $3\omega(\omega - 1)$
(c) $3\omega^2$
(d) $3\omega(1 - \omega)$

35. The inequality $|z - i| < |z + i|$ represents the region
(a) $\mathrm{Re}(z) > 0$
(b) $\mathrm{Re}(z) < 0$
(c) $\mathrm{Im}(z) > 0$
(d) $\mathrm{Im}(z) < 0$

36. If $iz^3 + z^2 - z + i = 0$, then
(a) $\mathrm{Re}\, z = 0$
(b) $\mathrm{Im}\, z = 0$
(c) $|z| = 1$
(d) none of these.

37. If $x + iy = \frac{1}{1 - \cos\theta + 2i\sin\theta}$, $\theta \neq 2n\pi$, $n \in \boldsymbol{I}$, then maximum value of x is
(a) 1
(b) 2
(c) $\frac{1}{2}$
(d) $\frac{1}{3}$

38. The equation $z^3 = \bar{z}$ has
(a) no solution
(b) two solutions
(c) five solutions
(d) infinite number of solutions

39. If $z = 5 + t + i\sqrt{25 - t^2}$, $(-5 \leq t \leq 5)$, then locus of z is a curve which passes through
(a) $5 + 0i$
(b) $-2 + 3i$
(c) $2 + 4i$
(d) $-2 - 3i$

40. If $\omega \neq 1$ is a cube root of unity and satisfies

$$\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} = 2\omega^2$$

and $\dfrac{1}{a+\omega^2}+\dfrac{1}{b+\omega^2}+\dfrac{1}{c+\omega^2} = 2\omega$,

then the value of $\dfrac{1}{a+1}+\dfrac{1}{b+1} + \dfrac{1}{c+1}$ is

(a) 2 (b) -2

(c) $-1 + \omega^2$ (d) none of these

41. If $|z - i\,\text{Re}(z)| = |z - \text{Im}(z)|$, then z lies on

(a) $\text{Re}(z) = 2$ (b) $\text{Im}(z) = 2$,

(c) $\text{Re}(z) + \text{Im}(z) = 2$ (d) none of these

42. If ω is a complex cube root of unity, then value of expression

$$\cos\left[\{(1-\omega)(1-\omega^2) + \ldots + (12-\omega)(12-\omega^2)\}\frac{\pi}{370}\right]$$

(a) -1 (b) 0

(c) 1 (d) $\sqrt{3}/2$

43. If roots of the equation $z^2 + az + b = 0$ are purely imaginary then

(a) $(b-\bar{b})^2 + (a+\bar{a})(\bar{a}b + a\bar{b}) = 0$

(b) $(b-\bar{b})^2 + (a-\bar{a})^2 = 0$

(c) $(b+\bar{b})^2 - (a-\bar{a})^2 = 0$

(d) none of these

44. The system of equations $|z + 1 - i| = \sqrt{2}$ and $|z| = 3$ has

(a) no solution

(b) one solution

(c) two solutions

(d) infinite number of solutions

45. If $8i\,z^3 + 12z^2 - 18z + 27i = 0$, then

(a) $|z| = \dfrac{3}{2}$ (b) $|z| = \dfrac{2}{3}$

(c) $|z| = 1$ (d) $|z| = \dfrac{3}{4}$

46. If the complex number z lies on the boundary of the circle of radius 5 and centre at 4, then the greatest value of $|z + 1|$

(a) 4 (b) 5

(c) 10 (d) 9

47. If $x + iy = \dfrac{3}{\cos\theta + i\sin\theta + 2}$, then $4x - x^2 - y^2$ reduces to

(a) 2 (b) 3

(c) 4 (d) 5

48. Suppose z_1, z_2, z_3 represent the vertices A, B, and C respectively of a ΔABC with centroid at G. If the mid point of AG is the origin, then

(a) $z_1 + z_2 + z_3 = 0$ (b) $2z_1 + z_2 + z_3 = 0$

(c) $z_1 + z_2 + 4z_3 = 0$ (d) $4z_1 + z_2 + z_3 = 0$

49. Suppose that three points z_1, z_2, z_3 are connected by the relation $az_1 + bz_2 + cz_3 = 0$, where $a + b + c = 0$, then the points are

(a) vertices of a right triangle

(b) vertices of an isosceles triangle

(c) vertices of an equilateral triangle

(d) collinear

50. If the complex number $\dfrac{z-1}{z+1}$ is purely imaginary, then

(a) $|z| = 1$ (b) $|z| < 1$

(c) $|z| > 1$ (d) $|z| \geq 2$

51. If z is a complex number such that $-\pi/2 \leq \arg z \leq \pi/2$, then which of the following inequality is true.

(a) $|z - \bar{z}| \leq |z|\,|\arg(z) - \arg(\bar{z})|$

(b) $|z - \bar{z}| \leq |\arg(z) - \arg(\bar{z})|$

(c) $|z - \bar{z}| > |z|\,|\arg(z) - \arg(\bar{z})|$

(d) none of these

52. If $|\omega| = 1$, then the set of points

$z = \omega + \dfrac{1}{\omega}$ is satisfies

(a) $|\text{Re}(z)| \leq 2$ (b) $|z| \leq 1$

(c) $|z| = 1$ (d) $|\text{Im}(z)| \geq 2$

53. Number of complex numbers satisfying $|z| = 1$ and $\left|\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}\right| = 1$, is

(a) 0 (b) 2

(c) 4 (d) 8

54. If z_1, z_2 are two complex numbers such that $|z_1| = |z_2| = \sqrt{2}$ and $|z_1 + z_2| = \sqrt{3}$, then $|z_1 - z_2|$ equals:

(a) $2\sqrt{2}$ (b) $\sqrt{5}$

(c) 3 (d) $2-\sqrt{2}$

55. Let z_1, z_2, z_3 be three non-zero complex numbers such that $z_1\bar{z}_2 = z_2\bar{z}_3 = z_3\bar{z}_1$, then z_1, z_2, z_3

(a) are vertices of an equilateral triangle

(b) are vertices of an isosceles triangle

(c) lie on a straight line

(d) none of these

56. If $|z_1| = |z_2| = |z_3| = 1$, then value of $|z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$ cannot exceed

(a) 3 (b) 6

(c) 9 (d) 12

57. Let z_1, z_2, z_3 be three complex number such that $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$, then $\left|z_1^2 + 2z_2^2 + z_3^2\right|$ equals

(a) 1 (b) 2
(c) 3 (d) 4

58. Let z_1, z_2, z_3 be three complex numbers such that $|z_1| = |z_2| = |z_3| = 1$ and

$$z = (z_1 + z_2 + z_3)\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right),$$ then $|z|$ cannot exceed

(a) 1 (b) 3
(c) 6 (d) 9

59. Suppose z is a complex number such that $z \neq -1$, $|z| = 1$, and $\arg(z) = \theta$. Let $\omega = \frac{z(1-\bar{z})}{\bar{z}(1+z)}$, then $\text{Re}(\omega)$ is equal to

(a) $1 + \cos(\theta/2)$ (b) $1 - \sin(\theta/2)$
(c) $-2\sin^2(\theta/2)$ (d) $2\cos^2(\theta/2)$

60. Let $a = \text{Im}\left(\frac{1+z^2}{2iz}\right)$, where z is any non-zero complex number. Then the set

$$A = \{a : |z| = 1 \quad \text{and} \quad z \neq \pm 1\}$$

is equal to

(a) $(-1, 1)$ (b) $[-1, 1]$
(c) $[0, 1)$ (d) $(-1, 0]$

61. Number of complex numbers such that

$|z| = 1$ and $z = 1 - 2\bar{z}$ is

(a) 0 (b) 1
(c) 2 (d) infinite

62. Let z_1, z_2 be two complex numbers such that $z_1 \neq 0$ and z_2/z_1 is purely real, then $\left|\frac{2iz_1 + 5z_2}{2iz_1 - 5z_2}\right|$ is equal to

(a) 3 (b) 2
(c) 1 (d) 0

63. If $z = i(1+\sqrt{3})$, then $z^4 + 2z^3 + 4z^2 + 5$ is equal to

(a) 5 (b) -5
(c) $2\sqrt{3}\,i$ (d) $-2\sqrt{3}\,i$

64. Suppose $a < 0$ and z_1, z_2, z_3, z_4 be the fourth roots of a. Then $z_1^2 + z_2^2 + z_3^2 + z_4^2$ is equal

(a) $-a^2$ (b) $|a| - a$
(c) $a + |a|$ (d) a^2

65. Suppose $\arg(z) = -5\pi/13$, then $\arg\left(\frac{z+\bar{z}}{1+z\bar{z}}\right)$ is

(a) $-5\pi/13$ (b) $5\pi/13$
(c) π (d) 0

66. The number of values of $\theta \in (0, \pi]$, such that $(\cos\theta + i\sin\theta)(\cos 3\theta + i\sin 3\theta)(\cos 5\theta + i\sin 5\theta)(\cos 7\theta + i\sin 7\theta)(\cos 9\theta + i\sin 9\theta) = -1$ is

(a) 11 (b) 13
(c) 14 (d) 16

67. If $z \in \mathbf{C} - \{0, -2\}$ is such that
$\log_{(1/7)}|z - 2| > \log_{(1/7)}|z|$
then

(a) $\text{Re}(z) > 1$ (b) $\text{Re}(z) < 1$
(c) $\text{Im}(z) > 1$ (d) $\text{Im}(z) < 1$

68. $\text{Im}\left(\frac{2z+1}{iz+1}\right) = 5$ represents

(a) a circle (b) a straight line
(c) a parabola (d) an ellipse

69. Let z_1, z_2 be two complex numbers such that $\text{Im}(z_1 + z_2) = 0$ and $\text{Im}(z_1 z_2) = 0$ then

(a) $z_1 = -\bar{z}_2$ (b) $z_1 = z_2$
(c) $z_1 = \bar{z}_2$ (d) none of these

70. If $n \in \mathbf{N}$, then $\frac{(1+i)^n}{(1-i)^{n-2}}$ is equal to

(a) i^{n+1} (b) $-2i^{n+1}$
(c) i^{n+2} (d) $-2i^{n+2}$

71. Let $\omega \neq 1$, be a cube root of unity, and $f : \mathbf{I} \to \mathbf{C}$ be defined by
$f(n) = 1 + \omega^n + \omega^{2n}$, then range of f is

(a) $\{0\}$ (b) $\{0, 3\}$
(c) $\{0, 1, 3\}$ (d) $\{0, 1\}$

72. If $z + \frac{1}{z} = 2\cos\theta$, $z \in \mathbf{C}$ then $z^{2n} - 2z^n \cos(n\theta)$ is equal to

(a) 1 (b) 0
(c) -1 (d) $-n$

73. If $\omega \neq 1$ is a cube root of unity, then

$$z = \sum_{k=1}^{60} \omega^k - \prod_{k=1}^{30} \omega^k \text{ is equal to}$$

(a) 0 (b) ω
(c) ω^2 (d) -1

74. Let $g(x)$ and $h(x)$ be two polynomials with real coefficients. If $P(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, then

(a) $g(1) = 0, h(1) = 1$
(b) $g(1) = 1, h(1) = 0$
(c) $g(1) = 0, h(1) = 0$
(d) $g(1) = 1, h(1) = 0$

75. If $x^2 - x + 1$ divides the polynomial $x^{n+1} - x^n + 1$, then n must be of the form

(a) $3k + 1$ (b) $6k + 1$
(c) $6k - 1$ (d) $3k - 1$

Assertion-Reason Type Questions

76. **Statement-1:** $z^2 + z\,|z - 1| + |z - 1|^2 = 0,\ z \in \mathbf{C}$ has no solution in C.

    Statement-2: $z^2 + az + a^2 = 0,\ z \in \mathbf{C},\ a > 0$ has no solution in C.

77. Let $z \in C$ satisfy the relations

    $|z| = 1$ and $z = 2i + \bar{z}$

    Statement-1: z is purely imaginary.

    Statement-2: $|\arg(z)| \le \pi/6$.

78. Let $\alpha = \cos\left(\frac{8\pi}{11}\right) + i\sin\left(\frac{3\pi}{11}\right)$

    Statement-1: $\text{Re}\,(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5) = -\frac{1}{2}$

    Statement-2: $\sum_{k=0}^{10} \alpha^k = 0$

79. Let $z = \cos\theta + i\sin\theta.\ \theta \in (0, \pi),\ \alpha = \pi/60$

    and $f(\theta) = \sum_{k=1}^{15} \sin[(2k-1)\theta]$.

    Statement-1: $f(\alpha) = \dfrac{1}{2\sin\alpha}$

    Statement-2: $z + z^3 + \ldots + z^{29} = \dfrac{z(1-z^{30})}{1-z^2}$

80. Let $\omega \ne 1$ be a cube root of unity and S be the set of all non-singular matrices of the form

    $$\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$$

    where each of a, b and c is either ω or ω^2.

    Statement-1: S contains exactly two distinct matrices.

    Statement-2: A square matrix A with complex entries is non-singular if and only if $|A| \ne 0$

81. **Statement-1:** If $|z_1|\,|z_2| = 1$, then

    $$|z_1 - z_2| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$$

    Statement-2: $|z| = |\bar{z}|\ \forall\ z \in C$.

82. Let $f(z)$ be a polynomial in z with complex coefficients. Suppose, when $f(z)$ is divided by $z - i$, the remainder is i and when $f(z)$ is divided by $z + i$, the remainder is $1 + i$.

    Statement-1: The remainder when $f(z)$ is divided by $z^2 + 1$ is $\frac{1}{2}(iz+1)+1$

    Statement-2: If $r(z)$ is remainder when $f(z)$ is divided by $p(z)$, then
    $r(z) = 0$ or $\deg(r(z)) < \deg(p(z))$.

83. Let $a, b \in \mathbf{C}$, $a \ne 0$. Suppose z_1, z_2, z_3 are the roots of the equation $(z + ab)^3 = a^3$.

    Statement-1: Length of a side of the triangle with vertices z_1, z_2 and z_3 is $\sqrt{3}\,|a|$

    Statement-2: Roots of $z^3 = a^3$ lie on a circle of radius $|a|$.

84. **Statement-1:** The equation $z^2 - z + \pi = 0$.

    has no solution in the unit disc $|z| < 1$.

    Statement-2: $|z_1 + z_2| \le |z_1| + |z_2|\ \forall\ z_1, z_2 \in \mathbf{C}$.

85. Let $z_1 = r_1e^{i\theta}$, $z_2 = r_2e^{i\varphi}$ where $\theta, \varphi \in (-\pi, \pi]$.

    Statement-1: $|z_1 - z_2|^2 \le (r_1 - r_2)^2 + (\theta - \phi)^2$

    Statement-2: $|\sin\theta| \le |\theta|\ \forall\ \theta \in \mathbf{R}$.

LEVEL 2

Straight Objective Type Questions

86. If the complex numbers z_1, z_2, z_3, are the vertices of a parallelogram $ABCD$, then the fourth vertex is

    (a) $\frac{1}{2}(z_1 + z_2)$ (b) $\frac{1}{4}(z_1 + z_2 - z_3 - z_4)$

    (c) $\frac{1}{3}(z_1 + z_2 + z_3)$ (d) $z_1 + z_3 - z_2$

87. If a, b and c are three integers such that at least two of them are unequal and ω ($\ne 1$) is a cube root of unity, then the least value of the expression $|a + b\omega + c\omega^2|$ is

    (a) 0 (b) 1

    (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

88. The shaded region in Fig. 2.55 is given by

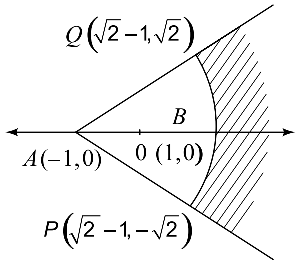


Fig. 2.55

(a) $\left\{z : |z-1| < 2, |\arg(z+1)| < \dfrac{\pi}{2}\right\}$

(b) $\left\{z : |z+1| < 2, |\arg(z+1)| < \dfrac{\pi}{2}\right\}$

(c) $\left\{z : |z-1| > 2, |\arg(z-1)| < \dfrac{\pi}{4}\right\}$

(d) $\left\{z : |z+1| > 2, |\arg(z+1)| < \dfrac{\pi}{4}\right\}$

89. Let $w = \alpha + i\beta$, $\beta \neq 0$ and $z \neq 1$. If $\dfrac{w - \bar{w}z}{1-z}$ is purely real, then the set of value of z is

(a) $\{z : |z| = 1\}$
(b) $\{z : \bar{z} = z\}$
(c) $\{z : |z| \neq 1\}$
(d) $\{z : |z| = 1, z \neq 1\}$

90. If z and w are non-zero complex numbers such that $zw = |z|^2$, then $|z - \bar{z}| + |w + \bar{w}| = 4$ represents a

(a) rectangle
(b) square
(c) rhombus
(d) trapezium

91. Let z be a complex number such that $z = 1 - t + i\sqrt{t^2 + t + 2}$, where $t \in \boldsymbol{R}$, then locus of z on the Argand plane is

(a) a parabola
(b) an ellipse
(c) a hyperbola
(d) a pair of straight line.

92. Let $z_1, z_2, \in \boldsymbol{C}$ and $a, b > 0$ be such that $a|z_1| = b|z_2|$, then $w = \dfrac{az_1 - bz_2}{az_1 + bz_2}$ lies

(a) in the 1st quadrant
(b) in the 3rd quadrant
(c) on the real axis
(d) on the imaginary axis

93. If $z \in \boldsymbol{C}$, then least value of the expression $|z| + |1 - z| + |z - 2|$ is

(a) 1
(b) 3/2
(c) 2
(d) cannot be determined

94. If $k > 0$, $k \neq 1$, and $z_1, z_2 \in \boldsymbol{C}$, then $\left|\dfrac{z - z_1}{z - z_2}\right| = k$ represents

(a) a circle
(b) an ellipse
(c) a parabola
(d) a hyperbola

95. If $z_1, z_2, z_3 \in \boldsymbol{C}$ are distinct and are such that $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$, then z_1, z_2, z_3

(a) are vertices of a right triangle
(b) an equilateral triangle
(c) an obtuse angled triangle
(d) none of these

96. If $w = \cos(\pi/n) + i\sin(\pi/n)$, then value of $1 + w + w^2 + \cdots + w^{n-1}$ is

(a) $1 + i$
(b) $1 + i\tan(\pi/2n)$
(c) $1 + i\cot(\pi/2n)$
(d) none of these

97. Let z_1, z_2 be two non-zero complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, then $\dfrac{z_1}{\bar{z}_1} + \dfrac{z_2}{\bar{z}_2}$ equals

(a) 0
(b) 1
(c) − 1
(d) none of these

98. If $|z_1| = 2$ and $(1 - i)z_2 + (1 + i)\bar{z}_2 = 8\sqrt{2}$, then

(a) minimum value of $|z_1 - z_2|$ is 1
(b) minimum value of $|z_1 - z_2|$ is 2
(c) maximum value of $|z_1 - z_2|$ is 8
(d) maximum value of $|z_1 - z_2|$ is 4

99. If z_1 lies on $|z| = r$, then equation of tangent at z_1 is

(a) $\dfrac{z}{z_1} + \dfrac{\bar{z}}{\bar{z}_1} = 2$
(b) $\dfrac{z}{\bar{z}_1} + \dfrac{\bar{z}}{z_1} = r$
(c) $\dfrac{z}{\bar{z}_1} + \dfrac{\bar{z}}{z_1} = 2$
(d) $\dfrac{z}{z_1} + \dfrac{\bar{z}}{\bar{z}_1} = r$

100. If $z \in \boldsymbol{C}$, then minimum value of $|z - 2 + 3i| + |z - 1 + i|$ is

(a) $\sqrt{5}$
(b) $2\sqrt{5}$
(c) $\sqrt{13} - \sqrt{2}$
(d) 0

101. If $a > 0$ and $|z - a^2| + |z - 2a| = 3$ represents an ellipse then a lies in

(a) (0, 3)
(b) (0, ∞)
(c) (1, 3)
(d) (3, ∞)

102. If the points $A(z)$, $B(-z)$ and $C(1 - z)$ are the vertices of an equilateral triangle, then value of z is

(a) $1 \pm \dfrac{i\sqrt{3}}{2}$
(b) $\dfrac{1}{2}(1 \pm i)$
(c) $\dfrac{1}{4}(1 \pm \sqrt{3}i)$
(d) $\dfrac{1}{3}(1 \pm \sqrt{3}i)$

103. If $|z + 1| + |z - 3| \leq 10$, then the range of values of $|z - 7|$ is

(a) [0, 10]
(b) [3, 13]
(c) [2, 12]
(d) [7, 9]

104. If $\omega \neq 1$ is a cube root of unity and $|z-1|^2 + 2|z-\omega|^2 = 3|z-\omega^2|^2$ then z lies on
   (a) a straight line
   (b) a parabola
   (c) an ellipse
   (d) a rectangular hyperbola

105. If $x > 0$, the least value of $n \in N$ such that $\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\sin^{-1}\left(\frac{1+x^2}{2x}\right)$ is
   (a) 2 (b) 4
   (c) 8 (d) 32

Previous Years' AIEEE/JEE Main Questions

1. z and w are two non-zero complex numbers such that $|z| = |w|$ and Arg z + Arg $w = \pi$, then z equals
   (a) $\overline{w}$ (b) $-\overline{w}$
   (c) w (d) $-w$ **[2002]**

2. If $|z-4| < |z-2|$, then
   (a) $\text{Re}(z) > 0$ (b) $\text{Re}(z) < 0$
   (c) $\text{Re}(z) > 3$ (d) $\text{Re}(z) > 2$ **[2002]**

3. The locus of the centre of a circle which touches the circles $|z - z_1| = a$ and $|z - z_2| = b$, $a \neq b$, externally is
   (a) an ellipse (b) a hyperbola
   (c) a circle (d) a pair of straight lines **[2002]**

4. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then
   (a) $x = 2n$, where n is any positive integer
   (b) $x = 4n + 1$, where n is any positive integer
   (c) $x = 2n + 1$, where n is any positive integer
   (d) $x = 4n$, where n is any positive integer **[2003]**

5. If z and w are two non-zero complex numbers such that $|zw| = 1$, and $\text{Arg}(z) - \text{Arg}(w) = \frac{\pi}{2}$, then $\overline{z}\,w$ is equal to
   (a) -1 (b) i
   (c) $-$i (d) 1 **[2003]**

6. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. If the origin, z_1 and z_2 form an equilateral triangle, then
   (a) $a^2 = 2b$ (b) $a^2 = 3b$
   (c) $a^2 = 4b$ (d) $a^2 = b$ **[2003]**

7. Let z, w be two complex numbers such that $\overline{z} + i\overline{w} = 0$ and $\arg(zw) = \pi$, then $\arg(z)$ equals
   (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$
   (c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{4}$ **[2004]**

8. If $z = x - iy$ and $z^{1/3} = p + iq$, then
$$\frac{1}{p^2+q^2}\left(\frac{x}{p}+\frac{y}{q}\right)$$
   is equal to
   (a) 2 (b) -1
   (c) 1 (d) -2 **[2004]**

9. If $|z^2 - 1| = |z|^2 + 1$, then z lies on
   (a) a circle (b) the imaginary axis
   (c) the real axis (d) an ellipse **[2004]**

10. If the cube roots of unity 1, ω, ω^2, then the roots of the equation $(x-1)^3 + 8 = 0$, are
   (a) $-1, 1-2\omega, 1-2\omega^2$
   (b) $-1, 1+2\omega, 1+2\omega^2$
   (c) $-1, -1+2\omega, -1-2\omega^2$
   (d) $-1, -1, -1$ **[2005]**

11. If $w = \frac{z}{z - \frac{1}{3}i}$ and $|w| = 1$, then z lies on
   (a) straight line (b) a parabola
   (c) an ellipse (d) a circle **[2005]**

12. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then arg z_1 − arg z_2 is equal to
   (a) 0 (b) $-\frac{\pi}{2}$
   (c) $\frac{\pi}{2}$ (d) π **[2005]**

13. The value of $\sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$ is
   (a) $-i$ (b) i
   (c) 1 (d) -1 **[2006]**

14. If $z^2 + z + 1 = 0$, where z is a complex number, then value of

$$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+\cdots+\left(z^6+\frac{1}{z^6}\right)^2$$ is

(a) 12 (b) 18
(c) 54 (d) 6 **[2006]**

15. If $|z + 4| \leq 3$, then the maximum value of $|z + 1|$ is

(a) 4 (b) 10
(c) 6 (d) 0 **[2007]**

16. The conjugate of a complex number is $\dfrac{1}{i-1}$. Then that number is

(a) $\dfrac{-1}{i-1}$ (b) $\dfrac{1}{i+1}$
(c) $\dfrac{-1}{i+1}$ (d) $\dfrac{1}{i-1}$ **[2008]**

17. If $\left|z-\dfrac{4}{z}\right| = 2$, then the maximum value of $|z|$ is equal to

(a) 1 (b) $2+\sqrt{2}$
(c) $\sqrt{3}+1$ (d) $\sqrt{5}+1$ **[2009]**

18. The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ equals

(a) 2 (b) ∞
(c) 0 (d) 1 **[2010]**

19. If $\omega(\neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$, there (A, B)

(a) (−1, 1) (b) (0,1)
(c) (1, 1) (d) (1, 0) **[2011]**

20. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re $(z) = 1$, then it is necessary that

(a) $\beta \in (1, \infty)$ (b) $\beta \in (0, 1)$
(c) $\beta\,(-1, 0)$ (d) $|\beta| = 1$ **[2011]**

21. If $z \neq 1$ and $\dfrac{z^2}{z-1}$ is real, then the point represented by the complex number z lies:

(a) on a circle with centre at the origin.
(b) either on the real axis or on a circle not passing through the origin.
(c) on the imaginary-axis.
(d) either on the real axis or on a circle passing through the origin. **[2012]**

22. If z is a complex number of unit modulus and argument θ, then $\arg\left(\dfrac{1+z}{1+\overline{z}}\right)$ equals

(a) $\dfrac{\pi}{2}-\theta$ (b) θ
(c) $\pi-\theta$ (d) $-\theta$ **[2013]**

23. If $z_1 \neq 0$ and z_2 be two complex numbers such that z_2/z_1 is purely imaginary number, then $\left|\dfrac{2z_1+3z_2}{2z_1-3z_2}\right|$ is equal to:

(a) 2 (b) 5
(c) 3 (d) 1 **[2013, online]**

24. Let $a = \text{Im}\left(\dfrac{1+z^2}{2iz}\right)$, where z is any non-zero complex number. The set $A = \{a : |z| = 1 \text{ and } z \neq \pm 1\}$ is equal to:

(a) (−1, 1) (b) [−1, 1]
(c) [0, 1) (d) (−1, 0] **[2013, online]**

25†. Let $z \in \mathbf{C}$ satisfy $|z| = 1$ and $z = 1 - \overline{z}$.

Statement-1: z is a real number

Statement-2: Principal argument of z is $\pm\,\pi/3$. **[2013, online]**

26. If a complex number z satisfies $z + \sqrt{2}\,|z + 1| + i = 0$, then $|z|$ is equal to:

(a) 2 (b) $\sqrt{3}$
(c) $\sqrt{5}$ (d) 1 **[2013, online]**

27. If z is a complex number such that $|z| \geq 2$, then minimum value of $\left|z+\dfrac{1}{2}\right|$:

(a) is strictly greater than 5/2
(b) is strictly greater that 3/2 but less than 5/2
(c) is equal to 5/2
(d) lies in the interval (1, 2) **[2014]**

28. Let w (Im $w \neq 0$) be a complex number. Then the set of all complex numbers z satisfying the equation $w - \overline{w}z = k\,(1 - z)$, for some real number k, is:

(a) $\{z : |z| = 1\}$ (b) $\{z : z = \overline{z}\}$
(c) $\{z : z \neq 1\}$ (d) $\{z : |z| = 1, z \neq 1\}$ **[2014, online]**

29†. If z_1, z_2 and z_3, z_4 are 2 pairs of complex conjugate numbers and $z_1, z_3 \notin \mathbf{R}$, then

$\arg\left(\dfrac{z_1}{z_4}\right)+\arg\left(\dfrac{z_2}{z_3}\right)$ equals:

---

† Slightly modified version.

(a) 0 (b) $\frac{\pi}{2}$

(c) $\frac{3\pi}{2}$ (d) π **[2014, online]**

30. Let $z \neq -i$ be any complex number such that $\frac{z-i}{z+i}$ is a purely imaginary number. Then $z+\frac{1}{z}$ is:
   (a) 0
   (b) any non-zero real number other than 1
   (c) any non-zero real number
   (d) a purely imaginary number **[2014, online]**

31. For all complex numbers z of the form $1 + i\alpha$, $\alpha \in \mathbf{R}$ if $z^2 = x + iy$, then
   (a) $y^2 - 4x + 2 = 0$ (b) $y^2 + 4x - 4 = 0$
   (c) $y^2 - 4x + 4 = 0$ (d) $y^2 + 4x + 2 = 0$
   [2014, online]

32. A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1 z_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a:
   (a) straight line parallel to the x-axis.
   (b) straight line parallel to the y-axis.
   (c) circle of radius 2.
   (d) circle of radius $\sqrt{2}$. **[2015]**

33. The largest value of r for which the region represented by the set $\{w \in \mathbf{C}: |w - 4 - i| \leq r\}$ is contained in the region represented by the set $\{z \in \mathbf{C}: |z - 1| \leq |z + i|\}$, is equal to:
   (a) $\sqrt{17}$ (b) $\sqrt{2}$
   (c) $\frac{3}{2}\sqrt{2}$ (d) $\frac{5}{2}\sqrt{2}$ **[2015, online]**

34. If z is a non-real complex number, then the minimum value of $\frac{\operatorname{Im} z^5}{(\operatorname{Im} z)^5}$ is:
   (a) – 1 (b) – 2
   (c) – 4 (d) – 5 **[2015, online]**

35. A value of θ for which $z = \frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is
   (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
   (c) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (d) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ **[2016]**

36. The point represented by $2 + i$ in the Argand plane moves 1 unit eastwards, then 2 units northwards and finally from there $2\sqrt{2}$ units in the south-westwards direction. Then its new position in the Argand plane is at the point represented by:
   (a) $1 + i$ (b) $2 + 2i$
   (c) $-2- 2i$ (d) $-1 - i$ **[2016, online]**

37. Let $z = 1 + ai$ be a complex number, $a > 0$, such that z^3 is a real number. Then the sum $1 + z + z^2 + \ldots + z^{11}$ is equal to:
   (a) $1365\sqrt{3}\,i$ (b) $-1365\sqrt{3}\,i$
   (c) $-1250\sqrt{3}\,i$ (d) $1250\sqrt{3}\,i$
   [2016, online]

Previous Years' B-Architecture Entrance Examination Questions

1. $\frac{5+i\sin\theta}{5-3i\sin\theta}$ is a real number when
   (a) $\theta = \pi/4$ (b) $\theta = -\pi$
   (c) $\theta = -\pi/2$ (d) $\theta = \pi/2$ **[2006]**

2. Two points P and Q in the Argand diagram represent complex numbers z and $3z + 2 + i$. If P moves arround the circle with centre at the origin and radius 2, then Q moves on the circle, whose centre and radius are
   (a) $-2 - i$, 6 (b) $2 - i$, 3
   (c) $2 + i$, 6 (d) $2 + i$, 3 **[2007]**

3. Let z be a complex number such that $|z| = 2$, then maximum possible value of $\left|z+\frac{2}{z}\right|$ is
   (a) 1 (b) 2
   (c) 3 (d) 4 **[2008]**

4. If $i = \sqrt{-1}$, then $4+3\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{127}+5\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{124}$ is equal to
   (a) $4\sqrt{3}\,i$ (b) $2\sqrt{3}\,i$
   (c) $1-\sqrt{3}\,i$ (d) $1+\sqrt{3}\,i$ **[2009]**

5. The real part of a complex number z having minimum principal argument and satisfying $|z - 5i| \leq 1$ is
   (a) $\frac{2}{5}\sqrt{6}$ (b) 0
   (c) $\frac{2}{\sqrt{5}}$ (d) $-\frac{1}{5}\sqrt{6}$ **[2010]**

6. Area of a triangle with vertices given by z, iz, $z + iz$, where z is a complex number, is

(a) 0 (b) $\frac{1}{2}|z|^2$

(c) $|z|^2$ (d) $2|z|^2$ **[2011]**

7. Two circles in the complex plane are

$C_1 : |z - i| = 2$

$C_2 : |z - 1 - 2i| = 4$

then

(a) C_1 and C_2 touch each other

(b) C_1 and C_2 intersect at two distinct points

(c) C_1 lies within C_2

(d) C_2 lies within C_1 **[2012]**

8. If $z = i(i + \sqrt{2})$, then value of $z^4 + 4z^3 + 6z^2 + 4z$ is

(a) –5 (b) 3

(c) 6 (d) – 9 **[2013]**

9. Suppose z is a complex number such that $z \neq -1$, $|z| = 1$ and $\arg(z) = \theta$. Let $w = \frac{z(1-\bar{z})}{\bar{z}(1+z)}$, then Re($w$) is equal to

(a) $1+\cos\left(\frac{\theta}{2}\right)$ (b) $1-\sin\left(\frac{\theta}{2}\right)$

(c) $-2\sin^2\left(\frac{\theta}{2}\right)$ (d) $2\cos^2\left(\frac{\theta}{2}\right)$ **[2014]**

10. If $|z_1| = |z_2| = |z_3| = 1$

and $z_1 + z_2 + z_3 = \sqrt{2} + i$, then the number

$z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1$ is:

(a) a positive real number

(b) a negative real number

(c) always zero

(d) a purely imaginary number **[2015]**

11. Let $S = \{z \in \mathbf{C}: z(iz_1 - 1) = z_1 + 1, [z_1] < 1\}$. Then, for all $z \in S$, which one of the following is always true?

(a) Re z + Im z < 0 (b) Re z < 0

(c) Re z – Im z > 1 (d) Re z – Im z < 0 **[2016]**

Answers

Concept-based

1. (b)	**2.** (b)	**3.** (c)	**4.** (d)
5. (b)	**6.** (c)	**7.** (a)	**8.** (b)
9. (d)	**10.** (c)	**11.** (c)	**12.** (b)
13. (b)	**14.** (d)	**15.** (a)	**16.** (b)
17. (b)	**18.** (a)	**19.** (a)	**20.** (d)

Level 1

21. (a)	**22.** (b)	**23.** (c)	**24.** (d)
25. (c)	**26.** (b)	**27.** (a)	**28.** (d)
29. (d)	**30.** (a)	**31.** (a)	**32.** (d)
33. (c)	**34.** (b)	**35.** (c)	**36.** (c)
37. (c)	**38.** (c)	**39.** (c)	**40.** (a)
41. (d)	**42.** (c)	**43.** (a)	**44.** (a)
45. (a)	**46.** (c)	**47.** (b)	**48.** (d)
49. (d)	**50.** (a)	**51.** (a)	**52.** (a)
53. (d)	**54.** (b)	**55.** (a)	**56.** (c)
57. (a)	**58.** (d)	**59.** (c)	**60.** (a)
61. (a)	**62.** (c)	**63.** (a)	**64.** (c)
65. (c)	**66.** (b)	**67.** (a)	**68.** (a)
69. (d)	**70.** (b)	**71.** (b)	**72.** (c)
73. (d)	**74.** (c)	**75.** (b)	**76.** (c)
77. (c)	**78.** (a)	**79.** (a)	**80.** (a)
81. (b)	**82.** (a)	**83.** (b)	**84.** (a)
85. (a)			

Level 2

86. (d)	**87.** (b)	**88.** (d)	**89.** (d)
90. (b)	**91.** (c)	**92.** (d)	**93.** (c)
94. (a)	**95.** (b)	**96.** (c)	**97.** (a)
98. (b)	**99.** (a)	**100.** (a)	**101.** (a)
102. (c)	**103.** (b)	**104.** (a)	**105.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (b)	**2.** (c)	**3.** (b)	**4.** (d)
5. (c)	**6.** (b)	**7.** (a)	**8.** (d)
9. (b)	**10.** (a)	**11.** (a)	**12.** (a)
13. (a)	**14.** (a)	**15.** (c)	**16.** (c)
17. (d)	**18.** (d)	**19.** (c)	**20.** (a)
21. (d)	**22.** (b)	**23.** (d)	**24.** (a)
25. (d)	**26.** (c)	**27.** (d)	**28.** (d)
29. (a)	**30.** (no answer)		**31.** (b)
32. (c)	**33.** (d)	**34.** (c)	**35.** (d)
36. (a)	**37.** (a)		

Previous Years' B-Architecture Entrance Examination Questions

1. (b)	**2.** (c)	**3.** (c)	**4.** (a)
5. (a)	**6.** (b)	**7.** (c)	**8.** (b)
9. (c)	**10.** (c)	**11.** (*)	

Hints and Solutions

Concept-based

1. $|(1+i)z| = |\,i\,|z||$

$\Rightarrow \quad \sqrt{2}\,|z| = |z| \quad \Rightarrow \quad (\sqrt{2}-1)\,|z| = 0$

$\Rightarrow \quad |z| = 0 \quad \Rightarrow \quad z = 0$

2. $\bar{z} = \bar{a} + (\overline{b+ic})^{2015} + (\overline{b-ic})^{2015}$

$= a + (b-ic)^{2015} + (b+ic)^{2015} = z$

$\therefore \quad z$ is real and thus, $\operatorname{Im}(z) = 0$

3. $|z^2| = |-|z|| \quad \Rightarrow \quad |z|^2 = |z|$

$\Rightarrow \quad |z| = 0$ or $|z| = 1$

If $|z| = 0$, we get $z = 0$. If $|z| = 1$, we get $z^2 = -1 \Rightarrow z = \pm\, i$

4. We can write the equation as

$$\left|\frac{z-\dfrac{2}{(1+i)}}{z+\dfrac{4}{(1+i)}}\right| = k \qquad (1)$$

But $\dfrac{2}{1+i} = \dfrac{1^2-i^2}{1+i} = 1 - i$

and $\dfrac{4}{1+i} = 2(1-i)$

Therefore, (1) can be written as

$$\left|\frac{z-(1-i)}{z+2(1-i)}\right| = k \qquad (2)$$

This will not represent a circle if $k = 1$. When $k=1$, (2) represents perpendicular bisector of the segment joining $-2(1-i)$ and $1-i$.

5. As $|z| \geq 5$, $\left|\dfrac{1}{z}\right| \leq \dfrac{1}{5}$. Now

$$\left|z-\frac{1}{z}\right| \geq \left||z|-\frac{1}{|z|}\right| = |z|-\frac{1}{|z|} \geq 5-\frac{1}{5} = \frac{24}{5}$$

The least value is attained when $z = 5$.

6. $i\left(1-\cos\dfrac{2\pi}{7}\right)+\sin\dfrac{2\pi}{7}$

$$= 2i\sin^2\frac{\pi}{7}+2\sin\frac{\pi}{7}\cos\frac{\pi}{7}$$

$$= 2\sin\left(\frac{\pi}{7}\right)\left[\cos\frac{\pi}{7}+i\sin\frac{\pi}{7}\right]$$

Also, $i-1 = \sqrt{2}\left(-\dfrac{1}{\sqrt{2}}+\dfrac{i}{\sqrt{2}}\right)$

$$= \sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

$$\therefore \quad z = \frac{\sqrt{2}}{2\sin\left(\dfrac{\pi}{7}\right)}\left[\cos\left(\frac{3\pi}{4}-\frac{\pi}{7}\right)+i\sin\left(\frac{3\pi}{4}-\frac{\pi}{7}\right)\right]$$

$$= \frac{\sqrt{2}}{2\sin\left(\dfrac{\pi}{7}\right)}\left[\cos\left(\frac{17\pi}{28}\right)+i\sin\left(\frac{17\pi}{28}\right)\right]$$

Thus, $\arg(z) = \dfrac{17\pi}{28}$

7. $(x+iy)^2\,(c+id) = a+ib$

$\Rightarrow |(x+iy)^2|\;|c+id| = |a+ib|$

$\Rightarrow |x+iy|^2\;|c+id| = |a+ib|$

$\Rightarrow (x^2+y^2)\,\sqrt{c^2+d^2} = \sqrt{a^2+b^2}$

8. $\bar{\Delta} = \dfrac{1}{-4i}\begin{vmatrix}1 & \bar{z}_1 & z_1\\ 1 & \bar{z}_2 & z_2\\ 1 & \bar{z}_3 & z_3\end{vmatrix} = \Delta$

$\Rightarrow \Delta$ is purely real $\Rightarrow \operatorname{Im}(\Delta) = 0$

9. $(a+ib)\,(x+iy) = (a+ib)\,(a-ib)\,i$

As $a \neq 0$

$x+iy = (a-ib)\,i = b+ia \Rightarrow (x, y) = (b, a)$

10. $\omega^{2017} = (\omega^3)^{672}\;\omega = \omega$

and $\omega^{2225} = (\omega^3)^{741}\;\omega^2 = \omega^2$

$\therefore \tan\,[(\omega^{2017}+\omega^{2225})\pi - \pi/3]$

$= \tan\,[(\omega+\omega^2)\pi - \pi/3]$

$= \tan\,(-\pi-\pi/3) = -\tan(\pi+\pi/3)$

$= -\tan\,(\pi/3) = -\sqrt{3}$

11. Let $z = -i\,t$ where $t > 0$, then

$i\bar{z} = i(i\,t) = -\,t$

$\therefore \arg(i\bar{z}) + \arg(z) = \pi - \pi/2 = \pi/2$

12. The inequality $a+ib < c+id$ is true if and only if $b = d = 0$ and $a > c$.

13. Let $z = x+iy$, then $z^2 = x^2-y^2+2ixy$

$\therefore \quad \operatorname{Re}(z^2) = 0 \quad \Rightarrow \quad x^2-y^2 = 0$

$\Rightarrow \quad |\operatorname{Re}(z)| = |\operatorname{Im}(z)|$

14. $\arg\left(\dfrac{z_1}{z_4}\right)+\arg\left(\dfrac{z_2}{z_3}\right)$

$$= \arg\left(\frac{z_1z_2}{z_4z_3}\right)+2k\pi = \arg\left(\frac{|z_1|^2}{|z_3|^2}\right)+2k\pi = 2k\pi$$

where $k = 0$ or 1

[$\because$ $z_1 = z_3 = i$, gives answer 2π and $z_1 = z_3 = 1$, gives answer 0]

15. $\dfrac{z-4}{2i} = \dfrac{x-4}{2i} + \dfrac{y}{2}$

For $0 \le \sin^{-1}\left(\dfrac{z-4}{2i}\right) \le \dfrac{\pi}{2}$, we must

$x - 4 = 0, \quad 0 \le \dfrac{y}{2} \le 1$

$\Rightarrow \quad x = 4, \quad 0 \le y \le 2.$

16. $z = \dfrac{-3i}{|z| + a} \quad \Rightarrow \quad z$ is purely imaginary.

17. Let $z = bi$, $b \in \mathbf{R}$, $\quad b \neq 0$,

Then $z^2 = -b^2 \in \mathbf{R}$,

Therefore $\text{Im}(z^2) = 0$

18. We have $z_k = \omega^k$ where

$\omega = \cos\left(\dfrac{\pi}{10}\right) + i \sin\left(\dfrac{\pi}{10}\right)$

Thus, $z_1\, z_2\, z_3\, z_4 = \omega . \omega^2 . \omega^3 . \omega^4 = \omega^{10}$

$= \cos\left(\dfrac{10\pi}{10}\right) + i \sin\left(\dfrac{10\pi}{10}\right)$

[by the De Movire's Theorem]

$= \cos\, \pi + i\, \sin\, \pi = -1.$

19. $\bar{z} = \dfrac{\bar{z}_1 + \bar{z}_2}{1 + \bar{z}_1\, \bar{z}_2} = \dfrac{1/z_1 + 1/z_2}{1 + 1/z_1\, z_2} = \dfrac{z_1 + z_2}{1 + z_1\, z_2} = z$

Thus, z is purely real.

20. Let $z = x + iy$, then $\bar{z}^2 = x^2 - y^2 - 2ixy$

$\therefore \quad \text{Re}(\bar{z}^2) = k^2 \Rightarrow \quad x^2 - y^2 = k^2$

which represents a hyperbola.

Level 1

21. $|1 - \omega| = |\omega - \omega^2| = |\omega^2 - 1|$

Alternatively plot the points on an argand diagram.

22. $x^3 = p = \left(p^{\frac{1}{3}}\right)^3 \Rightarrow x = p^{\frac{1}{3}},\ p^{\frac{1}{3}}\,\omega,\ p^{\frac{1}{3}}\,\omega^2.$

Let $\alpha = p^{\frac{1}{3}}$, $\beta = p^{\frac{1}{3}}\,\omega$, $\gamma = p^{\frac{1}{3}}\,\omega^2$.

$$\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha} = \frac{x + y\omega + \omega^2 z}{x\omega + y\omega^2 + z} = \frac{1}{\omega} = \omega^2$$

If $\alpha = p^{\frac{1}{3}}\,\omega$, $\beta = p^{\frac{1}{3}}$, $\gamma = p^{\frac{1}{3}}\,\omega^2$, then

$$\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha} = \frac{1}{\omega^2} = \omega.$$

23. $\dfrac{z - (2-i)}{1 + i - (2-i)} = \dfrac{AM}{MD}\, e^{\pm \pi i/2} \Rightarrow \dfrac{z - (2-i)}{-1 + 2i} = \dfrac{1}{2}(\pm\, i)$

$\Rightarrow z = (2 - i) + \dfrac{i}{2}(-\, 1 + 2i)$

or $z = (2 - i) - \dfrac{i}{2}(-\, 1 + 2i)$

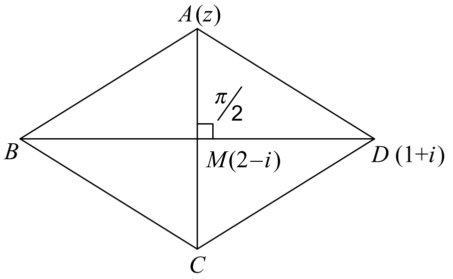


Fig. 2.56

$\Rightarrow z = 1 - \dfrac{3}{2}i$ or $z = 3 - \dfrac{1}{2}i$

24. Let $\quad \alpha = \omega$, $\beta = \omega^2$ so that $\alpha^{19} = \omega$ and $\beta^7 = \omega^2$.

$\therefore$ equation whose roots are α^{19}, β^7 is $x^2 + x + 1 = 0$.

25. Using $\omega^{10} = \omega$, $\omega^{23} = \omega^2$, we get

$$\sin\left\{\left(\omega^{10} + \omega^{23}\right)\pi - \frac{\pi}{4}\right\} = \sin\left(-\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

26. $(1 + \omega)^{2017} = (-\, \omega^2)^{2017} = -\, \omega^{4034} = -\, \omega^2$

$\therefore \quad A + B\omega = 1 + \omega \quad \Rightarrow \quad A = 1,\ B = 1.$

27. Use $\quad R_2 \to R_2 - R_1 - R_3$

28. $(1 + \omega - \omega^2)^7 = (-\, \omega^2 - \omega^2)^7 = -\, 2^7 \omega^{14} = -\, 128\omega^2$

29. Use $\quad C_2 \to C_2 + 3i\ C_3$.

30. Let $\alpha = \arg z < 0$, then $\arg(-z) = \pi + \alpha$,

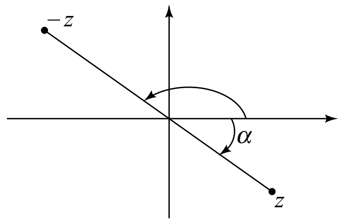


Fig. 2.57

31. $\quad |z_1| = |z_2| = |z_3| = 1$

$\Rightarrow \quad z_1\bar{z}_1 = z_2\bar{z}_2 = z_3\bar{z}_3 = 1$

$\therefore\ |z_1 + z_2 + z_3| = \left|\dfrac{1}{\bar{z}_1} + \dfrac{1}{\bar{z}_2} + \dfrac{1}{\bar{z}_3}\right| = \left|\dfrac{1}{z_1} + \dfrac{1}{z_2} + \dfrac{1}{z_3}\right| = 1$

32. nth roots of unity are given by

$$\cos\left(\frac{2m\pi}{n}\right) + i\, \sin\left(\frac{2m\pi}{n}\right) = e^{2m\pi i/n} \text{ for } m = 0, 1, 2, \cdots, n - 1.$$

Let $z_1 = e^{2m_1\pi i/n}$ and $z_2 = e^{2m_2\pi i/n}$

where $0 \le m_1,\ m_2 < n,\ m_1 \neq m_2$.

As the join of z_1 and z_2 subtend a right angle at the origin z_1/z_2 is purely imaginary we get

$$\frac{e^{2m_1\pi i/n}}{e^{2m_2\pi i/n}} = il \text{ for some real } l \Rightarrow \quad e^{2(m_1-m_2)\pi i/n} = il$$

$$\Rightarrow \cos\left[\frac{2(m_1-m_2)\pi}{n}\right] + i\sin\left[\frac{2(m_1-m_2)\pi}{n}\right] = il$$

$$\Rightarrow \cos\left[\frac{2(m_1-m_2)\pi}{n}\right] = 0 \Rightarrow \frac{2(m_1-m_2)\pi}{n} = \frac{\pi}{2}$$

$\Rightarrow n = 4(m_1 - m_2)$

Thus, n must be of the form $4k$.

33. $$\left|\frac{z_1-z_3}{z_2-z_3}\right| = \left|\frac{z_1-z_3}{z_2-z_3}\right| = \left|\frac{1-i\sqrt{3}}{2}\right|$$

$$= \sqrt{\frac{1}{4}(1+3)} = 1$$

$\Rightarrow \quad |z_1 - z_3| = |z_2 - z_3|$

Also, $\dfrac{z_1-z_3}{z_2-z_3} - 1 = \dfrac{1-i\sqrt{3}}{2} - 1$

$$\Rightarrow \quad \frac{z_1-z_2}{z_2-z_3} = \frac{-1-i\sqrt{3}}{2}$$

$$\Rightarrow \quad \frac{|z_1-z_2|}{|z_2-z_3|} = \sqrt{\frac{1}{4}(1+3)} = 1$$

$\Rightarrow \quad |z_1 - z_2| = |z_2 - z_3|$

Thus, $|z_1 - z_3| = |z_2 - z_3| = |z_2 - z_1|$

Hence, z_1, z_2 and z_3 are the vertices of an equilateral triangle.

34. Using $1 + \omega^2 = -\omega$, $\omega^4 = \omega$ and applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \omega-1 & \omega^2-1 \\ 1 & \omega^2-1 & \omega-1 \end{vmatrix}$$

$$= (\omega - 1)^2 - (\omega^2 - 1)^2$$
$$= (\omega + \omega^2 - 2)(\omega - 1 - \omega^2 + 1)$$
$$= (-3)(\omega - \omega^2) = 3\omega(\omega - 1)$$

35. $|z - i| = |z + i|$ represents the real axis.
As $z = i$ satisfies $|z - i| < |z + i|$, we get
$|z - i| < |z + i|$, represents Im $(z) > 0$

36. $iz^3 + z^2 - z + i = 0 \quad \Rightarrow iz^3 - i^2z^2 - (z - i) = 0$
$\Rightarrow \quad iz^2(z - i) - (z - i) = 0$
$\Rightarrow \quad (iz^2 - 1)(z - i) = 0$
$\Rightarrow \quad z^2 = 1/i$ or $z = i$
In any case $|z| = 1$.

37. $$2x = (x + iy) + (x - iy) = \frac{2(1-\cos\theta)}{(1-\cos\theta)^2 + 4\sin^2\theta}$$

$$\Rightarrow \quad x = \frac{1}{5+3\cos\theta}$$

Thus, maximum value of x is 1/2. It is attained at $\theta = \pi$.

38. $z^3 = \bar{z} \Rightarrow |z^3| = |\bar{z}| \Rightarrow |z^3| = |\bar{z}|$
$\Rightarrow \quad |z| = 0$ or $|z| = 1 \Rightarrow z = 0$ or $\bar{z} = 1/z$.

$\therefore \quad z^3 = \bar{z} \Rightarrow z^3 = \dfrac{1}{z} \Rightarrow z^4 = 1$

$\Rightarrow \quad z = 1, -1, i, -i$.
Thus, solutions of $z^3 = \bar{z}$ are 0, 1, –1, i, $-i$

39. Let $\quad z = x + iy$ so that
$x = 5 + t, \quad y = \sqrt{25 - t^2}$
$\Rightarrow \quad (x - 5)^2 + y^2 = 25,$
This clearly passes through $2 + 4i$

40. Note that ω, ω^2 are roots of

$$\frac{1}{a+x} + \frac{1}{b+x} + \frac{1}{c+x} = \frac{2}{x}$$

$\Leftrightarrow \quad x[bc + ca + ab + 2(a + b + c)x + 3x^2]$
$= 2[abc + (bc + ca + ab)x + (a + b + c)x^2 + x^3]$
$\Leftrightarrow \quad x^3 - (bc + ca + ab)x - 2abc = 0.$
If α is the third root of this equation, then
$$\alpha + \omega + \omega^2 = 0 \quad \Rightarrow \quad \alpha = 1.$$

41. $z = x + iy$, $|z - i\text{Re}(z)| = |z - \text{Im}(z)|$
$\Rightarrow |x + iy - ix|^2 = |x + iy - y|^2$
$\Rightarrow x^2 + (y - x)^2 = (x - y)^2 + y^2$
$\Rightarrow x = \pm y$.

42. We have
$$(r - \omega)(r - \omega^2) = r^2 - (\omega + \omega^2)r + 1$$
$$= r^2 + r + 1 = (r + 1)^2 - r$$

$$\therefore \quad \sum_{r=1}^{12}(r-\omega)(r-\omega^2) = \sum_{r=1}^{12}\left[(r+1)^2 - r\right]$$

$$= \frac{1}{6}(13)(13+1)(26+1) - 1 - \frac{1}{2}(12)(13) = 740$$

43. Let ix ($x \in \boldsymbol{R}$) be root of $z^2 + az + b = 0$, then
$-x^2 + aix + b = 0 \qquad (1)$
$\Rightarrow \; -x^2 - \bar{a}ix + \bar{b} = 0$
Subtracting we get
$(a + \bar{a})ix + b - \bar{b} = 0$

$$\Rightarrow \quad x = -\frac{b-\bar{b}}{i(a+\bar{a})} = \frac{i(b-\bar{b})}{a+\bar{a}}$$

Putting this in (1), we get

$$\frac{(b-\bar{b})^2}{(a+\bar{a})^2} - \frac{a(b-\bar{b})}{a+\bar{a}} + b = 0$$

$\Rightarrow (b - \bar{b})^2 - a(b - \bar{b})(a + \bar{a}) + b(a + \bar{a})^2 = 0$
$\Rightarrow (b - \bar{b})^2 - (a + \bar{a})\{ab - a\bar{b} - ab - \bar{a}b\} = 0$
$\Rightarrow (b - \bar{b})^2 + (a + \bar{a})(a\bar{b} + \bar{a}b) = 0$

44. The circle $|z - (-1 + i)| = \sqrt{2}$ completely lies inside the circle $|z| = 3$.

45. $\quad 8iz^3 + 12z^2 - 18z + 27i = 0$
$\Rightarrow \quad 8iz^3 - 12i^2z^2 - 18z + 27i = 0$
$\Rightarrow \quad 4iz^2(2z - 3i) - 9(2z - 3i) = 0$

$\Rightarrow \quad (4iz^2 - 9)\ (2z - 3i) = 0$

$\Rightarrow \quad z^2 = \dfrac{9}{4i}$ or $z = \dfrac{3i}{2}$

In any case $|z| = 3/2$.

46. As – 1 lies on the circle $|z - 4| = 5$, the real number $|z + 1|$ is maximum when z is the other end point of the diameter.

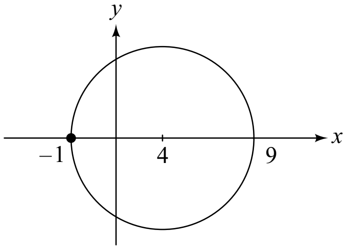


Fig. 2.58

47. $\dfrac{1}{x+iy} = \dfrac{\cos\theta+2}{3} + i\dfrac{\sin\theta}{3}$

$\Rightarrow \dfrac{x-iy}{x^2+y^2} = \dfrac{\cos\theta+2}{3} + i\dfrac{\sin\theta}{3}$

$\Rightarrow \dfrac{x}{x^2+y^2} = \dfrac{\cos\theta+2}{3}$

Also, $x^2+y^2 = \dfrac{9}{(2+\cos\theta)^2+\sin^2\theta} = \dfrac{9}{5+4\cos\theta}$

$\therefore\ 4x - (x^2 + y^2)$

$= \left(\dfrac{4(\cos\theta+2)}{3} - 1\right)(x^2+y^2)$

$= \dfrac{4\cos\theta+5}{3}\cdot\dfrac{9}{5+4\cos\theta} = 3$

48. Affix of G is $\frac{1}{3}(z_1+z_2+z_3)$.

As origin is the mid-point of AG,

$$0 = \frac{1}{2}\left[\frac{1}{3}(z_1+z_2+z_3)+z_1\right]$$

$\Rightarrow 4z_1 + z_2 + z_3 = 0.$

49. $z_3 = -\dfrac{1}{c}(az_1+bz_2) = \dfrac{az_1+bz_2}{a+b}$

50. $\dfrac{z-1}{z+1}$ = it where $t \in \mathbf{R}$. $\Rightarrow \dfrac{\bar z-1}{\bar z+1}$ = – it

$\Rightarrow \dfrac{z-1}{z+1} + \dfrac{\bar z-1}{\bar z+1} = 0$

$\Rightarrow (z - 1)(\bar z + 1) + (\bar z - 1)(z + 1) = 0$

$\Rightarrow 2(z\bar z - 1) = 0 \quad\Rightarrow\quad |z| = 1.$

51. $\left|\dfrac{z}{|z|} - \dfrac{\bar z}{|z|}\right| \le |\arg(z) - \arg(\bar z)|$

$\Rightarrow |z - \bar z| \le |z|\ |\arg(z) - \arg(\bar z)|$

52. As $|\omega| = 1 \Rightarrow |\omega|^2 = 1 \Rightarrow \omega\bar\omega = 1 \Rightarrow \dfrac{1}{\omega} = \bar\omega$.

Thus, $\quad z = \omega + 1/\omega = \omega + \bar\omega = 2\,\mathrm{Re}(\omega)$

Now, $\quad |\mathrm{Re}(z)| = |2\,\mathrm{Re}(\omega)| = |2\,\mathrm{Re}(\omega)| \le 2|\omega| = 2$

53. $|z| = 1 \Rightarrow z = \cos\theta + i\sin\theta$ for some $\theta \in [0, 2\pi)$

Now, $\quad |z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\bar z = 1.$

Thus, $\dfrac{z}{\bar z} + \dfrac{\bar z}{z} = z^2 + \dfrac{1}{z^2}$

$= (\cos\theta + i\sin\theta)^2 + (\cos\theta - i\sin\theta)^2$

$= 2(\cos^2\theta + i^2\sin^2\theta) = 2(\cos 2\theta)$

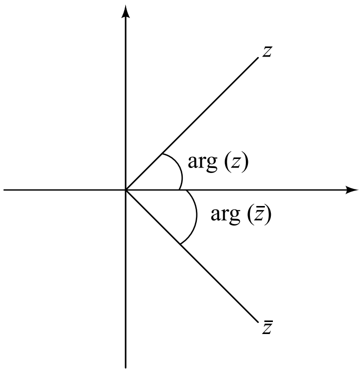


Fig. 2.59

$\therefore \quad \left|\dfrac{z}{\bar z} + \dfrac{\bar z}{z}\right| = 1 \Rightarrow |\cos 2\theta| = \dfrac{1}{2}$

$\Rightarrow \quad \cos 2\theta = \pm 1/2$

$\Rightarrow \quad 2\theta = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3, 7\pi/3, 8\pi/3, 10\pi/3, 11\pi/3$

$\Rightarrow \quad \theta = \pi/6, \pi/3, 2\pi/3. 5\pi/6, 7\pi/6, 4\pi/3, 5\pi/3, 11\pi/6$

Hence, there are 8 values of z

54. $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

$\Rightarrow \quad 3 + |z_1 - z_2|^2 = 2(2) + 2(2)$

$\Rightarrow \quad |z_1 - z_2|^2 = 5 \Rightarrow |z_1 - z_2| = \sqrt{5}$.

55. $|z_1\bar z_2| = |z_2\bar z_3| = |z_3\bar z_1|$

$\Rightarrow \quad |z_1|\,|z_2| = |z_2|\,|z_3| = |z_3|\,|z_1| \Rightarrow |z_1| = |z_2| = |z_3| = r$(say)

[$\because z_1, z_2, z_3 \ne 0$]

Thus, $\quad z_1\bar z_1 = z_2\bar z_2 = z_3\bar z_3$

Now, $\quad z_1\bar z_2 = z_2\bar z_3 = z_3\bar z_1$

$\Rightarrow \quad \dfrac{z_1}{z_2} = \dfrac{z_2}{z_3} = \dfrac{z_3}{z_1}$

$\Rightarrow \quad z_1^2 = z_2z_3, \quad z_2^2 = z_3z_1,\ z_3^2 = z_1z_2$

Hence, $z_1^2 + z_2^2 + z_3^2 = z_2z_3 + z_3z_1 + z_1z_2$

$\Rightarrow z_1, z_2, z_3$ are vertices of an equilateral triangle.

56. $|z_1 + z_2 + z_3| \ge 0$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\mathrm{Re}\left(z_1\bar z_2 + z_2\bar z_3 + z_3\bar z_1\right) \ge 0$

$\Rightarrow \quad \mathrm{Re}\left(z_1\bar z_2 + z_2\bar z_3 + z_3\bar z_1\right) \ge -\dfrac{3}{2}$

Now, $\quad |z_2 - z_3|^2 + |z_3 - z_1|^2 + |z_1 - z_2|^2$

$= 6 - 2\mathrm{Re}\left(z_1\bar z_2 + z_2\bar z_3 + z_3\bar z_1\right)$

$\leq \; 6 - (-3) = 9$

57. $z_1 + z_2 + z_3 = 0 \Rightarrow \bar{z}_1 + \bar{z}_2 + \bar{z}_3 = 0$

$$\Rightarrow \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0 \qquad [\because\ z_1\bar{z}_1 = 1 \text{ etc.}]$$

Now, $0 = (z_1 + z_2 + z_3)^2 = z_1^2 + z_2^2 + z_3^2 + 2\,(z_2z_3 + z_3z_1 + z_1z_2)$

$$\Rightarrow \quad 0 = z_1^2 + z_2^2 + z_3^2 + 2z_1z_2z_3\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$$

$$= z_1^2 + z_2^2 + z_3^2$$

Thus, $\left|z_1^2 + 2z_2^2 + z_3^2\right| = \left|z_1^2 + z_2^2 + z_3^2 + z_2^2\right|$

$$= \left|0 + z_2^2\right| = |z_2|^2 = 1$$

58. Note that $\dfrac{1}{z_1} = \bar{z}_1$ etc. Thus,

$z = (z_1 + z_2 + z_3)\ (\bar{z}_1 + \bar{z}_2 + \bar{z}_3)$

$= |z_1 + z_2 + z_3|^2 \leq (|z_1| + |z_2| + |z_3|)^2 = 9$

The maximum value is obtained when $z_1 = z_2 = z_3 = 1$.

59. $$\omega = \frac{z - z\bar{z}}{\bar{z} + \bar{z}\,z} = \frac{z-1}{1/z + 1} = \frac{(z-1)z}{z+1} \qquad [\because\ |z| = 1]$$

$$= \frac{z^2-1}{z+1} - \frac{z-1}{z+1} = z - 1 + \frac{1-z}{1+z}$$

$$\text{Re}(\omega) = \text{Re}(z) - 1 + \text{Re}\left(\frac{1-z}{1+z}\right)$$

But $$\text{Re}\left(\frac{1-z}{1+z}\right) = \frac{1}{2}\left(\frac{1-z}{1+z} + \frac{1-\bar{z}}{1+\bar{z}}\right)$$

$$= \frac{1}{2}\left(\frac{1-z}{1+z} + \frac{z-1}{z+1}\right) = 0$$

$\therefore \quad \text{Re}(\omega) = \cos\theta - 1 = -2\sin^2(\theta/2)$

60. Let $\omega = \dfrac{1+z^2}{2iz}$. $\text{Im}(\omega) = \dfrac{1}{2}(\omega - \bar{\omega})$

For $|z| = 1$, $$\bar{\omega} = \frac{1+\bar{z}^2}{-2i\bar{z}} = \frac{1+(1/z)^2}{-2i(1/z)}$$

$$= \frac{z^2+1}{-2iz} = -\omega$$

Thus, $\text{Im}(\omega) = \dfrac{1}{2i}(\omega + \omega) = -i\omega$

$$= -\frac{1}{2}\left(z + \frac{1}{z}\right) = -\frac{1}{2}(z + \bar{z})$$

$$= -\frac{1}{2}(2\cos\theta) = -\cos\theta$$

$[\because\ |z| = 1 \Rightarrow z = \cos\theta + i\sin\theta]$

As $z \neq 1$, $\theta \neq 0$ and as $z \neq -1$, $\theta \neq \pi$,

$\therefore \quad A = (-1, 1)$

61. $|z| = 1 \Rightarrow z\bar{z} = 1$. Therefore $z = 1 - 2\bar{z}$

$\Rightarrow \quad z = 1 - 2/z \quad \Rightarrow \quad z^2 - z + 2 = 0$

$\Rightarrow \quad z = \dfrac{1 \pm \sqrt{7}\,i}{2}$, but then $|z| \neq 1$.

62. Suppose $z_2/z_1 = a$, where $a \in \mathbf{R}$.

Now, $$\left|\frac{2iz_1 + 5z_2}{2iz_1 - 5z_2}\right| = \left|\frac{2i + 5(z_2/z_1)}{2i - 5(z_2/z_1)}\right| = \left|\frac{2i + 5a}{2i - 5a}\right| = 1$$

63. $z = i\left(1 + \sqrt{3}\right) = -1 + \sqrt{3}\,i = 2\omega$

where $\omega \neq 1$ is a cube root of unity.

$\therefore \quad z^4 + 2z^3 + 4z^2 + 5$

$= (2\omega)^4 + 2\,(2\omega)^3 + 4\,(2\omega)^2 + 5$

$= 16\omega^4 + 16\omega^3 + 16\omega^2 + 5$

$= 16\,(\omega + 1 + \omega^2) + 5 = 16\,(0) + 5 = 5$

64. Let $a = -\,b^4$ where $b > 0$.

Then $z^4 = a = b^4(-1)$

$$\Rightarrow z = b\left(\pm\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4}\right)$$

$\therefore \quad z_1^2 + z_2^2 + z_3^2 + z_4^2$

$$= 2b^2\left[\cos\left(\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$$

$= 0 = a + |a| \qquad [\because\ a < 0]$

65. As $\dfrac{z+\bar{z}}{1+z\bar{z}}$ is a positive real number,

$$\arg\left(\frac{z+\bar{z}}{1+z\bar{z}}\right) = 0.$$

66. Using $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= \cos(\alpha + \beta) + i\sin(\alpha + \beta)$,

we get

$\cos(25\theta) = -1, \quad \sin(25\theta) = 0$

$\Rightarrow \quad 25\theta = (2k + 1)\pi,\ k \in \mathbf{I}$.

Now, $0 < \dfrac{(2k+1)\pi}{25} \leq \pi$

$\Rightarrow \quad 1 \leq (2k + 1) \leq 25$

$\Rightarrow \quad k = 0, 1, 2, \ldots, 12$

67. $\log_{(1/7)}|z - 2| > \log_{(1/7)}|z|$

$\Rightarrow \quad |z - 2| < |z| \qquad (1)$

But $|z - 2| = |z|$ represents perpendicular bisector of the segment joining 0 and 2, that is, $|z - 2| = |z|$ represents the line $\text{Re}(z) = 1$. As 0 does not satisfy (1), we get (1) represents $\text{Re}(z) > 1$.

68. $$\frac{1}{2i}\left[\frac{2z+1}{iz+1} - \frac{2\bar{z}+1}{-i\bar{z}+1}\right] = 5$$

$\Rightarrow \quad (2z + 1)(i\bar{z} - 1) + (2\bar{z} + 1)(iz + 1)$

$\quad = 10\, i\,(iz + 1)(i\bar{z} - 1)$

$\Rightarrow \quad 4iz\bar{z} + (-2 + i)z + (2 + i)\bar{z}$

$\quad = 10\, i\,(-z\bar{z} - iz + i\bar{z} - 1)$

$\Rightarrow \quad 14iz\bar{z} + (8 + i)z + (-8 + i)\bar{z} - 10\,i = 0$

$\Rightarrow \quad z\bar{z} + \frac{1}{14}(1 - 8i)z + \frac{1}{14}(1 + 8i)\bar{z} - \frac{5}{7} = 0$

This represents a circle.

69. Take z_1 and z_2 as two real numbers.

70. $\frac{(1+i)^n}{(1-i)^{n-2}} = \left(\frac{1+i}{1-i}\right)^{n-2}(1+i)^2 = \left(\frac{-i^2+i}{1-i}\right)^{n-2}(2i)$

$\quad = i^{n-2}(2i) = 2i^{n-1} = -2i^{n+1}$

71. If $n = 3k$, $k \in \mathbf{I}$, then $f(n) = 1 + 1 + 1 = 3$

If $n = 3k + 1$, $k \in \mathbf{I}$, then $f(n) = 1 + \omega + \omega^2 = 0$

If $n = 3k + 2$ $k \in \mathbf{I}$, then $f(n) = 1 + \omega^2 + \omega = 0$

Thus, range of f is $\{0, 3\}$.

72. $z + \frac{1}{z} = 2\cos\theta \Rightarrow z^2 - 2z\cos\theta + 1 = 0$

$\Rightarrow \quad z = \cos\theta \pm i\sin\theta$

Now, $z^{2n} - 2z^n\cos(n\theta)$

$\quad = z^n[z^n - 2\cos(n\theta)]$

$\quad = z^n[\cos(n\theta) \pm i\sin(n\theta) - 2\cos(n\theta)]$

$\quad = -z^n\bar{z}^n = -1$

73. Using $\omega^k + \omega^{k+1} + \omega^{k+2} = 0\ \forall\ k \in \mathbf{I}$,

we get $\sum_{k=1}^{60}\omega^k = 0$

Next, $\prod_{k=1}^{30}\omega^k = \omega^m =$ where

$m = 1 + 2 + \ldots + 30 = \frac{1}{2}(30)(31) = 465$

$\therefore \quad \prod_{k=1}^{30}\omega^k = \omega^{465} = 1$

74. $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

Therefore, ω, ω^2 are zeros of $P(x)$

$0 = g(1) + \omega h(1)$ and $0 = g(1) + \omega^2 h(1)$

$\Rightarrow g(1) = 0,\ h(1) = 0.$

75. $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

Therefore, $-\omega$, $-\omega^2$ must be zeros of $x^{n+1} - x^n + 1$. Now,

$(-\omega)^{n+1} - (-\omega)^n + 1 = (-1)^{n+1}\omega^n(\omega + 1) + 1$

$\quad = (-1)^{n+1}\omega^n(-\omega^2) + 1$

$\quad = (-1)^{n+2}\omega^{n+2} + 1$

$\quad = 0$ if $n = 6k + 1$

76. Clearly $z \neq 0, 1$. We can write the given equation as $t^2 + t + 1 = 0$ where

$t = z/|z - 1|$

$\Rightarrow \quad t = \omega, \omega^2$ where $\omega \neq 1$ is a cube root of unity.

Thus, $\frac{z}{|z-1|} = \omega$

$\Rightarrow \quad |z| = |z - 1|\,|\omega| = |z - 1|$

$\Rightarrow$ z lies on the perpendicular bisector of the segment joining $z = 0$ and $z = 1$, that is, z lies on $\text{Re}(z) = 1/2$. Let $z = 1/2 + ai$ where $a \in \mathbf{R}$.

Note that $a \neq 0$ since $z = 1/2$ does not satisfy $z^2 + z|z - 1| + |z - 1|^2 = 0$

Putting $z = 1/2 + ai$ in $z^2 + z|z - 1| + |z - 1|^2 = 0$ and equating imaginary parts of both sides, we get

$a + a|z - 1| = 0 \Rightarrow |z - 1| = -1.$

A contradiction.

Thus $z^2 + z\,|z - 1| - |z + 1|^2 = 0$ has no solution in C, that is, statement-1 is true.

Statement-2 is false as $a\omega$, $a\omega^2$ are solutions of $z^2 + az + a^2 = 0$.

77. $z = 2i + \bar{z} \quad \Rightarrow \quad z - \bar{z} = 2i$

$\Rightarrow \quad 2i\,\text{Im}(z) = 2i \quad \Rightarrow \quad \text{Im}(z) = 1$

As $|z| = 1$, we get $z = \pm\, i$

Thus, statement-1 is true.

As $\arg(i) = \pi/2$ and $\arg(-i) = -\pi/2$,

$\therefore$ Statement-2 is false.

78. Note that

$\alpha = \cos\left(\frac{8\pi}{11}\right) + i\sin\left(\pi - \frac{8\pi}{11}\right)$

$\quad = \cos\left(\frac{8\pi}{11}\right) + i\sin\left(\frac{8\pi}{11}\right)$

$\therefore\ \alpha^{11} = \cos(8\pi) + i\sin(8\pi) = 1$

[De Moivre's Theorem]

Now, $\sum_{k=0}^{10}\alpha^k = \frac{1 - \alpha^{11}}{1 - \alpha} = 0$

$\therefore$ Statement-2 is true

We have $\bar{\alpha} = \frac{\alpha\bar{\alpha}}{\alpha} = \frac{1}{\alpha} = \alpha^{10}$ etc.

Thus,

$\text{Re}\,(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$

$$= \frac{1}{2}(\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5+\bar{\alpha}+\bar{\alpha}^2+\bar{\alpha}^3+\bar{\alpha}^4+\bar{\alpha}^5)$$

$$= \frac{1}{2}\sum_{k=1}^{10}\alpha^k = \frac{1}{2}(-1) = -\frac{1}{2} \qquad \text{[Use statement-2]}$$

∴ Statement-1 is also correct and Statement-2 is a correct explanation for it.

Statement-2 is true as it is formula for sun of a G.P.

79. $z^k = \cos(k\theta) + i\sin(k\theta)$

$$f(\theta) = \sum_{k=1}^{15}\operatorname{Im}(z^{2k-1})$$

$$= \operatorname{Im}\left(\sum_{k=1}^{15} z^{2k-1}\right)$$

$$= \operatorname{Im}\left(\frac{z(1-z^{30})}{1-z^2}\right)$$

But $z^{30} = \cos(30\ \theta) + i\sin(30\ \theta)$

When $\theta = \alpha = \pi/60$, $z^{30} = 0 + i = i$

$$\therefore f(\alpha) = \operatorname{Im}\left(\frac{z(1-z^{30})}{1-z^2}\right)$$

$$= \operatorname{Im}\left(\frac{1-i}{1/z-z}\right)$$

$$= \operatorname{Im}\left(\frac{1-i}{-2i\sin\alpha}\right)$$

$$= \frac{1}{2\sin\alpha}\operatorname{Im}(1+i) = \frac{1}{2\sin\alpha}$$

80. Statement-2 is true. See theory of chapter 5 on matrices.

Let $A = \begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$,

$$|A| = 1\begin{vmatrix} 1 & c \\ \omega & 1 \end{vmatrix} - a\begin{vmatrix} \omega & c \\ \omega^2 & 1 \end{vmatrix} + b\begin{vmatrix} \omega & 1 \\ \omega^2 & \omega \end{vmatrix}$$

$= 1 - c\omega - a(\omega - c\omega^2)$

$= (1 - a\omega)(1 - c\omega)$

Note that $|A| = 0$ if $a = \omega^2$ or $c = \omega^2$

Thus, $|A| \neq 0$ if $a = c = \omega$ and $b = \omega$ or ω^2

∴ S contains exactly two distinct elements.

81. $|z_1 - z_2| - \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$

$$= |z_1 - z_2| - \frac{|z_2 - z_1|}{|z_1||z_2|}$$

$$= |z_1 - z_2| - |z_2 - z_1| = 0$$

∴ Statement-1 is true.

Statement-2 is also true but a correct explanation for truth of statement-1.

82. Statement-2 is a true statement.
Suppose $f(z) = (z^2 + 1)\,q(z) + r(z)$.
If $r(z) \neq 0$, then $r(z) = az + b$
where $a, b \in \mathbf{C}$.
We have

$$i = f(i) = (i^2 + i)\,q(i) + ai + b$$

$\Rightarrow \quad ai + b = i$

Also, $1 + i = f(-i) = ((-i)^2 + 1)\,q(-i) - ai + b$

$\Rightarrow \quad -ai + b = 1 + i$

$\therefore \quad b = \frac{1}{2} + i, \ a = \frac{1}{2}i$

Thus, $az + b = \frac{1}{2}(iz + 1) + i$

83. Statement-2 is true, as $z^3 = a^3 \Rightarrow |z|^3 = |a|^3$
$\Rightarrow |z| = |a|$
Next,
$(z + ab)^3 = a^3$
$\Rightarrow \quad z + ab = a, aw, aw^2$
Let $z_1 = a - ab$, $z_2 = aw - ab$
and $z_3 = aw^2 - ab$

We gave $|z_2 - z_1| = |a||w - 1| = \sqrt{3}a$,

$|z_3 - z_2| = \sqrt{3}a$, $|z_1 - z_3| = \sqrt{3}a$

Thus, statement-1 is also true, but statement-2 is not a correct explanation of statement-1.

84. Statement-2 is true. See Theory. Let α be a root of $z^2 - z + \pi = 0$ and suppose $|\alpha| < 1$. We have

$$\pi = |\pi| = |\alpha - \alpha^2| \leq |\alpha| + |\alpha|^2 < 2$$

A contradiction.

Thus, statement-1 is also true and statement-2 is a correct explanation for it.

85. We know that $|\sin\theta| \leq \theta \quad \forall\ \theta \geq 0$.

If $\theta < 0$, then $|\sin(-\theta)| \leq -\theta \Rightarrow |\sin\theta| \leq |\theta|$

Thus, $|\sin\theta| \leq |\theta|\ \forall\ \theta \in \mathbf{R}$ and statement-2 is true.

Now, $|z_1 - z_2|^2 = (r_1\cos\theta - r_2\cos\varphi)^2 + (r_1\sin\theta - r_2\sin\varphi)^2$

$$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta - \varphi)$$

$$= (r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\frac{\theta - \varphi}{2}\right)$$

$$\leq (r_1 - r_2)^2 + 4(1)(1)\left(\frac{\theta - \phi}{2}\right)^2$$

Thus, Statement-1 is also true and Statement-2 is a correct explanation for it.

Level 2

86. As $ABCD$ is a parallelogram,

mid point of AC = mid point of BD

$$\Rightarrow \frac{1}{2}(z_1+z_3) = \frac{1}{2}(z_2+z_4)$$

$\Rightarrow z_4 = z_1 + z_3 - z_2$

87. $|a + b\omega + c\omega^2|^2 = (a + b\omega + c\bar{\omega})\ (a + b\bar{\omega} + c\omega)$

$= a^2 + b^2 + c^2 - bc - ca - ab$

$= \frac{1}{2}\left[(b-c)^2+(c-a)^2+(a-b)^2\right]$

As a, b, c are integers and at least two of them are unequal, we get,

$(b - c)^2 + (c - a)^2 + (a - b) \geq 2.$

Thus, $|a + b\omega + c\omega^2|^2 \geq 1 \Rightarrow |a + b\omega + c\omega^2| \geq 1$

Least value 1 is attained when $a = 2$, $b = 1$, $c = 1$.

88. We have $AP = AB = AQ = 2$

Thus, for the shaded region $|z + 1| > 2$

Also, $\angle BAQ = \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}-1+1}\right) = \frac{\pi}{4}$

and $\angle BAP = -\pi/4$

Hence, for the shaded region

$|z + 1| > 2$ and $|\arg(z + 1)| < \pi/4$

89. As $\frac{w-\bar{w}z}{1-z}$ is purely real,

$$\frac{w-\bar{w}z}{1-z} = \frac{\bar{w}-w\bar{z}}{1-\bar{z}}$$

$\Rightarrow (1 - \bar{z})\ (w - \bar{w}z) = (1 - z)\ (\bar{w} - w\bar{z})$

$\Rightarrow (w - \bar{w})\ (1 - z\bar{z}) = 0$

As $w \neq \bar{w}$, we get $z\bar{z} = 1$

Thus, set of values of z is

$\{z: |z| = 1, z \neq 1\}$.

90. $zw = |z|^2 \Rightarrow zw = z\bar{z} \Rightarrow w = \bar{z}$

Thus, $|z - \bar{z}| + |w + \bar{w}| = 4$

$\Rightarrow |z - \bar{z}| + |\bar{z} + z| = 4$

$\Rightarrow |2iy| + |2x| = 4$

$\Rightarrow |x| + |y| = 2$

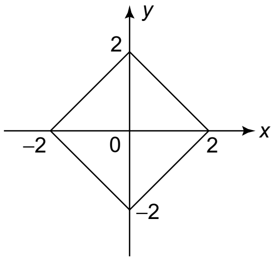


Fig. 2.60

This represents a square. See Fig. 2.60.

91. Let $z = x + iy$, then

$x = 1 - t,\ y = \sqrt{t^2+t+2}$

$\Rightarrow t = 1 - x$ and $y^2 = t^2 + t + 2 = (t + 1/2)^2 + 7/4$

$\Rightarrow y^2 = (x - 3/2)^2 + 7/4.$

This represents a hyperbola.

92. $w+\bar{w} = \frac{az_1-bz_2}{az_1+bz_2} + \frac{a\bar{z}_1-b\bar{z}_2}{a\bar{z}_1+b\bar{z}_2}$

$$= \frac{(az_1-bz_2)(a\bar{z}_1+b\bar{z}_2)+(a\bar{z}_1-b\bar{z}_2)(az_1+bz_2)}{|az_1+bz_2|^2}$$

$$= \frac{a^2\ z_1\ \bar{z}_1 - b^2\ z_2\ \bar{z}_2}{|az_1+bz_2|^2} = 0\ [\because\ a\ |z_1| = b\ |z_2|]$$

$\Rightarrow w$ lies on the imaginary axis.

93. $|z| + |1 - z| + |z - 2|$

$\geq \max\ \{|z|, |(1 - z) + (z - 2)|, |z + (1 - z)| + |z - 2|, |1 - z| + |z - (z - 2)|\} = 2$

The value 2 is attained when $z = 1$

94. See Theory.

95. $|z_2 - z_3|^2 + |z_1|^2 = |z_2 - z_3|^2 + |-z_2 - z_3|^2$

$\Rightarrow |z_2 - z_3|^2 + 1 = 2|(|z_2|^2 + |z_3|^2) = 4$

$\Rightarrow |z_2 - z_3| = \sqrt{3}$

Similarly, $|z_3 - z_1| = |z_1 - z_2| = \sqrt{3}$

Thus, z_1, z_2, z_3 are vertices of an equilateral triangle.

96. $1 + w + w^2 + \ldots + w^{n-1}$

$$= \frac{1-w^n}{1-w} = \frac{1-\cos\pi+i\sin\pi}{1-\cos(\pi/n)-i\sin(\pi/n)}$$

$$= \frac{2}{2\sin^2(\pi/2n) - 2i\sin(\pi/2n)\cos(\pi/2n)}$$

$$= \frac{2}{-2i\sin(\pi/2n)\left[\cos(\pi/2n)+i\sin(\pi/2n)\right]}$$

$$= \frac{\cos(\pi/2n) - i\sin(\pi/2n)}{-i\sin(\pi/2n)} = 1 + i\cot\left(\frac{\pi}{2n}\right)$$

97. $|z_1 + z_2|^2 = |z_1 - z_2|^2$

$\Rightarrow |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

$= |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2$

$\Rightarrow 2(z_1\bar{z}_2 + \bar{z}_1 z_2) = 0 \Rightarrow \frac{z_1}{\bar{z}_1} + \frac{z_2}{\bar{z}_2} = 0$

98. z_1 lies on the circle $|z| = 2$ and z_2 lies on the line $x + y = 4\sqrt{2}$

Distance of $x + y = 4\sqrt{2}$ from (0, 0) is 4

Thus, minimum distance between z_1 and z_2 is 2.

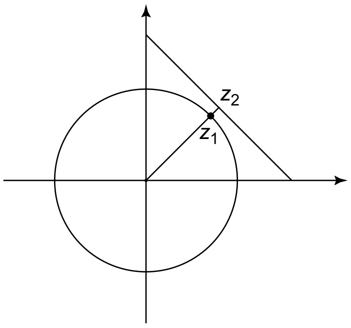


Fig. 2.61

Alternative Solution

Distance of $z_1 = 2(\cos\theta + i\sin\theta)$ from $x + y = 4\sqrt{2}$, is

$$\frac{\left|2\cos\theta + 2\sin\theta - 4\sqrt{2}\right|}{\sqrt{2}}$$

$= \sqrt{2}\ (2\sqrt{2} - (\cos\theta + \sin\theta))$

Maximum possible value of $\cos\theta + \sin\theta$ is $\sqrt{2}$.

99. Let $z_1 = r(\cos\theta + i\sin\theta)$.

Equation of tanget to $x^2 + y^2 = r^2$ at $(r\cos\theta, r\sin\theta)$ is $x\cos\theta + y\sin\theta = r$

$$\Rightarrow \left(\frac{z+\bar{z}}{2}\right)\left(\frac{z_1+\bar{z}_1}{2}\right) + \left(\frac{z-\bar{z}}{2i}\right)\left(\frac{z_1-\bar{z}_1}{2i}\right) = r^2$$

$\Rightarrow z\bar{z}_1 + \bar{z}z_1 = 2r^2$

$\Rightarrow \frac{z}{z_1} + \frac{\bar{z}}{\bar{z}_1} = 2.$

100. Using $|z_1| + |z_2| \ge |z_1 - z_2|$, we get

$|z - 2 + 3i| + |z - 1 + i| \ge |-1 + 2i| = \sqrt{5}$

101. $|z - a^2| + |z - 2a| = 3$ will represent an ellipse if

$|a^2 - 2a| < 3$

$\Leftrightarrow -3 < a^2 - 2a < 3$

$\Leftrightarrow -2 < (a - 1)^2 < 4$

$\Leftrightarrow (a - 1)^2 < 4 \Leftrightarrow -1 < a < 3 \Leftrightarrow a \in (0, 3)$

102. As $AB = BC = CA$, we get

$2|z| = |1| = |1 - 2z|$

$\Rightarrow |z| = \frac{1}{2}$ and $|z - \frac{1}{2}| = \frac{1}{2}$

$\Rightarrow z$ is the point of intersection of circles

$|z| = 1/2$ and $|z - 1/2| = 1/2$

$\Rightarrow z = \frac{1}{4}\left(1 \pm \sqrt{3}i\right)$

103. $|z + 1| + |z - 3| \le 10$ represents the ellipse with focii at (–1, 0), and (3, 0) and length of major axis 10. Its centre is (1, 0), and its equation is

$$\frac{(x-1)^2}{25} + \frac{y^2}{21} = 1$$

Any point on the ellipse is $P(1 + 5\cos\theta, \sqrt{21}\sin\theta)$.

Its distance from A (7, 0) is given by

$AP^2 = (5\cos\theta + 8)^2 + 21\sin^2\theta$

$= (2\cos\theta + 20)^2 - 20^2 + 85$

$\Rightarrow 18^2 - 20^2 + 85 \le AP^2 \le 22^2 - 20^2 + 85$

$\Rightarrow 9 \le AP^2 \le 169$

$\Rightarrow 3 \le AP \le 13$

104. $|z - 1|^2 + 2|z - w|^2 = 3|z - w^2|^2$

$\Rightarrow |z|^2 + 1 - z - \bar{z} + 2[|z|^2 + 1 - z\bar{w} - \bar{z}w]$

$= 3[|z|^2 + 1 - z\bar{w}^2 - \bar{z}w^2]$

$\Rightarrow (3w - 2w^2 - 1)z + (3w^2 - 2w - 1)\bar{z} = 0$

which represents a straight line.

105. As R.H.S is real, L.H.S must be real.

Also, $\left(\frac{1+i}{1-i}\right) = \left(\frac{i-i^2}{1-i}\right)^n = i^n$ is real when n is even.

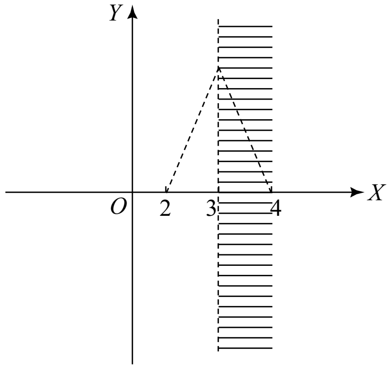
As $x > 0$ and $\frac{1+x^2}{2x} = \frac{1}{2}\left(x + \frac{1}{x}\right) > 1$ for $x \ne 1$.

Thus, we get only possible value of x is 1.

$\therefore$ RHS = 1, thus, least value of n is 4.

Previous Years' AIEEE/JEE Main Questions

1. Let $|z| = |w| = r$ and Arg $(w) = \theta$,
   so that Arg $(z) = \pi - \theta$. We have
   $$z = r[\cos(\pi - \theta) + i\sin(\pi - \theta)]$$
   $$= r[-\cos\theta + i\sin\theta]$$
   $$= -r(\cos\theta - i\sin\theta) = -\bar{w}$$

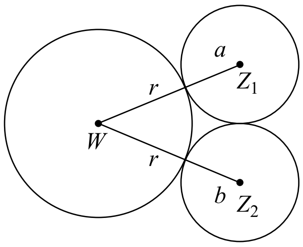
2. 

   If z satisfies $|z - 4| = |z - 2|$, then z lies on the perpendicular bisector of the segment joining $z = 2$ and $z = 4$.
   i.e., $|z - 4| = |z - 2| \Rightarrow \text{Re}(z) = 3$.

   As $z = 0$ does not satisfy $|z - 4| < |z - 2|$, we get $|z - 4| < |z - 2|$ represents the region $\text{Re}(z) > 3$.

3. Suppose $|z - w| = r$ touches $|z - z_1| = a$ and $|z - z_2| = b$ externally.

   Then $|w - z_1| = a + r$, $|w - z_2| = b + r$
   $\Rightarrow$ $|w - z_1| - |w - z_2| = a - b$
   $\Rightarrow$ w lies on a hyperbola with foci at z_1 and z_2



4. As $1 = -i^2$,
   $$1 = \left(\frac{1+i}{1-i}\right)^x = \left(\frac{-i^2+i}{1-i}\right)^x = \left(\frac{i(1-i)}{1-i}\right)^x = i^x$$
   $\Rightarrow x = 4n$ for some $n \in \mathbf{N}$.

5. $|\bar{z}w| = |\bar{z}||w| = |z||w| = |zw| = 1$
   $$\text{Arg}(\bar{z}w) = \arg(w) + \arg(\bar{z})$$
   $$= \arg(w) - \arg(z) = -\pi/2$$
   $$\therefore\ \bar{z}w = |\bar{z}w|[\cos(-\pi/2) + i\sin(-\pi/2)] = -i$$

6. $z_1 + z_2 = -a$, $z_1z_2 = b$
   As 0, z_1, z_2 for an equilateral triangle,
   $$0^2 + z_1^2 + z_2^2 = 0(z_1) + 0(z_2) + z_1z_2$$
   $$\Rightarrow\quad (z_1 + z_2)^2 = 3z_1z_2 \Rightarrow a^2 = 3b.$$

7. $\bar{z} + i\bar{w} = 0 \Rightarrow \overline{\bar{z} + i\bar{w}} = \bar{0} \Rightarrow z - iw = 0$
   Now, $\arg(zw) = \pi$
   $$\Rightarrow\quad \arg\left(\frac{z^2}{i}\right) = \pi$$
   $$\Rightarrow \arg(z^2) - \arg(i) = \pi$$
   $$\Rightarrow\ 2\arg(z) - \pi/2 = \pi$$
   $$\Rightarrow\quad \arg(z) = 3\pi/4$$

8. $z^{1/3} = p + iq$
   $$\Rightarrow\ x - iy = (p + iq)^3$$
   $$\Rightarrow\quad = p^3 + 3p^2(iq) + 3p(iq)^2 + (iq)^3$$
   $$\Rightarrow\quad x = p^3 - 3pq^2 \text{ and } -y = 3p^2q - q^3$$
   $$\Rightarrow\quad \frac{x}{p} = p^2 - 3q^2 \text{ and } \frac{-y}{q} = 3p^2 - q^2$$
   $$\Rightarrow\ \frac{x}{p} + \frac{y}{q} = -2(p^2 + q^2)$$
   $$\therefore\ \left(\frac{x}{p} + \frac{y}{q}\right)\Big/\left(p^2 + q^2\right) = -2$$

9. $|z^2 - 1| = |z|^2 + 1$ can be written as $|z^2 + (-1)| = |z^2| + |-1|$
   $$\Leftrightarrow\ \frac{z^2}{-1} \text{ is a non-negative real number.}$$
   $\Leftrightarrow$ z^2 is a non-positive real number.
   $\Leftrightarrow$ z lies on the imaginary axis.

   Alternative solution

   Let $z = x + iy$ then $|z^2 - 1| = |z|^2 + 1$, we get
   $$|(x^2 - y^2 - 1) + 2ixy| = x^2 + y^2 + 1$$
   $$\Rightarrow\ \sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2} = x^2 + y^2 + 1$$
   $$\Rightarrow\ (x^2 - y^2)^2 + 1 - 2(x^2 - y^2) + 4x^2y^2 = (x^2 + y^2 + 1)^2$$
   $$\Rightarrow\ (x^2 + y^2)^2 + 1 - 2(x^2 - y^2) = (x^2 + y^2)^2 + 1 + 2(x^2 + y^2)$$
   $$\Rightarrow\ -4x^2 = 0 \Rightarrow x = 0$$
   $\therefore$ z lies on the imaginary axis.

10. $(x - 1)^3 = -8$
    $$\Rightarrow\ \left(-\frac{x-1}{2}\right)^3 = 1$$
    $$\Rightarrow\ -\frac{x-1}{2} = 1, \omega, \omega^2$$

$\Rightarrow\ x = -1,\ 1 - 2\omega,\ 1 - 2\omega^2$

11. $|w| = 1 \Rightarrow |z| = \left|z - \frac{1}{3}i\right|$

$\Rightarrow$ z is equidistant from $z = 0$ and $z = \frac{1}{3}i$. Thus, z lies on the perpendicular bisector of the segment joining $z = 0$ and $z = i/3$. Therefore, z lies on a straight line.

12. $|z_1 + z_2| = |z_1| + |z_2|$

$\Leftrightarrow$ z_1, z_2 lie on a ray through the origin O and same side of the origin

$\Leftrightarrow$ $\arg(z_1) = \arg(z_2)$

13. $\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}$

$= i\left(\cos\frac{2k\pi}{11} - i\sin\frac{2k\pi}{11}\right)$

$= i\omega^k$

where $\omega = \cos\frac{2\pi}{11} - i\sin\frac{2\pi}{11}$

Thus, $S = \sum_{k=1}^{10}\left(\sin\frac{2k\pi}{11} + i\cos\frac{2k\pi}{11}\right)$

$= i\sum_{k=1}^{10}\omega^k = i\left(\frac{\omega - \omega^{11}}{1-\omega}\right)$

But $\omega^{11} = \cos 2\pi - i\sin 2\pi = 1$

$\therefore\ S = -i$

14. $z^2 + z + 1 = 0 \Rightarrow z = \omega$ or ω^2

where ω is complex cube root of unity.

Let $z = \omega$, so that $\frac{1}{z} = \omega^2$

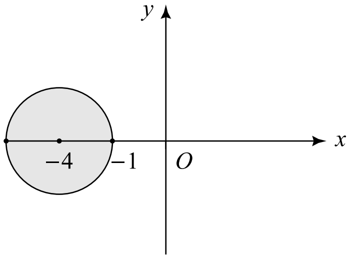
Thus,

$$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \cdots + \left(z^6 + \frac{1}{z^6}\right)^2$$

$= (-1)^2 + (-1)^2 + (2)^2 + (-1)^2 + (-1)^2 + (2)^2$

$= 12.$

15. $|z + 4| \le 3$ represents the interior and boundary of the circle with centre at $(-4, 0)$ and radius $= 3$. See Figure.



As -1 is an end point of a diameter of the circle, maximum possible value of $|z + 1|$ is 6 which is attained when $z = -7$

16. $\bar{z} = \frac{1}{i-1} \Rightarrow z = \frac{1}{\bar{i} - 1} = \frac{1}{-i-1} = \frac{-1}{i+1}$

17. $|z| - \frac{4}{|z|} \le \left||z| - \frac{4}{|z|}\right| \le \left|z - \frac{4}{z}\right| = 2$

$\Rightarrow\ |z|^2 - 2|z| - 4 \le 0$

$\Rightarrow\ (|z| - 1)^2 \le 5 \Rightarrow |z| \le \sqrt{5} + 1$

18. z is equidistant from $A(1 + 0i)$, $B(-1 + 0i)$ and $C(0 + i)$. Thus, z is circumcentre of ΔABC, that is, there is exaclty one such z.

19. $(1 + \omega)^7 = (-\omega^2)^7 = (-\omega^2)^6(-\omega^2) = -\omega^2$

$= 1 + \omega$

$\therefore\ A = 1,\ B = 1$

20. As $\alpha, \beta \in \mathbf{R}$, the roots of $z^2 + \alpha z + \beta = 0$ are of the form

$1 + ia$, $1 - ia$, where $a \in \mathbf{R}$, $a \ne 0$.

Now, $\beta = (1 + ia)(1 - ia) = 1 + a^2 > 1$

$\Rightarrow \beta \in (1, \infty)$

21. As $\frac{z^2}{z-1}$ is real, we get

$\frac{z^2}{z-1} = \frac{\bar{z}^2}{\bar{z}-1}$

$\Leftrightarrow\ z^2(\bar{z} - 1) = \bar{z}^2(z - 1)$

$\Leftrightarrow\ z\bar{z}(z - \bar{z}) - (z - \bar{z})(z + \bar{z}) = 0$

$\Leftrightarrow\ (z - \bar{z})(z\bar{z} - z - \bar{z}) = 0$

$\Leftrightarrow\ z - \bar{z} = 0$ or $z\bar{z} - z - \bar{z} = 0$

$\Rightarrow$ z lies on the real axis

or z lies on a circle through the origin.

22. $|z| = 1 \Rightarrow z\bar{z} = 1$

$\arg\left(\frac{1+z}{1+\bar{z}}\right) = \arg\left(\frac{1+z}{1+1/z}\right)$

$= \arg(z) = \theta$

23. Let $z_2/z_1 = ik$ where k is a real number.

$$\left|\frac{2z_1 + 3z_2}{2z_1 - 3z_2}\right| = \left|\frac{2/3 + z_2/z_1}{2/3 - z_2/z_1}\right| = \left|\frac{2/3 + ik}{2/3 - ik}\right| = 1$$

24. Let $\omega = \frac{1+z^2}{2iz}$. $\operatorname{Im}(\omega) = \frac{1}{2}(\omega - \bar{\omega})$

For $|z| = 1$, $\bar{\omega} = \frac{1+\bar{z}^2}{-2i\bar{z}} = \frac{1+(1/z)^2}{-2i(1/z)}$

$= \frac{z^2+1}{-2iz} = -\omega$

Thus, Im(ω) = $\frac{1}{2i}(\omega+\omega)=-i\omega$

$= -\frac{1}{2}\left(z+\frac{1}{z}\right)=-\frac{1}{2}(z+\bar{z})$

$=-\frac{1}{2}(2\cos\theta)=-\cos\theta$

[$\because$ $|z| = 1 \Rightarrow z = \cos\theta + i\sin\theta$]

As $z \neq 1$, $\theta \neq 0$ and as $z \neq -1$, $\theta \neq \pi$

$\therefore$ $a \in (-1, 1)$

25. $z=1-\bar{z} \quad \Rightarrow \quad z+\bar{z}=1$

$\Rightarrow$ 2 Re(z) = 1 $\Rightarrow$ Re (z) $=\frac{1}{2}=\cos\frac{\pi}{3}$

As $|z| = 1$, Re(z) = cos(π/3),

Im (z) = ± sin (π/3)

Thus, arg (z) = ± π/3

$\therefore$ statement-1 is false and statement-2 is true .

26. Im(z) + 1 = 0 Im(z) = –1

Let $z = a - i$

Now, $z+\sqrt{2}\,|z+1|+i=0$

$\Rightarrow a-i+\sqrt{2}\sqrt{(a+1)^2+1}+i=0$

$\Rightarrow a^2 = 2\,(a^2 + 2a + 2)$

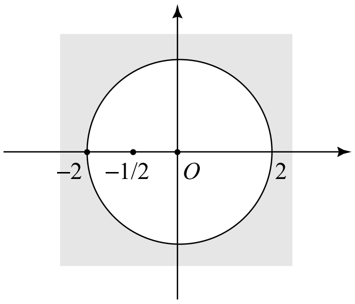
$\Rightarrow a^2 + 4a + 4 = 0 \quad \Rightarrow \quad (a + 2)^2 = 0$

$\therefore$ $a = -2$

Thus, $z = -2 - i \Rightarrow |z| = \sqrt{5}$

27. $\left|z+\frac{1}{2}\right|\geq|z|-\frac{1}{2}\geq\frac{3}{2}$

Minimum value 3/2 of $|z + 1/2|$ is attained when $z = -2$, and $3/2 \in (1, 2)$.



28. $w-\bar{w}z = k(1 - z)$

$\Rightarrow \quad w - k = z(\bar{w}-k)$

$\Rightarrow \quad |w - k| = |z||\bar{w}-k|$

$= |z||w - k|$

As Im (w) ≠ 0, $|w - k| \neq 0$

$\therefore$ $|z| = 1$

Also, $z \neq 1$ for otherwise $w=\bar{w}$

$\Rightarrow$ Im(w) = 0

29. arg $\left(\frac{z_1}{z_4}\right)$ + arg $\left(\frac{z_2}{z_3}\right)$

$=$ arg $\left(\frac{z_1z_2}{z_4z_3}\right)$ = arg $\left(\frac{|z_1|^2}{|z_3|^2}\right)$ = 0

30. Let $\frac{z-i}{z+i}=ik$, where $k \in \mathbf{R}$.

$\Rightarrow \quad z - i = ikz - k$

$\Rightarrow \quad z(1 - ik) = -k + i$

$\Rightarrow \quad z = \frac{-k+i}{1-ik}$

Note that $|z|^2=\frac{k^2+1}{1+k^2}=1$

$\Rightarrow \quad z\bar{z}=1 \quad \Rightarrow \quad \bar{z}=1/z$

Thus, $z+\frac{1}{z}=z+\bar{z}$, which is a real number.

Also, $z+\bar{z}=0$

$\Rightarrow$ 2Re(z) = 0 $\Rightarrow$ Re(z) = 0

$\Rightarrow$ $z = ai$ for some $a \in \mathbf{R}$.

But in this case

$\frac{z-i}{z+i}$ is a real number

Therefore, $z+\bar{z}\neq 0$.

31. $(1+ \alpha i)^2 = x + iy$

$\Rightarrow \quad 1 - \alpha^2 + 2\alpha i = x + iy$

$\Rightarrow \quad 1 - \alpha^2 = x, \quad 2\alpha = y$

$\Rightarrow \quad 1-(y/2)^2 = x$

$\Rightarrow \quad 4 - y^2 = 4x$

$\Rightarrow \quad y^2 + 4x - 4 = 0$

32. $\left|\frac{z_1-2z_2}{2-z_1\bar{z}_2}\right|=1$

$\Leftrightarrow |z_1-2z_2|^2=|2-z_1\bar{z}_2|^2$

$\Leftrightarrow |z_1|^2+4|z_2|^2-2\bar{z}_1z_2-2z_1\bar{z}_2$

$= 4 + |z_1|^2|\bar{z}_2|^2-2z_1\bar{z}_2-2\bar{z}_1z_2$

$\Leftrightarrow |z_1|^2\,|z_2|^2 - |z_1|^2 - 4|z_2|^2 + 4 = 0$

$\Leftrightarrow (|z_1|^2 - 4)\,(|z_1|^2 - 1) = 0$

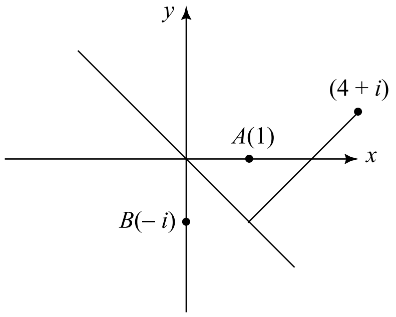
As $|z_2| \neq 1$, $|z_1|^2 - 4 = 0 \Rightarrow |z_1| = 2$

$\Rightarrow z_1$ lies on a circle of radius 2.

33. $|z - 1| = |z + i|$

$\Rightarrow$ z is equidistant from $A(1)$ and $B(-i)$

$\Rightarrow$ z lies on the line $y = -x$



As $z = 1$ satisfies the inequality

$|z - 1| \le |z + i|$,

we get $|z - 1| \le |z + i|$ represents the region lying above the line $x + y = 0$

Largest value of r is length of perpendicular from (4, 1) to the line $x + y = 0$, that is,

largest $r = \frac{|4+1|}{\sqrt{1+1}} = \frac{5}{\sqrt{2}} = \frac{5}{2}\sqrt{2}$.

34. Let $z = r(\cos\theta + i\sin\theta)$, so that

$\Rightarrow$ As z is non-real complex number, $\text{Im}(z) \ne 0$

$z^5 = r^5 [\cos(5\theta) + i\sin(5\theta)]$

$\Rightarrow \quad (\text{Im}(z))^5 = r^5 \cos^5\theta$

and $\quad \text{Im}(z^5) = r^5 \cos(5\theta)$

Thus, $\frac{\text{Im}(z^5)}{(\text{Im}(z))^5} = \frac{\cos(5\theta)}{\cos^5\theta}$

But $\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

$$\Rightarrow \quad \frac{\cos(5\theta)}{\cos^5\theta} = 16 - 20\sec^2\theta + 5\sec^4\theta$$
$$= 5(\sec^4\theta - 4\sec^2\theta + 4) - 4$$
$$= 5(\sec^2\theta - 2)^2 - 4 \ge -4$$

Thus, least value of

$\frac{\cos(5\theta)}{\cos^5\theta}$ is – 4 which is attained when $\theta = \pi/4$.

35. As $z = \frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, $\text{Re}(z) = 0$

that is, $z + \bar{z} = 0$

$$\Rightarrow \frac{2+3i\sin\theta}{1-2i\sin\theta} + \frac{2-3i\sin\theta}{1+2i\sin\theta} = 0$$

$\Rightarrow (2 + 3i\sin\theta)(1 + 2i\sin\theta) + (2 - 3i\sin\theta)(1 - 2i\sin\theta)$

$= 0$

$\Rightarrow 2(2 - 6\sin^2\theta) = 0 \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}}$

Thus, a value of θ for which z is purely imaginary is $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$

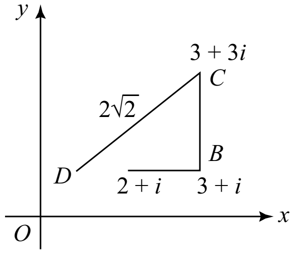
36. Refer figure

Coordinates of A are (2, 1) that of B are (3, 1) and C are (3, 3).

Coordinates of D are $(3 + 2\sqrt{2}\cos(5\pi/4), 3 + 2\sqrt{2}\sin(5\pi/4))$

$= (3 - 2, 3 - 2) = (1, 1)$

Thus, D is represented by $1 + i$



37. $z^3 = (1 + ai)^3$

$= 1 + 3ai - 3a^2 - ia^3$

As z^3 is real $3a - a^3 = 0$

$\Rightarrow a(\sqrt{3} - a)(a + \sqrt{3}) = 0$

$\Rightarrow a = \sqrt{3}$

Thus, $z = 1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

We have $S = 1 + z + z^2 + \ldots + z^{11}$

$$= \frac{1-z^{12}}{1-z}$$

But $z^{12} = 2^{12}(\cos(4\pi) + i\sin(4\pi))$

$= 2^{12}(1 + i0) = 2^{12}$

and $1 - z = -\sqrt{3}i$

Thus, $S = \frac{1-2^{12}}{-\sqrt{3}i}$

$$= \frac{4095}{\sqrt{3}}i = 1365\sqrt{3}i$$

Previous Years' B-Architecture Entrance Examination Questions

1. $\frac{5+i\sin\theta}{5-3i\sin\theta}$ is a real number

$$\Leftrightarrow \quad \frac{5+i\sin\theta}{5-3i\sin\theta} = \frac{5-i\sin\theta}{5+3i\sin\theta}$$

$\Leftrightarrow \quad (5 + i\sin\theta)(5 + 3i\sin\theta)$

$\quad = (5 - 3\,i\sin\theta)(5 - i\sin\theta)$

$\Leftrightarrow \quad 25 + 20\,i\sin\theta - 3\sin^2\theta$

$\quad = 25 - 20\,i\sin\theta - 3\sin^2\theta$

$\Leftrightarrow 40\,i\sin\theta = 0 \Leftrightarrow \sin\theta = 0$

This is possible when $\theta = -\pi$.

TIP Amongst the choices given, $-\pi$ is the only value which makes imaginary part of the numerator and denominator 0.

2. $|z| = 2$

Let $w = 3z + 2 + i$, then $z = (w - (2 + i))/3$

$|z| = 2 \quad \Rightarrow \quad |w - (2 + i)| = 6$

$\Rightarrow$ w lies on a circle with centre at $(2 + i)$ and radius 6.

3. Let $z = 2(\cos\theta + i\sin\theta)$. Now

$z + \frac{2}{z} = 2(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$

$\quad = 3\cos\theta + i\sin\theta$

$\Rightarrow \quad \left|z + \frac{2}{z}\right|^2 = 9\cos^2\theta + \sin^2\theta$

$= 1 + 4(1 + \cos 2\theta) \le 9$

$\therefore$ maximum possible value of $\left|z + \frac{2}{z}\right|$ is 3 which is attained when $z = 1$

4. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ then $\omega^3 = 1$

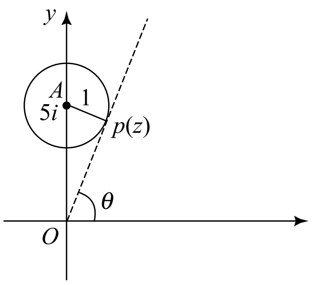
$z = 4 + 3\omega^{127} + 5\omega^{124}$

$\quad = 4 + 3(\omega^3)^{42}\omega + 5(\omega^3)^{41}\omega$

$\quad = 4 + 3\omega + 5\omega = 4 + 8\omega$

$\quad = 4 - 4 + 4\sqrt{3}i = 4\sqrt{3}i$

5. Principal argument is least at point P.



$\sin(\pi/2 - \theta) = \frac{AP}{OA}$

$\Rightarrow \quad \cos\theta = 1/5$

$\Rightarrow \quad \sin\theta = \sqrt{24}/5$

Now, $z = \sqrt{24}(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \text{Re}(z) = \frac{\sqrt{24}}{5} = \frac{2}{5}\sqrt{6}$

6. Area of triangle is

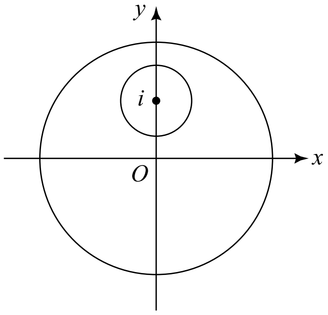
$$\Delta = \frac{1}{4}\left|\begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ z + iz & \bar{z} - i\bar{z} & 1 \end{vmatrix}\right|$$

Using $R_3 \to R_3 - R_1 - R_2$, we get

$$\Delta = \frac{1}{4}\left|\begin{vmatrix} z & \bar{z} & 1 \\ iz & -i\bar{z} & 1 \\ 0 & 0 & -1 \end{vmatrix}\right|$$

$$= \frac{1}{4}|-iz\bar{z} - iz\bar{z}| = \frac{1}{2}|z|^2$$

7. As $|i - (1 + 2i)| = \sqrt{2} < |4 - 2| = 2$, C_1 lies inside C_2



8. $z = i\left(i + \sqrt{2}\right) = -1 + \sqrt{2}i$

$\Rightarrow \quad (z + 1)^2 = -2 \quad \Rightarrow z^2 + 2z + 3 = 0$

We now divide

$z^4 + 4z^3 + 6z^2 + 4z$ by $z^2 + 2z + 3$.

$$\begin{array}{r|l} z^2 + 2z + 3 & z^4 + 4z^3 + 6z^2 + 4z \; \big| \; z^2 + 2z - 1 \\ & \underline{\overset{\ominus}{z^4} + \overset{\ominus}{2z^3} + \overset{\ominus}{3z^2}} \\ & \qquad 2z^3 + 3z^2 + 4z \\ & \qquad \underline{\overset{\ominus}{2z^3} + \overset{\ominus}{4z^2} + \overset{\ominus}{6z}} \\ & \qquad\qquad -z^2 - 2z \\ & \qquad\qquad \underline{\overset{\oplus}{-z^2} \overset{\oplus}{-2z} \overset{\oplus}{-3}} \\ & \qquad\qquad\qquad 3 \end{array}$$

$\therefore \quad z^4 + 4z^3 + 6z^2 + 4z$

$\quad = (z^2 + 2z + 3)(z^2 + 2z - 1) + 3$

$\quad = 0 + 3 = 3$

9. $|z| = 1$, $\arg(z) = \theta$

$\Rightarrow z = \cos\theta + i\sin\theta$

Now, $w = \frac{z(1 - \bar{z})}{\bar{z}(1 + z)} = \frac{z - z\bar{z}}{z + z\bar{z}}$

$$= \frac{z-1}{\bar{z}+1}$$

$$\bar{w} = \frac{\bar{z}-1}{z+1}$$

Now, $2\text{Re}(w) = w + \bar{w} = \dfrac{z-1}{\bar{z}+1} + \dfrac{\bar{z}-1}{z+1}$

$$= \frac{z^2 - 1 + \bar{z}^2 - 1}{(z+1)(\bar{z}+1)}$$

$$= \frac{2\cos 2\theta - 2}{z\bar{z} + z + \bar{z} + 1}$$

$$= \frac{2(\cos 2\theta - 1)}{2(\cos\theta + 1)}$$

$$= \frac{-2\sin^2(\theta)}{2\cos^2(\theta/2)} = -4\sin^2\left(\frac{\theta}{2}\right)$$

$\Rightarrow$ $\text{Re}(w) = -2\sin^2\left(\dfrac{\theta}{2}\right)$

10. $\left|\sqrt{2} + i\right|^2 = |z_1 + z_2 + z_3|^2$

$\Rightarrow 3 = |z_1|^2 + |z_2|^2 + |z_3|^2 + 2\text{Re}\,(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1)$

$\Rightarrow 3 = 1 + 1 + 1 + 2\ \text{Re}\,(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1)$

$\Rightarrow \text{Re}\,(z_1\bar{z}_2 + z_2\bar{z}_3 + z_3\bar{z}_1) = 0$

11. $z\ (iz_1 - 1) = z_1 + 1$

$\Rightarrow z_1(iz - 1) = 1 + z$

$\Rightarrow z_1 = \dfrac{1+z}{iz-1}$

$\Rightarrow \left|\dfrac{1+z}{iz-1}\right| = |z_1| < 1$

$\Rightarrow |1 + z|^2 < |1 - iz|^2$

$\Rightarrow 1 + z + \bar{z} + z\bar{z} < 1 - iz + i\bar{z} + z\bar{z}$

$\Rightarrow 2\ \text{Re}(z) < -\ i(2i\ \text{Im}(z))$

$\Rightarrow \text{Re}(z) - \text{Im}(z) < 0$

CHAPTER THREE

Quadratic Equations

QUADRATIC EQUATIONS

An equation of the form

$$ax^2 + bx + c = 0 \tag{1}$$

where $a \neq 0$, $a, b, c \in \mathbf{C}$, the set of complex numbers, is called a *quadratic equation*.

A *root* of the quadratic equation (1) is a complex number α such that

$$a\alpha^2 + b\alpha + c = 0$$

The quantity $D = b^2 - 4ac$ is known as the *discriminant* of the equation (1).

The roots of equation (1) are given by the *quadratic formula*

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

NATURE OF ROOTS

1. If $a, b, c \in \mathbf{R}$ and $a \neq 0$. Then the followings hold good:
   - (a) The equation (1) has real and distinct roots if and only if $D > 0$.
   - (b) The equation (1) has real and equal roots if and only if $D = 0$.
   - (c) The equation (1) has complex roots with non-zero imaginary parts if and only if $D < 0$.
   - (d) $p + iq$ $(p, q \in \mathbf{R}, q \neq 0)$ is a root of equation (1) if and only if $p - iq$ is a root of equation (1).
2. If $a, b, c \in \mathbf{Q}$ and D is a perfect square of a rational number, then equation (1) has rational roots.
3. If $a, b, c \in \mathbf{Q}$ and $p + \sqrt{q}$ $(p, q \in \mathbf{Q})$ is an irrational root of equation (1) then $p - \sqrt{q}$ is also a root of equation (1).
4. If $a = 1$, $b, c \in \mathbf{I}$, the set of integers, and the roots of equation (1) are rational numbers, then these roots must be integers.
5. If equation (1) is satisfied by more than two distinct complex numbers, then equation (1) becomes an identity, that is $a = b = c = 0$.

RELATION BETWEEN ROOTS AND COEFFICIENTS

If α and β are the roots of the quadratic equation (1), then

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}$$

Note that a quadratic equation whose roots are α and β is given by

$$(x - \alpha)(x - \beta) = 0$$

QUADRATIC EXPRESSION AND ITS GRAPH

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{R}$ and $a \neq 0$.

We have

$$\begin{aligned} f(x) &= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] \\ &= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2}\right] \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right] \end{aligned} \tag{2}$$

When is a Quadratic Expression always positive Or always negative?

It follows from (2) that $f(x) > 0$ (< 0) $\forall x \in \mathbf{R}$ if and only if $a > 0$ (< 0) and $D = b^2 - 4ac < 0$. See Fig. 3.1 and (Fig. 3.2). Also, it follows from (2) that $f(x) \geq 0$ (≤ 0) $\forall x \in \mathbf{R}$ if and only if $a > 0$ (< 0) and $D = b^2 - 4ac = 0$. In this case $f(x) > 0$ (< 0) for each $x \in \mathbf{R}$, $x \neq -b/2a$. Also, in this case the graph of $y = f(x)$ will touch the x-axis at $x = -b/2a$.

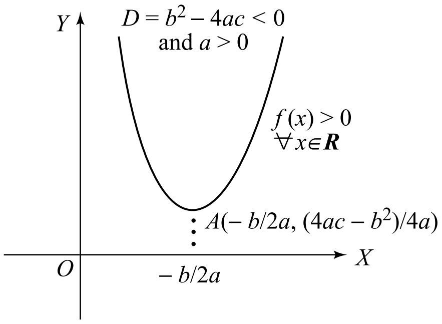

Y
$D = b^2 - 4ac < 0$
and $a > 0$
$f(x) > 0$
$\forall x \in \mathbf{R}$
$A(-b/2a, (4ac - b^2)/4a)$
O
$-b/2a$
X

Fig. 3.1

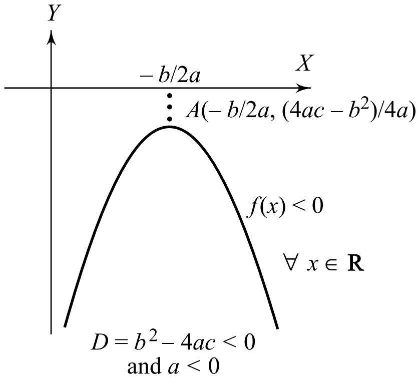

Y
$-b/2a$
X
$A(-b/2a, (4ac - b^2)/4a)$
$f(x) < 0$
$\forall x \in \mathbf{R}$
$D = b^2 - 4ac < 0$
and $a < 0$

Fig. 3.2

Sign of a Quadratic Expression

If $D = b^2 - 4ac > 0$, then (2) can be written as

$$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2\right]$$

$$= a\left[\left(x + \frac{b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b - \sqrt{b^2 - 4ac}}{2a}\right)\right]$$

$$= a\,(x - \alpha)\,(x - \beta)$$

where $\alpha = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ and $\beta = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

If $D = b^2 - 4ac > 0$ and $a > 0$ then

$$f(x)\begin{cases} > 0 & \text{for } x < \alpha \text{ or } x > \beta \\ < 0 & \text{for } \alpha < x < \beta \\ = 0 & \text{for } x = \alpha, \beta \end{cases}$$

See Fig. 3.3

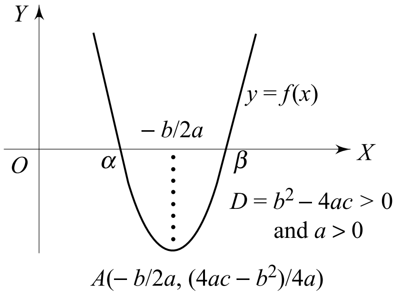

Y
$y = f(x)$
$-b/2a$
X
O
α
β
$D = b^2 - 4ac > 0$
and $a > 0$
$A(-b/2a, (4ac - b^2)/4a)$

Fig. 3.3

If $D = b^2 - 4ac > 0$ and $a < 0$ then

$$f(x)\begin{cases} < 0 & \text{for } x < \beta \text{ or } x > \alpha \\ > 0 & \text{for } \beta < x < \alpha \\ = 0 & \text{for } x = \alpha, \beta \end{cases}$$

See Fig. 3.4

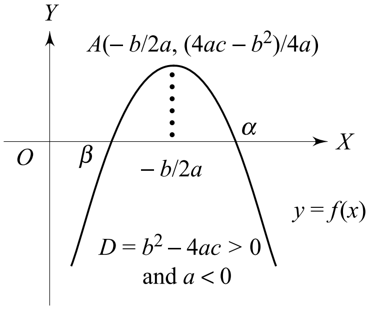

Y
$A(-b/2a, (4ac - b^2)/4a)$
α
X
O
β
$-b/2a$
$y = f(x)$
$D = b^2 - 4ac > 0$
and $a < 0$

Fig. 3.4

Note that if $a > 0$, then $f(x)$ attains the least value at $x = -b/2a$. This least value is given by

$$f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}$$

If $a < 0$, then $f(x)$ attains the a greatest value at $x = -b/2a$. This greatest value is given by

$$f\left(-\frac{b}{2a}\right) = \frac{4ac - b^2}{4a}$$

POSITION OF ROOTS OF A QUADRATIC EQUATION

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{R}$ be a quadratic expression and let k be a real number.

Conditions for Both the Roots to be more than a Real Number *k*.

If $a > 0$ and $b^2 - 4ac > 0$ then the parabola $y = ax^2 + bx + c$ opens upwards and intersect the x-axis in α and β where

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In this case both the roots α and β will be more than k if k lies to left of both α and β. See Fig. 3.5

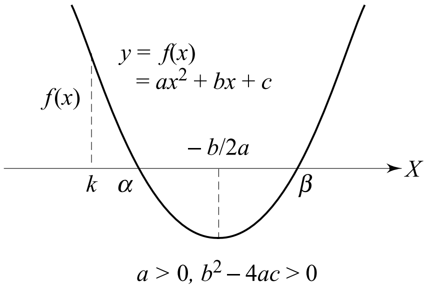

$y = f(x)$
$= ax^2 + bx + c$
$f(x)$
$-b/2a$
X
k
α
β
$a > 0,\ b^2 - 4ac > 0$

Fig. 3.5

From the Fig. 3.5, we note that both the roots are more than k if and only if

(i) $D > 0$ (ii) $k < \frac{-b}{2a}$ (iii) $f(k) > 0$

In case $a < 0$, both the roots will be more than k (see Fig. 3.6) if and only if

(i) $D > 0$ (ii) $k < \frac{-b}{2a}$ (iii) $f(k) < 0$

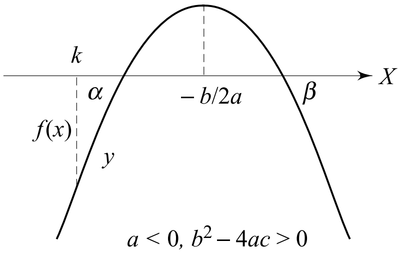


Fig. 3.6

Combining the above two sets, we get both the roots of $ax^2 + bx + c = 0$ are more than a real number k if only if

(i) $D > 0$ (ii) $k < \frac{-b}{2a}$ (iii) $af(k) > 0$

Conditions for Both the Roots to be less than a Real Number *k*

Both the roots of $ax^2 + bx + c = 0$ are less than a real number k if and only if

(i) $D > 0$ (ii) $k > \frac{-b}{2a}$ (iii) $af(k) > 0$

Note

1. Both the roots of $ax^2 + bx + c = 0$ are **positive** if and only if

   $D \geq 0$, $-\frac{b}{a} > 0$ and $\frac{c}{a} > 0$
2. Both the roots of $ax^2 + bx + c = 0$ are negative if and only if

   $D \geq 0$, $-\frac{b}{a} < 0$ and $\frac{c}{a} > 0$

CONDITIONS FOR A NUMBER *k* TO LIE BETWEEN THE ROOTS OF A QUADRATIC EQUATION

The real number k lies between the roots of the quadratic equation $f(x) = ax^2 + bx + c = 0$ if and only if a and $f(k)$ are of opposite signs, that is, if and only if

(i) $a > 0$ (ii) $D > 0$ (iii) $f(k) < 0$ [See Fig. 3.7]

or

(i) $a < 0$ (ii) $D > 0$ (iii) $f(k) > 0$ [See Fig. 3.8]

Combining, we may say k lies between the roots of $f(x) = ax^2 + bx + c = 0$ if and only if

(i) $D > 0$ (ii) $af(k) < 0$

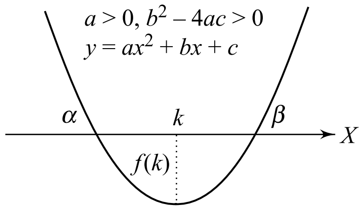


Fig. 3.7

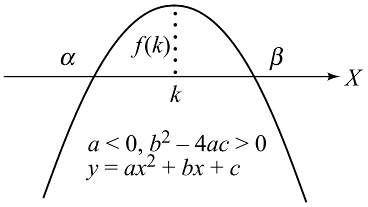


Fig. 3.8

CONDITIONS FOR EXACTLY ONE ROOT OF A QUADRATIC EQUATION TO LIE IN THE INTERVAL (k_1, k_2) WHERE $k_1 < k_2$

If $a > 0$, then exactly one root of $f(x) = ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if and only if $f(k_1) > 0$ and $f(k_2) < 0$. Also, exactly one root lies in the interval (k_1, k_2) if and only if $f(k_1) < 0$ and $f(k_2) > 0$. See Fig. 3.9. Thus, if $a > 0$, exactly one root of $f(x) = ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if and only if $f(k_1)\,f(k_2) < 0$.

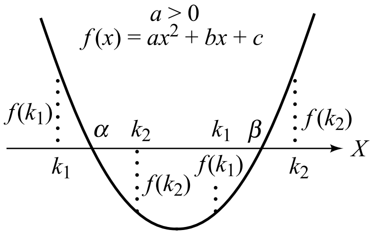


Fig. 3.9

Similarly, if $a < 0$, exactly one of the roots of $f(x) = ax^2 + bx + c = 0$ lies in the interval (k_1, k_2) if $f(k_1)\,f(k_2) < 0$.

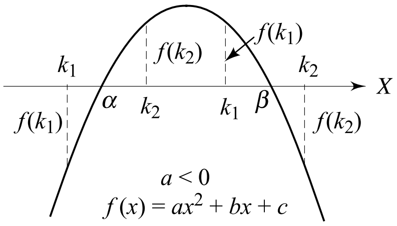


Fig. 3.10

Hence, if $f(k_1)\,f(k_2) < 0$, then exactly one root of $f(x) = 0$ lies in the interval (k_1, k_2).

CONDITIONS FOR BOTH THE ROOTS OF A QUADRATIC EQUATION TO LIE IN THE INTERVAL (k_1, k_2) WHERE $k_1 < k_2$.

If $a > 0$, both the roots of $f(x) = ax^2 + bx + c = 0$ lie in the interval (k_1, k_2) if and only if

(i) $D > 0$ (ii) $k_1 < -\dfrac{b}{2a} < k_2$

(iii) $f(k_1) > 0$ and $f(k_2) > 0$

See Fig. 3.11

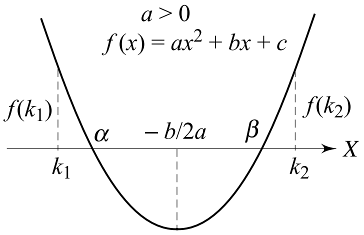


Fig. 3.11

In case $a < 0$, the conditions read as

(i) $D > 0$ (ii) $k_1 < -\dfrac{b}{2a} < k_2$

(iii) $f(k_1) < 0$ and $f(k_2) < 0$

See Fig. 3.12

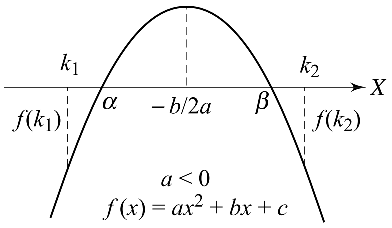


Fig. 3.12

Some Useful Tips

Let quadratic equation be

$$f(x) = ax^2 + bx + c = 0 \qquad (1)$$

where $a, b, c \in \mathbf{R}$, $a \neq 0$.

Let roots of (1) be α, β, and $D = b^2 - 4ac$.

1. Both the roots are positive

   $\Leftrightarrow D \geq 0,\ -\dfrac{b}{2a} = \alpha + \beta > 0,\ \dfrac{c}{a} = \alpha\beta > 0$
2. Both the roots are negative

   $\Leftrightarrow D \geq 0,\ -\dfrac{b}{2a} = \alpha + \beta < 0,\ \dfrac{c}{a} = \alpha\beta > 0$
3. Both the roots lie in the interval (p, q).

   $\Leftrightarrow D \geq 0, p < -\dfrac{b}{2a} < q, af(p) > 0, af(q) > 0$
4. If one of α, β is real, then other must be real.
5. If sum of the coefficients of a quadratic equation is 0, then 1 is a root of the quadratic equation.

Illustration 1

1 is a root of $(b - c)x^2 + (c - a)x + a - b = 0$ as $(b - c) + (c - a) + (a - b) = 0$.

Finding Range of a Rational Function

Let $f(x) = ax^2 + bx + c$, $g(x) = px^2 + qx + r$

where $a, b, c, p, q, r \in \mathbf{R}$ and one of $a, p \neq 0$.

To find range of

$$R(x) = \frac{f(x)}{g(x)},\ x \in \mathbf{R}, \text{ put}$$

$$y = \frac{ax^2 + bx + c}{px^2 + qx + r},$$

$\Rightarrow$ $(a - py)x^2 + (b - qy)x + (c - ry) = 0$.

Since x is real,

$(b - qy)^2 - 4(a - py)(c - ry) \geq 0$

We use this inequality to obtain range of $R(x)$.

Illustration 2

Find range of

$$y = \frac{x^2 - 2x + 5}{x^2 + 2x + 7}$$

$\Rightarrow$ $(x^2 + 2x + 7)y = x^2 - 2x + 5$

$\Rightarrow$ $(y - 1)x^2 + 2x(y + 1) + 7y - 5 = 0$

As x is real,

$4(y + 1)^2 - 4(y - 1)(7y - 5) \geq 0$

$\Rightarrow$ $(y^2 + 2y + 1) - (7y^2 - 12y + 5) \geq 0$

$\Rightarrow$ $3y^2 - 7y + 2 \leq 0$

$\Rightarrow$ $(3y - 1)(y - 2) \leq 0$

$\Rightarrow$ $1/3 \leq y \leq 2$

Some Useful Tips

Suppose range of $\dfrac{ax^2 + bx + c}{px^2 + qx + r}$ is the set S and $h(x)$ takes all real values, then range of $\dfrac{a(h(x))^2 + b(h(x)) + c}{p(h(x))^2 + q(h(x)) + r}$ is also S.

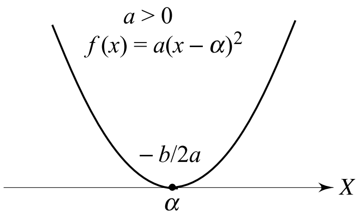
Illustration 3

Range of $\dfrac{\tan^2 x - 2\tan x + 5}{\tan^2 x + 2\tan x + 7}$ is $\dfrac{1}{3} \leq y \leq 2$.

However, range of $\dfrac{\sin^2 x - 2\sin x + 5}{\sin^2 x + 2\sin x + 7}$ is **not** [1/3, 2] as $-1 \leq \sin x < 1$ will curtail its range. For instance try $y = 2$.

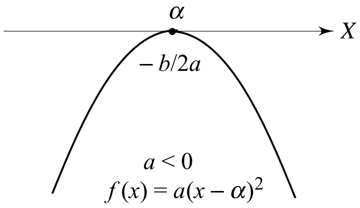
CONDITIONS FOR A QUADRATIC EQUATION TO HAVE A REPEATED ROOT

The quadratic equation $f(x) = ax^2 + bx + c = 0$, $a \neq 0$ has α as a repeated if and only if $f(\alpha) = 0$ and $f'(\alpha) = 0$. In this case $f(x) = a(x - \alpha)^2$. In fact $\alpha = -b/2a$. See Fig. 3.13 and Fig. 3.14.



$f(\alpha) = 0, \; f'(\alpha) = 0$

Fig. 3.13



$f(\alpha) = 0, \; f'(\alpha) = 0$

Fig. 3.14

CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE A COMMON ROOT

Suppose that the quadratic equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ (where $a, a' \neq 0$ and $ab' - a'b \neq 0$) have a common root. Let this common root be α. Then

$$a\alpha^2 + b\alpha + c = 0 \quad \text{and} \quad a'\alpha^2 + b'\alpha + c' = 0$$

Solving the above equations, we get

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

$$\Rightarrow \qquad \alpha^2 = \frac{bc' - b'c}{ab' - a'b} \quad \text{and } \alpha = \frac{a'c - ac'}{ab' - a'b}$$

Eliminating α we get

$$\frac{(a'c - ac')^2}{(ab' - a'b)^2} = \frac{bc' - b'c}{ab' - a'b}$$

$$\Rightarrow \qquad (a'c - ac')^2 = (bc' - b'c)(ab' - a'b)$$

This is the required condition for two quadratic equation to have a common root. See Fig. 3.15

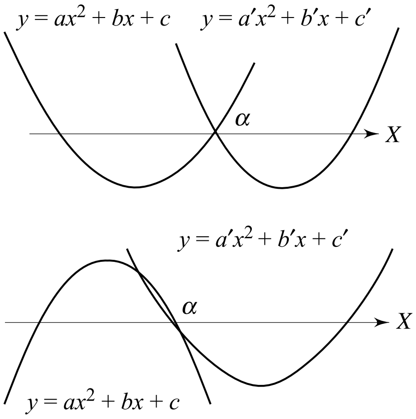


Fig. 3.15

TIP

How to Obtain the Common Root?

Make coefficients of x^2 in both the equations same and subtract one equation from the other to obtain a linear equation in x. Solve it for x to obtain the common root.

CONDITION FOR TWO QUADRATIC EQUATIONS TO HAVE THE SAME ROOTS

Two quadratic equations $ax^2 + bx + c = 0$. and $a'x^2 + b'x + c' = 0$ have the same roots if and only if

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

EQUATIONS OF HIGHER DEGREE

The equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$$

where $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbf{C}$, the set of complex numbers, and $a_0 \neq 0$, is said to be an equation of degree n. An equation of degree n has exactly n roots. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbf{C}$ be the n roots of (1). Then

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$$

Also $\qquad \alpha_1 + \alpha_2 + \ldots + \alpha_n = -\dfrac{a_1}{a_0}$

and $\qquad \alpha_1\alpha_2 \ldots \alpha_n = (-1)^n \dfrac{a_n}{a_0}$.

CUBIC AND BIQUADRATIC EQUATIONS

If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then

$$\alpha + \beta + \gamma = -\frac{b}{a}, \; \beta\gamma + \gamma\alpha + \alpha\beta = \frac{c}{a} \text{ and } \alpha\beta\gamma = -\frac{d}{a}.$$

Also, if $\alpha, \beta, \gamma, \delta$ are the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$\alpha + \beta + \gamma + \delta = \frac{-b}{a}, (\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = \frac{c}{a}$$

$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = \frac{-d}{a}, \alpha\beta\gamma\delta = \frac{e}{a}.$$

TRANSFORMATION OF EQUATIONS

We now list some of the rules to form an equation whose roots are given in terms of the roots of another equation. Let given equation be

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0 \qquad (1)$$

Rule 1: To form an equation whose roots are $k(\neq 0)$ times roots of the equations in (1), replace x by $\frac{x}{k}$ in (1).

Rule 2: To form an equation whose roots are the negatives of the roots in equation (1), replace x by $-x$ in (1). Alternatively, change the sign of the coefficients of $x^{n-1}, x^{n-3}, x^{n-5}, \ldots$ etc. in (1).

Rule 3: To form an equation whose roots are k more than the roots of equation in (1), replace x by $x - k$ in (1).

Rule 4: To form an equation whose roots are reciprocals of the roots in equation (1), replace x by $\frac{1}{x}$ in (1) and then multiply both the sides by x^n.

Rule 5: To form an equation whose roots are square of the roots of the equation in (1) proceed as follows:

Step 1 Replace x by $\sqrt{x}$ in (1)

Step 2 Collect all the terms involving $\sqrt{x}$ on one side.

Step 3 Square both the sides and simplify.

For instance, to form an equation whose roots are squares of the roots of $x^3 + 2x^2 - x + 2 = 0$, replace x by $\sqrt{x}$ to obtain

$$x\sqrt{x} + 2x - \sqrt{x} + 2 = 0$$

$$\Rightarrow \qquad \sqrt{x}(x - 1) = -2(x + 1)$$

Squaring we get

$$x(x - 1)^2 = 4(x + 1)^2$$

or

$$x^3 - 6x^2 - 7x - 4 = 0$$

Rule 6: To form an equation whose roots are cubes of the roots of the equation in (1) proceed as follows:

Step 1 Replace x by $x^{1/3}$.

Step 2 Collect all the terms involving $x^{1/3}$ and $x^{2/3}$ on one side.

Step 3 Cube both the sides and simplify.

DESCARTES RULE OF SIGNS FOR THE ROOTS OF A POLYNOMIAL

Rule 1: The maximum number of positive real roots of a polynomial equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$$

is the number of changes of the signs of coefficients from positive to negative and negative to positive. For instance, in the equation $x^3 + 3x^2 + 7x - 11 = 0$ the signs of coefficients are

$$+ + + -$$

As there is just one change of sign, the number of positive roots of $x^3 + 3x^2 + 7x - 11 = 0$ is at most 1.

Rule 2: The maximum number of negative roots of the polynomial equation $f(x) = 0$ is the number of changes from positive to negative and negative to positive in the signs of coefficients of the equation $f(-x) = 0$.

SOME HINTS FOR SOLVING POLYNOMIAL EQUATIONS

1. To solve an equation of the form $(x - a)^4 + (x - b)^4 = A$
   put $\quad y = x - \frac{a+b}{2}$
   In general to solve an equation of the form
   $$(x - a)^{2n} + (x - b)^{2n} = A$$
   where n is a positive integer, we put $y = x - \frac{a+b}{2}$.
2. To solve an equation of the form
   $$a_0 f(x)^{2n} + a_1 (f(x))^n + a_2 = 0 \qquad (1)$$
   we put $(f(x))^n = y$ and solve $a_0 y^2 + a_1 y + a_2 = 0$ to obtain its roots y_1 and y_2.
   Finally, to obtain solutions of (1) we solve,
   $$(f(x))^n = y_1$$
   and $\quad (f(x))^n = y_2$
3. An equation of the form
   $$(ax^2 + bx + c_1)(ax^2 + bx + c_2) \ldots (ax^2 + bx + c_n) = A$$
   where $c_1, c_2, \ldots, c_n, A \in \mathbf{R}$, can be solved by putting $ax^2 + bx = y$.
4. An equation of the form
   $$(x - a)(x - b)(x - c)(x - d) = Ax^2$$
   where $ab = cd$, can be reduced to a product of two quadratic polynomials by putting $y = x + \frac{ab}{2}$.
5. An equation of the form
   $$(x - a)(x - b)(x - c)(x - d) = A$$
   where $a < b < c < d$, $b - a = d - c$
   can be solved by change of variable
   $$y = \frac{(x-a) + (x-b) + (x-c) + (x-d)}{4}$$
   $$= x - \frac{1}{4}(a + b + c + d)$$
6. A polynomial $f(x, y)$ is said to be symmetric if $f(x, y) = f(y, x) \;\forall\; x, y$.
   A symmetric polynomials can be represented as a function of $x + y$ and xy.

Equations Reducible to Quadratic

1. To solve an equation of the type $ax^4 + bx^2 + c = 0$, put $x^2 = y$.
2. To solve an equation of the type $a\,(p(x))^2 + bp(x) + c = 0$ (where $p(x)$ is an expression in x) put $p(x) = y$.
3. To solve an equation of the type
$$ap(x) + \frac{b}{p(x)} + c = 0$$
where $p(x)$ is an expression in x, put $p(x) = y$. This reduces the equation to
$$ay^2 + cy + b = 0.$$
4. To solve an equation of the form
$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$$
put $\quad x + \dfrac{1}{x} = y$
and to solve
$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x - \frac{1}{x}\right) + c = 0$$
put $\quad x - \dfrac{1}{x} = y$
5. To solve reciprocal equation of the type
$$ax^4 + bx^3 + cx^2 + bx + a = 0, a \neq 0$$
we divide the equation by x^2, to obtain
$$a\left(x^2 + \frac{1}{x^2}\right) + b\left(x + \frac{1}{x}\right) + c = 0$$
and then put $\quad x + \dfrac{1}{x} = y.$
6. To solve equation of the type
$$(x + a)(x + b)(x + c)(x + d) + k = 0$$
where $a + b = c + d$, put $x^2 + (a + b)x = y$
7. To solve equation of the type
$$\sqrt{ax+b} = cx + d$$
or
$$\sqrt{ax^2+bx+c} = dx + e$$
square both the sides and solve for x.
8. To solve equation of the type
$$\sqrt{ax+b} \pm \sqrt{cx+d} = e$$
proceed as follows.

Step 1 Transfer one of the radical to the other side and square both the sides.

Step 2 Keep the expression with radical sign on one side and transfer the remaining expression on the other side.

Step 3 Now solve as in 7 above.

USE OF CONTINUITY AND DIFFERENTIABILITY TO FIND ROOTS OF A POLYNOMIAL EQUATION

Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2x^{n-2} + \dots + a_{n-1} x + a_n$, then f is continuous on **R**.

Since f is continuous on **R**, we may use the intermediate value theorem to find whether or not f has a real root in an interval (a, b).

If there exist a and b such that $a < b$ and $f(a) f(b) < 0$, then there exists at least one $c \in (a, b)$ such that $f(c) = 0$.

Also, if $\lim\limits_{x\to-\infty} f(x)$ and $f(a)$ are of opposite signs, then at least one root of $f(x) = 0$ lies in the interval $(-\infty, a)$. Also, if $f(a)$ and $\lim\limits_{x\to\infty} f(x)$ are of opposite signs, then at least one root of $f(x) = 0$ lies in the interval (a, ∞).

Result 1 If $f(x) = 0$ has a root α of multiplicity r (where $r > 1$), then we can write
$$f(x) = (x - \alpha)^r g(x)$$
where $g(\alpha) \neq 0$.

Also, $f'(x) = 0$ has α as a root with multiplicity $r - 1$.

Result 2 If $f(x) = 0$ has n real roots, then $f'(x) = 0$ has $(n - 1)$ real roots.

It follows immediately using Result 1 and Rolle's Theorem.

Result 3 If $f(x) = 0$ has n distinct real roots, we can write
$$f(x) = a_0 (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$$
where $\alpha_1, \alpha_2, \dots \alpha_n$ are n distinct roots of $f(x) = 0$.

We also have
$$\frac{f'(x)}{f(x)} = \sum_{k=1}^{n} \frac{1}{x - \alpha_k}$$

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: Solution set of
$\sqrt{3-x} = -x^2 - x - 1, x \in$ **R** is

(a) $(-1, \infty)$ (b) $(0, 1)$
(c) $(-1, 3/4)$ (d) ϕ

Ans. (d)

Solution: Note that LHS $= \sqrt{3-x} \geq 0 \ \forall \ x \leq 3$.

$$\text{and RHS} = -\left(x - \frac{1}{2}\right)^2 - \frac{5}{4} < 0 \ \forall \ x.$$

Thus, the given equation has no solution.

Example 2: Number of solutions of

$$x - \frac{5}{x-2} = 2 - \frac{5}{x-2} \qquad (1)$$

is

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (a)

Solution: The equation (1) is defined when $x \neq 2$. But when $x \neq 2$. We can cancel $5/(x-2)$ from both the sides of (1) to obtain $x = 2$
Thus, (1) has no solution.

Example 3: The number of real solutions of $x^2 + 5|x| + 4 = 0$ is

(a) 0 (b) 1
(c) 2 (d) 4

Ans. (a)

Solution: If $x \in \mathbf{R}$, then $x^2 + 5|x| + 4 \geq 4$.
Thus, the given equation has no solution.

Example 4: If $\frac{1}{3-4i}$ is a root of $ax^2 + bx + c = 0$, $(a, b, c \in \mathbf{R}, a \neq 0)$, then

(a) $b + 6c = 0$ (b) $b = 6c$
(c) $a + 25c = 0$ (d) $b^2 = 6c$

Ans. (a)

Solution: TIP: As $a, b, c \in \mathbf{R}$, the other root must $\frac{1}{3+4i}$.

$$\text{Thus, } \frac{1}{3-4i} + \frac{1}{3+4i} = -\frac{b}{a},$$

$$\text{and } \frac{1}{3-4i} \cdot \frac{1}{3+4i} = \frac{c}{a}$$

$$\Rightarrow \quad \frac{6}{25} = -\frac{b}{a} \text{ and } \frac{1}{25} = \frac{c}{a}$$

$$\text{Now, } \quad 6\left(\frac{c}{a}\right) = -\frac{b}{a} \Rightarrow b + 6c = 0.$$

Also, $a - 25c = 0$

Example 5: If $1 - p$ is a root of the quadratic equation $x^2 + px + 1 - p = 0$, then its roots are

(a) 0, 1 (b) –1, 1
(c) 0, – 1 (d) –2, 1

Ans. (c)

Solution: $(1 - p)^2 + p(1 - p) + (1 - p) = 0$
$\Rightarrow \quad (1 - p)[1 - p + p + 1] = 0 \Rightarrow 2(1 - p) = 0$
$\Rightarrow \quad p = 1$
Thus, equation becomes $x^2 + x = 0$
$\Rightarrow \quad x = 0$ or -1

Example 6: Suppose $p \in \mathbf{R}$. If the roots of $3x^2 + 2x + p(1 - p) = 0$ are of opposite signs, then p must lie in the interval

(a) $(-\infty, 0)$ (b) $(0, 1)$
(c) $(1, \infty)$ (d) $(2, \infty)$

Ans. (b)

Solution: We must have $\frac{1}{3}p(1-p) < 0$

$\Rightarrow \quad p(1 - p) < 0 \Rightarrow 0 < p < 1.$

Example 7: If α, β are the roots of $(x - a)(x - b) + c = 0$, $c \neq 0$, then roots of $(\alpha\beta - c)x^2 + (\alpha + \beta)x + 1 = 0$ are

(a) $1/a, 1/b$ (b) $-1/a, -1/b$
(c) $1/a, -1/b$ (d) $-1/a, 1/b$

Ans. (b)

Solution: α, β are roots of $x^2 - (a + b)x + ab + c = 0$.
$\therefore \quad \alpha + \beta = a + b, \ \alpha\beta = ab + c$
The equation

$$(\alpha\beta - c)x^2 + (\alpha + \beta)x + 1 = 0$$

becomes

$$abx^2 + (a + b)x + 1 = 0$$

$\Rightarrow \quad (ax + 1)(bx + 1) = 0$
$\Rightarrow \quad x = -1/a, -1/b$

Example 8: If α, β are roots $x^2 + px + q = 0$, then value of $\alpha^3 + \beta^3$ is

(a) $3pq + p^3$ (b) $3pq - p^3$
(c) $3pq$ (d) $p^3 - 3pq$

Ans. (b)

Solution: $\alpha + \beta = -p, \ \alpha\beta = q$.
Now, $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
$\qquad = -p^3 + 3pq = 3pq - p^3$

Example 9: If $a \in \mathbf{R}$ and both the roots of $x^2 - 6ax + 9a^2 + 2a - 2 = 0$ exceed 3, then a lies in the interval

(a) $(1, \infty)$ (b) $(2, \infty)$
(c) $(11/9, \infty)$ (d) ϕ

Ans. (d)

Solution: Given equation can be written as $(x - 3a)^2 = 2(1 - a)$
We must have $1 - a \geq 0$ or $a \leq 1$.
Also, $\quad x = 3a \pm \sqrt{2}\sqrt{1-a}$.
Both the roots will exceed 3, if smaller of the two roots viz $3a - \sqrt{2}\sqrt{1-a}$ exceeds 3, that is, $3a - \sqrt{2}\sqrt{1-a} > 3$

$$\Rightarrow \quad 3(a - 1) > \sqrt{2}\sqrt{1-a}. \qquad (1)$$

For $a = 1$, this is not possible. For $a < 1$, we can write (1) as

$$-3\left(\sqrt{1-a}\right)^2 > \sqrt{2}\sqrt{1-a}$$

which is no possible

Thus, there is not real value of a.

Example 10: If $x = \sqrt{7 - 4\sqrt{3}}$, then $x + \frac{1}{x}$ is equal to:

(a) 2 (b) $3\sqrt{7}$

(c) 4 (d) $4\sqrt{7}$

Ans. (c)

Solution: $7 - 4\sqrt{3} = 2^2 + (\sqrt{3})^2 - 4\sqrt{3}$

$$= (2 - \sqrt{3})^2$$

$$\therefore \quad x = \sqrt{7 - 4\sqrt{3}} = 2 - \sqrt{3}$$

$$\text{and} \quad \frac{1}{x} = \frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}$$

Thus, $x + \frac{1}{x} = 4$

Example 11: Suppose α, β are roots of $ax^2 + bx + c = 0$ then roots of $a(x-2)^2 - b(x-2)(x-3) + c(x-3)^2 = 0$ are

(a) $\frac{2+\alpha}{1+\beta}, \frac{2+\beta}{1+\alpha}$ (b) $\frac{2+\alpha}{3+\beta}, \frac{3+\alpha}{2+\beta}$

(c) $\frac{2+3\alpha}{2+\alpha}, \frac{2+3\beta}{2+\beta}$ (d) $\frac{2+3\alpha}{1+\alpha}, \frac{2+3\beta}{1+\beta}$

Ans. (d)

Solution: Write the equation as

$$a\left(-\frac{x-2}{x-3}\right)^2 + b\left(-\frac{x-2}{x-3}\right) + c = 0$$

Now, $-\frac{x-2}{x-3} = \alpha, \beta$

$$\Rightarrow \quad x = \frac{2+3\alpha}{1+\alpha}, \frac{2+3\beta}{1+\beta}$$

Example 12: If $x, y, z \in \mathbf{R}$ then the least value of the expression $E = 3x^2 + 5y^2 + 4z^2 - 6x + 20y - 8z - 3$ is:

(a) – 15 (b) – 30

(c) – 45 (d) – 60

Ans. (b)

Solution: We can write

$$\begin{aligned} E &= 3(x^2 - 2x + 1) + 5(y^2 + 4y + 4) \\ &\quad + 4(z^2 - 2z + 1) - 30 \\ &= 3(x-1)^2 + 5(y+2)^2 + 4(z-1)^2 - 30 \geq -30 \end{aligned}$$

The least value is attained when

$$x = 1, y = -2, z = 1.$$

Example 13: The number real of roots of $\sqrt{x-3}\,(x^2 + 7x + 10) = 0$, where $x \in \mathbf{R}$ is

(a) 0 (b) 1

(c) 2 (d) 3

Ans. (b)

Solution: For the equation to be defined, $x \geq 3$, and $x^2 + 7x + 10 \geq 40 \ \forall \ x \geq 3$. Therefore, the given equation has exactly one root, *viz.* 3.

Example 14: Suppose $a \in \mathbf{R}$. If $3x^2 + 2(a^2 + 1)x + (a^2 - 3a + 2) = 0$ possesses roots of opposite signs, then a lies in the interval:

(a) $(-\infty, -1)$ (b) $(-1, 1)$

(c) $(1, 2)$ (d) $(2, 3)$

Ans. (c)

Solution: As the roots are of opposite signs, the product of roots must be negative, that is,

$$\frac{a^2 - 3a + 2}{3} < 0 \Rightarrow (a-1)(a-2) < 0$$

$$\Rightarrow \quad a \in (1, 2)$$

Example 15: Suppose $a^2 = 5a - 8$ and $b^2 = 5b - 8$, then equation whose roots are $\frac{a}{b}$ and $\frac{b}{a}$ is

(a) $6x^2 - 5x + 6 = 0$ (b) $8x^2 - 9x + 8 = 0$

(c) $9x^2 - 8x + 9 = 0$ (d) $8x^2 + 9x + 8 = 0$

Ans. (b)

Solution: a, b are roots of $x^2 - 5x + 8 = 0$.

$$\therefore \quad a + b = 5, ab = 8$$

$$\text{Now,} \quad \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab} = \frac{(a+b)^2 - 2ab}{ab}$$

$$= \frac{25}{8} - 2 = \frac{9}{8}$$

$\therefore$ required equation is

$$x^2 - (9/8)x + 1 = 0 \text{ or } 8x^2 - 9x + 8 = 0$$

Example 16: Sum of the roots of the equation $x^2 + |2x - 3| - 4 = 0$ is

(a) 2 (b) –2

(c) $\sqrt{2}$ (d) $-\sqrt{2}$

Ans. (c)

Solution: **Case 1:** When $x < 3/2$

In this case the equation becomes

$$x^2 - (2x - 3) - 4 = 0$$

$$\Rightarrow \quad x^2 - 2x + 1 = 2$$

$$\Rightarrow \quad (x-1)^2 = 2 \Rightarrow x - 1 = \pm\sqrt{2}$$

$$\Rightarrow \quad x = 1 \pm \sqrt{2}$$

As $x < 3/2$, we take, $x = 1 - \sqrt{2}$

Case 2: When $x \geq 3/2$

In this case the equation becomes

$$x^2 + (2x - 3) - 4 = 0$$

$$\Rightarrow \quad (x+1)^2 = 8 \Rightarrow x + 1 \pm 2\sqrt{2}$$

As $x \geq 3/2$, $x = -1 + 2\sqrt{2}$

Sum of the roots is

$$(1-\sqrt{2})+(-1+2\sqrt{2})=\sqrt{2}$$

Example 17: If the quadratic equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$, where a, b, c are distinct real numbers and $abc \neq 0$, has equal roots, then a, b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

Ans. (c)

Solution:

TIP

If sum of the coefficients of a polynomial equation is zero, then 1 is a root of the equaiton.

As 1 is a root of the equation and equation has equal roots, the other root must be 1. Thus we have

$$1 = \text{product of roots} = \frac{c(a-b)}{a(b-c)}$$

$$\Rightarrow \quad a(b-c)=c(a-b)$$

$$\Rightarrow \quad b=\frac{2ac}{a+c}$$

$\Rightarrow$ a, b, c are in H.P.

Example 18: Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$, and $(a+c)^2 < b^2$. Then the quadratic equation $ax^2+bx+c=0$ has

(a) real and distinct roots (b) real and equal roots
(c) purely imaginary roots (d) none of these

Ans. (a)

Solution:

TIP

Suppose $f(x)=ax^2+bx+c$, $a, b, c \in \mathbf{R}$, $a \neq 0$. If $f(\alpha)f(\beta) < 0$, then exactly one root of the equation $f(x)=0$ lies between α and β and the equation $f(x)=0$ has real roots.

Let $f(x)=ax^2+bx+c$.
Then $f(1)=a+b+c$, $f(-1)=a-b+c$.
We have $f(1)f(-1)=(a+c)^2-b^2<0$

Thus, $f(x)=0$ has exactly one root in $(-1, 1)$. Hence, $ax^2+bx+c=0$ has real and distinct roots.

Example 19: If both the roots of the quadratic equation $x^2+2(a+2)x+9a-1=0$ are negative, then a lies in the set

(a) $[1, \infty)$ (b) $(1/9, 1] \cup [4, \infty)$
(c) $(-2, 4) \cup (6, \infty)$ (d) ϕ

Ans. (b)

Solution: TIP: Both the roots are negative, if and only if $D \geq 0$, sum of roots < 0 and product of roots > 0

Thus, $4(a+2)^2-4(9a-1) \geq 0$,

$$-2(a+2)<0,\ 9a-1>0$$

$$\Rightarrow \quad a^2-5a+4 \geq 0,\ a>-2,\ a>1/9$$

$$\Rightarrow \quad (a-1)(a-4) \geq 0,\ a>1/9$$

$$\Rightarrow \quad a \leq 1 \text{ or } a \geq 4 \text{ and } a>1/9$$

$$\Rightarrow \quad a \in (1/9, 1] \cup [4, \infty)$$

Example 20: Suppose $p \in \mathbf{R}$ and α, β are roots of $x^2-px+\frac{1}{2p^2}=0$, then the minimum value of $\alpha^4+\beta^4$ is

(a) $\sqrt{2}$ (b) $2-\sqrt{2}$
(c) $\sqrt{2}-2$ (d) -2

Ans. (c)

Solution: $\alpha+\beta=p$, $\alpha\beta=1/2p^2$

$$\Rightarrow \quad \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=p^2-1/p^2$$

$$\therefore \quad \alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2=\left(p^2-\frac{1}{p^2}\right)^2-\frac{1}{2p^4}$$

$$=p^4+\frac{1}{p^4}-2-\frac{1}{2p^4}$$

$$=p^4+\frac{1}{2p^4}-2$$

$$\geq 2\left(p^4\cdot\frac{1}{2p^4}\right)^{1/2}-2=\sqrt{2}-2$$

$[\because \text{AM} \geq \text{GM}]$

Thus, minimum possible value of $\alpha^4+\beta^4$ is $\sqrt{2}-2$, which is attained when $p=1$

LEVEL 1

Straight Objective Type Questions

Example 21: The number of real solutions of

$$x+\sqrt{x}+\sqrt{x-2}=3 \qquad (1)$$

is:

(a) 0 (b) 1
(c) 2 (d) 4

Ans. (a)

Solution: For $\sqrt{x}$, $\sqrt{x-2}$ to be defined $x \geq 0$, $x-2 \geq 0$

$\Rightarrow$ $x \geq 2$. But for $x \geq 2$

$$x+\sqrt{x}+\sqrt{x-2} \geq 2+\sqrt{2}>3$$

Thus, (1) has no real solution.

◉ **Example 22:** In Fig. 3.16 graph of $y = ax^2 + bx + c$ is given. Which one of the following is not true?

(a) $a < 0$ (b) $c > 0$

(c) $b^2 - 4ac > 0$ (d) $b < 0$

Ans. (d)

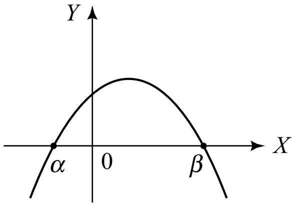


Fig. 3.16

◎ **Solution:** The given parabola opens downwards, $a < 0$. Also, $y(0) = c > 0$.
As roots of $ax^2 + bx + c = 0$ are distinct, $b^2 - 4ac > 0$.
However nothing can be said about b. It can be positive, zero or negative. For instance, consider equations $-x^2 - 5x + 6 = 0$, $-x^2 + 3x + 4 = 0$ and $-x^2 + 9 = 0$.

◉ **Example 23:** Let α and β be the roots of $px^2 + qx + r = 0$, $p \neq 0$. If p, q, r are in A.P. and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then value of $|\alpha - \beta|$ is

(a) $\frac{1}{9}\sqrt{61}$ (b) $\frac{2}{9}\sqrt{17}$

(c) $\frac{1}{9}\sqrt{34}$ (d) $\frac{2}{9}\sqrt{13}$

Ans. (d)

◎ **Solution:** $4 = \dfrac{\alpha + \beta}{\alpha\beta} = \dfrac{-q/p}{r/p} = -\dfrac{q}{r}$

$\Rightarrow \quad q + 4r = 0$

Also, $\quad 2q = p + r \qquad [\because p, q, r \text{ are in A.P.}]$

Thus, $\quad p = -9r,\ q = -4r$

$$|\alpha - \beta|^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$= \left(\frac{-q}{p}\right)^2 - \frac{4r}{p}$$

$$= \frac{16}{81} - 4\left(-\frac{1}{9}\right) = \frac{52}{81}$$

$\Rightarrow \quad |\alpha - \beta| = \frac{2}{9}\sqrt{13}$

◉ **Example 24:** If tan 25° and tan 20° are roots of the quadratic equation $x^2 + 2px + q = 0$, then $2p - q$ is equal to

(a) -2 (b) -1

(c) 0 (d) 1

Ans. (b)

◎ **Solution:** We have
$\tan 25° + \tan 20° = -2p$
and $\tan 25° \tan 20° = q$

Now, $\quad 1 = \tan 45° = \tan(25° + 20°)$

$$= \frac{\tan(25°) + \tan(20°)}{1 - \tan(25°)\tan(20°)} = \frac{-2p}{1 - q}$$

$\Rightarrow \quad 1 - q = -2p \Rightarrow 2p - q = -1$

◉ **Example 25:** Suppose $0 < \alpha < \beta$, and $2\alpha + \beta = \pi/2$. If $\tan\alpha$, $\tan\beta$ are roots of $ax^2 + bx + c = 0$, then

(a) $\tan\alpha = \dfrac{c - a}{b}$ (b) $\tan\beta = \dfrac{c + a}{b}$

(c) $\tan\alpha = \dfrac{b}{c - a}$ (d) $\tan\beta = -\dfrac{b}{c + a}$

Ans. (a)

◎ **Solution:** $\tan\alpha + \tan\beta = -b/a$ and $\tan\alpha \tan\beta = c/a$.

Now, $\quad \tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$

$\Rightarrow \quad \tan(\pi/2 - \alpha) = \dfrac{-b/a}{1 - c/a}$

$\Rightarrow \quad \cot\alpha = \dfrac{-b}{a - c}$

Thus, $\quad \tan\alpha = \dfrac{c - a}{b}$.

◉ **Example 26:** The value of a for which one root of the equation $x^2 - (a + 1)x + a^2 + a - 8 = 0$ exceeds 2 and other is less than 2, are given by

(a) $3 < a < 10$ (b) $a \geq 10$

(c) $-2 < a < 3$ (d) $a \leq -2$

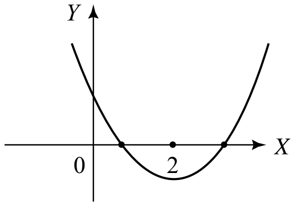
Ans. (c)

◎ **Solution:** As the roots are real and distinct

$$(a + 1)^2 - 4(a^2 + a - 8) > 0$$

$$3a^2 + 2a - 33 < 0$$

$\Rightarrow \quad (3a + 11)(a - 3) < 0 \Rightarrow -\dfrac{11}{3} < a < 3 \quad (1)$



Also, for

$$2^2 - 2(a + 1) + a^2 + a - 8 < 0$$

$\Rightarrow \quad a^2 - a - 6 < 0 \Rightarrow (a - 3)(a + 2) < 0$

$\Rightarrow \quad -2 < a < 3 \quad (2)$

From (1) and (2), we get $-2 < a < 3$.

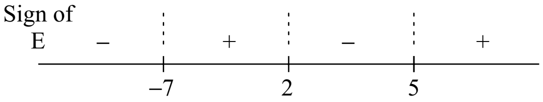
Example 27: The least integral value α of x such that $\dfrac{x-5}{x^2+5x-14} > 0$, satisfies the relation:

(a) $\alpha^2 + 3\alpha - 4 = 0$ (b) $\alpha^2 - 5\alpha + 4 = 0$
(c) $\alpha^2 - 7\alpha + 6 = 0$ (d) $\alpha^2 + 5\alpha - 6 = 0$

Ans. (d)

Solution: $0 < \dfrac{x-5}{x^2+5x-14} = \dfrac{x-5}{(x+7)(x-2)} = \text{E(say)}$

Sign of E in different intervals are shown below



Thus, $\dfrac{x-5}{(x+7)(x-2)} > 0$

$\Rightarrow \qquad -7 < x < 2 \text{ or } x > 5.$

Therefore the least integral value α of x is -6. This value of α satisfies the relation $\alpha^2 + 5\alpha - 6 = 0$.

Example 28: Let p and q be two non-zero real numbers and α, β are two numbers such that $\alpha^3 + \beta^3 = -p$, $\alpha\beta = q$, then the quadratic equation whose roots are α^2/β and β^2/α is

(a) $px^2 - qx + p^2 = 0$ (b) $qx^2 + px + q^2 = 0$
(c) $px^2 + qx + p^2 = 0$ (d) $qx^2 - px + q^2 = 0$

Ans. (b)

Solution: $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha} = \dfrac{\alpha^3+\beta^3}{\alpha\beta} = \dfrac{-p}{q}$

and $\left(\dfrac{\alpha^2}{\beta}\right)\left(\dfrac{\beta^2}{\alpha}\right) = \alpha\beta = q$

Thus, required equation is

$$x^2 - \left(-\frac{p}{q}\right)x + q = 0$$

or $\qquad qx^2 + px + q^2 = 0.$

Example 29: The set of values of α for which the quadratic equation

$$(\alpha+2)\,x^2 - 2\alpha x - \alpha = 0$$

has two roots on the number line symmetrically placed about the point 1 is

(a) $\{-1, 0\}$ (b) $\{0, 2\}$
(c) ϕ (d) $\{0, 1\}$

Ans. (c)

Solution: Let roots of the equation be

$1 - k$ and $1 + k$, where $k > 0$

Then $\qquad 2 = (1-k) + (1+k) = \dfrac{2\alpha}{\alpha+2} \Rightarrow 1 = \dfrac{\alpha}{\alpha+2}$

This is not possible for any value of α.

Example 30: Suppose $k \in \mathbf{R}$ and the quadratic equation $x^2 - (k-3)x + k = 0$ has at least one positive roots, then k lies in the set:

(a) $(-\infty, 0) \cup [9, 16]$ (b) $(-\infty, 0) \cup [16, 8)$
(c) $(-\infty, 0) \cup [9, \infty)$ (d) $(-\infty, 0) \cup (1, \infty)$

Ans. (c)

Solution: Both the roots α, β will be non-positive if

$D \geq 0,\ \alpha + \beta \leq 0,\ \alpha\beta \geq 0$

$\Rightarrow \qquad (k-3)^2 - 4k \geq 0,\ (k-3) \leq 0,\ k \geq 0$

$\Rightarrow \qquad (k-1)(k-9) \geq 0,\ k \leq 3,\ k \geq 0$

$\Rightarrow \qquad 0 \leq k \leq 1.$

Thus, quadratic equation will have at least one positive root if $k < 0$ or $k > 1$ and ($k \leq 1$ or $k \geq 9$)

$\Rightarrow \qquad k \in (-\infty, 0) \cup [9, \infty)$

Example 31: Two non-integer roots of the equation

$$(x^2+3x)^2 - (x^2+3x) - 6 = 0 \qquad (1)$$

are

(a) $\frac{1}{2}(-3+\sqrt{11}), \frac{1}{2}(-3-\sqrt{11})$

(b) $\frac{1}{2}(-3+\sqrt{7}), \frac{1}{2}(-3-\sqrt{7})$

(c) $\frac{1}{2}(-3+\sqrt{21}), \frac{1}{2}(-3-\sqrt{21})$

(d) none of these

Ans. (c)

Solution: TIP: It is an equation which is reducible to quadratic. Put $x^2 + 3x = y$.

The equation (1) becomes

$$y^2 - y - 6 = 0 \quad \Rightarrow \quad (y+2)(y-3) = 0$$

$\Rightarrow \qquad y = -2, y = 3$

When $\qquad y = 2,$

$$x^2 + 3x = -2 \quad \Rightarrow \quad x^2 + 3x + 2 = 0$$

$\Rightarrow \quad (x+1)(x+2) = 0 \quad \Rightarrow \quad x = -1, -2$

When $\qquad y = 3,\ x^2 + 3x = 3 \quad \Rightarrow \quad x^2 + 3x - 3 = 0$

$\Rightarrow \qquad x = \dfrac{1}{2}(-3 \pm \sqrt{9+12}) = \dfrac{1}{2}(-3 \pm \sqrt{21})$

Example 32: Two non-integer roots of

$$\left(\frac{3x-1}{2x+3}\right)^4 - 5\left(\frac{3x-1}{2x+3}\right)^2 + 4 = 0 \qquad (1)$$

are

(a) $-5/7, -2/5$ (b) $-2/5, 7/5$
(c) $5/7, 7/5$ (d) $-2/5, 3/5$

Ans. (a)

Solution: TIP: It is an equation which is reducible to quadratic. Put $\left(\dfrac{3x-1}{2x+3}\right)^2 = y$. The equation (1) becomes

$$y^2 - 5y + 4 = 0 \Rightarrow (y - 1)(y - 4) = 0 \Rightarrow y = 1, 4$$

When $y = 1, \left(\dfrac{3x-1}{2x+3}\right)^2 = 1 \Rightarrow \dfrac{3x-1}{2x+3} = \pm 1$

$\Rightarrow \quad x = 4, -2/5$

When $y = 4, \left(\dfrac{3x-1}{2x+3}\right)^2 = 4 \Rightarrow \dfrac{3x-1}{2x+3} = \pm 2$

$\Rightarrow \quad x = -7, -5/7$

Thus, required roots are – 2/5, – 5/7.

Example 33: Sum of the roots of the equation

is $$4^x - 3(2^{x+3}) + 128 = 0 \qquad (1)$$

(a) 5 (b) 6
(c) 7 (d) 8

Ans. (c)

Solution: Put $2^x = y$. Equation (1) becomes

$$y^2 - 3(8y) + 128 = 0 \quad \Rightarrow \quad y^2 - 24y + 128 = 0$$

$\Rightarrow \quad (y - 8)(y - 16) = 0 \quad \Rightarrow \quad y = 16, 8$

$\Rightarrow \quad 2^x = 16, 8 \quad \Rightarrow \quad x = 4, 3$

$\therefore$ Sum of the roots is 7.

Example 34: The only real value of x satisfying the equation is

$$6\sqrt{\frac{x}{x+4}} - 2\sqrt{\frac{x+4}{x}} = 11 \qquad (1)$$

where $x \in \mathbf{R}$

(a) 4/35 (b) – 4/35
(c) 16/3 (d) none of these.

Ans. (d)

Solution: For (1) to be defined either $x < -4$ or $x > 0$.

Put $\sqrt{\dfrac{x}{x+4}} = y$. Equation (1) becomes

$$6y - 2/y = 11 \quad \Rightarrow \quad 6y^2 - 11y - 2 = 0$$

$\Rightarrow \quad (6y + 1)(y - 2) = 0 \quad \Rightarrow \quad y = -1/6, y = 2.$

As $y = \sqrt{\dfrac{x}{x+4}} \geq 0$, we reject $y = -\dfrac{1}{6}$.

Thus, $y = 2 \quad \Rightarrow \quad \dfrac{x}{x+4} = y^2 = 4$

$\Rightarrow \quad x = 4x + 16 \quad \Rightarrow \quad x = -16/3 < -4$

Example 35: The number of real values of x satisfying the equation

$$2\left(x^2 + \frac{1}{x^2}\right) - 9\left(x + \frac{1}{x}\right) + 14 = 0 \qquad (1)$$

is

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (c)

Solution: Put $x + \dfrac{1}{x} = y$, so that $x^2 + \dfrac{1}{x} = y^2 - 2$. Then equation (1) becomes

$$2(y^2 - 2) - 9y + 14 = 0$$

$\Rightarrow \quad 2y^2 - 9y + 10 = 0 \quad \Rightarrow \quad (2y - 5)(y - 2) = 0$

$\Rightarrow \quad y = 5/2, 2$

When $x + \dfrac{1}{x} = \dfrac{5}{2}$, we get $x = 2, \dfrac{1}{2}$

When $x + \dfrac{1}{x} = 2$, we get $x = 1$.

Example 36: The non-integral roots of

$$x^4 - 3x^3 - 2x^2 + 3x + 1 = 0 \qquad (1)$$

are

(a) $\frac{1}{2}(3 + \sqrt{13}), \frac{1}{2}(3 - \sqrt{13})$

(b) $\frac{1}{2}(3 - \sqrt{13}), \frac{1}{2}(-3 - \sqrt{13})$

(c) $\frac{1}{2}(3 + \sqrt{17}), \frac{1}{2}(3 - \sqrt{17})$

(d) none of these.

Ans. (a)

Solution: Dividing (1) by x^2, we get

$$x^2 - 3x - 2 + \frac{3}{x} + \frac{1}{x^2} = 0$$

or $$\left(x^2 + \frac{1}{x^2}\right) - 3\left(x - \frac{1}{x}\right) - 2 = 0 \qquad (2)$$

Put $x - 1/x = y$, so that

$$x^2 + \frac{1}{x^2} - 2 = y^2 \text{ or } x^2 + \frac{1}{x^2} = y^2 + 2$$

Equation (2) now becomes

$$y^2 + 2 - 3y - 2 = 0 \quad \Rightarrow \quad y^2 - 3y = 0$$

$\Rightarrow \quad y(y - 3) = 0 \quad \Rightarrow \quad y = 0 \quad \text{or} \quad y = 3$

When $y = 0$, we get $x - \dfrac{1}{x} = 0 \quad \Rightarrow \quad x = \pm 1$

When $y = 3$, we get $x - \dfrac{1}{x} = 3$

$\Rightarrow \quad x^2 - 3x - 1 = 0 \quad \Rightarrow \quad x = \dfrac{1}{2}(3 \pm \sqrt{13})$

Example 37: Suppose $a, b, c \in \mathbf{R}$ and $b \neq c$. If α, β are roots of $x^2 + ax + b = 0$ and γ, δ are roots of $x^2 + ax + c = 0$, then value of

$$\frac{(\alpha - \gamma)(\alpha - \delta)}{(\beta - \gamma)(\beta - \delta)} \text{ is independent of:}$$

(a) a, b (b) b, c
(c) c, a (d) a, b, c

Ans. (d)

Solution: $x^2 + ax + c = (x - \gamma)(x - \delta)$

Thus,

$$\begin{aligned}\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)} &= \frac{\alpha^2+a\alpha+c}{\beta^2+a\beta+c}\\ &= \frac{-b+c}{-b+c} = 1\end{aligned}$$

[∵ α, β are roots of $x^2 + ax + b = 0$]

Example 38: Number of real solutions of

$$(x-1)(x+1)(2x+1)(2x-3) = 15 \qquad (1)$$

is

(a) 0
(b) 2
(c) 3
(d) 4

Ans. (b)

Solution: We write (1) as

$$\begin{aligned} (x-1)(2x+1)(x+1)(2x-3) &= 15\\ \Rightarrow \quad (2x^2-x-1)(2x^2-x-3) &= 15 \qquad (2)\end{aligned}$$

Put $2x^2 - x = y$, so that (2) becomes

$$\begin{aligned} (y-1)(y-3) &= 15 \;\Rightarrow\; y^2-4y+3 = 15\\ \Rightarrow \quad y^2-4y-12 &= 0 \;\Rightarrow\; (y-6)(y+2) = 0\\ \Rightarrow \quad y &= 6, -2.\end{aligned}$$

When $y = 6$, $2x^2 - x = 6 \;\Rightarrow\; 2x^2 - x - 6 = 0$

$$\begin{aligned}\Rightarrow \quad 2x^2-4x+3x-6 = 0 \;&\Rightarrow\; (2x+3)(x-2) = 0\\ \Rightarrow \quad x &= -3/2, 2\end{aligned}$$

When $y = -2$, $2x^2 - x = -2$

or $\qquad 2x^2 - x + 2 = 0$

As $D = 1 - 4(2)(2) = -15 < 0$, we get $2x^2 - x + 2 = 0$ does not have real roots.

Example 39: If $p, q \in \mathbf{R}$ and $2 + \sqrt{3}i$ a root of $x^2 + px + q = 0$, then

(a) $p = -2, q = \sqrt{3}$
(b) $p = -4, q = 7$
(c) $p = 3, q = 2$
(d) $p = -4, q = 2$

Ans. (b)

Solution: Other root of the equation is $2 - \sqrt{3}i$. Thus

$$-p = \left(2+\sqrt{3}i\right)+\left(2-\sqrt{3}i\right) \qquad (2)$$

and $\qquad q = \left(2+\sqrt{3}i\right)\left(2-\sqrt{3}i\right)$

$\Rightarrow \qquad p = -4, q = 7$

Example 40: The number of solutions of

$$\sqrt{3x^2+x+5} = x - 3, \text{ where } x \in \mathbf{R} \qquad (1)$$

is:

(a) 0
(b) 1
(c) 2
(d) 4

Ans. (a)

Solution: Note that we must have $3x^2 + x + 5 \geq 0$ and $x - 3 \geq 0$ or $x \geq 3$. Squaring both sides of (1), we get

$$\begin{aligned} 3x^2+x+5 &= x^2-6x+9\\ \Rightarrow \quad 2x^2+7x-4 &= 0 \;\Rightarrow\; (2x-1)(x+4) = 0\\ \Rightarrow \quad x &= 1/2, -4\end{aligned}$$

None of these satisfy the inequality $x \geq 3$. Thus, (1) has no solution.

Example 41: The number of solutions of

$$\sqrt{4-x} + \sqrt{x+9} = 5 \qquad (1)$$

is:

(a) 0
(b) 1
(c) 2
(d) 3

Ans. (c)

Solution: Note that $4 - x \geq 0$ and $x + 9 \geq 0 \Rightarrow -9 \leq x \leq 4$.

We can write (1) as $\sqrt{x+9} = 5 - \sqrt{4-x}$

Squaring both the sides we get

$$\begin{aligned} x+9 &= 25 - 10\sqrt{4-x} + 4 - x\\ \Rightarrow \quad 10\sqrt{4-x} &= 20 - 2x \;\Rightarrow\; 5\sqrt{4-x} = 10 - x\end{aligned}$$

Squaring both the sides we get

$$\begin{aligned} 25(4-x) &= 100 - 20x + x^2 \;\Rightarrow\; x^2 + 5x = 0\\ \Rightarrow \quad x(x+5) &= 0 \;\Rightarrow\; x = 0 \text{ or } x = -5\end{aligned}$$

Both $x = 0$ and $x = -5$ satisfy (1).

Example 42: Suppose $a \in \mathbf{R}$ and α, β are roots of $x^2 - 4ax + 5a^2 - 6a = 0$. The maximum possible distance between α and β is

(a) 6
(b) 5
(c) 3
(d) 1

Ans. (a)

Solution:
$$\begin{aligned}(\alpha-\beta)^2 &= (\alpha+\beta)^2 - 4\alpha\beta\\ &= 16a^2 - 4(5a^2 - 6a)\\ &= 4(6a - a^2)\\ &= 4[9 - (3-a)^2]\end{aligned}$$

As $a \in \mathbf{R}$, $(\alpha - \beta)^2$ is maximum when $a = 3$. Thus, maximum possible distance between α and β is 6.

Example 43: The value of a for which one root of the quadratic equation.

$$(a^2-5a+3)x^2 + (3a-1)x + 2 = 0 \qquad (1)$$

is twice the other, is

(a) $-2/3$
(b) $1/3$
(c) $-1/3$
(d) $2/3$

Ans. (d)

Solution: Let α and 2α be the roots of (1), then

$$(a^2-5a+3)\alpha^2 + (3a-1)\alpha + 2 = 0 \qquad (2)$$

and $\quad (a^2-5a+3)(4\alpha^2) + (3a-1)(2\alpha) + 2 = 0 \qquad (3)$

Multiplying (2) by 4 and subtracting it form (3) we get

$$(3a-1)(2\alpha)+6=0$$

Clearly $a \neq 1/3$. Therefore, $\alpha = -3/(3a-1)$

Putting this value in (2) we get

$$(a^2-5a+3)(9)-(3a-1)^2(3)+2(3a-1)^2=0$$

$$\Rightarrow \quad 9a^2-45a+27-(9a^2-6a+1)=0$$

$$\Rightarrow \quad -39a+26=0$$

$$\Rightarrow \quad a=2/3.$$

For $a = 2/3$, the equation becomes $x^2 + 9x + 18 = 0$, whose roots are $-3, -6$.

Example 44: Range of function $f(x) = \dfrac{x^2+x+2}{x^2+x+1}$, $x \in \mathbf{R}$ is

(a) $(1, \infty)$ (b) $(1, 3/2)$

(c) $(1, 7/3]$ (d) $(1, 7/5]$

Ans. (c)

Solution: Let $y = \dfrac{x^2+x+2}{x^2+x+1} = 1 + \dfrac{1}{x^2+x+1}$

$$= 1 + \frac{1}{(x+1/2)^2+3/4} > 1$$

$$\Rightarrow \quad \frac{1}{y-1} = x^2+x+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4} \geq \frac{3}{4}$$

Thus, $y > 1$ and $y - 1 \leq 4/3 \Rightarrow y \leq 7/3$

$\therefore \quad 1 < y \leq 7/3$

Example 45: If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are such that $\min f(x) > \max g(x)$, then relation between b and c, is:

(a) no relation (b) $0 < c < b/2$

(c) $|c| < \dfrac{|b|}{\sqrt{2}}$ (d) $|c| > \sqrt{2}\ |b|$.

Ans. (d)

Solution: $f(x) = (x + b)^2 + 2c^2 - b^2$

$\Rightarrow \quad \min f(x) = 2c^2 - b^2$

Also $\quad g(x) = -x^2 - 2cx + b^2 = b^2 + c^2 - (x + c)^2$

$\Rightarrow \quad \max g(x) = b^2 + c^2$

As $\min f(x) > \max g(x)$, we get $2c^2 - b^2 > b^2 + c^2$

$\Rightarrow \quad c^2 > 2b^2 \Rightarrow |c| > \sqrt{2}\ |b|$

Example 46: If a, b are the roots of $x^2 + px + 1 = 0$, and c, d are the roots of $x^2 + qx + 1 = 0$, then the value of

$$E = (a-c)(b-c)(a+d)(b+d)$$

is:

(a) $p^2 - q^2$ (b) $q^2 - p^2$

(c) $q^2 + p^2$ (d) none of these

Ans. (b)

Solution: We have

$$x^2 + px + 1 = (x-a)(x-b)$$

Thus,

$$E = (c-a)(c-b)(-d-a)(-d-b)$$

$$= (c^2+pc+1)[(-d)^2-pd+1]$$

$[\because\ a + b = -p]$

$$= (c^2+pc+1)(d^2-pd+1)$$

But $c^2 + qc + 1 = 0$ and $d^2 + qd + 1 = 0$

$$\therefore \quad E = (-qc+pc)(-qd-pd)$$

$$= cd(q-p)(q+p)$$

$$= cd(q^2-p^2) = q^2-p^2 \qquad [\because\ cd = 1]$$

Example 47: If $4^x - 3^{x-1/2} = 3^{x+1/2} - 2^{2x-1}$, then value of x is equal to

(a) 5/2 (b) 2

(c) 3/2 (d) 1

Ans. (c)

Solution: $4^x - 3^{x-1/2} = 3^{x+1/2} - 2^{2x-1}$

$$\Rightarrow \quad 4^x + 2^{2x-1} = 3^{x+1/2} + 3^{x-1/2}$$

$$\Rightarrow \quad 4^x + \frac{4^x}{2} = 3^x\sqrt{3} + \frac{3^x}{\sqrt{3}}$$

$$\Rightarrow \quad \left(\frac{3}{2}\right)4^x = \left(\frac{4}{\sqrt{3}}\right)3^x$$

$$\Rightarrow \quad \left(\frac{4}{3}\right)^x = \frac{2}{3}\times\frac{4}{\sqrt{3}} = \frac{8}{3\sqrt{3}} = \left(\frac{2}{3^{1/2}}\right)^3 = \left(\frac{4}{3}\right)^{3/2}$$

$$\Rightarrow \quad x = 3/2$$

Example 48: If $x^2 + 2ax + 10 - 3a > 0$ for each $x \in \mathbf{R}$, then

(a) $a < -5$ (b) $-5 < a < 2$

(c) $a > 5$ (d) $2 < a < 5$

Ans. (b)

Solution: $x^2 + 2ax + 10 - 3a > 0 \ \forall\ x \in \mathbf{R}$

$$\Rightarrow \quad (x+a)^2 - (a^2+10-3a) > 0 \ \forall\ x \in \mathbf{R}$$

$$\Rightarrow \quad a^2 + 3a - 10 < 0$$

$$\Rightarrow \quad (a+5)(a-2) < 0$$

$$\Rightarrow \quad -5 < a < 2$$

Example 49: The number of solutions of

$$\sqrt{x+1} - \sqrt{x-1} = 1 \qquad (x \in \mathbf{R})$$

is

(a) 1 (b) 2

(c) 4 (d) infinite

Ans. (a)

Solution: For the equation to make sense we must have $x + 1 \geq 0$ and $x - 1 \geq 0 \Rightarrow x \geq -1, x \geq 1$ i.e. $x \geq 1$.

We rewrite equation as

$$\sqrt{x+1} = 1 + \sqrt{x-1}$$

and square both the sides to obtain

$$x + 1 = 1 + x - 1 + 2\sqrt{x-1}$$

$$\Rightarrow \quad \frac{1}{2} = \sqrt{x-1} \Rightarrow \frac{1}{4} = x - 1 \text{ or } x = \frac{5}{4}$$

Also, $x = 5/4$ satisfies the given equation.

Example 50: If a, b, c are real and $a \neq b$, then the roots of the equation

$$2(a-b)x^2 - 11(a+b+c)x - 3(a-b) = 0 \qquad (1)$$

are

(a) real and equal (b) real and unequal
(c) purely imaginary (d) none of these

Ans. (b)

Solution: The discriminant D of the quadratic equation (1) is given by

$$D = 121(a+b+c)^2 + 24(a-b)^2$$

As a, b, c are real, $121(a+b+c)^2 \geq 0$

Also, as $a \neq b$, $(a-b)^2 > 0$

Thus, $D > 0$.

Therefore, equation (1) has real and unequal roots.

Example 51: Let $a > 0$, $b > 0$ and $c > 0$. Then both the roots of the equation

$$2ax^2 + 3bx + 5c = 0 \qquad (1)$$

(a) are real and negative
(b) have negative real parts
(c) have positive real parts
(d) none of these

Ans. (b)

Solution: The discriminant D of the equation (1) is given by

$$D = (3b)^2 - 4(2a)(5c) = 9b^2 - 40ac$$

If $D \geq 0$, then both the roots of (1) are real.

Also, since $ac > 0$, $D < 9b^2$.

$$\Rightarrow \quad \sqrt{D} < 3b \quad [\because b > 0]$$

Therefore, in this case both the roots $\dfrac{-3b-\sqrt{D}}{4a}$ and $\dfrac{-3b+\sqrt{D}}{4a}$ are negative.

If $D < 0$, both the roots of (1) are imaginary and are given by

$$x = \frac{-3b \pm i\sqrt{40ac - 9b^2}}{2}$$

Both these roots have negative real parts.

Example 52: If a, b, c are real, then both the roots of the equation

$$(x-b)(x-c) + (x-c)(x-a) + (x-a)(x-b) = 0 \qquad (1)$$

are always

(a) positive (b) negative
(c) real (d) none of these

Ans. (c)

Solution: We can write (1) as

$$3x^2 - 2(a+b+c)x + bc + ca + ab = 0 \qquad (2)$$

The discriminant D of (2) is given by

$$\begin{aligned} D &= 4(a+b+c)^2 - 4 \times 3(bc+ca+ab) \\ &= 4[a^2 + b^2 + c^2 + 2bc + 2ca + 2ab - 3bc - 3ca - 3ab] \\ &= 4[a^2 + b^2 + c^2 - bc - ca - ab] \\ &= 2[(b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ca) + (a^2 + b^2 - 2ab)] \\ &= 2[(b-c)^2 + (c-a)^2 + (a-b)^2] \end{aligned}$$

As a, b, c are real, we get $D \geq 0$. Thus, roots of (1) are real.

Example 53: If the roots of the equation $(4a - a^2 - 5)x^2 - (2a-1)x + 3a = 0$, $a \in \mathbf{R}$, are real and lie on opposite sides of unity, then the a lies in the interval:

(a) $(1, 5)$ (b) $(1, 4)$
(c) $(3, \infty)$ (d) $(-\infty, 4)$

Ans. (b)

Solution: **TIP:** If roots of a quadratic equation $f(x) = 0$ are on the opposite sides of a real number p, then $f(x + p) = 0$ has roots of opposite sings.

Put $\quad y = x - 1$, so that $x = y + 1$

The equation now becomes:

$$\begin{aligned} & (4a - a^2 - 5)(y+1)^2 - (2a-1)(y+1) + 3a = 0 \\ \Rightarrow \quad & (4a - a^2 - 5)y^2 + (8a - 2a^2 - 10 - 2a + 1)y \\ & + (4a - a^2 - 5 - 2a + 1 + 3a) = 0 \\ \Rightarrow \quad & (4a - a^2 - 5)y^2 + (6a - 2a^2 - 9)y \\ & + (5a - a^2 - 4) = 0 \end{aligned}$$

This equation must have roots of opposite signs. This is possible if and only if

$$\frac{5a - a^2 - 4}{4a - a^2 - 5} < 0 \text{ or } \frac{a^2 - 5a + 4}{a^2 - 4a + 5} < 0$$

As $a^2 - 4a + 5 = (a-2)^2 + 1 > 0 \ \forall \ a$,
we get $(a-1)(a-4) < 0 \Rightarrow 1 < a < 4$.

Example 54: If α, β are roots of $ax^2 + bx + c = 0$, then roots of

$$a^3x^2 + abcx + c^3 = 0 \text{ are}$$

(a) $\alpha\beta, \alpha + \beta$ (b) $\alpha^2\beta, \alpha\beta^2$
(c) $\alpha\beta, \alpha^2\beta^2$ (d) α^3, β^3

Ans. (b)

Solution: $\alpha + \beta = \dfrac{-b}{a}$, $\alpha\beta = \dfrac{c}{a}$.

Write $\quad a^3x^2 + abcx + c^3 = 0 \quad$ as

$$\begin{aligned} & x^2 + \frac{bc}{a^2}x + \left(\frac{c}{a}\right)^3 = 0 \\ \Rightarrow \quad & x^2 - (\alpha + \beta)\,\alpha\beta x + \alpha^3\beta^3 = 0 \\ \Rightarrow \quad & x^2 - (\alpha^2\beta + \alpha\beta^2)x + (\alpha^2\beta)(\alpha\beta^2) = 0 \\ \Rightarrow \quad & (x - \alpha^2\beta)(x - \alpha\beta^2) = 0 \\ \Rightarrow \quad & x = \alpha^2\beta, \alpha\beta^2 \end{aligned}$$

Example 55: If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $a, b, c \in \mathbf{R}$, then $P(x)\,Q(x) = 0$ has

(a) no real root
(b) exactly two real roots
(c) at least two real roots
(d) none of these

Ans. (c)

Solution: Let $D_1 = b^2 - 4ac$ and $D_2 = d^2 + 4ac$
We have $D_1 + D_2 = b^2 + d^2 \geq 0$
$\Rightarrow$ at least one of $D_1, D_2 \geq 0$
$\Rightarrow$ one of $P(x) = 0$ or $Q(x) = 0$
has real roots. Thus, $P(x)\,Q(x) = 0$
has at least two real roots.

Example 56: If the product of the roots of the equation $x^2 - 5kx + 2e^{4\ln k} - 1 = 0$ is 31, then sum of the root is

(a) – 10 (b) 5
(c) – 8 (d) 10

Ans. (d)

Solution: We have product of the roots

$$= 2e^{4\ln k} - 1 = 31 \text{ (given)}$$

$$\Rightarrow \quad e^{4\ln k} = 16$$

$$\Rightarrow \quad k^4 = 16 \Rightarrow k^4 - 16 = 0$$

$$\Rightarrow \quad (k - 2)(k + 2)(k^2 + 4) = 0$$

$$\Rightarrow \quad k = 2, -2$$

As $k > 0$, we get $k = 2$.
Sum of the roots $= 5k = 10$

Example 57: Let α, β be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is :

(a) $\frac{2}{9}(p - q)(2q - p)$ (b) $\frac{2}{9}(q - p)(2p - q)$
(c) $\frac{2}{9}(q - 2p)(2q - p)$ (d) $\frac{2}{9}(2p - q)(2q - p)$

Ans. (d)

Solution: $\alpha + \beta = p$, $\alpha\beta = r$

and $\quad \frac{\alpha}{2} + 2\beta = q.$

But $\quad \alpha + \beta = p$ and $\alpha + 4\beta = 2q$

$$\Rightarrow \quad \beta = \frac{1}{3}(2q - p) \text{ and } \alpha = \frac{2}{3}(2p - q)$$

Thus, $\quad \alpha\beta = r$

$$\Rightarrow \quad \frac{2}{9}(2p - q)(2q - p) = r$$

Example 58: Sum of all the values of x satisfying the equation

$$\log_{17}\log_{11}\left(\sqrt{x+11} + \sqrt{x}\right) = 0 \qquad (1)$$

is:

(a) 25 (b) 36
(c) 171 (d) 0

Ans. (a)

Solution: Equation (1) is defined if $x \geq 0$.
We can rewrite (1) as

$$\log_{11}\left(\sqrt{x+11} + \sqrt{x}\right) = 17^0 = 1$$

$$\Rightarrow \quad \sqrt{x+11} + \sqrt{x} = 11^1 = 11$$

$$\Rightarrow \quad \sqrt{x+11} = 11 - \sqrt{x}$$

Squaring both the sides we get

$$x + 11 = 121 - 22\sqrt{x} + x$$

$$\Rightarrow \quad 22\sqrt{x} = 110 \Rightarrow \sqrt{x} = 5 \text{ or } x = 25$$

This clearly satisfies (1).
Thus, sum of all the values satisfying (1) is 25.

Example 59: If $\frac{1}{\sqrt{\alpha}}$ and $\frac{1}{\sqrt{\beta}}$ are the roots of the equation $ax^2 + bx + 1 = 0$ ($a \neq 0, a, b, \in \mathbf{R}$), then the equation, $x(x + b^3) + (a^3 - 3abx) = 0$ has roots:

(a) $\alpha^{3/2}$ and $\beta^{3/2}$ (b) $\alpha\beta^{-3/2}$ and $\alpha^{1/2}\beta$
(c) $\sqrt{\alpha\beta}$ and $\alpha\beta$ (d) $\alpha^{-3/2}$ and $\beta^{-3/2}$

Ans. (a)

Solution: The equation $x(x + b^3) + (a^3 - 3abx) = 0$ can be written as

$$x^2 + (b^3 - 3ab)x + a^3 = 0$$

We have

$$\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}} = -\frac{b}{a}, \frac{1}{\sqrt{\alpha}\sqrt{\beta}} = \frac{1}{a}$$

$$\sqrt{\alpha} + \sqrt{\beta} = -b, \sqrt{\alpha}\sqrt{\beta} = a$$

Now

$$-(b^3 - 3ab) = (-b)^3 - 3a(-b)$$

$$= \left(\sqrt{\alpha} + \sqrt{\beta}\right)^3 - 3\left(\sqrt{\alpha} + \sqrt{\beta}\right)\sqrt{\alpha}\sqrt{\beta}$$

$$= \left(\sqrt{\alpha}\right)^3 + \left(\sqrt{\beta}\right)^3 = \alpha^{3/2} + \beta^{3/2}$$

and $a^3 = \alpha^{3/2}\beta^{3/2}$
$\therefore$ The roots of the equation

$$x^2 + (b^2 - 3ab)x + a^3 = 0 \text{ are } \alpha^{3/2}, \beta^{3/2}$$

Example 60: If p, q are roots of $x^2 + px + q = 0$, then

(a) $p = 1$ (b) $p = 1$ or 0
(c) $p = -2$ (d) $p = -2$ or 0

Ans. (b)

Solution: We have $p + q = -p$, $pq = q$
As $\quad pq = q,$
we get $\quad q(p - 1) = 0$
$\Rightarrow \quad q = 0$ or $p = 1$
If $q = 0$, we get $p = 0$
If $p = 1$, we get $p + q = -p \Rightarrow q = -2$.
Thus, $p = 1$ or $p = 0$.

Example 61: The equation

$$\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}, \ (x \in \mathbf{R})$$

(a) no solution
(b) one solution
(c) two solutions
(d) more than two solutions

Ans. (a)

Solution: The given equation is valid if $x + 1 \geq 0$, $x - 1 \geq 0$ and $4x - 1 \geq 0$ i.e. if $x \geq 1$.

Squaring both the sides we get

$$x + 1 + x - 1 - 2\sqrt{(x+1)(x-1)} = 4x - 1$$

$$\Rightarrow \quad 1 - 2x = 2\sqrt{(x+1)(x-1)}$$

Squaring again, we get

$$1 - 4x + 4x^2 = 4(x^2 - 1)$$

$$\Rightarrow \quad 4x = 5 \text{ or } x = 5/4.$$

Putting this value of x in the given equation, we get

$$\sqrt{\frac{5}{4}+1} - \sqrt{\frac{5}{4}-1} = \sqrt{4\left(\frac{5}{4}\right)-1}$$

$$\Rightarrow \quad \frac{3}{2} - \frac{1}{2} = 2 \text{ or } 1 = 2$$

which is not true.

Thus, the given equation has no solution.

Example 62: The sum of all the real roots of the equation

$$|x - 2|^2 + |x - 2| - 2 = 0 \qquad (1)$$

is

(a) 7 (b) 4
(c) 1 (d) none of these

Ans. (b)

Solution: Putting $y = |x - 2|$, we can rewrite (1) as

$$y^2 + y - 2 = 0 \text{ or } y^2 + 2y - y - 2 = 0$$

$$\Rightarrow \quad y(y + 2) - (y + 2) = 0 \Rightarrow (y - 1)(y + 2) = 0$$

As $\quad y = |x - 2| \geq 0,\ y + 2 \geq 2$

$$\therefore \quad y - 1 = 0 \Rightarrow y = 1$$

$$\Rightarrow \quad |x - 2| = 1 \Rightarrow x - 2 = \pm 1$$

$$\Rightarrow \quad x = 2 \pm 1 \Rightarrow x = 3 \text{ or } 1$$

$\therefore$ Sum of the roots = 4

Example 63: Let p and q be the roots of $x^2 - 2x + A = 0$ and let r and s be the roots of $x^2 - 18x + B = 0$. If $p < q < r < s$ are in A.P. then ordered pair (A, B) is equal to

(a) $(-3, 77)$ (b) $(77, -3)$
(c) $(-3, -77)$ (d) none of these

Ans. (a)

Solution: We have $p + q = 2$, $pq = A$ (1)

and $\quad r + s = 18$, $rs = B$ (2)

As p, q, r, s are in AP, we take

$$p = a - 3d,\ q = a - d,\ r = a + d,\ s = a + 3d.$$

As $p < q < r < s$, we have $d > 0$

Now, $\quad 2 = p + q = 2a - 4d$

and $\quad 18 = r + s = 2a + 4d$

Solving above equations, we get $a = 5$ and $d = 2$

$$\therefore \quad p = -1,\ q = 3,\ r = 7 \text{ and } s = 11$$

Thus, $\quad A = pq = -3$ and $B = rs = 77$.

Example 64: In a triangle PQR, $\angle R = \pi/2$. If $\tan(P/2)$ and $\tan(Q/2)$ are the roots of the equation $ax^2 + bx + c = 0$ where $a \neq 0$, then

(a) $a + b = c$ (b) $b + c = a$
(c) $a + c = b$ (d) $b = c$

Ans. (a)

Solution: We have

$$\tan\left(\frac{P}{2}\right) + \tan\left(\frac{Q}{2}\right) = \frac{-b}{a}$$

and
$$\tan\left(\frac{P}{2}\right)\tan\left(\frac{Q}{2}\right) = \frac{c}{a}$$

Now, $\quad P + Q = \dfrac{\pi}{2} \Rightarrow 1 = \tan\left(\dfrac{\pi}{4}\right) = \tan\left(\dfrac{P}{2} + \dfrac{Q}{2}\right)$

$$= \frac{\tan(P/2) + \tan(Q/2)}{1 - \tan(P/2)\tan(Q/2)}$$

$$\Rightarrow \quad 1 = \frac{-b/a}{1 - c/a} = \frac{-b}{a - c}$$

$$\Rightarrow \quad a - c = -b \Rightarrow c = a + b$$

Example 65: If α and β $(\alpha < \beta)$, are the roots of the equation $x^2 + bx + c = 0$ where $c < 0 < b$, then

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$

Ans. (b)

Solution: We have $\alpha + \beta = -b$, $\alpha\beta = c$

As $c < 0$, $b > 0$, we get $\alpha < 0 < \beta$

Also, $\quad \beta = -b - \alpha < -\alpha = |\alpha|$

Thus, $\quad \alpha < 0 < \beta < |\alpha|$

Example 66: For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the roots is square of the other, than p is equal to

(a) 1/3 (b) 1
(c) 3 (d) 2/3

Ans. (c)

Solution: Let α, α^2 be the roots of $3x^2 + px + 3 = 0$.

Then $\alpha + \alpha^2 = -p/3$ and $\alpha \cdot \alpha^2 = 1$

As $\quad \alpha^3 = 1$, we get $\alpha = 1$, ω or ω^2.

If $\quad \alpha = 1$, then $p = -3(\alpha + \alpha^2) = -6 < 0$.

Not possible as $p > 0$.

Thus, $\alpha \neq 1$.

We take $\alpha = \omega$ and $\alpha^2 = \omega^2$, so that $\dfrac{-p}{3} = \omega + \omega^2 = -1$

$$\Rightarrow \quad p = 3$$

Example 67: Let α, β be roots of $ax^2 + bx + c = 0$, where $ac \neq 0$. Roots of $cx^2 - bx + a = 0$ are

(a) $1/\alpha, 1/\beta$ (b) $-1/\alpha, -1/\beta$

(c) $1/\sqrt{\alpha}, 1/\sqrt{\beta}$ (d) $-1/\sqrt{\alpha}, -1/\sqrt{\beta}$

Ans. (b)

◎ **Solution:** $\alpha + \beta = -b/a$, $\alpha\beta = c/a$.

Now, $$\frac{b}{c} = -\frac{b/a}{c/a} = -\frac{\alpha+\beta}{\alpha\beta}$$

$$= \left(-\frac{1}{\alpha}\right) + \left(-\frac{1}{\beta}\right)$$

And $$\frac{a}{c} = \frac{1}{\alpha\beta} = \left(-\frac{1}{\alpha}\right)\left(-\frac{1}{\beta}\right)$$

Thus, roots of $cx^2 - bx + a = 0$ are $-1/\alpha, -1/\beta$.

◉ **Example 68:** If $b > a$, then the equation $(x - a)(x - b) - 1 = 0$ has

(a) both roots in $[a, b]$

(b) both roots in $(-\infty, a)$

(c) both roots in (b, ∞)

(d) one root in $(-\infty, a)$ and other in (b, ∞).

Ans. (d)

◎ **Solution:** Graph of $y = (x - a)(x - b) - 1$ is given in Fig. 3.17.

It is a parabola which open upwards. Also, $y < 0$ for $x = a$ and $x = b$.

$\therefore$ $y = (x - a)(x - b) - 1$ meets the x-axis at two points once in $(-\infty, a)$ and once in (b, ∞). Thus, one root lies in $(-\infty, a)$ and one in (b, ∞).

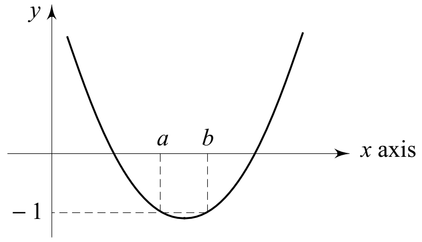


Fig. 3.17

◉ **Example 69:** Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in G.P. then the integral values of p and q respectively, are

(a) – 2, – 32 (b) – 2, 3

(c) – 6, 3 (d) – 6, – 32

Ans. (a)

◎ **Solution:** We have

$$\alpha + \beta = 1,\ \alpha\beta = p,$$

$$\gamma + \delta = 4,\ \gamma\delta = q$$

Let r be the common ratio of the GP $\alpha, \beta, \gamma, \delta$. Then

$$\alpha + \beta = 1 \Rightarrow \alpha + \alpha r = 1 \Rightarrow \alpha(1 + r) = 1$$

$$\gamma + \delta = 4 \Rightarrow \alpha r^2 + \alpha r^3 = 4$$

$\Rightarrow$ $$\alpha r^2(1 + r) = 4$$

Thus, $\dfrac{\alpha r^2(1+r)}{\alpha(1+r)} = 4 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$

When $r = 2$,

we get $\alpha(1 + r) = 1 \Rightarrow \alpha = 1/3$

In this case $p = \alpha\beta = \alpha(\alpha r) = \alpha^2 r$

$$= \frac{1}{9}(2) = \frac{2}{9}$$

which is not an integer.

Thus, $r = -2$. In this case, $\alpha(1 + r) = 1 \Rightarrow \alpha = -1$.

$\therefore$ $p = \alpha^2 r = (-1)^2(-2) = -2$

and $q = \gamma\delta = (\alpha r^2)(\alpha r^3) = \alpha^2 r^5 = -32$

Hence, $p = -2$, $q = -32$.

◉ **Example 70:** If a, b, and c are not all equal and α and β be the roots of the equation $ax^2 + bx + c = 0$, then value of $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$ is

(a) 0 (b) positive

(c) negative (d) non-negative

Ans. (d)

◎ **Solution:** We have $\alpha + \beta = -b/a$, $\alpha\beta = c/a$

Now, $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$

$$= 1 + \alpha + \beta + \alpha^2 + \beta^2 + \alpha\beta + \alpha^2\beta + \alpha\beta^2 + \alpha^2\beta^2$$

$$= 1 + (\alpha + \beta) + (\alpha + \beta)^2 - 2\alpha\beta + \alpha\beta[1 + \alpha + \beta + \alpha\beta]$$

$$= 1 - \frac{b}{a} + \frac{b^2}{a^2} - \frac{2c}{a} + \frac{c}{a} + \frac{c}{a}\left(-\frac{b}{a}\right) + \frac{c^2}{a^2}$$

$$= \frac{1}{a^2}\{a^2 + b^2 + c^2 - bc - ca - ab\}$$

$$= \frac{1}{2a^2}[(b - c)^2 + (c - a)^2 + (a - b)^2] \geq 0$$

◉ **Example 71:** If a, b, c are in A.P. and if the equations

$$(b - c)x^2 + (c - a)x + (a - b) = 0 \quad (1)$$

and $$2(c + a)x^2 + (b + c)x = 0 \quad (2)$$

have a common root, then

(a) a^2, b^2, c^2 are in A.P.

(b) a^2, c^2, b^2 are in A.P.

(c) c^2, a^2, b^2 are in A.P.

(d) none of these

Ans. (b)

◎ **Solution:** Clearly $x = 1$ is a root of (1). If α is the other root of (1), then

$$\alpha = 1 \cdot \alpha = \frac{a-b}{b-c} \text{ [product of roots]}$$

$$= 1 \ [\because a, b, c \text{ are in A. P.}]$$

Thus, the roots of (1) are 1, 1. Now, (1) and (2) will have a common root if 1 is also a root of (2).

$\Rightarrow \quad 2(c + a) + b + c = 0$

$\Rightarrow \quad 2(2b) + b + c = 0 \Rightarrow c = -5b \quad [\because a, b, c \text{ are in AP}]$

Also, $\quad a + c = 2b$

$\Rightarrow \quad a = 2b - c = 2b + 5b = 7b$

$\therefore \quad a^2 = 49b^2, c^2 = 25b^2$

This, show that a^2, c^2, b^2 are in A.P.

Example 72: Value of

$$x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \cdots \text{ up to } \infty}}} \qquad (1)$$

is

(a) 3 (b) 2
(c) 1 (d) none of these

Ans. (a)

Solution: We can write (1) as

$$x = \sqrt{6 + x} \Rightarrow x^2 = 6 + x$$

$\Rightarrow \quad x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0$

$\Rightarrow \quad x = 3, -2$

As $x > 0$, we take $x = 3$

Example 73: Two complex numbers α and β are such that $\alpha + \beta = 2$ and $\alpha^4 + \beta^4 = 272$, then the quadratic equation whose roots are α and β can be

(a) $x^2 - 2x - 16 = 0$ (b) $x^2 - 2x + 12 = 0$
(c) $x^2 - 2x - 8 = 0$ (d) none of these

Ans. (c)

Solution: $\alpha + \beta = 2 \Rightarrow \alpha^2 + \beta^2 + 2\alpha\beta = 4$

$\Rightarrow \quad \alpha^2 + \beta^2 = 4 - 2\alpha\beta \Rightarrow (\alpha^2 + \beta^2)^2 = 16 - 16\alpha\beta + 4\alpha^2\beta^2$

$\Rightarrow \quad \alpha^4 + \beta^4 + 2\alpha^2\beta^2 = 16 - 16\alpha\beta + 4\alpha^2\beta^2$

$\Rightarrow \quad 272 + 2(\alpha\beta)^2 = 16 - 16\alpha\beta + 4(\alpha\beta)^2$

$\Rightarrow \quad 2(\alpha\beta)^2 - 16(\alpha\beta) - 256 = 0$

$\Rightarrow \quad (\alpha\beta)^2 - 8(\alpha\beta) - 128 = 0$

$\Rightarrow \quad (\alpha\beta - 16)(\alpha\beta + 8) = 0$

$\Rightarrow \quad \alpha\beta = 16 \text{ or } \alpha\beta = -8$

Thus, required equation is either

$$x^2 - 2x + 16 = 0$$

or $\quad x^2 - 2x - 8 = 0$

$\therefore$ Required answer is (c).

Example 74: If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$, then the equation whose roots are $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ is

(a) $3x^2 - 25x + 3 = 0$ (b) $x^2 - 5x + 3 = 0$
(c) $x^2 + 5x - 3 = 0$ (d) $3x^2 - 19x + 3 = 0$

Ans. (d)

Solution: α, β are roots of $x^2 = 5x - 3$ or $x^2 - 5x + 3 = 0$. Thus,

$$\alpha + \beta = 5, \ \alpha\beta = 3.$$

Next, $\quad \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = \dfrac{(\alpha + \beta)^2}{\alpha\beta} - 2$

$$= \frac{25}{3} - 2 = \frac{19}{3}$$

and $\quad \left(\dfrac{\alpha}{\beta}\right)\left(\dfrac{\beta}{\alpha}\right) = 1.$

Thus, the quadratic equation whose roots are $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$ is

$$x^2 - \frac{19}{3}x + 1 = 0 \text{ or } 3x^2 - 19x + 3 = 0.$$

Example 75: If $a \neq b$ and difference between the roots of the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ is the same, then

(a) $a + b + 4 = 0$ (b) $a + b - 4 = 0$
(c) $a - b + 4 = 0$ (d) $a - b - 4 = 0$

Ans. (a)

Solution: Let α, β be the roots of $x^2 + ax + b = 0$ and γ, δ be the roots of $x^2 + bx + a = 0$.

We are given

$$|\alpha - \beta| = |\gamma - \delta|$$

$\Rightarrow \quad |\alpha - \beta|^2 = |\gamma - \delta|^2$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = (\gamma + \delta)^2 - 4\gamma\delta$

$\Rightarrow \quad a^2 - 4b = b^2 - 4a$

$\Rightarrow \quad a^2 - b^2 + 4(a - b) = 0$

$\Rightarrow \quad (a - b)(a + b + 4) = 0$

As $\quad a \neq b, a + b + 4 = 0.$

Example 76: Product of real roots of the equation

$t^2x^2 + |x| + 9 = 0.\ t \in \mathbf{R}$ is always

(a) positive (b) negative
(c) zero (d) does not exist.

Ans. (d)

Solution: Note that $t^2x^2 + |x| + 9 \geq 0 + 9 > 9 \ \forall \ x \in \mathbf{R}$,
Thus, $t^2x^2 + |x| + 9 = 0$ does not have real roots.

Example 77: Let $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$, $x \in \mathbf{R}$. The range of $f(x)$ is

(a) $[0, \infty)$ (b) $[1/3, 3]$
(c) $[3, \infty)$ (d) $[0, 3]$

Ans. (b)

Solution: Let $y = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4} = \dfrac{(x-1)^2 + 3}{(x+1)^2 + 3} > 0$ for each $x \in \mathbf{R}$.

$\Rightarrow \quad (y - 1)x^2 + 2(y + 1)x + 4(y - 1) = 0$

As x is real,

$$4(y + 1)^2 - 16(y - 1)^2 \geq 0$$

$\Rightarrow (y + 1 + 2y - 2)\,(y + 1 - 2y + 2) \ge 0$

$\Rightarrow 3(y - 1/3)\,(y - 3) \le 0$

$\Rightarrow 1/3 \le y \le 3.$

● **Example 78:** The values of $a \in \mathbf{R}$ for which one root of the equation $f(x) = x^2 - (a + 1)x + a^2 + a - 8 = 0$ exceeds 2 and the other is less than 2 are given by

(a) $3 < a < 10$ (b) $a > 0$
(c) $-2 < a < 3$ (d) $a \le -2$

Ans. (c)

TIP: If $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbf{R}$ with $a > 0$ and $f(\alpha) < 0$ for some $\alpha \in \mathbf{R}$ then $D = b^2 - 4ac > 0$, and hence $f(x) = 0$ has real and distinct roots.

◎ **Solution:** As one of the roots is less than 2 and other is more than 2, $f(2) < 0$.

$$\Rightarrow \quad 2^2 - (a + 1)(2) + a^2 + a - 8 < 0$$
$$\Rightarrow \quad a^2 - a - 6 < 0 \Rightarrow (a + 2)(a - 3) < 0$$
$$\Rightarrow \quad -2 < a < 3.$$

● **Example 79:** If one root of the equation $x^2 + px + 12 = 0$ is 4, while equation $x^2 + px + q = 0$ has equal roots, then the value of q is

(a) 3 (b) 12
(c) $\frac{49}{4}$ (d) 4

Ans. (c)

◎ **Solution:** $4^2 + 4p + 12 = 0 \quad \Rightarrow \quad p = -7$

Since $x^2 - 7x + q = 0$ has equal roots, we get $49 - 4q = 0$ $\Rightarrow q = \frac{49}{4}$

● **Example 80:** Suppose a is an integer and x_1 and x_2 are positive real roots of $x^2 + (2a - 1)x + a^2 = 0$, then value of $\left|\sqrt{x_1} - \sqrt{x_2}\right|$ is

(a) 1 (b) 2
(c) a (d) $1 - 4a$

Ans.(a)

◎ **Solution:** Two real roots are positive implies $2a - 1 \le 0$, that is, $a \le 1/2$. As a is an integer, we get $a \le 0$. Now,

$$\left|\sqrt{x_1} - \sqrt{x_2}\right|^2 = \left(\sqrt{x_1} - \sqrt{x_2}\right)^2$$
$$= x_1 + x_2 - 2\sqrt{x_1 x_2}$$
$$= (1 - 2a) - 2\sqrt{a^2} = 1 - 2a - 2\,|a|$$
$$= 1 - 2(a + |a|) = 1$$
$$\Rightarrow \quad \left|\sqrt{x_1} - \sqrt{x_2}\right| = 1$$

● **Example 81:** If a, b are two real number satisfying the relations $2a^2 - 3a - 1 = 0$ and $b^2 + 3b - 2 = 0$ and $ab \ne 1$, then value of $\frac{ab + a + 1}{b}$ is

(a) -1 (b) 0
(c) 1 (d) 2

Ans. (c)

◎ **Solution:** $b^2 + 3b - 2 = 0 \Rightarrow 1 + \frac{3}{b} - \frac{2}{b^2} = 0$

$$\Rightarrow \quad \frac{2}{b^2} - \frac{3}{b} - 1 = 0$$

Thus, a and $1/b$ are roots of $2x^2 - 3x - 1 = 0$

$$\therefore \quad a + \frac{1}{b} = \frac{3}{2} \text{ and } \frac{a}{b} = \frac{-1}{2}$$

$$\text{Now,} \quad \frac{ab + a + 1}{b} = a + \frac{1}{b} + \frac{a}{b} = \frac{3}{2} - \frac{1}{2} = 1$$

● **Example 82:** If α, β are roots of $x^2 - 2x - 1 = 0$, then value of $5\alpha^4 + 12\beta^3$ is

(a) 153 (b) 169
(c) 183 (d) none of these

Ans. (b)

◎ **Solution:** We have $\alpha^2 = 2\alpha + 1$

$$\Rightarrow \quad \alpha^4 = (2\alpha + 1)^2 = 4\alpha^2 + 4\alpha + 1$$
$$= 4(2\alpha + 1) + 4\alpha + 1 = 12\alpha + 5, \text{ and}$$
$$\beta^2 = 2\beta + 1 \Rightarrow \quad \beta^3 = 2\beta^2 + \beta = 2(2\beta + 1) + \beta$$
$$= 5\beta + 2$$

$$\text{Thus,} \quad 5\alpha^4 + 12\beta^3 = (60\alpha + 25) + (60\beta + 24)$$
$$= 60(\alpha + \beta) + 49 = 60\,(2) + 44 = 169$$

● **Example 83:** Suppose a, b, c are the lengths of three sides of a ΔABC, $a > b > c$, $2b = a + c$ and b is a positive integer. If $a^2 + b^2 + c^2 = 84$, then value of b is

(a) 7 (b) 6
(c) 5 (d) 4

Ans. (c)

◎ **Solution:** We have

$$ac = \frac{1}{2}\,[(a + c)^2 - (a^2 + c^2)] = \frac{1}{2}\,[4b^2 - (84 - b^2)]$$
$$= 5b^2/2 - 42$$

Thus, a and c are the roots of the equation

$$x^2 - 2bx + (5b^2/2 - 42) = 0$$

As a and c are distinct real numbers the discriminant of the above must be positive, that is, $4b^2 - 4(5b^2/2 - 42) > 0$

$$\Rightarrow \quad 6b^2 < 168 \text{ or } b^2 < 28.$$

$$\text{Also,} \quad ac > 0 \Rightarrow 5b^2 > 84.$$

$$\therefore \quad 84/5 < b^2 < 28.$$

As b is a positive integer, we get $b = 5$.

● **Example 84:** Suppose $a, b \in \mathbf{R}$, $a \ne 0$ and $2a + b \ne 0$. A root of the equation

$(a + b)(ax + b)(a - bx) = (a^2x - b^2)(a + bx)$ is

(a) $\frac{a + 2b}{2a + b}$ (b) $\frac{2a + b}{a + 2b}$

(c) $-\dfrac{a-2b}{a+2b}$ (d) $-\dfrac{a+2b}{2a+b}$

Ans. (d)

◎ **Solution:** We can write the quation as

$$[a^2b + (a + b)\,ab]x^2 + [(a^3 - b^3) - (a + b)\,(a^2 - b^2)]x - ab^2 - ab(a + b) = 0$$

$$\Rightarrow (2a + b)x^2 - (a - b)x - (a + 2b) = 0$$

Since the sum of coefficients is 0 one of the roots is 1 and the other root is $-\dfrac{a+2b}{2a+b}$.

◉ **Example 85:** Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ as its roots is

(a) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(b) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(c) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(d) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

Ans. (b)

◎ **Solution:** $q = \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)$
$= -p^3 + 3\alpha\beta p$

$$\Rightarrow \quad \alpha\beta = \frac{p^3 + q}{3p}$$

We have

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{(\alpha + \beta)^2}{\alpha\beta} - 2 = \frac{p^2}{(p^3 + q)/3p} - 2 = \frac{3p^3 - 2p^3 - 2q}{p^3 + q} = \frac{p^3 - 2q}{p^3 + q}$$

and $\quad \dfrac{\alpha}{\beta} \cdot \dfrac{\beta}{\alpha} = 1$

Thus, required quadratic equation is

$$x^2 - \frac{(p^3 - 2q)}{p^3 + q}\,x + 1 = 0$$

or $\quad (p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

◉ **Example 86:** Suppose $a, b \in \mathbf{R}$. If the roots of $x^2 - ax + b = 0$ are real and distinct and differ by at most 1, then $4b$ lies in the interval

(a) $(a^2 - 1, a^2)$ (b) $[a^2 - 1, a^2)$
(c) $(a^2 - 1, \infty)$ (d) (a^2, ∞)

Ans. (b)

◎ **Solution:** As roots of $x^2 - ax + b = 0$ are real and distinct, $a^2 - 4b > 0 \Rightarrow 4b < a^2$.

If α, β are roots of $x^2 - ax + b = 0$, then

$$|\alpha - \beta| \le 1 \Rightarrow (\alpha - \beta)^2 \le 1$$
$$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta \le 1$$
$$\Rightarrow \quad a^2 - 4b \le 1 \Rightarrow a^2 - 1 \le 4b.$$

Thus, $a^2 - 1 \le 4b < a^2$.

◉ **Example 87:** If a is the minimum root of the equation $x^2 - 3|x| - 2 = 0$, then value of $-1/a$ is

(a) $\left(\sqrt{17} - 3\right)/4$ (b) $\left(\sqrt{17} + 3\right)/4$
(c) 2 (d) -3

Ans. (a)

◎ **Solution:** $x^2 - 3|x| - 2 = 0$ can be written as $|x|^2 - 3|x| - 2 = 0$

$$\Rightarrow \quad |x| = \frac{1}{2}\left(3 \pm \sqrt{17}\right)$$

As $\quad |x| \ge 0$, we get $|x| = \dfrac{1}{2}\left(3 + \sqrt{17}\right)$

$$\Rightarrow \quad x = \pm\frac{1}{2}\left(3 + \sqrt{17}\right)$$

$$\therefore \quad a = -\frac{1}{2}\left(3 + \sqrt{17}\right) \Rightarrow -\frac{1}{a} = \frac{2}{\sqrt{17} + 3} = \frac{\sqrt{17} - 3}{4}$$

◉ **Example 88:** Let $f(x) = \dfrac{x^2 - 2x + 3}{x^2 - 2x - 8}$, $x \in \mathbf{R} - \{-2, 4\}$

The range of f is

(a) $\left(\dfrac{-2}{9}, 1\right]$ (b) $\mathbf{R} - \left(\dfrac{-2}{9}, 1\right)$
(c) $\left(-\infty, \dfrac{-2}{9}\right]$ (d) $\mathbf{R} - \left(\dfrac{-2}{9}, 1\right]$

Ans. (d)

◎ **Solution:** Let $y = \dfrac{x^2 - 2x + 3}{x^2 - 2x - 8}$, $x \in \mathbf{R} - \{-2, 4\}$

Note that $y \neq 1$.

Now,

$$(y - 1)x^2 - 2x(y - 1) - (8y + 3) = 0$$

As x is real

$$4(y - 1)^2 + 4(y - 1)\,(8y + 3) \ge 0$$
$$\Rightarrow \quad (y - 1)\,[y - 1 + 8y + 3] \ge 0.$$
$$\Rightarrow \quad (y - 1)\,(y + 2/9) \ge 0$$
$$\Rightarrow \quad y \le -2/9 \text{ or } y > 1. \; [\because y \neq 1]$$

Thus, $\quad y \in \mathbf{R} - (-2/9, 1]$.

◉ **Example 89:** Suppose a, b, c are three non-zero real numbers. The equation

$$x^2 + (a + b + c)x + (a^2 + b^2 + c^2) = 0 \qquad (1)$$

has

(a) two negative real roots,
(b) two positive real roots,
(c) two real roots with opposite signs,
(d) no real roots

Ans. (d)

◎ **Solution:** Discriminant D of (1) is given by

$$D = (a + b + c)^2 - 4(a^2 + b^2 + c^2)$$

$$= -\{(b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ca) + (a^2 + b^2 - 2ab) + (a^2 + b^2 + c^2)\}$$

$$= -[(b - c)^2 + (c - a)^2 + (a - b)^2 + (a^2 + b^2 + c^2)] < 0$$

Thus, (1) cannot have real roots.

◉ **Example 90:** Suppose $a, b \in \mathbf{R}$. If the equation

$$x^2 - (2a + b)x + (2a^2 + b^2 - b + 1/2) = 0 \qquad (1)$$

has two real roots, then

(a) $a = \frac{1}{2}, b = -1$ (b) $a = -\frac{1}{2}, b = 1$
(c) $a = 2, b = 1$ (d) $a = -2, b = -1$

Ans. (a)

◎ **Solution:** As (1) has real roots,

$$(2a + b)^2 - 4\,(2a^2 + b^2 - b + 1/2) \geq 0$$

$$\Rightarrow \qquad 4a^2 + 3b^2 - 4ab - 4b + 2 \leq 0$$

$$\Rightarrow \qquad (2a - b)^2 + 2(b - 1)^2 \leq 0 \;\Rightarrow\; b = 1, a = \frac{1}{2}$$

◉ **Example 91:** The equation $e^{\sin x} - e^{-\sin x} = 4$ has:

(a) no real roots
(b) exactly one real root
(c) exactly four real roots
(d) infinite number of real roots.

Ans. (a)

◎ **Solution:** Put $e^{\sin x} = y$. Note that $1/e \leq y \leq e$.

Also, the given equation can be written as

$$y - 1/y = 4 \quad \text{or} \quad y^2 - 4y - 1 = 0$$

$$\Rightarrow \qquad y = 2 \pm \sqrt{5}.$$

As $1/e \leq y \leq e$, none of the two values of y is possible.

◉ **Example 92:** If $x^2 - 3x + 2$ is a factor of $x^4 - ax^2 + b$ then the equation whose roots are a and b is

(a) $x^2 - 9x - 20 = 0$ (b) $x^2 - 9x + 20 = 0$
(c) $x^2 + 9x - 20 = 0$ (c) $x^2 + 9x + 20 = 0$.

Ans. (b)

◎ **Solution:** As $x^2 - 3x + 2 = (x - 1)(x - 2)$ is a factor of $x^4 - ax^2 + b$, $x = 1$ and $x = 2$ are zeros of $x^4 - ax^2 + b$, therefore,

$$1 - a + b = 0, \quad 16 - 4a + b = 0$$

$$\Rightarrow \qquad a = 5, b = 4.$$

Thus, equation whose roots a and b is $x^2 - 9x + 20 = 0$.

◉ **Example 93:** Let for $a, a_1 \neq 0, a \neq a_1$

$$f(x) = ax^2 + bx + \mathrm{c}, \quad g(x) = a_1x^2 + b_1x + c_1$$

and $\quad p(x) = f(x) - g(x)$. If $p(x) = 0$ only for $x = -1$

and $\quad p(-2) = 2$, then the value of $p(2)$ is

(a) 3 (b) 9
(c) 6 (d) 18

Ans. (d)

◎ **Solution:** As $a \neq a_1$, $p(x) = f(x) - g(x)$ is a quadratic polynomial.

As $\quad p(x) = 0$ only for $x = -1$, we get

$$p(x) = k(x + 1)^2 \quad \text{where } k = a - a_1$$

As $\quad p(-2) = 2$, we get $2 = k(-2 + 1)^2 = k$.

Thus, $\quad p(2) = 2(2 + 1)^2 = 18$

◉ **Example 94:** If a, b, c are non-zero rational numbers such that $a + b + c = 0$, then the roots of the equation

$$(b + c - a)\,x^2 + (c + a - b)\,x + (a + b - c) = 0$$

are

(a) both rational
(b) both irrational
(c) both purely imaginary
(d) one rational and one irrational

Ans. (a)

◎ **Solution:** As $a + b + c = 0$, we can write the equation

$$2ax^2 + 2bx + 2c = 0$$

or

$$ax^2 + bx + c = 0.$$

Note that $x = 1$ satisfies this equation and its other root is c/a.

◉ **Example 95:** If α and β are the roots of the equation $ax^2 + bx + c = 0$, then roots of

$$ax^2 - bx(x - 1) + c(x - 1)^2 = 0 \qquad (1)$$

are

(a) $\frac{1}{\alpha}, \frac{1}{\beta}$ (b) $\frac{1}{\alpha}, \beta$
(c) $\frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta}$ (d) $\alpha, \frac{1}{\beta}$

Ans. (c)

◎ **Solution:** We can write (1) as

$$a\left(-\frac{x}{x-1}\right)^2 + b\left(\frac{-x}{x-1}\right) + c = 0$$

$$\Rightarrow \qquad \frac{-x}{x-1} = \alpha, \beta \quad \Rightarrow \quad x = \frac{\alpha}{1+\alpha}, \frac{\beta}{1+\beta}$$

◉ **Example 96:** If $3p^2 = 5p + 2$ and $3q^2 = 5q + 2$ then the equation whose roots $3p - 2q$ and $3q - 2p$ is

(a) $x^2 - 5x + 100 = 0$ (b) $3x^2 - 5x - 100 = 0$
(c) $3x^2 + 5x + 100 = 0$ (d) $5x^2 - x + 7 = 0$

Ans. (b)

Solution: Note that p, q are roots of

$$3x^2 = 5x + 2 \quad \text{or} \quad 3x^2 - 5x - 2 = 0$$

$$p + q = 5/3, \; pq = -2/3.$$

Let $\alpha = 3p - 2q, \; \beta = 3q - 2p,$

then $\alpha + \beta = p + q = 5/3$

and $\alpha\beta = (3p - 2q)(3q - 2p)$

$$= -6(p^2 + q^2) + 13pq$$

$$= -6(p + q)^2 + 25pq = -100/3.$$

Thus, required quadratic equation is

$$x^2 - (5/3)x - 100/3 = 0$$

$$\Rightarrow \quad 3x^2 - 5x - 100 = 0$$

Example 97: The number of distinct real roots of the equation

$$(x + 3)^4 + (x + 5)^4 = 16 \qquad (1)$$

is

(a) 1 (b) 2

(c) 3 (d) 4

Ans. (b)

Solution: Put $x + 4 = t$, so that (1) becomes

$$(t - 1)^4 + (t + 1)^4 = 16$$

$$\Rightarrow \quad 2(t^4 + 6t^2 + 1) = 16$$

$$\Rightarrow \quad t^4 + 6t^2 - 7 = 0$$

$$\Rightarrow \quad t^2 = 1, -7 \quad \Rightarrow \quad t = \pm 1, \sqrt{7}i$$

Thus, the equation (1) has two real roots.

Example 98: If $a, b, c, \in$ **R** and the equation $x^2 + (a + b)x + c = 0$ has no real roots, then $c(a + b + c)$ more than

(a) 2 (b) – 2

(c) 0 (d) none of these.

Ans. (c)

Solution: We have $f(x) = x^2 + (a + b)x + c > 0 \; \forall \; x \in \mathbf{R}$

Thus, $f(0) > 0, f(1) > 0$

$\Rightarrow c(a + b + c) > 0$

Example 99: The value of

$$y = 6 + \log_{3/2}\left[\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\,....,}}}\right]$$

is

(a) 2 (b) 3

(c) 4 (d) 8

Ans. (c)

Solution: Let $x =$

$$\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\,....,}}}$$

then $x = \dfrac{1}{3\sqrt{2}}\sqrt{4 - x}$

$$\Rightarrow \quad 18x^2 + x - 4 = 0$$

$$\Rightarrow \quad (9x - 4)(2x + 1) = 0$$

$$\Rightarrow \quad x = 4/9, -1/2$$

As $x > 0$, we get $x = 4/9$.

Thus, $y = 6 + \log_{3/2}(4/9)$

$$= 6 + \log_{3/2}(3/2)^{-2} = 4$$

Example 100: The sum of the roots of the equation

$$x + 1 - 2\log_2(2^x + 3) + 2\log_4(10 - 2^{-x}) = 0 \qquad (1)$$

is

(a) $\log_2 11$ (b) $\log_2 12$

(c) $\log_2 13$ (d) $\log_2 14$

Ans. (a)

Solution: Put $2^x = t$, then (1) can be written as

$$1 + \log_2 t - 2\log_2(t + 3) + \frac{2}{2}\log_2\left(10 - \frac{1}{t}\right) = 0$$

$$\Rightarrow \quad 1 + \log_2 t - \log_2(t + 3)^2 + \log_2(10t - 1) - \log_2 t = 0$$

$$\Rightarrow \quad \log_2(10t - 1)/(t + 3)^2 = -1$$

$$\Rightarrow \quad 10t - 1 = \frac{1}{2}(t + 3)^2$$

$$\Rightarrow \quad t^2 - 14t + 11 = 0 \qquad (2)$$

If $t_1 = 2^{x_1}, t_2 = 2^{x_2}$ are roots of (2),

then, $t_1 t_2 = 2^{x_1} . 2^{x_2} = 11$

$$\Rightarrow \quad x_1 + x_2 = \log_2 11$$

Assertion-Reason Type Questions

Example 101: Suppose $a, b, c, p \in$ **R** $a \neq 0, c \neq 0$ and $b^2 - 4ac > 0$.

Statement-1: If the Roots of $f(x) = ax^2 + bx + c = 0$ are symmetrically placed on the real line about the point p, then $p = b/2a$ and $a(ap^2 + bp + c) < 0$

Statement-2: If the roots of $ax^2 + bx + c = 0$ are equal in magnitude but opposite in signs, then $b = 0$ and $ac < 0$.

Ans. (a)

Solution: If $\alpha, -\alpha$ are roots of $ax^2 + bx + c = 0$,

then $0 = \alpha + (-\alpha) = -b/a \Rightarrow b = 0$

Also, $\alpha(-\alpha) = c/a$.

As $c \neq 0, -\alpha \neq 0$ and therefore $c/a = -\alpha^2 < 0 \Rightarrow ac < 0$.

Thus, Statement-2 is true.

If $p - \alpha$ and $p + \alpha$ are roots of $f(x) = ax^2 + bx + c = 0$, then $\alpha, -\alpha$ are roots of $f(x + p) = a(x + p)^2 + b(x + p) + c = 0$, that is, $\alpha, -\alpha$ are roots of

$$ax^2 + (2ap + b)x + ap^2 + bp + c = 0.$$

By Statement-2: $2ap + b = 0 \Rightarrow p = -b/2a$ and $a(ap^2 + bp + c) < 0$.

Example 102: Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$, $b^2 - 4ac > 0$.
Statement-1: If α, β are roots of $f(x) = ax^2 + bx + c = 0$, such that $\alpha < -1$ and $\beta > 1$, then $a + |b| + c < 0$.
Statement-2: If α, β are roots of $f(x) = ax^2 + bx + c = 0$ such that $\alpha < -1$, $\beta > 1$, then $ac < 0$.
Ans. (d)

Solution: If $\alpha < -1$, $\beta > 1$, then $\alpha < 0$, $\beta > 0$

$$\Rightarrow \qquad \alpha\beta < 0 \Leftrightarrow \frac{c}{a} < 0 \Leftrightarrow ac < 0.$$

Thus, Statement-2 is true.

As $ax^2 + bx + c = 0$ and $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ have the same roots, roots of

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \text{ are } \alpha, \beta.$$

$$\therefore \qquad g(x) = x^2 + \frac{b}{a}x + \frac{c}{a} = (x - \alpha)(x - \beta).$$

Note that $g(x) < 0$ for $\alpha < x <$ and $\alpha < -1 < 1 < \beta$

$$\therefore \qquad g(-1) < 0, g(1) < 0$$

$$\Rightarrow \qquad 1 - \frac{b}{a} + \frac{c}{a} < 0, 1 + \frac{b}{a} + \frac{c}{a} < 0$$

$$\Rightarrow \qquad 1 + \left|\frac{b}{a}\right| + \frac{c}{a} < 0.$$

We cannot conclude $a + |b| + c < 0$ unless $a > 0$.

Example 103: Suppose $a, b, c \in \mathbf{R}$
Statement-1: If $a > 0$, $b^2 - 4ac < 0$, and $f(x) = \sqrt{a + bx + cx^2}$ then domain of f is $\mathbf{R}$
Statement-2: $ax^2 + bx + c > 0 \ \forall\, x \in \mathbf{R}$ if $a > 0$ and $b^2 - 4ac > 0$.
Ans. (a)

Solution: Statement-2 is true. See Theory.
As $ax^2 + bx + c > 0 \ \forall\, x \in \mathbf{R}$, we get $c = a(0)^2 + b(0) + c > 0$.
Thus, $cx^2 + bx + a > 0 \ \forall\, x \in \mathbf{R}$.
Therefore, domain of f is **R.**
Hence, Statement-1 is true and
Statement-2 is a correct explanation for it.

Example 104: Suppose $a, b, c, d \in \mathbf{R}$.
Statement-1: If $a < b < c < d$, then the quadratic equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ has real and distinct roots.
Statement-2: Let $f(x) = ax^2 + bx + c$:
If $f(\alpha) f(\beta) < 0$ for some $\alpha, \beta \in \mathbf{R}$, then $f(x) = 0$ has real and distinct roots.
Ans. (a)

Solution: As $f(x) = ax^2 + bx + c$ is a continuous function and $f(\alpha) f(\beta) < 0$, there is at least one $\gamma \in \mathbf{R}$ lying between α and β such that $f(\gamma) = 0$. As $a, b, c \in \mathbf{R}$ and one root of $f(x) = 0$ is real, other must be real. Also, roots are distinct. Therefore, Statement-2 is true.
Now, let $f(x) = (x - a)(x - c) + 2(x - b)(x - d)$,
then $f(a) = 2(a - b)(a - d) > 0$,

$$f(b) = (b - a)(b - c) > 0$$

Thus, $f(x) = 0$ has a real root lying in the interval (a, b). As coefficients are real and one of the roots is real, the quadratic equation must have real and distinct roots.

Example 105: Suppose $a, b, c\ \alpha \in \mathbf{R}$
Statement-1: The quadratic equation
$(x - \sin\alpha)(x - \cos\alpha) + \sqrt{3} = 0$ has imaginary roots.
Statement-2: If $a > 0$ and $b^2 - 4ac < 0$, then the graph of $y = ax^2 + bx + c$ lies above the x-axis.
Ans. (a)

Solution: For truth of Statement-2, see Theory.
We can write
$(x - \sin\alpha)(x - \cos\alpha) - \sqrt{3} = 0$
as $\qquad f(x) = 0$
where $\qquad f(x) = x^2 - (\sin\alpha + \cos\alpha) + \sin\alpha\cos\alpha + \sqrt{3}$
coefficient of $x^2 = 1 > 0$, and

$$\begin{aligned} b^2 - 4ac &= (\sin\alpha + \cos\alpha)^2 - 4(\sin\alpha\cos\alpha + \sqrt{3}) \\ &= (\sin\alpha - \cos\alpha)^2 - 4\sqrt{3} \\ &\leq (\sqrt{2})^2 - 4\sqrt{3} < 0 \end{aligned}$$

as maximum possible value of $\sin\alpha - \cos\alpha$ is $\sqrt{2}$.
Thus graph of $y = f(x)$ lies above the x-axis and hence $f(x) = 0$ has imaginary roots.

Example 106: Suppose $a, b, c \in \mathbf{R}$ and $a \neq 0$.
Statement-1 If $ac > 0$, $ab < 0$, $b^2 - 4ac > 0$ the equation $ax^4 + bx^2 + c = 0$ has four distinct roots.
Statement-2 If $ac > 0$, $ab < 0$, then the equation $ax^2 + bx + c = 0$ has distinct positive roots.
Ans. (c)

Solution: Statement-2 is false as $x^2 - x + 1 = 0$ satisfies the condition $ac > 0$ and $ab < 0$ but $x^2 - x + 1 = 0$ does not have positive roots.
Statement-1 is true. Put $x^2 = y$, so that $ay^2 + by + c = 0$.
As $b^2 - 4ac > 0$. This equation has two distinct roots.

As $-\frac{b}{a} = -\frac{ab}{a^2} > 0$ and $\frac{c}{a} = \frac{ac}{a^2} > 0$ both the roots of $ay^2 + by + c = 0$ are positive. If the roots are $\alpha, \beta > 0$, then $x^2 = \alpha, \beta$
$\Rightarrow x = \pm\sqrt{\alpha}, \pm\sqrt{\beta}$.

Thus, $ax^4 + bx^2 + c = 0$ has four distinct roots.

Example 107: Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$, and let $f(x) = ax^2 + bx + c$.

Statement-1: If $f(x) = 0$ has imaginary roots and $a + c > 0$, then $f(x) > 0\ \forall\ x \in \mathbf{R}$.

Statement-2: If $f(x) > 0\ \forall\ x \in \mathbf{R}$, then $g(x) = f(x) + f'(x) + f''(x) > 0\ \forall\ x \in \mathbf{R}$.

Ans. (b)

Solution: $f(x) > 0\ \forall\ x \in \mathbf{R}$,

$$\Leftrightarrow \qquad a > 0,\ b^2 - 4ac < 0.$$

We have
$$g(x) = ax^2 + bx + c + (2ax + b) + 2a$$
$$= ax^2 + (b + 2a)x + c + b + 2a$$

Note that
$$(b + 2a)^2 - 4a\,(c + b + 2a)$$
$$= (b^2 - 4ac) - 4a^2 < 0$$

and $a > 0$, thus $g(x) > 0\ \forall\ x \in \mathbf{R}$

$\therefore$ Statement-2 is true.

As $f(x) = 0$ has imaginary roots, $f(x) > 0\ \forall \in \mathbf{R}$ or $f(x) < 0\ \forall\ x \in \mathbf{R}$

We have $f(1) + f(-1) = 2(a + c) > 0$,

therefore $f(x) > 0\ \forall\ x \in \mathbf{R}$.

Thus, statement-1 is true but statement-2 is not a correct explanation for it.

Example 108: Suppose $a, b, c \in \mathbf{R}$ are three distinct real numbers.

Statement-1:

$$a\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\frac{(x-c)(x-a)}{(b-c)(b-a)}$$
$$+\ c\frac{(x-a)(x-b)}{(c-a)(c-b)} = x\ \forall\ x \in \mathbf{R}.$$

Statement-2: If $Ax^2 + Bx + C = 0$ for $x = a, b, c$, then $A = B = C = 0$.

Ans. (a)

Solution: If three distinct values of x satisfy a quadratic equation, then it is identically equal to zero. Thus, Statement-2 is true.

Let

$$f(x) = a\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\frac{(x-c)(x-a)}{(b-c)(b-a)}$$
$$+\ c\frac{(x-a)(x-b)}{(c-a)(c-b)} - x.$$

Then $f(x) = 0$ is a quadratic and $f(a) = f(b) = f(c) = 0$.

Thus, $f(x) \equiv 0$.

Therefore, Statement-1 is true and statement-2 is a correct reason for it.

Example 109: Suppose $a, b, c \in \mathbf{I}$, $a \neq 0$ and $f(x) = ax^2 + bx + c$.

Statement-1: If $f(x) = 0$ has no rational roots, then $\left|f\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^2}\ \forall\ p, q \in \mathbf{I}$, $q \neq 0$ and

Statement-2: If $\alpha + \sqrt{\beta}$, $\alpha, \beta \in \mathbf{Q}$ and $\beta \neq 0$ is a zero of $f(x)$, then its other zero must be $\alpha - \sqrt{\beta}$.

Ans. (b)

Solution: Statement-2 is true. (See Theory)

Let $\quad x = p/q$, $p, q \in \mathbf{I}$ and $q \neq 0$.

Then
$$f\left(\frac{p}{q}\right) = \frac{1}{q^2}\,(ap^2 + bpq + cq^2)$$

As $\quad f(x)= 0$ has no rational roots, $f(p/q) \neq 0$
$$\Rightarrow ap^2 + bpq + cq^2 \neq 0$$

As $ap^2 + bpq + cq^2$ is an integer, we get $|ap^2 + bpq + cq^2| \geq 1$.

Thus, $q^2\left|f\left(\frac{b}{q}\right)\right| \geq 1 \Rightarrow \left|f\left(\frac{p}{q}\right)\right| \geq \frac{1}{q^2}$.

Example 110: Suppose $a, b, c \in \mathbf{R}$ and $a < b < c$.

Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{(x-a)(x-c)}{x-b} & \text{if } x \neq b \\ 0 & \text{if } x = b \end{cases}$$

Statement-1: Range of f is $\mathbf{R}$.

Statemetnt-2: For each $y \in \mathbf{R}$, $y = f(x)$ has exactly two distinct solutions.

Ans. (c)

Solution: Statement-2 is false as $f(a) = f(b) = f(c) = 0$

$\Rightarrow \qquad f(x) = 0$ has three distinct solutions.

Note that Statement-1, put $t = x - b$, so that

$$f(x) = \frac{(t + b - a)(t + b - c)}{t}$$
$$= t - (a + c - 2b) + \frac{(a-b)(c-b)}{t}$$
$$= x - (a + c - b) + \frac{(a-b)(c-b)}{x-b}$$

note that f is continuous on $(-\infty, b)$. Also, $\lim\limits_{x \to -\infty} f(x) = -\infty$

and $\lim\limits_{x \to b-} f(x) = \infty$

Thus, range of f is $(-\infty, \infty) = \mathbf{R}$.

$\therefore$ Statement-1 is true.

LEVEL 2

Straight Objective Type Questions

Example 111: Suppose $a, b, c, d \in \mathbf{R}$ and $2ac = b + d$. Consider the quadratic equations $x^2 + 2ax + b = 0$ and $x^2 + 2cx + d = 0$. Then

(a) none of these equations have real roots
(b) both the equations have real roots
(c) exactly one of the equation has real roots
(d) at least one of the equation has real roots

Ans. (d)

Solution: Let D_1 be the discriminant of $x^2 + 2ax + b = 0$ and D_2 be the discriminant of $x^2 + 2cx + d = 0$

Then

$$\begin{aligned} D_1 + D_2 &= 4(a^2 - b) + 4(c^2 - d) \\ &= 4(a^2 + c^2) - 8ac \\ &= 4(a - c)^2 \geq 0 \end{aligned}$$

$\Rightarrow$ at least one of D_1, D_2 is non-negative. Thus, at least one of the equations has real roots.

Example 112: If the roots of the equation

$$\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c} \qquad (1)$$

are equal in magnitude but opposite in sign, then their product is

(a) $\frac{1}{2}(a^2 + b^2)$ (b) $-\frac{1}{2}(a^2 + b^2)$
(c) $\frac{1}{2}ab$ (d) $-\frac{1}{2}ab$

Ans. (b)

Solution: The equation (1) can be written as

$$c(x + b + x + a) = (x + a)(x + b)$$

or $$x^2 + (a + b - 2c)x + ab - ac - bc = 0 \qquad (2)$$

Let α and $-\alpha$ be the roots of (2) then

$$0 = \alpha + (-\alpha) = a + b - 2c \Rightarrow c = \frac{1}{2}(a + b)$$

Also, $$\begin{aligned} 2\alpha(-\alpha) &= 2ab - 2(a + b)c = 2ab - (a + b)^2 \\ &= -(a^2 + b^2) \end{aligned}$$

$$\Rightarrow \quad \alpha(-\alpha) = -\frac{1}{2}(a^2 + b^2)$$

Example 113: If c, d are roots of $x^2 - 10ax - 11b = 0$ and a, b are root of $x^2 - 10cx - 11d = 0$, then value of $a + b + c + d$ is

(a) 1210 (b) – 1
(c) 2530 (d) – 11

Ans. (a)

Solution: $$c + d = 10a \qquad (1)$$
$$a + b = 10c \qquad (2)$$

Subtracting (1) from (2) we get

$$(a - c) + (b - d) = 10(c - a)$$
$$\Rightarrow \quad b - d = 11(c - a) \qquad (3)$$

As c is a root of $x^2 - 10ax - 11b = 0$,
we get

$$c^2 - 10ac - 11b = 0 \qquad (3)$$

Similarly, $$a^2 - 10ac - 11d = 0 \qquad (4)$$

Subtracting (4) from (3), we get

$$c^2 - a^2 = 11(b - d)$$
$$\Rightarrow \quad (c - a)(c + a) = (11)11(c - a)$$
$$\Rightarrow \quad c + a = 121$$
$$\therefore \quad a + b + c + d = 10(a + c) \qquad \text{[from (1) and (2)]}$$
$$= 10(121) = 1210$$

Example 114: If α, β are the roots of the equation $ax^2 + bx + c = 0$, then the value of $\alpha^3 + \beta^3$ is

(a) $\dfrac{3abc + b^3}{a^3}$ (b) $\dfrac{a^3 + b^3}{3abc}$
(c) $\dfrac{3abc - b^3}{a^3}$ (d) $\dfrac{-(3abc + b^3)}{a^3}$

Ans. (c)

Solution: We have

$$\alpha + \beta = \frac{-b}{a}, \ \alpha\beta = \frac{c}{a}$$

Now, $$\begin{aligned} \alpha^3 + \beta^3 &= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \\ &= \left(\frac{-b}{a}\right)^3 - \frac{3c}{a}\left(\frac{-b}{a}\right) = \frac{3abc - b^3}{a^3} \end{aligned}$$

Example 115: Suppose $a \in \mathbf{I}$ and the equation $(x - a)(x - 5) = 3$ has integral roots, then the set of values which a can take is:

(a) ϕ (b) $\{-11, -13\}$
(c) $\{3, 7\}$ (d) $\{-3, -7\}$

Ans. (c)

Solution: Let $m \in \mathbf{I}$ be a roots of $(x - a)(x - 5) = 3$, then

$$(m - a)(m - 5) = 3$$

As $m - a$ and $m - 5$ are integers, $m - 5 = \pm 1$ or $m - 5 = \pm 3$
$\Rightarrow m = 2, 4, 6, 8.$
Thus, $a = 3, 7$.

Example 116: If the ratio of the roots of the equation $x^2 + bx + c = 0$ is the same as that of the ratio of the roots of $x^2 + qx + r = 0$, then

(a) $br^2 = qc^2$ (b) $cq^2 = rb^2$
(c) $q^2c^2 = b^2r^2$ (d) $bq = rc$

Ans. (b)

Solution: Let α, β be the roots of the equation $x^2 + bx + c = 0$ and γ, δ be the roots of the equation $x^2 + qx + r = 0$. We are given

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} \Rightarrow \frac{\alpha-\beta}{\alpha+\beta} = \frac{\gamma-\delta}{\gamma+\delta}$$

$$\Rightarrow \quad \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 = \left(\frac{\gamma-\delta}{\gamma+\delta}\right)^2$$

$$\Rightarrow \quad \frac{(\alpha+\beta)^2 - 4\alpha\beta}{(\alpha+\beta)^2} = \frac{(\gamma+\delta)^2 - 4\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad \frac{\alpha\beta}{(\alpha+\beta)^2} = \frac{\gamma\delta}{(\gamma+\delta)^2}$$

$$\Rightarrow \quad \frac{c}{(-b)^2} = \frac{r}{(-q)^2} \Rightarrow cq^2 = rb^2$$

Example 117: If a and b are the non-zero distinct roots of $x^2 + ax + b = 0$, then the least value of $x^2 + ax + b$ is

(a) 2/3 (b) 9/4
(c) – 9/4 (d) 1

Ans. (c)

Solution: We have $a + b = -a$, $ab = b$

As $b \neq 0$, we get $a = 1$

$\therefore$ $1 + b = -1 \Rightarrow b = -2$

Thus, $x^2 + ax + b = x^2 + x - 2$

$$= \left(x + \frac{1}{2}\right)^2 - \frac{9}{4} \geq -\frac{9}{4}$$

$\therefore$ least value of $x^2 + ax + b$ is $-9/4$ which is attained at $x = -1/2$.

Example 118: If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has

(a) at least one root in [0, 1]
(b) one root in [2, 3] and other is [– 2, – 1]
(c) imaginary roots
(d) none of these

Ans. (a)

Solution: Let $f(x) = ax^3 + bx^2 + cx$, $x \in [0, 1]$. Since f is a polynomial function, f is differentiable on the whole real line and in particular on [0, 1].

Also, $f(0) = 0$ and $f(1) = a + b + c = 0$.

Thus, all the conditions in the hypothesis of the Rolle's theorem are satisfied. By the Rolle's theorem there exists at least one $\alpha \in (0, 1)$ such that

$$f'(\alpha) = 0$$

But $f'(x) = 3ax^2 + 2bx + c$

Hence, $3ax^2 + 2bx + c = 0$ has a root in [0, 1].

Example 119: If $a < b < c < d$, then the equation

$3(x - a)(x - c) + 5(x - b)(x - d) = 0$

has

(a) real and distinct roots
(b) real and equal roots
(c) purely imaginary roots
(d) none of these

Ans. (a)

Solution: Let $f(x) = 3(x - a)(x - c) + 5(x - b)(x - d)$

Since f is a polynomial, f is continuous on **R**. Also, since $a < b < c < d$,

$$f(a) = 5(a - b)(a - d) > 0$$
$$f(b) = 3(b - a)(b - c) < 0$$
$$f(c) = 5(c - b)(c - d) < 0$$
$$f(d) = 3(d - a)(d - c) > 0$$

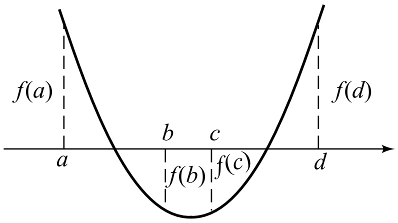


Fig. 3.18

As f is continuous on ***R***, $y = f(x)$ crosses x-axis at least once between a and b and once between c and d. See Fig. 3.18.

Thus, f has two distinct real roots, one lying between a and b and one lying between c and d.

Example 120: For real x, the function $\dfrac{(x-a)(x-c)}{x-b}$ will assume all real values provided

(a) $a < b < c$ (b) $b < c < a$
(c) $c < a < b$ (d) none of these

Ans. (a)

Solution: Let $y = \dfrac{(x-a)(x-c)}{x-b} = \dfrac{x^2 - (a+c)x + ac}{x-b}$

$$\Rightarrow \quad y(x - b) = x^2 - (a + c)x + ac$$

$$\Rightarrow \quad x^2 - (a + c + y)x + ac + by = 0 \qquad (1)$$

Since x is real, the discriminant of (1)

$$(a + c + y)^2 - 4(ac + by) \geq 0$$

$$\Rightarrow \quad y^2 + 2(a + c)y + (a + c)^2 - 4ac - 4by \geq 0$$

$$\Rightarrow \quad y^2 + 2(a + c - 2b)y + (a - c)^2 \geq 0 \qquad (2)$$

Since y takes all real values, (2) is possible if and only if

$$4(a + c - 2b)^2 - 4(a - c)^2 < 0$$

[$\because$ coeff. of $y^2 = 1 > 0$]

$$\Leftrightarrow \quad (a + c - 2b + a - c)(a + c - 2b - a + c) < 0$$

$$\Leftrightarrow \quad (2a-2b)(2c-2b) < 0$$

$$\Leftrightarrow \quad (a-b)(c-b) < 0$$

$\Leftrightarrow$ b lies between a and c. Thus, one of the possibilities $a < b < c$.

Example 121: Let $a, b, c \in \boldsymbol{R}$ and $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies

(a) $\gamma = \frac{1}{2}(\alpha + \beta)$ (b) $\gamma = \alpha + \frac{1}{2}\beta$

(c) $\gamma = \alpha + \beta$ (d) $\alpha < \gamma < \beta$

Ans. (d)

Solution: By the hypothesis,

$$a^2\alpha^2 + b\alpha + c = 0 \qquad (1)$$

and $$a^2\beta^2 - b\beta - c = 0 \qquad (2)$$

where $0 < \alpha < \beta$, and $a \neq 0$.

Let $f(x) = a^2x^2 + 2bx + 2c$

Since f is a polynomial function, f is continuous on $\boldsymbol{R}$ and hence on $[\alpha, \beta]$. Also,

$$\begin{aligned} f(\alpha) &= a^2\alpha^2 + 2b\alpha + 2c \\ &= 2(a^2\alpha^2 + b\alpha + c) - a^2\alpha^2 \\ &= 2(0) - a^2\alpha^2 = -a^2\alpha^2 < 0 \qquad \text{[using (1)]} \end{aligned}$$

and $$\begin{aligned} f(\beta) &= a^2\beta^2 + 2b\beta + 2c \\ &= 3a^2\beta^2 - 2(a^2\beta^2 - b\beta - c) \\ &= 3a^2\beta^2 - 2(0) = 3a^2\beta^2 > 0 \quad \text{[using (2)]} \end{aligned}$$

Since f is continuous on $[\alpha, \beta]$ and $f(\alpha) < 0$, $f(\beta) > 0$, f must vanish at least once in (α, β).

Example 122: Suppose $p, q, r, s \in \boldsymbol{R}$ and α, β be the roots of $x^2 + px + q = 0$ and α^4, β^4 be the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always

(a) two imaginary roots
(b) two positive roots
(c) two negative roots
(d) one positive and one negative root

Ans. (d)

Solution: We have

$$\alpha + \beta = -p,\ \alpha\beta = q,\ \alpha^4 + \beta^4 = r,\ \alpha^4\beta^4 = s$$

Discriminant D of the equation

$$x^2 - 4qx + 2q^2 - r = 0 \qquad (1)$$

is given by $$\begin{aligned} D &= 16q^2 - 4(2q^2 - r) = 8q^2 + 4r = 4[2q^2 + r] \\ &= 4[2\alpha^2\beta^2 + \alpha^4 + \beta^4] = 4(\alpha^2 + \beta^2)^2 \geq 0 \end{aligned}$$

Thus, the equation (1) has real roots.

The roots of (1) are given by

$$\begin{aligned} x &= \frac{4q \pm \sqrt{D}}{2} = 2\alpha\beta \pm (\alpha^2 + \beta^2) \\ &= (\alpha + \beta)^2, -(\alpha - \beta)^2 \end{aligned}$$

Hence, (1) has one positive and one negative root.

Example 123: The equation

$$x^{(3/4)(\log_2 x)^2 + \log_2 x - 5/4} = \sqrt{2} \qquad (1)$$

has

(a) exactly two real roots
(b) no real root
(c) one irrational root
(d) none of these

Ans. (c)

Solution: Taking log of the sides in (1), we get

$$\left[\frac{3}{4}(\log_2 x)^2 + \log_2 x - \frac{5}{4}\right]\log_2 x = \log_2(\sqrt{2})$$

$$\Rightarrow \quad \left(\frac{3}{4}y^2 + y - \frac{5}{4}\right)y = \frac{1}{2}\log_2 2 \text{ where } y = \log_2 x$$

$$\Rightarrow \quad 3y^3 + 4y^2 - 5y - 2 = 0$$

$$\Rightarrow \quad 3y^3 - 3y^2 + 7y^2 - 7y + 2y - 2 = 0$$

$$\Rightarrow \quad (y-1)(3y^2 + 7y + 2) = 0$$

$$\Rightarrow \quad (y-1)[3y^2 + 6y + y + 2] = 0$$

$$\Rightarrow \quad (y-1)(y+2)(3y+1) = 0$$

$$\Rightarrow \quad y = 1, -2, -1/3$$

$$\Rightarrow \quad \log_2 x = 1, -2, -1/3 \Rightarrow x = 2, \frac{1}{4}, 2^{-1/3}$$

Thus, (1) has one irrational root viz. $2^{-1/3}$.

Example 124: Let $f(x)$ be a quadratic expression which is positive for all x. If $g(x) = f(x) + f'(x) + f''(x)$ then for all real x,

(a) $g(x) < 0$ (b) $g(x) > 0$

(c) $g(x) = 0$ (d) $g(x) \geq 0$

Ans. (b)

Solution: Let $f(x) = ax^2 + bx + c$

As $f(x) > 0\ \forall\ x \in \boldsymbol{R}$, we must have $a > 0$ and $b^2 - 4ac < 0$.

Also, $f'(x) = 2ax + b$ and $f''(x) = 2a$

Thus, $g(x) = ax^2 + (2a + b)x + 2a + b + c$.

Since $a > 0$ and discriminant

$$\begin{aligned} &(2a + b)^2 - 4a(2a + b + c) \\ &= 4a^2 + 4ab + b^2 - 8a^2 - 4ab - 4ac \\ &= -4a^2 + (b^2 - 4ac) < 0 \qquad [\because b^2 - 4ac < 0] \end{aligned}$$

we get $$g(x) > 0\ \forall\ x \in \mathbf{R}$$

Example 125: If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then the quadratic equation whose roots are α^3, β^3 is

(a) $a^3y^2 + (b^3 - 3abc)y + c^3 = 0$
(b) $a^3y^2 + (3abc - b^3)y - c^3 = 0$
(c) $a^2y^2 + 2aby + c^2 = 0$
(d) none of these

Ans. (a)

Solution: Put $y = \alpha^3 \Rightarrow \alpha = y^{1/3}$.

As α is a root of $ax^2 + bx + c = 0$, we get

$$ay^{2/3} + by^{1/3} + c = 0 \Rightarrow y^{1/3}(ay^{1/3} + b) = -c \qquad (1)$$

Cubing both the side, we get

$$y(ay^{1/3}+b)^3 = -c^3$$

$$\Rightarrow \quad y[a^3y + b^3 + 3aby^{1/3}(ay^{1/3}+b)] = -c^3$$

$$\Rightarrow \quad y[a^3y + b^3 + 3ab(-c)] = -c^3 \quad \text{[using (1)]}$$

$$\Rightarrow \quad a^3y^2 + (b^3 - 3abc)y + c^3 = 0$$

◉ **Example 126:** If $a, b, c \in \mathbf{R}$ and the equations $ax^2 + bx + c = 0$ and $x^3 + 3x^2 + 3x + 2 = 0$ have two common roots, then

(a) $a = b = -c$ (b) $a = -b = c$

(c) $a = b = c$ (d) none of these

Ans. (c)

◎ **Solution:** $x^3 + 3x^2 + 3x + 2 = 0$

$$\Rightarrow \quad (x+1)^3 = -1$$

$$\Rightarrow \quad x + 1 = -1, -\omega, -\omega^2$$

$$\Rightarrow \quad x = -2, -1, -\omega, -1-\omega^2$$

As $a, b, c \in \mathbf{R}$, the roots of $ax^2 + bx + c = 0$ are both real or both imaginary.

$\therefore$ roots of $ax^2 + bx + c = 0$ must be $-1-\omega, -1-\omega^2$.

Thus, $$-1-\omega-1-\omega^2 = -\frac{b}{a}$$

and $$(-1-\omega)(-1-\omega^2) = \frac{c}{a}$$

$$\Rightarrow \quad 1 = \frac{b}{a} \text{ and } 1 = \frac{c}{a}$$

$$\Rightarrow \quad a = b = c$$

◉ **Example 127:** Let a, b, c be non-zero real number such that

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx$$

Then the quadratic equation $ax^2 + bx + c = 0$ has

(a) no root in (0, 2)

(b) at least one root in (1, 2)

(c) a double root (0, 2)

(d) none of these

Ans. (b)

◎ **Solution:** Let $f(x) = (1 + \cos^8 x)(ax^2 + bx + c)$

We are given

$$\int_0^1 f(x)\,dx = \int_0^2 f(x)\,dx$$

$$\Rightarrow \quad \int_0^1 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$\Rightarrow \quad \int_1^2 f(x)\,dx = 0$$

If $f(x) > 0\ (f(x) < 0)\ \forall\ x \in [1, 2]$,

then $$\int_1^2 f(x)\,dx > 0 \left(\int_1^2 f(x)\,dx < 0\right)$$

But $$\int_1^2 f(x)\,dx = 0$$

$\therefore$ $f(x)$ is partly positive and partly negative on [1, 2].

$\Rightarrow$ there exist $\alpha, \beta \in [1, 2]$ such that

$$f(\alpha) > 0 \text{ and } f(\beta) < 0.$$

As f is continuous on [1, 2] there exists γ lying between α and β (and hence between 1 and 2) such that $f(\gamma) = 0$

$$\Rightarrow \quad (1 + \cos^8 \gamma)(a\gamma^2 + b\gamma + c) = 0$$

$$\Rightarrow \quad a\gamma^2 + b\gamma + c = 0 \qquad [\because\ 1 + \cos^8 \gamma \geq 1]$$

Thus, $ax^2 + bx + c = 0$ has at least one root in [1, 2].

◉ **Example 128:** The number of real solutions of the equation

$$27^{1/x} + 12^{1/x} = 2(8^{1/x}) \qquad (1)$$

is

(a) 0 (b) 1

(c) infinite (d) none of these

Ans. (a)

◎ **Solution:** We can write (1) as

$$\left(\frac{27}{8}\right)^{1/x} + \left(\frac{12}{8}\right)^{1/x} = 2$$

or $$\left(\frac{3}{2}\right)^{3/x} + \left(\frac{3}{2}\right)^{1/x} = 2 \qquad (1)$$

As $3/2 > 1$, $(3/2)^t > 1$ if $t > 0$

and $(3/2)^t < 1$ if $t < 0$.

Thus, $$\left(\frac{3}{2}\right)^{3/x} + \left(\frac{3}{2}\right)^{1/x} < 2 \text{ if } x < 0$$

and $$\left(\frac{3}{2}\right)^{3/x} + \left(\frac{3}{2}\right)^{1/x} > 2 \text{ if } x > 0.$$

Therefore, (1) is possible if $1/x = 0$.

But this is not true for any real value of x.

◉ **Example 129:** If $0 < a < b < c < d$, then the quadratic equation

$$ax^2 + \{1 - a(b+c)\}x + abc - d = 0 \qquad (1)$$

has

(a) real and distinct roots out of which one lies between c and d.

(b) real and distinct roots out of which one lies between a and b

(c) real and distinct roots out of which one lies between b and c

(d) non-real roots

Ans. (a)

◎ **Solution:** We can rewrite (1) as

$$ax^2 - a(b+c)x + abc + x - d = 0$$

or $$a(x-b)(x-c) + x - d = 0$$

Let $f(x) = a(x-b)(x-c) + x - d$.

As $a > 0$, $y = f(x)$ represents a parabola which open upwards. See Fig. 3.19.

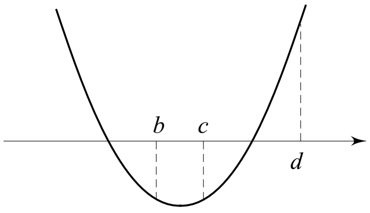


Fig. 3.19

Also, $f(b) = b - d < 0$

$f(c) = c - d < 0$, and $f(d) = a(d - b)(d - c) > 0$

Thus, $f(x) = 0$ has a root between $-\infty$ and b and a root between c and d.

Example 130: If $ax^2 + 2bx - 3c = 0$ has no real roots and $c < \frac{4}{3}(a + b)$, then range of c is

(a) (0, b) (b) $(-1, b)$
(c) $(-\infty, -b^2/3a)$ (d) $(-\infty, -b/12a)$

Ans. (c)

Solution: As $f(x) = ax^2 + 2bx - 3c = 0$ has no real roots, $f(x) > 0\ \forall\ x \in \mathbf{R}$ or $f(x) < 0\ \forall\ x \in \mathbf{R}$

Since $4a + 4b - 3c > 0, f(2) > 0.$

$\therefore \quad f(x) = ax^2 + 2bx - 3c > 0$

$\Rightarrow \quad f(0) = -3c > 0 \Rightarrow c < 0$

Also, $a > 0$ and $b^2 + 3ca < 0$

$\Rightarrow \quad c < -b^2/3a \Rightarrow c \in (-\infty, -b^2/3a)$

Example 131: Suppose $a, b, c \in \mathbf{R}$ and $a \neq b$. If $ax + \frac{b}{x} + c$, $x \in \mathbf{R} - \{0\}$, assume all real values, then a and b satisfy the relation

(a) $ab \leq 0$ (b) $ab \geq 0$
(c) $ab \geq 1$ (d) $ab \leq 1$

Ans. (a)

Solution: Let $y = ax + \frac{b}{x} + c$, then

$$ax^2 + (c - y)x + b = 0.$$

As x is real, $(c - y)^2 - 4ab \geq 0$
This is true for each $y \in \mathbf{R}$ if $ab \leq 0$

EXERCISE

Concept-based Straight Objective Type Questions

1. The number of real solutions of $x^2 - 2x + 2 + |x - 1| = 0$ is:
(a) 0 (b) 1
(c) 2 (d) infinite

2. If roots of $7x^2 - 11x + k = 0$, $k \neq 0$ are reciprocal of each other, then k is equal to
(a) –1 (b) 7/11
(c) 7 (d) 11/7

3. If $a + b \neq 0$ and the roots of $x^2 - px + q = 0$ differ by –1, then $p^2 - 4q$ equals:
(a) –1 (b) 0
(c) 1 (d) 2

4. If the equations $x^2 - ax + b = 0$ and $x^2 + bx - a = 0$ have a common root then
(a) $a = b$ (b) $a + b = -1$
(c) $a - b = -1$ (d) $a - b = 1$

5. If both the roots of $x^2 + x + a = 0$ exceed a, then a belongs to:
(a) $(-\infty, -1)$ (b) $(-\infty, -2)$
(c) (0, 1) (d) $(1, \infty)$

6. The number of real solutions of $x^2 - 4|x| - 2 = 0$ is
(a) 1 (b) 2
(c) 3 (d) 4

7. Suppose α and β are roots of the equation $x^2 + px + \frac{3}{4}p = 0$. If $|\alpha - \beta| = \sqrt{10}$, then p belongs to the set
(a) $\{-2, 5\}$ (b) $\{-3, 2\}$
(c) $\{2, -5\}$ (d) $\{3, -5\}$

8. The number of solutions of $\sqrt{5 + x} + \sqrt{x} = 2$ is
(a) 0 (b) 1
(c) 2 (d) infinite

9. Suppose $a \in \mathbf{R}$. The set of values of a for which the quadratic equation $x^2 - 2(a + 1)x + a^2 - 4a + 3 = 0$ has two negative roots is
(a) $(-\infty, -1)$ (b) (1, 3)
(c) $(-\infty, 1) \cup (3, \infty)$ (d) ϕ

10. Suppose $\alpha > 0$, $\beta > 0$ and $\alpha + \beta = \pi/4$. If $\tan \alpha$, $\tan \beta$ are roots of $x^2 - ax + b = 0$, then

(a) $a + b = 1$ (b) $a + b = 1, 0 < b < 1$
(c) $a = b$ (d) $0 < a + b < 2$

11. If $b > a$, and $c > 0$ then the equation $(x - a)(x - b) - c = 0$ has:
    (a) both roots in $(-\infty, a)$
    (b) both roots in $(a, b]$
    (c) one root in $(-\infty, a)$ and other root in (b, ∞)
    (d) one root in $(-\infty, a)$ and other root in $[a, b]$

12. If the quadratic equation $x^2 + 2(k + 1)x + 9k - 5 = 0$ has exactly one positive root, then k lies in the set
    (a) $[5/9, \infty)$ (b) $(-\infty, 1) \cup (6, \infty)$
    (c) $(-\infty, 5/9]$ (d) $[1, 6]$

13. If $k \in \mathbf{R}$ lies between the roots of $ax^2 + 2bx + c = 0$, then
    (a) $ak^2 + 2bk + c < 0$
    (b) $a^2k^2 + 2abk + ac < 0$
    (c) $a^2k^2 + 2abk + ac > 0$
    (d) $ak^2 + 2bk + c > 0$

14. If both the roots of the quadratic equation $x^2 - 4ax + 2a^2 - 3a + 5$ are less than 2, then a lies in the set
    (a) $(9/2, \infty)$
    (b) $(-\infty, 9/2)$
    (c) $(-1, \infty)$
    (d) $(2, \infty)$

15. Fig. 3.20 Shows graph of $y = ax^2 + bx + c$. Then which one of the following is not true.
    (a) $a > 0$
    (b) $c < 0$
    (c) $b^2 - 4ac > 0$
    (d) $b > 0$

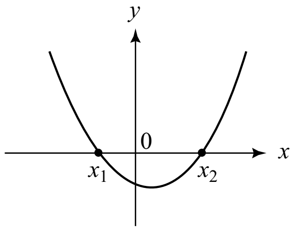


Fig. 3.20

16. Greatest value of the expression $\dfrac{8}{9x^2 - 6x + 5}$ is
    (a) 2 (b) 5
    (c) $8\frac{1}{3}$ (d) 9.2

17. If $a, b \in \{1, 2, 3, 4\}$ the number of quadratic equation of the form $ax^2 + bx + c = 0$ which have non-real complex roots is:
    (a) 27 (b) 35
    (c) 52 (d) 56

18. The number of real roots of the equation $(x - 1)^2 + (x - 2)^2 + (x - 3)^2 = 0$ is
    (a) 0 (b) 2
    (c) 3 (d) infinite

19. Suppose $0 < a < b < c$. If the roots α, β of $ax^2 + bx + c = 0$ are imaginary, then
    (a) $|\alpha| = \sqrt{\dfrac{c}{a}}$ (b) $|\beta| = \sqrt{\dfrac{a}{c}}$
    (c) $\alpha + \beta = 0$ (d) $\alpha - \beta = -b/2a$

20. If $a > 0$ and both the roots of $ax^2 + bx + c = 0$ are more than 1, then
    (a) $a + b + c > 0$ (b) $a + b + 4c = 0$
    (c) $a + b + c < 0$ (d) $a + 4b + c = 0$

LEVEL 1

Straight Objective Type Questions

21. In Fig. 3.21 graph of $y = ax^2 + 2bx + c$ is given. Which one of the following is not true?

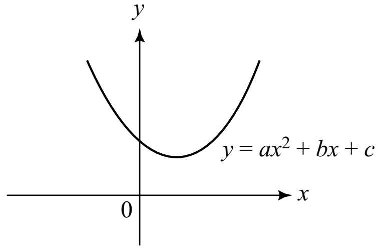


Fig. 3.21

    (a) $a > 0$ (b) $b > 0$
    (c) $c > 0$ (d) $b^2 < ac$

22. If $ax^2 + 2bx + c = 0$ and $ax^2 + 2cx + b = 0$, $b \neq c$ have a common root, then $\dfrac{a + b + c}{a}$ is equal to
    (a) -2 (b) -1
    (c) $3/4$ (d) $-1/4$

23. Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$ and $4a - 6b + 9c < 0$. If $ax^2 + bx + c = 0$ does not have real roots, then $\dfrac{b + c}{a}$ is less than
    (a) 0 (b) 1
    (c) -1 (d) -2

24. Suppose α, β are roots of $4x^2 - 16x + c = 0$, where $c \in \mathbf{N}$, the set of natural numbers. The number of values of c for which $1 < \alpha < 2 < \beta < 3$ is

(a) 2
(b) 3
(c) 4
(d) 9

25. Suppose $a, b \in \mathbf{R}$, $ab \neq 0$. If all the three quadratic equations $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common negative root, then (a, b) is equal to

(a) $(-7, -8)$
(b) $(-8, -7)$
(c) $(7, 8)$
(d) $(8, 7)$

26. Suppose $a, b \in \mathbf{R}$. If the equation $\frac{a}{x} = \frac{1}{x-b} + \frac{1}{x+b}$ is not satisfied by any real value of x, then

(a) $0 < a < b < 2$
(b) $0 < a < 2$
(c) $0 < b < 2$
(d) $0 < b < a < 2$

27. Suppose $a \in \mathbf{R}$, $a \neq -1/2$. Let α, β be roots of $(2a + 1)x^2 - ax + a - 2 = 0$ If $\alpha < 1 < \beta$, then

(a) $\frac{1}{7}\left(6 - 2\sqrt{23}\right) < a < 1$
(b) $\frac{1}{7}\left(6 - 2\sqrt{23}\right) < a < \frac{1}{2}$
(c) $\frac{1}{2} < a < \frac{1}{7}\left(6 + 2\sqrt{23}\right)$
(d) none of these

28. Suppose α, β are roots of $8x^2 - 10x + 3 = 0$, then $\sum_{n=0}^{\infty}\left(\alpha^n + \beta^n\right)$ is

(a) 7/4
(b) 3/7
(c) 6
(d) 7

29. Suppose $a, b, c \in \mathbf{R}$ $a \neq 0$. If $a + |b| + 2c = 0$, then roots of $ax^2 + bx + c = 0$ are

(a) real and distinct
(b) real and equal
(c) purely imaginary
(d) non-real complex numbers

30. Suppose $0 < b < c$ and $f(x) = \frac{x^2 - bc}{2x - (b + c)}$, $x \in \mathbf{R}$, then $f(x)$ cannot lie in

(a) (b, c)
(b) $(-\infty, b)$
(c) (c, ∞)
(d) $(0, b) \cup (b, c)$

31. If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, then value of $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ is

(a) $(r - p)^2 - (q - s)^2$
(b) $(r - p)^2 + (q - s)^2$
(c) $(r - p)^2 + (q - s)^2 - 2rp(r - p)(q - s)$
(d) none of these

32. The number of real solutions of $x^2 + 5|x| + 4 = 0$ is

(a) 4
(b) 2
(c) 1
(d) 0

33. The number of real solutions of $x^2 - 3|x| + 2 = 0$ is

(a) 4
(b) 2
(c) 1
(d) 0

34. If $2 + \sqrt{5}\,i$ is a root of $x^2 - px + q = 0$ where p and q are real, then the ordered pair (p, q) is equal to

(a) $(4, 9)$
(b) $(9, 4)$
(c) $(3, 3)$
(d) $(2, 3)$

35. If the quadratic equation $2x^2 + ax + b = 0$ and $2x^2 + bx + a = 0$, $(a \neq b)$ have a common root, the value of $a + b$ is

(a) -3
(b) -2
(c) -1
(d) 0

36. If a, b, c, d and p are distinct real numbers such that

$(a^2 + b^2 + c^2)p^2 + 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0$,

then a, b, c, d

(a) are in A.P.
(b) are in G.P.
(c) are in H.P.
(d) none of these

37. The number of real roots of the equation $\sin(e^x) = 5^x + 5^{-x}$ is

(a) 0
(b) 1
(c) 2
(d) infinitely many

38. The value of a for which the equations $x^3 + ax + 1 = 0$ and $x^4 + ax^2 + 1 = 0$ have a common root is

(a) 2
(b) -2
(c) 0
(d) none of these

39. The roots of the equation $|x^2 - x - 6| = x + 2$ are

(a) $-2, 1, 4$
(b) $0, 2, 4$
(c) $0, 1, 4$
(d) $-2, 2, 4$

40. The number of real roots of the equation

$x^2 - 3|x| + 2 = 0$

(a) 4
(b) 1
(c) 2
(d) infinite

41. The least value of $n \in \mathbf{N}$ for which $(n - 4)x^2 + 8x + n + 2 > 0$ $\forall$ $x \in \mathbf{R}$, is

(a) 11
(b) 10
(c) 8
(d) 7

42. Let $f(x)$ be a quadratic expression such that $f(x) < 0$ $\forall$ $x \in \boldsymbol{R}$. If $g(x) = f(x) + f'(x) + f''(x)$ then for $x \in \boldsymbol{R}$.

(a) $g(x) < 0$
(b) $g(x) \leq 0$
(c) $g(x) > 0$
(d) $g(x) \geq 0$

43. If $x + 1$ is a factor of $x^4 + (p - 3)x^3 - (3p - 5)x^2 + (2p + 9)x + 12$, then value of p is

(a) -2 (b) 2
(c) 1 (d) -1

44. If both the roots of $x^2 - (p - 4)x + 2e^{2\log p} - 4 = 0$ are negative, then p belongs to

(a) $(-\sqrt{2}, 4)$ (b) $(\sqrt{2}, 4)$
(c) $(4, \infty)$ (d) none of these

45. Let f and g be two real valued functions and $S = \{x|f(x) = 0\}$ and $T = \{x|g(x) = 0\}$, then $S \cap T$ represent the set of roots of

(a) $f(x)\, g(x) = 0$ (b) $f(x)^2 + g(x)^2 = 0$
(c) $f(x) + g(x) = 0$ (d) $\dfrac{f(x)}{g(x)} = 0$

46. If domain of $f(x) = \sqrt{x^2 + bx + 4}$ is $\mathbf{R}$, then maximum possible integral value of b is

(a) 2 (b) 3
(c) 4 (d) 5

47. If $p \in (-1, 1)$, then roots of the quadratic equation $(p - 1)x^2 + px + \sqrt{1 - p^2} = 0$ are

(a) purely imaginary
(b) non-real complex numbers
(c) real and equal
(d) real and distinct.

48. If a, b, c are positive real numbers, then the number of positive real roots of the equation $ax^2 + bx + c = 0$ is

(a) 0 (b) 1
(c) 2 (d) infinite

49. If the roots of the equation $x^2 + p^2 = 8x + 6p$ are real, then p belongs to the interval

(a) $[2, 8]$ (b) $[-8, 2]$
(c) $[-2, 8]$ (d) $[-8, -2]$

50. If sum of the roots of the equation $(a + 1)x^2 + (2a + 3)x + (3a + 4) = 0$ is -3, then the product of the roots is

(a) 1 (b) 4
(c) 2 (d) -2

51. If $3 - 4i$ is a root of $x^2 - px + q = 0$ where $p, q \in \boldsymbol{R}$, then value $\dfrac{2p - q}{p + q}$ is

(a) $-12/31$ (b) $-13/31$
(c) $-15/31$ (d) none of these

52. If $x = 1 + i$ is a root of $x^3 - ix + 1 - i = 0$, then the quadratic equation whose roots are the remaining two roots of $x^3 - ix + 1 - i = 0$ is

(a) $x^2 + (1 + i)x + 1 + i = 0$
(b) $x^2 + (1 + i)x + i = 0$
(c) $x^2 + 2(1 + i)x - 2 = 0$
(d) none of these

53. If α and β be the roots of the equation $x^2 + px - \dfrac{1}{2p^2} = 0$, where $p \in \boldsymbol{R}$. Then the minimum possible value of $\alpha^2 + \beta^2$ is

(a) 2 (b) $2\sqrt{2}$
(c) $2 + \sqrt{2}$ (d) none of these

54. The equation $\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1$ has

(a) no solution
(b) exactly one solution
(c) exactly two solutions
(d) more than two solution

55. The equation $|x - x^2 - 1| = |2x - 3 - x^2|$ has

(a) no solution
(b) exactly one solution
(c) exactly two solutions
(d) more than two solutions

56. If $\sin \alpha$, $\cos \alpha$ are the roots of the equation $ax^2 + bx + c = 0$, $(a \neq 0)$, then

(a) $a^2 - b^2 + 2ac = 0$
(b) $a^2 + b^2 - 2ac = 0$
(c) $(a - c)^2 = b^2 + c^2$
(d) none of these

57. If $x \in \boldsymbol{R}$, and $k = \dfrac{x^2 - x + 1}{x^2 + x + 1}$, then

(a) $1/3 \leq k \leq 3$ (b) $k \geq 5$
(c) $k \leq 0$ (d) none of these

58. If the equation $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have a common root and $b \neq c$, then their other roots will satisfy the equation

(a) $x^2 + (b + c)x + bc = 0$ (b) $x^2 - ax + bc = 0$
(c) $x^2 + ax + bc = 0$ (d) none of these

59. If the inequality $\dfrac{mx^2 + 3x + 4}{x^2 + 2x + 2} < 5$

is satisfied for all $x \in \boldsymbol{R}$, then
(a) $m < 5$ (b) $m > 5$
(c) $m < 71/24$ (d) $m > 71/24$

60. If a, b, c are distinct real numbers, then the quadratic expression

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)}$$

is identically equal to

(a) 1 (b) x
(c) x^2 (d) none of these

61. If $ax^2 + bx + c$, $a, b, c \in \boldsymbol{R}$, $a \neq 0$ has no real zero and $a - b + c < 0$, then value of ac is

(a) positive (b) zero
(c) negative (d) non-negative

62. Suppose $p \in \mathbf{R}$. Let α, β be roots of $x^2 - 2x - (p^2 - 1) = 0$ and γ, δ $(\gamma < \delta)$ be roots of $x^2 - 2(p + 1)x + p(p - 1) = 0$. If $\alpha, \beta \in (\gamma, \delta)$, then p lies in the interval

(a) $(-1/4, 1)$ (b) $(-1, 1)$
(c) $(0, \infty)$ (d) $(\sqrt{2}, \infty)$

63. If α, β are the roots of the equation $ax^2 + 2bx + c = 0$ and $\alpha + h, \beta + h$ are the roots of the equation $Ax^2 + 2Bx + C = 0$, then

(a) $\dfrac{b^2 - ac}{B^2 - AC} = \dfrac{a}{A}$ (b) $b^2 - ac = B^2 - AC$
(c) $h = \dfrac{bA - aB}{2Aa}$ (d) $h = \dfrac{Ac + aC}{Aa + Bb}$

64. The quadratic equation $x^2 + 7x = 14(q^2 + 1)$, where q is an integer has

(a) real and distinct roots
(b) integral roots
(c) imaginary roots
(d) none of these

65. Let $a, b, c \in \boldsymbol{R}$ and $a > 0$. If the quadratic equation $ax^2 + bx + c = 0$ has two real roots α and β such that $\alpha < -1$ and $\beta > 1$, then value of $\dfrac{c}{a} + \left|\dfrac{b}{a}\right|$ is

(a) less than 2 (b) less than 1
(c) less than 0 (d) less than -1

66. Let $a, b, c \in \boldsymbol{R}$ and $a \neq 0$ be such that $(a + c)^2 < b^2$, then the quadratic equation $ax^2 + bx + c = 0$ has

(a) imaginary roots
(b) real roots
(c) two real roots lying between $(-1, 1)$
(d) none of these

67. The integral values of a for which the quadratic equation

$(x - a)(x - 10) + 1 = 0$ has integeral roots are
(a) $-1, 3$ (b) $2, 3$
(c) $12, 8$ (d) $-8, -12$

68. The number of real solution of $4^{x + 1.5} + 9^{x + 0.5} = (10)(6^x)$ is

(a) zero (b) one
(c) two (d) infinite

69. The number of real solution of $2^{\sin^2 x} + 5(2^{\cos^2 x}) = 7$ is

(a) zero (b) 1
(c) finitely many (d) infinitely many

70. The number of values of k for which the equation $x^2 - 2x + k = 0$ has two distinct roots lying in the interval $(0, 1)$ is

(a) 0 (b) 1
(c) 2 (d) infinitely many

71. If roots of the quadratic equation $ax^2 + bx + c = 0$ are real and are of the form $\dfrac{\alpha}{\alpha - 1}, \dfrac{\alpha + 1}{\alpha}$, then value of $(a + b + c)^2$ is

(a) $4ac - b^2$ (b) $b^2 - 4ac$
(c) $c^2 + a^2 - 2b^2$ (d) none of these

72. Let x be an integer and $x^2 + x + 1$ is divisible by 3. When x is divided by 3, it leaves remainder

(a) 0 (b) 1
(c) 2 (d) any of (a), (b) and (c)

73. If α, β are the roots of the equation $x^2 + ax + b = 0$, then maximum value of $-x^2 + ax + b + \dfrac{1}{4}(\alpha - \beta)^2$ is

(a) $\dfrac{1}{4}(a^2 - 4b)$ (b) $\dfrac{1}{4}(b^2 - 4a)$
(c) $\dfrac{a^2}{2}$ (d) none of these

74. If both the roots of the equation $x^2 + bx + c = 0$ lie in the interval $(0, 1)$, then

(a) $b = -1, c = 2$ (b) $b > -2, c < 1$
(c) $b = -5, c < 2$ (d) none of these

75. Let $a, b, c \in \mathbf{R}$ be such that $a + b + c < 0$, $a - b + c < 0$ and $c > 0$. If α and β are roots of the equation $ax^2 + bx + c = 0$, then value of $[\alpha] + [\beta]$ is

(a) 2 (b) 1
(c) -1 (d) 0

76. If roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ lie in the interval $(-2, 4)$, then

(a) $-1 < m < 3$ (b) $1 < m < 5$
(c) $1 < m < 3$ (d) $-1 < m < 5$

77. The value of $\sqrt{8 + 2\sqrt{8 + 2\sqrt{8 + 2\sqrt{8 + \cdots}}}}$ is

(a) 10 (b) 6
(c) 8 (d) none of these

78. The number of solutions of the equation

$$\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}\,x + 4 \text{ is}$$

(a) 1 (b) 2
(c) 0 (d) infinite

79. The number of solutions of $|x + 2| = 2(3 - x)$ is
(a) 1 (b) 2
(c) 3 (d) 0

80. If the equation $ax^2 + bx + c = 0$, $a > 0$ has two distinct roots α and β such that $\alpha < -2$ and $\beta > 3$, then
(a) $c > 0$ (b) $c = 0$
(c) $c < 0$ (d) $c = a - b$

81. Two non-integer roots of
$$(x^2 - 5x)^2 - 7(x^2 - 5x) + 6 = 0$$
are
(a) $\frac{1}{2}(5+\sqrt{29}), \frac{1}{2}(5-\sqrt{29})$
(b) $\frac{1}{2}(-5+\sqrt{29}), \frac{1}{2}(-5+\sqrt{29})$
(c) $\frac{1}{2}(-5+\sqrt{14}), \frac{1}{2}(-5-\sqrt{41})$
(d) none of these

82. The number of real roots of
$$\left(\frac{x-1}{x+1}\right)^4 - 13\left(\frac{x-1}{x+1}\right)^2 + 36 = 0, x \neq -1$$
is
(a) 0 (b) 2
(c) 3 (d) 4

83. The number of negative roots of
$$9^{x+2} - 6(3^{x+1}) + 1 = 0$$
is
(a) 0 (b) 1
(c) 2 (d) 4

84. The number of rational roots of
$$81\left(\frac{2x-5}{3x+1}\right)^4 - 45\left(\frac{2x-5}{3x+1}\right)^2 + 4 = 0,\ x \neq 1/3$$
is
(a) 1 (b) 2
(c) 3 (d) 4

85. The number irrational roots of
$$(x^2 + 3x + 2)^2 - 8(x^2 + 3x) - 4 = 0$$
is
(a) 0 (b) 2
(c) 3 (d) 4

86. The number of roots of the equation
$$\sqrt{\frac{x}{x-3}} + \sqrt{\frac{x-3}{x}} = \frac{5}{2}, x \neq 0, x \neq 3$$
is
(a) 0 (b) 2
(c) 3 (d) 4

87. The number of irrational roots of the equation
$$4\left(x-\frac{1}{x}\right)^2 + 8\left(x+\frac{1}{x}\right) = 29$$
is
(a) 0 (b) 2
(c) 4 (d) infinite

88. Irrational roots of the equation
$$2x^4 + 9x^3 + 8x^2 + 9x + 2 = 0$$
are
(a) $-2-\sqrt{3}, 2+\sqrt{3}$ (b) $2-\sqrt{3}, 2+\sqrt{3}$
(c) $-2+\sqrt{3}, -2-\sqrt{3}$ (d) none of these

89. Sum of the roots of the equation
$$4\left(x-\frac{1}{x}\right)^2 - 4\left(x-\frac{1}{x}\right) + 1 = 0$$
is
(a) 5 (b) 1
(c) $-5/2$ (d) -1

90. The number of irrational roots of the equation
$$(x - 1)(x - 2)(3x - 2)(3x + 1) = 21$$
is
(a) 0 (b) 2
(c) 3 (d) 4

91. Product of roots of the equation
$$x - \sqrt{3x-6} = 2$$
is
(a) 2 (b) 5
(c) 7 (d) 10

92. The number of real roots of
$$2\sqrt{2x+1} = 2x - 1$$
is
(a) 1 (b) 2
(c) 3 (d) 4

93. Product of roots of the equation
$$\sqrt{13-x^2} = x + 5$$
is
(a) -6 (b) 7
(c) 6 (d) -7

94. The number of roots of the equation
$$\sqrt{x^2-4} - (x-2) = \sqrt{x^2-5x+6} \text{ is}$$
(a) 0 (b) 1
(c) 2 (d) 3

95. The product of the roots of the equation

$\sqrt{x^2-4x+3}+\sqrt{x^2-7x+12} = 3\sqrt{x-3}$ is

(a) 15 (b) – 15
(c) 20 (d) – 20

96. Suppose a and b satisfy the equations $18a^2 + 77a + 2 = 0$ and $2b^2 + 77b + 18 = 0$ then value of $\dfrac{ab+a+1}{b}$ is

(a) – 25 (b) – 25/6
(c) 6/25 (d) – 1/25

97. Suppose α, β are roots of $x^2 - 7x + 8 = 0$, with $\alpha > \beta$, then value of $\dfrac{16}{\alpha}+3\beta^2-19\beta$ is

(a) – 10 (b) 10
(c) – 23 (d) 17

98. If α is a root of $x^4 + x^2 - 1 = 0$, the value of $(\alpha^6 + 2\alpha^4)^{2012}$ is

(a) 0 (b) – 1
(c) 1 (d) none of these

99. Sum and product of all the roots of the equation $(x^2 - x - 1)(x^2 - x - 2)(x^2 - x - 3) \ldots (x^2 - x - 2012) = 0$ is

(a) 2012, 2012! (b) – 2012, 2012!
(c) – 2012, – 2012! (d) 2012, – 2012!

100. Suppose three distinct non-zero real numbers satisfy $a^2(a + k) = b^2(b + k) = c^2(c + k)$, where k is some real number, then value of $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$ is

(a) 0 (b) k
(c) $-k$ (d) $2k$

Assertion-Reason Type Questions

101. Suppose $a, b, c, \alpha, \beta \in \mathbf{R}$ $a \neq 0$ and $0 < \alpha < \beta$. Let $f(x) = a^2x^2 + bx + c$, $g(x) = a^2x^2 - bx - c$

Statement-1: If α is a root of $f(x) = 0$ and β is a root of $g(x) = 0$, then there exists $\gamma \in (\alpha, \beta)$ such that γ is a root of $a^2x^2 + 2bx + 2c = 0$.

Statement-2: If a function $h : [\alpha, \beta] \to \mathbf{R}$ is continuous on $[\alpha, \beta]$ and $h(\alpha)\, h(\beta) < 0$, then there exists $\gamma \in (\alpha, \beta)$ such that $h(\gamma) = 0$.

102. Suppose $p \in \mathbf{R}$ and $f(x) = x^2 + 2px + p$

Statement-1: If $f(x) < 0\ \forall\ x \in [1, 2]$ then $p \in (-\infty, -4/5)$.

Statement-2: a lies between the roots of $f(x) = 0$, if and only if $f(a) < 0$.

103. Suppose $a \in \mathbf{R}$.

Statement-1: The equation $\sqrt{a\left(2^x-2\right)+1} = 1 - 2^x$ has a real solution for all $a \in (0, 1]$.

Statement-2: $ax^2 + bx + c = 0$, $a \neq 0$ has two positive roots if $ab > 0$ and $ac > 0$.

104 Suppose $a, b, c \in \mathbf{R}$ and $f(x) = 2(a-x)\left(x+\sqrt{x^2+b^2}\right)$ $x \in \mathbf{R}$.

Statement-1: Maximum value of $f(x)$ is $a^2 + b^2$.

Statement-2: If $a < 0$, the maximum value of $f(x) = ax^2 + bx + c$ is $(b^2 - 4ac)/4a$.

105. Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$.

Statement-1: If $ax^2 + bx + c = x$ does not have real roots, then

$a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c = x$ has two real and two imaginary roots.

Statement-2: If $\alpha + i\beta$, where $\alpha, \beta \in \mathbf{R}$, $\beta \neq 0$ is a root of $ax^2 + bx + c = x$, then its other root must be $\alpha - i\beta$

106. Suppose $a, b, c \in \mathbf{R}$, $a \neq 0$ and $a - b + c < 0$.

Statement-1: If $ax^2 + bx + c = 0$ has imaginary roots, and

$$\left|a\left(x^2+\frac{y}{a}\right)+(b+1)\,x+c\right| = \left|ax^2+bx+c\right|+\left|x+y\right|$$

then (x, y) lies in the half plane $\{(x, y) |\ x + y \leq 0\}$

Statement-2: $|a + b| = |a| + |b| \Leftrightarrow ab \geq 0$

107. $a, b, c \in \mathbf{R}$ and $c < \min\{a, b\}$.

Let $f(x) = \dfrac{(x+a)(x+b)}{x+c}$, $x > -c$.

Statement-1: Minimum value of f is $\left(\sqrt{a-c}+\sqrt{b-c}\right)^2$

Statement-2: If $a < 0$, then $ax^2 + bx + c$ has no minimum value.

108. Suppose $a, b, c, p, q, r \in \mathbf{R}$, $a, p \neq 0$.

Let $f(x) = ax^2 + bx + c$ and $g(x) = px^2 + qx + r$

Statement-1: If $f(x) = g(x)$ for three distinct real values of x then $a = p$, $b = q$ and $c = r$.

Statement-2: If $a \neq p$, then $f(x) - g(x) = 0$ has two real roots.

109. Suppose $p \in \mathbf{R}$.

Statement-1: If $\sin^4 x + p\sin^2 + 1 = 0$ has a solution, then $p \in (-\infty, -2]$

Statement-2: $y + \dfrac{1}{y} \geq 2 \ \ \forall \ y > 0$.

110. Suppose $a, b, c \in \mathbf{R}$ $a \neq 0$ and $c > 0$.

Let $f(x) = ax^2 + bx + c$.

Statement-1: If $f(x) = 0$, does not have real roots, then

$a > \max \left\{\dfrac{b^2}{4c}, b - c\right\}$

Statement-2: If $b^2 - 4ac < 0$ then $f(x) > 0 \ \forall \ x \in \mathbf{R}$.

LEVEL 2

Straight Objective Type Questions

111. If $\alpha \in (0, \pi/2)$, then the expression

$$\sqrt{x^2 + x} + \frac{\tan^2 \alpha}{\sqrt{x^2 + x}}$$

is always greater than or equal to

(a) $2 \tan \alpha$ (b) 2
(c) 1 (d) $\sec^2 \alpha$

112. If $a, b \in \mathbf{R}$, and the equation

$$x^2 + (a - b)x - a - b + 1 = 0$$

has real roots for all $b \in \mathbf{R}$, then a lies in the interval

(a) $(1, \infty)$ (b) $(0, \infty)$
(c) $(-\infty, 1)$ (d) $(-1, 1)$

113. If α and β $(\alpha < \beta)$ are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then

(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) none of these.

114. Suppose $a, b, c \in \mathbf{R}$ and the equation $x^2 + (a + b)x + c = 0$ has no real roots, then which one of the following is **not** true.

(a) $c + c(a + b + c) > 0$
(b) $c - c(a + b - c) > 0$
(c) $(a + b)^2 - 4c < 0$
(d) $c(a + b + c) < 0$

115. If $[x]$ denotes the greatest integer $\leq x$, then number of solutions of the equation $x^2 - 2 - 2[x] = 0$ is

(a) 4 (b) 2
(c) 3 (d) none of these

116. If $[x]$ denotes the greatest integer $\leq x$, and a, b are two odd integers, then number of solutions of $[x]^2 + a[x] + b = 0$ is

(a) 1 (b) 0
(c) 2 (d) infinite

117. Let α, β, γ be distinct real numbers lying in $(0, \pi/2)$, then the equation

$$\frac{1}{x - \sin\alpha} + \frac{1}{x - \sin\beta} + \frac{1}{x - \sin\gamma} = 0, \text{ has}$$

(a) two distinct real roots
(b) two equal roots
(c) two imaginary roots
(d) one real and one imaginary root.

118. If roots of the equation $x^2 + ax + b = 0$ are α, β, then the roots of $x^2 + (2\alpha + a)x + \alpha^2 + a\alpha + b = 0$ are

(a) $1, \beta - \alpha$ (b) $0, \alpha - \beta$
(c) $0, \beta - \alpha$ (d) $0, 1$.

119. Let a, b, c be the sides of a triangle with $a \neq c$ and $\lambda \in \boldsymbol{R}$. If the roots of $x^2 + 2(a + b + c)x + 3\lambda\,(ab + bc + ca) = 0$ are real, then λ lies in

(a) $(-\infty, 4/3)$ (b) $(5/3, \infty)$
(c) $(1/3, 5/3)$ (c) $(4/3, 5{,}3)$

120. If $\tan\theta$ and $\cot\theta$ are roots of $x^2 + 2ax + b = 0$, then least value of $|a|$ is

(a) $\dfrac{1}{2}$ (b) 1
(c) 2 (d) cannot be found.

121. Let α, β, γ be the roots of $x^3 + x^2 - 5x - 1 = 0$, then value of $[\alpha] + [\beta] + [\gamma]$, where $[x]$ denotes the greatest integer $\leq x$, is

(a) 1 (b) 2
(c) -2 (d) -3.

[**Hint:** Show that $f(-3) < 0$, $f(-2) > 0$, $f(-1) > 0$, $f(0) < 0$, $f(1) < 0$, $f(2) > 0$]

122. If tan A and tan B are roots of the quadratic equation $x^2 - px + q = 0$, then the value of $\sin^2$ (A + B) is

(a) $\dfrac{p^2}{p^2 + q^2}$ (b) $\dfrac{p^2}{(q + p)^2}$

(c) $\dfrac{p^2}{(1 - q)^2 + p^2}$ (d) $1 - \dfrac{p^2}{(1 - q)^2}$

123. The equation

$$\sqrt{x+3+4\sqrt{x-1}}+\sqrt{x+8+6\sqrt{x-1}}=1 \text{ has}$$

(a) no solution
(b) only one solution
(c) only two solutions
(d) infinite number of solutions

124. The number of irrational solutions of the equation

$$\sqrt{x^2+\sqrt{x^2+11}}+\sqrt{x^2-\sqrt{x^2+11}}=4 \text{ is}$$

(a) 0 (b) 2
(c) 4 (d) infinite

125. Let a, b, c, p, q be five different non-zero real numbers and x, y, z be three numbers satisfying the system of equations

$$\frac{x}{a}+\frac{y}{a-p}+\frac{z}{a-q}=1$$

$$\frac{x}{b}+\frac{y}{b-p}+\frac{z}{b-q}=1$$

and $$\frac{x}{c}+\frac{y}{c-p}+\frac{z}{c-q}=1$$

then x equals

(a) $\frac{abc}{pq}$ (b) $\frac{pq}{abc}$
(c) $\frac{abc}{p+q}$ (d) none of these.

126. Let $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbf{R}$. Suppose $|f(x)| \le 1\ \forall\ x \in [0, 1]$, then $|a|$ cannot exceed

(a) 5 (b) 6
(c) 7 (d) 8

127. If $a(p + q)^2 + 2bpq + c = 0$ and
$a(p + r)^2 + 2bpr + c = 0$,
then $|q - r|$ equals

(a) $\frac{2}{|a|}\sqrt{(2a+b)bp^2-ac}$ (b) $\frac{2}{|a|}\sqrt{p^2-4ac}$
(c) $p^2+\frac{c}{a}$ (d) none of these.

128. Let $a, b, c \in \mathbf{R}$ be such that $b^2 \ge 4ac$. All the four roots of the equation $ax^4 + bx^2 + c = 0$ will be real if

(a) $a > 0, b < 0, c > 0$ or $a < 0, b > 0, c < 0$
(b) $a > 0, b > 0, c > 0$ or $a < 0, b < 0, c < 0$
(c) $a > 0, b < 0, c > 0$ or $a > 0, b > 0, c < 0$
(d) none of these.

129. If the equations
$x^2 + mx + 1 = 0$ and
$(b - c)x^2 + (c - a)x + (a - b) = 0$ have *a* common root, then

(a) $m = -2$ (b) $m = -1$
(c) $m = 0$ (d) $m = 1$

130. If $x, y, z \in \mathbf{R}$, $x + y + z = 4$ and $x^2 + y^2 + z^2 = 6$, then the maximum possible value of z is

(a) 1 (b) 2
(c) $\frac{3}{2}$ (d) $\frac{4}{3}$

Previous Years' AIEEE/JEE Main Questions

1. If $\alpha \ne \beta$ and $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$, then the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is

(a) $3x^2 - 25x + 3 = 0$ (b) $x^2 - 5x + 3 = 0$
(c) $x^2 + 5x - 3 = 0$ (d) $3x^2 - 19x + 3 = 0$

[2002]

2. If $a \ne b$ and differences between the roots of the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ is the same, then

(a) $a + b + 4 = 0$ (b) $a + b - 4 = 0$
(c) $a - b + 4 = 0$ (d) $a - b - 4 = 0$

[2002]

3. If $a, b, c \in \mathbf{R}$ and $2a + 3b + 6c = 0$, then the equation $ax^2 + bx + c = 0$ has

(a) at least one root in [0, 1]
(b) at least one root in [2, 3]
(c) at least one root in [– 1, 0]
(d) at least one root in (– ∞, 1) **[2002]**

4. Product of real roots of the equation $t^2x^2 + |x| + 9 = 0$, $t \in \mathbf{R}$ is always

(a) positive (b) negative
(c) zero (d) does not exit **[2002]**

5. The value of a for which one root of the quadratic equation

$$(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$$

is twice as large as the other is

(a) $-\frac{2}{3}$ (b) $\frac{1}{3}$
(c) $-\frac{1}{3}$ (d) $\frac{2}{3}$ **[2003]**

6. If sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then $\frac{a}{c}$, $\frac{b}{a}$ and $\frac{c}{b}$ are in
(a) G.P. (b) H.P.
(c) A.G.P. (d) A.P. **[2003]**

7. The number of real solution of the equation $x^2 - 3|x| + 2 = 0$ is
(a) 4 (b) 1
(c) 3 (d) 2 **[2004]**

8. Let two numbers have arithmetic mean 9 and geometric mean 4. Then these numbers are roots of the quadratic equation
(a) $x^2 + 18x - 16 = 0$ (b) $x^2 - 18x + 16 = 0$
(c) $x^2 + 18x + 16 = 0$ (d) $x^2 - 18x - 16 = 0$
[2004]

9. If $1 - p$ is a root of quadratic equation $x^2 + px + 1 - p = 0$, then its roots are
(a) $0, -1$ (b) $-1, 1$
(c) $0, 1$ (d) $-1, 2$ **[2004]**

10. If one root of the equation $x^2 + px + 12 = 0$ is 4, while equation $x^2 + px + q = 0$ has equal roots, then the value of q is
(a) 3 (b) 12
(c) $\frac{49}{4}$ (d) 4 **[2004]**

11. If $2a + 3b + 6c = 0$, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval
(a) (2, 3) (b) (1, 2)
(c) (0, 1) (d) (1, 3) **[2004]**

12. In a triangle PQR, $\angle R = \frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of $ax^2 + bx + c = 0$, $a \neq 0$, then
(a) $b = c$ (b) $b = a + c$
(c) $a = b + c$ (d) $c = a + b$ **[2005]**

13. The value of a for which sum of the squares of the roots of the equation of the roots of the equation
$$x^2 - (a - 2)x - a - 1 = 0$$
assume least value is
(a) 3 (b) 2
(c) 1 (d) 0 **[2005]**

14. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers then $b^2 - 4c$ equals
(a) 2 (b) 1
(c) -2 (d) 3 **[2005]**

15. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k lies in the interval
(a) $(-\infty, 4)$ (b) $[4, 5]$
(c) $(5, 6)$ (d) $(6, \infty)$ **[2005]**

16. If the equation $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x = 0$, $a_1 \neq 0$, $n \geq 2$ has a positive root $x = \alpha$, then the equation
$n\, a_n x^{n-1} + (n - 1)\, a_{n-1} x^{n-2} + \ldots + a_1 = 0$
has a positive root, which is
(a) greater than or equal to α
(b) equal to α
(c) greater than α
(d) smaller than α **[2005]**

17. If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15° respectively, then the value of $2 + q - p$ is
(a) 1 (b) 2
(c) 3 (d) 0 **[2006]**

18. All the values of m for which both the roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4, lie in the interval
(a) $1 < m < 4$ (b) $-2 < m < 0$
(c) $m > 3$ (d) $-1 < m < 3$ **[2006]**

19. If x is real, the maximum value of $\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is
(a) $\frac{17}{7}$ (b) $\frac{1}{4}$
(c) 41 (d) 1 **[2006]**

20. If the difference between the roots of $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then set of possible values of a lie in the interval
(a) $(-3, 3)$ (b) $(-3, \infty)$
(c) $(3, \infty)$ (d) $(-\infty, -3)$ **[2007]**

21. The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one common root. The other roots of first and second equation are integers in the ratio 4 : 3. Then common root is
(a) 1 (b) 4
(c) 3 (d) 2 **[2008]**

22. If the roots of the equation $bx^2 + cx + a = 0$ be imaginary, then for all real values of x, the expression $3b^2x^2 + 6bcx + 2c^2$ is
(a) greater then $-4ab$ (b) less then $-4ab$
(c) greater than $4ab$ (d) less than $4ab$
[2009]

23. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009}$ equals
(a) 1 (b) 2
(c) -2 (d) -1 **[2010]**

24. Let for $a, a_1 \neq 0$, $a \neq a_1$
$f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and
$p(x) = f(x) - g(x)$.
If $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then value of $p(2)$ is:
(a) 3 (b) 9
(c) 6 (d) 18 **[2011]**

25. The equation $e^{\sin x} - e^{-\sin x} = 4$ has
    (a) no real roots
    (b) exactly one real root
    (c) exactly four real roots
    (d) infinite number of real roots. **[2012]**

26. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in \mathbf{R}$ have a common root, then $a : b : c$ is
    (a) 3 : 2 : 1 (b) 1 : 3 : 2
    (c) 3 : 1 : 2 (d) 1 : 2 : 3 **[2013]**

27. If α and β are the roots of the equation $x^2 + px + \frac{3}{4}p = 0$, such that $|\alpha - \beta| = \sqrt{10}$, then p belongs to the set
    (a) $\{2, -5\}$ (b) $\{-3, 2\}$
    (c) $\{-2, 5\}$ (d) $\{-3, 5\}$ **[2013, online]**

28. If p and q are non-zero real number such that $\alpha^3 + \beta^3 = -p$ and $\alpha\beta = q$, then a quadratic equation whose roots are $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$ is
    (a) $px^2 - qx + p^2 = 0$
    (b) $qx^2 + px + q^2 = 0$
    (c) $px^2 + qx + p^2 = 0$
    (d) $qx^2 - px + q^2 = 0$ **[2013, online]**

29. The values of a for which one root of the equation $x^2 - (a + 1)x + a^2 + a - 8 = 0$ exceeds 2 and the other is less than 2 are given by
    (a) $3 < a < 10$ (b) $a > 10$
    (c) $-2 < a < 3$ (d) $a \le -2$ **[2013, online]**

30. The least integral value α of x such that $\frac{x-5}{x^2+5x-14} > 0$, satisfies
    (a) $\alpha^2 + 3\alpha - 4 = 0$ (b) $\alpha^2 - 5\alpha + 4 = 0$
    (c) $\alpha^2 - 7\alpha + 6 = 0$ (d) $\alpha^2 + 5\alpha - 6 = 0$
    [2013, online]

31. Let α and β be the roots of the equation $px^2 + qx + r = 0$, $p \ne 0$. If p, q, r are in A.P. and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then value of $|\alpha - \beta|$ is
    (a) $\frac{1}{9}\sqrt{61}$ (b) $\frac{2}{9}\sqrt{17}$
    (c) $\frac{1}{9}\sqrt{34}$ (d) $\frac{2}{9}\sqrt{13}$ **[2014]**

32. If equations $ax^2 + bx + c = 0$, ($a, b, c \in \mathbf{R}$, $a \ne 0$) and $2x^2 + 3x + 4 = 0$ have a common root, then $a : b : c$ equals:
    (a) 1 : 2 : 3 (b) 2 : 3 : 4
    (c) 4 : 3 : 2 (d) 3 : 2 : 1 **[2014, online]**

33. The sum of the roots of the equation $x^2 + |2x - 3| + 4 = 0$, is:
    (a) 2 (b) -2
    (c) $\sqrt{2}$ (d) $-\sqrt{2}$ **[2014, online]**

34. If α and β are the roots of $x^2 - 4\sqrt{2}\,kx + 2e^{4\ln k} - 1 = 0$ for some k, and $\alpha^2 + \beta^2 = 66$, then $\alpha^3 + \beta^3$ is equal to:
    (a) $248\sqrt{2}$ (b) $280\sqrt{2}$
    (c) $-32\sqrt{2}$ (d) $-280\sqrt{2}$ **[2014, online]**

35. The equation $\sqrt{3x^2 + x + 5} = x - 3$, where x is real, has:
    (a) has no solution
    (b) exactly one solution
    (c) exactly two solutions
    (d) exactly four solutions **[2014, online]**

36. Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$, for $n \ge 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to:
    (a) 6 (b) -6
    (c) 3 (d) -3 **[2015]**

37. If $2 + 3i$ is one of the roots of the equation $2x^3 - 9x^2 + kx - 13 = 0$, $k \in \mathbf{R}$, then the real root of the equation:
    (a) does not exist.
    (b) exists and is equal to $\frac{1}{2}$
    (c) exists and is equal to $-\frac{1}{2}$
    (d) exists and is equal to 1 **[2015, online]**

38. If the two roots of the equation, $(a - 1)(x^4 + x^2 + 1) + (a + 1)(x^2 + x + 1)^2 = 0$ are real and distinct, then the set of all values of a is:
    (a) $\left(-\frac{1}{2}, 0\right)$ (b) $(-\infty, -2) \cup (2, \infty)$
    (c) $\left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$ (d) $\left(0, \frac{1}{2}\right)$ **[2015, online]**

39. The sum of all real values of x satisfying the equation $(x^2 - 5x + 5)^{x^2 + 4x - 60} = 1$ is
    (a) 3 (b) -4
    (c) 6 (d) 5 **[2016]**

40. If the equations $x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$ have a common root different from -1, then $|b|$ is equal to:
    (a) 2 (b) 3
    (c) $\sqrt{3}$ (d) $\sqrt{2}$ **[2016, online]**

41. If x is a solution of the equation, $\sqrt{2x+1} - \sqrt{2x-1} = 1$, $\left(x \ge \frac{1}{2}\right)$, then $\sqrt{4x^2 - 1}$ is equal to:

(a) $\dfrac{3}{4}$ (b) $\dfrac{1}{2}$

(c) $2\sqrt{2}$ (d) 2 **[2016, online]**

42. Let x, y, z be positive real numbers such that $x + y + z = 12$ and $x^3y^4z^5 = (0.1)(600)^3$. Then $x^3 + y^3 + z^3$ is equal to

(a) 342 (b) 216

(c) 258 (d) 270 **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. If the roots of the quadratic equation $x^2 + 2px + q = 0$ are tan 30° and tan 15° respectively, then q is

(a) $1 + p$ (b) $1 - p$

(c) $1 - 2p$ (d) $1 + 2p$ **[2006]**

2. The set of values of α for which the quadratic equation $(\alpha + 2)\,x^2 - 2\alpha x - \alpha = 0$ has two roots on the number line symmetrically placed about the point 1 is

(a) $\{-1, 0\}$ (b) $\{0, 2\}$

(c) ϕ (d) $\{0, 1\}$ **[2007]**

3. The number of solutions of the equation $x^2 - 4|x| - 2 = 0$ is:

(a) 1 (b) 2

(c) 3 (d) 4 **[2008]**

4. The quadratic equation whose roots are a/b and b/a, $a \neq b \neq 0$, where $a^2 = 5a - 3$, and $b^2 = 5b - 3$, is:

(a) $3x^2 - 19x + 3 = 0$

(b) $3x^2 + 19x - 3 = 0$

(c) $3x^2 + 19x + 3 = 0$

(d) $3x^2 - 19x - 3 = 0$ **[2009]**

5. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are α, β, then the roots of the quadratic equation $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$, are

(a) $\dfrac{\alpha+1}{\alpha}, \dfrac{\beta+1}{\beta}$ (b) $\dfrac{\alpha}{\alpha-1}, \dfrac{\beta}{\beta-1}$

(c) $\dfrac{\alpha}{\alpha+1}, \dfrac{\beta}{\beta+1}$ (d) $1 - \alpha, 1 - \beta$ **[2010]**

6. If $x^2 - 3x + 2$ is a factor of $x^4 - ax^2 + b = 0$ then the equation whose roots are a and b is

(a) $x^2 + 9x + 20 = 0$ (b) $x^2 - 9x - 20 = 0$

(c) $x^2 - 9x + 20 = 0$ (d) $x^2 + 9x - 20 = 0$ **[2011]**

7. Let $a, b, c \in \mathbf{R}$, $a > 0$ and the function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = ax^2 + bx + c$.

Statement-1: $b^2 < 4ac \Rightarrow f(x) > 0$ for every value of x.

Statement-2: f is strictly decreasing in the interval $(-\infty, -b/2a)$ and strictly increasing in the interval $(-b/2a, \infty)$. **[2012]**

8. If the quadratic equation $3x^2 + 2(a^2 + 1)x + a^2 - 3a + 2 = 0$ posseses roots of opposite signs, then a lies in the interval:

(a) $(-\infty, -1)$ (b) $(-1, 1)$

(c) $(1, 2)$ (d) $(2, 3)$ **[2013]**

9. If the roots of the equation $\dfrac{1}{x+p} + \dfrac{1}{x+q} = \dfrac{1}{r}$ are equal in magnitude and opposite in sign, then product of roots is:

(a) $p^2 + q^2$ (b) $\dfrac{1}{2}(p^2 + q^2)$

(c) $-\dfrac{1}{2}(p^2 + q^2)$ (d) $-\dfrac{1}{2}(p^2 - q^2)$ **[2014]**

10. The values of k for which each root of the equation, $x^2 - 6kx + 2 - 2k + 9k^2 = 0$ is greater than 3, always satisfy the inequality:

(a) $7 - 9y > 0$ (b) $11 - 9y < 0$

(c) $29 - 11y > 0$ (d) $29 - 11y < 0$ **[2015]**

11. The number of integral values of m for which the equation, $(1 + m^2)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$, has no real root, is:

(a) 2 (b) 3

(c) infinitely many (d) 1 **[2016]**

Answers

Concept-based

1. (a) **2.** (c) **3.** (c) **4.** (d)
5. (b) **6.** (b) **7.** (a) **8.** (a)
9. (d) **10.** (b) **11.** (c) **12.** (c)
13. (b) **14.** (b) **15.** (d) **16.** (a)
17. (c) **18.** (a) **19.** (a) **20.** (a)

Level 1

21. (b) **22.** (c) **23.** (c) **24.** (b)
25. (a) **26.** (b) **27.** (a) **28.** (c)
29. (a) **30.** (a) **31.** (d) **32.** (d)
33. (a) **34.** (a) **35.** (b) **36.** (b)
37. (a) **38.** (b) **39.** (d) **40.** (a)

41. (d)	**42.** (a)	**43.** (b)	**44.** (d)
45. (b)	**46.** (c)	**47.** (d)	**48.** (a)
49. (c)	**50.** (b)	**51.** (b)	**52.** (b)
53. (a)	**54.** (d)	**55.** (b)	**56.** (a)
57. (a)	**58.** (c)	**59.** (c)	**60.** (a)
61. (a)	**62.** (a)	**63.** (c)	**64.** (a)
65. (d)	**66.** (b)	**67.** (c)	**68.** (c)
69. (d)	**70.** (a)	**71.** (b)	**72.** (b)
73. (c)	**74.** (d)	**75.** (c)	**76.** (a)
77. (d)	**78.** (a)	**79.** (a)	**80.** (c)
81. (a)	**82.** (d)	**83.** (b)	**84.** (d)
85. (d)	**86.** (b)	**87.** (b)	**88.** (c)
89. (b)	**90.** (b)	**91.** (d)	**92.** (a)
93. (c)	**94.** (d)	**95.** (a)	**96.** (b)
97. (a)	**98.** (c)	**99.** (a)	**100.** (a)
101. (a)	**102.** (a)	**103.** (c)	**104.** (c)
105. (d)	**106.** (a)	**107.** (b)	**108.** (b)
109. (a)	**110.** (a)		

Level 2

111. (a)	**112.** (a)	**113.** (b)	**114.** (d)
115. (d)	**116.** (b)	**117.** (a)	**118.** (c)
119. (a)	**120.** (b)	**121.** (d)	**122.** (c)
123. (a)	**124.** (b)	**125.** (a)	**126.** (d)
127. (a)	**128.** (a)	**129.** (a)	**130.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (d)	**2.** (a)	**3.** (a)	**4.** (d)
5. (d)	**6.** (b)	**7.** (a)	**8.** (b)
9. (a)	**10.** (c)	**11.** (c)	**12.** (d)
13. (c)	**14.** (b)	**15.** (a)	**16.** (d)
17. (c)	**18.** (d)	**19.** (c)	**20.** (a)
21. (d)	**22.** (a)	**23.** (a)	**24.** (d)
25. (a)	**26.** (d)	**27.** (c)	**28.** (b)
29. (c)	**30.** (d)	**31.** (d)	**32.** (b)
33. (c)	**34.** (b)	**35.** (a)	**36.** (c)
37. (b)	**38.** (c)	**39.** (a)	**40.** (c)
41. (a)	**42.** (b)		

Previous Years' B-Architecture Entrance Examination Questions

1. (d)	**2.** (c)	**3.** (b)	**4.** (a)
5. (c)	**6.** (c)	**7.** (a)	**8.** (c)
9. (c)	**10.** (b)	**11.** (c)	

Hints and Solutions

Concept-based

1. $(x-1)^2 + 1 + |x-1| \geq 1 \ \forall\ x \in \mathbf{R}$

2. $1 = \alpha\left(\dfrac{1}{\alpha}\right) = \dfrac{k}{7}$

3. $\alpha - \beta = -1 = \Rightarrow (\alpha+\beta)^2 - 4\alpha\beta = 1$ or $p^2 - 4q = 1$

4. If α is a common root, then $\alpha^2 - a\alpha + b = 0$, $\alpha^2 + b\alpha - a = 0$

   Subtracting, we get $-(a+b)\ \alpha + b + a = 0$

   $\Rightarrow \alpha = 1$. Thus, $1 - a + b = 0 \Rightarrow a - b = 1$

5. $x^2 + x + a = 0 \Rightarrow \left(x + \dfrac{1}{2}\right)^2 = \dfrac{1}{4} - a$

   We must have $\dfrac{1}{4} - a \geq 0 \Rightarrow a \leq \dfrac{1}{4}$.

   Roots of the equation are $-\dfrac{1}{2} \pm \sqrt{\dfrac{1}{4} - a}$

   These will exceed a, if $-\dfrac{1}{2} - \sqrt{\dfrac{1}{4} - a} > a$

   $\Rightarrow -\sqrt{\dfrac{1}{4} - a} > a + \dfrac{1}{2}$

   This is possible if $a + 1/2 < 0$ and $\sqrt{\dfrac{1}{4} - a} < \left(a + \dfrac{1}{2}\right)$

   $\Rightarrow \dfrac{1}{4} - a < a^2 + a + \dfrac{1}{4}$

   $\Rightarrow a^2 + 2a > 0 \Rightarrow a < -2$ as $a < 0$.

6. $(|x| - 2)^2 = 6 \Rightarrow |x| = 2 \pm \sqrt{6}$.

   As $|x| \geq 0$, $|x| = 2 + \sqrt{6} \Rightarrow x = \pm(2 + \sqrt{6})$

7. $10 = \left(\sqrt{10}\right)^2 = |\alpha - \beta|^2 = (\alpha+\beta)^2 - 4\alpha\beta = p^2 - 3p$

   $\Rightarrow p^2 - 3p - 10 = 0 \Rightarrow (p-5)(p+2) = 0$

   $\Rightarrow p = -2, 5$

8. For the equation to be defined, $x \geq 0$, $5 + x \geq 0$. For $x \geq 0$, $x + 5 \geq 5 \Rightarrow \sqrt{x} + \sqrt{x+5} \geq \sqrt{5} > 2$

9. $D = 4(a+1)^2 - 4(a^2 - 4a + 3) \geq 0$,

   $2(a+1) < 0$ and $a^2 - 4a + 3 > 0$

   First two imply $a \geq 1/3$, $a < -1$. Not possible.

10. $\tan\alpha + \tan\beta = -a$, $\tan\alpha\tan\beta = b$.

    $1 = \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{a}{1-b}$

    Also, $0 < \alpha < \pi/4$, $0 < \beta < \pi/4$

$\Rightarrow 0 < \tan\alpha < 1,\ 0 < \tan\beta < 1$

$\Rightarrow 0 < \tan\alpha\tan\beta < 1 \Rightarrow 0 < b < 1.$

11. Let $f(x) = (x - a)(x - b) - c$

    Use coefficient of $x^2 > 0$,

    $f(a) = -c = f(b) < 0.$

12. $D \ge 0$ and product of roots ≤ 0.

    $\Rightarrow (k + 1)^2 - (9k - 5) \ge 0$ and $9k - 5 \le 0$

    $\Rightarrow k^2 - 7k + 6 \ge 0$ and $\le 5/9$

    $\Rightarrow (k - 1)(k - 6) \ge 0$ and $k \le 5/9$

    Thus, $k \le 5/9$

13. Use : a and $a(k - \alpha)(k - \beta)$ have the opposite signs, where α, β are roots of

    $ax^2 + bx + c = 0.$

14. Write the equation as

    $(x - 2a)^2 = 4a^2 - (2a^2 - 3a + 5)$

    $= (2a + 5)(a - 1)$

    $\Rightarrow x = 2a \pm \sqrt{(2a+5)(a-1)}$

    We must have $(2a + 5)(a - 1) \ge 0$

    and $2a + \sqrt{(2a+5)(a-1)} < 2$

    $\Rightarrow a \le -5/2$ or $a \ge 1$

    and $\sqrt{(2a+5)(a-1)} < -2(a - 1)$

    Clearly, $a \ge 1$ is not possible

    For $a \le -5/2$, we must have

    $\sqrt{-(2a+5)(1-a)} < 2(1 - a)$

    $\Rightarrow -(2a + 5) < 4(1 - a) \Rightarrow a < 9/2$

15. As parabola open upwards, $a > 0$.

    Also, for $x = 0$ $ax^2 + bx + c = c < 0$.

    Since the equation has distinct roots, $b^2 - 4ac > 0$. However, we cannot be sure of sign of b.

    If can be positive, negative or zero. For instance, consider $x^2 + 2x - 3 = 0$, $x^2 - 2x - 3 = 0$ and $x^2 - 3 = 0$.

16. $9x^2 - 6x + 5 = (3x - 1)^2 + 4$ attains least value when $x = 1/3$.

17. $ax^2 + bx + c = 0$ has real roots if

    $b^2 - 4ac \ge 0$, i.e. if $b^2 \ge 4ac$.

    Minimum possible value of b is 2.

    For $b = 2$, $(a, c) = (1, 1)$

    For $b = 3$, $ac \le 9/4 \Rightarrow (a, c) = (1, 1), (1, 2)$ or $(2, 1)$

    For $b = 4$, $ac \le 4 \Rightarrow (a, c) = (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (2, 2)$

    Thus, there $4^3 - 12 = 52$ such quadratic equations.

18. LHS $> 0\ \forall\ x \in \mathbf{R}$.

19. As the roots are imaginary $b^2 - 4ac < 0$ and

    $\alpha, \beta = \frac{1}{2a}\left(-b \pm i\sqrt{4ac - b^2}\right)$

    $|\alpha| = \frac{1}{2a}\sqrt{b^2 + 4ac - b^2} = \sqrt{\frac{c}{a}} = |\beta|$

20. Suppose $\alpha, \beta > 1$, $\alpha < \beta$ be roots of $ax^2 + bx + c = 0$, then

    $ax^2 + bx + c > 0$ if $x < \alpha$ or $x > \beta$

    < 0 if $\alpha < x < \beta$

    Thus, $a + b + c > 0$.

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21. $a > 0$, $c > 0$, $4b^2 - 4ac < 0$. However nothing can be said about b. For instance, consider parabolas $y = x^2 - x + 1$, $y = x^2 + x + 1$ or $y = x^2 + 1$

22. If α is a common root, then $a\alpha^2 + 2b\alpha + c = 0$, $a\alpha^2 + 2c\alpha + \beta = 0$

    Subtracting, we get $2(b - c)\alpha + c - b = 0$

    $\Rightarrow \alpha = \frac{1}{2}$. Thus $\frac{1}{4}a + b + c = 0$

    $\Rightarrow \frac{a+b+c}{a} = \frac{3}{4}$

23. Let $f(x) = ax^2 + bx + c$. As $f(x) = 0$, does not have real roots, $f(x) > 0\ \forall\ x \in \mathbf{R}$ or $f(x) < 0\ \forall\ x \in \mathbf{R}$. But

    $f\left(-\frac{2}{3}\right) = \frac{1}{9}(4a - 6b + 9c) < 0$, therefore

    $f(x) < 0\ x\ \forall\ x \in \mathbf{R}. \Rightarrow f(1) < 0$

    $\Rightarrow a + b + c < 0.$

    $\Rightarrow \frac{b+c}{a} < -1$

24. We have $16^2 - 16c > 0 \Rightarrow c < 16$

    If $f(x) = 4x^2 - 16x + c$, then $1 < \alpha < 2 < \beta < 3$ implies $f(1) > 0, f(2) < 0, f(3) > 0$.

    $\Rightarrow c > 12,\ c < 16,\ c > 12$

    Thus, $c = 13, 14, 15$.

25. If α is a common root of the three equations, then

    $\alpha^2 + a\alpha + 12 = 0$, $\alpha^2 + b\alpha + 15 = 0$

    and $\alpha^2 + (a + b)\alpha + 36 = 0$

    $\Rightarrow b\alpha = -24,\ a\alpha = -21$

    Thus, $\alpha^2 - 21 + 12 = 0 \Rightarrow \alpha = \pm 3$

    As $\alpha > 0$, we get $\alpha = 3$.

    Therefore, $a = -7$, $b = -8$.

26. $\frac{a}{x} = \frac{2x}{x^2 - b^2} \Rightarrow a(x^2 - b^2) = 2x^2$

    $\Rightarrow (a - 2)x^2 - ab^2 = 0$

    This equation will have no solution if $a - 2$ and a have opposite signs, that is , $a(a - 2) < 0$

    $\Rightarrow 0 < a < 2.$

27. As the roots are real and distinct, $\frac{a^2}{(2a+1)^2} - \frac{4(a-2)}{2a+1} > 0$

    $\Rightarrow a^2 - 4(a - 2)(2a + 1) > 0$

    $\Rightarrow \frac{6 - 2\sqrt{23}}{7} < a < \frac{6 + 2\sqrt{23}}{7}$

Write the equation as

$$x^2 - \frac{a}{2a+1}x + \frac{a-2}{2a+1} = 0$$

Note that its roots are α and β.

As $\alpha < 1 < \beta$, we must have

$$1^2 - \frac{a}{2a+1} + \frac{a-2}{2a+1} < 0$$

$\Rightarrow 2a + 1 - a + a - 2 < 0 \Rightarrow a < 1/2$

Thus, $\frac{1}{7}\left(6 - 2\sqrt{23}\right) < a < \frac{1}{2}$

28. $8x^2 - 10x + 3 = 0 \Rightarrow (2x - 1)(4x - 3) = 0$

$\Rightarrow \alpha, \beta = 1/2. 3/4$

$$\therefore \sum_{n=0}^{\infty}\left(\alpha^n + \beta^n\right) = \frac{1}{1-\alpha} + \frac{1}{1-\beta} = 6$$

29. $b^2 - 4ac = (-a - 2c)^2 - 4ac$

$= a^2 + 4c^2 > 0$

30. Let $y = \dfrac{x^2 - bc}{2x - (b+c)}$, then

$x^2 - bc = 2xy - (b + c)y$

$\Rightarrow x^2 - 2xy + (b + c)y + - bc = 0$

As x is real,

$y^2 - (b + c)y + bc \geq 0$

$\Rightarrow (y - b)(y - c) \geq 0$

$\Rightarrow y$ cannot lie in (b, c).

31. $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$

$= (-p\alpha - q + r\alpha + s)(-p\beta - q + r\beta + s)$

$= [(r - p)\alpha - (q - s)][(r - p)\beta - (q - s)]$

$$= (r-p)^2\left[\frac{q-s}{r-p} - \alpha\right]\left[\frac{q-s}{r-p} - \beta\right]$$

$$= (r-p)^2\left[\left(\frac{q-s}{r-p}\right)^2 + p\left(\frac{q-s}{r-p}\right) + q\right]$$

$= (q - s)^2 + p(q - s)(r - p) + q(r - p)^2$

32. Use $x^2 \geq 0$, $|x| \geq 0$

33. $x^2 - 3|x| + 2 = 0 \quad \Rightarrow \quad |x| = 1, 2 \quad \Rightarrow \quad x = \pm 1, \pm 2$

34. As $p, q \in \mathbf{R}$, other root of the equation is $2 - i\sqrt{5}$. Thus,

$p = 4$, $q = \left(2 + \sqrt{5}i\right)\left(2 - \sqrt{5}i\right) = 9$

35. Subtract one equation from the other to obtain $x = 1$.

36. Write the expression as

$(ap + b)^2 + (bp + c)^2 + (cp + d)^2 \leq 0.$

$\Rightarrow ap + b = 0, bp + c = 0, cp + d = 0$

$$\Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = -p.$$

37. Use $5^x + 5^{-x} \geq 2 \ \forall \ x \in \mathbf{R}$.

38. Multiply the first equation by x and subtract from the second to obtain $x = 1$.

39. $x + 2 \geq 0 \quad \Rightarrow \quad x \geq -2$

Now, $|(x + 2)(x - 3)| = x + 2$

$|x + 2|\,|x - 3| = x + 2$

$\Rightarrow \quad x + 2 = 0$ or $|x - 3| = 1$

$\Rightarrow \quad x = -2, x = 3 \pm 1$ *i.e.*, $x = -2, 2, 4$

40. $|x|^2 - 3|x| + 2 = 0 \quad \Rightarrow \quad |x| = 1, 2 \quad \Rightarrow \quad x = \pm 1, \pm 2$

41. Use $n - 4 > 0$ and $D < 0$

42. Let $f(x) = ax^2 + bx + c$

$f(x) < 0 \ \forall \ x \in \mathbf{R} \quad \Leftrightarrow \quad a < 0, b^2 - 4ac < 0.$

Now, $g(x) = ax^2 + bx + c + (2ax + b) + 2a$

$= ax^2 + (2a + b)x + (2a + b + c)$

Here $\quad a < 0$ and $(2a + b)^2 - 4a(2a + b + c)$

$= (b^2 - 4ac) - 4a^2 < 0.$

43. Use the fact that expression becomes zero when $x = -1$.

44. Note that $p > 0$, and

$p - 4 < 0, 2p^2 - 4 > 0,$

and $\quad (p - 4)^2 - 4(2p^2 - 4) \geq 0$

$\Rightarrow \quad p < 4, p > \sqrt{2}$ and $7p^2 + 8p - 32 \leq 0.$

$$\Rightarrow \quad \sqrt{2} < p < \frac{4}{7}\left(\sqrt{15} - 1\right)$$

45. $f(x)^2 + g(x)^2 = 0 \quad \Leftrightarrow \quad f(x) = 0$ and $g(x) = 0$

46. Use D < 0

47. Show that D > 0

48. $ax^2 + bx + c > 0 \ \forall \ x > 0.$

49. $x^2 - 8x + p^2 - 6p = 0$ has real roots

$\Rightarrow 64 - 4(p^2 - 6p) \geq 0$

$\Rightarrow p^2 - 6p - 16 \leq 0 \Rightarrow (p - 8)(p + 2) \leq 0$

$\Rightarrow p \in [-2, 8]$

50. $-\dfrac{2a+3}{a+1} = -3 \quad \Rightarrow \quad a = 0.$

$\therefore$ Product of roots $= \dfrac{3a+4}{a+1} = 4$

51. Use that other root of the equation is $3 + 4i$ to obtain $p = 6$, $q = 25$

52. Let other roots be β, γ then

$(1 + i) + \beta + \gamma = 0$ and $(1 + i)\beta\gamma = -(1 - i)$

$\Rightarrow \quad \beta + \gamma = -(1 + i), \beta\gamma = i$

53. $\alpha + \beta = -p$, $\alpha\beta = -\dfrac{1}{2p^2}$.

$\Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 + \dfrac{1}{p^2}$

$$\Rightarrow \alpha^2 + \beta^2 = \left(p - \frac{1}{p}\right)^2 + 2 \geq 2.$$

54. Put $\sqrt{x-1} = t$ so that $x = t^2 + 1$

$$\therefore \sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1$$

$\Rightarrow \sqrt{t^2 - 4t + 4} + \sqrt{t^2 - 6t + 9} = 1$

$\Rightarrow |t - 2| + |t - 3| = 1 \Rightarrow 2 \le t \le 3$

$\Rightarrow 2 \le \sqrt{x-1} \le 3$

$\Rightarrow 4 \le x - 1 \le 9 \quad \Rightarrow \quad 5 \le x \le 10.$

55. $|x - x^2 - 1| = |2x - 3 - x^2| \Rightarrow \quad |x^2 - x + 1| = |x^2 - 2x + 3|$

$\Rightarrow \left|\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}\right| = |(x - 1)^2 + 2|$

$\Rightarrow x^2 - x + 1 = x^2 - 2x + 3 \quad \Rightarrow \quad x = 2$

56. $\sin \alpha + \cos \alpha = -\frac{b}{a}$, $\sin \alpha \cos \alpha = \frac{c}{a}$

Now $1 = \sin^2\alpha + \cos^2\alpha$
$= (\sin \alpha + \cos \alpha)^2 - 2\sin \alpha \cos \alpha$

$\Rightarrow \quad 1 = \frac{b^2}{a^2} - \frac{2c}{a} \quad \Rightarrow \quad a^2 = b^2 - 2ac$

$\Rightarrow \quad a^2 - b^2 + 2ac = 0.$

57. $k = \frac{x^2 - x + 1}{x^2 + x + 1}$

$\Rightarrow \quad (k - 1)x^2 + (k + 1)x + (k - 1) = 0$
As x is real $(k + 1)^2 - 4(k - 1)^2 \ge 0$
$\Rightarrow (k + 1 + 2k - 2)\ (k + 1 - 2k + 2) \ge 0$
$\Rightarrow (3k - 1)\ (3 - k) \ge 0 \Rightarrow 1/3 \le k \le 3.$

58. $\alpha^2 + b\alpha + ca = 0$ and $\alpha^2 + c\alpha + ab = 0$
$\Rightarrow (b - c)\alpha + a(c - b) = 0 \quad \Rightarrow \quad \alpha = a.$
Now, show that $a + b + c = 0$ and use the fact that other roots are b and c.

59. $mx^2 + 3x + 4 < 5\ (x^2 + 2x + 2)$
$\Rightarrow (m - 5)x^2 - 7x - 6 < 0\ \forall\, x \in \mathbf{R}$
$\Rightarrow m - 5 < 0$ and $(-7)^2 + 24(m - 5) < 0$
$\Rightarrow m < 5$ and $24m - 71 < 0 \Rightarrow m < 71/24$

60. Use the fact that expression becomes equal to 1 for $x = a, b, c$.

61. As $ax^2 + bx + c$ has no real zeros.
$ax^2 + bx + c > 0\ \forall\, x \in \mathbf{R}$ or $ax^2 + bx + c < 0\ \forall\, x \in \mathbf{R}$.
Since $a(-1)^2 + b(-1) + c < 0$, we get $ax^2 + bx + c < 0\ \forall\, x \in \mathbf{R}$
Thus, $a < 0$ and $c < 0 \Rightarrow ac > 0$.

62. $x^2 - 2x - p^2 + 1 = 0$
$\Rightarrow (x - 1)^2 = p^2$
$\Rightarrow x = 1 \pm p$
Let $\alpha = 1 - p$, $\beta = 1 + p$
Let $f(x) = x^2 - 2(p + 1)x + p(p - 1)$
As $f(x) = 0$ has distinct real roots,
$(p + 1)^2 - p(p - 1) > 0$

$\Rightarrow 3p + 1 > 0 \Rightarrow p > -\frac{1}{3}$

Also, $f(1 - p) < 0$,
and $f(1 + p) < 0$

$\Rightarrow (1 - p)^2 - 2(1 + p)\ (1 - p) + p(p - 1) < 0$
and $(1 + p)^2 - 2(1 + p)^2 + p(p - 1) < 0$

$\Rightarrow -1/4 < p < 1$ and $p > -\frac{1}{3}$

Thus, $p \in \left(-\frac{1}{4}, 1\right)$

63. $\alpha + \beta = -\frac{b}{a}$, $\alpha + \beta + 2h = -\frac{B}{A}$

64. $D = 49 + 56(q^2 + 1) > 0.$

65. Note that α, β are roots of

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

As $\alpha < -1$, $\beta > 1$, we get

$1 - \frac{b}{a} + \frac{c}{a} < 0$

and $1 + \frac{b}{a} + \frac{c}{a} < 0$

$\Rightarrow \quad 1 + \left|\frac{b}{a}\right| + \frac{c}{a} < 0$

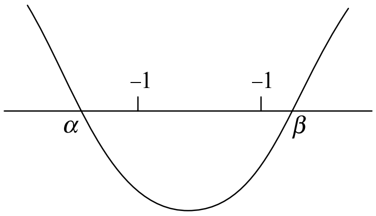


Fig. 3.22

66. Let $f(x) = ax^2 + bx + c$.
Now, $(a + c)^2 < b^2$
$\Leftrightarrow \quad (a - b + c)\ (a + b + c) < 0$
$\Rightarrow f(-1)\ f(1) < 0.$
Thus, $ax^2 + bx + c = 0$ has a real root between -1 and 1.
The other root also must be real.

67. If x is an integer and
$(x - a)\ (x - 10) = -1$
$\Rightarrow x - 10 = 1$ and $x - a = -1$ or $x - 10 = -1$ and $x - a = 1$
$\Rightarrow x = 11$ and $a = x + 1 = 12$ or $x = 9$ and $a = x - 1 = 8$

68. $4^{1.5}\left(\frac{2}{3}\right)^x + 9^{0.5}\left(\frac{3}{2}\right)^x = 10$

Put $\left(\frac{3}{2}\right)^x = t$, so that

$3t + 8/t = 10 \quad \Rightarrow \quad 3t^2 - 10t + 8 = 0$

$\Rightarrow (3t - 4)\ (t - 2) = 0 \quad \Rightarrow \quad t = \frac{4}{3}, 2$

$\Rightarrow \left(\frac{3}{2}\right)^x = \frac{4}{3}, 2$

$$\Rightarrow x = \frac{\log(4/3)}{\log(3/2)}, \frac{\log 2}{\log(3/2)}$$

69. Put $2^{\sin^2 x} = t$, so that equation becomes

$$t + \frac{5\times 2}{t} = 7 \Rightarrow t^2 - 7t + 10 = 0 \quad \Rightarrow \quad t = 2, 5$$

But $1 \le t \le 2$. Thus, $\sin^2 x = 1$

70. Let the two distinct roots lying between 0 and 1 be α, β such that $\alpha < \beta$.
As $f(x) = x^2 - 2x + k$ is a differentiable and $f(\alpha) = f(\beta) = 0$, therefore by the Roll's theorem there exists $\gamma \in (\alpha, \beta) \subset (0, 1)$ such that $f'(\gamma) = 0$
$\Rightarrow \quad 2\gamma - 2 = 0 \quad \Rightarrow \quad \gamma = 1$. Not possible.

71. $\dfrac{\alpha}{\alpha-1}+\dfrac{\alpha+1}{\alpha}=-\dfrac{b}{a}$, and $\dfrac{\alpha}{\alpha-1}.\dfrac{\alpha+1}{\alpha}=\dfrac{c}{a}$

Now, $(a + b + c)^2 = a^2\left(1+\dfrac{b}{a}+\dfrac{c}{a}\right)^2$

$$= a^2\left[1-\left\{\frac{\alpha}{\alpha-1}+\frac{\alpha+1}{\alpha}\right\}+\frac{\alpha+1}{\alpha-1}\right]^2$$

$$= a^2\left(\frac{\alpha}{\alpha-1}-\frac{\alpha+1}{\alpha}\right)^2$$

$$= a^2\left[\left(\frac{\alpha}{\alpha-1}+\frac{\alpha+1}{\alpha}\right)^2-4\frac{\alpha}{\alpha-1}.\frac{\alpha+1}{\alpha}\right]$$

$$= a^2\left[\frac{b^2}{a^2}-\frac{4c}{a}\right]=b^2-4ac$$

72. Let $x = 3m + r$, $0 \le r \le 2$.
$x^2 + x + 1 = (9m^2 + 6mr + 3m) + (r^2 + r + 1)$
Note that $x^2 + x + 1$ will be divisible by 3 if and only if $r = 1$.

73. $f(x) = \dfrac{1}{4}(\alpha - \beta)^2 - x^2 - (\alpha + \beta)x + \alpha\beta$

$$= \frac{(\alpha-\beta)^2+4\alpha\beta}{4}-\left\{x-\frac{\alpha+\beta}{2}\right\}^2+\frac{(\alpha+\beta)^2}{4}$$

$$= \frac{1}{2}(\alpha+\beta)^2-\left\{x+\frac{a}{2}\right\}^2 \le \frac{a^2}{2}$$

Thus, maximum value $f(x)$ is $\dfrac{a^2}{2}$ which is attained when $x = -\dfrac{a}{2}$

74. Let α, β be roots of $f(x) = x^2 + bx + c$, then $0 < \alpha, \beta < 1 \Rightarrow 0 < \alpha + \beta < 2$ and $\alpha\beta < 1$
$\Rightarrow \quad -b < 2$ and $c < 1$
$\Rightarrow \quad b > -2$ and $c < 1$
However, these conditions are not sufficient to guarantee that the roots are real. For instance, $x^2 + 0x + 0.5 = 0$ does not have real roots.

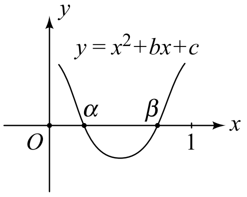


Fig. 3.23

75. Let $f(x) = ax^2 + bx + c$.
Note that $f(0) = c > 0$, $f(-1) = a - b + c < 0$ and $f(1) = a + b + c < 0$
Thus, $y = ax^2 + bx + c$ represents a parabola that open downwards. See Fig.3.21
Thus $[\alpha] = -1$ and $[\beta] = 0$
$\Rightarrow [\alpha] + [\beta] = -1$

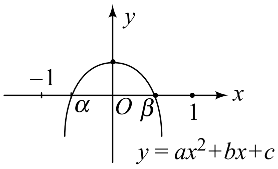


Fig. 3.24

76. $x^2 - 2mx + m^2 - 1 = 0 \Rightarrow (x - m)^2 = 1$
$\Rightarrow x - m = \pm 1 \quad \Rightarrow \quad x = m \pm 1$
Therefore, $-2 < m - 1 < m + 1 < 4$
$\Rightarrow \quad -1 < m < 3$.

77. $y = \sqrt{8+2\sqrt{8+2\sqrt{8+2\sqrt{8+\cdots}}}}$

$\Rightarrow \quad y^2 = 8 + 2y \quad \Rightarrow \quad y^2 - 2y - 8 = 0$
$\Rightarrow \quad y = 4, -2$. As $y > 0$, $y = 4$

78. $\sin\left(\dfrac{\pi x}{2\sqrt{3}}\right)=\left(x-\sqrt{3}\right)^2+1\ge 1$

But $\sin\left(\dfrac{\pi x}{2\sqrt{3}}\right) \le 1$. Thus, $\sin\left(\dfrac{\pi x}{2\sqrt{3}}\right) = 1$

Therefore, $x = \sqrt{3}$

79. $2(3 - x) = |x + 2| \ge 0 \quad \Rightarrow \quad x \le 3$
If $x \le -2$, the equation becomes
$2(3 - x) = -x - 2 \quad \Rightarrow \quad x = 8$. Not possible
If $-2 < x \le 3$, the equation becomes
$2(3 - x) = x + 2 \quad \Rightarrow \quad 3x = 4 \quad \Rightarrow \quad x = 4/3$.

80. Use product of roots < 0

81. Put $x^2 - 5x = y$;
so that $y^2 - 7y + 6 = 0 \quad \Rightarrow \quad y = 1, 6$
$\therefore\ x^2 - 5x = 1$ or $x^2 - 5x = 6$
$\Rightarrow x^2 - 5x - 1 = 0$ or $x^2 - 5x - 6 = 0$
But $x^2 - 5x - 6 = 0$ has integral roots and roots of $x^2 - 5x - 1 = 0$ are

$$x = \frac{1}{2}\left(5\pm\sqrt{29}\right)$$

82. Put $\left(\dfrac{x-1}{x+1}\right)^2 = y$ to obtain
$y^2 - 13y + 36 = 0 \quad \Rightarrow \quad y = 4, 9$
$\therefore\ \dfrac{x-1}{x+1} = \pm 2, \pm 3$
Thus, the given equation has four real roots.

83. $9^{x+2} - 6(3^{x+1}) + 1 = 0$
Put $3^{x+1} = t$ so that $t^2 - 6t + 1 = 0$

$$\Rightarrow t = \frac{6\pm\sqrt{32}}{2}=3\pm 2\sqrt{2}$$

$\Rightarrow x + 1 = \log_3 (3 + 2\sqrt{2}) > 1$

or $x + 1 = \log_3 (3 - 2\sqrt{2}) < 0$

84. Put $\left(\frac{2x-5}{3x+1}\right)^2 = t$, so that equation becomes

$81t^2 - 45t + 4 = 0$

$\Rightarrow 81t^2 - 36t - 9t + 4 = 0$

$\Rightarrow (9t - 1)(9t - 4) = 0$

$\Rightarrow t = \frac{1}{9}, \frac{4}{9} \Rightarrow \frac{2x-5}{3x+1} = \pm\frac{1}{3}, \pm\frac{2}{3}$

Thus, the given equation has four rational roots.

85. Put $x^2 + 3x + 2 = t$ to obtain $t^2 - 8(t - 2) - 4 = 0$

$\Rightarrow t^2 - 8t + 12 = 0 \Rightarrow (t - 2)(t - 6) = 0$

$\Rightarrow t = 2, 6$

But both $x^2 + 3x + 2 = 2$

and $x^2 + 3x + 2 = 6$ have integral roots.

86. Put $\sqrt{\frac{x}{x-3}} = t$ to obtain $t + \frac{1}{t} = \frac{5}{2}$

$\Rightarrow \quad t = 2, \frac{1}{2}$

$\therefore \sqrt{\frac{x}{x-3}} = 2, \frac{1}{2} \quad \Rightarrow \quad \frac{x}{x-3} = 4, \frac{1}{4}$

87. Put $x + 1/x = t$ to obtain

$4\{t^2 - 4\} + 8t = 29$ or $4t^2 + 8t - 45 = 0$

$\Rightarrow t^2 + 2t = \frac{45}{4} \Rightarrow (t + 1)^2 = \frac{45}{4} + 1 = \left(\frac{7}{2}\right)^2$

$\Rightarrow t = -1 \pm \frac{7}{2} = -\frac{9}{2}, \frac{5}{2}$

Thus, $\quad x + \frac{1}{x} = -\frac{9}{2}, \frac{5}{2}$

$\Rightarrow \quad x^2 + \frac{9}{2}x + 1 = 0, \; x^2 - \frac{5}{2}x + 1 = 0$

The first equation has irrational roots and the second equation has rational roots.

88. Divide the given equation by x^2 to obtain

$$2\left(x^2 + \frac{1}{x^2}\right) + 9\left(x + \frac{1}{x}\right) + 8 = 0$$

Put $x + \frac{1}{x} = t$ to get $2(t^2 - 2) + 9t + 8 = 0$

$\Rightarrow t = -\frac{1}{2}, -4$

$\therefore \; x + \frac{1}{x}, = -\frac{1}{2} - 4$

$\Rightarrow \quad \frac{x^2+1}{x} = -\frac{1}{2}, \; \frac{x^2+1}{x} = -4$

$\Rightarrow 2x^2 + x + 2 = 0, \quad x^2 + 4x + 1 = 0$

The first equation has non-real complex roots and the second equation has irrational roots. *viz.* $-2 \pm \sqrt{3}$.

89. Put $x - \frac{1}{x} = t$ to obtain $4t^2 - 4t + 1 = 0$

$\Rightarrow \quad t = \frac{1}{2}, \frac{1}{2}$

Now, $x - \frac{1}{x} = \frac{1}{2} \quad \Rightarrow \quad \frac{x^2 - 1}{x} = \frac{1}{2}$

$\Rightarrow \quad 2x^2 - x - 2 = 0$

$\therefore$ Sum of roots of the given equation $= \frac{1}{2} + \frac{1}{2} = 1$.

90. $(3x - 2)(x - 1)(3x + 1)(x - 2) = 21$

$\Rightarrow (3x^2 - 5x + 2)(3x^2 - 5x - 2) = 21$

$\Rightarrow (3x^2 - 5x)^2 - 4 = 21 \quad \Rightarrow \quad 3x^2 - 5x = \pm 5$

$\Rightarrow 3x^2 - 5x - 5 = 0$ or $3x^2 - 5x + 5 = 0$

The first equation has irrational roots and the second has imaginary roots.

91. Note that $x \ge 2$ and

$\Rightarrow x - 2 = \sqrt{3x-6} \quad \Rightarrow \quad x^2 - 4x + 4 = 3x - 6$

$\Rightarrow x^2 - 7x + 10 = 0 \quad \Rightarrow \quad x = 2, 5$

Both of these satisfy the given equation.

92. Note that $x \ge -\frac{1}{2}$. Put $\sqrt{2x+1} = t$, so that

$2t = t^2 - 1 - 1$ or $t^2 - 2t + 1 = 3$

$\Rightarrow t = 1 \pm \sqrt{3}$.

But $t \ge 0$, therefore $t = 1 + \sqrt{3}$

$\Rightarrow 2x + 1 = 1 + 2 + 2\sqrt{3} \quad \Rightarrow \quad x = \frac{1}{2}\left(3 + 2\sqrt{3}\right)$

93. Note that $-\sqrt{13} \le x \le \sqrt{13}$ and

$13 - x^2 = x^2 + 10x + 25$

$\Rightarrow x^2 + 5x + 6 = 0 \Rightarrow x = -2, -3$.

94. $\sqrt{x^2 - 4} - (x - 2) = \sqrt{x^2 - 5x + 6} \qquad (1)$

is defined for $x = 2$ or $x \le -2$ or $x \ge 3$. Write (1) as

$\sqrt{x-2}\left[\sqrt{x+2} - \sqrt{x-2}\right] = \sqrt{x-2}\,\sqrt{x-3}$

$\Rightarrow x = 2$ or $\sqrt{x+2} - \sqrt{x-2} = \sqrt{x-3}$

$\Rightarrow x + 2 + x - 2 - 2\sqrt{x^2 - 4} = x - 3$

$\Rightarrow x + 3 = 2\sqrt{x^2 - 4} \Rightarrow x^2 + 6x + 9 = 4(x^2 - 4)$

$\Rightarrow 3x^2 - 6x - 25 = 0$

$\Rightarrow \quad (x - 1)^2 = \frac{28}{3} \Rightarrow x = 1 \pm \sqrt{\frac{28}{3}}$

Note that $1 - \sqrt{\frac{28}{3}} < -2$ and $1 + \sqrt{\frac{28}{3}} > 3$

95. $\sqrt{x^2 - 4x + 3} + \sqrt{x^2 - 7x + 12} = 3\sqrt{x-3} \qquad (1)$

is defined for $x = 3$ or $x \ge 4$.

Write (1) as

$$\sqrt{x-3}\,[\sqrt{x-1}+\sqrt{x-4}] = 3\sqrt{x-3}$$

$$\Rightarrow x = 3 \text{ or } \sqrt{x-1}+\sqrt{x-4} = 3$$

$$\Rightarrow x = 3 \text{ or } \sqrt{x-1} = 3-\sqrt{x-4}$$

$$\Rightarrow x = 3 \text{ or } x - 1 = 9-6\sqrt{x-4}+x-4$$

$$\Rightarrow x = 3 \text{ or } 6\sqrt{x-4} = 6 \Rightarrow x = 3 \text{ or } x = 5.$$

Thus, product of roots = 15.

96. Note that $2b^2 + 77b + 18 = 0$

$\Rightarrow \quad 2 + 77/b + 18(1/b)^2 = 0$

Thus, a and $1/b$ are roots of the equation

$$18x^2 + 77x + 2 = 0$$

$\therefore \quad a + 1/b = -77/18$ and $\dfrac{a}{b} = \dfrac{2}{18} = \dfrac{1}{9}$

Now, $\dfrac{ab+a+1}{b} = a+\dfrac{1}{b}+\dfrac{a}{b} = -\dfrac{77}{18}+\dfrac{1}{9} = -\dfrac{25}{6}$

97. $\beta^2 = 7\beta - 8.$

Thus, $\dfrac{16}{\alpha}+3\beta^2-19\beta = \dfrac{16}{\alpha}+3(7\beta-8)-19\beta$

$$= \frac{16}{\alpha}+2\beta-24$$

Also, $\alpha = \dfrac{7+\sqrt{17}}{2}$, $\beta = \dfrac{7-\sqrt{17}}{2}$

$$\therefore \quad \frac{1}{\alpha} = \frac{2}{7+\sqrt{17}} = \frac{2\left(7-\sqrt{17}\right)}{49-17} = \frac{7-\sqrt{17}}{16}$$

Thus $\dfrac{16}{\alpha}+3\beta^2-19\beta$

$$= \left(7-\sqrt{17}\right)+\left(7-\sqrt{17}\right)-24 = -10$$

98. As α is a root of $x^4 + x^2 - 1 = 0$, we get

$$\alpha^4 + \alpha^2 - 1 = 0 \Rightarrow \alpha^6 + \alpha^4 - \alpha^2 = 0$$

$$\Rightarrow \quad \alpha^6 + \alpha^4 - (1 - \alpha^4) = 0 \Rightarrow \alpha^6 + 2\alpha^2 = 1$$

$$\therefore \quad (\alpha^6 + 2\alpha^2)^{2012} = 1$$

99. Use the fact that sum and product of the roots of $x^2 - x - k = 0$ are 1 and $-k$ respectively.

100. Suppose $a^2(a + k) = b^2(b + k) = c^2(c + k) = t$, then a, b, c are roots of $x^3 + kx^2 - t = 0$ thus, $bc + ca + ab = 0$ and $abc = t$

$$\Rightarrow \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{bc+ca+ab}{abc} = 0$$

101 For truth of Statement-2, see theory

$f(\alpha) = a^2\alpha^2 + b\alpha + c = 0$ and

$g(\beta) = a^2\beta^2 - b\beta - c = 0$

We have $a^2\alpha^2 + 2b\alpha + 2c$

$= a^2\alpha^2 - 2a^2\alpha^2 < 0$

and $a^2\beta^2 + 2b\beta + 2c$

$= a^2\beta^2 + 2a^2\beta^2 > 0$

As $a^2x^2 + 2bx + 2c$ is continuous on $[\alpha, \beta]$, by statement-2 there exists γ such that $a^2\gamma^2 + 2b\gamma + 2c = 0$

102. As $y = f(x)$ open upwards, a lies between the roots of $f(x) = 0$, if and only if $f(a) < 0$. Therefore, statement-2 is true.

If $f(x) < 0$ for $1 \le x \le 2$, then 1 and 2 lie between roots of $f(x) = 0$. Thus, we must have

$4p^2 - 4p > 0 \Rightarrow p(p - 1) > 0$

$\Rightarrow p < 0$ or $p > 1$.

Also, $f(1) = 1 + 2p + p < 0 \Rightarrow p < -1/3$ and

$f(2) = 4 + 4p + p < 0 \Rightarrow p < -4/5$.

$\therefore p \in (-\infty, -4/5)$

103. Statement-2 is false as $x^2 + 5x + 4 = 0$ meets both the conditions but does not have positive roots.

As LHS ≥ 0, $1 - 2^x \ge 0 \Rightarrow 2^x \le 1 \Rightarrow x \le 0$.

Squaring, we get

$a(2^x - 2) + 1 = 1 - 2(2^x) + 2^{2x}$

$\Rightarrow y^2 - (a + 2)y + 2a = 0$

$\Rightarrow (y - a)(y - 2) = 0 \Rightarrow y = 2, a.$

$y = 2^x = 2 \Rightarrow x = 1$. Not possible.

Thus, $y = a$. This implies

$0 < 2^x = a \le 1$

Therefore $0 < a \le 1$, and $x = \log_2 a$

104. Put $t = x + \sqrt{x^2+b^2}$

$$\Rightarrow \frac{1}{t} = \frac{1}{\sqrt{x^2+b^2}+x} = \frac{\sqrt{x^2+b^2}-x}{b^2}$$

$$\therefore t - \frac{b^2}{t} = 2x$$

Thus,

$$f(x) = (2a - t + \frac{b^2}{t})t = b^2 + 2at - t^2$$

$$= a^2 + b^2 - (t - a)^2$$

Therefore, maximum value of $f(x)$ is $a^2 + b^2$ which is attained when $t = a$ or $x = \dfrac{1}{2}\left(a-\dfrac{b^2}{a}\right)$

Statement-2 is false as maximum value of $f(x) = ax^2 + bx + c$, $a < 0$ is $(4ac - b^2)/4a$.

105. For truth of statement-2 see theory

Let $P(x) = ax^2 + bx + c$.

As $P(x) = x$ does not have real roots, $P(x) > x \ \forall \ x \in \mathbf{R}$ or $P(x) < x \ \forall \ x \in \mathbf{R}$.

Suppose $P(x) < x \;\forall\; x \in \mathbf{R}$.

$\Rightarrow P(P(x)) < P(x) < x \;\forall\; x \in \mathbf{R}$

$\Rightarrow a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c - x = 0$

cannot have real roots.

106. Statement-2 is true as $|a + b| = |a| + |b|$

$\Leftrightarrow a, b$ are both non-negative or both non-positive.

As $ax^2 + bx + c = 0$ has imaginary roots, $ax^2 + bx + c > 0 \;\forall\; x \in \mathbf{R}$ or $ax^2 + bx + c < 0 \;\forall\; x \in \mathbf{R}$.

Since $a(-1)^2 + b(-1) + c = a - b + c < 0$,

we get $ax^2 + bx + c < 0 \;\forall\; x \in \mathbf{R}$.

Now, $|ax^2 + bx + c| + |x + y|$

$$= \left|a\left(x^2 + \frac{y}{a}\right) + (b+1)x + c\right|$$

$= |(ax^2 + bx + c) + (x + y)|$

$\Leftrightarrow (ax^2 + bx + c)(x + y) \geq 0$

$\Leftrightarrow x + y \leq 0$ as $ax^2 + bx + c < 0 \;\forall\; x \in \mathbf{R}$.

107. Statement-2 is true. See Theory

Put $x + c = t$, so that

$$f(x) = \frac{[t + (a-c)][t + (b-c)]}{t}$$

$$= t + \frac{(a-c)(b-c)}{t} + [(a - c) + (b - c)]$$

$$= \left[\sqrt{t} - \frac{\sqrt{(a-c)(b-c)}}{\sqrt{t}}\right]^2 + \left[\sqrt{a-c} + \sqrt{b-c}\right]^2$$

Thus, minimum value of $f(x)$ is $\left(\sqrt{a-c} + \sqrt{b-c}\right)^2$

and it is attained when $t = \sqrt{(a-c)(b-c)}$ or

$x = -c + \sqrt{(a-c)(b-c)} > -c$.

108. If $a \neq p$, $f(x) - g(x) = 0$ is a quadratic equation whose coefficient of $x^2 \neq 0$.

$\therefore f(x) - g(x) = 0$ has two real roots

Thus, statement-2 is true

Let $h(x) = f(x) - g(x)$

$\qquad = (a - b)x^2 + (b - q)x + (c - r)$

If $h(x) = f(x) - g(x) = 0$

for three distinct real values of x, then $a - p = 0$, $b - q = 0$, $c - r = 0$

$\Rightarrow a = p, b = q, c = r$.

$\therefore$ Statement-1 is also true.

However, statement-2 is not a correct explanation for Statement-1.

109. As A.M. ≥ G.M, we get

$$\frac{1}{2}\left(y + \frac{1}{y}\right) \geq \sqrt{y\left(\frac{1}{y}\right)} = 1$$

$$\Rightarrow y + \frac{1}{y} \geq 2.$$

Thus, statement-2 is true.

Since, $\sin^2 x = 0$,

does not satisty $\sin^4 x + p \sin^2 x + 1 = 0$, we get

$0 < \sin^2 x \leq 1$, for all x satisfying $\sin^4 x + p \sin^2 x + 1 = 0$

For such a value of $\sin x$,

$$-p = \sin^2 x + \frac{1}{\sin^2 x} \geq 2$$

$\Rightarrow p \leq -2$ or $p \in (-\infty, -2]$

Let $f(t) = t^2 + pt + 1$

As $f(0) = 1 > 0$ and $f(1) = 1 - p + 1 \leq 0$,

we get there is at least one value of $t \in (0, 1]$ such that $f(t) = 0$.

110. As $b^2 - 4ac < 0$, $f(x) > 0 \;\forall\; x \in \mathbf{R}$ or $f(x) < 0 \;\forall\; x \in \mathbf{R}$.

As $f(0) = c > 0$, $f(x) > 0 \;\forall\; x \in \mathbf{R}$.

Thus, statement-2 is true.

As $f(x) = 0$ does not have real roots, $b^2 - 4ac < 0$. By the statement-2, $f(x) > 0 \;\forall\; x \in \mathbf{R}$.

$\Rightarrow f(-1) = a - b + c > 0 \Rightarrow a > b - c$.

Also $b^2 - 4ac < 0$

$\Rightarrow a > b^2/4c$

Hence, $a > \max\left\{\frac{b^2}{4c}, b - c\right\}$.

Level 2

111. As A.M ≥ G.M, we get

$$\sqrt{x^2 + x} + \frac{\tan^2\alpha}{\sqrt{x^2 + x}} \geq 2\left[\sqrt{x^2 + x}\,\frac{\tan^2\alpha}{\sqrt{x^2 + x}}\right]^{1/2}$$

$$= 2\tan\alpha$$

112. $(a - b)^2 + 4(a + b - 1) \geq 0 \;\forall\; b \in \mathbf{R}$.

$\Rightarrow b^2 - 2b(a - 2) + a^2 + 4a - 4 \geq 0 \;\forall\; b \in \mathbf{R}$.

$\Rightarrow 4(a - 2)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow -8a + 8 < 0 \Rightarrow a > 1$

$\therefore a \in (1, \infty)$

113. $\alpha + \beta = -b < 0$ and $\alpha\beta = \mathrm{c} < 0$.

As $\alpha\beta < 0$, and $\alpha < \beta$,

$\alpha < 0 < \beta < -\alpha = |\alpha|$

114. As $x^2 + (a + b)x + c = 0$ has no real roots,

and the coefficient of $x^2 = 1 > 0$,

$f(x) = x^2 + (a + b)x + c > 0 \;\forall\; x \in \mathbf{R}$.

$\Rightarrow (a + b)^2 - 4\mathrm{c} < 0$.

Also $f(0) = c > 0$, $f(1) = a + b + c + 1 > 0$

$\Rightarrow c(a + b + c) + c > 0$

Next, $f(0) f(-1) > 0$

$\Rightarrow c(1 - a - b + c) > 0$

$\Rightarrow c - c(a + b - c) > 0$

115. As $x - 1 < [x] \leq x$, we get $-x \leq -[x] < -x + 1$

$= x^2 - 2x - 2 \leq x^2 - 2[x] - 2 < x^2 - 2x$

$= (x - 1)^2 - 3 \leq x^2 - 2[x] - 2 < (x - 1)^2 - 1$

Note that $(x-1)^2-3 \le 0$ for $-\sqrt{3}+1 \le x \le \sqrt{3}+1$
and $(x-1)^2-1 > 0$ for $x < 0$ or $x > 2$,
The roots of $x^2-2[x]-2=0$
lie in $\left(-\sqrt{3}+1, 0\right)$ or $(2, 1+\sqrt{3})$
If $x \in \left(-\sqrt{3}+1, 0\right)$, $[x]=-1$,
and $x^2-2[x]-2=0 \Rightarrow x^2=0$ or $x=0$.
This is not possible.
Suppose $x \in (2, 1+\sqrt{3})$, then $[x]=2$,
$x^2-2[x]-2=0 \Rightarrow x^2=6$ or $x=\sqrt{6}$.
Note that $x=\sqrt{6}$ satisfies $x^2-2[x]-2=0$

116. If $[x]$ is odd, then
$[x]^2+a[x]+b$ is odd and if $[x]$ is even, then $[x]^2+a[x]+b$ is also odd
Thus, $[x]^2+a[x]+b$ can never take value 0.

117. Let $a=\sin\alpha$, $b=\sin\beta$, $c=\sin\gamma$.
Note that a, b, c are distinct and $0 < a, b, c < 1$.
We can write the given equation as
$$f(x)=(x-b)(x-c)+(x-c)(x-a)+(x-a)(x-b)=0.$$
Assume that $a < b < c$.
Note that $f(a)=(a-b)(a-c) > 0$
$f(b)=(b-c)(b-a) < 0$
and $f(c)=(c-a)(c-b) > 0$.
Thus, $f(x)=0$ has a root in (a, b) and a root in (b, c).

118. Let $f(x)=x^2+(2\alpha+a)x+\alpha^2+a\alpha+b$.
$f(0)=\alpha^2+a\alpha+b=0$
Let other root of $f(x)=0$ be r, then
$0+r=-(2\alpha+a)=-2\alpha+\alpha+\beta=\beta-\alpha$.
Thus, two roots are 0 and $\beta-\alpha$.

119. As roots are real
$(a+b+c)^2-3\lambda(ab+bc+ca) \ge 0$
$$= 3\lambda \le \frac{(a+b+c)^2}{ab+bc+ca}$$
Since, a, b, c are sides of triangle.
$b+c-a > 0$, $c+a-b > 0$, $a+b-c > 0$
$\Rightarrow 2(ab+bc+ca) > a^2+b^2+c^2$
$\Rightarrow (a+b+c)^2 < 4(ab+bc+ca)$
Thus, $3\lambda < 4 \Rightarrow \lambda \in (-\infty, 4/3)$.

120. $\tan\theta+\cot\theta=2a$ and $\tan\theta\cot\theta=b$
Now, $2a=\tan\theta+\cot\theta \ge 2$ if $\tan\theta > 0$
and $2a=\tan\theta+\cot\theta \le -2$ if $\tan\theta < 0$.
$\Rightarrow 2|a| \ge 2 \Rightarrow |a| \ge 1$
Thus, least value of $|a|$ is 1.

121. Let $f(x)=x^3+x^2-5x-1$, then
$f(-3) < 0, f(-2) > 0, f(-1) > 0, f(0) < 0,$
$f(1) < 0, f(2) > 0.$
Thus, a root of $f(x)=0$ lies in each of the three intervals:
$(-3, -2), (-1, 0), (1, 2)$.
$\therefore$ $[\alpha]+[\beta]+[\gamma]=-3+(-1)+1=-3$

122. $\tan A \tan B = p$, $\tan A \tan B = q$.
As, $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}=\dfrac{p}{1-q}$, we get
$$\sin^2(A+B)=\frac{\tan^2(A+B)}{1+\tan^2(A+B)}$$
$$=\frac{p^2/(1-q)^2}{1+p^2/(1-q)^2}=\frac{p^2}{(1-q)^2+p^2}$$

123. Put $\sqrt{x-1}=t \Rightarrow x-1=t^2$, so that equation becomes;
$$\sqrt{t^2+1+3+4t}+\sqrt{t^2+1+8+6t}=1$$
$\Rightarrow |t+2|+|t+3|=1$
Not possible as $t \ge 0$.
Thus, the equation has no solution.

124. Put $\sqrt{x^2+11}=t \Rightarrow x^2+11=t^2$, so that equation becomes
$$\sqrt{t^2+t-11}+\sqrt{t^2-t-11}=4.$$
Since $(t^2+t-11)-(t^2-t-11) \equiv 2t$,
we get
$$\sqrt{t^2+t-11}-\sqrt{t^2-t-11}=\frac{t}{2}$$
$$\therefore\ 2\sqrt{t^2+t-11}=4+\frac{t}{2}$$
$$\Rightarrow 4(t^2+t-11)=16+\frac{1}{4}t^2+4t \Rightarrow t^2=16$$
$$\Rightarrow x^2+11=16 \Rightarrow x=\pm\sqrt{5}$$

125. Note that a, b, c are roots of
$$\frac{x}{t}+\frac{y}{t-p}+\frac{z}{t-q}=1$$
$\Leftrightarrow$ $t^3-(x+y+z+p+q)t^2$
$+[(p+q)x+qy+pz]t-pqx=0.$
$\therefore$ product of roots $=pqx$
$$\Rightarrow abc=pqx \Rightarrow x=\frac{abc}{pq}$$

126. Putting $x=0, 1, 1/2$, we get
$-1 \le c \le 1$, $-1 \le a+b+c \le 1$,
$$-1 \le \frac{1}{4}a+\frac{1}{2}b+c \le 1,$$

$\Rightarrow -4 \le -a - 2b - 4c \le 4$

As $-4 \le 4a + 4b + 4c \le 4$, we get

$-8 \le 3a + 2b \le 8$.

Also, $-8 \le a + 2b \le 8$

$\Rightarrow -16 \le 2a \le 16 \Rightarrow |a| \le 8$

127. $p + q$ and $p + r$ are roots of

$at^2 + 2bp\,(t - p) + c = 0$

or $at^2 + 2bpt + c - 2bp^2 = 0$.

We have

$|q - r|^2 = |(p + q) - (p + r)|^2$

$= [(p + q) + (p + r)]^2 - 4(p + q)(p + r)$

$$= \left(\frac{-2bp}{a}\right)^2 - \frac{4\left(c - 2bp^2\right)}{a}$$

$$= \frac{4}{a^2}\left[b^2p^2 - ac + 2abp^2\right]$$

$$= \frac{4}{|a|^2}\,[(2a + b)\,bp^2 - ac]$$

$\Rightarrow |q - r|$

$$= \frac{2}{|a|}\sqrt{(2a+b)bp^2 - ac}$$

128. Put $x^2 = y$. The given equation will have four real roots if

$f(y) = ay^2 + by + c = 0$

has two non-negative roots.

This is possible if

$-\dfrac{b}{a} \ge 0$, $af(0) \ge 0$, $b^2 - 4ac \ge 0$

$\Rightarrow ab \le 0,\ ac \ge 0$.

This condition is met if $a > 0$, $b < 0$, $c > 0$ or $a < 0$, $b > 0$, $c < 0$

129. As $x = 1$ satisfies

$(b - c)\,x^2 + (c - a)\,x + (a - b) = 0$,

$x = 1$ must satisfy $x^2 + mx + 1 = 0$

$\Rightarrow 1 + m + 1 = 0 \Rightarrow m = -2$.

130. Eliminating x from $x + y + z = 4$,

and $x^2 + y^2 + z^2 = 6$, we get

$y^2 + z^2 + (4 - y - z)^2 = 6$

$\Rightarrow 2y^2 - 2(4 - z)y + (4 - z)^2 + z^2 - 6 = 0$

As y is real, we get

$(4 - z)^2 - 2[(4 - z)^2 + z^2 - 6] \ge 0$

$\Rightarrow 3z^2 - 8z + 4 \le 0$

$\Rightarrow (3z - 2)\,(z - 2) \le 0$

$\Rightarrow 2/3 \le z \le 2$.

Thus, maximum value of z is 2.

Previous Years' AIEEE/JEE Main Questions

1. α, β are root of $x^2 = 5x - 3$ or $x^2 - 5x + 3 = 0$

$\Rightarrow \alpha + \beta = 5,\ \alpha\beta = 3$.

Now, $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta} = \dfrac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

$$= \frac{25 - 6}{3} = \frac{19}{3}$$

and $\left(\dfrac{\alpha}{\beta}\right)\left(\dfrac{\beta}{\alpha}\right) = 1$

Thus, equation whose roots are α/β and β/α is

$$x^2 - (19/3)x + 1 = 0$$

or $\quad 3x^2 - 19x + 3 = 0$

2. $|\alpha - \beta| = |\alpha_1 - \beta_1|$

$\Rightarrow (\alpha - \beta)^2 = (\alpha_1 - \beta_1)^2$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = (\alpha_1 + \beta_1)^2 - 4\alpha_1\beta_1$

$\Rightarrow a^2 - 4b = b^2 - 4a$

$\Rightarrow (a - b)(a + b + 4) = 0 \Rightarrow a + b + 4 = 0$

$[\because\ a \ne b]$

3. Let $f(x) = \dfrac{1}{3}ax^3 + \dfrac{1}{2}bx^2 + cx$.

As f is a polynomial f is continuous on [0, 1] and f is differentiable on (0, 1). Also

$$f(0) = 0 \text{ and } f(1) = \frac{1}{6}(2a + 3b + 6c) = 0$$

By the Rolle's theorem there exist some $\alpha \in (0, 1)$ such that $f'(\alpha) = 0$.

Thus, $ax^2 + bx + c = 0$ has at least one root in [0, 1].

4. For each $x \in \mathbf{R}$, $t \in \mathbf{R}$

$t^2x^2 + |x| + 9 \ge 9 > 0$

Thus, $t^2x^2 + |x| + 9 = 0$ does not have real roots.

5. Let α and 2α be the roots of the given equation, then

$(a^2 - 5a + 3)\alpha^2 + (3a - 1)\alpha + 2 = 0 \qquad (1)$

$(a^2 - 5a + 3)(4\alpha^2) + (3a - 1)(2\alpha) + 2 = 0 \qquad (2)$

Multiplying (1) by 4 and subtracting it form (2) we get

$(3a - 1)\,(2\alpha) + 6 = 0$

Clearly $\alpha \neq 1/3$. Therefore, $\alpha = -3/(3a-1)$

Putting this value in (1) we get

$$(a^2 - 5a + 3)(9) - (3a - 1)^2(3) + 2(3a - 1)^2 = 0$$

$$\Rightarrow \quad 9a^2 - 45a + 27 - (9a^2 - 6a + 1) = 0$$

$$\Rightarrow \quad -39a + 26 = 0$$

$$\Rightarrow \quad a = 2/3.$$

For $a = 2/3$, the equation becomes $x^2 + 9x + 18 = 0$, whose roots are –3, –6.

6. Let α, β be the roots of $ax^2 + bx + c = 0$

We are given

$$\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$$

$$= \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$\Rightarrow \quad \frac{-b}{a} = \frac{(-b/a)^2 - 2(c/a)}{(c/a)^2}$$

$$\Rightarrow \quad -\frac{b}{a}\cdot\frac{c^2}{a^2} = \frac{b^2}{a^2} - \frac{2c}{a}$$

$$\Rightarrow \quad 2a^2c = ab^2 + bc^2$$

Divide by abc to obtain

$$2\frac{a}{b} = \frac{c}{a} + \frac{b}{c}$$

$$\Rightarrow \quad \frac{c}{a}, \frac{a}{b}, \frac{b}{c} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{a}{c}, \frac{b}{a}, \frac{c}{b} \text{ are in H.P.}$$

7. $$x^2 - 3|x| + 2 = 0$$

$$\Rightarrow \quad |x|^2 - 3|x| + 2 = 0$$

$$\Rightarrow \quad (|x| - 1)(|x| - 2) = 0$$

$$\Rightarrow \quad |x| = 1, 2$$

$$\Rightarrow \quad x = \pm 1, \pm 2$$

8. Let two numbers be α and β. We are given

$$\frac{a+\beta}{2} = 9, \ \sqrt{a\beta} = 4$$

$$\Rightarrow \quad \alpha + \beta = 18 \text{ and } \alpha\beta = 16$$

∴ Required equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

or $$x^2 - 18x + 16 = 0$$

9. Let other root be α. Then

$$\alpha + (1 - p) = -p \text{ and } \alpha(1 - p) = 1 - p$$

$$\Rightarrow \quad \alpha = -1 \text{ and } -(1 - p) = 1 - p$$

$$\Rightarrow \quad \alpha = 1 \text{ and } 1 - p = 0$$

Thus, equation is $x^2 + x = 0$

$\Rightarrow \quad x = 0, -1$

10. Let α be the other root of $x^2 + px + 12 = 0$

∴ $(\alpha)(4) = 12$ and $\alpha + 4 = p$

$\Rightarrow \quad \alpha = 3$ and $3 + 4 = p$

Thus, $p = 7$,

As $x^2 + px + q = 0$ has equal roots, we must have $p^2 - 4q = 0$

$$\Rightarrow \quad q = \frac{1}{4}p^2 = \frac{49}{4}$$

11. See solution to Question No. 3

12. We have

$$\tan\left(\frac{P}{2}\right) + \tan\left(\frac{Q}{2}\right) = \frac{-b}{a} \text{ and } \tan\left(\frac{P}{2}\right)\tan\left(\frac{Q}{2}\right) = \frac{c}{a}$$

Also $P + Q = \pi - R = \pi/2$

$$\Rightarrow \quad \frac{P}{2} + \frac{Q}{2} = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan\left(\frac{P}{2} + \frac{Q}{2}\right) = \tan\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \frac{\tan(P/2) + \tan(Q/2)}{1 - \tan(P/2)\tan(Q/2)} = 1$$

$$\Rightarrow \quad \frac{-bla}{1 - cla} = 1 \Rightarrow \frac{-b}{a - c} = 1$$

$$\Rightarrow \quad -b = a - c \quad \text{or} \quad c = a + b$$

13. Let α, β be roots of $x^2 - (a - 2)x - a - 1 = 0$

Then $\alpha + \beta = a - 2$, $\alpha\beta = -(a + 1)$

Now, $$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (a - 2)^2 + 2(a + 1)$$

$$= a^2 - 4a + 4 + 2a + 2$$

$$= a^2 - 2a + 1 + 3$$

$$= (a - 1)^2 + 3$$

Thus, $\alpha^2 + \beta^2$ is least when $a = 1$

14. We know that if α and β are roots of $x^2 - bx + c = 0$, then $(\alpha - \beta)^2$ = discriminant $= b^2 - 4c$.

In the present case $\beta = \alpha + 1$.

$\therefore \quad 1 = b^2 - 4c$

15. We can write the given equation as

$$(x - k)^2 = 5 - k.$$

Thus, for roots to be real we must have $5 - k \geq 0$ or $k \leq 5$.

Also, for $k \leq 5$,

$$x = k \pm \sqrt{5-k}$$

For both the roots to be less than 5, we must have that larger of the two roots is less than 5. Therefore.

$$\Rightarrow \quad k + \sqrt{5-k} < \quad \Rightarrow 5\sqrt{5-k} < 5 - k \qquad (1)$$

This does not hold for $k = 5$. For $k < 5$, we can write (1) as

$1 < \sqrt{5-k}$ or $5 - k > 1$ or $k < 4$

$\therefore \quad k \in (-\infty, 4)$.

16. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x$. Then, $f(0) = 0$ and $f(\alpha) = 0$.

By the Rolle's theorem $\exists\ \beta \in (0, \alpha)$ such that $f'(\beta) = 0$.

Thus, $na_n x^{n-1} + (n - 1)a_{n-1}x^{n-2} + ... + a_1 = 0$

has a positive root β which is smaller than α.

17. Let $\alpha = \tan 30°$, $\beta = \tan 15°$, then $\alpha = \dfrac{1}{\sqrt{3}}$ and $\beta = 2 - \sqrt{3}$

$$\therefore \quad -p = \frac{1}{\sqrt{3}} + (2 - \sqrt{3})$$

$$\text{and} \quad q = \frac{1}{\sqrt{3}}(2 - \sqrt{3}) = \frac{2}{\sqrt{3}} - 1$$

$$\text{Thus,} \quad q - p = \frac{3}{\sqrt{3}} + 2 - \sqrt{3} - 1 = 1$$

$\therefore \quad 2 + q - p = 3$

18. $x^2 - 2mx + m^2 - 1 = 0$

$\Rightarrow \quad (x - m)^2 = 1 \quad \Rightarrow x = m \pm 1$

As both the roots lie between –2 and 4, we get

$-2 < m \pm 1 < 4$

$\Rightarrow \quad -1 < m < 3.$

19. Let $y = \dfrac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$

$\Rightarrow \quad 3(y - 1)x^2 + 9(y - 1)x + 7y - 17 = 0$

As x is real

$81(y - 1)^2 - 12(y - 1)(7y - 17) \geq 0$

$\Rightarrow \quad 3(y - 1)[27(y - 1) - 4(7y - 17)] \geq 0$

$\Rightarrow \quad 3(y - 1)(41 - y) \geq 0$

$\Rightarrow \quad (y - 1)(y - 41) \leq 0$

$\Rightarrow \quad 1 \leq y \leq 41$

20. $\alpha + \beta = -a$, $\alpha\beta = 1$

$$|\alpha - \beta| < \sqrt{5}$$

$\Rightarrow \quad |\alpha - \beta|^2 < 5$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta < 5$

$\Rightarrow \quad a^2 - 4 < 5$

$\Rightarrow \quad a^2 < 9$

$\Rightarrow \quad -3 < a < 3$

$\Rightarrow \quad a \in (-3, 3)$

21. Let roots of $x^2 - 6x + a = 0$ be α, $4m$ and that of $x^2 - cx + 6 = 0$ be α, $3m$ where m is an integer We have,

$\alpha + 4m = 6$, $\alpha(4m) = a$

and $\quad \alpha + 3m = c$, $\alpha(3m) = 6$

$\Rightarrow \quad \alpha m = 2 \Rightarrow \quad a = 8$

Also, $\alpha + \dfrac{8}{\alpha} = 6 \quad \Rightarrow \alpha^2 - 6\alpha + 8 = 0$

$\Rightarrow \quad (\alpha - 2)(\alpha - 4) = 0 \quad \Rightarrow \alpha = 2, 4.$

As m is an integer, $\alpha \neq 4$ ($\because$ $\alpha m = 2$)

$\therefore \quad \alpha = 2$

22. As roots of $bx^2 + cx + a = 0$ are imaginary, $c^2 - 4ab < 0$. Now,

$3b^2x^2 + 6bcx + 2c^2$

$$= 3\left[b^2x^2 + 2bcx + \frac{2}{3}c^2\right]$$

$$= 3\left[(bx + c)^2 - \frac{1}{3}c^2\right]$$

$= 3(bx + c)^2 - c^2 \geq -c^2 > -4ab$

$\therefore$ $3b^2x^2 + 6bcx + 2c^2 > -4ab\ \forall\ x \in \mathbf{R}.$

23. $x^2 - x + 1 = 0$

$\Rightarrow \quad x = -\,\omega, -\,\omega^2$ where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

Let $\quad \alpha = -\omega$ and $\beta = -\,\omega^2$

We have $\alpha^{2009} + \beta^{2009}$

$= (-\,\omega)^{2009} + (-\,\omega^2)^{2009}$

$= (-\,1)\,[\omega^{2007}\,\omega^2 + (\omega^2)^{2007}\,\omega^4]$

$= (-\,1)\,(\omega^2 + \omega) = (-\,1)\,(-\,1) = 1$

24. As $a \neq a_1$, $p(x) = f(x) - g(x)$ is a quadratic polynomial. As $p(x) = 0$ only for $x = -1$, we get $p(x)$ must be of the form $p(x) = k(x+1)^2$ where $k = a - a_1$.

As $\quad p(-2) = 2$, we get

$$2 = k(-2+1)^2 \ \Rightarrow\ k = 2.$$

Thus, $\quad p(x) = 2(x+1)^2$

$\therefore \quad p(2) = 2(2+1)^2 = 18$

25. We can write the given equation as

$y - 1/y - 4 = 0 \quad$ where $y = e^{\sin x}$

$\Rightarrow y^2 - 4y - 1 = 0$

$\Rightarrow\ y = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$

As $y > 0$, $y = 2 + \sqrt{5}$

$\Rightarrow \sin x = \ln(2+\sqrt{5}) > \ln(4) > 1$

This is not possible.

Thus, equation has no real roots.

26. $x^2 + 2x + 3 = 0$

$\Rightarrow \quad (x+1)^2 = 2i^2$

$\Rightarrow \quad x = -1 \pm \sqrt{2}i$

As $a, b, c \in \mathbf{R}$, if $-1 \pm \sqrt{2}i$ (or $-1 - \sqrt{2}i$) is a root of $ax^2 + bx + c = 0$, the other root must be $-1 - \sqrt{2}i$ (or $-1 \pm \sqrt{2}i$)

Thus, $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$ have both roots in common, therefore

$$\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$$

$\Rightarrow \quad a : b : c = 1 : 2 : 3$

27. $(\sqrt{10})^2 = |\alpha - \beta|^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$\Rightarrow 10 = p^2 - 3p \quad \Rightarrow p^2 - 3p - 10 = 0$

$\Rightarrow (p-5)\,(p+2) = 0 \quad p = -\,2,\ 5$

Thus, p lies in the set $\{-2, 5\}$

28. $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{a^3 + \beta^3}{\alpha\beta} = \frac{-p}{q}$

and $\left(\frac{\alpha^2}{\beta}\right)\left(\frac{\beta^2}{\alpha}\right) = \alpha\beta = q$

Thus, required equation is

$$x^2 - \left(-\frac{p}{q}\right)x + q = 0$$

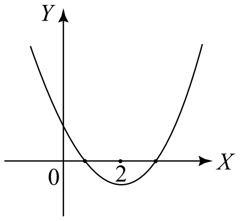
or $\qquad qx^2 + px + q^2 = 0$

29. As the roots are real and distinct

$(a+1)^2 - 4(a^2 + a - 8) > 0$

$3a^2 + 2a - 33 < 0$

$\Rightarrow \quad (3a+11)(a-3) < 0 \Rightarrow -\frac{11}{3} < a < 3 \qquad (1)$



Also, for

$2^2 - 2(a+1) + a^2 + a - 8 < 0$

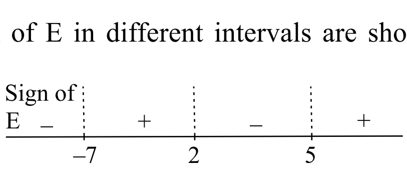
$\Rightarrow \quad a^2 - a - 6 < 0 \Rightarrow (a-3)\,(a+2) < 0$

$\Rightarrow \quad -\,2 < a < 3 \qquad (2)$

From (1) and (2), we get $-\,2 < a < 3$

30. $0 < \frac{x-5}{x^2+5x-14} = \frac{x-5}{(x+7)(x-2)} = \text{E(say)}$

Sign of E in different intervals are shown below



Thus, $\frac{x-5}{(x+7)(x-2)} > 0$

$\Rightarrow \quad -7 < x < 2$ or $x > 5$.

Therefore the least integral value α of x is $-\,6$. This value of α satisfies the relation $\alpha^2 + 5\alpha - 6 = 0$.

31. $4 = \dfrac{\alpha+\beta}{\alpha\beta} = \dfrac{-q/p}{r/p} = \dfrac{-q}{r}$

$\Rightarrow \quad q + 4r = 0$

Also, $2q = p + r$ $\quad$ [$\because$ p, q, r are in A.P.]

Thus, $\quad p = -9r,\ q = -4r$

$$|\alpha - \beta|^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$= \left(\frac{-q}{p}\right)^2 - \frac{4r}{p}$$

$$= \frac{16}{81} - 4\left(-\frac{1}{9}\right) = \frac{52}{81}$$

$\Rightarrow \quad |\alpha - \beta| = \dfrac{2}{9}\sqrt{13}$

32. As roots of $2x^2 + 3x + 4 = 0$ are non-real complex numbers, the given quadratic equations have both roots in common. [See Solution Question No. 26.]

Thus, $\dfrac{a}{2} = \dfrac{b}{3} = \dfrac{c}{4} \quad \Rightarrow a : b : c = 2 : 3 : 4$

33. **Case 1:** When $x < 3/2$

In this case the equation becomes

$x^2 - (2x - 3) - 4 = 0$

$\Rightarrow \quad x^2 - 2x + 1 = 2$

$\Rightarrow \quad (x - 1)^2 = 2 \quad \Rightarrow \quad x - 1 = \pm\sqrt{2}$

$\Rightarrow \quad x = 1 \pm \sqrt{2}$

As $x < 3/2$, we take $x = 1 - \sqrt{2}$

Case 2: When $x \geq 3/2$

In this case the equation becomes

$x^2 + (2x - 3) - 4 = 0$

$\Rightarrow \quad (x + 1)^2 = 8 \Rightarrow x + 1 \pm 2\sqrt{2}$

As $x \geq 3/2$, $x = -1 + 2\sqrt{2}$

sum of roots $= \left(1 - \sqrt{2}\right) + \left(-1 + 2\sqrt{2}\right) = \sqrt{2}$

34. $\alpha + \beta = 4\sqrt{2}\,k$, $\alpha\beta = 2e^{4\ln k} - 1 = 2k^4 - 1$

Now,

$66 = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 32k^2 - 2(2k^4 - 1)$

$\Rightarrow 33 = 16k^2 - 2k^4 + 1$

$\Rightarrow 2k^4 - 16k^2 + 32 = 0$

$\Rightarrow k^4 - 8k^2 + 16 = 0$

$\Rightarrow (k^2 - 4)^2 = 0 \Rightarrow k = 2 \qquad [\because\ k > 0]$

We have

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta\,(\alpha + \beta)$$

$$= (\alpha + \beta)\ [(\alpha + \beta)^2 - 3\alpha\beta]$$

$$= \left(4\sqrt{2}\right)(2)\left[66 - \left\{2(2^4) - 1\right\}\right]$$

$$= 280\sqrt{2}$$

35. Note that we must have $3x^2 + x + 5 \geq 0$ and $x - 3 \geq 0$ or $x \geq 3$. Squaring both sides of (1), we get

$$3x^2 + x + 5 = x^2 - 6x + 9$$

$\Rightarrow \qquad 2x^2 + 7x - 4 = 0$

$\Rightarrow \qquad (2x - 1)(x + 4) = 0$

$\Rightarrow \qquad x = 1/2,\ -4$

None of these satisfy the inequality $x \geq 3$. Thus, (1) has no solution.

36. $\alpha + \beta = 6$, $\alpha\beta = -2$.

$a_{10} - 2a_8 = \alpha^{10} + \beta^{10} + \alpha\beta\,(\alpha^8 + \beta^8)$

$= (\alpha + \beta)\ (\alpha^9 + \beta^9) = 6a_9$

$\Rightarrow \quad \dfrac{a_{10} - 2a_8}{2a_9} = 3$

37. As the coefficients of

$2x^3 - 9x^2 - kx - 13 = 0$

are all real, non-real complex roots occur in conjugate pair, so let the roots of (1) be $2 + 3i$, $2 - 3i$ and α, where $\alpha \in$ **R.** Now,

$(2 + 3i) + (2 - 3i) + \alpha = 9/2 \Rightarrow \alpha = 1/2$

$\therefore$ real root of the equation is 1/2.

[Verification: $(2 + 3i)\ (2 - 3i)\alpha = 13/2 \Rightarrow \alpha = 1/2$]

38. Write

$$x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2$$

$$= (x^2 + 1)^2 - x^2$$

$$= (x^2 + x + 1)\ (x^2 - x + 1)$$

The given equation, now can be written as

$(x^2 + x + 1)\ [(a - 1)(x^2 - x + 1) + (a + 1)\ (x^2 + x + 1)] = 0$

$\Rightarrow (x^2 + x + 1)\ [2ax^2 + 2x + 2a] = 0$

As x is real, we get

$ax^2 + x + a = 0.$

This equation will have real and distinct roots if

$a \neq 0$ and $1 - 4a^2 > 0 \Rightarrow a \neq 0,\ a^2 < 1/4$

$\Rightarrow a \in (-1/2, 0) \cup (0, 1/2)$

39. If $x^2 - 5x + 5 \neq 1$, then $x^2 + 4x - 60 = 0$

$\Rightarrow (x + 10)(x - 6) = 0 \Rightarrow x = -10, 6$

When $x = -10$ and 6, $x^2 - 5x + 5 \neq 1$

If $x^2 - 5x + 5 = 1$, $(x^2 - 5x + 5)^{x^2+4x-60} = 1$

In this case, $x^2 - 5x + 4 = 0$

$\Rightarrow (x - 1)(x - 4) = 0 \Rightarrow x = 1, 4$

If $x^2 - 5x + 5 = -1$, then $x^2 + 4x - 60$ must be even integer.

But $x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

For $x = 2$, $x^2 + 4x - 60 = -48$ is even.

For $x = 3$, $x^2 + 4x - 60 = -39$ is odd.

Thus, sum of desired values is $-10 + 6 + 1 + 4 + 2 = 3$.

40. Let α be common roots of

$x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$.

Then $\alpha^2 + b\alpha - 1 = 0$ and $\alpha^2 + \alpha + b = 0$

$\Rightarrow (\alpha^2 + b\alpha - 1) - (\alpha^2 + \alpha + b) = 0$

$\Rightarrow (b - 1)\alpha - (b + 1) = 0$

$\Rightarrow \alpha = \frac{b+1}{b-1}$ if $b \neq 1$

As $\alpha \neq -1$, $b \neq 0$

Also, $\left(\frac{b+1}{b-1}\right)^2 + b\left(\frac{b+1}{b-1}\right) - 1 = 0$

$\Rightarrow (b + 1)^2 + b(b + 1)(b - 1) - (b - 1)^2 = 0$

$\Rightarrow 4b + b(b + 1)(b - 1) = 0$

As $b \neq 0$, $4 + b^2 - 1 = 0 \Rightarrow b^2 = 3$

$\Rightarrow |b| = \sqrt{3}$

41. We are given

$\sqrt{2x+1} - \sqrt{2x-1} = 1 \qquad (1)$

Also, $(2x + 1) - (2x - 1) = 2 \qquad (2)$

Dividing (2) by (1),

$\sqrt{2x+1} + \sqrt{2x-1} = 2 \qquad (3)$

From (1) and (3), we get

$\sqrt{2x+1} = \frac{3}{2},\ \sqrt{2x-1} = \frac{1}{2}$

$\therefore \sqrt{4x^2 - 1} = \frac{3}{4}$

42. $x + y + z = 12$, $x^3y^4z^5 = (0.1)(600)^3$

We have

$x^3y^4z^5 = (0.1)(600)(600)^2$

$= (60)(600)^2$

$= 3^3 2^8 5^5$

$\therefore$ we may take

$x = 3, y = 4, z = 5$

Thus, $x^3 + y^3 + z^3 = 3^3 + 4^3 + 5^3$

$= 27 + 64 + 125 = 216$

Previous Years' B-Architecture Entrance Examination Questions

1. $\tan 30^\circ + \tan 15^\circ = -2p$

$\tan 30^\circ \tan 15^\circ = q$

$$1 = \tan 45^\circ = \frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ}$$

$\Rightarrow \quad 1 = \frac{-2p}{1-q} \quad \Rightarrow \quad 1 - q = -2p$

$\Rightarrow \quad q = 1 + 2p.$

2. Let roots of the equation be $1 - k$ and $1 + k$, where $k > 0$.

Then $2 = (1 - k) + (1 + k) = \frac{2\alpha}{\alpha+2}$

$\Rightarrow \quad 1 = \frac{\alpha}{\alpha+2}$

This is not possible for any value of α.

3. $|x|^2 - 4|x| - 2 = 0$

$\Rightarrow (|x| - 2)^2 = 6 \quad \Rightarrow \quad |x| - 2 = \pm\sqrt{6}$

$\Rightarrow |x| = 2 \pm \sqrt{6}$

As $|x| > 0$, $|x| = 2 + \sqrt{6}$

$\Rightarrow x = \pm(2 + \sqrt{6})$

$\therefore$ there are two values of x.

4. See Solution to Question 1 in Previous Years' AIEEE/JEE Questions.

5. $ax^2 - bx(x - 1) + c(x - 1)^2 = 0$

$$\Rightarrow \quad a\left(-\frac{x}{x-1}\right)^2 + b\left(-\frac{x}{x-1}\right) + c = 0$$

$$\Rightarrow \quad -\frac{x}{x-1} = \alpha, \beta$$

$$\Rightarrow \quad x = \frac{\alpha}{\alpha+1}, \frac{\beta}{\beta+1}$$

6. $x^4 - ax^2 + b = (x^2 - 3x + 2)\,p(x)$

$$= (x-1)(x-2)\,p(x)$$

where $p(x)$ is a polynomial of degree 2.

$\therefore\ 1 - a + b = 0$ and $16 - 4a + b = 0$

$\Rightarrow\ a = 5,\ b = 4$

Thus, equation whose roots are a and b is

$x^2 - 9x + 20 = 0$

7. $f'(x) = 2ax + b = a\left(x + \dfrac{b}{2a}\right)$

As $a > 0$,

$$f'(x) < 0 \text{ for } x < -b/2a$$

and $f'(x) > 0$ for $x > -b/2a$

Thus, f is strictly decreasing in the interval $(-\infty, -b/2a)$ and strictly increasing in the interval $(-b/2a, \infty)$.

Also, Min $f(x) = f\left(\dfrac{-b}{2a}\right) = \dfrac{4ac - b^2}{4a} > 0$

if $b^2 < 4ac$

As Min $f(x) > 0$, $f(x) > 0\ \forall\ x \in \mathbf{R}$

$\therefore$ Both the statements are true and statement- 2 is a correct explanation for statement-1.

8. As the roots are of opposite signs, the product of roots must be negative, that is,

$$\frac{a^2 - 3a + 2}{3} < 0 \Rightarrow (a-1)(a-2) < 0$$

$\Rightarrow\ a \in (1, 2)$

9. The given equation can be written as

$r(2x + p + q) = (x + p)(x + q)$

or $x^2 + (p + q - 2r) + pq - r(p + q) = 0$

Let roots of this equation be α, $-\alpha$.

Then

$$0 = \alpha + (-\alpha) = -(p + q - 2r) \Rightarrow 2r = p + q$$

and $\alpha(-\alpha) = pq - r(p + q)$

$$= pq - \frac{1}{2}(p+q)^2 = -\frac{1}{2}(p^2 + q^2)$$

10. Write the equation as

$$(x - 3k)^2 = 2(k - 1)$$

$$\Rightarrow \quad x = 3k \pm \sqrt{2}\sqrt{k-1}$$

Both the roots will be greater than 3 if smaller root is greater than 3, that is, if

$3k - \sqrt{2}\sqrt{k-1} > 3$

$\Rightarrow\ 3(k-1) > \sqrt{2}\sqrt{k-1}$

$\Rightarrow\ k > 1$ and $9(k - 1) > 2 \Rightarrow\ k > 11/9$

$\Rightarrow\ 11 - 9k < 0.$

11. $(1 + 3m)^2 - (1 + m^2)(1 + 8m) < 0$

$\Rightarrow 1 + 6m + 9m^2 - (1 + m^2 + 8m + 8m^3) < 0$

$\Rightarrow -2m + 8m^2 - 8m^3 < 0$

$\Rightarrow 2m(4m^2 - 4m + 1) > 0$

$\Rightarrow m(2m - 1)^2 > 0$

which is true for each $m \in \mathbf{N}$.

CHAPTER FOUR

Determinants

EVALUATION OF DETERMINANTS

A determinant of order two is written as

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad (a_{ij} \in \mathbf{C}\ \forall\ i, j)$$

and is equal to $a_{11}\, a_{22} - a_{12}\, a_{21}$.

A determinant of order three is written as

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (a_{ij} \in \mathbf{C}\ \forall\ i, j)$$

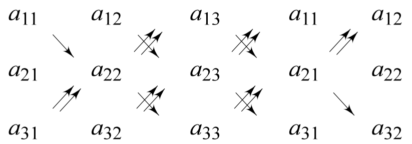
and is equal to

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\,(a_{22}\, a_{33} - a_{23}\, a_{32}) - a_{12}\,(a_{21}\, a_{33} - a_{23}\, a_{31}) + a_{13}(a_{21}\, a_{32} - a_{22}\, a_{31})$$

$$= a_{11}\, a_{22}\, a_{33} + a_{12}\, a_{23}\, a_{31} + a_{13}\, a_{32}\, a_{21} - a_{13}\, a_{31}\, a_{22} - a_{32}\, a_{23}\, a_{11} - a_{12}\, a_{21}\, a_{33}$$

A determinant of order 3 can also be evaluated by using the following diagram, due to **Sarrus:**


a_{11} a_{12} a_{13} a_{11} a_{12}
a_{21} a_{22} a_{23} a_{21} a_{22}
a_{31} a_{32} a_{33} a_{31} a_{32}

The product of the three terms on each of the three single arrows are prefixed by a positive sign and the product of the three terms on each of the three double arrows are prefixed by a negative sign.

Remark

This method does not work for higher order determinants.

MINORS AND COFACTORS

Minor

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n. Then the minor M_{ij} of the element a_{ij} of the matrix A is the determinant of the square sub-matrix of order $(n - 1)$ obtained by deleting ith row and jth column of matrix A.

Illustration 1

Minor of element a_{23} in the determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

Cofactor

Let $A = (a_{ij})_{n\times n}$ be a square matrix of order n. Then the cofactor of the element a_{ij} of the matrix A is denoted by C_{ij} and is equal to $(-1)^{i+j}\, M_{ij}$ where M_{ij} is the minor of the element a_{ij} of the matrix A.

Note that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{aligned} &= a_{11}\, M_{11} - a_{12}\, M_{12} + a_{13}\, M_{13} \\ &= a_{11}\, C_{11} + a_{12}C_{12} + a_{13}C_{13} \end{aligned}$$

If $$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

then $$\begin{aligned} \Delta &= a_{i1}\, C_{i1} + a_{i2}\, C_{i2} + a_{i3}\, C_{i3} \qquad i = 1, 2, 3 \\ &= a_{1j}\, C_{1j} + a_{2j}\, C_{2j} + a_{3j}\, C_{3j} \qquad j = 1, 2, 3 \end{aligned}$$

Note

$$a_{i1}\, C_{j1} + a_{i2}\, C_{j2} + a_{i3}\, C_{j3} = 0 \qquad i \neq j$$

and $$a_{1i}\, C_{1j} + a_{2i}\, C_{2j} + a_{3i}\, C_{3j} = 0 \qquad i \neq j$$

Remark

The above results remain true for determinants of every order.

PROPERTIES OF DETERMINANTS

1. Reflection Property

The determinant remains unaltered if its rows are changed into columns and the columns into rows.
In other words, if A is a square matrix, then $|A| = |A'|$ where A' is transpose of A.

2. All-zero Property

If all the elements of a row (column) are zero, then the determinant is zero.

3. Proportionality [Repetition] Property

If the elements of a row (column) are proportional [identical] to the element of the some other row (column), then the determinant is zero.

4. Switching Property

The interchange of any two rows (columns) of the determinant changes its sign.

5. Scalar Multiple Property

If all the elements of a row (column) of a determinant are multiplied by a non-zero constant, then the determinant gets multiplied by the same constant.

6. Property of Invariance

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + \alpha b_1 + \beta c_1 & b_1 & c_1 \\ a_2 + \alpha b_2 + \beta c_2 & b_2 & c_2 \\ a_3 + \alpha b_3 + \beta c_3 & b_3 & c_3 \end{vmatrix}$$

That is, a determinant remains unaltered under an operation of the form $C_i \to C_i + \alpha C_j + \beta C_k$, where $j, k \neq i$, or an operation of the form $R_i \to R_i + \alpha R_j + \beta R_k$, where $j, k \neq i$.

7. Sum Property

$$\begin{vmatrix} a_1 + b_1 & c_1 & d_1 \\ a_2 + b_2 & c_2 & d_2 \\ a_3 + b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

Remark

It is one of the most under used property. But evaluation of some of the determinants beame very easy when we use it.

Illustration 2

To evaluate

$$\Delta = \begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix}$$

write $\Delta = \Delta_1 + a\Delta_2$ where

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 0 & 1+b & c \\ 0 & b & 1+c \end{vmatrix}$$

$$= (1 + b)\,(1 + c) - bc = 1 + b + c$$

and

$$\Delta_2 = \begin{vmatrix} 1 & b & c \\ 1 & 1+b & c \\ 1 & b & 1+c \end{vmatrix}$$

Using $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$,
we get

$$\Delta_2 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1$$

$\therefore$ $\Delta = 1 + a + b + c$

8. Factor Property

If a determinant Δ becomes zero when we put $x = \alpha$, then $(x - \alpha)$ is a factor of Δ.

9. Triangle Property

If all the elements of a determinant above or below the main diagonal consists of zeros, then the determinant is equal to the product of diagonal elements. That is,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 b_2 c_3$$

10. Product of Two Determinants

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Here we have multiplied rows by rows. We can also multiply rows by columns, or columns by rows, or columns by columns.

11. Conjugate of a Determinant

If $a_i, b_i, c_i \in \mathbf{C}$ (i = 1, 2, 3), and

$$Z = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ then } \overline{Z} = \begin{vmatrix} \overline{a}_1 & \overline{b}_1 & \overline{c}_1 \\ \overline{a}_2 & \overline{b}_2 & \overline{c}_2 \\ \overline{a}_3 & \overline{b}_3 & \overline{c}_3 \end{vmatrix}$$

12. Differentiation of a Determinant

If each $a_i(x)$ is differentiable function

and $$\Delta(x) = \begin{vmatrix} a_1(x) & a_2(x) \\ a_3(x) & a_4(x) \end{vmatrix}$$

then $$\Delta'(x) = \begin{vmatrix} a_1'(x) & a_2(x) \\ a_3'(x) & a_4(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & a_2'(x) \\ a_3(x) & a_4'(x) \end{vmatrix}$$

If we write $\Delta(x) = [C_1, C_2]$, where C_i denotes ith column, then

$\Delta'(x) = [C_1', C_2] + [C_1, C_2']$ where C_1' denotes the column which contains the derivative of all the functions in the ith column C_i. Similarly, if

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \quad \text{then} \quad \Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \end{bmatrix}$$

Next, if $$\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix}$$

then $\Delta'(x)$

$$= \begin{vmatrix} a_{11}'(x) & a_{12}(x) & a_{13}(x) \\ a_{21}'(x) & a_{22}(x) & a_{23}(x) \\ a_{31}'(x) & a_{32}(x) & a_{33}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a_{12}'(x) & a_{13}(x) \\ a_{21}(x) & a_{22}'(x) & a_{23}(x) \\ a_{31}(x) & a_{32}'(x) & a_{33}(x) \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}'(x) \\ a_{21}(x) & a_{22}(x) & a_{23}'(x) \\ a_{31}(x) & a_{32}(x) & a_{33}'(x) \end{vmatrix}$$

$$= [C_1', C_2, C_3] + [C_1, C_2', C_3] + [C_1, C_2, C_3']$$

Similarly, if

$$\Delta(x) = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \text{ then } \Delta'(x) = \begin{bmatrix} R_1' \\ R_2 \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2' \\ R_3 \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \\ R_3' \end{bmatrix}$$

Corollary (Differentiation and Integration of Determinant)

If $$\Delta(x) = \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

where a_{21}, a_{22}, a_{23}, a_{31}, a_{32} and a_{33} are constants, then

$$\Delta'(x) = \begin{vmatrix} a_{11}'(x) & a_{12}'(x) & a_{13}'(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ and}$$

$$\int \Delta(x)dx = \begin{vmatrix} \int a_{11}(x)dx & \int a_{12}(x)dx & \int a_{13}(x)dx \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

In general, for any positive integer m

$$\Delta^m(x) = \begin{vmatrix} a_{11}^m(x) & a_{12}^m(x) & a_{13}^m(x) \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

13. Determinant of Cofactor Matrix

If $$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ then } \Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \Delta^2$$

where C_{ij} denotes the co-factor of the element a_{ij} in Δ.

SOME TIPS FOR QUICK EVALUATION OF DETERMINANTS

1. If Δ is a skew symmetric determinant of odd order, then $\Delta = 0$

Illustration 3

Let $$\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

Using the reflection property, write

$$\Delta = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Taking –1 common from R_1, R_2 and R_3 we get

$$\Delta = (-1)^3 \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = -\Delta$$

$$\Rightarrow \quad 2\Delta = 0 \quad \Rightarrow \quad \Delta = 0$$

2. If a_1, a_2, a_3 are in A.P.; b_1, b_2, b_3 are in A.P. and c_1, c_2, c_3 are also in A.P. Then

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

Use $C_1 \to C_1 + C_3 - 2C_2$

3. If a_1, a_2, a_3 are in G.P.; b_1, b_2, b_3 are also in G.P., with the same common ratio, then

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$$

[c_1, c_2, c_3 can be any three complex numbers.]

Some Frequently used Determinants

1. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix}$

$$= (a-b)(b-c)(c-a)$$

2. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

3. $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(bc+ca+ab)$

4. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

$$= (a+b+c)(bc+ca+ab-a^2-b^2-c^2)$$

$$= -\frac{1}{2}(a+b+c)\left[(b-c)^2+(c-a)^2+(a-b)^2\right]$$

LINEAR EQUATIONS

The system of linear homogeneous equations

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$
$$a_3 x + b_3 y + c_3 z = 0$$

has a non-trivial solution (i.e., at least one of the x, y, z is different from zero) if and only if $\Delta = 0$, where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

If $\Delta \neq 0$, then the only solution of the above system of equations is $x = 0$, $y = 0$ and $z = 0$.

Corollary If at least one of x, y, z is non-zero and x, y and z are connected by the three given equations, then the elimination of x, y and z leads to the relation

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

CRAMER'S RULE

If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

then the solution of the system of linear equations

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

is given by $x = \dfrac{\Delta_1}{\Delta}$, $y = \dfrac{\Delta_2}{\Delta}$, $z = \dfrac{\Delta_3}{\Delta}$

where

$$\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and}$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: Let

$$\Delta(x, y) = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Then $\Delta(-3, 2)$ equals

(a) 13 (b) – 6

(c) 12 (d) – 5

Ans. (b)

Solution: Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ we get

$$\Delta(x, y) = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix} = xy$$

$$\therefore \quad \Delta(-3, 2) = -6$$

Example 2: Suppose a, b, c are distinct real numbers and

$$\Delta = \begin{vmatrix} a & a^2 & b+c \\ b & b^2 & c+a \\ c & c^2 & a+b \end{vmatrix} = 0$$

Then $a + b + c$ equals

(a) -1 (b) 2
(c) 0 (d) -5

Ans. (b)

Solution: Using $C_3 \to C_3 + C_1$ and taking $(a + b + c)$ common from C_3 we get

$$\Delta = (a + b + c)\,\Delta_1 \qquad (1)$$

where

$$\Delta_1 = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Using $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta_1 = \begin{vmatrix} a-b & a^2-b^2 & 0 \\ b-c & b^2-c^2 & 0 \\ c & c^2 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a+b & 0 \\ 1 & b+c & 0 \\ c & c^2 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}$$

[Expand along C_3]

$$= (a-b)(b-c)(c-a) \qquad (2)$$

From (1) and (2)

$$\Delta = (a+b+c)(a-b)(b-c)(c-a)$$

As $\Delta = 0$ and a, b, c are distinct,

we get $a + b + c = 0$

Example 3: Suppose $A = (a_{ij})_{3\times 3}$, where $a_{ij} \in \mathbf{R}$. If det (adj A) = 25, then |det (A)| equals:

(a) 5 (b) 12.5
(c) $5\sqrt{5}$ (d) $5^{2/3}$

Ans. (a)

Solution: Using det (adj A) = $(\det(A))^2$, we get

$$(\det(A))^2 = 25 \quad \Rightarrow \quad |\det(A)| = 5$$

Example 4: Let

$$\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}, \text{ then}$$

Δ equals:

(a) 0 (b) abc
(c) $a^2 + b^2 + c^2$ (d) $bc + ca + ab$

Ans. (a)

Solution: Interchanging the rows and columns, we get

$$\Delta = \begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$$

Taking -1 common from each of R_1, R_2, R_3 we get

$$\Delta = (-1)^3\begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = -\Delta \Rightarrow 2\Delta = 0 \text{ or } \Delta = 0$$

Alternative Solution

Δ is a skew symmetric determinant of odd order, therefore $\Delta = 0$

Example 5: Let $A = (a_{ij})_{3\times 3}$, where $a_{ij} \in \mathbf{C}$ the set of complex numbers. If det (A) = $2 - 3i$, then det (A^{-1}) equals:

(a) $\frac{1}{13}(2 - 3i)$ (b) $\frac{1}{13}(2 + 3i)$
(c) $2 - 3i$ (d) $2 + 3i$

Ans. (b)

Solution: $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{2-3i}$

$$= \frac{2+3i}{2^2+3^2} = \frac{1}{13}(2+3i)$$

Example 6: In a triangle ABC, if

$$\Delta = \begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$$

then $\sin^2 A + \sin^2 B + \sin^2 C$ is:

(a) $\frac{3\sqrt{3}}{2}$ (b) $\frac{5}{4}$
(c) $\frac{9}{4}$ (d) 2

Ans. (a)

Solution: Evaluating along C_1, we get

$$\Delta = (c^2 - ab) + (b^2 - ac) + (a^2 - bc) = 0$$

$$\Rightarrow \quad (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

$$\Rightarrow \quad a = b = c \Rightarrow A = B = C = \pi/3$$

$$\therefore \quad \sin^2 A + \sin^2 B + \sin^2 C = \frac{3\sqrt{3}}{2}.$$

Example 7: If

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = Ax + B,$$

then A is equal to:

(a) 12 (b) 18

(c) 24 (d) 30

Ans. (c)

Solution: Applying $R_2 \to R_2 - R_1 - R_3$, we get

$$Ax + B = \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

Expanding along R_2, we get

$$Ax + B = (-1)(-4)\begin{vmatrix} x+1 & x-2 \\ 2x-1 & 2x-1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 3 & x-2 \\ 0 & 2x-1 \end{vmatrix}$$

$$= 4(3)\,(2x-1) = 24x - 12$$

Thus, $A = 24$

Example 8: Suppose a, b, c are three integers such $a < b < c$ and p is a prime number.

Let $$\Delta = \begin{vmatrix} a & a^2 & p+a^3 \\ b & b^2 & p+b^3 \\ c & c^2 & p+c^3 \end{vmatrix}$$

If $\Delta = 0$, then which one of the following is **not** true

(a) $a = -1, b = 1$ (b) $b = 1, c = p$

(c) $a = 0, c = p$ (d) $abc + p = 0$

Ans. (c)

Solution: Write $\Delta = p\Delta_1 + \Delta_2$
where

$$\Delta_1 = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} \quad \text{and} \quad \Delta_2 = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix}$$

write

$$\Delta_2 = abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = abc(-1)^2\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= abc\,\Delta_1$$

Thus, $\Delta = (p + abc)\,\Delta_1$.

Using $R_2 \to R_2 - R_3$, $R_1 \to R_1 - R_2$, we get

$$\Delta_1 = \begin{vmatrix} a-b & a^2-b^2 & 0 \\ b-c & b^2-c^2 & 0 \\ c & c^2 & 1 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}$$

$$= (a-b)\,(b-c)\,(c-a)$$

As $a < b < c$, $\Delta_1 \neq 0$. Therefore,

$$\Delta = 0 \quad \Rightarrow \quad p + abc = 0$$

$$\Rightarrow \qquad p = -abc$$

As p is prime and a, b, c are integers such that $a < b < c$, we must have $-a = 1, b = 1, c = p$.

$$\Rightarrow \qquad a = -1, b = 1, c = p.$$

Example 9: Suppose

$$P(x) = \begin{vmatrix} x & -51 & -71 \\ 51 & x & -73 \\ 71 & 73 & x \end{vmatrix}$$

Product of zeros of $P(x)$ is

(a) 0 (b) 195

(c) –195 (d) –264333

Ans. (a)

Solution:

TIP

Suppose $P(x) = x^3 + ax^2 + bx + c$, then product of zeros of $P(x)$ is $-c = -P(0)$.

Now, $$-c = -P(0) = -\begin{vmatrix} 0 & -51 & -71 \\ 51 & 0 & -73 \\ 71 & 73 & 0 \end{vmatrix}$$

$$= 0$$

[∵ Skew symmetric determinant of odd order]

Example 10: Let

$$P(x) = \begin{vmatrix} x & -3+4i & 3-4i \\ x & -7i & 5+6i \\ -x & 7-2i & -7-2i \end{vmatrix}$$

The number of values of x for which $P(x) = 0$ is

(a) 0 (b) 1

(c) 2 (d) 3

Ans. (b)

Solution: Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$, we get

$$P(x) = \begin{vmatrix} x & -3+4i & 3-4i \\ 0 & 3-11i & 2+10i \\ 0 & 4+2i & -4-6i \end{vmatrix}$$

As $P(x)$ is a linear polynomial, $P(x) = 0$ for exactly one value of x.

Example 11: Let

$$\Delta(\theta) = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}, \; 0 \le \theta \le 2\pi$$

Solution of $\Delta(\theta) = 3$ is

(a) $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ (b) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

(c) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ (d) $\left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi\right\}$

Ans. (b)

Solution: Using $C_1 \to C_1 + C_3$, we get

$$\Delta(\theta) = \begin{vmatrix} 2 & \sin\theta & 1 \\ 0 & 1 & \sin\theta \\ 0 & -\sin\theta & 1 \end{vmatrix}$$

$$= 2\,(1 + \sin^2\theta)$$

$$\therefore \quad \Delta(\theta) = 3 \;\Rightarrow\; 2\sin^2\theta = 1$$

$$\Rightarrow \quad \sin\theta = \pm\frac{1}{\sqrt{2}} \;\Rightarrow\; \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Example 12: Suppose $a \in \mathbf{R}$ and $x \neq 0$. Let

$$\Delta(x) = \begin{vmatrix} 1-x & a & a^2 \\ a & a^2 - x & a^3 \\ a^2 & a^3 & a^4 - x \end{vmatrix}$$

Number of values of x for which $\Delta(x) = 0$ is

(a) 0 (b) 1

(c) 2 (d) 3

Ans. (b)

Solution: Using the sum property, write

$$\Delta(x) = \Delta_1 - x\Delta_2$$

where

$$\Delta_1 = \begin{vmatrix} 1 & a & a^2 \\ a & a^2 - x & a^3 \\ a^2 & a^3 & a^4 - x \end{vmatrix}$$

and

$$\Delta_2 = \begin{vmatrix} 1 & a & a^2 \\ 0 & a^2 - x & a^3 \\ 0 & a^3 & a^4 - x \end{vmatrix}$$

$$= (a^2 - x)(a^4 - x) - a^6$$

$$= -(a^2 + a^4)x + x^2$$

In Δ_1, use $C_2 \to C_2 - aC_1$, $C_3 \to C_3 - a^2C_1$ to obtain

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ a & -x & 0 \\ a^2 & 0 & -x \end{vmatrix} = x^2$$

Thus, $\Delta(x) = x^2 + (a^2 + a^4)\,x^2 - x^3$

$$= (1 + a^2 + a^4)\,x^2 - x^3$$

As $x \neq 0$, $\Delta(x) = 0 \;\Rightarrow\; x = \dfrac{1}{1 + a + a^2}$.

Thus, $\Delta(x) = 0$ for exactly one value of x.

Example 13: If

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+\lambda)^2 & (b+\lambda)^2 & (c+\lambda)^2 \\ (a-\lambda)^2 & (b-\lambda)^2 & (c-\lambda)^2 \end{vmatrix} = k\lambda \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$\lambda \neq 0$, then k is equal to:

(a) $4\lambda abc$ (b) $-4\lambda abc$

(c) $4\lambda^2$ (d) $-4\lambda^2$

Ans. (c)

Solution: Using $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} a^2 & b^2 & c^2 \\ 2a\lambda + \lambda^2 & 2b\lambda + \lambda^2 & 2c\lambda + \lambda^2 \\ -4a\lambda & -4b\lambda & -4c\lambda \end{vmatrix}$$

Take -4λ common from R_3, and applying $R_2 \to R_2 - 2\lambda R_3$, we get

$$\Delta = -4\lambda^3 \begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}$$

$$= \lambda(4\lambda^2) \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$$\therefore \quad k = 4\lambda^2$$

Example 14: Let

$$f(\theta) = \begin{vmatrix} 1 & \cos\theta & -1 \\ -\sin\theta & 1 & -\cos\theta \\ -1 & \sin\theta & 1 \end{vmatrix}$$

Suppose A and B are respectively maximum and minimum value of $f(\theta)$. Then (A, B) is equal to:

(a) (2, 1) (b) (2, 0)

(c) $(\sqrt{2}, 1)$ (d) $\left(2, \frac{1}{\sqrt{2}}\right)$

Ans. (b)

Solution: Using $C_1 \to C_1 + C_3$, we get

$$f(\theta) = \begin{vmatrix} 0 & \cos\theta & -1 \\ -(\sin\theta + \cos\theta) & 1 & -\cos\theta \\ 0 & \sin\theta & 1 \end{vmatrix}$$

Evaluating along C_1, we get

$$f(\theta) = (\sin\theta + \cos\theta)\begin{vmatrix} \cos\theta & -1 \\ \sin\theta & 1 \end{vmatrix}$$

$$= (\sin\theta + \cos\theta)^2$$

$$= \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\theta + \frac{1}{\sqrt{2}}\cos\theta\right)\right]^2$$

$$= 2\sin^2\left(\frac{\pi}{4} + \theta\right)$$

As $\quad 0 \le \sin^2\left(\frac{\pi}{4} + \theta\right) \le 1$, we get

$$0 \le f(\theta) \le 2$$

$$A = 2 \quad \text{for} \quad \theta = \frac{\pi}{4}$$

and $\quad B = 0 \quad \text{for} \quad \theta = -\frac{\pi}{4}$

Thus, $\quad (A, B) = (2, 0)$

Example 15: If a, b, c are non-zero real numbers and if the system of equations

$(a - 1)x = y + z,$

$(a - 1)y = z + x,$

$(a - 1)z = x + y,$

has a non-trivial solution, then $ab + bc + ca$ equals:

(a) $a + b + c$ (b) abc

(c) 1 (d) –1

Ans. (b)

Solution: As the given system of equations has a non-trivial solution,

$$\Delta = \begin{vmatrix} a-1 & -1 & -1 \\ -1 & b-1 & -1 \\ -1 & -1 & c-1 \end{vmatrix} = 0$$

Write

$$\Delta = \begin{vmatrix} a & -1 & -1 \\ 0 & b-1 & -1 \\ 0 & -1 & c-1 \end{vmatrix} - \begin{vmatrix} 1 & -1 & -1 \\ 1 & b-1 & -1 \\ 1 & -1 & c-1 \end{vmatrix}$$

$$= a[(b-1)(c-1) - 1] - \begin{vmatrix} 1 & 0 & 0 \\ 1 & b & 0 \\ 1 & 0 & c \end{vmatrix}$$

[use $C_2 \to C_2 + C_1$
$C_3 \to C_2 + C_1$]

$$= a(bc - b - c) - bc$$

As $\Delta = 0$, we get

$$ab + bc + ca = abc$$

LEVEL 1

Straight Objective Type Questions

Example 16: Let

$$P(x) = \begin{vmatrix} 7 & 6 & x-10 \\ 2 & x-10 & 5 \\ x-10 & 3 & 4 \end{vmatrix}$$

sum of zeros of $P(x)$ is

(a) 30 (b) 28

(c) 27 (d) 25

Ans. (a)

Solution: Using $R_1 \leftrightarrow R_3$, write

$$P(x) = -\begin{vmatrix} x-10 & 3 & 4 \\ 2 & x-10 & 5 \\ 7 & 6 & x-10 \end{vmatrix}$$

TIP

Observe $P(x)$ is of the form
$P(x) = -(x - 10)^3 + a(x - 10) + b$
where a, b are some real numbers.

∴ sum of zeros of $P(x)$

$$= -\frac{\text{coefficient of } x^2}{\text{coefficient of } x^3} = 30.$$

Example 17: Let

$$P(x) = \begin{vmatrix} x^2-13 & 4 & 2 \\ 3 & x^2-13 & 7 \\ 6 & 5 & x^2-13 \end{vmatrix}$$

If $x = -2$ is a zero of $P(x)$, then sum of the remaining five zeros is

(a) –2 (b) 0

(c) 2 (d) 3

Ans. (c)

Solution:

TIP

Observe
$P(x) = (x^2 - 13)^3 + a(x^2 - 13) + b.$
where a, b are some real numbers.

As coefficient of x^5 in $P(x)$ is 0, sum of six zeros of $P(x)$ is 0.

⇒ sum of the remaining five zeros + (–2) = 0

$\Rightarrow$ sum of the remaining five zeros = 2.

Example 18: Suppose α, β are two real numbers and $f(n) = \alpha^n + \beta^n$. Let

$$\Delta = \begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

If $\Delta = k(\alpha - 1)^2 (\beta - 1)^2 (\alpha - \beta)^2$, then k is equal to

(a) 1 (b) $4\alpha\beta$
(c) 9 (d) $\alpha^2\beta^2$

Ans. (a)

Solution:

$$\Delta = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \Delta_1^2$$

where $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$

Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha - 1 & \beta - \alpha \\ 1 & \alpha^2 - 1 & \beta^2 - \alpha^2 \end{vmatrix}$$

$$= (\alpha - 1)(\beta - \alpha)\begin{vmatrix} 1 & 1 \\ \alpha + 1 & \beta + \alpha \end{vmatrix}$$

$$= (\alpha - 1)(\beta - 1)(\beta - \alpha)$$

Thus, $\Delta = (\alpha - 1)^2 (\beta - 1)^2 (\alpha - \beta)^2$

$\therefore$ $k = 1$

Example 19: Suppose a, b and c are distinct real numbers. Let

$$\Delta = \begin{vmatrix} a & a+c & a-b \\ b-c & b & a+b \\ c+b & c-a & c \end{vmatrix} = 0$$

Then the straight line $a(x - 5) + b(y - 2) + c = 0$ passes through the fixed point

(a) (5, 2) (b) (6, 2)
(c) (6, 3) (d) (5, 3)

Ans. (c)

Solution: Applying $C_1 \to aC_1 + bC_2 + cC_3$, we get

Let $\Delta = \dfrac{1}{a}\begin{vmatrix} a(a+b+c) & a+c & a-b \\ b(a+b+c) & b & a+b \\ c(a+b+c) & c-a & c \end{vmatrix}$

$$= \frac{1}{a}a(a+b+c)\Delta_1 \text{, where}$$

$$\Delta_1 = \begin{vmatrix} a & a+c & a-b \\ b & b & a+b \\ c & c-a & c \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta_1 = \begin{vmatrix} a & c & -b \\ b & 0 & a \\ c & -a & 0 \end{vmatrix} = a(a^2 + b^2 + c^2)$$

$\therefore$ $\Delta = (a + b + c)(a^2 + b^2 + c^2) = 0$

As a, b, c are distinct real numbers, $a^2 + b^2 + c^2 \neq 0$, therefore $a + b + c = 0$.

$\Rightarrow$ the line $a(x - 5) + b(y - 2) + c = 0$ passes through (6, 3).

Example 20: Suppose a, b, c and x are real numbers. Let

$$\Delta = \begin{vmatrix} 1+a & 1+ax & 1+ax^2 \\ 1+b & 1+bx & 1+bx^2 \\ 1+c & 1+cx & 1+cx^2 \end{vmatrix}$$

Then Δ is independent of

(a) a, b, c (b) x
(c) a, b, c, x (d) none of these

Ans. (c)

Solution: Write $\Delta = \Delta_1 + \Delta_2$, where

$$\Delta_1 = \begin{vmatrix} 1 & 1+ax & 1+ax^2 \\ 1 & 1+bx & 1+bx^2 \\ 1 & 1+cx & 1+cx^2 \end{vmatrix}$$

and $\Delta_2 = \begin{vmatrix} a & ax & ax^2 \\ b & bx & bx^2 \\ c & cx & cx^2 \end{vmatrix} = 0$

[$\because$ C_1 and C_2 are propotional]

In Δ_1, use $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ to obtain

$$\Delta_1 = \begin{vmatrix} 1 & ax & ax^2 \\ 1 & bx & bx^2 \\ 1 & cx & cx^2 \end{vmatrix} = 0$$

[$\because$ C_2 and C_3 are proportional]

Thus, $\Delta = 0$ and hence independent of a, b, c, x.

Example 21: Suppose $a, b, c > 1$ and

$$f(x) = \begin{vmatrix} a^{-x} & a^x & x \\ b^{-3x} & b^{3x} & 3x^3 \\ c^{-5x} & c^{5x} & 5x^5 \end{vmatrix}, x \in \mathbf{R}$$

then f is

(a) a constant function
(b) a polynomial of degree 5
(c) an odd function
(d) an even function

Ans. (d)

◎ **Solution:**

$$f(-x) = \begin{vmatrix} a^{x} & a^{-x} & -x \\ b^{3x} & b^{-3x} & -3x^{3} \\ c^{5x} & c^{-5x} & -5x^{5} \end{vmatrix}$$

$$= (-1)(-1)\begin{vmatrix} a^{-x} & a^{x} & x \\ b^{-3x} & b^{3x} & 3x^{3} \\ c^{-5x} & c^{5x} & 5x^{5} \end{vmatrix}$$

$$= f(x)$$

Thus, f is an even function.

◉ **Example 22:** Suppose n, m are natural numbers and

$$f(x) = \begin{vmatrix} 1 & (1+x)^{m} & (1+mx)^{mn} \\ (1+mx)^{n} & 1 & (1+nx)^{mn} \\ (1+nx)^{m} & (1+x)^{n} & 1 \end{vmatrix}$$

constant term of the polynomial $f(x)$ is:

(a) 1 (b) $m + n$
(c) $m - n$ (d) 0

Ans. (d)

◎ **Solution:** Constant term of polynomial $f(x)$ is $f(0)$, and

$$f(0) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

◉ **Example 23:** Suppose a, b, c are sides of a scalene triangle. Let

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Then

(a) $\Delta \le 0$ (b) $\Delta < 0$
(c) $\Delta > 0$ (d) $\Delta \ge 0$

Ans. (b)

◎ **Solution:** Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (a + b + c)\,\Delta_1$$

where

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$$= -(b-c)^2 - (a-b)(a-c)$$

$$= -(a^2 + b^2 + c^2 - bc - ca - ab)$$

$$\Rightarrow \qquad \Delta_1 = -\frac{1}{2}\,[(b-c)^2 + (c-a)^2 + (a-b)^2] < 0$$

As $a + b + c > 0$, we get

$$\Delta = (a + b + c)\,\Delta_1 < 0$$

◉ **Example 24:** Suppose A, B, C are angles of a triangle, and let

$$\Delta = \begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}$$

Then value of Δ is

(a) – 1 (b) – 4
(c) 0 (d) 4

Ans. (b)

◎ **Solution:** Taking e^{iA}, common from R_1, e^{iB} from R_2 and e^{iC} from R_3, we get

$$\Delta = e^{i(A+B+C)}\,\Delta_1$$

where

$$\Delta_1 = \begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC} \end{vmatrix}$$

But $A + B + C = \pi$, so that $e^{i(A+B+C)} = e^{i\pi}$

$$= \cos \pi + i \sin \pi = -1. \text{ Also,}$$

$$A + C = \pi - B \quad \Rightarrow \quad e^{-i(A+C)} = e^{-\pi i}\, e^{iB} = -e^{iB}.$$

Thus,

$$\Delta_1 = \begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

$$= e^{i(A+B+C)}\begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix}$$

Using $C_1 \to C_1 + C_2$, we get

$$\Delta_1 = (-1)\begin{vmatrix} 0 & -1 & -1 \\ 0 & 1 & -1 \\ -2 & -1 & 1 \end{vmatrix} = (-1)\,(-2)\,(2) = 4$$

Therefore, $\Delta = (-1)\,\Delta_1 = -4$

◉ **Example 25:** Suppose x_1, x_2, x_3 are real numbers such that $x_1 x_2 x_3 \neq 0$. Let

$$\Delta = \begin{vmatrix} x_1 + a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & x_2 + a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & x_3 + a_3 b_3 \end{vmatrix}$$

Then $\dfrac{\Delta}{x_1x_2x_3} - 1$ equals:

(a) $\dfrac{a_1b_1}{x_1} + \dfrac{a_2b_2}{x_2} + \dfrac{a_3b_3}{x_3}$

(b) -1

(c) $\dfrac{a_1a_2a_3 + b_1b_2b_3}{x_1x_2x_3}$

(d) 0

Ans. (a)

Solution: Using the sum property, write

$$\Delta = x_1\,\Delta_1 + b_1\,\Delta_2$$

where

$$\Delta_1 = \begin{vmatrix} 1 & a_1b_2 & a_1b_3 \\ 0 & x_2 + a_2b_2 & a_2b_3 \\ 0 & a_3b_2 & x_3 + a_3b_3 \end{vmatrix}$$

$$= (x_2 + a_2\,b_2)\,(x_3 + a_3\,b_3) - a_3\,b_2\,a_2\,b_3$$

$$= x_2\,x_3 + x_2\,a_3\,b_3 + x_3\,a_2\,b_2$$

and $$\Delta_2 = \begin{vmatrix} a_1 & a_1b_2 & a_1b_3 \\ a_2 & x_2 + a_2b_2 & a_2b_3 \\ a_3 & a_3b_2 & x_3 + a_3b_3 \end{vmatrix}$$

Using $C_2 \to C_2 - b_2\,C_1$, $C_3 \to C_3 - b_3\,C_1$, we get

$$\Delta_2 = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & x_2 & 0 \\ a_3 & 0 & x_3 \end{vmatrix} = a_1x_2x_3$$

Thus,

$$\Delta = x_1[x_2\,x_3 + x_2\,a_3\,b_3 + x_3\,a_2\,b_2] + a_1\,b_1\,x_2\,x_3$$

$$\Rightarrow \quad \frac{\Delta}{x_1x_2x_3} - 1 = \frac{a_1b_1}{x_1} + \frac{a_2b_2}{x_2} + \frac{a_3b_3}{x_3}$$

Example 26: Let $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t$

$$= \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda - 3 \\ \lambda - 1 & -2\lambda & \lambda - 4 \\ \lambda - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$$

where p, q, r, s and t are constants. Then value of t is

(a) 0 (b) -1

(c) 2 (d) 3

Ans. (a)

Solution: Putting $\lambda = 0$, we obtain

$$t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix} = 0$$

as it is a skew symmetric determinant of odd order.

Example 27: Let $\Delta = \begin{vmatrix} 1 & -4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix}$. Solution set of $\Delta = 0$ is

(a) $\{-2, 3\}$ (b) $\{-3, 4\}$

(c) $\{4, -6\}$ (d) $\{-2, -1\}$

Ans. (d)

Solution: Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ we get

$$\Delta = \begin{vmatrix} 1 & -4 & 20 \\ 0 & 2 & -15 \\ 0 & 2\,(x+2) & 5\,(x^2-4) \end{vmatrix} = 2(x+2) \begin{vmatrix} 1 & -15 \\ 1 & 5\,(x-2) \end{vmatrix}$$

$$= 10(x + 2)\,(x + 1)$$

Now, $\Delta = 0 \Rightarrow x = -2, -1$.

Example 28: Let $\Delta = \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$, then Δ is equal to

(a) 0 (b) $a + b + c$

(c) $\dfrac{1}{2}(a^2 + b^2 + c^2)$ (d) none of these

Ans. (a)

Solution: Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = (b - a)\,(c - a) \begin{vmatrix} 1 & a & a^2 - bc \\ 0 & 1 & a + b + c \\ 0 & 1 & a + b + c \end{vmatrix} = 0$$

[$\because$ R_2 and R_3 are identical]

Example 29: Suppose a, b, $c > 0$ and a, b, c are the pth, qth, rth terms of a G.P. Let

$$\Delta = \begin{vmatrix} 1 & p & \log a \\ 1 & q & \log b \\ 1 & r & \log c \end{vmatrix}$$

then numerical value of Δ is

(a) -1 (b) 2

(c) 0 (d) none of these

Ans. (c)

Solution: Let $a = AR^{p-1}$, $b = AR^{q-1}$ and $c = AR^{r-1}$

$\Rightarrow$ $\log a = \alpha + (p - 1)\beta$, $\log b = \alpha + (q - 1)\beta$ and $\log c = \alpha + (r - 1)\beta$

where $\alpha = \log a$, $\beta = \log R$.

Now,

$$\Delta = \begin{vmatrix} 1 & p & \alpha + (p-1)\,\beta \\ 1 & q & \alpha + (q-1)\,\beta \\ 1 & r & \alpha + (r-1)\,\beta \end{vmatrix}$$

Using $C_3 \to C_3 - (\alpha - \beta)C_1 - \beta C_3$, we get $\Delta = 0$.

Example 30: Let $\omega = \frac{1}{2}\left(-1+\sqrt{3}i\right)$ and

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} \text{ then } \Delta \text{ equals}$$

(a) 3ω (b) $3\omega(\omega - 1)$
(c) $3\omega^2$ (d) $3\omega(1 - \omega)$

Ans. (b)

Solution: Using $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$, and applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega \end{vmatrix} = 3(\omega^2 - \omega) = 3\omega(\omega - 1)$$

Example 31: Let a, b, and c be respectively the pth, qth and rth terms of a harmonic progression and

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ bc & ca & ab \end{vmatrix}$$

Then numerical value of Δ is

(a) 0 (b) – 1
(c) 1 (d) none of these

Ans. (a)

Solution: $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are the pth, qth, rth terms of an A.P.

Let $\frac{1}{a} = A + (p-1)D$, $\frac{1}{b} = A + (q-1)D$,

and $\frac{1}{c} = A + (r-1)D$.

Now

$$\Delta = abc \begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ A+(p-1)D & A+(q-1)D & A+(r-1)D \end{vmatrix}$$

Using $R_3 \to R_3 - (A - D)R_1 - DR_2$, we get
$\Delta = 0$.

Example 32: Let $\omega \neq 1$ be complex cube root of unity and n be a natural number and

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$$

Then Δ equals

(a) 0 (b) 1
(c) ω (d) ω^2

Ans. (a)

Solution: If n is a multiple of 3, we get each element of Δ becomes 1.

$\therefore$ $\Delta = 0$.

If $n = 3k + 1$, then

$$\Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0 \quad [\text{use } C_1 \to C_1 + C_2 + C_3]$$

If $n = 3k + 2$, then

$$\Delta = \begin{vmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{vmatrix} = 0 \quad [\text{use } C_1 \to C_1 + C_2 + C_3]$$

Example 33: Let

$$\Delta = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} \text{ and } \Delta_1 = \begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix},$$

then

(a) $\Delta = 2\Delta_1$ (b) $\Delta = -2\Delta_1$
(c) $\Delta = 4\Delta_1$ (d) $\Delta = -4\Delta_1$

Ans. (a)

Solution: Using $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(p+q+r) & 2(x+y+z) \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Taking 2 common from R_1 and applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we obtain

$$\Delta = 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = 2(-1)(-1)\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = -2\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$$

$$= 2\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} = 2\Delta_1$$

Example 34: If $x = -2$, and $\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$ then numerical value of Δ is

(a) 8 (b) – 8
(c) 4 (d) – 4

Ans. (b)

Solution: Taking x common from C_2 and C_3 we obtain

$$\Delta = x^2 \begin{vmatrix} x+y & 1 & 1 \\ 5x+4y & 4 & 2 \\ 10x+8y & 8 & 3 \end{vmatrix} = x^3 \begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix}$$

[using $C_1 \to C_1 - yC_2$]

Next using $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$, we obtain

$$D = x^3 \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 5 & 3 \end{vmatrix} = x^3 \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = x^3$$

As $x = -2, \Delta = -8.$

Example 35: If $a = \omega \neq 1$, is a cube root of unity, $b = -785$, $c = 2008\, i$, and

$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$$

then Δ equals

(a) $-i$ (b) i
(c) 1 (d) $1 - wi$

Ans. (c)

Solution: Write

$$\Delta = a \begin{vmatrix} 1 & a+b & a+b+c \\ 2 & 3a+2b & 4a+3b+2c \\ 3 & 6a+3b & 10a+6b+3c \end{vmatrix}$$

and apply $C_2 \to C_2 - bC_1$, $C_3 \to C_3 - cC_1$ to obtain

$$\Delta = a \begin{vmatrix} 1 & a & a+b \\ 2 & 3a & 4a+3b \\ 3 & 6a & 10a+6c \end{vmatrix}$$

Take a common from C_2 and apply $C_3 \to C_3 - bC_2$ to obtain

$$\Delta = a^2 \begin{vmatrix} 1 & 1 & a \\ 2 & 3 & 4a \\ 3 & 6 & 10a \end{vmatrix} = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

Apply $C_3 \to C_3 - C_2$, $C_2 \to C_2 - C_1$ to obtain

$$\Delta = a^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 3 & 4 \end{vmatrix} = a^3 = \omega^3 = 1.$$

Example 36: Let x, y, z be positive and $x, y, z \neq 1$. Let

$$\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix},$$

then numerical value of Δ is

(a) -1 (b) 0
(c) 1 (d) none of these

Ans. (b)

Solution: Using change of base formula

$$\log_a b = \frac{\log b}{\log a}, \; a, b > 0, a, b \neq 1,$$

we can write

$$\Delta = \frac{1}{\log x \log y \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0.$$

Example 37: Let $\omega \neq 1$ be a cube root of unity and

$$\Delta = \begin{vmatrix} 1-\omega-\omega^2 & 2 & 2 \\ 2\omega & \omega-\omega^2-1 & 2\omega \\ 2\omega^2 & 2\omega^2 & \omega^2-1-\omega \end{vmatrix}$$

then Δ equals

(a) $-\omega$ (b) $3\omega(1-\omega)$
(c) 0 (d) $1-\omega^2$

Ans. (c)

Solution: Using $R_1 \to R_1 + R_2 + R_3$ and $1 + \omega + \omega^2 = 0$, we obtain

$$\Delta = 0.$$

Example 38: Suppose $x = -\frac{1}{3}\left(1+\sqrt{7}i\right)$ and

$$y = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}.$$

Let $$\Delta = \begin{vmatrix} 1 & x & x \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix},$$

then Δ equals

(a) $-\sqrt{7}$ (b) 7
(c) i (d) $-1.$

Ans. (c)

Solution: Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & x & x \\ 0 & y & y-x \\ 0 & 0 & y \end{vmatrix} = y^2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$$

[Using De Moivre's Theorem]

Example 39: Let $x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ and

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

then numerical value of Δ is

(a) 0 (b) -1
(c) 8 (d) -4

Ans. (c)

◎ **Solution:** Use $C_3 \to C_3 - xC_2$ and $C_2 \to C_2 - xC_1$.

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^3 & 0 \\ 1 & x^2-x & 1-x^3 \end{vmatrix}$$

$$= (1-x^3)^2 = [1 - \cos\pi - i\sin\pi]^2 = 8$$

[De Moivre's Theorem]

◉ **Example 40:** Let $f(x) = [2^{-x^2}[2^{x^2}]]$, $x \in \mathbf{R}$. ([ ] denotes the greatest integer function.) Let $x_1 = 0$, $x_2 = \log_2 3$ and $x_3 = \sqrt{2}$. Suppose $0 < x_4 < 1$.

$$\Delta = \begin{vmatrix} f(x_1) & f(x_2) & f(x_3) \\ f(x_4) & f(x_4) & f(x_2) \\ f(x_2) & f(x_3) & f(x_1) \end{vmatrix}, \text{ then } \Delta \text{ is equal to:}$$

(a) -1 (b) 0
(c) 1 (d) $2\log_2{}^3$

Ans. (b)

◎ **Solution:** $0 \le [2^{x^2}] \le 2^{x^2} \Rightarrow 0 \le \dfrac{[2^{x^2}]}{2^{x^2}} \le 1$. Therefore, range $f(x)$ is $\{0, 1\}$

Also $[2^{x^2}] = 2^{x^2} \Leftrightarrow 2^{x^2}$ is a positive integer.

Thus, $f(x_1) = 1, f(x_2) = 1, f(x_3) = 1$ and $f(x_4) = 0$.
Therefore,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

◉ **Example 41:** If $\omega \ne 1$ is a cube root of unity and

$$\Delta = \begin{vmatrix} x+\omega^2 & \omega & 1 \\ \omega & \omega^2 & 1+x \\ 1 & x+\omega & \omega^2 \end{vmatrix} = 0, \text{ then value of } x \text{ is}$$

(a) 0 (b) 1
(c) -1 (d) none of these

Ans. (a)

◎ **Solution:** Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+\omega^2+\omega+1 & \omega & 1 \\ \omega+\omega^2+1+x & \omega^2 & 1+x \\ 1+x+\omega+\omega^2 & x+\omega & \omega^2 \end{vmatrix}$$

$$= \begin{vmatrix} x & \omega & 1 \\ x & \omega^2 & 1+x \\ x & x+\omega & \omega^2 \end{vmatrix} \text{ using } 1+\omega+\omega^2 = 0$$

Δ is clearly equal to 0 for $x = 0$.

◉ **Example 42:** Let $f: \mathbf{N} \to \mathbf{N}$ be defined by

$$f(x) = (x+1)^2 + x - \left[\sqrt{(x+1)^2 + (x+1)}\right]^2$$

([ ] denotes the greatest integer function). Suppose *a, b, c* are three distinct natural numbers. Let

$$\Delta = \begin{vmatrix} f(a) & a^2 & a \\ f(b) & b^2 & b \\ f(c) & c^2 & c \end{vmatrix}$$

The Δ is equal to:

(a) $-(a+b+c)$ (b) $a+b+c$
(c) -1 (d) 0

Ans. (d)

◎ **Solution:** If $n \in \mathbf{N}$, then

$$n^2 < n^2 + n < (n+1)^2$$

$$\Rightarrow \quad n < \sqrt{n^2+n} < n+1$$

$$\Rightarrow \quad \left[\sqrt{n^2+n}\right] = n$$

$$\Rightarrow \quad \left[\sqrt{(x+1)^2+(x+1)}\right] = x+1 \ \forall\, x \in \mathbf{N}.$$

$\therefore \quad f(x) = (x+1)^2 + x - (x+1)^2 = x \ \forall\, x \in \mathbf{N}$.

Thus,

$$\Delta = \begin{vmatrix} a & a^2 & a \\ b & b^2 & b \\ c & c^2 & c \end{vmatrix} = 0$$

[$\because$ C_1 and C_3 are identical]

◉ **Example 43:** If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then the other two roots are

(a) 3, 7 (b) 2, 7
(c) 3, 6 (d) 2, 6

Ans. (b)

◎ **Solution:** Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}$$

Taking $x + 9$ common from R_1 and applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ we get

$$\Delta = (x+9)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ 7 & -1 & x-7 \end{vmatrix}$$

$$= (x+9)(x-2)(x-7)$$

Thus, $\Delta = 0 \Rightarrow x = -9, 2, 7$

$\therefore$ the other two roots are 2 and 7.

Example 44: If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$, then

(a) $\Delta_1 = 3(\Delta_2)^2$ (b) $\dfrac{d}{dx}(\Delta_1) = 3\Delta_2$

(c) $\dfrac{d}{dx}(\Delta_1) = 3\Delta_2^2$ (d) $\Delta_1 = 3\Delta_2^{3/2}$

Ans. (b)

Solution: We have

$$\frac{d\Delta_1}{dx} = \begin{vmatrix} 1 & b & b \\ 0 & x & b \\ 0 & a & x \end{vmatrix} + \begin{vmatrix} x & 0 & b \\ a & 1 & b \\ a & 0 & x \end{vmatrix} + \begin{vmatrix} x & b & 0 \\ a & x & 0 \\ a & b & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix}$$

$$= 3\Delta_2.$$

Example 45: If $x \varepsilon \mathbf{R}$ and $n \varepsilon \mathbf{I}$, then the determinant

$$\Delta = \begin{vmatrix} \sin(n\pi) & \sin x - \cos x & \log\tan x \\ \cos x - \sin x & \cos[(2n+1)\pi/2] & \log\cot x \\ \log\cot x & \log\tan x & \tan(n\pi) \end{vmatrix}$$

equals

(a) 0 (b) $\log\tan x - \log\cot x$

(c) $\tan(\pi/4 - x)$ (d) none of these

Ans. (a)

Solution: We can write Δ as

$$\Delta = \begin{vmatrix} 0 & \sin x - \cos x & \log\tan x \\ -(\sin x - \cos x) & 0 & -\log\tan x \\ -\log\tan x & \log\tan x & 0 \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} 0 & -(\sin x - \cos x) & -\log\tan x \\ \sin x - \cos x & 0 & \log\tan x \\ \log\tan x & -\log\tan x & 0 \end{vmatrix}$$

[taking – 1 common from R_1, R_2 and R_3]

$= -\Delta$ [using the reflection property]

$\Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0.$

Example 46: Let $\Delta_1 = \begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$ and

$\Delta_2 = \begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}$, then $\Delta_1 - \Delta_2$ equals

(a) $(x-1)(y-1)(z-1)$

(b) $(x-y)(y-z)(z-x)$

(c) $abc\,(x-y)(y-z)(z-x)$

(d) 0

Ans. (d)

Solution: Multiplying R_3 of Δ_1 by xyz, we get

$$\Delta_1 = \frac{1}{xyz}\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix} = \Delta_2$$

[taking x, y, z common from C_1, C_2, C_3 respectively]

$\Rightarrow \quad \Delta_1 - \Delta_2 = 0.$

Example 47: If

$$\Delta(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x^2-1) \end{vmatrix}$$

then $\Delta(100)$ equals

(a) 0 (b) – 100

(c) 100! (d) – 100!

Ans. (a)

Solution: Taking x common from C_2, $x + 1$ from C_3 and $x - 1$ from R_3, we get

$$\Delta(x) = x(x+1)(x-1)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$, we get

$$\Delta(x) = x(x+1)(x-1)\begin{vmatrix} 0 & 0 & 1 \\ x & -1 & x \\ 2x & -2 & x \end{vmatrix} = 0$$

[$\because$ C_1 and C_2 are proportional]

Thus, $\quad \Delta(100) = 0$

Example 48: If $\Delta(x) = \begin{vmatrix} 1 & 1 & 1 \\ (e^x + e^{-x})^2 & (\pi^x + \pi^{-x})^2 & 2 \\ (e^x - e^{-x})^2 & (\pi^x - \pi^{-x})^2 & -2 \end{vmatrix}$

then $\Delta(x)$ equals

(a) x^2 (b) $x^2 - 1$

(c) $e^{x^2} - \pi^{x^2}$ (d) 0

Ans. (d)

Solution: Using $R_2 \to R_2 - R_3$ and $(a^x + a^{-x})^2 - (a^x - a^{-x})^2 = 4a^x a^{-x} = 4$, we get

$$\Delta(x) = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 4 \\ (e^x - e^{-x})^2 & (\pi^x - \pi^{-x})^2 & -2 \end{vmatrix} = 0$$

[since R_1 and R_2 are proportional]

Example 49: If $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1-\cos x \\ 1+\sin x & \cos x & 1+\sin x-\cos x \\ \sin x & \sin x & 1 \end{vmatrix}$, then $\int_0^{\pi/2} \Delta(x)\, dx$ equals

(a) 1/4 (b) 1/2

(c) 0 (d) – 1/2

Ans. (d)

Solution: Using $C_1 \to C_1 - C_2 - C_3$, we get

$$\Delta(x) = \begin{vmatrix} 0 & \cos x & 1-\cos x \\ 0 & \cos x & 1+\sin x-\cos x \\ -1 & \sin x & 1 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} \cos x & 1-\cos x \\ \cos x & 1+\sin x-\cos x \end{vmatrix}$$

$$= (-1)\cos x\,[1 + \sin x - \cos x - 1 + \cos x]$$

$$= -\cos x \sin x = -\frac{1}{2}(\sin 2x)$$

$$\therefore \quad \int_0^{\pi/2} \Delta(x)\, dx = -\frac{1}{2}\frac{(-1)}{2}\cos 2x \Big]_0^{\pi/2}$$

$$= \frac{1}{4}(\cos \pi - \cos 0) = -\frac{1}{2}$$

Example 50: The determinant

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} \text{ equals zero, if}$$

(a) a, b, c are in A.P.

(b) a, b, c are in H.P.

(c) $x - \alpha$ is a factor of $ax^2 + 2bx + c$

(d) $x - \alpha$ is a factor of $ax^2 + bx + c$

Ans. (c)

Solution: Applying $C_3 \to C_3 - \alpha C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & -\left(a\alpha^2+2b\alpha+c\right) \end{vmatrix}$$

$$= -(a\alpha^2 + 2b\alpha + c)\begin{vmatrix} a & b \\ b & c \end{vmatrix} \quad [\text{expanding along } C_3]$$

$$= -(a\alpha^2 + 2b\alpha + c)(ac - b^2)$$

$= 0$ if $x - \alpha$ is a factor of $ax^2 + 2bx + c$.

Example 51: If $\theta \,\varepsilon\, \mathbf{R}$, then the determinant

$$\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin(\theta+2\pi/3) & \cos(\theta+2\pi/3) & \sin(2\theta+4\pi/3) \\ \sin(\theta-2\pi/3) & \cos(\theta-2\pi/3) & \sin(2\theta-4\pi/3) \end{vmatrix}$$

equals

(a) $-\sin\theta - \cos\theta$ (b) $\sin 2\theta$

(c) $1 + \sin 2\theta - \cos 2\theta$ (d) 0

Ans. (d)

Solution: Applying $R_2 \to R_2 + R_3$, and using $\sin(A + B) + \sin(A - B) = 2\sin A \cos B$ and $\cos(A + B) + \cos(A - B) = 2\cos A \cos B$, we get

$$\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ 2\sin\theta\cos(2\pi/3) & 2\cos\theta\cos(2\pi/3) & 2\sin 2\theta\cos(4\pi/3) \\ \sin(\theta-2\pi/3) & \cos(\theta-2\pi/3) & \sin(2\theta-4\pi/3) \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin(\theta-2\pi/3) & \cos(\theta-2\pi/3) & \sin(2\theta-4\pi/3) \end{vmatrix}$$

since $\cos(2\pi/3) = \cos(\pi - \pi/3) = -\cos(\pi/3) = -1/2$

and $\cos(4\pi/3) = \cos(\pi + \pi/3) = -\cos(\pi/3) = -1/2$

$\Rightarrow \quad \Delta = 0 \quad [\because R_1 \text{ and } R_2 \text{ are proportional.}]$

Example 52: If α, β, γ are the roots of $x^3 + px^2 + q = 0$, where $q \neq 0$, and

$$\Delta = \begin{vmatrix} 1/\alpha & 1/\beta & 1/\gamma \\ 1/\beta & 1/\gamma & 1/\alpha \\ 1/\gamma & 1/\alpha & 1/\beta \end{vmatrix}$$

then Δ equals

(a) $-p/q$ (b) $1/q$

(c) p^2/q (d) 0

Ans. (d)

Solution: We have $\beta\gamma + \gamma\alpha + \alpha\beta = 0$. We can write Δ as

$$\Delta = \frac{1}{\alpha^3\beta^3\gamma^3}\begin{vmatrix} \beta\gamma & \gamma\alpha & \alpha\beta \\ \gamma\alpha & \alpha\beta & \beta\gamma \\ \alpha\beta & \beta\gamma & \gamma\alpha \end{vmatrix}$$

$$= \frac{1}{\alpha^3\beta^3\gamma^3}\begin{vmatrix} \beta\gamma+\gamma\alpha+\alpha\beta & \gamma\alpha & \alpha\beta \\ \gamma\alpha+\alpha\beta+\beta\gamma & \alpha\beta & \beta\gamma \\ \alpha\beta+\beta\gamma+\gamma\alpha & \beta\gamma & \gamma\alpha \end{vmatrix}$$

$$[\text{using } C_1 \to C_1 + C_2 + C_3]$$

$$= \frac{1}{\alpha^3\beta^3\gamma^3}\begin{vmatrix} 0 & \gamma\alpha & \alpha\beta \\ 0 & \alpha\beta & \beta\gamma \\ 0 & \beta\gamma & \gamma\alpha \end{vmatrix} = 0 \text{ [all zero property]}$$

Example 53: If A, B, C are the angles of a triangle, and

$$\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

then Δ equals

(a) 0 (b) − 1

(c) 2 cos A cos B cos C (d) none of these

Ans. (a)

◎ **Solution:** Applying $C_1 \to aC_1 + bC_2 + cC_3$, we get

$$\Delta = \frac{1}{a}\begin{vmatrix} -a + b\cos C + c\cos B & \cos C & \cos B \\ a\cos C - b + c\cos A & -1 & \cos A \\ a\cos B + b\cos A - c & \cos A & -1 \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0$$

[using projection formulae and all zero property.]

◉ **Example 54:** If a, b, c are three complex numbers such that $a^2 + b^2 + c^2 = 0$ and

$$\Delta = \begin{vmatrix} b^2 + c^2 & ab & ac \\ ab & c^2 + a^2 & bc \\ ac & bc & a^2 + b^2 \end{vmatrix} = ka^2\, b^2\, c^2,$$

then the value of k is

(a) 1 (b) 2

(c) − 2 (d) 4

Ans. (*d*)

◎ **Solution:** Using $a^2 + b^2 + c^2 = 0$, we can write Δ as

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$$

[taking a, b, c common from C_1, C_2, C_3 respectively]

$$= a^2\, b^2\, c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

[taking a, b, c common from R_1, R_2, R_3 respectively]

$$= a^2\, b^2\, c^2\begin{vmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & -1 \end{vmatrix} = -\,2a^2\, b^2\, c^2\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} = 4a^2\, b^2\, c^2$$

[applying $C_1 \to C_1 + C_3$ and $C_3 \to C_2 + C_3$]

Thus, $k = 4$.

◉ **Example 55:** If θ, $\phi\ \varepsilon\ \boldsymbol{R}$, then the determinant

$$\Delta = \begin{vmatrix} \cos\theta & -\sin\theta & 1 \\ \sin\theta & \cos\theta & 1 \\ \cos(\theta + \phi) & -\sin(\theta + \phi) & 0 \end{vmatrix}$$

lies in the interval

(a) $\left[-\sqrt{2}, \sqrt{2}\right]$ (b) $[-1, 1]$

(c) $\left[-\sqrt{2}, 1\right]$ (d) $\left[-1, \sqrt{2}\right]$

Ans. (a)

◎ **Solution:** Applying $R_3 \to R_3 - \cos\varphi\, R_1 + \sin\varphi\, R_2$, we get

$$\Delta = \begin{vmatrix} \cos\theta & -\sin\theta & 1 \\ \sin\theta & \cos\theta & 1 \\ 0 & 0 & \sin\varphi - \cos\phi \end{vmatrix}$$

$$= (\sin\varphi - \cos\varphi)\,(\cos^2\theta + \sin^2\theta)$$

$$= \sqrt{2}\left\{\frac{1}{\sqrt{2}}\sin\varphi - \frac{1}{\sqrt{2}}\cos\varphi\right\} = \sqrt{2}\ \sin(\phi - \pi/4)$$

As $-1 \le \sin(\phi - \pi/4) \le 1$, $-\sqrt{2} \le \sqrt{2}\ \sin(\varphi - \pi/4) \le \sqrt{2}$ or $-\sqrt{2} \le \Delta \le \sqrt{2}$.

◉ **Example 56:** If $\Delta_1 = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$ and

$\Delta_2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$, then $\Delta_1 - \Delta_2$ equal

(a) 0 (b) $3abc$

(c) $6abc$ (d) $2(a^3 + b^3 + c^3)$

Ans. (a)

◎ **Solution:** Using $C_1 \to C_1 + C_3$ and $C_2 \to C_2 - C_3$ in Δ_1, we get

$$\Delta_1 = \begin{vmatrix} a+b+c & -b & a \\ a+b+c & -c & b \\ a+b+c & -a & c \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 - C_3$, we get

$$\Delta_1 = \begin{vmatrix} c & -b & a \\ a & -c & b \\ b & -a & c \end{vmatrix} = -\begin{vmatrix} a & -b & c \\ b & -c & a \\ c & -a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$\Rightarrow \Delta_1 = \Delta_2 \Rightarrow \Delta_1 - \Delta_2 = 0.$

◉ **Example 57:** If x, y, z are different from zero and

$$\Delta = \begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0,$$

then the value of the expression $\frac{a}{x} + \frac{b}{y} + \frac{c}{z}$ is

(a) 0 (b) −1

(c) 1 (d) 2

Ans. (d)

◎ **Solution:** Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} a & b-y & c-z \\ -x & y & 0 \\ -x & 0 & z \end{vmatrix} = 0$$

Expanding along C_3, we get

$$(c-z)\begin{vmatrix} -x & y \\ -x & 0 \end{vmatrix} + z\begin{vmatrix} a & b-y \\ -x & y \end{vmatrix} = 0$$

$$\Rightarrow \quad (c-z)(xy) + z(ay + bx - xy) = 0$$

$$\Rightarrow \quad cxy - xyz + ayz + bxz - xyz = 0$$

$$\Rightarrow \quad ayz + bzx + cxy = 2xyz$$

$$\Rightarrow \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

Example 58: If $p + q + r = a + b + c = 0$, then the determinant

$$\Delta = \begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} \text{ equals}$$

(a) 0 (b) 1
(c) $pa + qb + rc$ (d) none of these

Ans. (a)

Solution: We have

$$\Delta = pqr(a^3 + b^3 + c^3) - abc(p^3 + q^3 + r^3)$$

But $a + b + c = 0 \Rightarrow (a + b)^3 = -c^3$

$$\Rightarrow \quad a^3 + b^3 + 3ab(a + b) + c^3 = 0$$

$$\Rightarrow \quad a^3 + b^3 + c^3 = -3ab(-c) = 3abc$$

Similarly, $p^3 + q^3 + r^3 = 3pqr$

Thus, $\Delta = pqr(3abc) - abc(3pqr) = 0$

Example 59: The system of equations

$$\lambda x + y + z = 0$$
$$-x + \lambda y + z = 0$$
$$-x - y + \lambda z = 0$$

will have a non-trivial solution if real values of λ are

(a) 0, 1 (b) 0, – 1
(c) 0, 2 (d) 0

Ans. (d)

Solution: The system of equation will have a non-trivial solution if and only if

$\Delta = 0$ where

$$\Delta = \begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = \lambda^3 + 3\lambda = \lambda(\lambda^2 + 1)$$

Thus, $\Delta = 0$, $\lambda \in \mathbf{R} \Rightarrow \lambda = 0$

Example 60: The values of k for which the system of equations

$$x + ky - 3z = 0 \qquad (1)$$
$$3x + ky - 2z = 0 \qquad (2)$$
$$2x + 3y - 4z = 0 \qquad (3)$$

has a non-trivial solution is (are)

(a) $\frac{21}{10}$ (b) $\frac{31}{10}$
(c) – 5 (d) 4

Ans. (a)

Solution:

From (1) and (2), $2x + z = 0 \Rightarrow 2x = -z$.

From (3), we get $3y = 5z$

$$\therefore \quad x = -\frac{1}{2}z, \; y = \frac{5}{3}z$$

Substituting this in (1) we get

$$-\frac{1}{2}z + \frac{5k}{3}z - 3z = 0. \Rightarrow \left(\frac{5k}{3} - \frac{7}{2}\right)z = 0$$

Now, $\quad z \neq 0 \;\Rightarrow\; k = \frac{21}{10}$.

Example 61: If $a \in \mathbf{R}$ and the system of equations $x + ay = 0$, $az + y = 0$, $ax + z = 0$ has infinite number of solution then the value of a is

(a) – 1 (b) 1
(c) 0 (d) no real value

Ans. (a)

Solution: We have $x = -ay$, therefore

$$-a^2y + z = 0 \Rightarrow z = a^2y.$$

Thus, $\quad az + y = 0 \Rightarrow (a^3 + 1)z = 0.$

For the system of equations to have infinite number of solutions $a^3 + 1 = 0$

$\Rightarrow \quad a = -1 \qquad [\therefore a$ is real$]$

Example 62: Given $2x - y + 2z = 2$,

$$x - 2y + z = -4,$$
$$x + y + \lambda z = 4$$

then the value of λ such that the given system of equations has no solution, is

(a) 3 (b) 1
(c) 0 (d) – 3

Ans. (b)

Solution: From the first two equations, we get

$$x + y = 6 - z$$

Putting this value in the last equation we get

$$6 + (-1 + \lambda)z = 4 = (\lambda - 1)z = -2.$$

For the system of equations to have no solutions, $\lambda = 1$

Example 63: If the system of linear equations

$$x + 2ay + az = 0$$
$$x + 3by + bz = 0$$
$$x + 4cy + cz = 0$$

has a non-zero solution, then a, b, c

(a) are in G.P.
(b) are in H.P.

(c) satisfy $a + 2b + 3c = 0$
(d) are in A.P.

Ans. (b)

◎ **Solution:** The system will have a non-zero solution if

$$\Delta = \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

Using $C_2 \to C_2 - 2C_3$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & a \\ 1 & b & b \\ 1 & 2c & c \end{vmatrix} = 0$$

$$\Rightarrow \ 1(bc - 2bc) + a(2c - b) = 0 \Rightarrow 2ac = bc + ab$$

$$\Rightarrow \ \frac{2ac}{a+c} = b \Rightarrow a, b, c \text{ are in H.P.}$$

◉ **Example 64:** The system of homogenous equations

$$(a - 1)x + (a + 2)y + az = 0$$
$$(a + 1)x + ay + (a + 2)z = 0$$
$$ax + (a + 1)y + (a - 1)z = 0$$

has a non-trivial solution if a equals

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
(c) 2 (d) – 1.

Ans. (b)

◎ **Solution:** The system of equations will have a non-trivial solution if $\Delta = 0$ where

$$\Delta = \begin{vmatrix} a-1 & a+2 & a \\ a+1 & a & a+2 \\ a & a+1 & a-1 \end{vmatrix}$$

Using $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we obtain

$$\Delta = \begin{vmatrix} -1 & 2 & a \\ -1 & -2 & a+2 \\ 1 & 2 & a-1 \end{vmatrix}$$

Using $R_3 \to R_3 + R_2$, we get

$$\Delta = \begin{vmatrix} -1 & 2 & a \\ -1 & -2 & a+2 \\ 0 & 0 & 2a+1 \end{vmatrix} = (2a + 1)\,(2 + 2) = 4(2a + 1)$$

Now, $\Delta = 0 \Rightarrow a = -1/2$.

◉ **Example 65:** The system of equations

$$\alpha x + y + z = \alpha - 1$$
$$x + \alpha y + z = \alpha - 1$$
$$x + y + \alpha z = \alpha - 1$$

has no solution, if α equals

(a) – 2 (b) 1
(c) – 2 (d) either – 2 or 1

Ans. (a)

◎ **Solution:** Let

$$\Delta = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (\alpha + 2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = (\alpha + 2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha - 1 & 0 \\ 0 & 1 & \alpha - 1 \end{vmatrix}$$

[using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$\therefore$ $\Delta = (\alpha + 2)\,(\alpha - 1)^2$

For no solution, $\Delta = 0 \Rightarrow \alpha = -2, 1$.

For $\alpha = 1$, the system of equations has infinite number of solutions.

For $\alpha = -2$, on adding the three equations we obtain

$$0 = -9$$

Thus, system of equations has no solution for $\alpha = -2$.

◉ **Example 66:** Suppose $a, b, c, \alpha \in \mathbf{R}$ and $abc\,\alpha \neq 0$. If the system of equations:

$$(a + \alpha)x + \alpha y + \alpha z = 0 \tag{1}$$
$$\alpha x + (b + \alpha)y + \alpha z = 0 \tag{2}$$
$$\alpha x + \alpha y + (\alpha + c)z = 0 \tag{3}$$

has a non-trivial solution, then $\alpha\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$ is equal to

(a) – 1 (b) 0
(c) abc (d) $bc + ca + ab$

Ans. (a)

◎ **Solution:** From (1) and (2)

$$ax - by = 0$$

and from (2) and (3)

$$ax - cz = 0$$

$\therefore$ $ax = by = cz$

$$\Rightarrow \ \frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c}$$

Putting in (1) we get

$$\frac{a+\alpha}{a} + \frac{\alpha}{b} + \frac{\alpha}{c} = 0$$

$$\Rightarrow \ \alpha\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = -1$$

◉ **Example 67:** Suppose $a, b, \in \mathbf{R}$ and $a, b \neq 1$. If the system of equations:

$$ax + y + z = 0 \tag{1}$$
$$x + by + z = 0 \tag{2}$$
$$x + y + 2z = 0 \tag{3}$$

has a non-trivial solution, then

(a) $a + b = 2$ (b) $a + b = ab$

(c) $a + \frac{1}{b} = 2$ (d) $a + b = 0$

Ans. (a)

Solution: From (1) and (3)

$$(a - 1)x - z = 0$$

and from (2) and (3)

$$(b - 1)y = z$$

$$\therefore \quad \frac{x}{\frac{1}{(1-a)}} = \frac{y}{\frac{1}{(1-b)}} = \frac{z}{1}$$

Putting in (1) we get

$$\frac{a}{1-a} + \frac{1}{1-b} + 1 = 0$$

$$\Rightarrow \quad \frac{a}{1-a} = -\frac{1}{1-b}$$

$$\Rightarrow \quad 1 - b = -1 + a$$

$$\Rightarrow \quad a + b = 2$$

Example 68: If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

then $f(x)$ is a polynomial of degree

(a) 3 (b) 2

(c) 1 (d) 0

Ans. (b)

Solution: Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$$

[using $a^2 + b^2 + c^2 = -2$]

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix} = (1 - x)^2$$

which is a polynomial of degree 2.

Example 69:

$$\text{Let} \quad f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$$

then value of $\int_{\pi/4}^{\pi/2} f(x)dx$ is

(a) 0 (b) $\pi/48$

(c) $-\frac{\pi}{2} - \frac{\pi}{15\sqrt{2}}$ (d) none of these

Ans. (d)

Solution: Applying $R_2 \to R_2 - R_3$, we get

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ -\sin^2 x & 0 & 0 \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$$

$$= -(-\sin^2 x) \begin{vmatrix} \cos x & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$$

$$= \sin^2 x [\cos x \operatorname{cosec}^2 x - \cos^2 x (\sec^2 x + \cot x \operatorname{cosec}^2 x)]$$

$$= \cos x - \sin^2 x - \frac{\cos^3 x}{\sin x}$$

$$= \cos x - \frac{1}{2} (1 - \cos 2x) - \left(\frac{1}{\sin x} - \sin x\right) \cos x$$

$$\text{Thus, } \int_{\pi/4}^{\pi/2} f(x)\, dx = \int_{\pi/4}^{\pi/2} \cos x\, dx - \frac{1}{2} \int_{\pi/4}^{\pi/2} dx + \frac{1}{2} \int_{\pi/4}^{\pi/2} \cos 2x\, dx - \int_{\pi/4}^{\pi/2} \left(\frac{1}{\sin x} - \sin x\right) \cos x dx$$

$$= 1 - \frac{1}{\sqrt{2}} - \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + \frac{1}{4}(0 - 1) - \left[\left(\log|t| - \frac{t^2}{2}\right)\right]_{1/\sqrt{2}}^{1}$$

where $t = \sin x$

$$= 1 - \frac{1}{\sqrt{2}} - \frac{\pi}{8} - \frac{1}{2}\log 2$$

Example 70: If $a + b + c = 0$, then a root of the equation

$$\Delta = \begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0 \text{ is}$$

(a) 1 (b) -1

(c) $a^2 + b^2 + c^2$ (d) 0

Ans. (d)

Solution: Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix}$$

$$= \begin{vmatrix} -x & c & b \\ -x & b-x & a \\ -x & a & c-x \end{vmatrix} \quad [\because a + b + c = 0]$$

Δ clearly equals 0 when $x = 0$.

It is unnecessary to evaluate the determinant further.

Example 71: A root of the equation

$$\Delta = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0 \text{ is}$$

(a) $\frac{1}{2}(a+b+c)$ (b) 0

(c) –1 (d) 1

Ans. (b)

Solution: When we substitute $x = 0$, Δ becomes

$$\begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

which is equal to 0 as Δ is skew symmetric determinant of odd order.

Alternative Solution

Evaluating Δ along R_1, we get

$$-(x-a)\begin{vmatrix} x+a & x-c \\ x+b & 0 \end{vmatrix} + (x-b)\begin{vmatrix} x+a & 0 \\ x+b & x+c \end{vmatrix} = 0$$

$\Rightarrow$ $(x-a)(x+b)(x-c) + (x-b)(x+a)(x+c) = 0$

$\Rightarrow$ $x^3 + (-a+b-c)x^2 + (-ab+ac-bc)x + abc + x^3 + (a-b+c)x^2 + (-ab+ac-bc) - abc = 0$

$\Rightarrow$ $2x^3 - 2(ab-ac+bc)x = 0$

$\Rightarrow$ $2x[x^2 - (ab-ac+bc)] = 0$

$\Rightarrow$ $x = 0$ or $x = \pm\sqrt{ab-ac+bc}$

$\therefore$ one of the roots of the equation is 0.

Example 72: If α, β, γ are three real numbers such that $\alpha+\beta+\gamma = 0$, then

$$\Delta = \begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} \text{ equals}$$

(a) – 1 (b) 0

(c) 1 (d) $\cos\alpha\cos\beta\cos\gamma$

Ans. (b)

Solution: Let A, B and C be three real numbers such that $\alpha = B - C$, $\beta = C - A$ and $\gamma = A - B$, clearly $\alpha+\beta+\gamma = 0$. We have

$$\Delta = \begin{vmatrix} 1 & \cos(A-B) & \cos(C-A) \\ \cos(A-B) & 1 & \cos(B-C) \\ \cos(C-A) & \cos(B-C) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2 A+\sin^2 A & \cos A\cos B+\sin A\sin B & \cos C\cos A+\sin A\sin C \\ \cos A\cos B+\sin A\sin B & \cos^2 B+\sin^2 B & \cos B\cos C+\sin B\sin C \\ \cos C\cos A+\sin C\sin A & \cos B\cos C+\sin B\sin C & \cos^2 C+\sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix}\begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} = (0)\,(0) = 0.$$

Example 73: If a, b, c are the sides of a ΔABC opposite angles A, B, C respectively, and

$$\Delta = \begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos(B-C) \\ c\sin A & \cos(B-C) & 1 \end{vmatrix}, \text{ then } \Delta \text{ equals}$$

(a) $\sin A - \sin C \sin B$ (b) abc

(c) 1 (d) 0

Ans. (d)

Solution: By the law of sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (say)}$$

$\Rightarrow a = k\sin A$, $b = k\sin B$, $c = k\sin C$. Now

$$\Delta = \begin{vmatrix} a^2 & ab/k & ac/k \\ ab/k & 1 & \cos(B-C) \\ ac/k & \cos(B-C) & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos(B-C) \\ \sin C & \cos(B-C) & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & \sin(A+C) & \sin(A+B) \\ \sin(A+C) & 1 & \cos(B-C) \\ \sin(A+B) & \cos(B-C) & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} \sin A & \cos A & 0 \\ \cos C & \sin C & 0 \\ \cos B & \sin B & 0 \end{vmatrix}\begin{vmatrix} \sin A & \cos A & 0 \\ \cos C & \sin C & 0 \\ \cos B & \sin B & 0 \end{vmatrix}$$

$$= a^2(0) = 0$$

Example 74: If a, b, c are distinct, and $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$ $= (b-c)(c-a)(a-b)(a+b+c)$

then $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$

vanishes if

(a) $x = \frac{1}{3}(a+b+c)$

(b) $x = \frac{2}{3}(a+b+c)$

(c) $x = a+b+c$

(d) none of these

Ans. (a)

◎ **Solution:** Multiplying C_1 by $(x-a)$, C_2 by $(x-b)$ and C_3 by $(x-c)$, we get

$$\Delta = \frac{1}{ABC}\begin{vmatrix} A & B & C \\ A^3 & B^3 & C^3 \\ ABC & ABC & ABC \end{vmatrix}$$

where $A = x-a$, $B = x-b$, $C = x-c$.

$$\Delta = \begin{vmatrix} A & B & C \\ A^3 & B^3 & C^3 \\ 1 & 1 & 1 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ A & B & C \\ A^3 & B^3 & C^3 \end{vmatrix}$$

$$= (B-C)(C-A)(A-B)(A+B+C)$$

$$= (c-b)(a-b)(b-a)[3x-(a+b+c)]$$

Note that Δ become 0 when $x = \frac{1}{3}(a+b+c)$

◉ **Example 75:** The equation

$$\Delta = \begin{vmatrix} x-a & x-b & x-c \\ x-b & x-c & x-a \\ x-c & x-a & x-b \end{vmatrix} = 0$$

is satisfied when

(a) $x = \frac{1}{3}(a+b+c)$ (b) $x = \frac{1}{2}(a+b+c)$

(c) $x = a+b+c$ (d) $x = 0$

Ans. (a)

◎ **Solution:** Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 3x-(a+b+c) & x-b & x-c \\ 3x-(a+b+c) & x-c & x-a \\ 3x-(a+b+c) & x-a & x-b \end{vmatrix}$$

Note that Δ becomes 0 when $x = \frac{1}{3}(a+b+c)$.

◉ **Example 76:** If α, β, γ are different from 1 and are the roots of $ax^3 + bx^2 + cx + d = 0$ and $(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta) = 25/2$, then the determinant

$$\Delta = \begin{vmatrix} \dfrac{\alpha}{1-\alpha} & \dfrac{\beta}{1-\beta} & \dfrac{\gamma}{1-\gamma} \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} \text{ equals}$$

(a) $\frac{25\,d}{2a}$ (b) $\frac{25\,d}{a}$

(c) $\frac{-25d}{a+b+c+d}$ (d) none of these

Ans. (d)

◎ **Solution:** Taking α, β, γ common from C_1, C_2, C_3 respectively, we get

$$\Delta = \alpha\beta\gamma\begin{vmatrix} \dfrac{1}{1-\alpha} & \dfrac{1}{1-\beta} & \dfrac{1}{1-\gamma} \\ 1 & 1 & 1 \\ \alpha & \beta & \gamma \end{vmatrix}$$

$$= \alpha\beta\gamma\begin{vmatrix} \dfrac{1}{1-\alpha} & \dfrac{1}{1-\beta}-\dfrac{1}{1-\alpha} & \dfrac{1}{1-\gamma}-\dfrac{1}{1-\alpha} \\ 1 & 0 & 0 \\ \alpha & \beta-\alpha & \gamma-\alpha \end{vmatrix}$$

[using $C_2 \to C_2 - C_1$, and $C_3 \to C_3 - C_1$]

$$= \frac{\alpha\beta\gamma(-1)(\beta-\alpha)(\gamma-\alpha)}{(1-\alpha)(1-\beta)(1-\gamma)}\begin{vmatrix} 1-\gamma & 1-\beta \\ 1 & 1 \end{vmatrix}$$

$$= \frac{\alpha\beta\gamma(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}{(1-\alpha)(1-\beta)(1-\gamma)}$$

As α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$,

$ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$

and $\alpha\beta\gamma = -d/a$

Thus, $\Delta = \dfrac{(-d/a)(25/2)}{(a+b+c+d)/a} = -\dfrac{25d}{2(a+b+c+d)}$

◉ **Example 77:** Let $P = [a_{ij}]$ be a 3×3 matrix and $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is

(a) 2^{10} (b) 2^{11}

(c) 2^{12} (d) 2^{13}

Ans. (d)

◎ **Solution:**

$$\det(Q) = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$= (2^2)(2^3)(2^4)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2^2a_{31} & 2^2a_{32} & 2^2a_{33} \end{vmatrix}$$

$$= 2^9\,(2)\,(2^2)\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 2^{12}\det(P) = 2^{13}$$

◉ **Example 78:** If x is a positive integer, and $\Delta(x) = \begin{vmatrix} x! & (x+1)! & (x+2)! \\ (x+1)! & (x+2)! & (x+3)! \\ (x+2)! & (x+3)! & (x+4)! \end{vmatrix}$, then $\Delta(x)$ is equal to

(a) $2x!\,(x+1)!$
(b) $2x!\,(x+1)!\,(x+2)!$
(b) $2x!\,(x+3)!$
(d) $2(x+1)!\,(x+2)!\,(x+3)!$

Ans. (b)

◎ **Solution:** Taking $x!$ common from R_1, $(x+1)!$ from R_2 and $(x+2)!$ from R_3 to obtain

$$\Delta(x) = x!\,(x+1)!\,(x+2)!\,\Delta_1(x) \text{ where}$$

$$\Delta_1(x) = \begin{vmatrix} 1 & x+1 & (x+1)(x+2) \\ 1 & x+2 & (x+2)(x+3) \\ 1 & x+3 & (x+3)(x+4) \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta_1(x) = \begin{vmatrix} 1 & x+1 & (x+1)(x+2) \\ 0 & 1 & 2(x+2) \\ 0 & 1 & 2(x+3) \end{vmatrix} = 2$$

◉ **Example 79:** Let a, b, c be such that $b(a+c) \neq 0$.

$$\text{If } \Delta = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$$

$$+ \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0$$

Then the value of n is

(a) any odd integer
(b) any integer
(c) zero
(d) any even integer

Ans. (a)

◎ **Solution:** Let $\Delta_2 = \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix}$

$$= \begin{vmatrix} a+1 & a-1 & (-1)^{n+2}a \\ b+1 & b-1 & (-1)^{n+1}b \\ c-1 & c+1 & (-1)^n c \end{vmatrix}$$

[using the reflection property]

$$= (-1)^2\begin{vmatrix} (-1)^{n+2}a & a+1 & a-1 \\ (-1)^{n+1}b & b+1 & b-1 \\ (-1)^n c & c-1 & c+1 \end{vmatrix}$$

Thus,

$$\Delta = \begin{vmatrix} a+(-1)^{n+2}a & a+1 & a-1 \\ -b+(-1)^{n+1}b & b+1 & b-1 \\ c+(-1)^n c & c-1 & c+1 \end{vmatrix}$$

The first column consists of all 0's if n is any odd integer.
$\therefore\ \Delta = 0$ if n is any odd integer.

◉ **Example 80:** Let ω be the complex number $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$. Then the number of distinct complex number z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is

(a) 1 (b) 0
(c) 2 (d) 3

Ans. (a)

◎ **Solution:** Denote the given determinant by Δ.
Using $C_1 \to C_1 + C_2 + C_3$ and $1 + \omega + \omega^2 = 0$, we get

$$\Delta = z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = z\begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & z+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & z+\omega-\omega^2 \end{vmatrix}$$

$$= z[(z+\omega^2-\omega)(z+\omega-\omega^2) - (1-\omega)(1-\omega^2)]$$
$$= z[z^2 - (\omega^2-\omega)^2 - (1-\omega-\omega^2+1)]$$
$$= z[z^2 - (\omega^4+\omega^2-2\omega^3) - 3] = z(z^2) = z^3$$

Thus, $z^3 = 0 \Rightarrow z = 0$

Thus, there is just one value of z, satisfying the given equation.

Assertion-Reason Type Questions

Example 81: Let $\Delta_n = \begin{vmatrix} a_n & a_{n+3} & a_{n+6} \\ a_{n+1} & a_{n+4} & a_{n+7} \\ a_{n+2} & a_{n+5} & a_{n+8} \end{vmatrix}$

Statement-1: If $a_k > 0 \ \forall \ k \geq 1$ and $a_1, a_2, a_3, \ldots$ are in G.P. then

$$\Delta_n = 0 \ \forall \ n \geq 1.$$

Statement-2: If $a_1, a_2, a_3 \ldots$ are in A.P. then

$$\Delta_n = 0 \ \forall \ n \geq 1.$$

Ans. (b)

Solution: Let $a_k = ar^{k-1} \ \forall \ k \geq 1$, then

$$\Delta_n = a_n a_{n+3}\, a_{n+6} \begin{vmatrix} 1 & 1 & 1 \\ r & r & r \\ r^2 & r^2 & r^2 \end{vmatrix} = 0$$

Next, if $a_k = b + (k-1)d$, then using $C_2 \to C_3 - C_1$ and $C_2 \to C_2 - C_1$, we get

$$\Delta_n = \begin{vmatrix} a_n & 3d & 3d \\ a_{n+1} & 3d & 3d \\ a_{n+2} & 3d & 3d \end{vmatrix} = 0$$

[$\because$ C_2 and C_3 are identical]

Example 82: Let $f(x) = \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}^2$.

Statement-1: If $\sin 2x = 1$, then $f(x) = 2/3$

Statement-2: $f(x) = 0$ if $\sin x = \cos x$

Ans. (d)

Solution: Multiplication of two determinants leads us

$$f(x) = \begin{vmatrix} 1 & -y & y \\ -y & 1 & y \\ y & y & 1 \end{vmatrix}$$

where $y = \sin x \cos x$

Using $C_1 \to C_1 - C_2$, $C_2 \to C_2 + C_3$, we get

$$f(x) = (1+y)^2 \begin{vmatrix} 1 & 0 & y \\ -1 & 1 & y \\ 0 & 1 & 1 \end{vmatrix} = (1+y)^2 \begin{vmatrix} 1 & 0 & y \\ 0 & 1 & 2y \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (1+y)^2 (1 - 2y)$$

When $\sin 2x = 1$, $y = 1/2$ and $f(x) = 0$

When $\sin x = \cos x$, $1 - 2y = 1 - 2\sin^2 x = \cos 2x = 0$

$\therefore \quad f(x) = 0.$

Example 83: Suppose $x > 0$, $y > 0$, $z > 0$ and

$$\Delta(a, b, c) = \begin{vmatrix} x\log 2 & 3 & 15 + \log(a^x) \\ y\log 3 & 5 & 25 + \log(b^y) \\ z\log 5 & 7 & 35 + \log(c^z) \end{vmatrix}$$

Statement-1: $\Delta(8, 27, 125) = 0$

Statement-2: $\Delta\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right) = 0$

Ans. (b)

Solution: Using $\log(b^c) = c \log b$ and applying $C_3 \to C_3 - 5C_2$ we get

$$\Delta(a, b, c) = \begin{vmatrix} x\log 2 & 3 & x\log a \\ y\log 3 & 5 & y\log b \\ z\log 5 & 7 & z\log c \end{vmatrix}$$

$\Delta(8, 27, 125) = \Delta(2^3, 3^3, 5^3) = 0$ as in this case C_1 and C_3 are proportional. Similarly, $\Delta(1/2, 1/3, 1/5) = \Delta(2^{-1}, 3^{-1}, 5^{-1}) = 0$.

Example 84: Let $a \neq 0$, $p \neq 0$ and

$$\Delta = \begin{vmatrix} a & b & c \\ 0 & p & q \\ p & q & 0 \end{vmatrix}$$

Statement-1: If the equations

$$ax^2 + bx + c = 0 \text{ and } px + q = 0$$

have a common root, then $\Delta = 0$.

Statement-2: If $\Delta = 0$, then the equations

$$ax^2 + bx + c = 0 \text{ and } px + q = 0$$

have a common root.

Ans. (b)

Solution: If λ is a common root of

$$ax^2 + bx + c = 0 \text{ and } px + q = 0, \text{ then}$$

$$a\lambda^2 + b\lambda + c = 0,\ p\lambda + q = 0 \text{ and } p\lambda^2 + q\lambda = 0$$

Eliminating λ, we obtained $\Delta = 0$.

For statement-2, expanding Δ along C_1 we obtain

$-aq^2 + p(bq - cp) = 0$

or $$a\left(-\frac{q}{p}\right)^2 + b\left(-\frac{q}{p}\right) + c = 0$$

Thus, $ax^2 + bx + c = 0$ and $px + q = 0$ have a common root.

Example 85: Statement-1:

$$\Delta = \begin{vmatrix} \sin\pi & \cos(x + \pi/4) & \tan(x - \pi/4) \\ \sin(x - \pi/4) & -\cos(\pi/2) & \log(x/y) \\ \cot(\pi/4 + x) & \log(y/x) & \tan\pi \end{vmatrix} = 0$$

Statement-2: A skew symmetric determinant of odd order equals 0.
Ans. (a)

◎ **Solution:** For statement-2, see theory.
Now, using

$$\cos\left(x+\frac{\pi}{4}\right) = \cos\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-x\right)\right]$$

$$= \sin\left(\frac{\pi}{4}-x\right) = -\sin\left(x-\frac{\pi}{4}\right);$$

$$\cot\left(\frac{\pi}{4}+x\right) = \cot\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-x\right)\right]$$

$$= \tan\left(\frac{\pi}{4}-x\right) = -\tan\left(x-\frac{\pi}{4}\right),$$

and $\log(x/y) = -\log(y/x)$

we find Δ is a skew symmetric determinant of odd order.

◉ **Example 86:** Let $f: Q \to [-1, 1]$ by
$f(x) = \sin x$

Let x_1, x_2, x_3 be three distinct rational numbers. Let $a = f(x_1)$, $b = f(x_2)$, $c = f(x_3)$.

Statement-1:

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$$

Statement-2: f is a one-to-one function.
Ans. (a)

◎ **Solution:** Let $x_1, x_2 \in \mathbf{Q}$ be such that

$$f(x_1) = f(x_2)$$

$\Rightarrow \quad \sin x_1 = \sin x_2$

$\Rightarrow \quad x_1 = n\pi + (-1)^n x_2,\ n \in \mathbf{I}$

$\Rightarrow \quad x_1 + (-1)^{n+1} x_2 = n\pi,\ n \in \mathbf{I}.$

But LHS is rational and RHS is irrational except when $n = 0$.

$\therefore \quad x_1 = x_2$

Thus, f is one-to-one.

Also, $\quad \Delta = (a-b)(b-c)(c-a) \neq 0$

as x_1, x_2, x_3 are distinct and hence a, b, c are distinct.

◉ **Example 87:** Let

$$f(x) = \begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & -2x & x-4 \\ x-3 & x+4 & 3x \end{vmatrix}$$

$$= ax^4 + bx^3 + cx^2 + dx + e$$

Statement-1: $a = -1$
Statement-2: $e = f(0)$

Ans. (a)

◎ **Solution:** $e = f(0) = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix}$
$= 0$

[skew symmetric determinant of odd order]

$\therefore$ Statement-2 is true.

To obtain a, replace x by $\frac{1}{x}$, so that

$$\begin{vmatrix} \left(\frac{1}{x}\right)^2+\frac{3}{x} & \frac{1}{x}-1 & \frac{1}{x}+3 \\ \frac{1}{x}+1 & -\frac{2}{x} & \frac{1}{x}-4 \\ \frac{1}{x}-3 & \frac{1}{x}+4 & \frac{3}{x} \end{vmatrix}$$

$$a\left(\frac{1}{x}\right)^4 + b\left(\frac{1}{x}\right)^3 + c\left(\frac{1}{x}\right)^2 + d\left(\frac{1}{x}\right) + e$$

Take x^2 common from C_1, x from C_2 and C_3

$$= \frac{1}{x^4}\begin{vmatrix} 1+3x & 1-x & 1+3x \\ x+x^2 & -2 & 1-4x \\ x-3x^2 & 1+4x & 3x \end{vmatrix}$$

$$= \frac{1}{x^4}\left[a + bx + cx^2 + dx^3 + ex^4\right]$$

Cancel $1/x^4$ and put $x = 0$ to obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 1 & 0 \end{vmatrix} = a$$

$\Rightarrow \quad a = -1$

◉ **Example 88:** **Statement-1:** The system of line equations

$x + (\sin\alpha)\,y + (\cos\alpha)\,z = 0$
$x + (\cos\alpha)\,y + (\sin\alpha)\,z = 0$
$x - (\sin\alpha)\,y - (\cos\alpha)\,z = 0$

has a non-trivial solution for only one value of α lying in the interval $(0, \pi/2)$.

Statement-2: The equation in α

$$\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \sin\alpha & \cos\alpha & \sin\alpha \\ \cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

has only one solution lying in the interval $(0, \pi/2)$.
Ans. (b)

◎ **Solution:** Using $C_1 \to C_1 - C_3$ in Δ, we get

$$\Delta = \begin{vmatrix} 0 & \sin\alpha & \cos\alpha \\ 0 & \cos\alpha & \sin\alpha \\ 2\cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix}$$

$$= 2\cos\alpha\,(\sin^2\alpha - \cos^2\alpha)$$

$$= -2\cos\alpha\cos 2\alpha$$

$$\Delta = 0 \text{ for } \alpha = \pi/4 \in (0, \pi/2).$$

This is the only value of α lying in $(0, \pi/2)$ for which $\Delta = 0$.

The system of linear equations will have a non-trivial solution if and only if

$$\Delta_1 = \begin{vmatrix} 1 & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

Using $R_2 \to R_2 + R_1$, we get

$$\Delta_1 = \begin{vmatrix} 2 & 0 & 0 \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(-\cos^2\alpha + \sin^2\alpha) = 0$$

$$\Rightarrow \quad -2\cos 2\alpha = 0$$

This is true for only one value of $\alpha \in (0, \pi/2)$ viz, $\alpha = \pi/4$.

Thus, statement-1 is also true. However, statement-2 is not a correct reason for statement-1.

LEVEL 2

Straight Objective Type Questions

Example 89: Consider the system of equations in x, y, z:

$$(\sin 3\theta)\, x - y + z = 0$$
$$(\cos 2\theta)\, x + 4y + 3z = 0$$
$$2x + 7y + 7z = 0$$

The values of θ for which the system has a non-trivial solution are given by

(a) $\pi\left(n + \frac{1}{3}(-1)^n\right)$ (b) $\pi\left(n + \frac{1}{4}(-1)^n\right)$

(b) $\pi\left(n + \frac{1}{6}(-1)^n\right)$ (d) $\frac{n\pi}{2}$

where n is an integer

Ans. (c)

Solution: For the system of equations to have a non-trivial solution we must have

$$\Delta = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

Using $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 + 7R_1$ we get

$$\Delta = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta + 4\sin 3\theta & 0 & 7 \\ 2 + 7\sin 3\theta & 0 & 14 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(\cos 2\theta + 4\sin 3\theta) - (2 + 7\sin 3\theta) = 0$$

$$\Rightarrow \quad \sin 3\theta + 2\cos 2\theta - 2 = 0$$

$$\Rightarrow \quad 3\sin\theta - 4\sin^3\theta - 4\sin^2\theta = 0$$

$$\Rightarrow \quad \sin\theta(2\sin\theta + 3)(2\sin\theta - 1) = 0$$

$$\Rightarrow \quad \sin\theta = 0 \text{ or } \sin\theta = 1/2 \qquad [\because |\sin\theta| \le 1]$$

$$\Rightarrow \quad \theta = m\pi \text{ or } \theta = n\left(\pi + \frac{(-1)^n}{6}\right) \text{ where } m \text{ and } n \text{ are integers}$$

Example 90:

Let $$\Delta(x) = \begin{vmatrix} 3 + 2\sin^4 x & 2\cos^4 x & \sin^2 2x \\ 2\sin^4 x & 3 + 2\cos^4 x & \sin^2 2x \\ 2\sin^4 x & 2\cos^4 x & 3 + \sin^2 2x \end{vmatrix}$$

then $\int_{-\pi/2}^{\pi/2} x\Delta(x)dx$ equals

(a) π^2 (b) $\pi(\pi - 1)$
(c) 1 (d) 0

Ans. (d)

Solution: Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta(x) = f(x)\begin{vmatrix} 1 & 2\cos^4 x & \sin^2 2x \\ 1 & 3 + 2\cos^4 x & \sin^2 2x \\ 1 & 2\cos^4 x & 3 + \sin^2 2x \end{vmatrix}$$

where $$f(x) = 3 + 2\sin^4 x + 2\cos^4 x + \sin^2 2x$$
$$= 3 + 2(\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x)$$
$$= 3 + 2(\cos^2 x + \sin^2 x)^2 = 5$$

Applying $C_2 \to C_2 - (2\cos^4 x)\, C_1$ and $C_3 \to C_3 - (\sin^2 2x)\, C_1$, we get

$$\Delta(x) = 5\begin{vmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 45$$

Thus, $\int_{-\pi/2}^{\pi/2} x\Delta(x)dx = 45\int_{-\pi/2}^{\pi/2} xdx = 0$

Example 91: If $f(x)$, $g(x)$and $h(x)$ are three polynomials of degree 3 then

$$\phi(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix}$$

is a polynomial of degree

(a) 3 (b) 4
(c) 5 (d) none of these

Ans. (d)

◎ **Solution:** We have

$$\phi'(x) = \begin{vmatrix} f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix} + \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'''(x) & g'''(x) & h'''(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix} + \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \\ f^{iv}(x) & g^{iv}(x) & h^{iv}(x) \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \\ 0 & 0 & 0 \end{vmatrix} = 0$$

since f, g, h are polynomials of degree 3, $f^{iv}(x) = g^{iv}(x) = h^{iv}(x) = 0$

$\Rightarrow \phi(x)$ must be a constant.

◉ **Example 92:** The determinant

$$\Delta = \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

equals

(a) $9b^2(a+b)$ (b) $9a^2(a+b)$
(c) $9(a+b)^3$ (d) $9ab(a+b)$

Ans. (a)

◎ **Solution:** Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = 3(a+b)\begin{vmatrix} 1 & a+b & a+2b \\ 1 & a & a+b \\ 1 & a+2b & a \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$\Delta = 3(a+b)\begin{vmatrix} 1 & a+b & a+2b \\ 0 & -b & -b \\ 0 & b & -2b \end{vmatrix}$$

$$= 3(a+b)\begin{vmatrix} -b & -b \\ b & -2b \end{vmatrix} = 9b^2(a+b)$$

◉ **Example 93:** Let

$$\Delta(x) = \begin{vmatrix} \sin x & \cos x & \sin 2x + \cos 2x \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

then $\Delta'(x)$ vanishes at least once in

(a) $(0, \pi/2)$ (b) $(\pi/2, \pi)$
(c) $(0, \pi/4)$ (d) $(-\pi/2, 0)$

Ans. (a)

◎ **Solution:** The function $\Delta(x)$ is continuous on $[0, \pi/2]$ and differentiable on $(0, \pi/2)$. Also $\Delta(0) = 0$ and $\Delta(\pi/2) = 0$. Thus, by the Rolle's theorem there exists at least one $c \,\varepsilon\, (0, \pi/2)$ such that $\Delta'(c) = 0$.

◉ **Example 94:** Let $\Delta(x) = \begin{vmatrix} \cos^2 x & \cos x \sin x & -\sin x \\ \cos x \sin x & \sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$

then $\int_0^{\pi/2} [\Delta(x) + \Delta'(x)]\, dx$ equals

(a) $\pi/3$ (b) $\pi/2$
(c) 2π (d) $3\pi/2$

Ans. (b)

◎ **Solution:** Applying $C_1 \to C_1 - \sin x\, C_3$ and $C_2 \to C_2 + \cos x\, C_3$, we get

$$\Delta(x) = \begin{vmatrix} 1 & 0 & -\sin x \\ 0 & 1 & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$$

Applying $R_3 \to R_3 - \sin x\, R_1 + \cos x\, R_2$, we get

$$\Delta(x) = \begin{vmatrix} 1 & 0 & -\sin x \\ 0 & 1 & \cos x \\ 0 & 0 & \cos^2 x + \sin^2 x \end{vmatrix} = 1$$

$$\Rightarrow \qquad \Delta'(x) = 0$$

Thus, $\int_0^{\pi/2} [\Delta(x) + \Delta'(x)]\, dx = \int_0^{\pi/2} dx = \dfrac{\pi}{2}$.

◉ **Example 95:** The determinant

$$\Delta = \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix} \text{ equals}$$

(a) $15\sqrt{2} - 25\sqrt{3}$ (b) $25\sqrt{3} - 15\sqrt{2}$
(c) $3\sqrt{5}$ (d) $-15\sqrt{2} + 7\sqrt{3}$

Ans. (a)

◎ **Solution:** Taking $\sqrt{5}$ common from C_2 and C_3, we get

$$\Delta = \left(\sqrt{5}\right)^2 \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2 & 1 \\ \sqrt{15}+\sqrt{26} & \sqrt{5} & \sqrt{2} \\ 3+\sqrt{65} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

Applying $C_1 \to C_1 - \sqrt{13}\; C_3 - \sqrt{3}\; C_2$, we get

$$\Delta = (5)\begin{vmatrix} -\sqrt{3} & 2 & 1 \\ 0 & \sqrt{5} & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{5} \end{vmatrix} = 5\left(-\sqrt{3}\right)\left(5 - \sqrt{6}\right)$$

$$= 5\left(\sqrt{18}\right) - 25\sqrt{3} = 15\sqrt{2} - 25\sqrt{3}$$

◉ **Example 96:** The values of λ for which the system of equations

$$x + y - 3 = 0$$
$$(1 + \lambda)x + (2 + \lambda)y - 8 = 0$$
$$x - (1 + \lambda)y + (2 + \lambda) = 0$$

has a non-trivial solution, are

(a) $-5/3, 1$ (b) $2/3, -3$
(c) $-1/3, -3$ (d) 0

Ans. (a)

◎ **Solution:** The given system of equations has a non trivial solution if

$$\Delta = \begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 + 3C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1+\lambda & 1 & -5+3\lambda \\ 1 & -2-\lambda & 5+\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (5 + \lambda) + (2 + \lambda)(3\lambda - 5) = 0$$
$$\Rightarrow \quad 5 + \lambda + 6\lambda - 10 + 3\lambda^2 - 5\lambda = 0$$
$$\Rightarrow \quad 3\lambda^2 + 2\lambda - 5 = 0 \Rightarrow (3\lambda + 5)(\lambda - 1) = 0$$
$$\Rightarrow \quad \lambda = -5/3 \text{ or } \lambda = 1.$$

◉ **Example 97:** The values of m for which the system of equations $3x + my = m$ and $2x - 5y = 20$ has a solution satisfying the conditions $x > 0$, $y > 0$ are given by the set

(a) $\{m : m < -13/2\}$
(b) $\{m : m > 17/2\}$
(c) $\{m : m < -13/2 \text{ or } m > 17/2\}$
(d) none of these

Ans. (d)

◎ **Solution:** Here $\Delta = \begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix} = -15 - 2m$

$$\Delta_x = \begin{vmatrix} m & m \\ 20 & -5 \end{vmatrix} = -25m$$

$$\text{and } \Delta_y = \begin{vmatrix} 3 & m \\ 2 & 20 \end{vmatrix} = 60 - 2m$$

If $\Delta = 0$, then $m = -15/2$. But for this value of m, $\Delta_x \neq 0$ and $\Delta_y \neq 0$. Thus, in this case, the system of equations is not consistent.

If $\Delta \neq 0$, then $x = \dfrac{25m}{2m+15}$ and $y = \dfrac{2m-60}{2m+15}$, by the Cramer's rule.

Now, $x > 0, y > 0 \Leftrightarrow 25m > 0, 2m - 60 > 0, 2m + 15 > 0$
or $25m < 0, 2m - 60 < 0, 2m + 15 < 0$
$\Rightarrow m > 30$ or $m < -15/2$.

◉ **Example 98:** If $a + b + c \neq 0$, the system of equations

$$(b + c)(y + z) - ax = b - c$$
$$(c + a)(z + x) - by = c - a$$
$$(a + b)(x + y) - cz = a - b$$

has

(a) a unique solution
(b) no solution
(c) infinite number of solutions
(d) finitely many solutions

Ans. (a)

◎ **Solution:** We can write the above system of equations

$$(a + b + c)(y + z) - a(x + y + z) = b - c$$
$$(a + b + c)(z + x) - b(x + y + z) = c - a$$
$$(a + b + c)(x + y) - c(x + y + z) = a - b$$

Adding the above equations, we obtain

$$2(a + b + c)(x + y + z) - (a + b + c)(x + y + z) = 0$$
$$\Rightarrow \quad (a + b + c)(x + y + z) = 0$$
$$\Rightarrow \quad x + y + z = 0 \qquad [\because a + b + c \neq 0]$$
$$\Rightarrow \quad y + z = -x$$
$$\therefore \quad (b + c)(-x) - ax = b - c \Rightarrow x = \frac{c-b}{a+b+c}$$

Similarly, $y = \dfrac{a-c}{a+b+c}$, $z = \dfrac{b-a}{a+b+c}$.

◉ **Example 99:** Let a, b, c be positive real numbers. The following system of equations in x, y and z.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,\ \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,\ \frac{-x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

has

(a) no solution
(b) unique solution
(c) infinitely many solutions
(d) finitely many solutions

Ans. (d)

◎ **Solution:** Adding all the equations, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3$$

Subtracting first equation from it we get $\dfrac{2z^2}{c^2} = 2 \Rightarrow z^2 = c^2$ $\Rightarrow z = \pm c$. Similarly, $x = \pm a$, $y = \pm b$. Thus, the given system of equations has eight solutions.

◉ **Example 100:** If the system of equations

$$x - ky - z = 0,\ kx - y - z = 0,\ x + y - z = 0$$

has a non-zero solution, then the possible values of k are

(a) $-1, 2$ (b) $1, 2$
(c) $0, 1$ (d) $-1, 1$

Ans. (d)

◎ **Solution:** As the given system has a non-zero solution,

$$0 = \begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1+k & -k-1 & -1 \\ 1+k & -2 & -1 \\ 0 & 0 & -1 \end{vmatrix}$$

[using $C_1 \to C_1 - C_2$, $C_2 \to C_2 + C_3$]

$$\Rightarrow \quad 0 = (-1)\,[(1+k)(-2) - (1+k)(-k-1)]$$

$$\Rightarrow \quad 0 = (1+k)(-2+k+1) \Rightarrow k = -1, 1$$

◉ **Example 101:** If the system of equations

$\lambda x_1 + x_2 + x_3 = 1$, $x_1 + \lambda x_2 + x_3 = 1$, $x_1 + x_2 + \lambda x_3 = 1$

is inconsistent, then λ equals

(a) 5 (b) $-2/3$

(c) -3 (d) -2

Ans. (d)

◎ **Solution:** Let $\Delta = \begin{vmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = \begin{vmatrix} \lambda+2 & 1 & 1 \\ \lambda+2 & \lambda & 1 \\ \lambda+2 & 1 & \lambda \end{vmatrix}$

[$C_1 \to C_1 + C_2 + C_3$]

$$= (\lambda+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{vmatrix} = (\lambda+2)\begin{vmatrix} 1 & 0 & 0 \\ 1 & \lambda-1 & 0 \\ 1 & 0 & \lambda-1 \end{vmatrix}$$

[using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$$= (\lambda+2)(\lambda-1)^2$$

If $\Delta = 0$, then $\lambda = -2$ or $\lambda = 1$.

But when $\lambda = 1$, the system of equation becomes $x_1 + x_2 + x_3 = 1$ which has infinite number of solutions. When $\lambda = -2$, by adding three equations, we obtain $0 = 3$ and thus, the system of equations is inconsistent.

◉ **Example 102:** If $p \neq a$, $q \neq b$, $r \neq c$ and the system of equations

$$px + ay + az = 0$$
$$bx + qy + bz = 0$$
$$cx + cy + rz = 0$$

has a non-trivial solution, then the value of

$$\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} \text{ is}$$

(a) -1 (b) 0

(c) 1 (d) 2

Ans. (d)

◎ **Solution:** As the given system of equations has a non-trivial solution

$$\Delta = \begin{vmatrix} p & a & a \\ b & q & b \\ c & c & r \end{vmatrix} = 0$$

Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} p & a-p & a-p \\ b & q-b & 0 \\ c & 0 & r-c \end{vmatrix} = 0$$

Expanding along C_3, we get

$$(a-p)\begin{vmatrix} b & q-b \\ c & 0 \end{vmatrix} + (r-c)\begin{vmatrix} p & a-p \\ b & q-b \end{vmatrix} = 0$$

$\Rightarrow (a-p)(-c)(q-b) + (r-c)\,\{p(q-b) - b(a-p)\} = 0$

$\Rightarrow (p-a)(q-b)\,c + p\,(r-c)(q-b) + b\,(r-c)(p-a) = 0$

Dividing by $(p-a)(q-b)(r-c)$ we get

$$\frac{c}{r-c} + \frac{p}{p-a} + \frac{b}{q-b} = 0$$

$$\Rightarrow \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = \frac{q-b}{q-b} + \frac{r-c}{r-c} = 2$$

◉ **Example 103:** Let λ and α be real. Relation between λ and α for which the system of equations

$$\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$$
$$x + (\cos\alpha)y + (\sin\alpha)z = 0$$
$$-x + (\sin\alpha)y - (\cos\alpha)z = 0$$

has a non-trivial solution is

(a) $\lambda = \sin 2\alpha + \cos 2\alpha$

(b) $\lambda = |\sin 2\alpha|$

(c) $\lambda = |\sin 2\alpha - \cos 2\alpha|$

(d) $\lambda = \cos 2\alpha$

Ans. (a)

◎ **Solution:** The system of equations will have a non-trivial solution if and only if

$$0 = \begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix}$$

$$= \begin{vmatrix} 0 & (\lambda+1)\sin\alpha & (1-\lambda)\cos\alpha \\ 0 & \cos\alpha+\sin\alpha & \sin\alpha-\cos\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix}$$

[using $R_1 \to R_1 + \lambda R_3$, $R_2 \to R_2 + R_3$]

$$= (\lambda+1)\sin\alpha\,(\sin\alpha - \cos\alpha) + (\lambda-1)\cos\alpha(\cos\alpha + \sin\alpha)$$

$$= \lambda(\sin^2\alpha + \cos^2\alpha) + [\sin^2\alpha - \cos^2\alpha - 2\sin\alpha\cos\alpha]$$

Thus, $\lambda = \sin 2\alpha + \cos 2\alpha$

◉ **Example 104:** If $f_r(x)$, $g_r(x)$, $h_r(x)$, $r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$, $r = 1, 2, 3$

and $$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

then $F'(a)$ is

(a) -1 (b) a

(c) 0 (d) none of these

Ans. (c)

◎ **Solution:** Differentiating w.r.t. x we get $F'(x)$

$$= \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

Since $f_r(a) = g_r(a) = h_r(a)$ for $r = 1, 2, 3$, we get each of the three determinants on the right side becomes zero when x is replaced by a. (In each case two rows become identical.) Thus, $F'(a) = 0$.

◉ **Example 105:** The number of real values of a for which the system of equations

$$x + ay - z = 0,\ 2x - y + az = 0,\ ax + y + 2z = 0$$

has a non-trivial solution, is

(a) 3 (b) 1

(c) 0 (d) infinite

Ans. (a)

◎ **Solution:** Since the given system of equations has a non-trivial solution,

$$\Delta = \begin{vmatrix} 1 & a & -1 \\ 2 & -1 & a \\ a & 1 & 2 \end{vmatrix} = 0$$

Using $C_1 \to C_1 + C_3$, $C_2 \to C_2 + aC_3$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & -1 \\ 2+a & -1+a^2 & a \\ 2+a & 1+2a & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad (-1)\begin{vmatrix} 2+a & -1+a^2 \\ 2+a & 1+2a \end{vmatrix} = 0$$

$$\Rightarrow \quad (2 + a)(1 + 2a + 1 - a^2) = 0$$

$$\Rightarrow \quad a = -2,\ 1 \pm\sqrt{3}.$$

Thus, there are three real values of a for which the system of equations has a non-trivial solution.

◉ **Example 106:** The solution set of

$$x + 2y + z = 1$$

$$2x - 3y + w = 2$$

subject $\quad x \geq 0, y \geq 0, z \geq 0, w \geq 0$ is

(a) $x = \frac{1}{7}(y - w), \quad y \geq w \geq 0, z \geq 0$

(b) $x = \frac{1}{8}(y + w), \quad z = \frac{1}{3}(y - w), y \geq w \geq 0$

(c) $x = \frac{1}{7}(y - w), \quad z = \frac{1}{3}(y + w), y \geq 0$

(d) $x = 1, y = 0, \quad z = 0, w = 0.$

Ans. (d)

◎ **Solution:** We have

$$1 - (x + 2y) = z \geq 0$$

and $\quad 2x - 3y - 2 = -w \leq 0.$

$\Rightarrow \quad x + 2y \leq 1$ and $2x - 3y \leq 2$, $x \geq 0$, $y \geq 0$

The only point where the two shaded region intersect is (1, 0). Thus, $x = 1$, $y = 0$ for these values $z = 0$, $w = 0$.

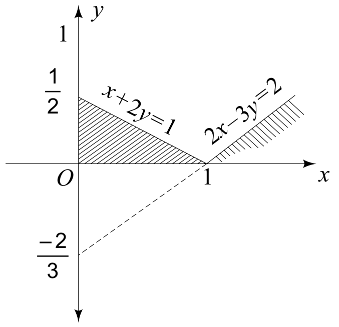


Fig. 4.1

◉ **Example 107:** The number of values of k for which the system of equations

$$(k + 1)x + 8y = 4k$$

$$kx + (k + 3)y = 3k - 1$$

has infinitely many solutions is

(a) 0 (b) 1

(c) 2 (d) infinite

Ans. (b)

◎ **Solution:** Let $\Delta = \begin{vmatrix} k+1 & 8 \\ k & k+3 \end{vmatrix}$

$$= k^2 - 4k + 3 = (k - 1)(k - 3)$$

If $\Delta \neq 0$, the system of equations has a unique solution.

For the system of equations to have an infinite number of solutions

$$\Delta = 0 \Rightarrow k = 3, 1$$

For $k = 3$, the system equations becomes

$$4x + 8y = 12 \text{ and } 3x + 6y = 8$$

$$\Rightarrow \quad x + 2y = 3,\ x + 2y = \frac{8}{3}$$

Thus, in this case the system of equations has no solution.

For $k = 1$, the system of equations becomes

$$2x + 8y = 4 \text{ and } x + 4y = 2$$

$$\Rightarrow \quad x + 4y = 2,\ x + 4y = 2$$

This system has infinite number of solutions.

Example 108: Suppose $a, b, c \in \mathbf{R}$ and let

$$f(x) = \begin{vmatrix} 0 & a-x & b-x \\ -a-x & 0 & c-x \\ -b-x & -c-x & 0 \end{vmatrix}$$

Then coefficient of x^2 in $f(x)$ is

(a) $-(a+b+c)$ (b) $a+b+c$

(c) 0 (d) $ab+bc+ca$

Ans. (c)

Solution: Expanding along C_1

$$f(x) = (a+x)\begin{vmatrix} a-x & b-x \\ -c-x & 0 \end{vmatrix} - (b+x)\begin{vmatrix} a-x & b-x \\ 0 & c-x \end{vmatrix}$$

$$= (a+x)(b-x)(c+x) - (b+x)(a-x)(c-x)$$

$$= -[(x+a)(x-b)(x+c) + (x-a)(x+b)(x-c)]$$

$$= -[2x^3 + x^2(a-b+c-a+b-c) + \ldots]$$

$$= -2x^3 + 0x^2 + \ldots$$

$\therefore$ coefficient of x^2 in $f(x)$ is 0.

Example 109: If the system of equations

$$x - ky - z = 0,\ kx - y - z = 0,\ x + y - z = 0$$

has a non-zero solution, then possible values of k are

(a) $-1, 2$ (b) $1, 2$

(c) $0, 1$ (d) $-1, 1$

Ans. (d)

Solution: Subtracting the last equation from the first two equations, we get

$$-(k+1)y = 0$$

$$(k-1)x - 2y = 0.$$

If $k = -1$, we can choose y in infinite number of ways, corresponding to which we can choose x and z in infinite number of ways.

If $k = 1$, then $y = 0$ and x and hence z can be chosen in infinite number of ways.

Example 110: Let $a_2, a_3 \in \mathbf{R}$ be such that $|a_2 - a_3| = 6$. Let

$$f(x) = \begin{vmatrix} 1 & a_3 & a_2 \\ 1 & a_3 & 2a_2 - x \\ 1 & 2a_3 - x & a_2 \end{vmatrix}, x \in \mathbf{R}$$

The maximum value of $f(x)$ is

(a) 6 (b) 9

(c) 12 (d) 36

Ans. (b)

Solution: Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$f(x) = \begin{vmatrix} 1 & a_3 & a_2 \\ 0 & 0 & a_2 - x \\ 0 & a_3 - x & 0 \end{vmatrix}$$

$$= -(a_2 - x)(a_3 - x) = -[x^2 - (a_2 + a_3)x + a_2a_3]$$

$$= \frac{1}{4}(a_2 - a_3)^2 - \left(x - \frac{a_2 + a_3}{2}\right)^2 \le 9$$

$f(x)$ attains maximum value 9 when

$$x = \frac{1}{2}(a_2 + a_3).$$

EXERCISES

Concept-based Straight Objective Type Questions

1. Suppose $A = (a_{ij})_{n \times n}$, where $a_{ij} \in \mathbf{R}$.

   If $\det(\text{adj}(A)A^{-1}) = 3$, then $\det(\text{adj}(A))$ equals:

   (a) $\sqrt{3}$ (b) 3

   (c) $3\sqrt{3}$ (d) 9

2. If a, b, c are in A.P. and p is a real number, and

   $$\Delta = \begin{vmatrix} p+c & p+2 & p+a \\ p+b & p+5 & p+b \\ p+a & p+8 & p+c \end{vmatrix}$$

   then Δ equals:

   (a) $-p^3$ (b) p^3

   (c) $p^3 - 2abc$ (d) 0

3. Suppose $x \neq 1$ and

   $$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

   then $\Delta = 0$

   (a) for exactly two distinct complex numbers

   (b) for exactly four distinct complex numbers

   (c) for exactly two distinct real numbers

   (d) none of these

4. Let $D = \begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$, then $\dfrac{D}{(10!)^3} - 260$ equals:

(a) 1 (b) 2
(c) 3 (d) 4

5. Let

$$P(x) = \begin{vmatrix} x+1 & 2 & 3 \\ 1 & x+2 & 3 \\ 1 & 2 & x+3 \end{vmatrix},$$

the product of zeros of $P(x)$ is

(a) 0 (b) 6
(c) – 6 (d) 12

6. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be vertices of an equilateral triangle whose side is 4 units. Let

$$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

then Δ^2 is equal to

(a) 64 (b) 128
(c) 192 (d) 256

7. Suppose a, b, c are in A.P. If p, q, r are also in A.P., then value of

$$\Delta = \begin{vmatrix} x^2+a & x+p & c \\ x^2+b & x+q & b \\ x^2+c & x+r & a \end{vmatrix},$$

is dependent on

(a) x (b) a, b, c
(c) p, q, r (d) none of these

8. Suppose a, b are two non-zero numbers. Let

$$\Delta = \begin{vmatrix} 2 & a+b & a^2+b^2 \\ a+b & a^2+b^2 & a^3+b^3 \\ a^2+b^2 & a^3+b^3 & a^4+b^4 \end{vmatrix},$$

then Δ is equal to:

(a) 0 (b) ab
(c) $a^6 + b^6$ (d) $a^3b^5 + a^5b^3$

9. Suppose a, b, $c > 1$. Let

$$\Delta = \begin{vmatrix} \log a & \log b & \log c \\ \log(2007a) & \log(2007b) & \log(2007c) \\ \log(2017a) & \log(2017b) & \log(2017c) \end{vmatrix}$$

then Δ is equal to:

(a) 0 (b) log (4024 abc)
(c) $\log\left(\dfrac{2017}{2007}\right)$ (d) none of these

10. Suppose a, b, $c \in \mathbf{R}$. Let

$$\Delta = \begin{vmatrix} (a+2016)^2 & (b+2016)^2 & (a+2016)^2 \\ (a-2016)^2 & (b-2016)^2 & (c-2016)^2 \\ a^2 & b^2 & c^2 \end{vmatrix}$$

If $\Delta = k\,(2016)^3\,(a - b)\,(b - c)(c - a)$, then k is equal to:

(a) – 1 (b) – 4
(c) 4 (d) 1

11. Let $P(x) = \begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x+3 \\ -3 & 2x & x+2 \end{vmatrix}$,

sum of the zeros of $P(x)$ is

(a) – 6 (b) – 7
(c) 13/5 (d) –12/5

12. Let n be an integer and x, y, $z > 1$. Suppose

$$\Delta = \begin{vmatrix} x^{n+1} & x^{n+2} & x^{n+3} \\ y^{n+1} & y^{n+2} & y^{n+3} \\ z^{n+1} & z^{n+2} & z^{n+3} \end{vmatrix}$$

If $\Delta = (x - y)\,(y - z)(z - x)\,x^2\,y^2\,z^2$, then n is equal to:

(a) –1 (b) 0
(c) 1 (d) 2

13. Suppose a, b, c are distinct real numbers. Let

$$P(x) = \begin{vmatrix} 0 & x^3-a & x^4-b \\ x^3+a & 0 & x^5+c \\ x^4+b & x^5-c & 0 \end{vmatrix}$$

A value of x satisfying $P(x) = 0$ is

(a) $-(a + b + c)$ (b) $a + b + c$
(c) $a + b - c$ (d) 0

14. Suppose a, b, $c \in \mathbf{R}$ and $abc \neq 0$. Let

$$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix}$$

If $\Delta = 0$, then $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$ is equal to:

(a) 0 (b) –1
(c) –2 (d) –3

15. Suppose n and m are natural numbers such that

$$\Delta = \begin{vmatrix} x^m & x^{m+2} & x^{2m} \\ 1 & x^n & 2^n \\ x^{m+5} & x^{n+6} & x^{2m+5} \end{vmatrix}$$

Then a possible relationship between n and m is

(a) $n = m + 2$ (b) $n = m + 1$
(c) $n = m$ (d) $n = m - 1$

LEVEL 1

Straight Objective Type Questions

16. Suppose $a, b, c \in \mathbf{R}$ and $a + b + c \neq 0$. Let

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

If $\Delta = 0$, then

(a) $a = b = c$ (b) $a^3 + b^3 - c^3 = 0$

(c) $a = b + c$ (d) $a = b = c = 0$

17. Distance of line

$$y = \begin{vmatrix} x+1 & x & x \\ x & x+2 & x \\ x & x & x+3 \end{vmatrix}$$

from the origin is

(a) $\dfrac{6}{11}$ (b) $\dfrac{7}{13}$

(c) $\dfrac{6}{\sqrt{122}}$ (d) $\dfrac{7}{\sqrt{122}}$

18. Suppose $a, b, c \in \mathbf{R}$. Let

$$\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Then Δ equals

(a) $3(a + b + c)(bc + ca + ab)$

(b) $a + b + c$

(c) $3(a + b + c)(a^2 + b^2 + c^2)$

(d) 0

19. Suppose a, b, c are in A.P. Let

$$\Delta = \begin{vmatrix} p^2+2^{n+1}+2a & q^2+2^{n+2}+3b & r^2+a \\ 2^n+a & 2^{n+1}+b & 2b \\ p^2+2^n+a & q^2+2^{n+1}+2b & r^2-c \end{vmatrix}$$

Then Δ equals:

(a) -1 (b) 0

(c) $p^2q^2r^2 - 3abc$ (d) $p^2q^2r^2 - 4(a + b + c)$

20. Suppose a, b, c, d, e and f are in G.P. with common ratio > 1. Let p, q, r be three real numbers. Let

$$\Delta = \begin{vmatrix} a^2 & d^2 & p \\ b^2 & e^2 & q \\ c^2 & f^2 & r \end{vmatrix}$$

Then Δ depends on

(a) a, b, c (b) d, e, f

(c) p, q, r (d) none of these

21. Suppose point (x, y, z) in space satisfies the equation

$$\begin{vmatrix} x^2+1 & xy & xz \\ yx & y^2+1 & yz \\ zx & zy & z^2+1 \end{vmatrix} = 5$$

Then (x, y, z) lies on a

(a) plane (b) straight line

(c) sphere (d) none of these

22. Suppose A, B, and C are angles of a triangle. Let

$$\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

then Δ equals:

(a) 0 (b) -1

(c) -2 (d) -3

23. Let

$$\Delta = \begin{vmatrix} a & x & x \\ x & b & x \\ x & x & c \end{vmatrix}$$

and $f(x) = (x - a)(x - b)(x - c)$

Determinant Δ is equal to:

(a) $f(x) - x^3$ (b) $f'(x)$

(c) $xf'(x) - f(x)$ (d) $f'(x) - xf''(x)$

24. Straight line

$$\begin{vmatrix} 2-x-y & 4 & 4 \\ 2x & x-y-2 & 2x \\ 2y & 2y & y-2-x \end{vmatrix} = 0$$

passes through the fixed point

(a) $(-2, -2)$ (b) $(-2, 0)$

(c) $(0, -2)$ (d) $(-1, -1)$

25. Suppose $a \in \mathbf{R}$. Let

$$f(x) = \begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix}$$

Then $f(2x) - f(x)$ is equal to

(a) $3xa^2$ (b) $3x^2a$

(c) xa^2 (d) a^2x

26. If α, β, γ are the roots of $x^3 + ax^2 + b = 0$, then the determinant

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} \text{ equals}$$

(a) $-a^3$ (b) $a^3 - 3b$
(c) $a^2 - 3b$ (d) a^3

27. If α, β, γ are the roots of $x^3 + bx + c = 0$, then the determinant

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} \text{ equals}$$

(a) $-b^3$ (b) $b^3 - 3c$
(c) $b^2 - 3c$ (d) 0

28. If a, b, c are distinct and different from zero and

$$\Delta = \begin{vmatrix} bc & ca & ab \\ ca & ab & bc \\ ab & bc & ca \end{vmatrix} = 0, \text{ then}$$

(a) $a^{-1} + b^{-1} + c^{-1} = 0$
(b) $a^{-1} + b^{-1} - c^{-1} = 0$
(c) $a^{-1} - b^{-1} + c^{-1} = 0$
(d) $a^{-1} - b^{-1} - c^{-1} = 0$

29. The determinant $\Delta = \begin{vmatrix} \lambda a & \lambda^2 + a^2 & 1 \\ \lambda b & \lambda^2 + b^2 & 1 \\ \lambda c & \lambda^2 + c^2 & 1 \end{vmatrix}$ equals

(a) $\lambda(a - b)(b - c)(c - a)$
(b) $\lambda(a^2 + b^2 + c^2)$
(c) $\lambda(a + b + c)$
(d) $\lambda^2(a - b)(b - c)(c - a)$

30. If α, β, γ are real numbers, then the determinant

$$\Delta = \begin{vmatrix} \sin^2\alpha & \cos 2\alpha & \cos^2\alpha \\ \sin^2\beta & \cos 2\beta & \cos^2\beta \\ \sin^2\gamma & \cos 2\gamma & \cos^2\gamma \end{vmatrix} \text{ equals}$$

(a) 0
(b) -1
(c) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$
(d) none of these

31. If $bc + ca + ab = 18$, and

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = \lambda \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

the value of λ is
(a) -1 (b) 0
(c) 9 (d) 18

32. If $x \neq 0$, the determinant

$$\Delta = \begin{vmatrix} a_0 & a_1 & a_2 \\ -x & x & 0 \\ 0 & -x & x \end{vmatrix}$$

vanishes if
(a) $a_0 + a_1 + a_2 = 0$ (b) $a_0 + a_1 = 2a_2$
(c) $a_0 + a_2 = 2a_1$ (d) none of these

33. If $x \in \mathbf{R}$, the determinant

$$\Delta = \begin{vmatrix} 1 & \cos x & 0 \\ -1 & 1 - \cos x & \sin x + \cos x \\ 0 & -1 & 1 - \sqrt{2}\sin(x + \pi/4) \end{vmatrix} \text{ equals}$$

(a) 0 (b) -1
(c) 1 (d) none of these

34. The factors of $\Delta = \begin{vmatrix} x & a & b \\ a & x & b \\ a & b & x \end{vmatrix}$ are

(a) $x - a$, $x - b$ and $x + a + b$
(b) $x - a$, $x - b$ and $x - a - b$
(c) $x + a$, $x + b$ and $x - a - b$
(d) none of these

35. If $\theta \,\varepsilon\, \mathbf{R}$, maximum value of

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 + \sin\theta & 1 \\ 1 & 1 & 1 + \cos\theta \end{vmatrix} \text{ is}$$

(a) 1/2 (b) $\sqrt{3}/2$
(c) $\sqrt{2}$ (d) $3\sqrt{2}/4$

36. If $\Delta = \begin{vmatrix} x + a & b & c \\ a & x + b & c \\ a & b & x + c \end{vmatrix} = 0$, then x equals

(a) $a + b + c$ (b) $-(a + b + c)$
(c) $0, a + b + c$ (d) $0, -(a + b + c)$

37. The determinant

$$\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 12 & 10 & 2 \end{vmatrix} \text{ equals}$$

(a) $2\sin^2\theta$
(b) $12\sec^2\theta - 10\tan^2\theta$
(c) $12\sec^2\theta - 10\tan^2\theta + 5$
(d) 0

38. If $\Delta = \begin{vmatrix} -a & 2b & 0 \\ 0 & -a & 2b \\ 2b & 0 & -a \end{vmatrix} = 0$, then

(a) $1/b$ is a cube root of unity
(b) a is one of the cube roots of unity

(c) b is one of the cube roots of 8
(d) a/b is a cube root of 8

39. The determinant

$$\Delta = \begin{vmatrix} 1 & 1+i & i \\ 1+i & i & 1 \\ i & 1 & 1+i \end{vmatrix} \text{ equals}$$

(a) $7 + 4i$
(b) $-7 + 4i$
(c) $-7 - 4i$
(d) $2(i - 1)$

40. If a, b, c are non-zero real numbers, then

$$\Delta = \begin{vmatrix} 1 & ab & \dfrac{1}{a}+\dfrac{1}{b} \\ 1 & bc & \dfrac{1}{b}+\dfrac{1}{c} \\ 1 & ca & \dfrac{1}{c}+\dfrac{1}{a} \end{vmatrix} \text{ equals}$$

(a) 0
(b) $bc + ca + ab$
(c) $a^{-1} + b^{-1} + c^{-1}$
(d) $abc - 1$

41. If $a, b, c > 1$, then $\Delta = \begin{vmatrix} \log_a(abc) & \log_a b & \log_a c \\ \log_b(abc) & 1 & \log_b c \\ \log_c(abc) & \log_c b & 1 \end{vmatrix}$ equals

(a) 0
(b) $\log_a b + \log_b c + \log_c a$
(c) $\log_{abc}(a + b + c)$
(d) none of these

42. If $\Delta = \begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = \lambda \begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$

then λ equals

(a) 0
(b) 1
(c) x
(d) $1 - x^2$

43. Let $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$, $0 \le \theta \le 2\pi$.The

(a) $\Delta = 0$
(b) $\Delta \in (2, \infty)$
(c) $\Delta \in (2, 4)$
(d) $\Delta \in [2, 4]$

44. The determinant $\Delta = \begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ equals

(a) $(b - c)(c - a)(a - b)$
(b) $abc(b - c)(c - a)(a - b)$
(c) $(a + b + c)(b - c)(c - a)(a - b)$
(d) 0

45. If $\Delta = \begin{vmatrix} 6i & -3i & w \\ 4 & 3i & -w \\ 20 & 3 & iw \end{vmatrix} = x + iy$, then

(a) $x = 3, y = 1$
(b) $x = 1, y = 3$
(c) $x = 0, y = 3$
(d) $x = 0, y = 0$

46. If $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$, the number of distinct real roots of $\Delta = 0$

$$\Delta = \begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} \text{ is}$$

(a) 0
(b) 2
(c) 1
(d) 3

47. If $\omega \ne 1$ is a complex cube root of unity, and

$$x + iy = \begin{vmatrix} 1 & i & -\omega \\ -i & 1 & \omega^2 \\ \omega & -\omega^2 & 1 \end{vmatrix} \text{ then}$$

(a) $x = -1, y = 0$
(b) $x = 1, y = -1$
(c) $x = 1, y = 1$
(d) none of these

48. If $e^{ix} = \cos x + i\sin x$ and

$$x + iy = \begin{vmatrix} 1 & e^{\pi i/4} & e^{\pi i/3} \\ e^{-\pi i/4} & 1 & e^{2\pi i/3} \\ e^{-\pi i/3} & e^{-2\pi i/3} & e^{-2\pi i/3} \end{vmatrix}, \text{ then}$$

(a) $x = -1, y = \sqrt{2}$
(b) $x = 1, y = -\sqrt{2}$
(c) $x = -\sqrt{2}, y = \sqrt{2}$
(d) none of these

49. If a, b, c ε **R**, the number of real roots of the equation

$$\begin{vmatrix} x & c & -b \\ -c & x & a \\ b & -a & x \end{vmatrix} = 0, \text{ is}$$

(a) 0
(b) 1
(c) 2
(d) 3

50. If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{vmatrix} = -7$ and

$$\Delta_1 = \begin{vmatrix} x^3-1 & 0 & x-x^4 \\ 0 & x-x^4 & x^3-1 \\ x-x^4 & x^3-1 & 0 \end{vmatrix}, \text{ then}$$

(a) $\Delta = 7$
(b) $\Delta = 343$
(c) $\Delta = -49$
(d) $\Delta = 49$

51. If $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha+\cos\beta \\ \sin\beta & \cos\alpha & \sin\beta+\cos\beta \\ \sin\gamma & \cos\alpha & \sin\gamma+\cos\beta \end{vmatrix}$ then Δ equals
   (a) $\sin\alpha\sin\beta\sin\gamma$
   (b) $\cos\alpha\sec\beta\tan\gamma$
   (c) $\sin\alpha\sin\beta\sin\gamma+\cos\alpha\cos\beta\cos\gamma$
   (d) 0

52. Suppose $a, b, c \in \mathbf{R}$ and $a, b, c > 0$,
   Let $\Delta = \begin{vmatrix} \log a & \log b & \log c \\ \log(7a) & \log(49b) & \log(343c) \\ \log(3a) & \log(9b) & \log(27c) \end{vmatrix}$
   then Δ is equals to
   (a) 0 (b) –1
   (c) 1 (d) 30

53. The values of θ lying between $\theta = 0$ and $\theta = \pi/2$ and satisfying the equation
$$\Delta = \begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\sin 4\theta \\ \cos^2\theta & \sin^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$
   are given by
   (a) $\pi/24, 5\pi/24$ (b) $7\pi/24, 11\pi/24$
   (c) $5\pi/24, 7\pi/24$ (d) $11\pi/24, \pi/24$

54. The number of distinct roots of the equation
$$P(x) = \begin{vmatrix} x^2-1 & x^2+2x+1 & 2x^2+3x+1 \\ 2x^2+x-1 & 2x^2+5x-3 & 4x^2+4x-3 \\ 6x^2-x-2 & 6x^2-7x+2 & 12x^2-5x-2 \end{vmatrix} = 0$$
   is
   (a) 6 (b) 5
   (c) 3 (d) 4

55. If $a_r = \cos\dfrac{2r\pi}{9} + i\sin\dfrac{2r\pi}{9}$, then value of the determinant
$$\Delta = \begin{vmatrix} 1 & a_8 & a_7 \\ a_3 & a_2 & a_1 \\ a_6 & a_5 & a_4 \end{vmatrix} \text{ is}$$
   (a) – 1 (b) 1
   (c) 0 (d) – 2

56. If $a \neq p$, $b \neq q$, $c \neq r$ and the system of equations
   $px + by + cz = 0$
   $ax + qy + cz = 0$
   $ax + by + rz = 0$
   has a non-zero solution, then value of
   $\dfrac{p+a}{p-a} + \dfrac{q+b}{q-b} + \dfrac{r+c}{r-c}$ is
   (a) 2 (b) – 2
   (c) 1 (d) 1

57. For a fixed positive integer n, if
$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$
   then $\dfrac{D}{n!(n+1)!(n+2)!}$ is equal to
   (a) – 4 (b) – 2
   (c) 2 (d) 4

58. If a, b, c are in A.P., and $\Delta = \begin{vmatrix} x+2 & x+7 & a \\ x+5 & x+11 & b \\ x+8 & x+15 & c \end{vmatrix}$
   then Δ equals
   (a) 0 (b) 1
   (c) $-(a+b+c)$ (d) $a+b+c$.

59. If $\Delta = \begin{vmatrix} 1+y & 1-y & 1-y \\ 1-y & 1+y & 1-y \\ 1-y & 1-y & 1+y \end{vmatrix} = 0$, then value of y are
   (a) 0, 3 (b) 2, – 1
   (c) – 1, 3 (d) 0, 2

60. The determinant
$$\Delta = \begin{vmatrix} al+a'l' & am+a'm' & an+a'n' \\ bl+b'l' & bm+b'm' & bn+b'n' \\ cl+c'l' & cm+c'm' & cn+c'n' \end{vmatrix}$$
   is equal to
   (a) $(abc + a'b'c')(lmn + l'm'n')$
   (b) $abc\,lmn + a'b'c'\,l'm'n'$
   (c) $(a^2+b^2+c^2)(l^2+m^2+n^2) + (a'^2+b'^2+c'^2)(l'^2+m'^2+n'^2)$
   (d) 0

61. If $a = i$, $b = \omega$, $c = \omega^2$ where ω is complex cube root of unity, then
$$\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 3a & 4a+3b & 5a+4b+3c \\ 6a & 9a+6b & 11a+9b+6c \end{vmatrix} \text{ is equal to}$$
   (a) $-\omega$ (b) $-\omega^2$
   (c) i (d) $-i$

62. If $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ {}^{m}C_1 & {}^{m+3}C_1 & {}^{m+6}C_1 \\ {}^{m}C_2 & {}^{m+3}C_2 & {}^{m+6}C_2 \end{vmatrix} = 2^{\alpha}\,3^{\beta}\,5^{\gamma}$, then
   $\alpha + \beta + \gamma$ is equal
   (a) 3 (b) 5
   (c) 7 (d) none of these

63. Suppose $a, b, c, x, y \in \mathbf{R}$. Let
$$\Delta = \begin{vmatrix} 1 & 2+ax & 3+ay \\ 1 & 2+bx & 3+by \\ 1 & 2+cx & 3+cy \end{vmatrix}$$

Then Δ is independent of

(a) a, b, c (b) x, y

(c) a, b, c, y (d) a, b, c, x, y

64. If A, B and C are the angles of a triangle, then the determinant

$$\Delta = \begin{vmatrix} 0 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} \text{ is equal to}$$

(a) $\sin^2 A$ (b) $\sin^2 B$

(c) $\sin^2 C$ (d) 0

65. Let $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$, then $\lim\limits_{x\to 0} \dfrac{f(x)}{x^2}$ is given by

(a) 0 (b) -1

(c) 2 (d) 3

66. If ω is a complex cube root of unity, then value of

$$\Delta = \begin{vmatrix} a_1 + b_1\omega & a_1\omega^2 + b_1 & c_1 + b_1\omega \\ a_2 + b_2\omega & a_2\omega^2 + b_2 & c_2 + b_2\omega \\ a_3 + b_3\omega & a_3\omega^2 + b_3 & c_3 + b_3\omega \end{vmatrix}$$

is

(a) 0 (b) -1

(c) 2 (d) none of these

67. If $pqr \neq 0$ and the system of equations

$$(p + a)x + by + cz = 0$$
$$ax + (q + b)y + cz = 0$$
$$ax + by + (r + c)z = 0$$

has a non-trivial solution, then value of $\dfrac{a}{p} + \dfrac{b}{q} + \dfrac{c}{r}$ is

(a) -1 (b) 0

(c) 1 (d) 2

68. The system of equations

$$ax + by + (a\alpha + b)z = 0$$
$$bx + cy + (b\alpha + c)z = 0$$
$$(a\alpha + b)x + (b\alpha + c)y = 0$$

has a non-zero solutions if a, b, c are in

(a) A.P. (b) G.P.

(c) H.P. (d) A.G.P.

69. If the system of equations

$$ax + ay - z = 0$$
$$bx - y + bz = 0$$
$$-x + cy + cz = 0$$

(where $a, b, c \neq -1$) has a non-trivial solution, then value of $\dfrac{1}{1+a} + \dfrac{1}{1+b} + \dfrac{1}{1+c}$ is

(a) 2 (b) -1

(c) -2 (d) 0

70. The values of λ for which the system of equations

$$(\lambda + 5)x + (\lambda - 4)y + z = 0$$
$$(\lambda - 2)x + (\lambda + 3)y + z = 0$$
$$\lambda x + \lambda y + z = 0$$

has a non-trivial solution is (are)

(a) $-1, 2$ (b) $0, -1$

(c) 0 (d) none of these

71. Given the system of equations

$$(b + c)(y + z) - ax = b - c$$
$$(c + a)(z + x) - by = c - a$$
$$(a + b)(x + y) - cz = a - b$$

(where $a + b + c \neq 0$); then $x : y : z$ is given by

(a) $b - c : c - a : a - b$ (b) $b + c : c + a : a + b$

(c) $a : b : c$ (d) $\dfrac{a}{b} : \dfrac{b}{c} : \dfrac{c}{a}$

72. If $a, b, c \;\varepsilon\; \mathbf{R}$ and $a + b + c \neq 0$ and the system of equations

$$ax + by + cz = 0$$
$$bx + cy + az = 0$$
$$cx + ay + bz = 0$$

has a non-zero solution, then $a : b : c$ is given by

(a) $1 : \alpha : \beta$
where α, β are roots of $ax^2 + bx + c = 0$

(b) $1 : r : r^2$ where r is some positive real number

(c) $1 : k : 2k$ where k is some positive real number

(d) none of these

73. If p is a constant and

$$f(x) = \begin{vmatrix} x^3 & x^3 & 3x^2 \\ 1 & -6 & 4 \\ p & p^2 & p^3 \end{vmatrix}, \text{ then } f'''(x) \text{ is}$$

(a) proportional to x^3 (b) proportional to x^2

(c) proportional to x (d) a constant

74. If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and

$$D = \begin{vmatrix} a_1 + pb_1 & b_1 + qc_1 & c_1 + ra_1 \\ a_2 + pb_2 & b_2 + qc_2 & c_2 + ra_2 \\ a_3 + pb_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix} \text{ then}$$

(a) $D = \Delta$ (b) $D = \Delta(1 - pqr)$

(c) $D = \Delta(1 + pqr)$ (d) $D = \Delta(1 + p + q + r)$

75. Number of real values of λ for which the system of equations

$$(\lambda + 3)x + (\lambda + 2)y + z = 0$$
$$3x + (\lambda + 3)y + z = 0$$
$$2x + 3y + z = 0$$

has a non-trivial solutions is

(a) 0 (b) 1

(c) 2 (d) infinite

Assertion-Reason Type Questions

76. **Statement-1:** If

$$\Delta(x) = \begin{vmatrix} \sin x - \cos x & \sin\left(x - \frac{\pi}{3}\right) & \cos\left(x - \frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3} - x\right) & \tan\left(x - \frac{\pi}{4}\right) & \sec\left(x - \frac{\pi}{4}\right) \\ \cos\left(\frac{2\pi}{3} + x\right) & \sec\left(\frac{2\pi}{3} + x\right) & \cot\left(x + \frac{\pi}{4}\right) \end{vmatrix}$$

then $\Delta\left(\frac{\pi}{4}\right) = 0$

Statement-2: If A is a skew symmetric matrix of odd order, then $|A| = 0$.

77. **Statement-1:** Let $p < 0$ and $\alpha_1, \alpha_2, \ldots \alpha_9$ be the nine roots of $x^9 = p$, then

$$\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{vmatrix} = 0$$

Statement-2: If two rows of a determinant are identical, then determinant equals zero.

78. **Statement-1:**

$$w = \begin{vmatrix} \sqrt{6} & 2i & 3 + \sqrt{6}i \\ \sqrt{12} & \sqrt{3} + \sqrt{8}i & 3\sqrt{2} + \sqrt{6}i \\ \sqrt{18} & \sqrt{2} + \sqrt{12}i & \sqrt{27} + 2i \end{vmatrix}$$

is a purely imaginary number.

Statement-2: $|z| = |\bar{z}|$ for each complex number z.

79. **Statement-1:** If $a_i, b_i \in \mathbf{N}$, for $i = 1, 2, 3$ and

$$\Delta(x) = \begin{vmatrix} (1+x)^{a_1 b_1} & (1+x)^{a_1 b_2} & (1+x)^{a_1 b_3} \\ (1+x)^{a_2 b_1} & (1+x)^{a_2 b_2} & (1+x)^{a_2 b_3} \\ (1+x)^{a_3 b_1} & (1+x)^{a_3 b_2} & (1+x)^{a_3 b_3} \end{vmatrix},$$

then coefficient of x in expansion of $\Delta(x)$ is 0.

Statement-2: If $P(x) = (1 + x)^n$, $n \in \mathbf{N}$ then coefficient of x in the expansion of $P(x)$ is $P'(0)$.

80. **Statement-1:** If $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq bc + ca + ab$, then the system of homogenous equations

$$ax + by + cz = 0$$
$$bx + cy + az = 0$$
$$cx + ay + bz = 0$$

has infinite number of solutions.

Statement-2: If $|A| = 0$, the system of equations $AX = B$ has infinite number of solutions.

81. Suppose $a, b, c \in \mathbf{R}$. Consider the system of linear equations:

$$ax + by + cz = 0$$
$$bx + cy + az = 0$$
$$cx + ay + bz = 0$$

Statement-1: If $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = bc + ca + ab$, then the system of equations has infinite number of solutions.

Statement-2: If $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq bc + ca + ab$, then the system of equations has unique solutions.

82. Suppose $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbf{R}$.

Let

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 2a_1b_1 & a_1b_2 + a_2b_1 & a_1b_3 + a_3b_1 \\ a_1b_2 + a_2b_1 & 2a_2b_2 & a_2b_3 + a_3b_2 \\ a_1b_3 + a_3b_1 & a_2b_3 + a_3b_2 & 2a_3b_3 \end{vmatrix}$$

Statement-1: $\Delta_2 \neq 0$ for some values of $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbf{R}$.

Statement-2: If (a_1, b_1), (a_2, b_2), (a_3, b_3) are non-collinear than $\Delta_1 \neq 0$.

83. Suppose a_1, b_1, c_1, d_1 a_2, b_2, c_2, d_2 are eight integers.

Statement-1: $\left(a_1^2 + b_1^2 + c_1^2 + d_1^2\right)\left(a_2^2 + b_2^2 + c_2^2 + d_2^2\right)$ can be written in the form $a^2 + b^2 + c^2 + d^2$ where a, b, c, d are some integers.

Statement-2: If $a, b, c, d \in \mathbf{Z}$, then

$$a^2 + b^2 + c^2 + d^2 = \begin{vmatrix} a + ib & c + id \\ -c + id & a - ib \end{vmatrix}$$

LEVEL 2

Straight Objective Type Questions

84. If $l_1^2 + m_1^2 + n_1^2 = 1$ etc., and $l_1l_2 + m_1m_2 + n_1n_2 = 0$ etc., and

$$\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

then

(a) $|\Delta| = 3$ (b) $|\Delta| = 2$

(c) $|\Delta| = 1$ (d) $\Delta = 0$

85. If a, b, c are non-zero real number and

$$\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$$

then Δ equals

(a) abc (b) $a^2\,b^2\,c^2$

(c) $bc + ca + ab$ (d) 0

86. Let $\Delta(x) = \begin{vmatrix} x^2-5x+3 & 2x-5 & 3 \\ 3x^2+x+9 & 6x+1 & 9 \\ 7x^2-6x+9 & 14x-6 & 21 \end{vmatrix}$

$= ax^3 + bx^2 + cx + d$,

then a equals

(a) – 1 (b) 0

(c) 2 (d) none of these

87. The number of distinct values of t for which the system

$(a + t)x + by + cz = 0$

$ax + (b + t)y + cz = 0$

$ax + by + (c + t)z = 0$

has a non-trivial solution is

(a) 1 (b) 2

(c) 3 (d) none of these

88. If $a^2 + b^2 + c^2 = 1$, then

$$\begin{vmatrix} a^2+(b^2+c^2)\cos\theta & ab\,(1-\cos\theta) & ac\,(1-\cos\theta) \\ ba\,(1-\cos\theta) & b^2+(c^2+a^2)\cos\theta & bc\,(1-\cos\theta) \\ ca(1-\cos\theta) & cb(1-\cos\theta) & c^2+(a^2+b^2)\cos\theta \end{vmatrix}$$ equals

(a) $\cos^2\theta$ (b) 0

(c) 1 (d) $\sin^2\theta$

89. Suppose $\alpha, \beta, \gamma, \theta \in \mathbf{R}$ and

$$A(\alpha, \beta, \gamma, \theta) = \begin{vmatrix} \cos(\alpha+\theta) & \sin(\alpha+\theta) & 1 \\ \cos(\beta+\theta) & \sin(\beta+\theta) & 1 \\ \cos(\gamma+\theta) & \sin(\gamma+\theta) & 1 \end{vmatrix}$$

Numerical value of $A\left(-\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{2\pi}{13}\right)$ is

(a) 0 (b) – 1

(c) 2 (d) none of these

90. If a, b, c are positive integers such that $a > b > c$ and

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = -2$$

then $3a + 7b - 10c$ equals

(a) 10 (b) 11

(c) 12 (d) 13

91. If $A, B, C, P, Q, R \in \mathbf{R}$, and

$$\Delta = \begin{vmatrix} \cos(A+P) & \cos(A+Q) & \cos(A+R) \\ \cos(B+P) & \cos(B+Q) & \cos(B+R) \\ \cos(C+P) & \cos(C+Q) & \cos(C+R) \end{vmatrix}$$

(a) Δ depends on P, Q, R

(b) Δ depends on A, B, C

(c) Δ depends on A, B, C, P, Q, R

(d) none of these

92. Let $f(x) = \begin{vmatrix} 2\cos x & 1 & 0 \\ 1 & 2\cos x & 1 \\ 0 & 1 & 2\cos x \end{vmatrix}$ then

(a) $f\left(\frac{\pi}{3}\right) = 1$ (b) $f'\left(\frac{\pi}{3}\right) = -\sqrt{3}$

(c) $f\left(\frac{\pi}{2}\right) = -1$ (d) none of these

93. Consider the set A consisting of all determinants of order 3 with entries 0 and 1 only. Let B be the subset of A consisting of all the determinants with value 1. Let C be the subset of A consisting of all the determinants with value –1. Then

(a) $C = \phi$

(b) B has as many elements as C

(c) $A = B \cap C$

(d) $A = B \cup C$

94. let $\omega = e^{2\pi i/3}$ and consider the system of linear equations

$$x + y + z = a$$
$$x + \omega y + \omega^2 z = b$$
$$x + \omega^2 y + \omega z = c$$

If x, y, z is a solution of the above system of equations, then value of $\dfrac{|a|^2 + |b|^2 + |c|^2}{|x|^2 + |y|^2 + |z|^2}$ is

(a) 9 (b) 6
(c) 3 (d) 1

95. If a, b, c are distinct and $\begin{vmatrix} a & a^2 & a^3 - 1 \\ b & b^2 & b^3 - 1 \\ c & c^2 & c^3 - 1 \end{vmatrix} = 0$ then abc equals

(a) 0 (b) 1
(c) –1 (d) –2

96. If the adjoint of a 3 × 3 matrix P is $\begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{pmatrix}$ then the possible value(s) of determinant P is (are)

(a) ± 2 (b) ± 3
(d) ± 1 (d) 0

97. If $\begin{vmatrix} 1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{vmatrix} = 0$, then x equals

(a) 17, 21 (b) 0,19
(c) 0, 35 (d) 21, 35

Previous Years' AIEEE/JEE Main Questions

1. If the equation $ax^2 + 2bx + c = 0$ has equal roots then the determinant

$$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$

is

(a) positive (b) negative
(c) 0 (d) dependent on a.

[2002]

2. If $l, m, n > 0$ and l, m, n are the pth, qth, rth terms of a G.P., then determinant

$$\Delta = \begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} \text{ equals}$$ **[2002]**

(a) 0 (b) – 1
(c) $p + q + r$ (d) none of these

3. If 1, ω, ω^2 are the cube roots of unity then

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix} \text{ equals}$$

(a) 1 (b) 2
(c) ω^2 (d) 0 **[2003]**

4. If the system of linear equations $x + 2ay + az = 0$, $x + 3by + bz = 0$, $x + 4cy + cz = 0$ has a non-zero solution, then a, b, c

(a) are in G.P
(b) are in H.P.
(c) satisfy $a + 2b + 3c = 0$
(d) are in A.P. **[2003]**

5. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in G.P., then the value of the determinant

$$\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+1} & \log a_{n+2} & \log a_{n+3} \\ \log a_{n+2} & \log a_{n+3} & \log a_{n+4} \end{vmatrix}$$

is

(a) 2 (b) 1
(c) 0 (d) – 2 **[2004, 2005]**

6. If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

then $f(x)$ is a polynomial of degree

(a) 3 (b) 2
(1) 1 (d) 0 **[2005]**

7. The system of equations

$$\alpha x + y + z = \alpha - 1$$
$$x + \alpha y + z = \alpha - 1$$
$$x + y + \alpha z = \alpha - 1$$

has no solution, if α is

(a) not – 2
(b) 1
(c) – 2
(d) either – 2 or 1 **[2005]**

8. If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $x \neq 0$, $y \neq 0$, then D is
   (a) divisible by neither x nor y
   (b) divisible by both x and y
   (c) divisible by x but not y
   (d) divisible by y but not x **[2007]**

9. Let a, b, c be such that $b(a + c) \neq 0$. If

$$\Delta = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^{n}c \end{vmatrix} = 0$$

   then the value of n is
   (a) any odd integer
   (b) any integer
   (c) zero
   (d) any even integer

[2009]

10. If a, b, c are sides of a scalene triangle, then the value of $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is:
   (a) non-negative
   (b) negative
   (c) positive
   (d) non-positive

[2013 online]

11. **Statement-1:** The system of linear equations

$$x + (\sin \alpha)y + (\cos \alpha)z = 0$$
$$x + (\cos \alpha)y + (\sin \alpha)z = 0$$
$$x - (\sin \alpha)y - (\cos \alpha)z = 0$$

   has a non-trivial solution for only one value of α lying in the interval $(0, \pi/2)$.

   Statement-2: The equation in α

$$\Delta = \begin{vmatrix} \cos\alpha & \sin\alpha & \cos\alpha \\ \sin\alpha & \cos\alpha & \sin\alpha \\ \cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

   has only one solution lyining in the interval $(0, \pi/2)$. **[2013 online]**

12. If α, $\beta \neq 0$, and $f(n) = \alpha^n + \beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

   $= K(1 - \alpha)^2 (1 - \beta)^2 (\alpha - \beta)^2$, then K is equal to:
   (a) $\dfrac{1}{\alpha\beta}$ (b) 1
   (c) –1 (d) $\alpha\beta$

[2014]

13. If a, b, c are non-zero real numbers and if the system of equations
   $(a - 1)x = y + z$,
   $(b - 1)y = z + x$,
   $(c - 1)z = x + y$,
   has a non-trivial solution, then $ab + bc + ca$ equals:
   (a) $a + b + c$ (b) abc
   (c) 1 (d) –1 **[2014 online]**

14. Let for $i = 1, 2, 3$, $p_i(x)$ be a polynomial of degree 2 in x, $p_i'(x)$ and $p_i''(x)$ be the first and second order derivatives of $p_i(x)$ respectively. Let,

$$A(x) = \begin{bmatrix} p_1(x) & p_1'(x) & p_1''(x) \\ p_2(x) & p_2'(x) & p_2''(x) \\ p_3(x) & p_3'(x) & p_3''(x) \end{bmatrix}$$

   and $B(x) = [A(x)]^T A(x)$. Then determinant of $B(x)$.
   (a) is a polynomial of degree 6 in x
   (b) is a polynomial of degree 3 in x
   (c) is a polynomial of degree 2 in x
   (d) does not depend on x

[2014 online]

15. If

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+\lambda)^2 & (b+\lambda)^2 & (c+\lambda)^2 \\ (a-\lambda)^2 & (b-\lambda)^2 & (c-\lambda)^2 \end{vmatrix} = \mathrm{k}\lambda \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}, \lambda \neq 0,$$

   then k is equal to:
   (a) $4\lambda abc$ (b) $-4\lambda abc$
   (c) $4\lambda^2$ (d) $-4\lambda^2$

[2014 online]

16. If $\Delta_r = \begin{vmatrix} r & 2r-1 & 3r-2 \\ \dfrac{n}{2} & n-1 & a \\ \dfrac{1}{2}n(n-1) & (n-1)^2 & \dfrac{1}{2}(n-1)(3n+4) \end{vmatrix}$

   then the value of $\displaystyle\sum_{r=1}^{n-1} \Delta_r$:
   (a) depends only on a
   (b) depends only on n
   (c) depends both on a and n
   (d) is independent of both a and n.

[2014 online]

17. The set of all values of λ for which the system of linear equations:

$$\begin{aligned} 2x_1 - 2x_2 + x_3 &= \lambda x_1 \\ 2x_1 - 3x_2 + 2x_3 &= \lambda x_2 \\ -x_1 + 2x_2 &= \lambda x_3 \end{aligned}$$

has a non-trivial solution,

(a) is an empty set
(b) is a singleton
(c) contains two elements
(d) contains more than two elements **[2015]**

18. If $\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = ax - 12,$

then a is equal to:

(a) 12 (b) 24
(c) – 12 (d) – 24 **[2015 online]**

19*. The least value of the product xyz for which the determinant $\Delta = \begin{vmatrix} x & 1 & 1 \\ 1 & y & 1 \\ 1 & 1 & z \end{vmatrix}$ is non-negative, is:

(a) $-2\sqrt{2}$ (b) $-16\sqrt{2}$
(c) – 8 (d) – 1 **[2015 online]**

20. The system of linear equations

$$\begin{aligned} x + \lambda y - z &= 0 \\ \lambda x - y - z &= 0 \\ x + y - \lambda z &= 0 \end{aligned}$$

has a non-trivial solution for

(a) infinitely many values of λ
(b) exactly one value of λ
(c) exactly two values of λ
(d) exactly three values of λ **[2016]**

21. The number of distinct real roots of the equation.

$$\Delta = \begin{vmatrix} \cos x & \sin x & \sin x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$$

in the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is:

(a) 1 (b) 4
(c) 2 (d) 3 **[2016 online]**

Previous Years' B-Architecture Entrance Examination Questions

1. If the system of equations

$$\begin{aligned} x + y + z &= 0 \\ ax + by + z &= 0 \\ bx + y + z &= 0 \end{aligned}$$

has a non-trivial solution, then

(a) $b^2 = 2b + 1$ (b) $b^2 = 2b - 1$
(c) $b - a = 0$ (d) $b^2 = 2b$ **[2006]**

2. The system of linear equations

$$\begin{aligned} (\lambda + 3)x + (\lambda + 2)y + z &= 0 \\ 3x + (\lambda + 3)y + z &= 0 \\ 2x + 3y + z &= 0 \end{aligned}$$

has a non-trivial solution

(a) if $\lambda = 1$ (b) if $\lambda = -1$
(c) for no real value of λ
(d) if $\lambda = 0$ **[2007]**

3. If

$$\Delta_k = \begin{vmatrix} 2(3^{k-1}) & 3(4^{k-1}) & 4(5^{k-1}) \\ \alpha & \beta & \gamma \\ 3^n - 1 & 4^n - 1 & 5^n - 1 \end{vmatrix}$$

then the value of $\sum_{k=1}^{n} \Delta_k$ depends

(a) only on α and β not on γ
(b) on all α, β and γ
(c) on none of α, β and γ
(d) only on α, not on β and γ **[2007]**

4. Let $A = (a_{ij})$ be a 3 × 3 matrix whose determinant is 5. The determinant of the matrix $B = (2^{i-j} a_{ij})$ is

(a) 5 (b) 10
(c) 20 (d) 40 **[2008]**

5. Let $\Delta_r = \begin{vmatrix} 2^{r-1} & 2(3^{r-1}) & 4(5^{r-1}) \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$, for

$r = 1, 2, \ldots, n$. Then $\sum_{r=1}^{n} \Delta_r$ is

(a) independent of α, β, γ and n
(b) independent of n only
(c) depends on α, β, γ and n
(d) independent of α, β, γ only **[2010]**

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* Question is incorrect as xyz can take any real value.

6. If x_1, x_2, x_3,…, x_{13} are in A.P. then the value of $\begin{vmatrix} e^{x_1} & e^{x_4} & e^{x_7} \\ e^{x_4} & e^{x_7} & e^{x_{10}} \\ e^{x_7} & e^{x_{10}} & e^{x_{13}} \end{vmatrix}$ is:

   (a) 27 (b) 0

   (c) 1 (d) 9 **[2011]**

7. The value of the determinant $\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$ is equal to:

   (a) $5\sqrt{3}(\sqrt{6}-5)$ (b) $5\sqrt{3}(\sqrt{6}-\sqrt{5})$

   (c) $5(\sqrt{6}-5)$ (d) $\sqrt{3}(\sqrt{6}-\sqrt{5})$ **[2012]**

8. If the system of linear equations, $x + 2ay + az = 0$, $x + 3by + bz = 0$ and $x + 4cy + cz = 0$ has a non-zero solution, then a, b, c satisfy:

   (a) $2b = a + c$ (b) $b^2 = ac$

   (c) $2ac = ab + bc$ (d) $2ab = ac + bc$ **[2013]**

9. In a ΔABC, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0,$ then $\sin^2 A + \sin^2 B + \sin^2 C$ is

   (a) $\frac{3}{2}\sqrt{3}$ (b) $\frac{9}{4}$

   (c) $\frac{5}{4}$ (d) 2 **[2014]**

10. The system of linear equations

$$x - y + z = 1$$
$$x + y - z = 3$$
$$x - 4y + 4z = \alpha$$

   has:

   (a) a unique solution when $\alpha = 2$

   (b) a unique solution when $\alpha \neq -2$

   (c) an infinite number of solutions, when $\alpha = 2$

   (d) an infinite number of solutions, when $\alpha = -2$

   [2015]

11. For all values of $\theta \in \left(0, \frac{\pi}{2}\right)$, the determinant of the matrix $\Delta = \begin{bmatrix} -2 & \tan\theta + \sec^2\theta & 3 \\ -\sin\theta & \cos\theta & \sin\theta \\ -3 & -4 & 3 \end{bmatrix}$ always lies in the interval:

   (a) [3, 5] (b) (4, 6)

   (c) $\left(\frac{5}{2}, \frac{19}{4}\right)$ (d) $\left[\frac{7}{2}, \frac{21}{4}\right]$ **[2016]**

Answers

Concept-based

1.	(d)	**2.**	(d)	**3.**	(a)	**4.**	(d)
5.	(a)	**6.**	(c)	**7.**	(d)	**8.**	(a)
9.	(a)	**10.**	(b)	**11.**	(d)	**12.**	(c)
13.	(d)	**14.**	(d)	**15.**	(a)		

Level 1

16.	(a)	**17.**	(c)	**18.**	(a)	**19.**	(b)
20.	(d)	**21.**	(c)	**22.**	(a)	**23.**	(c)
24.	(d)	**25.**	(a)	**26.**	(d)	**27.**	(d)
28.	(a)	**29.**	(a)	**30.**	(a)	**31.**	(d)
32.	(a)	**33.**	(c)	**34.**	(a)	**35.**	(a)
36.	(d)	**37.**	(d)	**38.**	(d)	**39.**	(d)
40.	(a)	**41.**	(a)	**42.**	(d)	**43.**	(d)
44.	(d)	**45.**	(d)	**46.**	(c)	**47.**	(a)
48.	(d)	**49.**	(b)	**50.**	(d)	**51.**	(d)
52.	(a)	**53.**	(b)	**54.**	(d)	**55.**	(c)
56.	(d)	**57.**	(c)	**58.**	(a)	**59.**	(a)
60.	(d)	**61.**	(d)	**62.**	(a)	**63.**	(d)
64.	(a)	**65.**	(b)	**66.**	(a)	**67.**	(a)
68.	(b)	**69.**	(a)	**70.**	(d)	**71.**	(a)
72.	(d)	**73.**	(d)	**74.**	(c)	**75.**	(a)
76.	(a)	**77.**	(d)	**78.**	(b)	**79.**	(a)
80.	(c)	**81.**	(c)	**82.**	(d)	**83.**	(a)

Level 2

84.	(c)	**85.**	(d)	**86.**	(b)	**87.**	(b)
88.	(a)	**89.**	(c)	**90.**	(d)	**91.**	(d)
92.	(b)	**93.**	(b)	**94.**	(c)	**95.**	(b)
96.	(a)	**97.**	(c)				

Previous Years' AIEEE/JEE Main Questions

1.	(c)	**2.**	(a)	**3.**	(d)	**4.**	(b)
5.	(c)	**6.**	(b)	**7.**	(c)	**8.**	(b)
9.	(a)	**10.**	(b)	**11.**	(b)	**12.**	(b)
13.	(b)	**14.**	(d)	**15.**	(c)	**16.**	(d)
17.	(c)	**18.**	(b)	**19.**	(?)	**20.**	(d)
21.	(c)						

Previous Years' B-Architecture Entrance Examination Questions

1.	(b)	**2.**	(c)	**3.**	(c)	**4.**	(a)
5.	(a)	**6.**	(b)	**7.**	(c)	**8.**	(c)
9.	(b)	**10.**	(d)	**11.**	(a)		

Hints and Solutions

Concept-based

1. det (adj (A)) A^{-1}) = det (adj(A)) det(A^{-1})

$$= (\det(A))^2 \, \frac{1}{\det(A)} = \det(A)$$

2. Use $R_1 \to R_1 + R_3 - 2R_2$

3. Using $R_2 \to R_2 - x^2R_1$ and $R_3 \to R_3 - xR_1$, we get

$$\Delta = \begin{vmatrix} 1 & x & \\ 0 & 1-x^3 & x(1-x^3) \\ 0 & 0 & 1-x^3 \end{vmatrix}$$

$$= (1 - x^3)^2 = (1 - x)^2\,(1 + x + x^2)^2$$

As $x \neq 1, \Delta = 0 \Rightarrow 1 + x + x^2 = 0 \Rightarrow x = \omega, \omega^2$

4. Taking 10!, 11! and 12! common from C_1, C_2, C_3 respectively, we get

$$D = (10!\ 11!\ 12!)\, D_1$$

where

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 11 & 12 & 13 \\ (11)(12) & (12)(13) & (13)(14) \end{vmatrix}$$

Using $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$, we get

$$D_1 = \begin{vmatrix} 1 & 0 & 0 \\ 11 & 1 & 1 \\ (11)(12) & 2(12) & 2(13) \end{vmatrix} = 2$$

Thus, $\dfrac{D}{(10!)^3} = 2(11)\ (12) = 264$

5. Product of zeros $= -P(0) = 0$

6. $\dfrac{1}{2}|\Delta|$ = area of triangle $= \dfrac{\sqrt{3}}{4}(4)^2 = 4\sqrt{3}$

$\Rightarrow\ |\Delta| = 8\sqrt{3}\ \Rightarrow\ \Delta^2 = 192$

7. Use $R_1 \to R_1 + R_3 - 2R_2$.

8. Write

$$\Delta = \begin{vmatrix} 1+1 & a+b & a^2+b^2 \\ a+b & a^2+b^2 & a^3+b^3 \\ a^2+b^2 & a^3+b^3 & a^4+b^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ a^2 & b^2 & 0 \end{vmatrix}\begin{vmatrix} 1 & 1 & 0 \\ a & b & 0 \\ a^2 & b^2 & 0 \end{vmatrix} = 0$$

9. Use $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ and $\log x - \log y = \log (x/y)$ to obtain

$$\Delta = \begin{vmatrix} \log a & \log b & \log c \\ \log(2007) & \log(2007) & \log(2007) \\ \log(2017) & \log(2017) & \log(2017) \end{vmatrix} = 0$$

[$\because$ R_2 and R_3 are proportional]

10. Using $R_2 \to R_2 - R_1$, we get

$$\Delta = -4(2016)\begin{vmatrix} (a+2016)^2 & (b+2016)^2 & (c+2016)^2 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Applying $R_1 \to R_1 - 2\ (2016)\ R_2 - R_3$, we get

$$\Delta = -4(2016)^3\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= -4\ (2016)^3\,(a - b)\,(b - c)\,(c - a)$$

11. Using $R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$, we get

$$P(x) = \begin{vmatrix} x & -6 & -1 \\ x+2 & -3(x+2) & x+2 \\ -3-x & 2(x+3) & x+3 \end{vmatrix}$$

$$= (x + 2)\,(x + 3)\begin{vmatrix} x & -6 & -1 \\ 1 & -3 & 1 \\ -1 & 2 & 1 \end{vmatrix}$$

Using $R_1 \to R_1 + R_3$, $R_2 \to R_2 - R_3$, we get

$$P(x) = (x + 2)\,(x + 3)\begin{vmatrix} x-1 & -4 & 0 \\ x & -5 & 0 \\ -1 & 2 & 1 \end{vmatrix}$$

$$= (x + 2)\,(x + 3)\,(-\,5x + 13)$$

sum of zeros of $P(x)$ is $-2 - 3 + 13/5 = -12/5$.

12. Write Δ as

$$\Delta = x^{n+1}\,y^{n+1}\,z^{n+1}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (xyz)^{n+1}\,(x - y)\,(y - z)(z - x)$$

$$\therefore \qquad n+1 = 2 \qquad \Rightarrow \quad n = 1.$$

13. When $x = 0$, $P(0)$ is a skew-symmetric determinant of odd order.

14. Write $\Delta = \Delta_1 + a\Delta_2$ where

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix}$$

and $$\Delta_2 = \begin{vmatrix} 1 & 0 & 0 \\ 1+b & 1+2b & 1 \\ 1+c & 1+c & 1+3c \end{vmatrix}$$

$$= (1 + 2b)\,(1 + 3c) - (1 + c)$$

$$= 2\,(b + c + 3c)$$

In Δ_1, apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$ to obtain

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1+b & b & -b \\ 1+c & 0 & 2c \end{vmatrix} = 2bc$$

Thus, $\Delta = 2(bc + ac + ab + 3abc)$

$$\therefore \quad \Delta = 0 \quad \Rightarrow \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = -3$$

15. Taking x^m common from R_1 and x^{m+5} from R_3, we get

$$\Delta = x^{2m+5}\begin{vmatrix} 1 & x^2 & x^m \\ 1 & x^n & 2^n \\ 1 & x^{n-m} & x^m \end{vmatrix}$$

If $n - m = 2$, then R_1 and R_3 are identical and hence $\Delta = 0$.

Level 1

16. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix}$$

$$= -2(a+b+c)\,[(b-c)^2 + (b-a)\,(c-a)]$$
$$= -(a+b+c)\,[(b-c)^2 + (c-a)^2 + (a-b)^2]$$

Now, use $\Delta = 0$, $a + b + c \neq 0$.

17. Write $y = \Delta_1 + \Delta_2$ where

$$\Delta_1 = \begin{vmatrix} 1 & x & x \\ 0 & x+2 & x \\ 0 & x & x+3 \end{vmatrix} = 5x+6$$

$$\text{and} \quad \Delta_2 = x\begin{vmatrix} 1 & x & x \\ 1 & x+2 & x \\ 1 & x & x+3 \end{vmatrix} = x\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 6x$$

Thus, $y = 11x + 6$

Its distance from the origin $= \dfrac{6}{\sqrt{122}}$

18. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

Now, use $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & a+2c \end{vmatrix}$$

$$\Delta = (a+b+c)\,[(a+2b)\,(a+2c) - (a-c)\,(a-b)]$$
$$= 3(a+b+c)\,(bc+ca+ab)$$

19. Using $R_1 \to R_1 - R_3 - R_2$ we get R_1 consists of all zeros.

20. If R is the common ratio of the G.P., then $b = aR$, $c = aR^2$, $d = aR^3$, $e = aR^4$, $f = aR^5$, and

$$\Delta = (a^2)(a^2R^6)\begin{vmatrix} 1 & 1 & p \\ R^2 & R^2 & q \\ R^4 & R^4 & r \end{vmatrix} = 0$$

[$\because$ C_1 and C_2 are identical]

21. Multiply C_1 by x, C_2 by y and C_3 by z to obtain

$$5 = \frac{1}{xyz}\begin{vmatrix} (x^2+1)x & xy^2 & xz^2 \\ yx^2 & (y^2+1)y & yz^2 \\ zx^2 & zy^2 & z(z^2+1) \end{vmatrix}$$

Taking x common from R_1, y from R_2 and z from R_3, we get

$$5 = \frac{xyz}{xyz}\begin{vmatrix} x^2+1 & y^2 & z^2 \\ x^2 & y^2+1 & z^2 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$5 = (x^2 + y^2 + z^2 + 1)\,\Delta_1$$

$$\text{where } \Delta_1 = \begin{vmatrix} 1 & y^2 & z^2 \\ 1 & y^2+1 & z^2 \\ 1 & y^2 & z^2+1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & y^2 & z^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Thus, $x^2 + y^2 + z^2 = 4$, which represents a sphere.

22. Apply $R_1 \to aR_1 + bR_2 + cR_3$ and use $a = b\cos C + c\cos B$ etc.

23. Write $\Delta = \Delta_1 + x\Delta_2$ where

$$\Delta_1 = \begin{vmatrix} a-x & x & x \\ 0 & b & x \\ 0 & x & c \end{vmatrix} = (a-x)(bc-x^2)$$

$$\text{and} \quad \Delta_2 = \begin{vmatrix} 1 & x & x \\ 1 & b & x \\ 1 & x & c \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x \\ 0 & b-x & 0 \\ 0 & 0 & c-x \end{vmatrix} \qquad [\text{use } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1]$$

$$= (b-x)\,(c-x)$$

Thus, $\Delta = (a - x)(bc - x^2) + x(b - x)(c - x)$

$$= 2x^3 - (a + b + c)x^2 + abc$$

Also, $xf'(x) - f(x)$

$$= x[(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b)] - f(x)$$

$$= 2x^3 - (a + b + c)x^2 + abc$$

24. Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$0 = (x+y+2)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x-y-2 & 2x \\ 2y & 2y & y-2-x \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$0 = (x+y+2)\begin{vmatrix} 1 & 0 & 0 \\ 2x & -(x+y+2) & 0 \\ 2y & 0 & -(x+y+2) \end{vmatrix}$$

$$\Rightarrow \quad 0 = (x + y + 2)^3$$

$$\Rightarrow \quad x + y + 2 = 0$$

It passes through (–1, –1).

25. Write $f(x) = \Delta_1 + x\Delta_2$ where

$$\Delta_1 = \begin{vmatrix} a & x & x \\ 0 & x+a & x \\ 0 & x & x+a \end{vmatrix} = a[(x + a)^2 - x^2]$$

$$= 2\,xa^2 + a^3$$

and $$\Delta_2 = \begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = a^2$$

Thus, $$f(x) = 2xa^2 + a^3 + xa^2$$

$$= 3xa^2 + a^3$$

$\therefore\ f(2x) - f(x) = 3xa^2$

26. $\alpha + \beta + \gamma = -a,\ \beta\gamma + \gamma\alpha + \alpha\beta = 0,\ \alpha\beta\gamma = -b.$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta & \gamma \\ 1 & \gamma & \alpha \\ 1 & \alpha & \beta \end{vmatrix} \qquad (1)$$

Using $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta = -a\begin{vmatrix} 1 & \beta & \gamma \\ 0 & \gamma-\beta & \alpha-\gamma \\ 0 & \alpha-\gamma & \beta-\alpha \end{vmatrix}$$

$$= -a[(\gamma - \beta)(\beta - \alpha) - (\alpha - \gamma)^2]$$

$$= -a[\beta\gamma + \gamma\alpha + \alpha\beta - (\alpha^2 + \beta^2 + \gamma^2)] \quad (2)$$

$$= -a[0 - a^2] = a^3$$

27. In this case $\alpha + \beta + \gamma = 0$.

Thus, from (1) in solution to question 26, we get $\Delta = 0$.

28. We can write

$$\Delta = (abc)^2\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$$

where $\alpha = \dfrac{1}{a},\ \beta = \dfrac{1}{b},\ \gamma = \dfrac{1}{c}$

From (2) in solution question 26, we have

$$\Delta = (abc)^2(\alpha + \beta + \gamma)[(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2]$$

$$\Rightarrow \alpha + \beta + \gamma = 0 \quad \Rightarrow \quad a^{-1} + b^{-1} + c^{-1} = 0$$

29. Using $C_2 \to C_2 - \lambda^2 C_3$, we can write

$$\Delta = \lambda\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Using $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = \lambda\begin{vmatrix} a & a^2 & 1 \\ b-a & b^2-a^2 & 0 \\ c-b & c^2-b^2 & 0 \end{vmatrix}$$

$$= \lambda(b - a)(c - b)\begin{vmatrix} 1 & b+a \\ 1 & c+b \end{vmatrix}$$

$$= \lambda(b - a)(c - b)(c - a)$$

$$= \lambda(a - b)(b - c)(c - a)$$

30. Use $C_2 \to C_2 - C_3 + C_1$ to show that C_2 consists of all zeros.

31. We know

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$$

$$\text{Let } \Delta_1 = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Using $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-b^2 & c^3-b^3 \end{vmatrix}$$

$$= (b - a)(c - b)\begin{vmatrix} b+a & b^2+a^2+ba \\ c+b & c^2+b^2+cb \end{vmatrix}$$

Applying $C_2 \to C_2 - bC_1$, we get

$$\Delta_1 = (b-a)\,(c-b)\begin{vmatrix} b+a & a^2 \\ c+b & c^2 \end{vmatrix}$$

$$= (b-a)\,(c-b)\begin{vmatrix} b+a & a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

[using $R_2 \to R_2 - R_1$]

$$= (b-a)\,(c-b)\,(c-a)\begin{vmatrix} b+a & a^2 \\ 1 & c+a \end{vmatrix}$$

$$= (a-b)\,(b-c)\,(c-a)\,(bc+ca+ab)$$

Thus, $\lambda = 18$

32. Show that $\Delta = (a_0 + a_1 + a_2)x^2$
33. Using $R_2 \to R_2 + R_1$, we get

$$\Delta = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & 1 & \sin + \cos x \\ 0 & -1 & 1 - (\sin x + \cos x) \end{vmatrix}$$

Using $R_3 \to R_3 + R_2$, we get

$$\Delta = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & 1 & \sin + \cos x \\ 0 & 0 & 1 \end{vmatrix} = 1$$

34. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (x+a+b)\begin{vmatrix} 1 & a & b \\ 1 & x & b \\ 1 & b & x \end{vmatrix}$$

Using $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = (x+a+b)\begin{vmatrix} 1 & a & b \\ 0 & x-a & 0 \\ 0 & b-a & x-b \end{vmatrix}$$

$$= (x+a+b)\,(x-a)\,(x-b)$$

35. Use $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ to obtain

$$\Delta = \sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$$

36. Use $C_1 \to C_1 + C_2 + C_3$ to obtain

$$\Delta = (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & c \\ 1 & b & x+c \end{vmatrix}$$

Now use $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ to obtain

$$\Delta = x^2(x+a+b+c)$$

Thus, $\Delta = 0 \Rightarrow x = 0$ or $x = -(a+b+c)$

37. Use $C_1 \to C_1 - C_2$.
38. Show $\Delta = -a^3 + 8b^3$

Thus, $\Delta = 0 \Rightarrow \left(\frac{a}{b}\right)^3 = 8$

$\Rightarrow a/b$ is a cube root of 8.

39. Using $C_1 \to C_1 + C_2 + C_3$, we can write $\Delta = 2(1+i)\Delta_1$ where

$$\Delta_1 = \begin{vmatrix} 1 & 1+i & i \\ 1 & i & 1 \\ 1 & 1 & 1+i \end{vmatrix} = \begin{vmatrix} 1 & 1+i & i \\ 0 & -1 & 1-i \\ 0 & -i & 1 \end{vmatrix} = i$$

[using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

Thus, $\Delta = 2(i-1)$

40. Write $\Delta = abc\begin{vmatrix} 1 & \frac{1}{c} & \frac{1}{a}+\frac{1}{b} \\ 1 & \frac{1}{a} & \frac{1}{b}+\frac{1}{c} \\ 1 & \frac{1}{b} & \frac{1}{c}+\frac{1}{a} \end{vmatrix}$

and use $C_2 \to C_2 + C_1$, to show that $\Delta = 0$

41. Applying $C_1 \to C_1 - C_2 - C_3$ and reduce Δ to determinant in Example 36.
42. Applying $R_1 \to R_1 - xR_2$ we get

$$\Delta = \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

Take $1 - x^2$ common from R_1 and apply $R_2 \to R_2 - xR_1$ to obtain

$$\Delta = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$$

43. Applying $R_1 \to R_1 + R_3$ to obtain

$$\Delta = \begin{vmatrix} 0 & 0 & 2 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} = 2(\sin^2\theta + 1)$$

As $0 \le \sin^2\theta \le 1$, we get $\Delta \in [2, 4]$.

44. Write $\Delta = \Delta_1 - \Delta_2$ where

$$\Delta_1 = \begin{vmatrix} b^2-ab & b & bc-ac \\ ab-a^2 & a & b^2-ab \\ bc-ac & c & ab-a^2 \end{vmatrix} \text{ and}$$

$$\Delta_2 = \begin{vmatrix} b^2-ab & c & bc-ac \\ ab-a^2 & b & b^2-ab \\ bc-ac & a & ab-a^2 \end{vmatrix}$$

In Δ_1 use $C_1 \to C_1 - (b-a)C_2$ to show that $\Delta_1 = 0$.

In Δ_2 use $C_3 \to C_3 - (b - a)C_2$ to show that $\Delta_2 = 0$.

Thus, $\Delta = 0$.

45. We can write

$$\Delta = -3iw \begin{vmatrix} 6i & 1 & 1 \\ 4 & -1 & -1 \\ 20 & i & i \end{vmatrix} = 0$$

$\therefore \quad x = 0, y = 0$

46. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (\sin x + 2\cos x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = (\sin x + 2\cos x) \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix}$$

$$= (\sin x + 2\cos x)\,(\sin x - \cos x)^2$$

$$\Delta = 0 \Rightarrow \tan x = -2, \tan x = 1.$$

As $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$, $-1 \le \tan x \le 1$.

Thus, $\tan x = 1 \Rightarrow x = \pi/4$.

47. Applying $C_1 \to C_1 + iC_2$, we get

$$x + iy = \begin{vmatrix} 0 & i & -\omega \\ 0 & 1 & \omega^2 \\ \omega - i\omega^2 & -\omega^2 & 1 \end{vmatrix}$$

$$= (\omega - i\omega^2)\,(i\omega^2 + \omega)$$

$$= \omega^2 - i^2\omega^4 = \omega^2 + \omega = -1$$

$$\Rightarrow x = -1, y = 0.$$

48. Taking conjugate, we get

$$x - iy = \begin{vmatrix} 1 & e^{-\pi i/4} & e^{-\pi i/3} \\ e^{\pi i/4} & 1 & e^{-2\pi i/3} \\ e^{\pi i/3} & e^{2\pi i/3} & e^{2\pi i/3} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & e^{\pi i/4} & e^{\pi i/3} \\ e^{-\pi i/4} & 1 & e^{2\pi i/3} \\ e^{-\pi i/3} & e^{-2\pi i/3} & e^{2\pi i/3} \end{vmatrix}$$

Now,

$$2iy = (x + iy) - (x - iy)$$

$$= \begin{vmatrix} 1 & e^{\pi i/4} & 0 \\ e^{-\pi i/4} & 1 & 0 \\ e^{-\pi i/3} & e^{-2\pi i/3} & e^{-2\pi i/3} - e^{2\pi i/3} \end{vmatrix}$$

$$= (e^{-2\pi i/3} - e^{2\pi i/3})\,[1 - 1] = 0$$

$$\Rightarrow y = 0.$$

49. Show that the determinant equals $x^3 + (a^2 + b^2 + c^2)x = 0$.

$\Rightarrow x = 0$ as $x \in \mathbf{R}$

$\therefore$ Number of real roots of the equation is one.

50. Note that

$$\Delta_1 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$$

where C_{ij} cofactor of (i, j)th element of

$$\begin{bmatrix} 1 & x & x^2 \\ x & x^2 & 1 \\ x^2 & 1 & x \end{bmatrix}$$

$\therefore \quad \Delta_1 = \Delta^2 = 49$

51. Applying $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos\beta \\ \sin\beta & \cos\alpha & \cos\beta \\ \sin\gamma & \cos\alpha & \cos\beta \end{vmatrix} = 0$$

52. Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, and $\log x - \log y = \log\left(\dfrac{x}{y}\right)$, we get

$$\Delta = \begin{vmatrix} \log a & \log b & \log c \\ \log 7 & \log(7^2) & \log(7^3) \\ \log(3) & \log(3^2) & \log(3^3) \end{vmatrix}$$

$$= (\log 7)\,(\log 3)\,\Delta_1$$

$$\text{where } \Delta_1 = \begin{vmatrix} \log a & \log b & \log c \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = 0$$

$\therefore \quad \Delta = 0$

53. $C_3 \to C_3 + C_1 + C_2$, gives

$$\Delta = (2 + 4\sin 4\theta) \begin{vmatrix} 1 + \cos^2\theta & \sin^2\theta & 1 \\ \cos^2\theta & 1 + \sin^2\theta & 1 \\ \cos^2\theta & \sin^2\theta & 1 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = 2(1 + 2\sin 4\theta) \begin{vmatrix} 1 + \cos^2\theta & \sin^2\theta & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= 2(1 + 2\sin 4\theta)$$

$$\Delta = 0 \Rightarrow \sin 4\theta = -1/2$$

Now, $0 \le \theta \le \dfrac{\pi}{2} \Rightarrow 0 \le 4\theta \le 2\pi$

$$\therefore \quad 4\theta = \frac{7\pi}{6}, \frac{11\pi}{6} \quad \Rightarrow \quad \theta = \frac{7\pi}{24}, \frac{11\pi}{24}$$

54.

$$P(x) = \begin{vmatrix} (x-1)(x+1) & (x+1)^2 & (2x+1)(x+1) \\ (2x-1)(x+1) & (2x-1)(x+3) & (2x+3)(2x-1) \\ (2x+1)(3x-2) & (2x-1)(3x-2) & (4x+1)(3x-2) \end{vmatrix}$$

$$= (x+1)(2x-1)(3x-2)\begin{vmatrix} x-1 & x+1 & 2x+1 \\ x+1 & x+3 & 2x+3 \\ 2x+1 & 2x-1 & 4x+1 \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - 2C_1$, we get

$$P(x) = (x+1)(2x-1)(3x-2)\begin{vmatrix} x-1 & 2 & 3 \\ x+1 & 2 & 1 \\ 2x+1 & -2 & -1 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$, we get

$$P(x) = (x+1)(2x-1)(3x-2)\begin{vmatrix} x-1 & 2 & 3 \\ 2 & 0 & -2 \\ 3x & 0 & 2 \end{vmatrix}$$

$$P(x) = -4(x+1)(2x-1)(3x-2)(3x+2)$$

$\therefore$ $P(x) = 0$ has four distinct roots.

55. Put $a_r = w^r$ where $w = \cos\dfrac{2\pi}{9} + i\sin\dfrac{2\pi}{9}$.

Note that $w^9 = 1 \Rightarrow \dfrac{1}{w} = w^8$. Now,

$$\Delta = \begin{vmatrix} 1 & w^8 & w^7 \\ w^3 & w^2 & w \\ w^6 & w^5 & w^4 \end{vmatrix} = 0$$

as C_2 and C_3 are proportional.

56. As the system has a non-zero solution

$$\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} p & b & c \\ a-p & q-b & 0 \\ a-p & 0 & r-c \end{vmatrix} = 0$$

Expanding along R_3, we get

$-(a-p)(q-b)c + (r-c)[p(q-b) - b(a-p)] = 0.$

Dividing $(p-a)(q-b)(r-c)$, we get

$$\frac{c}{r-c} + \frac{p}{p-a} + \frac{b}{q-b} = 0.$$

$$\Rightarrow \quad \frac{c-r}{r-c} + \frac{r}{r-c} + \frac{p}{p-a} + \frac{b-q}{q-b} + \frac{q}{q-b} = 0$$

$$\Rightarrow \quad \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2.$$

Now use

$$\frac{p+a}{p-a} = \frac{2p-(p-a)}{p-a} = \frac{2p}{p-a} - 1 \text{ etc.}$$

57. We can write D as

$$D = n!\,(n+1)!\,(n+2)!\begin{vmatrix} 1 & n+1 & (n+1)(n+2) \\ 1 & n+2 & (n+3)(n+2) \\ 1 & n+3 & (n+4)(n+3) \end{vmatrix}$$

Applying $R_3 \to R_3 - R_3$, $R_2 \to R_2 - R_1$, we get

$$D = n!\,(n+1)!\,(n+2)!\begin{vmatrix} 1 & n+1 & (n+1)(n+2) \\ 0 & 1 & 2(n+2) \\ 0 & 1 & 2(n+3) \end{vmatrix}$$

$$\Rightarrow \quad \frac{D}{n!(n+1)!(n+2)!} = 2$$

58. Use $R_1 \to R_1 + R_3 - 2R_2$

59. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (3-y)\begin{vmatrix} 1 & 1-y & 1-y \\ 1 & 1+y & 1-y \\ 1 & 1-y & 1+y \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = (3-y)\begin{vmatrix} 1 & 1-y & 1-y \\ 0 & 2y & 0 \\ 0 & -2y & 2y \end{vmatrix} = (3-y)(4y^2)$$

$\therefore \quad \Delta = 0 \Rightarrow y = 0$ or $y = 3$.

60. Write $\Delta = l\Delta_1 + l'\Delta_2$, where

$$\Delta_1 = \begin{vmatrix} a & am + a'm' & an + a'n' \\ b & bm + b'm' & bn + b'n' \\ c & cm + c'm' & cn + c'n' \end{vmatrix} \text{ and}$$

$$\Delta_2 = \begin{vmatrix} a' & am + a'm' & an + a'n' \\ b' & bm + b'm' & bn + b'n' \\ c' & cm + c'm' & cn + c'n' \end{vmatrix}$$

In Δ_1 apply $C_2 \to C_2 - mC_1$, $C_3 \to C_3 - nC_1$ to obtain

$$\Delta_1 = \begin{vmatrix} a & a'm' & a'n' \\ b & b'm' & b'n' \\ c & c'm' & c'n' \end{vmatrix} = 0$$

Similarly $\Delta_2 = 0$. Thus, $\Delta = 0$

61. Imitate Example 35.

62. We have

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ m & m+3 & m+6 \\ \frac{1}{2}m(m-1) & \frac{1}{2}(m+3)(m+2) & \frac{1}{2}(m+6)(m+5) \end{vmatrix}$$

Applying $C_3 \to C_3 - C_2$, $C_2 \to C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ m & 3 & 3 \\ \frac{1}{2}m(m-1) & 3(m+1) & 3(m+4) \end{vmatrix}$$

$= 3^2(m + 4 - m - 1) = 3^3$

$\therefore \quad \alpha + \beta + \gamma = 3$

63. Use $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 - 3C_1$ to obtain C_2 and C_3 are proportional.

64. Write $\Delta = \dfrac{1}{a}\begin{vmatrix} 0 & \cos C & \cos B \\ a\cos C & -1 & \cos A \\ a\cos B & \cos A & -1 \end{vmatrix}$

Applying $C_1 \to C_1 + bC_2 + cC_3$, we get

$$\Delta = \frac{1}{a}\begin{vmatrix} a & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = \sin^2 A$$

65. For $x \neq 0$,

$$\frac{f(x)}{x^2} = \begin{vmatrix} \cos x & 1 & 1 \\ 2\dfrac{\sin x}{x} & 1 & 2 \\ \tan x & 1 & 1 \end{vmatrix}$$

$$\lim_{x \to 0} \frac{f(x)}{x^2} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -1$$

[using $R_1 \to R_1 - R_3$]

66. Applying $C_2 \to C_2 - \omega^2 C_1$ we find the second column of Δ becomes 0.

67. The system will have a non-trivial solution if

$$\Delta = \begin{vmatrix} p+a & b & c \\ a & q+b & c \\ a & b & r+c \end{vmatrix} = 0$$

Write

$$\Delta = pqr\begin{vmatrix} 1+\dfrac{a}{p} & \dfrac{b}{q} & \dfrac{c}{r} \\ \dfrac{a}{p} & 1+\dfrac{b}{q} & \dfrac{c}{r} \\ \dfrac{a}{p} & \dfrac{b}{q} & 1+\dfrac{c}{r} \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = pqr\left(1+\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)\begin{vmatrix} 1 & \dfrac{b}{q} & \dfrac{c}{r} \\ 1 & 1+\dfrac{b}{q} & \dfrac{c}{r} \\ 1 & \dfrac{b}{q} & 1+\dfrac{c}{r} \end{vmatrix}$$

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = pqr\left(1+\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right) \qquad (1)$$

$$\therefore \qquad \Delta = 0, pqr \neq 0 \Rightarrow \frac{a}{p}+\frac{b}{q}+\frac{c}{r} = -1.$$

68. The system will have a non-zero solution if

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - \alpha R_1 - R_2$ to obtain

$$\Delta = \begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ 0 & 0 & -(a\alpha^2+2b\alpha+c) \end{vmatrix}$$

$= -(a\alpha^2 + 2b\alpha + c)(ac - b^2)$

Note that $\Delta = 0$ if a, b, c are in G.P.

69. The system will have a non-trivial solution if

$$\Delta = \begin{vmatrix} a & a & -1 \\ b & -1 & b \\ -1 & c & c \end{vmatrix} = 0.$$

Using $C_3 \to C_3 - C_2$, $C_2 \to C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} a & 0 & -(1+a) \\ b & -(1+b) & (b+1) \\ -1 & c+1 & 0 \end{vmatrix}$$

$= -a(b + 1)(c + 1) - (1 + a)[b(c + 1) - (1 + b)]$

$$\therefore \quad \Delta = 0 \Rightarrow -\frac{a}{1+a} - \frac{b}{1+b} + \frac{1}{1+c} = 0$$

$$\Rightarrow \quad \frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} = 2.$$

70. The system will have a non-trivial solution if $\Delta = 0$ where

$$\Delta = \begin{vmatrix} \lambda+5 & \lambda-4 & 1 \\ \lambda-2 & \lambda+3 & 1 \\ \lambda & \lambda & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - \lambda C_3$, $C_1 \to C_1 - \lambda C_3$, we get

$$\Delta = \begin{vmatrix} 5 & -4 & 1 \\ -2 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 7 \neq 0$$

Thus, the system cannot have a non-trivial solution.

71. Adding above the system of equations, we get $(a + b + c)\ (x + y + z) = 0 \Rightarrow x + y + z = 0$

$\therefore\quad (b + c)\ (-x) - ax = b - c$

$\Rightarrow\quad x = -\ \dfrac{b-c}{a+b+c}$ etc.

72. The above system of equations will have a non-trivial solution if

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

But $\Delta = -\dfrac{1}{2}\ (a + b + c)$
$[(b - c)^2 + (c - a)^2 + (a - b)^2]$

$\therefore\quad \Delta = 0,\ a + b + c \neq 0$

$\Rightarrow\quad a = b = c.$

Thus, $a : b : c = 1 : 1 : 1$

73. $f'''(x) = \begin{vmatrix} 6 & 6 & 0 \\ 1 & -6 & 4 \\ p & p^2 & p^3 \end{vmatrix}$

$\therefore\quad f'''(x)$ is a constant.

74. Write

$D = \Delta_1 + p\Delta_2$ where

$$\Delta_1 = \begin{vmatrix} a_1 & b_1 + q\,c_1 & c_1 + ra_1 \\ a_2 & b_2 + qc_2 & c_2 + ra_2 \\ a_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix} \text{ and}$$

$$\Delta_2 = \begin{vmatrix} b_1 & b_1 + q\,c_1 & c_1 + ra_1 \\ b_2 & b_2 + qc_2 & c_2 + ra_2 \\ b_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix}$$

Now show that $\Delta_1 = \Delta$ and $\Delta_2 = pqr\ \Delta$.

75. The system will have a non-trivial solution if

$$\Delta = \begin{vmatrix} \lambda+3 & \lambda+2 & 1 \\ 3 & \lambda+3 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 0.$$

Using $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} \lambda & -1 & 0 \\ 1 & \lambda & 0 \\ 2 & 3 & 1 \end{vmatrix} = \lambda^2 + 1$$

Note that there is no real value of λ for which $\Delta = 0$.

76. Let A be a skew symmetric matrix of order n, then $|A'| = (-1)^n\ |A| \Rightarrow |A| = (-1)^n\ |A|$

If n is odd, then $|A| = -\ |A| \Rightarrow 2|A| = 0 \Rightarrow |A| = 0.$ We have

$$\Delta\left(\frac{\pi}{4}\right) = \begin{vmatrix} 0 & \sin(\pi/12) & \cos(\pi/12) \\ -\sin(\pi/12) & 0 & \sec(\pi/12) \\ -\cos(\pi/12) & -\sec(\pi/12) & 0 \end{vmatrix} = 0$$

[using statement-2]

77. Statement-2 is true, see theory.

The roots of $x^9 = p$ are $p^{1/9}\ \omega^r$

where $\omega = \cos\ \dfrac{2\pi}{9} + i\ \sin\ \dfrac{2\pi}{9}$.

value of determinant Δ depends on $\alpha_1, \alpha_2, \ldots \alpha_9$. If we put $\alpha_k = p^{1/9}\ \omega^k$, then

$$\Delta = (p^{1/9})^3 \begin{vmatrix} 1 & \omega^8 & \omega^7 \\ 1 & \omega & \omega^2 \\ 1 & \omega & \omega^2 \end{vmatrix} = 0$$

However, if $\alpha_1 = p^{1/9}$, $\alpha_2 = p^{1/9}\ \omega^8$,
$a_3 = p^{1/9}\ \omega^7$, $\alpha_4 = p^{1/9}\ \omega$, $\alpha_5 = p^{1/9}\ \omega^5$
$\alpha_6 = p^{1/9}\ \omega^4$, $\alpha_7 = p^{1/9}\ \omega^2$, $\alpha_8 = p^{1/9}\ \omega^3$,
$\alpha_9 = p^{1/9}\ \omega^6$, then

$$\Delta = (p^{1/9})^3 \begin{vmatrix} 1 & \omega^8 & \omega^7 \\ \omega & \omega^5 & \omega^4 \\ \omega^2 & \omega^3 & \omega^6 \end{vmatrix}$$

$$= p^{1/3}\ (2\omega^2 - \omega^7 - \omega^6) \neq 0.$$

78. Statement-2 is true.

Using $R_2 \to R_2 - \sqrt{2}\ R_1$ and $R_3 \to R_3 - \sqrt{3}\ R_1$, we get

$$w = \begin{vmatrix} \sqrt{6} & 2i & 3+\sqrt{6}i \\ 0 & \sqrt{3} & (\sqrt{6}-2\sqrt{3})i \\ 0 & \sqrt{2} & (2-3\sqrt{2})i \end{vmatrix}$$

$$= \sqrt{6}\left[2\sqrt{3} - 3\sqrt{6} - 2\sqrt{3} + 2\sqrt{6}\right]i = -6i$$

79. We first show statement-2 is true.

$$P(x) = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

$$\Rightarrow\quad P'(x) = \binom{n}{1} + 2\binom{n}{2}x + \ldots + n\binom{n}{n}x^{n-1}$$

$\Rightarrow\quad P'(0) = \dbinom{n}{1}$ = coefficient of x in the expansion of $P(x)$

$\therefore\quad$ Statement-2 is true.

Note that $\Delta(x)$ consists of 6 terms of the form $(1 + x)^n$.

Thus, coefficients of x in $\Delta(x) = \Delta'(0)$

$$\text{But } \Delta'(0) = \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$+\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{vmatrix} = 0$$

80. Statement-2 is not always true. For instance, the system of equations

$$x + 2y + 3z = 1$$
$$2x + 3y + 4z = 2$$
$$3x + 4y + 5z = 4$$

has no solution but $|A| = 0$

For statement-1, let

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$, we obtain

$$\Delta = (a + b + c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$

Also, note that $x = y = z$ satisfies each of the three equations.

Thus, the system of equations has infinite number of solutions.

81. Let

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$
$$= -\frac{1}{2}(a+b+c)\left[(b-c)^2+(c-a)^2+(a-b)^2\right]$$

If $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = bc + ca + ab$, then $(b - c)^2 + (c - a)^2 + (a - b)^2 = 0$

$\Rightarrow \quad a = b = c \neq 0 \qquad [\because\ a + b + c \neq 0]$

Thus, the system of equation reduces to $x + y + z = 0$ which is satisfied by infinite number of solutions.

$\therefore$ Statement-1 is true.

If $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq bc + ca + ab$, then at least one following is true:

$a^2 \neq bc, \quad b^2 \neq ca, \quad c^2 \neq ab.$

Suppose $b^2 - ac \neq 0$

Write first two equations as:

$$ax + by = (a + b)z$$
$$bx + cy = (b + c)z$$

Eliminating y, we get

$$(ac - b^2)x = (ac - b^2)z$$
$$\Rightarrow \quad x = z$$

Thus, from (1) equation, we get

$$by = bz \quad \Rightarrow \quad y = z$$
$$\therefore \quad x = y = z$$

Therefore, statement-2 is false.

82. If (a_1, b_1), (a_2, b_2) (a_3, b_3) are non-collinear, then $\Delta_1 \neq 0$. Therefore statement-2 is true.

We can write Δ_2 as

$$\Delta_2 = \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}\begin{vmatrix} b_1 & a_1 & 0 \\ b_2 & a_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix} = 0$$

for all values of $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbf{R}$.

83. As $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

$$= (a + ib)\,(a - ib) + (c - id)\,(c + id)$$
$$= a^2 + b^2 + c^2 + d^2$$

we get statement-2 is true.

Now,

$$\left(a_1^2 + b_1^2 + c_1^2 + d_1^2\right)\left(a_2^2 + b_2^2 + c_2^2 + d_2^2\right)$$
$$= \begin{vmatrix} a_1+ib_1 & c_1+id_1 \\ -c_1+id_1 & a_1-ib_1 \end{vmatrix}\begin{vmatrix} a_2+ib_2 & c_2+id_2 \\ -c_2+id_2 & a_2-ib_2 \end{vmatrix}$$
$$= \begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} = a^2 + b^2 + c^2 + d^2$$

where $a = a_1a_2 - b_1b_2 + c_1c_2 - d_1d_2$,
$b = a_2b_1 + a_1b_2 + c_1d_2 + c_2d_1$,
$c = -\,a_2c_1 + b_2d_1 - b_2c_1 + a_2d_1$,
$d = -\,b_2c_1 + a_2d_1 - b_1c_2 - b_1d_2$

Level 2

84. We have

$$\Delta^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$
$$= \begin{vmatrix} a_1 & b_{21} & b_{31} \\ b_{12} & a_2 & b_{32} \\ b_{13} & b_{23} & a_3 \end{vmatrix}$$

where $a_k = l_k^2 + m_k^2 + n_k^2 = 1$ for $k = 1, 2, 3$

and $b_{ij} = l_i\,l_j + m_im_j + n_in_j = 0\ \forall\ i \neq j$

Thus, $\Delta^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow |\Delta| = 1.$

85. Using $R_1 \to aR_1$, $R_2 \to bR_2$, $R_3 \to cR_3$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ a^2bc^2 & abc & bc+ab \\ a^2b^2c & abc & ac+bc \end{vmatrix}$$
$$= \frac{a^2b^2c^2}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix}$$

using $C_3 \to C_3 + C_1$, we get C_2 and C_3 are propotional. Thus, $\Delta = 0$

86. Using $C_1 \to C_1 - (x^2/3)C_3$, $C_2 \to C_2 - (2x/3)\, C_3$, we get

$$\Delta(x) = \begin{vmatrix} -5x+3 & -5 & 3 \\ x+9 & 1 & 9 \\ -6x+9 & -6 & 21 \end{vmatrix}$$

Using $C_1 \to C_1 - xC_2$, we get

$$\Delta(x) = \begin{vmatrix} 3 & -5 & 3 \\ 9 & 1 & 9 \\ 9 & -6 & 21 \end{vmatrix} = a \text{ constant.}$$

Thus, $a = 0$.

Alternate Solution

Replacing x by $1/x$, we get

$$\begin{vmatrix} \dfrac{1}{x^2} & -5\left(\dfrac{1}{x}\right)+3 & 2\left(\dfrac{1}{x}\right)-5 & 3 \\ 3\left(\dfrac{1}{x^2}\right) & +\dfrac{1}{x}+9 & 6\left(\dfrac{1}{x}\right)+1 & 9 \\ 7\left(\dfrac{1}{x^2}\right) & -6\left(\dfrac{1}{x}\right)+9 & 14\left(\dfrac{1}{x}\right)-6 & 21 \end{vmatrix}$$

$$= a\left(\frac{1}{x^3}\right)+b\left(\frac{1}{x^2}\right)+c\left(\frac{1}{x}\right)+d$$

$$\Rightarrow \begin{vmatrix} 1-5x+3x^2 & 2-5x & 3 \\ 3+x+9x^2 & 6+x & 9 \\ 7-6x+9x^2 & 14-6x & 21 \end{vmatrix} = a + bx + cx^2 + dx^3$$

Putting $x = 0$, we get

$$a = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 7 & 14 & 21 \end{vmatrix} = 0$$

[$\because$ C_1 and C_2 are proportional]

87. The given system of equations will have a non-trivial solution if

$$\Delta = \begin{vmatrix} a+t & b & c \\ a & b+t & c \\ a & b & c+t \end{vmatrix} = 0$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (a + b + c + t) \begin{vmatrix} 1 & b & c \\ 1 & b+t & c \\ 1 & b & c+t \end{vmatrix} = 0$$

Using $C_2 \to C_2 - bC_1$, $C_3 \to C_3 - cC_1$, we get

$\Delta = (a + b + c + t)t^2 = 0 \Rightarrow t = 0, -(a + b + c)$

Thus, there are just two distinct values of t.

88. First multiply C_1 by a, C_2 by b, C_3 by c, followed by multiplying R_1 by $1/a$, R_2 by $1/b$ and R_3 by $1/c$, we get

$$\Delta = \begin{vmatrix} a^2+(b^2+c^2)\cos\theta & b^2(1-\cos\theta) & c^2(1-\cos\theta) \\ a^2(1-\cos\theta) & b^2+(c^2+a^2)\cos\theta & c^2(1-\cos\theta) \\ a^2(1-\cos\theta) & b^2(1-\cos\theta) & c^2+(a^2+b^2)\cos\theta \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$ and $a^2 + b^2 + c^2 = 1$, we get

$$\Delta = \begin{vmatrix} 1 & b^2(1-\cos\theta) & c^2(1-\cos\theta) \\ 1 & b^2+(1-b^2)\cos\theta & c^2(1-\cos\theta) \\ 1 & b^2(1-\cos\theta) & c^2+(1-c^2)\cos\theta \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & b^2(1-\cos\theta) & c^2(1-\cos\theta) \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix} = \cos^2\theta$$

89. We have

$$\frac{dA}{d\theta} = \begin{vmatrix} -\sin(\alpha+\theta) & \sin(\alpha+\theta) & 1 \\ -\sin(\beta+\theta) & \sin(\beta+\theta) & 1 \\ -\sin(\gamma+\theta) & \sin(\gamma+\theta) & 1 \end{vmatrix}$$

$$+\begin{vmatrix} \cos(\alpha+\theta) & \cos(\alpha+\theta) & 1 \\ \cos(\beta+\theta) & \cos(\beta+\theta) & 1 \\ \cos(\gamma+\theta) & \cos(\gamma+\theta) & 1 \end{vmatrix}$$

$= 0 + 0 = 0$

$\Rightarrow A$ is independent θ. Thus, $A(\alpha, \beta, \gamma, \theta) = A(\alpha, \beta, \gamma, 0)$

$$\Rightarrow A\left(-\frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{2\pi}{13}\right) = A\left(-\frac{\pi}{2}, 0, \frac{\pi}{2}, 0\right)$$

$$= \begin{vmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 2$$

90. We have $\Delta = (a - b)(b - c)(c - a) = -2$.

As $a > b > c$, $a - b$, $b - c$ are positive integers and $c - a$ is a negative integers.

Only possibilities are

$a - b = 2$, $b - c = 1$, $c - a = -1$ (1)

or $a - b = 1$, $b - c = 2$, $c - a = -1$ (2)

or $a - b = 1$, $b - c = 1$, $c - a = -2$ (3)

(1) and (2) lead us to $0 = 2$.

$\therefore\ a - b = 1,\ b - c = 1,\ c - a = -2.$

Now, $3a + 7b - 10c = 3(a - c) + 7(b - c) = 13$

91. Write

$$\Delta = \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \begin{vmatrix} \cos P & -\sin P & 0 \\ \cos Q & -\sin Q & 0 \\ \cos R & -\sin R & 0 \end{vmatrix}$$

$= 0$

Alternate Solution

Write $\Delta = \cos P\ \Delta_1 - \sin P\ \Delta_2$

$$\text{where } \Delta_1 = \begin{vmatrix} \cos A & \cos(A+Q) & \cos(A+R) \\ \cos B & \cos(B+Q) & \cos(B+R) \\ \cos C & \cos(C+Q) & \cos(C+R) \end{vmatrix}$$

$$\text{and } \Delta_2 = \begin{vmatrix} \sin A & \cos(A+Q) & \cos(A+R) \\ \sin B & \cos(B+Q) & \cos(B+R) \\ \sin C & \cos(C+Q) & \cos(C+R) \end{vmatrix}$$

In Δ_1, use $C_2 \to C_2 - \cos Q\ C_1$

$C_3 \to C_3 - \cos R\ C_1$

$$\Delta_1 = \begin{vmatrix} \cos A & \sin A \sin Q & \sin A \sin R \\ \cos B & \sin B \sin Q & \sin B \sin R \\ \cos C & \sin C \sin Q & \sin C \sin R \end{vmatrix}$$

$= 0$ [$\because$ C_2 and C_3 are propotional]

Similarly, $\Delta_2 = 0$

Thus, $\Delta = 0$

92. Expanding along C_1, we get

$$f(x) = 2\cos x \begin{vmatrix} 2\cos x & 1 \\ 1 & 2\cos x \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & 2\cos x \end{vmatrix}$$

$= 8\cos^3 x - 4\cos x = 4\cos x \cos 2x$

$$\Rightarrow f\left(\frac{\pi}{3}\right) = -1,\ f\left(\frac{\pi}{2}\right) = 0.$$

$f'(x) = -4[\sin x \cos 2x + 2\cos x \sin 2x]$

$= -4[\sin 3x + \cos x \sin 2x]$

$$\Rightarrow f'\left(\frac{\pi}{3}\right) = -4\left[\sin \pi + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{2\pi}{3}\right)\right]$$

$= -\sqrt{3}$

93. For each $\Delta \in B$, there exist $\Delta_1 \in C$ where Δ_1 is obtained by interchanging 1st and 2nd row of Δ, similarly, for each $\Delta \in C$ there exists $\Delta_1 \in B$.

$\therefore$ B and C has same number of elements.

94. $$|a|^2 = |x|^2 + |y|^2 + |z|^2 + \bar{x}y + x\bar{y} + x\bar{z} + \bar{x}z + y\bar{z} + \bar{y}z$$

$\therefore\ |a|^2 + |b|^2 + |c|^2 = 3(|x|^2 + |y|^2 + |z|^2) +$

$(\bar{x}y + x\bar{y} + x\bar{z} + \bar{x}z + y\bar{z} + \bar{y}z)(1 + \omega + \omega^2)$

$$\Rightarrow \frac{|a|^2 + |b|^2 + |c|^2}{|x|^2 + |y|^2 + |z|^2} = 3$$

95. Write the determinant as $= abc\ \Delta_1 - \Delta_2$

$$\text{where } \Delta_1 = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$$

$$\text{and } \Delta_2 = \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \Delta_1$$

$\therefore\ (a - b)(b - c)(c - a)(abc - 1) = 0$

Since, a, b, c are distinct, we get

$abc - 1 = 0$ or $abc = 1$

96. $$|P|^2 = |\text{adj } P| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix} = 4$$

$\Rightarrow |P| = \pm 2$

97. Using $R_2 \to R_2 + 5R_1$, $R_3 \to R_3 - 4R_1$, we get

$$\begin{vmatrix} 1 & -3 & 4 \\ 0 & x-13 & 22 \\ 0 & 13 & x-22 \end{vmatrix} = 0$$

$\Rightarrow (x - 13)(x - 22) - (13)(22) = 0$

$\Rightarrow x(x - 35) = 0 \Rightarrow x = 0, 35$

Previous Years' AIEEE/JEE Main Questions

1. As $ax^2 + 2bx + c = 0$ has equal roots,

$b^2 - ac = 0$

Using $R_3 \to R_3 - xR_1 - R_2$, we get

$$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ 0 & 0 & \alpha \end{vmatrix}$$

where $\alpha = 0 - x(ax + b) - (bx + c)$

$= -(ax^2 + 2bx + c)$

$\therefore \quad \Delta = \alpha(ac - b^2) = 0$

2. As l, m, n are pth, qth, rth terms of a G.P., log l, log m, log n are pth, qth, rth terms of an A.P. Let first term of the A.P. be a and its common difference be d.

   Now,

   $$\Delta = \begin{vmatrix} a+(p-1)d & p & 1 \\ a+(q-1)d & q & 1 \\ a+(r-1)d & r & 1 \end{vmatrix}$$

   Using $C_1 \to C_1 - dC_2 - (a - d)C_3$, we get

   $$\Delta = \begin{vmatrix} 0 & p & 1 \\ 0 & q & 1 \\ 0 & r & 1 \end{vmatrix} = 0$$

3. If n is a multiple of 3, we get each element of Δ becomes 1.

   $\therefore \quad \Delta = 0$

   If $n = 3k + 1$, then

   $$\Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0 \quad [\text{use } C_1 \to C_1 + C_2 + C_3]$$

   If $n = 3k + 2$, then

   $$\Delta = \begin{vmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{vmatrix} = 0 \quad [\text{use } C_1 \to C_1 + C_2 + C_3]$$

4. As the system has a non-zero solution

   $$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

   $$\begin{vmatrix} 1/a & 2 & 1 \\ 1/b & 3 & 1 \\ 1/c & 4 & 1 \end{vmatrix} = 0$$

   Using $R_1 \to R_1 - 2R_2 + R_3$, we get

   $$\Rightarrow \begin{vmatrix} 1/a - 2/b + 1/c & 0 & 0 \\ 1/b & 3 & 1 \\ 1/c & 4 & 1 \end{vmatrix} = 0$$

   $$\frac{1}{a} - \frac{2}{b} + \frac{1}{c} = 0$$

   $\Rightarrow \quad a, c, b$ are in H.P.

5. Let R be the common ratio of the G.P. Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$, we get that the given determinant

   $$= \begin{vmatrix} \log a_n & \log R & \log R \\ \log a_{n+1} & \log R & \log R \\ \log a_{n+2} & \log R & \log R \end{vmatrix} = 0$$

6. Applying $C_1 \to C_1 + C_2 + C_3$, we get

   $$f(x) = \begin{vmatrix} 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & (1+c^2)x \\ 1+2x+(a^2+b^2+c^2)x & 1+b^2x & (1+c^2)x \\ 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

   $$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$$

   Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

   $$f(x) = \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix}$$

   $= (1 - x)^2$

   which is a polynomial of degree 2.

7. If A denotes the coefficient matrix, then

   $$|A| = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = (\alpha+2)\begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix}$$

   [using $C_1 \to C_1 + C_2 + C_3$]

   $$= (\alpha+2)\begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & 0 \\ 0 & 0 & \alpha-1 \end{vmatrix}$$

   $= (\alpha + 2)(\alpha - 1)^2$

   If $|A| \neq 0$, the system has a unique solution.

   If $|A| = 0$, then $\alpha = -2$ or $\alpha = 1$. For $\alpha = 1$, the system of equations becomes $x + y + z = 0$ which has infinite number of solutions.

   For $\alpha = -2$, the system of equation becomes

   $$-2x + y + z = -3$$
   $$x - 2y + z = -3$$

$$x + y - 2z = -3$$

which on adding becomes $0 = -9$

Thus, the system has no solution if $\alpha = -2$.

8. $$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 0 & x & 0 \\ 0 & 0 & y \end{vmatrix} = xy$$

which is divisible by both x and y.

9. Let $$\Delta_2 = \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix}$$

Using the reflection property, we get

$$\Delta_2 = \begin{vmatrix} a+1 & a-1 & (-1)^{n+2}a \\ b+1 & b-1 & (-1)^{n+1}b \\ c-1 & c+1 & (-1)^n c \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} (-1)^{n+2}a & a+1 & a-1 \\ (-1)^{n+1}b & b+1 & b-1 \\ (-1)^n c & c-1 & c+1 \end{vmatrix}$$

Thus,

$$\Delta = \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + \begin{vmatrix} (-1)^{n+2}a & a+1 & a-1 \\ (-1)^{n+1}b & b+1 & b-1 \\ (-1)^n c & c-1 & c+1 \end{vmatrix}$$

$$= \begin{vmatrix} a+(-1)^{n+2}a & a+1 & a-1 \\ -b+(-1)^{n+1}b & b+1 & b-1 \\ c+(-1)^n c & c-1 & c+1 \end{vmatrix}$$

The first column consists of all 0's if n is any odd integer.

$\therefore\ \ \Delta = 0$ if n is any odd integer.

10. Using $C_1 \to C_1 + C_2 + C_3$, we get
$\Delta = (a + b + c)\,\Delta_1$
where

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$$

$= -(b - c)^2 - (a - b)(a - c)$

$= -(a^2 + b^2 + c^2 - bc - ca - ab)$

$$\Rightarrow \quad \Delta_1 = -\frac{1}{2}[(b-c)^2 + (c-a)^2 + (a-b)^2] < 0$$

As $a + b + c > 0$, we get

$\Delta = (a + b + c)\,\Delta_1 < 0$

11. Using $C_1 \to C_1 - C_3$ in Δ, we get

$$\Delta = \begin{vmatrix} 0 & \sin\alpha & \cos\alpha \\ 0 & \cos\alpha & \sin\alpha \\ 2\cos\alpha & -\sin\alpha & -\cos\alpha \end{vmatrix}$$

$= 2\cos\alpha\,(\sin^2\alpha - \cos^2\alpha)$

$= -2\cos\alpha\cos 2\alpha$

$\Delta = 0$ for $\alpha = \pi/4 \in (0, \pi/2)$.

This is the only value of α lying in $(0, \pi/2)$ for which $\Delta = 0$.

The system of linear equation will have a non-trivial solution if and only if

$$\Delta_1 = \begin{vmatrix} 1 & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

Using $R_2 \to R_2 + R_1$, we get

$$\Delta_1 = \begin{vmatrix} 2 & 0 & 0 \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$\Rightarrow\ \ 2(-\cos^2\alpha + \sin^2\alpha) = 0$
$\Rightarrow\ \ -2\cos 2\alpha = 0$

This is true for only one value of $\alpha \in (0, \pi/2)$ viz, $\alpha = \pi/4$.

Thus, statement-1 is also true. However statement-2 is not a correct reason for statement-1.

12. $$\Delta = \begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} = \Delta_1^2$$

where $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$

Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \alpha - 1 & \beta - \alpha \\ 1 & \alpha^2 - 1 & \beta^2 - \alpha^2 \end{vmatrix}$$

$$= (\alpha - 1)(\beta - \alpha) \begin{vmatrix} 1 & 1 \\ \alpha + 1 & \beta + \alpha \end{vmatrix}$$

$$= (\alpha - 1)\ (\beta - 1)\ (\beta - \alpha)$$

Thus, $\Delta = (\alpha - 1)^2\ (\beta - 1)^2 (\alpha - \beta)^2$

$\therefore\ K = 1$

13. As the given system of equations has a non-trivial solutions.

$$\Delta = \begin{vmatrix} a - 1 & -1 & -1 \\ -1 & b - 1 & -1 \\ -1 & -1 & c - 1 \end{vmatrix} = 0$$

Write

$$\Delta = \begin{vmatrix} a & -1 & -1 \\ 0 & b - 1 & -1 \\ 0 & -1 & c - 1 \end{vmatrix} - \begin{vmatrix} 1 & -1 & -1 \\ 1 & b - 1 & -1 \\ 1 & -1 & c - 1 \end{vmatrix}$$

$$= a[(b - 1)(c - 1) - 1] - \begin{vmatrix} 1 & 0 & 0 \\ 1 & b & 0 \\ 1 & 0 & c \end{vmatrix}$$

[use $C_2 \to C_2 + C_1$, C_3 — $C_3 + C_1$]

$= a\ (bc - b - c) - bc$

As $\Delta = 0$, we get

$ab + bc + ca = abc$

14. We have $|B(x)| = |(A(x))^T A(x)|$

$= |A(x)^T|\ |A(x)|$

$= |A(x)|\ |A(x)| = |A(x)|^2$

Let $p_i(x) = a_i x^2 + b_i x + c_i\ \ (i = 1, 2, 3)$

Applying $C_1 \to C_1 - \frac{1}{2}x^2 C_3, C_2 \to C_2 - xC_3$.

we get

$$|A(x)| = 2 \begin{vmatrix} b_1 x + c_1 & b_1 & a_1 \\ b_2 x + c_2 & b_2 & a_2 \\ b_3 + c_3 & b_3 & a_3 \end{vmatrix}$$

Applying $C_1 \to C_1 - xC_2$, we get

$$|A(x)| = 2 \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$$

$\therefore\ |B(x)|$ is independent of x.

15. Using $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} a^2 & b^2 & c^2 \\ 2a\lambda + \lambda^2 & 2b\lambda + \lambda^2 & 2c\lambda + \lambda^2 \\ -4a\lambda & -4b\lambda & -4c\lambda \end{vmatrix}$$

Take -4λ common from R_3, and applying $R_2 \to R_2 - 2\lambda R_3$, we get

$$\Delta = -4\lambda^3 \begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix}$$

$$= \lambda(4\lambda^2) \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

$\therefore\ k = 4\lambda^2$

16. As $\sum_{r=1}^{n-1} r = \frac{1}{2}n(n-1), \sum_{r=1}^{n-1} (2r - 1) = (n - 1)^2$

and $\sum_{r=1}^{n-1} (3r - 2) = \frac{3}{2}n(n-1) - 2(n-1)$

$$= \frac{1}{2}(n-1)(3n-4)$$

Thus,

$$\sum_{r=1}^{n-1} \Delta_r = \begin{vmatrix} n(n-1)/2 & (n-1)^2 & (n-1)(3n-4)/2 \\ n/2 & n-1 & a \\ n(n-1)/2 & (n-1)^2 & (n-1)(3n+4)/2 \end{vmatrix}$$

Using $R_3 \to R_3 - R_1$, we get

$$\sum_{r=1}^{n-1} \Delta_r = \begin{vmatrix} n(n-1)/2 & (n-1)^2 & (n-1)(3n-4)/2 \\ n/2 & n-1 & a \\ 0 & 0 & 4(n-1) \end{vmatrix}$$

Expanding along R_3, we get

$$\sum_{r=1}^{n-1}\Delta_r = 4(n-1)\begin{vmatrix} n(n-1)/2 & (n-1)^2 \\ n/2 & n-1 \end{vmatrix}$$

$$= 4(n-1)\left(\frac{n}{2}\right)(n-1)\begin{vmatrix} n-1 & n-1 \\ 1 & 1 \end{vmatrix} = 0$$

17. The system will have a non-trivial solution if

$$\Delta = \begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

Using $R_1 \to R_1 + R_3$, $R_2 \to R_2 + 2R_3$,

$$\Delta = \begin{vmatrix} 1-\lambda & 0 & 1-\lambda \\ 0 & 1-\lambda & 2-2\lambda \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)^2\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

Using $C_3 \to C_3 - C_1$, we get

$$(1-\lambda)^2\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 2 & -\lambda+1 \end{vmatrix} = 0$$

$\Rightarrow (1 - \lambda)^2 [- \lambda + 1 - 4] = 0 \Rightarrow \lambda = 1, - 3$

Thus, the set contains two elements

18. Using $R_2 \to R_2 - R_1 - R_3$, we get

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ -4 & 0 & 0 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

Expanding along R_2, we get

$$\Delta = -(-4)\begin{vmatrix} x+1 & x-2 \\ 2x-1 & 2x-1 \end{vmatrix}$$

Using $C_2 \to C_2 - C_1$, we get

$$\Delta = 4\begin{vmatrix} x+1 & -3 \\ 2x-1 & 0 \end{vmatrix}$$

$= 4(3)\ (2x - 1)$

$= 24x - 12$

$\therefore\ a = 24$

TIP : Put $x = 1$ to obtain

$$\begin{vmatrix} 2 & 2 & -1 \\ 4 & 3 & 0 \\ 6 & 1 & 1 \end{vmatrix} = a - 12$$

$\Rightarrow 12 = a - 12$

$\Rightarrow a = 24$

19. Let x be any real number and $y = 1$, $z = 1$, then

$$\Delta = \begin{vmatrix} x & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \geq 0$$

and $xyz = x$ can take any real value, thus, minimum value of xyz does not exist.

20. The given system of equations has a non-trivial solution if

$$\begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda - \lambda - \lambda - 1 + 1 + \lambda^3 = 0$

$\Rightarrow \lambda\ (\lambda - 1)\ (\lambda + 1) = 0$

$\Rightarrow \lambda = 0, -1, 1$

When $\lambda = 0$, $x = -y = z = k \neq 0$

When $\lambda = -1$, $x = 0$, $y = -z = k \neq 0$

When $\lambda = 1$, $y = 0$, $x = -z = k \neq 0$

Thus, there are three values of λ, for which the system of equations has a non-trivial solutions.

21. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (2\sin x + \cos x)\begin{vmatrix} 1 & \sin x & \sin x \\ 1 & \cos x & \sin x \\ 1 & \sin x & \cos x \end{vmatrix}$$

Using $C_2 \to C_2 - (\sin x)\ C_1$

and $C_3 \to C_3 - (\sin x)\ C_1$,

we get

$$\Delta = (2\sin x + \cos x)\begin{vmatrix} 1 & 0 & 0 \\ 1 & \cos x - \sin x & 0 \\ 1 & 0 & \cos x - \sin x \end{vmatrix}$$

$= (2\sin x + \cos x)\ (\cos x - \sin x)^2$

$\therefore\ \Delta = 0 \Rightarrow \tan x = -\dfrac{1}{2}, 1$

As $-\pi/4 \leq x \leq \pi/4$, $-1 \leq \tan x \leq 1$

and $\tan x$ is one-to-one in the interval $[-\pi/4, \pi/4]$.

Thus, there are two values of x.

Previous Years' B-Architecture Entrance Examination Questions

1. From first and third equation

$$(b-1)x = 0$$

If $b \neq 1$, $x = 0$.

In this case $y + z = 0$, $by + z = 0$

$\Rightarrow \quad (b-1)y = 0 \quad \Rightarrow y = 0$

Thus, $z = 0$

That is, if $b \neq 1$, then the system has only trivial solution.

$\therefore \quad b = 1$

From first and second equation,

$(a-1)\,x = 0$.

If $a \neq 1$, $x = 0$ and $y + z = 0$

In this case the system has infinite number of solution.

If $a = 1$, all the three equations become identical to $x + y + z = 0$ and the system has infinite number of solutions.

$b = 1 \Rightarrow (b-1)^2 = 0 \quad \Rightarrow b^2 = 2b - 1$

2. As the system has a non-trivial solution,

$$\Delta = \begin{vmatrix} \lambda+3 & \lambda+2 & 1 \\ 3 & \lambda+3 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

Using $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} \lambda+1 & \lambda-1 & 0 \\ 1 & \lambda & 0 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

$\Rightarrow \Delta = \lambda(\lambda + 1) - (\lambda - 1) = 0$

$\Rightarrow (\lambda - 1)^2 = 0 \quad \Rightarrow \lambda = 1$

For $\lambda = 1$, the system becomes

$$4x + 3y + z = 0 \qquad \text{(i)}$$

$$3x + 4y + z = 0 \qquad \text{(ii)}$$

$$2x + 3y + z = 0 \qquad \text{(iii)}$$

From (i) and (iii) $x = 0$

From (i) and (ii) $3y + z = 0$

and $\quad 4y + z = 0$

$\Rightarrow \quad y = 0, z = 0$

Thus, the system has non-trivial solution for no value of λ.

3. $$\sum_{k=1}^{2} \Delta_k = \begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ 3^n - 1 & 4^n - 1 & 5^n - 1 \end{vmatrix}$$

where $a = \sum_{k=1}^{n} 2(3^{k-1})$

$$= \frac{2(3^n - 1)}{3 - 1} = 3^n - 1,$$

$$b = \sum_{k=1}^{n} 3(4^{k-1}) = \frac{3(4^n - 1)}{4 - 1} = 4^n - 1$$

and $c = \sum_{k=1}^{n} 4(5^{k-1}) = \dfrac{4(5^n - 1)}{5 - 1} = 5^n - 1$

Thus, $\sum_{k=1}^{n} \Delta_k = \begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ a & b & c \end{vmatrix} = 0$

4. $$\det(B) = \begin{vmatrix} a_{11} & 2^{-1} a_{12} & 2^{-2} a_{13} \\ 2a_{21} & a_{22} & 2^{-1} a_{23} \\ 2^2 a_{31} & 2a_{32} & a_{33} \end{vmatrix}$$

$$= (2^{-2})(2^{-1}) \begin{vmatrix} 2^2 a_{11} & 2a_{12} & a_{13} \\ 2^2 a_{21} & 2a_{22} & a_{23} \\ 2^2 a_{31} & 2a_{32} & a_{33} \end{vmatrix}$$

$$= (2^{-3})(2^2)(2) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$= \det(A) = 5$

5. $$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$$

where $\quad a = \sum_{r=1}^{n} 2^{r-1} = 2^n - 1$

$$b = \sum_{r=1}^{n} 2(3^{r-1}) = \frac{2(3^n - 1)}{3 - 1}$$

$= 3^n - 1$

and $\quad c = \sum_{r=1}^{n} 4(5^{r-1}) = \dfrac{4(5^n - 1)}{5 - 1}$

$= 5^n - 1$

Thus, $$\sum_{r=1}^{n}\Delta_r = \begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ a & b & c \end{vmatrix} = 0$$

6. Taking e^{x_1} common from R_1, e^{x_4} common from R_2 and e^{x_7} from R_3, we get

$\Delta = e^{x_1 + x_4 + x_7}\Delta_1$

where

$$\Delta_1 = \begin{vmatrix} 1 & e^{3d} & e^{6d} \\ 1 & e^{3d} & e^{6d} \\ 1 & e^{3d} & e^{6d} \end{vmatrix} = 0$$

where d is common difference of A.P.

$\therefore\ \Delta = 0$

7. Taking $\sqrt{5}$ common from C_2 and C_3, we get

$\Delta = 5\Delta_1$

where $$\Delta_1 = \begin{vmatrix} \sqrt{13}+\sqrt{3} & 2 & 1 \\ \sqrt{15}+\sqrt{26} & \sqrt{5} & \sqrt{2} \\ 3+\sqrt{65} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

Applying, $C_1 \to C_1 - \sqrt{3}C_2 - \sqrt{13}C_3$, we get

$$\Delta_1 = \begin{vmatrix} -\sqrt{3} & 2 & 1 \\ 0 & \sqrt{5} & \sqrt{2} \\ 0 & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$$= -\sqrt{3}(5-\sqrt{6})$$

Thus, $\Delta = 5\sqrt{3}(\sqrt{6}-5)$

8. As the given system has a non-zero solution

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

$$\Rightarrow abc\begin{vmatrix} 1/a & 2 & 1 \\ 1/b & 3 & 1 \\ 1/c & 4 & 1 \end{vmatrix} = 0$$

Using $R_1 \to R_1 - 2R_2 + R_3$, we get

$$abc\begin{vmatrix} \frac{1}{a}-\frac{2}{b}+\frac{1}{c} & 0 & 0 \\ 1/b & 3 & 1 \\ 1/c & 4 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad abc\left(\frac{1}{a}-\frac{2}{b}+\frac{1}{c}\right)(-1) = 0$$

$\Rightarrow\quad 2ac = ab + bc$

9. Using $R_1 \to R_1 - R_2$, $R_3 \to R_3 - R_2$, we get

$$\begin{vmatrix} 0 & a-c & b-a \\ 1 & c & a \\ 0 & b-c & c-a \end{vmatrix} = 0$$

$\Rightarrow\quad (a-c)^2 + (b-a)(b-c) = 0$

$\Rightarrow\quad a^2 - 2ac + c^2 + b^2 - ab - bc + ac = 0$

$\Rightarrow\quad a^2 + b^2 + c^2 - bc - ca - ab = 0$

$\Rightarrow\quad (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$

$\Rightarrow\quad a = b = c \quad \Rightarrow A = B = C$

$\Rightarrow\quad \sin A = \sin B = \sin C = \sqrt{3}/2$

Thus, $\sin^2 A + \sin^2 B + \sin^2 C = 9/4$

10. From the first two equations $x = 2$.

$\therefore\quad y - z = 1$ and $-4y + 4z = \alpha - 2$

$\Rightarrow\quad 4y - 4z + (-4y + 4z) = 4 + (\alpha - 2)$

$\Rightarrow\quad 0 = \alpha + 2$

Thus, the system has no solution for $\alpha \neq -2$.

If $\alpha = -2$, the system is consistent and has infinite number of solutions, since $y - z = 1$ has infinite number of solutions.

11. Using $C_1 \to C_1 + C_3$, we get

$$\Delta = \begin{vmatrix} 1 & \tan\theta + \sec^2\theta & 3 \\ 0 & \cos\theta & \sin\theta \\ 0 & -4 & 3 \end{vmatrix}$$

$= 3\cos\theta + 4\sin\theta$

$$\frac{d\Delta}{d\theta} = -3\cos\theta + 4\sin\theta$$

$$\frac{d\Delta}{d\theta} = 0 \Rightarrow \frac{\sin\theta}{4} = \frac{\cos\theta}{3} = \frac{1}{5}$$

$\Rightarrow \sin\theta = 4/5$, $\cos\theta = 3/5$

Max $\Delta = \max\{\Delta(0), \Delta(\sin^{-1}(4/5)), \Delta(1)\}$

$$= \max\left\{3, \frac{24}{5}, 4\right\} = \frac{24}{5}$$

min $\Delta = 3$.

$\therefore\ \Delta \in [3, 5]$

CHAPTER FIVE

Matrices

THE ALGEBRA OF MATRICES

A *matrix* is a rectangular array of numbers. The numbers may be real or complex. It may be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

or as $A = (a_{ij})_{m \times n}$.

A matrix with m rows and n columns is called as $m \times n$ matrix and the *size* (or *dimension*) of this matrix is said be $m \times n$.

Two matrices are said to be *equal* provided they are of the same size and corresponding elements are equal. For example

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 11 \end{bmatrix}$$

if and only if

$a = -1, \quad b = 2, \quad c = 5, \quad d = 7, \quad e = 3$ and $f = 11$.

Definitions A matrix $A = (a_{ij})_{m \times n}$ is said to be a

(i)	*square matrix*	if $m = n$
(ii)	*row matrix*	if $m = 1$
(iii)	*column matrix*	if $n = 1$
(iv)	*null or zero matrix*	if $a_{ij} = 0 \ \forall\ i$ and j
(v)	*diagonal matrix*	if $m = n$ and $a_{ij} = 0 \ \forall\ i \neq j$
(vi)	*Scalar matrix*	if $m = n$ and $a_{ij} = 0 \ \forall\ i \neq j$ and $a_{ii} = \lambda \ \forall i$
(vii)	*Unit or identity matrix*	if $m = n$ and $a_{ij} = 0 \ \forall\ i \neq j$ and $a_{ii} = 1 \ \forall\ i$
(viii)	*Upper triangular matrix*	if $m = n$ and $a_{ij} = 0 \ \forall\ i > j$
(ix)	*Lower triangular matrix*	if $m = n$ and $a_{ij} = 0 \ \forall\ i < j$

A matrix is said to be triangular if it is either lower or upper triangular matrix.

Addition of Matrices Two matrices A and B can be added if and only if they are of the same size. If two matrices are of the same size, addition is carried out term by term. For instance

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} + \begin{bmatrix} p & q & r \\ s & t & u \end{bmatrix} = \begin{bmatrix} a+p & b+q & c+r \\ d+s & e+t & f+u \end{bmatrix}$$

Addition is *not defined* for matrices of different sizes.

The additive inverse of a matrix A, denoted by $-A$, is the matrix whose elements are the negatives of the corresponding elements of A. For example,

$$-\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \\ -e & -f \end{bmatrix}$$

If A and B are two matrices of the same size, then the *differences* between A and B is defined by

$$A - B = A + (-B)$$

Thus, the subtraction is carried out term-by-term. For instance

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - \begin{bmatrix} p & q & r \\ s & t & u \end{bmatrix} = \begin{bmatrix} a-p & b-q & c-r \\ d-s & e-t & f-u \end{bmatrix}$$

Properties of Addition If A, B and C are three matrices of the same size, then

$A + B = B + A$ [Commutative law]

$(A + B) + C = A + (B + C)$ [Associative law]

$A + O = O + A$ [Additive property of zero]

$A + (-A) = O$

where O is the null or zero matrix of the same size as that of A.

Scalar Multiplication If A is a matrix and α is a scalar, then αA is defined as the matrix obtained by multiplying every element of A by α. For example

$$3\begin{bmatrix} 1 & -2 & 5 \\ 7 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -6 & 15 \\ 21 & 24 & 3 \end{bmatrix}$$

Properties of Scalar Multiplication If A and B are two matrices of the same size, and α, β are two scalars, then

$$(\alpha+\beta)A = \alpha A + \beta A$$
$$(\alpha\beta)A = \alpha(\beta A)$$
$$\alpha(A+B) = \alpha A + \alpha B$$

Matrix Multiplication Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{r\times s}$ be two matrices. We say that A and B are *comparable for the product AB* if $n = r$, that is, if the number of columns of A is equal to the number of rows of B.

Definition: Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{n\times p}$ be two matrices. Their product AB is the matrix $C = (c_{ij})_{m\times p}$ such that $c_{ij} = a_{i1}\,b_{1j} + a_{i2}\,b_{2j} + a_{i3}\,b_{3j} + \ldots + a_{in}\,b_{nj}$ for $1 \le i \le m$, $1 \le j \le p$. Note that c_{ij}, the (i, j)th element of AB, has been obtained by multiplying ith row of A, namely $(a_{i1}\;a_{i2}\;a_{i3}\;\ldots\;a_{in})$

with the jth column of B, namely $\begin{pmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ . \\ . \\ . \\ b_{nj} \end{pmatrix} = (b_{1j}\;b_{2j}\ldots b_{nj})'$

where A' denotes the transpose of matrix A.

Properties of Matrix Multiplication

If $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$ and $C = (c_{ij})_{p\times q}$ then

1. $(AB)C = A(BC)$ [Associative law]
2. $AI_n = I_mA = A$
3. AB may not be equal to BA
4. $k(AB) = (kA)B = A(kB)$ where k is a scalar
5. If A is a square matrix, then
   $A^m\,A^n = A^{m+n}\;\forall\; m, n \in \mathbf{N}$
   $(A^m)^n = A^{mn}\quad \forall\; m, n \in \mathbf{N}$.
6. If A is an invertible matrix then
   $(A^{-1}\,B\,A)^m = A^{-1}\,B^m\,A$
   and $A^{-m} = (A^{-1})^m\;\forall\; m \in \mathbf{N}$.

TRANSPOSE OF A MATRIX

Definition: Let $A = (a_{ij})_{m\times n}$ be a matrix. The transpose of A, denoted by A' or by A^t is the matrix $A' = (b_{ij})_{n\times m}$ where $b_{ij} = a_{ji}\;\forall\; i$ and j.

By $\bar{A}$ we mean a matrix $B = (b_{ij})_{m\times n}$ where $b_{ij} = \bar{a}_{ij}$ where $\bar{a}$ denotes the conjugate of a. By A^* we mean

$$A^* = (\bar{A})' = (\bar{A}')$$

Properties of Transpose of Matrix

1. $(A+B)' = A' + B'$
2. $(kA)' = kA'$ where k is a scalar
3. $(AB)' = B'A'$ [Reversal law]
4. If A is an invertible matrix, then $(A^{-1})' = (A')^{-1}$.

ADJOINT AND INVERSE OF A MATRIX

Let $A = (a_{ij})_{n\times n}$ be square matrix. The adjoint of A is defined to be the matrix adj $A = (b_{ij})_{n\times n}$ where $b_{ij} = A_{ji}$ where A_{ji} is the cofactor of (j, i)th element of A.

Properties of Adjoint

1. $A(\text{adj } A) = (\text{adj } A)A = |A|I_n$
2. $\text{adj}(kA) = k^{n-1}\,(\text{adj } A)$
3. $\text{adj}(AB) = (\text{adj } B)\,(\text{adj } A)$

Definition: A square matrix A is said to be *singular* if $|A| = 0$ and *non-singular* if $|A| \ne 0$.

Definition: Inverse of a square matrix $A = (a_{ij})_{n\times n}$ is the matrix $B = (b_{ij})_{n\times n}$ such that

$$AB = BA = I_n.$$

In fact $\quad A^{-1} = \dfrac{1}{|A|}\,(\text{adj } A) \qquad \text{if } |A| \ne 0.$

Properties of Inverse

Let A and B be two invertible matrices of the same size.

1. Inverse of a matrix if it exists is unique.
2. $AA^{-1} = A^{-1}A = I_n$
3. $(A^{-1})^{-1} = A$
4. $(kA)^{-1} = k^{-1}\,A^{-1}$ if $k \ne 0$
5. $(AB)^{-1} = B^{-1}\,A^{-1}$ [Reversal law]
6. For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, adj $A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
   and $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ if $ad - bc \ne 0$
7. If A is a triangular matrix, then A^{-1}, if it exists is a triangular matrix of the same kind.
   In fact if $A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $a_{11}a_{22}a_{33} \ne 0$,
   then
   $$A^{-1} = \frac{1}{a_{11}a_{22}a_{33}}\begin{pmatrix} a_{22}a_{33} & 0 & 0 \\ -a_{21}a_{33} & a_{33}a_{11} & 0 \\ A_{13} & -a_{32}a_{11} & a_{11}a_{22} \end{pmatrix}$$
   where A_{13} = cofactor of (1, 3)th element in A i.e.
   $$A_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
8. If $A = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)$ then A^{-1} exists if and only if $\lambda_i \ne 0\;\forall\; i$ and
   $A^{-1} = \text{diag}\,(\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1})$
   Also, $A^m = \text{diag}\,(\lambda_1^m, \lambda_2^m \ldots \lambda^m_n)$ if $m \in \mathbf{N}$
9. If a square matrix A satisfies the equation $a_0 + a_1x + a_2\,x^2 + \cdots + a_r\,x^r = 0$, then A is invertible if $a_0 \ne 0$ and its inverse is given by

$$A^{-1} = -\frac{1}{a_0}\,[a_1 I + a_2 A + \cdots + a_r A^{r-1}]$$

10. If $AB = I$, then $BA = I$ and $B = A^{-1}$
11. If $AB = CA = I$, then $B = C$.

SOME DEFINITIONS AND RESULTS

A **square matrix** A is said to be

1. *Symmetric* if $A' = A$
2. *Skew-symmetric* if $A' = -A$
3. *Hermitian if* $A^* = A$
4. *Skew Hermitian* if $A^* = -A$
5. *Orthogonal* if $AA' = A'A = I$ or if $A' = A^{-1}$
6. *Unitary* if $AA^\theta = A^\theta A = I$

Illustration 1

1. $A = \begin{pmatrix} 2 & 5 & -1 \\ 5 & 7 & -2 \\ -1 & -2 & 0 \end{pmatrix}$ is a symmetric matrix
2. $A = \begin{pmatrix} 0 & 5 & 2 \\ -5 & 0 & 7 \\ -2 & -7 & 0 \end{pmatrix}$ is a skew symmetric matrix
3. $A = \begin{pmatrix} 3 & 2-i \\ 2+i & 7 \end{pmatrix}$ is a Hermitian matrix
4. $A = \begin{pmatrix} 2i & 6-5i \\ -6-5i & 4i \end{pmatrix}$ is a skew Hermitian matrix
5. $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is an orthogonal matrix
6. $A = \frac{1}{\sqrt{5}}\begin{pmatrix} 2i & 1 \\ 1 & 2i \end{pmatrix}$ is a unitary matrix.

TIP

If A is a skew symmetric matrix, the $I + A$ and $I - A$ are non-singular matrices.

Some Results

1. The main diagonal elements of a skew-symmetric matrix are zeros, i.e. $a_{ii} = 0 \ \forall\ i$.
2. Determinant of a skew-symmetric matrix of odd order is zero and determinant of a skew-symmetric matrix of even order is a perfect square.
3. Every square matrix can be written uniquely as a sum of symmetric and a skew-symmetric matrix, i.e. if A is a square matrix, then there exists a symmetric matrix P and a skew-symmetric matrix Q such that
$$A = P + Q$$
In fact
$$P = \frac{1}{2}(A + A') \text{ and } Q = \frac{1}{2}(A - A').$$
4. For every square matrix A, matrices $A + A'$, AA' and $A'A$ are symmetric and matrices $A - A'$ and $A' - A$ are skew-symmetric.

Results 1. If A is an orthogonal matrix, then $|A| \neq 0$. In fact $|A| = \pm 1$ and $\bar{A}$, A', A^* are also orthogonal.

2. If A and B are two orthogonal matrices, then AB and BA are both orthogonal matrices.

A square matrix A is said to be

1 *nilpotent* if $A^m = O$ for some $m \in \mathbf{N}$.
2. *idempotent* if $A^2 = A$
3. *involutory* if $A^2 = I$.

Illustration 2

1. $A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ is a nilpotent matrix as $A^2 = 0$
2. $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ is an idempotent matrix as $A^2 = A$
3. $A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$ is an involutory as $A^2 = I$.

Remark

❑ The least value of m for which $A^m = O$, is called the *index* of the nilpotent matrix.

Definition: If $A = (a_{ij})_{n \times n}$ is a square matrix, the trace of A denoted by tr (A) is sum of all the main diagonal elements, i.e.

$$\text{tr}(A) = a_{11} + a_{22} + \ldots + a_{nn}$$

Properties of Trace

Let A, B be two $n \times n$ matrices, and $k \in \mathbf{R}$.

1. $\text{tr}(kA) = k\,\text{tr}(A)$
2. $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$
3. $\text{tr}(AB) = \text{tr}(BA)$

RANK OF A MATRIX

If $A = (a_{ij})_{m \times n}$ is a matrix, and B is its submatrix of order r, then $|B|$, the determinant is called as *r-rowed minor of A*.

Definition Let $A = (a_{ij})_{m \times n}$ be a matrix. A positive integer r is said to be rank of A if

(i) A possesses at least one r-rowed minor which is different from zero; and
(ii) every $(r + 1)$ rowed minor of A is zero.

From (ii), it automatically follows that all minors of higher order are zeros.

We denote rank of A by $\rho(A)$

Result The rank of a matrix does not change when the following *elementary row operations* are applied to the matrix.

(a) Two rows are interchanged ($R_i \leftrightarrow R_j$)

(b) A row is multiplied by a non-zero constant, ($R_i \to kR_i$, with $k \neq 0$)

(c) a constant multiple of another row is added to a given row ($R_i \to R_i + kR_j$) where $i \neq j$

Note : The arrow → means "replaced by".

Note that the application of these elementary row operations does not change a singular matrix to a non-singular matrix or a non-singular matrix to a singular matrix. Therefore, the order of the largest non-singular square submatrix is not affected by application of any of the elementary row operations. Thus, the rank of a matrix does not change by application of any of the elementary row operations.

A matrix obtained from a given matrix by applying any of the elementary row operations is said to be equivalent to it. If A and B are two equivalent matrices, we write $A \sim B$.

Note that if $A \sim B$, then $\rho(A) = \rho(B)$.

By using the elementary row operations, we shall try to transform the given matrix in the following form

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & 0\cdots & * \end{pmatrix}$$

where * stands for zero or non-zero element. That is, we shall try to make a_{ii} as 1 and all the elements below a_{ii} as zero.

We illustrate the above procedure by the following illustration.

Illustration 3

Find the rank of the matrix

$$A = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 5 & 7 \\ 5 & 3 & 8 \end{pmatrix}$$

Solution

Step 1: As a first step we must get a 1 in the first column of A. For this we subtract Row 1 from the Row 2.

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 5 & 7 \\ 5 & 3 & 8 \end{pmatrix} [R_2 \to R_2 - R_1] \begin{pmatrix} 2 & -3 & 1 \\ 1 & 8 & 6 \\ 5 & 3 & 8 \end{pmatrix}$$

Step 2: We must get a 1 in the upper left corner. For this we interchange Row 1 and Row 2.

$$\begin{pmatrix} 2 & -3 & 1 \\ 1 & 8 & 6 \\ 5 & 3 & 8 \end{pmatrix} [R_1 \leftrightarrow R_2] \begin{pmatrix} 1 & 8 & 6 \\ 2 & -3 & 1 \\ 5 & 3 & 8 \end{pmatrix}$$

Step 3: We must get zeros at the two remaining two places in the first column. For this we multiply R_1 by – 2 and add it to R_2 and multiply R_1 by – 5 and add it to R_3.

$$\begin{pmatrix} 1 & 8 & 6 \\ 2 & -3 & 1 \\ 5 & 3 & 8 \end{pmatrix} [R_2 \to R_2 - 2R_1,$$

$$R_3 \to R_3 - 5R_1] \begin{pmatrix} 1 & 8 & 6 \\ 0 & -19 & -11 \\ 0 & -37 & -22 \end{pmatrix}$$

Step 4: We must have 1 in the second column. This 1 should *not be* in the first row. Also, you should not be tempted to use R_1 to obtain this 1. For, if we try to use R_1, then two zeros obtained in the first column will be destroyed. We multiply R_2 by – 2 and add it to R_3.

$$\begin{pmatrix} 1 & 8 & 6 \\ 0 & -19 & -11 \\ 0 & -37 & -22 \end{pmatrix} [R_3 \to R_3 - 2R_1] \begin{pmatrix} 1 & 8 & 6 \\ 0 & -19 & -11 \\ 0 & 1 & 0 \end{pmatrix}$$

Step 5: We now obtain a 1 at the $(2, 2)^{\text{th}}$ place. For this we interchange R_2 and R_3.

$$\begin{pmatrix} 1 & 8 & 6 \\ 0 & -19 & -11 \\ 0 & 1 & 0 \end{pmatrix} [R_2 \leftrightarrow R_3] \begin{pmatrix} 1 & 8 & 6 \\ 0 & 1 & 0 \\ 0 & -19 & -11 \end{pmatrix}$$

Step 6: We must get a zero at $(3, 2)^{\text{th}}$ place. For this we multiply R_2 by 19 and add it to R_3.

$$\begin{pmatrix} 1 & 8 & 6 \\ 0 & 1 & 0 \\ 0 & -19 & -11 \end{pmatrix} [R_3 \to R_3 + 19R_2] \begin{pmatrix} 1 & 8 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & -11 \end{pmatrix}$$

This matrix is in the desired triangular form. Recall if $A \sim B$ then $\rho(A) = \rho(B)$. Thus, rank of the given matrix A is equal to the rank of the matrix

$$B = \begin{pmatrix} 1 & 8 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & -11 \end{pmatrix}. \text{ Since } |B| = \begin{vmatrix} 1 & 8 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & -11 \end{vmatrix}$$

$$= -11 \neq 0, \rho(B) = 3.$$

Hence, $\rho(A) = 3$.

Some Tips

- ❑ After obtaining 1 at (1, 1)th place and zeros at the remaining places in the first column, forget the first row. Do not use the first row to manipulate elements in the second or any other column. If you try to do so the zeros in the first column will be destroyed.
- ❑ After obtaining 1 at (2, 2)th place and zeros at the remaining places in the second column, forget the second row. Do not use it for manipulating elements in the remaining columns. The same remark applies to the remaining columns.
- ❑ If A is a non-singular matrix of order $n \times n$, then $\rho(A) = n$.

SYSTEM OF LINEAR EQUATIONS

Let us consider the following m linear equations in n unknowns:

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \qquad \dots(1) \\ &. \\ &. \\ &. \\ &a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{aligned}$$

where $b_1, b_2, \dots b_m$ are not all zero.

The $m \times n$ matrix $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & . & . & . \\ . & . & & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is called the *coefficient matrix* of the system of linear equations. Using it, we can now write these equations as follows:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & . & . & . \\ . & . & & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{pmatrix}$$

We can abbreviate the above matrix equation as $AX = B$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & & . \\ . & . & . & . \\ . & . & & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{pmatrix}$$

By a **solution** of (1) we mean a set of values $x_1, x_2, \dots, x_n$ such that (1) reduces to an identity.

The **augmented matrix** for system of equations

$$AX = B$$

is the matrix $(A \mid B)$. This matrix is obtained by adding $(n + 1)$th column to A. The elements of this column are the constants $b_1, \dots, b_m$.

Result The system of equations

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \qquad \dots(1) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{aligned}$$

is consistent (that is, possesses a solution) if and only if the coefficient matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and the augmented matrix

$$(A \mid B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

have the same rank.

We split the remaining result in two cases.

Case 1. If the system of equations in (1) is consistent and $m \geq n$, then

(*i*) if $\rho(A) = \rho(A \mid B) = n$, then the system of equations has a unique solution and $(m - n)$ equations are superfluous in the sense that any solution that satisfies other n equations will satisfy these $(m - n)$ equations.

(*ii*) if $\rho(A) = \rho(A \mid B) = r < n$, then the $(n - r)$ unknowns are assigned arbitrary values and the remaining r unknowns can be found in terms of those $(n - r)$ unknowns which have already been assigned values.

Case 2. If the system of equations in (1) is consistent and $m < n$, then (*i*) if $\rho(A) = \rho(A \mid B) = m$, then $(n - m)$ unknowns can be assigned arbitrary values and the values of the remaining m unknowns can be found in terms of those $(n - m)$ unknowns which have already been assigned values. Also, $(m - r)$ equations are superfluous.

(*ii*) if $\rho(A) = \rho(A \mid B) = r < m$ then $(n - r)$ unknowns can be assigned arbitrary values and the values of the remaining r unknowns can be found in terms of those $(n - r)$ unknowns which have already been assigned values. Also, $(m - r)$ equations are superfluous.

Finding Inverse by Elementary Row Operations

To find inverse of a square matrix A we begin with the augmented matrix $[A \mid I_n]$. If a sequence of elementary row operations transforms this matrix to $[I_n \mid B]$, then B is A^{-1}. However, if at any step we obtain all zeros in a row on the left of the vertical line, the matrix A is not invertible.

SOLUTION OF A SYSTEM OF EQUATION *AX* = *B* WHEN *A* IS A SQUARE MATRIX

Unique Solution

The system of equations

$$AX = B$$

has a unique solution if $|A| \neq 0$ and it is given by $X = A^{-1} B$

No Solution

If $|A| = 0$ and $(\text{Adj } A)B \neq O$, the system of equations has no solution.

Infinite Number of Solution or no Solution

If $|A| = 0$, and $(\text{Adj } A)B = O$, the system of equations has *infinite number of solutions* or *no solution*.

Illustration 4

Consider the system of equations

$$x + y + z = 1$$
$$2x + 2y + 2z = 2$$
$$3x + 3y + 3z = k$$

Here

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 2 \\ k \end{pmatrix}, \text{ and}$$

$$adj\ A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note that

$$(\text{adj } A)\, B = O$$

Also, note that the system of equations has

(1) infinite number of solutions if $k = 3$

(2) no solution if $k \neq 3$.

Remark

- ☐ Note this case carefully. Mostly students commit mistake in this case.

SOLUTION OF A SYSTEM OF HOMOGENEOUS LINEAR EQUATIONS *AX* = 0

The system

$$AX = 0$$

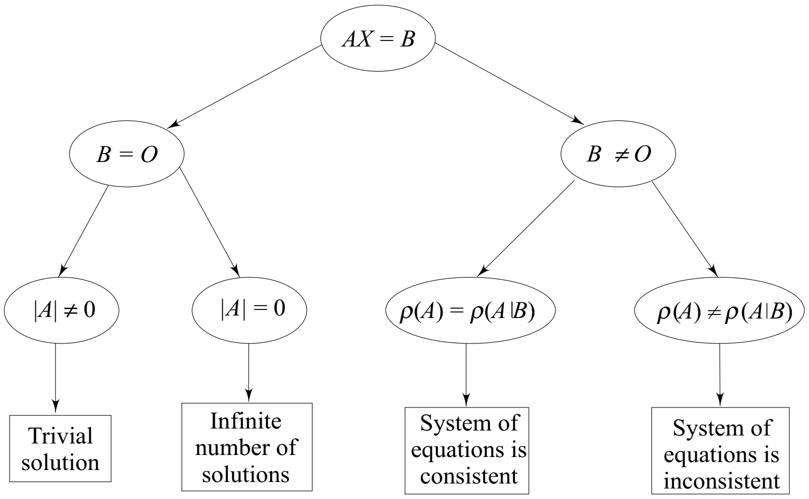
has a unique solution if $|A| \neq 0$ and it is the trivial solution viz.

$$x_1 = x_2 = \cdots = x_n = 0.$$

If $|A| = 0$, the system has infinite number of solutions.

Also if $AX = 0$ has at least one non-zero solution, then $|A| = 0$.

The following tree diagram is helpful.



SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: The number of 2 × 2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for which

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & 1/b \\ 1/c & 1/d \end{bmatrix}, (a, b, c, d \in \mathbf{R}) \text{ is}$$

(a) 0 (b) 1

(c) 2 (d) infinite

Ans. (a)

Solution: If $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Thus, $$\frac{d}{ad-bc} = \frac{1}{a} \Leftrightarrow ad = ad - bc$$

$$\Leftrightarrow bc = 0 \Leftrightarrow b = 0 \text{ or } c = 0$$

Therefore,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} 1/a & 1/b \\ 1/c & 1/d \end{bmatrix}$$

is never possible.

TIP

If $n > 1$, then a $n \times n$ matrix cannot have inverse in the above form.

Example 2: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b, c, d \in \mathbf{R}$

If $A^5 = A^3 + I$, then A is

(a) a symmetric matrix

(b) a skew symmetric matrix

(c) an invertible matrix

(d) none of these

Ans. (c)

Solution: $A^5 = A^3 + I$

$\Rightarrow$ $A(A^4 - A^2) = (A^4 - A^2)\,A = I$

$\Rightarrow$ A is invertible and $A^{-1} = A^4 - A^2$.

TIP

If $P(x)$ is a polynomial such that $P(0) \neq 0$ and $P(A) = O$, then A is invertible.

Example 3: Let A and B be two 3 × 3 invertible matrices. If $A + B = AB$, then

(a) $A^{-1} + B^{-1} = O$

(b) $A^{-1} + B^{-1} = B^{-1}A^{-1}$

(c) $I - A^{-1}$ is invertible

(d) $B^{-1} + I$ is invertible

Ans. (c)

Solution: $A + B = AB \Rightarrow A = AB - B = (A - I)B$. Premultiplying by A^{-1} we get

$$I = A^{-1}A = (I - A^{-1})B$$

$\Rightarrow$ $I - A^{-1}$ is invertible.

Example 4: Let A be a 3 × 3 matrix and

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \, x, y, z \in \mathbf{R} \right\}$$

Define $f : S \to S$ by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Suppose $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x = y = z = 0.$

Then

(a) f is one-to-one (b) f cannot be onto

(c) A is not invertible (d) $A = O$

Ans. (a)

Solution: Let $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ be such

that $$f\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = f\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\Rightarrow \quad A\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = A\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\Rightarrow \quad A\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad f\begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad x_1 - x_2 = 0,\ y_1 - y_2 = 0,\ z_1 - z_2 = 0$$

$$\Rightarrow \quad x_1 = x_2,\ y_1 = y_2,\ z_1 = z_2$$

or
$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

Thus, f is one-to-one.

Example 5: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbf{R}$.

If $|a|, |b|, |c|, |d| \le k$, where $k > 0$, then

(a) $\det(A) \ge 2k^2$ (b) $\det(A) \ge k^2$

(c) $\det(A) \le 2k^2$ (d) $\det(A) \le k$

Ans. (c)

Solution: $\det(A) = ad - bc \le |ad - bc|$

$$\Rightarrow \quad \det(A) \le |a||d| + |b||c| \le 2k^2$$

Example 6: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbf{R}$.

Then

(a) $\det(A) \le \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$

(b) $\det(A) \le (a + b)(c + d)$

(c) $\det(A) \le ac + bd$

(d) $\det(A) \le (|a| - |b|)(|c| - |d|)$

Ans. (a)

Solution: $\det(A) = ad - bc \le |ad - bc|$
$$\le |a||d| + |b||c|$$

Now,
$$\begin{aligned} &(a^2 + b^2)(c^2 + d^2) - (|a||d| + |b||c|)^2 \\ &= a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 \\ &\quad - (a^2d^2 + b^2c^2 + 2|a||d||b||c|) \\ &= (|ac| - |bd|)^2 \ge 0 \end{aligned}$$

$$\Rightarrow \quad |a||d| + |b||c| \le \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$$

Example 7: Let A be a 3×3 matrix such that $\det(A) = -2$. Then $\det(-2A^{-1})$ is equal to

(a) 4 (b) −4

(c) 8 (d) −2

Ans. (a)

Solution: As A^{-1} is a 3×3 matrix,
$$\begin{aligned} \det(-2A^{-1}) &= (-2)^3 \det(A^{-1}) \\ &= (-2)^3 (\det(A))^{-1} \\ &= (-8)\left(-\frac{1}{2}\right) = 4 \end{aligned}$$

Example 8: For $1 \le i, j \le 3$, let
$$a_{ij} = \int_{-\pi/2}^{\pi/2} \cos(ix)\cos(jx)\,dx$$
and let $A = (a_{ij})_{3\times3}$. Then

(a) A is a singular matrix

(b) $AX = B$ has a unique solution for every 3×3 matrix B

(c) A is a skew-symmetric matrix.

(d) $A^2 = I$

Ans. (b)

Solution: For $1 \le i \le 3$,
$$a_{ii} = 2\int_0^{\pi/2} \cos(ix)\cos(ix)\,dx$$
[integrand is an even function]
$$= \int_0^{\pi/2} [1 + \cos(2ix)]\,dx$$
$$= \left(x + \frac{\sin(2ix)}{2i}\right)\Bigg]_0^{\pi/2} = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

For $1 \le i, j \le 3$, $i \ne j$,
$$a_{ij} = 2\int_0^{\pi/2} \cos(ix)\cos(jx)\,dx$$
$$= \int_0^{\pi/2} [\cos((i + j)x) + \cos((i - j)x)]\,dx$$
$$= \frac{\sin((i + j)\pi/2)}{i + j} + \frac{\sin((i - j)\pi/2)}{i - j}$$

$$\therefore \quad a_{12} = -\frac{1}{3} + 1 = \frac{2}{3} = a_{21}$$
$$a_{13} = 0 = a_{31}, \text{ and}$$
$$a_{23} = \frac{1}{5} + 1 = \frac{6}{5} = a_{32}$$

Thus,
$$A = \begin{pmatrix} 1 & 2/3 & 0 \\ 2/3 & 1 & 6/5 \\ 0 & 6/5 & 1 \end{pmatrix}$$

Using $C_2 \to C_2 - \frac{2}{3}C_1$, we get
$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 2/3 & 5/9 & 6/5 \\ 0 & 6/5 & 1 \end{vmatrix}$$
$$= \frac{5}{9} - \frac{36}{25} = -\frac{199}{225} \ne 0$$

As $\det(A) \ne 0$, A^{-1} exists. Therefore, $AX = B$ has a unique solution viz. $X = A^{-1}B$.

Example 9: The number of values of λ for which there exist a non-zero 3×3 matrix A such that $A' = \lambda A$ is:

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (c)

Solution: We have

$$A = (A')' = (\lambda A)' = \lambda A' = \lambda^2 A.$$

As $A \neq 0$, $\lambda^2 = 1 \Rightarrow \lambda = \pm 1$.

For $\lambda = 1$ all non-zero symmetric matrices can work as A. For $\lambda = -1$, all non-zero skew symmetric matrices can work as A.

Example 10: Let $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, where $a > 0$. Sum of the series $S = \text{trace}(A) + \text{trace}\left(\frac{1}{2}A\right) + \text{trace}\left(\frac{1}{2^2}A^2\right) + \text{trace}\left(\frac{1}{2^3}A^3\right) + \dots$ is

(a) 3 (b) 4
(c) 6 (d) 8

Ans. (b)

Solution: $A = I + B$, where $B = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$

As $B^2 = O$, we get $B^r = O \quad \forall\, r \geq 2.$

Thus, $A^r = (I + B)^r = I + rB$

$$= \begin{bmatrix} 1 & ra \\ 0 & 1 \end{bmatrix} \quad \forall\, r \geq 1.$$

$$\Rightarrow \quad \text{trace}\left(\frac{1}{2^r}A^r\right) = \frac{1}{2^r}(2) = \frac{1}{2^{r-1}} \quad \forall\, r \geq 1.$$

$$\therefore \quad S = 2 + 1 + \frac{1}{2^2} + \frac{1}{2^3} + \dots = \frac{2}{1 - \frac{1}{2}} = 4$$

LEVEL 1

Straight Objective Type Questions

Example 11: If $\begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} x & y^2 \\ z & 0 \end{bmatrix}$, $y < 0$ then $x - y + z$ is equal to

(a) 5 (b) 2
(c) 1 (d) – 3

Ans. (a)

Solution: By the equality of two matrices, $x = 1$, $y^2 = 4$, $z = 2$

$\Rightarrow \quad x = 1, y = -2, z = 2$ as $y < 0$.

$\therefore \quad x - y + z = 1 + 2 + 2 = 5$

Example 12: If $A = [1 \; -2 \;\; 3]$, $B = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$, then AB is equal to

(a) $\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$

(c) $[2 \; 6 \; -3]$ (d) none of these

Ans. (d)

Solution: $AB = [1 \quad -2 \quad 3]\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = [2 + 6 - 3] = [5]$

Example 13: If $A = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$, then $A'A$ is equal to

(a) I (b) $-iA$
(c) $-I$ (d) iA

Ans. (c)

Solution: We have

$$A'A = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

Example 14: If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then $A_\alpha A_\beta$ is equal to

(a) $A_{\alpha+\beta}$ (b) $A_{\alpha\beta}$
(c) $A_{\alpha-\beta}$ (d) none of these

Ans. (a)

Solution: We have

$$A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \cos\alpha\sin\beta + \sin\alpha\cos\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = A_{\alpha+\beta}$$

Example 15: Let A and B be two 2 × 2 matrices. Consider the statements

(i) $AB = O \Rightarrow A = O$ or $B = O$
(ii) $AB = I_2 \Rightarrow A = B^{-1}$
(iii) $(A + B)^2 = A^2 + 2AB + B^2$

Then

(a) (i) is false, (ii) and (iii) are true
(b) (i) and (iii) are false, (ii) is true
(c) (i) and (ii) are false, (iii) is true
(d) (ii) and (iii) are false, (i) is true

Ans. (b)

Solution: (i) is false.

If $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Thus, $AB = O \not\Rightarrow A = O$ or $B = O$

(iii) is false since matrix multiplication is not commutative.

(ii) is true as product AB is an identity matrix, if and only if B is inverse of the matrix A.

Example 16: If $A - 2B = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$ and $2A - 3B = \begin{bmatrix} -2 & 5 \\ 0 & 7 \end{bmatrix}$, then matrix B is equal to

(a) $\begin{bmatrix} -4 & -5 \\ -6 & -7 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 6 \\ -3 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 6 & -1 \\ 0 & 1 \end{bmatrix}$

Ans. (a)

Solution: We have

$$B = (2A - 3B) - 2(A - 2B) = \begin{bmatrix} -4 & -5 \\ -6 & -7 \end{bmatrix}$$

Example 17: If A and B two are 3×3 matrices, then which one of the following is **not** true:

(a) $(A + B)' = A' + B'$
(b) $(AB)' = A'B'$
(c) $\det(AB) = \det(A)\det(B)$
(d) $A(\text{adj } A) = |A|\, I_3$

Ans. (b)

Solution: If A and B are two 3×3 matrices, then

$(AB)' = B'A'$ [Reversal Law]

and not $(AB)' = A'B'$.

Example 18: If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, then

(a) A is an orthogonal matrix
(b) A is a symmetric matrix
(c) A is a skew-symmetric matrix
(d) none of these

Ans. (a)

Solution: We have

$$AA' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \cos\theta\sin\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Similarly, $A'A = I_2$

Thus, A is an orthogonal matrix.

Example 19: If

$A = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ then the product AB is equal to

(a) O (b) A
(c) B (d) I

Ans. (a)

Solution: We have

$$AB = \begin{bmatrix} 0 - abc + abc & a^2c + 0 - a^2c & -a^2b + a^2b + 0 \\ 0 - b^2c + b^2c & abc + 0 - abc & -ab^2 + ab^2 + 0 \\ 0 - bc^2 + bc^2 & ac^2 + 0 - ac^2 & -abc + abc + 0 \end{bmatrix} = O$$

Example 20: If A is an invertible matrix and B is an orthogonal matrix, of the order same as that of A, then $C = A^{-1}BA$ is

(a) an orthogonal matrix
(b) symmetric matrix
(c) skew-symmetric matrix
(d) none of these

Ans. (d)

Solution: Let $B = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) \\ -\sin(\pi/2) & \cos(\pi/2) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

and $A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$

Note that B is an orthogonal matrix.

$$C = A^{-1}BA = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 10 \\ -1 & -3 \end{pmatrix}$$

Note that C is neither symmetric, nor skew-symmetric and nor-orthogonal.

Example 21: Let $E(\alpha) = \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix}$. If α and β differs by an odd multiple of $\pi/2$, then $E(\alpha)\,E(\beta)$ is a

(a) null matrix
(b) unit matrix
(c) diagonal matrix
(d) orthogonal matrix

Ans. (a)

◎ **Solution:** We have

$$E(\alpha)\,E(\beta)$$

$$= \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix} \begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta\cos(\alpha-\beta) & \cos\alpha\sin\beta\cos(\alpha-\beta) \\ \sin\alpha\cos\beta\cos(\alpha-\beta) & \sin\alpha\sin\beta\cos(\alpha-\beta) \end{bmatrix}$$

As α and β differ by an odd multiple of $\pi/2$, $\alpha - \beta = (2n+1)\pi/2$ for some integer n. Thus, $\cos[(2n+1)\pi/2] = 0$

$$\therefore \quad E(\alpha)\,E(\beta) = O$$

◉ **Example 22:** If $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then matrix A equals

(a) $\begin{bmatrix} 7 & 5 \\ -11 & -8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 7 & 1 \\ 34 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 3 \\ 13 & 8 \end{bmatrix}$

Ans. (a)

◎ **Solution:** If $XAY = I$, then $A = X^{-1}Y^{-1} = (YX)^{-1}$

In this case $$YX = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -11 & -7 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 8 & 5 \\ -11 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & 5 \\ -11 & -8 \end{bmatrix}$$

◉ **Example 23:** The matrix A satisfying

$$A \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix} \text{ is}$$

(a) $\begin{bmatrix} 3 & 2 \\ 6 & -3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -16 \\ 6 & -30 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -16 \\ 6 & 30 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -3 \\ 6 & 2 \end{bmatrix}$

Ans. (b)

◎ **Solution:** We know that if $AC = B$, then $A = BC^{-1}$.

$$\therefore \quad A = \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -16 \\ 6 & -30 \end{bmatrix}$$

◉ **Example 24:** If product $A \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ is the matrix $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, then A^{-1} is given by

(a) $\begin{bmatrix} 0 & -1 \\ 2 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & -1 \\ -2 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$ (d) none of these

Ans. (c)

◎ **Solution:** If $AB = C$, then $B^{-1}A^{-1} = C^{-1}$

$$\Rightarrow \quad A^{-1} = BC^{-1}$$

Here $$A \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix}$$

◉ **Example 25:** If A and B are two skew-symmetric matrices of order n, then

(a) AB is a skew-symmetric matrix
(b) AB is a symmetric matrix
(c) AB is a symmetric matrix if A and B commute
(d) none of these

Ans. (c)

◎ **Solution:** We are given

$$A' = -A \text{ and } B' = -B$$

Now, $$(AB)' = B'A' = (-B)(-A) = BA$$
$$= AB \text{ if } A \text{ and } B \text{ commute.}$$

◉ **Example 26:** Which of the following statements is false:

(a) If $|A| = 0$, then $|\text{adj } A| = 0$
(b) Adjoint of a diagonal matrix of order 3 × 3 is a diagonal matrix
(c) Product of two upper triangular matrices is a upper triangular matrix
(d) adj (AB) = adj (A) adj (B)

Ans. (d)

◎ **Solution:** We have

$$\text{adj}(AB) = \text{adj}(B)\,\text{adj}(A)$$

and *not* $$\text{adj}(AB) = \text{adj}(A)\,\text{adj}(B)$$

◉ **Example 27:** If A and B are symmetric matrices, then $AB - BA$ is a

(a) symmetric matrix
(b) skew-symmetric matrix
(c) diagonal matrix
(d) null matrix

Ans. (b)

◎ **Solution:** We are given $A' = A$, $B' = B$

Now $$(AB - BA)' = (AB)' - (BA)' = B'A' - A'B'$$
$$= BA - AB = -(AB - BA)$$

i.e. $$(AB - BA)' = -(AB - BA)$$

Hence, $AB - BA$ is a skew-symmetric matrix.

◉ **Example 28:** Let A and B be two 3 × 3 matrices, such that $A + B = 2B'$ and $3A + 2B = I$, then

(a) $A - B = O$ (b) $A + B = I$

(c) $A - B = I$ (d) $A + 2B = O$

Ans. (a)

◎ **Solution:** $A + B = 2B'$

$3A + 2B = I$

⇒ $B = 6B' - I$

⇒ $B' = (6B' - I)' = 6B - I$

Therefore, $B = 6(6B - I) - I$

⇒ $(1 - 36)B = -7I$

⇒ $B = \frac{1}{5}I$

∴ $3A = I - \frac{2}{5}I = \frac{3}{5}I$

⇒ $A = \frac{1}{5}I$

∴ $A = B$

◉ **Example 29:** Let A and B be two non-zero 3×3 matrices such that $AB = O$. Then

(a) Both A and B are non-singular
(b) Exactly one of A, B is singular
(c) Both A and B are singular
(d) Both $A + B$ and AB are singular

Ans. (c)

◎ **Solution:** Suppose A is non-singular, then

$$A^{-1}(AB) = A^{-1}(O) \Rightarrow B = O$$

A contradiction.

∴ A is singular

Similarly, B is singular.

◉ **Example 30:** Suppose A is a 3×3 skew-symmetric matrix. Let $B = (I + A)^{-1}(I - A)$. Then

(a) B is orthogonal
(b) B is skew symmetric
(c) $B^2 = O$
(d) B is a diagonal matrix

Ans. (a)

◎ **Solution:** We have

$$\begin{aligned} BB' &= (I + A)^{-1}(I - A)\,[(I + A)^{-1}(I - A)]' \\ &= (I + A)^{-1}(I - A)\,(I - A)'\,((I + A)')^{-1} \\ &= (I + A)^{-1}(I - A)\,(I + A)\,(I - A)^{-1} \\ &\qquad\qquad [\because\ A' = -A] \\ &= (I + A)^{-1}(I + A)\,(I - A)\,(I - A)^{-1} \\ &\qquad\qquad [I + A \text{ and } I - A \text{ commute}] \\ &= (I)(I) = I \end{aligned}$$

Thus, B is orthogonal.

◉ **Example 31:** Let $a_n = 3^n + 5^n$, $n \in \mathbf{N}$, and let

$$A = \begin{pmatrix} a_n & a_{n+1} & a_{n+2} \\ a_{n+1} & a_{n+2} & a_{n+3} \\ a_{n+2} & a_{n+3} & a_{n+4} \end{pmatrix}$$

Then

(a) 0 is a root of the equation $\det(A - xI) = 0$
(b) $\det(A) = a_n\, a_{n+2}\, a_{n+4}$
(c) $\det(A) < 0$
(d) $\det(A) = a_n + a_{n+2} + a_{n+4}$

Ans. (a)

◎ **Solution:** Using $C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 3^2C_1$, we get

$$\det(A) = \begin{vmatrix} a_n & 2(5^n) & 16(5^n) \\ a_{n+1} & 2(5^{n+1}) & 16(5^{n+1}) \\ a_{n+2} & 2(5^{n+2}) & 16(5^{n+2}) \end{vmatrix}$$

$$= 0 \qquad [\because\ C_2 \text{ and } C_3 \text{ are proportional}]$$

Thus, $x = 0$ is a root of $\det(A - xI) = 0$

◉ **Example 32:** First row of a matrix A is [1 3 2]. If

$$\text{adj}\, A = \begin{bmatrix} -2 & 4 & \alpha \\ -1 & 2 & 1 \\ 3\alpha & -5 & -2 \end{bmatrix}$$

then a possible value of $\det(A)$ is

(a) 1
(b) 2
(c) –1
(d) –2

Ans. (a)

◎ **Solution:** We know that

$$\begin{aligned} \det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\ &= (1)(-2) + (3)(-1) + (2)(3\alpha) \\ &= 6\alpha - 5 \end{aligned}$$

and $(\det(A))^2 = \det(\text{adj}\, A) = 11\alpha - 10$

Thus, $(6\alpha - 5)^2 = 11\alpha - 10$

⇒ $36\alpha^2 - 71\alpha + 35 = 0$

⇒ $\alpha = 1,\ 35/36$

Therefore, a possible value of $\det(A)$ is 1.

◉ **Example 33:** Suppose ABC is a triangle with sides a, b, c and semiperimeter s. Then matrix

$$A = \begin{bmatrix} s & s-c \\ s(s-b)^2 & (s-a)^2(s-c) \\ s(s-c) & (s-a)^2 \end{bmatrix}_{3\times 2} \begin{bmatrix} s-a \\ s-b \end{bmatrix}_{2\times 1} - \begin{bmatrix} bc \\ ca(s-a)(s-b) \\ ab(s-a) \end{bmatrix}_{3\times 1}$$

is equal to:

(a) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$
(b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
(c) $\begin{bmatrix} s \\ s \\ s \end{bmatrix}$
(d) $\begin{bmatrix} s-a \\ s-b \\ s-c \end{bmatrix}$

Ans. (b)

◎ **Solution:**

$$A = \begin{bmatrix} s(s-a)+(s-c)(s-b)-bc \\ (s-a)(s-b)(s(s-b)+(s-a)(s-c)-ca) \\ (s-a)(s(s-c)+(s-a)(s-b)-ab) \end{bmatrix}$$

But $s(s-a)+(s-c)(s-b)-bc$

$$= s(s-a)+s(s-b-c)$$
$$= s(2s-a-b-c)=0 \text{ etc.}$$

Thus, $A = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Example 34: The number of matrices

$$A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \text{ (where } a, b, c, d \in \mathbf{R}\text{)}$$

such that $A^{-1} = -A$ is :

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (d)

Solution: As A^{-1} exists, $\det(A) = ad - bc \neq 0$.

Also, $\det(-A) = \det\begin{bmatrix}-a & -b\\ -c & -d\end{bmatrix} = ad - bc$

$$= \det(A)$$

since $\det(A^{-1}) = \dfrac{1}{\det(A)}$, $A^{-1} = -A$ implies

$$\frac{1}{ad-bc} = ad - bc$$

$\Rightarrow \quad (ad-bc)^2 = 1 \quad \Rightarrow \quad ad - bc = \pm 1.$

If $ad - bc = 1$, then $A^{-1} = -A$

gives

$$A^{-1} = \begin{bmatrix}d & -b\\ -c & a\end{bmatrix} = \begin{bmatrix}-a & -b\\ -c & -d\end{bmatrix}$$

$\Rightarrow \quad a + d = 0.$

$\therefore \quad a(-a) - bc = 1$

$\Rightarrow \quad bc = -(1+a^2) \Rightarrow bc \neq 0$ and

$$b = -\frac{1+a^2}{c}$$

$\therefore \quad A = \begin{bmatrix}a & -\dfrac{(1+a^2)}{c}\\ c & -a\end{bmatrix}$, where $c \neq 0$.

Thus, there are infinite number of such matrices.

TIP

It is unnecessary to consider the case $ad - bc = -1$

Example 35: Let A be a 3×3 matrix with entries from the set of real numbers, If the system of equations $A^2X = 0$ has a non-trivial solution, then

(a) $AX = 0$ has a non-trivial solution.
(b) $AX = 0$ does not have a non-trivial solution.
(c) A is a non-singular matrix.
(d) none of these.

Ans. (a)

Solution: As $A^2X = 0$ has a non-trivial solution,

$$\det(A^2) = 0 \quad \Rightarrow \quad (\det(A))^2 = 0$$

$\Rightarrow \quad \det(A) = 0$

$\therefore \quad AX = 0$ has a non-trivial solution.

Example 36: Suppose $a, b \in \mathbf{R}$ and $a \neq b$.

Let $\quad A = \begin{bmatrix}a & b\\ b & -a\end{bmatrix}$.

Let M be a matrix such that $MA = A^{2m}$ for some $m \in \mathbf{N}$, then M is equal to

(a) $(a^2+b^2)^m I$ (b) $(a^2+b^2)^{m-1} A$
(c) $-(a^2+b^2)^{m-1} A$ (d) $(a^2+b^2)^m A$

Ans. (b)

Solution: We have

$$A^2 = \begin{bmatrix}a^2+b^2 & 0\\ 0 & a^2+b^2\end{bmatrix} = (a^2+b^2)I$$

$\Rightarrow \quad A^{2m} = ((a^2+b^2)I)^m = (a^2+b^2)^m I$

Also, $\quad A^{-1} = \dfrac{-1}{a^2+b^2}\begin{bmatrix}-a & -b\\ -b & a\end{bmatrix}$

$$= \frac{1}{a^2+b^2}\begin{bmatrix}a & b\\ b & -a\end{bmatrix}$$

Now, $\quad MA = A^{2m} = (a^2+b^2)^m I$

$\Rightarrow \quad M = (a^2+b^2)^m A^{-1}$

$$= (a^2+b^2)^{m-1} A$$

Example 37: Let $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$, be a 2 × 2 matrix where $a, b, c, d \in \{0, 1\}$. The number of such matrices which have inverse is

(a) 5 (b) 6
(c) 7 (d) 8

Ans. (b)

Solution: $\det(A) = ad - bc$

Note that $\det(A)$ can take value –1, 0 or 1. We have

$\det(A) = 1 \Leftrightarrow ad = 1, bc = 0$

$\Leftrightarrow a = 1, d = 1$ or $(b = 0, c = 0, b = 0; c = 1; b = 1, c = 0)$

and $\quad \det(A) = -1 \Leftrightarrow ad = 0 \quad$ or $\quad bc = 1$

This is also possible in 3 cases.

$\therefore \quad A^{-1}$ exists in 6 cases.

Example 38: If $D = \text{diag}(d_1, d_2, \ldots, d_n)$ where $d_i \neq 0$, for $i = 1, 2, \ldots, n$, then D^{-1} is equal to

(a) D
(b) $2D$
(c) $\text{diag}(d_1^{-1}, d_2^{-1}, \ldots, d_n^{-1})$
(d) Adj D

Ans. (c)

Solution: See Theory.

Example 39: The inverse of a symmetric matrix (if it exists) is

(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) a diagonal matrix
(d) none of these

Ans. (a)

Solution: Let A be an invertible symmetric matrix.

We have $AA^{-1} = A^{-1}A = I_n$

$\Rightarrow \quad (AA^{-1})' = (A^{-1}A)' = (I_n)'$

$\Rightarrow \quad (A^{-1})'A' = A'(A^{-1})' = I_n$

$\Rightarrow \quad (A^{-1})'A = A(A^{-1})' = I_n$

$(A^{-1})' = A^{-1}$ [inverse of a matrix is unique]

Example 40: The inverse of a skew-symmetric matrix (if it exists) is

(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) a diagonal matrix
(d) none of these

Ans. (b)

Solution: We have $A' = -A$

Now, $AA^{-1} = A^{-1}A = I_n$

$\Rightarrow \quad (AA^{-1})' = (A^{-1}A)' = (I_n)'$

$\Rightarrow \quad (A^{-1})'A' = A'(A^{-1})' = I_n$

$\Rightarrow \quad (A^{-1})'(-A) = (-A)(A^{-1})' = I_n$

Thus, $(A^{-1})' = -(A^{-1})$ [inverse of a matrix is unique]

Example 41: The inverse of a skew-symmetric matrix of odd order is

(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) diagonal matrix
(d) does not exist

Ans. (d)

Solution: Let A be a skew-symmetric, matrix of order n. By definition

$A' = -A$

$\Rightarrow \quad |A'| = |-A| \quad \Rightarrow \quad |A| = (-1)^n|A|$

$\Rightarrow \quad |A| = -|A| \qquad [\because n \text{ is odd}]$

$\Rightarrow \quad 2|A| = 0 \quad \Rightarrow \quad |A| = 0$

$\therefore$ A^{-1} does not exist.

Example 42: If A is an orthogonal matrix, then $|A|$ is

(a) 1 (b) -1
(c) ± 1 (d) 0

Ans. (c)

Solution: As A is an orthogonal matrix,

$A'A = AA' = I_n$

$\Rightarrow \quad |A'A| = |AA'| = |I_n|$

$\Rightarrow \quad |A'|\,|A| = 1 \quad \Rightarrow \quad |A|\,|A| = 1$

$\Rightarrow \quad |A|^2 = 1 \quad \Rightarrow \quad |A| = \pm 1$

Example 43: If $A = \begin{bmatrix} 1 & 0 & 2 \\ 5 & 1 & x \\ 1 & 1 & 1 \end{bmatrix}$ is a singular matrix, then x is equal to

(a) 3 (b) 5
(c) 9 (d) 11

Ans. (c)

Solution: As A is a singular matrix,

$$|A| = 0 \quad \Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ 5 & 1 & x-10 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

[using $C_3 \to C_3 - 2C_1$]

$$\Rightarrow \quad \begin{vmatrix} 1 & x-10 \\ 1 & -1 \end{vmatrix} = 0 \quad \Rightarrow \quad -1 - x + 10 = 0$$

$\Rightarrow \quad x = 9.$

Example 44: The value of x for which the matrix

$$A = \begin{bmatrix} 2/x & -1 & 2 \\ 1 & x & 2x^2 \\ 1 & 1/x & 2 \end{bmatrix} \text{ is singular is}$$

(a) ± 1 (b) ± 2
(c) ± 3 (d) none of these

Ans. (a)

Solution: We have

$$|A| = \left(\frac{2}{x}\right)\begin{vmatrix} x & 2x^2 \\ 1/x & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2x^2 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & x \\ 1 & 1/x \end{vmatrix}$$

$$= \frac{2}{x}(0) + 2 - 2x^2 + 2\left(\frac{1}{x} - x\right)$$

$$= \frac{2x(1-x^2) + 2(1-x^2)}{x} = \frac{2(x+1)^2(1-x)}{x}$$

Now, $|A| = 0 \quad \Rightarrow \quad x = \pm 1.$

Example 45: If square matrix A is such that $3A^3 + 2A^2 + 5A + I = O$, then A^{-1} is equal to

(a) $3A^2 + 2A + 5I$
(b) $-(3A^2 + 2A + 5I)$
(c) $3A^2 - 2A - 5I$
(d) none of these

Ans. (b)

Solution: We have $A(3A^2 + 2A + 5I) = -I$

$\therefore \quad A^{-1} = -(3A^2 + 2A + 5I).$ [$\therefore$ Inverse of a matrix is unique]

Example 46: If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then which one of the following holds for all $n \geq 1$, by the principle of mathematical induction.

(a) $A^n = nA + (n-1)I$
(b) $A^n = 2^{n-1}A + (n-1)I$
(c) $A^n = nA - (n-1)I$
(d) $A^n = 2^{n-1}A - (n-1)I.$

Ans. (c)

◎ **Solution:** For $n = 2$, $A^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

For $n = 2$, RHS of (a) $= 2A + I = 3\begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix} \neq A^2$

For $n = 2$, RHS of (b) $= 2A + I \neq A^2$
So possible answer is (c) or (d).

In fact $A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$ which equals $nA - (n-1)I.$

Alternatively. Write $A = I + B$

where $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

As $B^2 = 0$, we get $B^r = 0\ \forall\ r \geq 2$.
By the binomial theorem

$$A^n = I + nB = I + n(A - I) = nA - (n-1)I.$$

◉ **Example 47:** Let A, B, C be three square matrices of the same order, such that whenever $AB = AC$ then $B = C$, if A is

(a) singular
(b) non-singular
(c) symmetric
(d) skew-symmetric

Ans. (b)

◎ **Solution:** If A is non-singular, A^{-1} exists.

Thus, $\quad AB = AC \quad \Rightarrow \quad A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow \quad (A^{-1}A)B = (A^{-1}A)C \quad \Rightarrow \quad IB = IC$

$\Rightarrow \quad B = C.$

◉ **Example 48:** Let $B = \begin{bmatrix} 2 & 6 & 4 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix}$. If the product BA has inverse $C = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix}$, then A^{-1} equals

(a) $\begin{bmatrix} -3 & -5 & 5 \\ 0 & 9 & 14 \\ 2 & 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} -3 & 5 & 5 \\ 0 & 0 & 9 \\ 2 & 14 & 16 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & -5 & -5 \\ 0 & 0 & 2 \\ 2 & 14 & 6 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & -3 & -5 \\ 0 & 9 & 2 \\ 2 & 14 & 6 \end{bmatrix}$

Ans. (c)

◎ **Solution:** We have $(BA)^{-1} = C \quad \Rightarrow \quad A^{-1}B^{-1} = C$
$\Rightarrow A^{-1} = CB$

$$\therefore \quad A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 6 & 4 \\ 1 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -5 & -5 \\ 0 & 9 & 2 \\ 2 & 14 & 6 \end{bmatrix}$$

◉ **Example 49:** If ω is a complex cube root of unity, then the matrix $A = \begin{bmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{bmatrix}$ is a

(a) singular matrix
(b) non-singular matrix
(c) skew-symmetric matrix
(d) none of these

Ans. (a)

◎ **Solution:** We have

$$|A| = \begin{vmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega & 1 \\ \omega & 1 & \omega^2 \end{vmatrix} = \begin{vmatrix} 1+\omega^2+\omega & \omega^2 & \omega \\ \omega^2+\omega+1 & \omega & 1 \\ \omega+1+\omega^2 & 1 & \omega^2 \end{vmatrix} = 0$$

using $C_1 \to C_1 + C_2 + C_3$ and $1 + \omega + \omega^2 = 0$

$\therefore \quad A$ is a singular matrix.

◉ **Example 50:** If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & x & 1 \end{bmatrix}$ and

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & y \\ 5/2 & -3/2 & 1/2 \end{bmatrix}, \text{ then}$$

(a) $x = 1, y = -1$ (b) $x = -1, y = 1$
(c) $x = 2, y = -1/2$ (d) $x = 1/2, y = 1/2$

Ans. (a)

◎ **Solution:** We have

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & x & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & y \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & y+1 \\ 0 & 1 & 2(y+1) \\ 4(1-x) & 3(x-1) & 2+xy \end{bmatrix}$$

$\Rightarrow \quad 1 - x = 0, \quad x - 1 = 0,$
$\quad y + 1 = 0, \quad y + 1 = 0, 2 + xy = 1$

$\therefore \quad x = 1, \quad y = -1.$

◉ **Example 51:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a, b, c, d \in \mathbf{R}$. If $A - \alpha I$ is invertible for all $\alpha \in \mathbf{R}$, then

(a) $bc > 0$ (b) $bc = 0$
(c) $bc > \min\left(0, \dfrac{1}{2}ad\right)$ (d) $a = 0$

Ans. (c)

◎ **Solution:** As $A - \alpha I$ is invertible for all $\alpha \in \mathbf{R}$.

$$\det(A - \alpha I) \neq 0 \qquad \forall\ \alpha \in \mathbf{R}.$$

$$\Rightarrow \quad (a - \alpha)(d - \alpha) - bc \neq 0 \qquad \forall\ \alpha \in \mathbf{R}.$$

$$\Rightarrow \quad \alpha^2 - (a + d)\alpha + ad - bc \neq 0 \qquad \forall\ \alpha \in \mathbf{R}.$$

Therefore

$$(a + d)^2 - 4(ad - bc) < 0$$

$$\Rightarrow \quad (a - d)^2 + 4\,bc < 0$$

Therefore, $bc < 0$.

Also, $\quad a^2 + d^2 - 2ad + 4bc < 0$

$$\Rightarrow \quad 0 \leq a^2 + d^2 < 2ad - 4bc$$

$$\Rightarrow \quad bc < \frac{1}{2}ad.$$

Thus, $\quad bc < \min\left(0, \frac{1}{2}ad\right)$

◉ **Example 52:** If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A^2 - 5A - I$ equals

(a) O (b) I

(c) $2I$ (d) none of these

Ans. (b)

◎ **Solution:** $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

We have

$$A^2 - 5A = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$$

$$\Rightarrow \quad A^2 - 5A - I = I$$

◉ **Example 53:** Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ satisfies $A^2 + aA + bI = O$, then a, b are respectively equal to

(a) $-4, 2$ (b) $-3, 3$

(c) $-4, 1$ (d) $-3, 1$

Ans. (c)

◎ **Solution:** $A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$

Now

$$O = A^2 + aA + bI = \begin{bmatrix} 11 + 3a + b & 8 + 2a \\ 4 + a & 3 + a + b \end{bmatrix}$$

$$\Rightarrow \quad 3a + b = -11,\ 4 + a = 0,\ a + b = -3$$

$$\Rightarrow \quad a = -4,\ b = 1.$$

◉ **Example 54:** Let $A^2 - A + I = O$, then inverse of A is

(a) $I - A$ (b) $A - I$

(c) $A + I$ (d) A

Ans. (b)

◎ **Solution:** $A^2 - A + I = 0 \Rightarrow I = A - A^2 = A(I - A)$.

Thus, $A^{-1} = I - A$.

◉ **Example 55:** Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ and

$$(10)B = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & 2 & 3 \end{pmatrix}$$

If B is the inverse of matrix A, then α equals

(a) 2 (b) -1

(c) -2 (d) 5

Ans. (d)

◎ **Solution:** $10I = 10(AB) = A(10B)$

$$\Rightarrow \quad \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 0 & 5 - \alpha \\ 0 & 10 & \alpha - 5 \\ 0 & 0 & 5 + \alpha \end{pmatrix}$$

$$\Rightarrow \quad \alpha = 5.$$

◉ **Example 56:** If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then

(a) $\alpha = a^2 + b^2\ \beta = 2ab$

(b) $\alpha = a^2 + b^2,\ \beta = a^2 - b^2$

(c) $\alpha = 2ab,\ \beta = a^2 + b^2$

(d) $\alpha = a^2 + b^2,\ \beta = -2ab$

Ans. (a)

◎ **Solution:** $A^2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{bmatrix}$$

$$\Rightarrow \quad \alpha = a^2 + b^2,\ \beta = 2ab$$

◉ **Example 57:** Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form

$$A = \begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$$

where a, b, c are either ω or ω^2. Then number of distinct matrices in the set S is

(a) 2 (b) 6

(c) 4 (d) 8

Ans. (a)

◎ **Solution:** We have

$$|A| = 1\begin{vmatrix}1 & c\\ \omega & 1\end{vmatrix} - a\begin{vmatrix}\omega & c\\ \omega^2 & 1\end{vmatrix} + b\begin{vmatrix}\omega & 1\\ \omega^2 & \omega\end{vmatrix}$$

$$= 1 - c\omega - a\omega + ac\omega^2 = (1 - a\omega)(1 - c\omega)$$

A is non-singular ⇔ $|A| \neq 0 \Leftrightarrow a = c = \omega$.

Note that b can take any value ω or ω^2.

Thus, there are two non-singular matrices.

◉ **Example 58:** If a matrix A is both symmetric and skew-symmetric, then

(a) A is a diagonal matrix
(b) A is a scalar matrix
(c) A is a zero matrix
(d) none of these

Ans. (c)

◎ **Solution:** We are given $A' = A$ and $A' = -A$.

$$\therefore \quad A = -A \Rightarrow 2A = O \Rightarrow A = O$$

◉ **Example 59:** Let $A = \begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}, B = \begin{bmatrix}5 & 2\\ 7 & 4\end{bmatrix}, C = \begin{bmatrix}2 & 5\\ 3 & 8\end{bmatrix}$.

Let D be a matrix such that $CD = AB$, then D equals

(a) I (b) O
(c) $-A$ (d) none of these

Ans. (d)

◎ **Solution:** $CD = AB \quad \Rightarrow \quad D = C^{-1}(AB)$

Now $$AB = \begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 2\\ 7 & 4\end{bmatrix} = \begin{bmatrix}3 & 0\\ 43 & 22\end{bmatrix}$$ and

$$C^{-1} = \begin{bmatrix}8 & -5\\ -3 & 2\end{bmatrix}$$

$$\therefore \quad D = \begin{bmatrix}8 & -5\\ -3 & 2\end{bmatrix}\begin{bmatrix}3 & 0\\ 43 & 22\end{bmatrix} = \begin{bmatrix}-191 & -110\\ 77 & 44\end{bmatrix}$$

◉ **Example 60:** If $A^2 = A$, the $(I + A)^4$ equals

(a) $I + 15A$ (b) $I + 7A$
(c) $I + 8A$ (d) $I + 11A$

Ans. (a)

◎ **Solution:** As I and A commute, we can apply binomial theorem to expand $(I + A)^4$. We have

$$(I + A)^4 = I + 4A + 6A^2 + 4A^3 + A^4.$$

But $\quad A^3 = A^2 = A, A^4 = A^3 = A$

$$\therefore \quad (I + A)^4 = I + 4A + 6A + 4A + A = I + 15A$$

◉ **Example 61:** The matrix $A = \begin{bmatrix}0 & 0 & -7\\ 0 & -7 & 0\\ -7 & 0 & 0\end{bmatrix}$ is a

(a) scalar matrix (b) diagonal matrix
(c) unit matrix (d) non-singular

Ans. (d)

◎ **Solution:** Note that $A^2 = 49I$

$$\Rightarrow \quad A^{-1} = \frac{1}{49}A \qquad [\therefore A \text{ is non-singular}]$$

◉ **Example 62:** Let $A = [5 \quad 3]$ and $B = [3 \quad 7]$

The number of non-zero matrices C such that $AC = BC$ is

(a) 0 (b) 1
(c) infinitely many (d) none of these

Ans. (c)

◎ **Solution:** Note that every matrix of the form $\begin{bmatrix}2x\\ x\end{bmatrix}, x \in \mathbf{R}$ satisfies the required condition.

◉ **Example 63:** If $A = \begin{bmatrix}0 & 2y & z\\ x & y & -z\\ x & -y & z\end{bmatrix}$ satisfies $A' = A^{-1}$, then

(a) $x = \pm 1/\sqrt{6}$, $y = \pm 1/\sqrt{6}$, $z = \pm 1/\sqrt{3}$
(b) $x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{6}$, $z = \pm 1/\sqrt{3}$
(c) $x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{6}$, $z = \pm 1/\sqrt{3}$
(d) $x = \pm 1/\sqrt{2}$, $y = \pm 1/3$, $z = \pm 1/\sqrt{2}$

Ans. (b)

◎ **Solution:** $A' = A^{-1} \Leftrightarrow AA' = I$

Now, $$AA' = \begin{bmatrix}0 & 2y & z\\ x & y & -z\\ x & -y & z\end{bmatrix}\begin{bmatrix}0 & x & x\\ 2y & y & -y\\ z & -z & z\end{bmatrix}$$

$$= \begin{bmatrix}4y^2 + z^2 & 2y^2 - z^2 & -2y^2 + z^2\\ 2y^2 - z^2 & x^2 + y^2 + z^2 & x^2 - y^2 - z^2\\ -2y^2 + z^2 & x^2 - y^2 - z^2 & x^2 + y^2 + z^2\end{bmatrix}$$

Thus, $\quad AA' = I$

$$\Rightarrow \quad 4y^2 + z^2 = 1, \quad 2y^2 - z^2 = 0$$

$$x^2 + y^2 + z^2 = 1, \quad x^2 - y^2 - z^2 = 0$$

$$\therefore \quad x = \pm 1/\sqrt{2}, \quad y = \pm 1/\sqrt{6}, \quad z = \pm 1/\sqrt{3}$$

◉ **Example 64:** Suppose A is square matrix such that $A^3 = I$, then $(A + I)^3 + (A - I)^3 - 6A$ equals

(a) I (b) $2I$
(c) A (d) $3A$

Ans. (b)

◎ **Solution:** Since $AI = IA$, we have

$$(A + I)^3 + (A - I)^3 - 6A$$

$$= 2A^3 + 2(3AI^2) - 6A = 2I + 6A - 6A = 2I$$

Example 65: The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system of equations $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ has exactly five distinct solutions is

(a) 0 (b) 511
(c) 1024 (d) 5

Ans. (a)

Solution: We know that the system of equation

$$AX = B$$

has either (i) unique solution
or (ii) infinite number of solutions
or (iii) no solution.

Hence, there cannot exist any matrix A such that

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

has five distinct solutions.

Example 66: The number of 3 × 3 non-singular matrices, with four entries as 1 and all other entries as 0 is

(a) 6 (b) at least 7
(c) less than 4 (d) 5

Ans. (b)

Solution: The matrix

$$\begin{bmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 1 \end{bmatrix}$$

where exactly one of a, b, c, d, e, f is 1 and rest of them are zeros, is invertible.

There are six such matrices.

Also, the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible.

Thus, there are at least 7 such matrices which are invertible.

Example 67: Consider the system of linear equations:

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + x_3 = 3$$
$$3x_1 + 5x_2 + 2x_3 = 1$$

The system has

(a) a unique solution
(b) no solution
(c) infinite number of solutions
(d) exactly 3 solutions

Ans. (b)

Solution: Adding the first two equations and subtracting the third from the sum, we obtain

$$(x_1 + 2x_2 + x_3) + (2x_1 + 3x_2 + x_3) - (3x_1 + 5x_2 + 2x_3) = 3 + 3 - 1$$

$$\Rightarrow \qquad 0 = 5$$

Thus, the system of equations has no solution.

Example 68: Let a, b and c be three real numbers satisfying

$$[a \ b \ c]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0\ 0\ 0] \qquad (1)$$

If the point $P(a, b, c)$ with reference to (1), lies on the plane $2x + y + z = 1$, then the value of $7a + b + c$ is

(a) 0 (b) 12
(c) 7 (d) 6

Ans. (d)

Solution: We have

$$a + 8b + 7c = 0, \quad 9a + 2b + 3c = 0, \quad 7a + 7b + 7c = 0$$

Subtracting the last equation from first and multiplying the last equation by (9/7) and subtracting from the second equation we obtain $-6a + b = 0$, $7b + 6c = 0$

$$\Rightarrow \qquad \frac{a}{1} = \frac{b}{6} = \frac{c}{-7} = k \text{ (say)}$$

Thus, $a = k$, $b = 6k$, $c = -7k$.

As $(a, b, c) = (k, 6k, -7k)$ lies on $2x + y + z = 1$, we get $k = 1$.

Thus, $7a + b + c = 7k + 6k - 7k = 6k = 6$

Example 69: Let P and Q be 3 × 3 matrices with $P \neq Q$ If $P^3 = Q^3$ and $P^2 Q = Q^2 P$, then determinant of$(P^2 + Q^2)$ is equal to

(a) 1 (b) 0
(c) – 1 (d) –2

Ans. (d)

Solution: $P^3 = Q^3$ and $P^2Q = Q^2P$ gives

$$P^3 - P^2Q = Q^3 - Q^2P$$

$$\Rightarrow \qquad P^2(P - Q) = -Q^2(P - Q)$$

$$\Rightarrow \quad (P^2 + Q^2)(P - Q) = 0$$

If det $(P^2 + Q^2) \neq 0$, then $P^2 + Q^2$ is invertible and hence $P = Q$. Therefore, det $(P^2 + Q^2) = 0$.

Example 70: Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ equals

(a) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ (d) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Ans. (c)

◎ **Solution:** We have

$$A(u_1+u_2)=Au_1+Au_2=\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$$

We solve the above equation for $u_1 + u_2$.
We consider the augmented matrix

$$(A|B)=\left(\begin{array}{ccc|c}1&0&0&1\\2&1&0&1\\3&2&1&0\end{array}\right)$$

Applying $R_3 \to R_3 - 2R_2 + R_1$ and $R_2 \to R_2 - 2R_1$, we get

$$(A|B)\sim\left(\begin{array}{ccc|c}1&0&0&1\\0&1&0&-1\\0&0&1&-1\end{array}\right) \quad\Rightarrow\quad u_1+u_2=\begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

◉ **Example 71:** If P is a 3×3 matrix such that $P' = 2P + I$, then there exists a column matrix

$$X=\begin{pmatrix}x\\y\\z\end{pmatrix}\neq O \text{ such that}$$

(a) $PX = O$ (b) $PX = X$
(c) $PX = 2X$ (d) $PX = -X$

Ans. (d)

◎ **Solution:** We have

$$P = (P')' = (2P + I)' = 2P' + \mathrm{I}$$
$$P = 2(2P + I) + \mathrm{I} \quad\Rightarrow\quad P = -I$$

Thus, there exists $X \neq 0$ such that $PX = -X$

◉ **Example 72:** Let $A = \begin{pmatrix}4&3\\2&5\end{pmatrix}$ and $x, y \in \mathbf{R}$ are such that $A^2 + xA + yI = 0$ then (x, y) equals

(a) $(-9, -14)$ (b) $(9, -14)$
(c) $(-9, 14)$ (d) $(9, 14)$

Ans. (c)

◎ **Solution:** We have $A^2 = \begin{pmatrix}22&27\\18&31\end{pmatrix}$

$$\therefore \quad A^2 + xA + yI = O$$
$$\Rightarrow \quad 22 + 4x + y = 0, \quad 27 + 3x = 0$$
$$18 + 2x = 0, \quad 31 + 5x + y = 0$$
$$\Rightarrow \quad x = -9, y = 14$$

◉ **Example 73:** The system of equations

$$\begin{pmatrix}3&-2&1\\5&-8&9\\2&1&a\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}b\\3\\-1\end{pmatrix}$$

has no solution if a and b are

(a) $a = -3, b \neq 1/3$ (b) $a = 2/3, b \neq 1/3$
(c) $a \neq 1/4, b = 1/3$ (d) $a \neq -3, b \neq 1/3$

Ans. (a)

◎ **Solution:** The augmented matrix is given by

$$(A|B)=\left(\begin{array}{ccc|c}3&-2&1&b\\5&-8&9&3\\2&1&a&-1\end{array}\right)$$

Applying $R_1 \to 2R_1 - R_2$, we get

$$(A|B)\sim\left(\begin{array}{ccc|c}1&4&-7&2b-3\\5&-8&9&3\\2&1&a&-1\end{array}\right)$$

Applying $R_2 \to R_2 - 5R_1$, $R_3 \to R_3 - 2R_1$, we get

$$(A|B)\sim\left(\begin{array}{ccc|c}1&4&-7&2b-3\\0&-28&44&18-10b\\0&-7&a+14&5-4b\end{array}\right)$$

The system of equations will have no solution if

$$\frac{-28}{-7}=\frac{44}{a+14}\neq\frac{18-10b}{5-4b}$$
$$\Rightarrow \quad a + 14 = 11 \quad \text{and} \quad 20 - 16b \neq 18 - 10b$$
$$\Rightarrow \quad a = -3 \text{ and } b \neq -1/3$$

◉ **Example 74:** Suppose $I + A$ is non-singular. Let $B = (I + A)^{-1}$ and $C = I - A$, then

(a) $BC = CB$ (b) $BC = O$
(c) $BC = I$ (d) none of these

Ans. (a)

◎ **Solution:** Write $C = 2I - (I + A)$, then $BC = 2(I + A)^{-1} - I = CB$

◉ **Example 75:** Let $A = \begin{pmatrix}a&b\\c&d\end{pmatrix}$ be such that $A^3 = O$, but $A \neq O$, then

(a) $A^2 = O$ (b) $A^2 = A$
(c) $A^2 = I - A$ (d) none of these

Ans. (a)

◎ **Solution:** As $A^3 = O$, we get $|A^3| = 0$

$$|A|^3 = 0 \quad\Rightarrow\quad |A| \Rightarrow ad - bc = 0.$$

Also,

$$A^2=\begin{pmatrix}a^2+bc&(a+d)\,b\\(a+d)c&bc+d^2\end{pmatrix}$$
$$=\begin{pmatrix}a^2+ad&(a+d)b\\(a+d)c&ad+d^2\end{pmatrix}$$
$$= (a + d)A$$

If $a + d = 0$, we get $A^2 = O$.

Suppose $a + d \neq 0$, then

$$A^3 = A^2A = (a + d)A^2$$
$$\Rightarrow \quad O = (a + d)A^2 \Rightarrow A^2 = O$$

Assertion-Reason Type Questions

Example 76: Let A be a 2×2 matrix.
Statement-1: $adj(adj\,A) = A$
Statement-2: $|adj\,A| = |A|$

Ans. (b)

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $adj\,A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$\Rightarrow \qquad adj\,(adj\,A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$$

∴ Statement-1 is true.
Also, $|adj\,A| = ad - bc = |A|$
∴ Statement-2 is true
But statement-2 is not a correct explanation for statement-1.

Example 77: Let A be a 2×2 matrix with non-zero entries and let $A^2 = I$, where I is 2×2 identity matrix. Define

$Tr\,(A)$ = sum of diagonal elements of A, and
$|A|$ = determinant of matrix A.

Statement-1: $Tr\,(A) = 0$
Statement-2: $|A| = 1$.
Ans. (c)

Solution: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now, $\quad A^2 = I \;\Rightarrow\; |A^2| = |I\,|$

$\Rightarrow \quad |A|^2 = 1 \;\Rightarrow\; |A| = \pm 1$.

Suppose $|A| = 1$.

In this case, $A^2 = I \;\Rightarrow\; A = A^{-1}$

$$\Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Rightarrow \quad a = d, \quad b = 0, \quad c = 0$$

$$Tr\,(A) = 0 \;\Rightarrow a + d = 0$$

$$\Rightarrow \quad 2a = 0 \;\Rightarrow\; a = 0$$

In this case $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Thus, if we assume statement-2 is true then we get statement-2 is false.

In case $|A| = -1$, then

$$A = -A^{-1}$$

$$\Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Rightarrow \; a + d = 0$$

$$\therefore \quad |A| = -1 \;\Rightarrow\; Tr(A) = 0.$$

Therefore statement-1 is true and statement-2 is false.

Example 78: Let A be a 2×2 matrix with real entries. Let I be the 2×2 identity matrix. Denote by $Tr\,(A)$, the sum of diagonal entries of A. Assume that $A^2 = I$.

Statement-1: If $A \neq I$ and $A \neq -I$, then $\det(A) = -1$.
Statement-1: If $A \neq I$ and $A \neq -I$, then $Tr\,(A) \neq 0$.
Ans. (c)

Solution: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now, $\quad A^2 = I \;\Rightarrow\; \det(A^2) = 1$

$$\Rightarrow (\det A)^2 = 1 \;\Rightarrow\; \det A = \pm 1.$$

Also, $\quad A^2 = I \;\Rightarrow\; A = A^{-1}$

$$\Rightarrow \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det A \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If $\det A = 1$, then

$$a = d,\, b = -b,\, c = -c \;\Rightarrow\; a = d,\, b = c = 0.$$

In this case $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

$$|A| = 1 \;\Rightarrow\; a^2 = 1 \;\Rightarrow\; a = \pm 1.$$

∴ $\quad A = I$ or $A = -I$. A contradiction.

Thus, $\det(A) = -1$.

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -1 \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

$$\Rightarrow \quad a = -d \;\Rightarrow\; Tr\,(A) = a + d = 0.$$

∴ Statement-1 is true and Statement-2 is false.

Example 79: Let

$$A(\theta) = \begin{pmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{pmatrix}$$

Statement-1: $A(\pi/3)^3 = -I$
Statement-2: $A(\theta)\,A(\varphi) = A(\theta + \varphi)$

Ans. (a)

Solution: $A(\theta)\,A(\varphi) = \begin{pmatrix} a & b \\ -b & c \end{pmatrix}$ where

$$a = (\cos\theta + \sin\theta)(\cos\varphi + \sin\varphi) - 2\sin\theta\sin\varphi$$
$$= (\cos\theta\cos\varphi - \sin\theta\sin\varphi) + (\sin\theta\cos\varphi + \cos\theta\sin\varphi)$$
$$= \cos(\theta + \varphi) + \sin(\theta + \varphi)$$

$$b = \sqrt{2}\,[\sin\varphi(\cos\theta + \sin\theta) + \sin\theta(\sin\varphi - \cos\phi)]$$
$$= \sqrt{2}\sin(\theta + \varphi)$$

and $\quad c = -2\sin\theta\sin\varphi + (\cos\theta - \sin\theta)(\cos\varphi - \sin\varphi)$
$$= \cos\theta\cos\varphi - \sin\theta\sin\varphi - (\sin\theta\cos\varphi + \cos\varphi\sin\theta)$$
$$= \cos(\theta + \varphi) - \sin(\theta + \varphi)$$

Thus, $A(\theta)\,A(\varphi) = A(\theta + \varphi)$

$$A(\theta)^2 = A(2\theta)$$
$$A(\theta)^3 = A(2\theta)\,A(\theta) = A(3\theta).$$
$$\therefore \quad A(\pi/3)^3 = A(\pi) = -I.$$

◉ **Example 80:** Suppose $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies the equation $X^2 - 4X + 3I = O$

Statement-1: If $a + d \neq 4$, then there are just two such matrices X.

Statement-2: There are infinite number of matrices X, satisfying $X^2 - 4X + 3I = 0$.

Ans. (b)

◎ **Solution:** $X^2 - 4X + 3I = O \;\Rightarrow\; (X - I)(X - 3I) = O$

$$\Rightarrow \begin{pmatrix} (a-1)(a-3)+bc & b(a+d-4) \\ c(a+d-4) & (d-1)(d-3)+bc \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

If $a + d \neq 4$, then $b = 0$, $c = 0$.

$$(a-1)(a-3) = 0,\ (d-1)(d-3) = 0.$$
$$\Rightarrow \quad a = 1, 3,\ d = 1, 3$$

As $a + d \neq 4$, $a = 1$, $d = 1$ or $a = 3$, $d = 3$.

$$\text{Thus,} \quad X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } X = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

If $a + d = 4$, we get

$$\begin{pmatrix} (a-1)(a-3)+bc & 0 \\ 0 & (1-a)(3-a)+bc \end{pmatrix} = O.$$

∴ There are infinite number of matrices satisfying $X^2 - 4X + 3I = 0$ as there are infinite number of values of a, b, c satisfying the relation $a^2 - 4a + 3 + bc = 0$.

◉ **Example 81:** Suppose $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$.

Let X be a 2×2 matrix such that

$$X'AX = B.$$

Statement-1: X is non-singular and $\det(X) = \pm 2$

Statement-2: X is a singular matrix.

Ans. (c)

◎ **Solution:** If $X = O$, then $X'AX = O \;\Rightarrow\; B = O$.

A contradiction.

Let $\quad \det(X) = a$, then $\det(X') = a$

$\therefore \quad \det(X'AX) = \det(B)$

$\Rightarrow \quad a(-1)a = -4 \;\Rightarrow\; a = \pm 2$.

As $\det(X) \neq 0$, X cannot be a singular matrix.

◉ **Example 82:** Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Statement-1: If $X'AX = O$ for each X, then A must be a skew-symmetric matrix.

Statement-2: If A is symmetric and $X'AX = O$ for each X, then $A = O$

Ans. (b)

◎ **Solution:** Let $E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

As $\quad X'AX = O \;\forall\; X$,

$$E_1'AE_1 = O \;\Rightarrow\; a_{11} = 0$$
$$E_2'AE_2 = O \;\Rightarrow\; a_{22} = 0.$$

Next, $\quad (E_1 + E_2)'\,A(E_1 + E_2) = O$

$\Rightarrow \quad a_{12} + a_{21} = 0$

Thus, $A = \begin{bmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{bmatrix}$ is a skew-symmetric matrix.

In case A is symmetric, $a_{12} = a_{21}$

$\therefore \quad 2a_{12} = 0 \;\Rightarrow\; a_{12} = 0$.

Thus, $A = O$, in this case.

◉ **Example 83:** Suppose A, B and C are three 3×3 matrices, such that AB is invertible.

Statement-1: If $AB = AC$ then $B = C$.

Statement-2: A is a invertible.

Ans. (a)

◎ **Solution:** AB is invertible

$$\Rightarrow \quad \det(AB) \neq 0$$
$$\Rightarrow \quad \det(A)\det(B) \neq 0$$
$$\Rightarrow \quad \det(A) \neq 0$$

$\Rightarrow \quad A$ is invertible

Premultiplying both the sides of

$$AB = AC$$

by A^{-1}, we get

$$A^{-1}(AB) = A^{-1}(AC)$$
$$\Rightarrow \quad (A^{-1}A)B = (A^{-1}A)C$$
$$\Rightarrow \quad IB = IC \quad \text{or} \quad B = C.$$

◉ **Example 84:** Suppose α, β, γ are real numbers such that

$\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -3/2 \qquad (1)$

Statement-1: The matrix

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha & -2 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{pmatrix}$$

is a singular matrix.

Statement-2: $\cos\alpha + \cos\beta + \cos\gamma = 0$

and $\quad \sin\alpha + \sin\beta + \sin\gamma = 0$

Ans. (a)

◎ **Solution:** Write (1) as

$$\begin{aligned} &\cos^2\alpha + \cos^2\beta + \cos^2\gamma \\ &+ \sin^2\alpha + \sin^2\beta + \sin^2\gamma \\ &+ 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma \\ &+ 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha \\ &+ 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta = 0 \end{aligned}$$

$$\Rightarrow \quad (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$$

$$\Rightarrow \quad \cos\alpha + \cos\beta + \cos\gamma = 0$$

$$\text{and } \sin\alpha + \sin\beta + \sin\gamma = 0$$

Using $R_1 \to R_1 + R_2 + R_3$, we get R_1 consists of all zeros, therefore $\det(A) = 0$.

Thus, A is a singular matrix.

◉ **Example 85:** Let $A = (a_{ij})_{3\times 3}$ be a 3×3 matrix with $a_{ij} \in \mathbf{R}$ for $1 \le i \le 3$, $1 \le j \le 3$. Let X be a 3×1 matrix.

Statement-1: If A is skew-symmetric, then $I + A$ is non-singular.

Statement-2: If $AX = O$ implies $X = O$, then A is non-singular.

Ans. (a)

◎ **Solution:** If A is not non-singular, then $AX = O$ has infinitely many solutions.

Thus, if $AX = O \Rightarrow X = O$, then A must be non-singular.

Next, consider

$$(I + A)X = O$$

$$\Rightarrow \quad AX = -X \quad \Rightarrow \quad (AX) = (-X)'$$

$$\Rightarrow \quad X'A' = -X' \quad \Rightarrow \quad X'(-A) = -X'$$

$$\Rightarrow \quad X'A = X' \quad \Rightarrow \quad X'AX = X'X$$

$$\Rightarrow \quad X'(-X) = X'X \quad \Rightarrow \quad X'X = O \quad \Rightarrow \quad X = O.$$

$\therefore$ $I + A$ is non-singular.

LEVEL 2

Straight Objective Type Questions

◉ **Example 86:** Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $a, b \in \mathbf{N}$.

(a) Then there exists infinitely many B's such that $AB = BA$
(b) there cannot exist B such that $AB = BA$
(c) there exist more than one but finite number of B's such that $AB = BA$
(d) there exists exactly one B such that $AB = BA$

Ans. (a)

◎ **Solution:** If $b = a$, $B = aI$ and B commutes with every matrix A.

◉ **Example 87:** Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $|A^2| = 25$ then $|\alpha|$ equals

(a) 5^2 (b) 1
(c) 1/5 (d) 5

Ans. (c)

◎ **Solution:** $|A^2| = 25 \Rightarrow |A|^2 = 25$

$$\therefore \quad (25\alpha)^2 = 25 \Rightarrow 25\alpha^2 = 1 \quad \Rightarrow \quad |\alpha| = 1/5$$

◉ **Example 88:** If $A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ and $A + A' = I$, then α equals

(a) $2n\pi \pm \dfrac{\pi}{2}$, $n \in \mathbf{I}$ (b) $2n\pi \pm \dfrac{\pi}{3}$, $n \in \mathbf{I}$
(c) $2n\pi \pm \dfrac{2\pi}{3}$, $n \in \mathbf{I}$ (d) $2n\pi \pm \dfrac{4\pi}{3}$, $n \in \mathbf{I}$

Ans. (b)

◎ **Solution:**

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix}$$

$$\Rightarrow \quad 2\cos\alpha = 1 \quad \Rightarrow \quad \cos\alpha = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad \alpha = 2n\pi \pm \frac{\pi}{3},\ n \in \mathbf{I}$$

◉ **Example 89:** Let $A = \begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$

$\alpha \ne (2n + 1)\pi$, $n \in \mathbf{I}$, $(I + A)(I - A)^{-1}$ equals

(a) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ (b) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

(c) $\begin{bmatrix} \tan\alpha & 0 \\ 0 & \tan\alpha \end{bmatrix}$ (d) $\begin{bmatrix} \tan\alpha & 0 \\ 0 & -\tan\alpha \end{bmatrix}$

Ans. (a)

◎ **Solution:** $I - A = \begin{bmatrix} 1 & \tan(\alpha/2) \\ -\tan(\alpha/2) & 1 \end{bmatrix}$

$$\text{then} \quad (I - A)^{-1} = \frac{1}{\sec^2(\alpha/2)} \begin{bmatrix} 1 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 1 \end{bmatrix}$$

Now, $(I + A)(I - A)^{-1}$

$$= \frac{1}{\sec^2(\alpha/2)} \begin{bmatrix} 1 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} & \dfrac{-2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} \\ \dfrac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} & \dfrac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

◉ **Example 90:** Let $P = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP'$, then $P'Q^{2015}P$ is

(a) $\begin{bmatrix} 1 & -2015 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2015 & 1 \\ 0 & 2015 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2015 & 2015 \\ 0 & 2015 \end{bmatrix}$

Ans. (c)

◎ **Solution:** $P = \begin{bmatrix} \cos(\pi/6) & \sin(\pi/6) \\ -\sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$

$$\Rightarrow \quad P' = \begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix}$$

Since $PP' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow P' = P^{-1}$

We have $Q = PAP' = PAP^{-1}$

$\Rightarrow$ $Q^{2015} = (PAP^{-1})^{2015} = PA^{2015}P^{-1}$

Thus, $P'Q^{2015}P = P^{-1}(PA^{2015}P^{-1})P$

$= (P^{-1}P)A^{2015}(P^{-1}P)$

Now, $A = I + B$ where $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Since, $B^2 = O$, we get $B^r = O \ \forall\ r \geq 2$.

Thus, $A^{2015} = I + 2015B = \begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$

◉ **Example 91:** If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $6A^{-1} = A^2 + cA + dI$, then (c, d) is

(a) (–6, 11) (b) (–11, 6)

(c) (11, 6) (d) (6, 11)

Ans. (a)

◎ **Solution:** $6A^{-1} = A^2 + cA + dI$

$\Rightarrow$ $6I = A^3 + cA^2 + dA$

We have

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix}$$

and $A^3 = A^2A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{bmatrix}$$

Now,

$6I = A^3 + cA^2 + dA$

$6 = 1 + c + d, \quad 0 = 19 + 5c + d$

$6 = -11 - c + d$

$6 = 46 + 14c + 4d, \quad 0 = -38 - 10c - 2d$

$\Rightarrow$ $d = 11, \quad c = -6$

◉ **Example 92:** If a, b, c are non-zero, then number of solutions of

$$\frac{2x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \qquad (1)$$

$$-\frac{x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 0 \qquad (2)$$

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{2z^2}{c^2} = 0 \qquad (3)$$

is

(a) 6 (b) 8

(c) 9 (d) infinite

Ans. (d)

◎ **Solution:** From (2), (3), we get

$$\frac{3y^2}{b^2} - \frac{3z^2}{c^2} = 0 \Rightarrow \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

Putting this in (1) we get

$$\frac{2x^2}{a^2} - \frac{2y^2}{b^2} = 0 \Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2}$$

$$\therefore \quad \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = k^2 \quad \text{(say)}$$

$\Rightarrow$ $x = \pm\, ka, \quad y = \pm\, kb, \quad z = \pm\, kc,$

where $k \in \mathbf{R}$.

◉ **Example 93:** If A and B are two matrices such that $AB = B$ and $BA = A$, then $A^2 + B^2$ is equal to

(a) $2AB$ (b) $2BA$

(c) $A + B$ (d) AB

Ans. (c)

◎ **Solution:** We have

$A^2 + B^2 = (BA)^2 + (AB)^2$

$= (BA)(BA) + (AB)(AB)$

$= B(AB)A + A(BA)B$

$= B(BA) + A(AB) = BA + AB = A + B.$

Example 94: If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then $A^2 - 5A + 6I$ is equal to

(a) $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & -5 \\ -1 & -1 & 4 \\ -3 & -10 & 4 \end{bmatrix}$

(c) O (d) I

Ans. (a)

Solution: $A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$

Now, $A^2 - 5A + 6I$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+6 & -1-0+0 & 2-5+0 \\ 9-10+0 & -2-5+6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2-0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Example 95: The inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ -a & 0 & 0 \\ b & -c & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ -a & 0 & 0 \\ ac & b & 1 \end{bmatrix}$ (d) none of these

Ans. (a)

Solution: Using the formula for inverse of a 3 × 3 triangular matrix given in theory, A^{-1} is the matrix given in (a).

Example 96: If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$ then A^{-1} is

(a) A (b) A^2
(c) A^3 (d) A^4

Ans. (c)

Solution: To show that $A^{-1} = B \Leftrightarrow AB = I$. We have

$$A^2 = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A(A^2) = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 1 & * & * \\ -2 & * & * \\ * & * & * \end{bmatrix}$$

TIP

[Need not evaluate the remaining terms as $A^3 \neq I_3$]

Next

$$A^4 = (A^2)(A^2) = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Thus, $A^{-1} = A^3$.

EXERCISES

Concept-based Straight Objective Type Questions

1. Let A be a 2×2 invertible matrix. For which of the following functions $\det(f(A)) = f(\det(A))$ is **not** true?
(a) $f(x) = x^3$ (b) $f(x) = x^{-1}, x \neq 0$
(c) $f(x) = 1 + x$ (d) $f(x) = x^{-3}, x \neq 0$

2. Suppose $a, b, c, d \in \mathbf{C}$, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then which one of the following is **not** true?
(a) $|\det(A)| \le |a| + |b| + |c| + |d|$
(b) $|\det(A)| \le (|a| + |b|)(|c| + |d|)$
(c) $|\det(A)| \le (|a| + |c|)(|b| + |d|)$
(d) $|\det(A)| \le \sqrt{|a|^2 + |b|^2}\sqrt{|c|^2 + |d|^2}$

3. Let A, B be two 3 × 3 matrices with entries from real numbers. Which one of the following is true?

(a) $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
(b) $(AB)^2 = O \quad \Rightarrow \quad AB = O$
(c) $(A + B)(A - B) = A^2 - B^2$
(d) $(A + B)A = BA + A^2$

4. Suppose A and B are two 3×3 matrices with entries from complex numbers such that $ABA = I$. Which one of the following is **not** true?
(a) B is invertible (b) $B^{-1} = A^2$
(c) A is not invertible (d) $A^4B^2 = I$

5. Let A be a 3×3 matrix with entries from the set of real numbers. Suppose the equation $AX = B$ has a solution for every 3×1 matrix B with entries from the set of real numbers. Then
(a) $A'Y = B$ has no solution
(b) $A'Y = O \Rightarrow Y = O$
(c) $AX = O$ has a non-trivial solution
(d) A is a singular matrix

6. Suppose $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$ for some $\theta \in \mathbf{R}$. Let B, C, D be three 2×2 matrices such that $AB = AC - AD$, then
(a) $C' = B' - D'$ (b) $C + D = B$
(c) $C' = B' + D'$ (d) none of these

7. Suppose A, B are two 3×3 matrices such that A^{-1} exists. Then $(A - B)A^{-1}(A + B)$ is equal to
(a) $(A + B)A^{-1})(A - B)$ (b) $A^{-1}B + B^2$
(c) $(I - BAB^{-1})(A - B)$ (d) $(I + BAB^{-1})(A + B)$

8. Let A be 3×3 matrix such that A is orthogonal and idempotent, then
(a) A must be symmetric
(b) $\det(A) = -1$
(c) $A + A^{-1} = 1$
(d) none of these

9. Suppose A and B are two orthogonal matrices of the same size such that $\det(A) + \det(B) = 0$. Then
(a) $A + B = -I$ (b) $A + B = I$
(c) $\det(A + B) = 0$ (d) $A + B = O$

10. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbf{R}$. $ad \neq 0$
If $(a + d)A - A^2 = A'$, then
(a) $a = d$ (b) $a = d = 1$
(c) $a + d = 0$ (d) $a + d = 1$

LEVEL 1

Straight Objective Type Questions

11. Let $S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k \in \mathbf{N}$. Then $(S_2)^n (S_k)^{-1}$ (where $n \in \mathbf{N}$) is equal to:
(a) S_{2n+k} (b) S_{2n-k}
(c) S_{2^n+k-1} (d) S_{2^n-k}

12. Let S be the set of all 2×2 real matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = 3$ and $A' = A^2 - 3A$. Then
(a) S contains infinite number of elements
(b) $S = Q$
(c) S contains exactly two elements
(d) S contains exactly 2^4 elements

13. Let $A = (a_{ij})_{3\times2}$ be a 3×2 matrix with real entries and $B = AA'$. Then
(a) B^{-1} is a 3 × 3 matrix
(b) B^{-1} is a 2 × 2 matrix
(c) B^{-1} does not exist
(d) B^{-1} exists if and only if exactly one row of A consists of zeros

14. Let $A = (a_{ij})_{3\times3}$ be a matrix with $a_{ij} \in \mathbf{C}$. Let B be a matrix obtained by interchanging two columns of A. Then $\det(A + B)$ is equal to
(a) $\det(A) + \det(B)$
(b) 0
(c) $2\det(A)$
(d) $\det(A) - \det(B)$

15. If $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and

$$B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$ then B equals

(a) $(\cos\theta)I + (\sin\theta)J$
(b) $(\sin\theta)I + (\cos\theta)J$
(b) $(\cos\theta)I - (\sin\theta)J$
(d) $-(\cos\theta)I + (\sin\theta)J$

16. If A is both diagonal and skew-symmetric, then
(a) A is a symmetric matrix
(b) A is a null matrix
(c) A is a unit matrix
(d) none of these matrix

17. If $A^2 - 3A + 2I = 0$, then A^{-1} equals

    (a) $\frac{1}{2}(A - 3I)$ (b) $\frac{1}{2}(3I - A)$
    (c) $A + 3I$ (d) none of these

18. If A is a square matrix of order 3 such that $A^2 = 2A$, then $|A|^2$ is equal to

    (a) $2\,|A|$ (b) $8\,|A|$
    (c) $16\,|A|$ (d) 0

19. If A is a square matrix then which one of the following is *not* a symmetric matrix

    (a) $A + A'$ (b) AA'
    (c) $A'A$ (d) $A - A'$

20. If $A = (a_{ij})_{3\times 3}$ where $a_{ij} = \cos(i + j)$, then

    (a) A is symmetric
    (b) A is skew-symmetric
    (c) A is a triangular matrix
    (d) A is a singular matrix

21. If $A = (a_{ij})_{3\times 3}$ is a matrix satisfying the equation $x^3 - 3x + 1 = 0$, then

    (a) A is a unit matrix
    (b) A is singular matrix
    (c) A is non-singular matrix
    (d) none of these

22. If A and B are two square matrices of the same size, then $(A + B)^2 = A^2 + 2AB + B^2$ can hold if and only if

    (a) $AB = BA$ (b) $AB + BA = O$
    (c) $|AB| \neq 0$ (d) $|AB| = 0$

23. If $\begin{bmatrix} i & 0 \\ 3 & -i \end{bmatrix} + X = \begin{bmatrix} i & 2 \\ 3 & 4+i \end{bmatrix} - X$, then X is equal to

    (a) $\begin{bmatrix} 0 & -1 \\ 3 & i \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 0 & 2+i \end{bmatrix}$

    (c) $\begin{bmatrix} 1 & 0 \\ 0 & 2-i \end{bmatrix}$ (d) $\begin{bmatrix} i & 2 \\ 0 & 2+i \end{bmatrix}$

24. If $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then $AB + BA$ is

    (a) null matrix (b) unit matrix
    (c) invertible matrix (d) none of these

25. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$, then A is a nilpotent matrix of index

    (a) 2 (b) 3
    (c) 4 (d) 5

26. If A is a 2×2 unitary matrix, then $|A|$ is equal to

    (a) 1 (b) -1
    (c) ± 1 (d) none of these

27. If $A = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$, then $A^{-1} - A^2$ is equal to a

    (a) null matrix (b) invertible matrix
    (c) unit matrix (d) none of these

28. If C is a 3×3 matrix satisfying the relation $C^2 + C = I$, then C^{-2} is given by

    (a) $2\,C$ (b) $3\,C - I$
    (c) C (d) $2I + C$

29. If A, B and C are three square matrices of the same size such that $B = CA\,C^{-1}$, then CA^3C^{-1} is equal to

    (a) B (b) B^2
    (c) B^3 (d) B^9

30. If X is a 2×3 matrix such that $|X'X| \neq 0$, and $A = I - X(X'X)^{-1}X'$ then A^2 is equal to

    (a) A (b) I
    (c) A^{-1} (d) none of these

31. The matrix $A = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ is orthogonal if and only if

    (a) $p^2 + q^2 = 1$ (b) $p^2 = q^2$
    (c) $p^2 = q^2 + 1$ (d) none of these

32. The values of λ for which the matrix
    $A = \begin{pmatrix} \lambda & 0 & \lambda \\ \lambda & 0 & -\lambda \\ 0 & 1 & 0 \end{pmatrix}$ is orthogonal is

    (a) ± 1 (b) $\pm 1/\sqrt{3}$
    (c) $\pm 1/2$ (d) $\pm 1/\sqrt{2}$

33. The values of a for which the matrix
    $A = \begin{pmatrix} a & a^2-1 & -3 \\ a+1 & 2 & a^2+4 \\ -3 & 4a & -1 \end{pmatrix}$ is symmetric are

    (a) -1 (b) -2
    (c) 3 (d) 2

34. Let $A_t = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & t \\ 4 & 7-t & -6 \end{pmatrix}$, then the value(s) of t for which inverse of A_t does not exist.

    (a) $-2, 1$ (b) $3, 2$
    (c) $2, -3$ (d) $3, -1$

35. If $A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$, where $a^2 + b^2 + c^2 + d^2 = 1$, then A^{-1} is equal to

(a) $\begin{bmatrix} a-ib & -c+id \\ c+id & a+ib \end{bmatrix}$
(b) $\begin{bmatrix} a-ib & c-id \\ -c-id & a+ib \end{bmatrix}$
(c) $\begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$
(d) none of these

36. If $A = \begin{bmatrix} \frac{1}{2}\left(e^{ix}+e^{-ix}\right) & \frac{1}{2}\left(e^{ix}-e^{-ix}\right) \\ \frac{1}{2}\left(e^{ix}-e^{-ix}\right) & \frac{1}{2}\left(e^{ix}+e^{-ix}\right) \end{bmatrix}$ then A^{-1} exists

(a) for all real x
(b) for positive real x only
(c) for negative real x only
(d) none of these

37. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, then A^2 is equal

(a) O
(b) I
(c) $-I$
(d) none of these

38. If A is 2×2 matrix such that $A^2 = O$, then $tr\,(A)$ is

(a) 1
(b) -1
(c) 0
(d) none of these

39. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that A satisfies the relation $A^2 - (a + d)A = O$, then inverse of A is

(a) I
(b) A
(c) $(a + d)A$
(d) none of these

40. If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, then A^{-3} is

(a) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$
(b) $\frac{1}{27}\begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$
(c) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$
(d) $\frac{1}{27}\begin{bmatrix} -1 & 26 \\ 0 & -27 \end{bmatrix}$

41. If A is a skew Hermitian matrix, then the main diagonal elements of A are all

(a) purely real
(b) positive
(c) negative
(d) purely imaginary

42. If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ then $A^3 - 3A^2 - A + 9I$ is equal to

(a) O
(b) I
(c) A
(d) A^2

43. If $3A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ x & 2 & y \end{bmatrix}$ and $AA' = I$, then $x + y$ is equal to

(a) -3
(b) -2
(c) -1
(d) 0

44. If the system of equations $ax + y = 3$, $x + 2y = 3$, $3x + 4y = 7$ is consistent, then value of a is given by

(a) 2
(b) 1
(c) -1
(d) 0

45. If the system of equations $x + 2y - 3z = 1$, $(p + 2)z = 3$, $(2p + 1)\,y + z = 2$ is inconsistent, then the value of p is

(a) -2
(b) $-1/2$
(c) 0
(d) 2

46. The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution if

(a) $k \neq 0$
(b) $-1 < k < 1$
(c) $-2 < k < 2$
(d) $k = 0$

47. If $A = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is an invertible matrix, then x cannot take value

(a) -1
(b) 2
(c) 3
(d) none of these

48. If A and B are two square matrices of the same order, then which of the following is true.

(a) $(AB)' = A'B'$
(b) $(AB)' = B'A'$
(c) $|AB| = 0 \Rightarrow |A| = 0$ and $|B| = 0$
(d) none of these

49. The values of α for which the system of equations $x + y + z = 1$, $x + 2y + 4z = \alpha$, $x + 4y + 10z = \alpha^2$ is consistent, are given by

(a) $1, -2$
(b) $-1, 2$
(c) $1, 2$
(d) none of these

50. The system of homogeneous equations

$$tx + (t+1)\,y + (t-1)\,z = 0$$
$$(t+1)\,x + ty + (t+2)\,z = 0$$
$$(t-1)\,x + (t+2)\,y + tz = 0$$

has a non-trivial solution for

(a) three values of t
(b) two values of t
(c) one value of t
(d) infinite number of values of t

51. If A and B are two 3×3 matrices and $|A| \neq 0$, then which of the following are **not** true?

(a) $|AB| = 0 \Rightarrow |B| = 0$
(b) $|AB| \neq 0 \Rightarrow |B| \neq 0$
(c) $|A^{-1}| = |A|^{-1}$
(d) $|A + A| = 2|A|$

52. If $A = \begin{pmatrix} i & -i \\ -i & i \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, then A^8 equals

(a) $128\,B$ (b) $32\,B$
(c) $16\,B$ (d) $64\,B$

53. If $A = \begin{pmatrix} 2 & 3-i & -i \\ 3+i & \pi & 7+i \\ i & 7-i & e \end{pmatrix}$, then A is

(a) symmetric (b) Hermitian
(c) skew Hermitian (d) none of these

Assertion-Reason Type Questions

54. **Statement-1:** If A and B are two 3×3 matrices such that $AB = O$, then $A = O$ or $B = O$.

Statement-2: If A, B and X are three 3×3 matrices such that $AX = B$, $|A| \neq 0$, then $X = A^{-1}B$.

55. **Statement-1:** If $A = \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix}$, then $A^{100} = \begin{bmatrix} 1 & 100\pi \\ 0 & 1 \end{bmatrix}$.

Statement-2: If B is a 2×2 matrix such that $B^2 = O$, then $(I + B)^n = I + nB$ for each $n \in \mathbf{N}$.

56. **Statement-1:** As $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ satisfies the equation $x^3 - 5x^2 + 7x - 3 = 0$, then A is invertible.

Statement-2: If a square matrix A satisfies the equation $a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = 0$, and $a_n \neq 0$, then A is invertible.

57. **Statement-1:** If a, b, c are in A.P. the system of equations

$3x + 4y + 5z = a$ (1)
$4x + 5y + 6z = b$ (2)
$5x + 6y + 7z = c$ (3)

is consistent.

Statement-2: If $|A| \neq 0$, the system of equations $AX = B$ is consistent.

58. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Statement-1: If A is symmetric, then $X'AY = Y'AX$ for each pair of X and Y.

Statement-2: If $X'AY = Y'AX$ for each pair of X and Y, then A is symmetric.

59. For any $n \geq 2$. Let $M_n(\mathbf{R})$ denote the set of all. $n \times n$ matrices over the set of real numbers.

Statement-1: If $A \in M_n(\mathbf{R})$, $A \neq O$ with det $(A) = 0$, then det (Adj A) = 0

Statement-2: For $A \in M_n(\mathbf{R})$, det (Adj A) = (det $A)^{n-1}$

60. Let B be a 3×3 matrix such that $(I - B)^{-1}$ exists.

Statement-1: If $BX = X$ then $X = O$

Statement-2: $I + B + B^2 + \ldots + B^k = (I - B^{k+1})(I - B)^{-1}$ for all $k \in \mathbf{N}$.

LEVEL 2

Straight Objective Type Questions

61. If $\begin{pmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{pmatrix}\begin{pmatrix} 1 & \tan\theta \\ -\tan\theta & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, then

(a) $a = b = 1$
(b) $a = \cos 2\theta$, $b = \sin 2\theta$
(c) $a = \sin 2\theta$, $b = \cos 2\theta$
(d) $a = 1$, $b = \sin 2\theta$

62. If a, b, $c \neq 0$ and $a + b + c = 0$, then the matrix

$$\begin{bmatrix} 1+\frac{1}{a} & 1 & 1 \\ 1 & 1+\frac{1}{b} & 1 \\ 1 & 1 & 1+\frac{1}{c} \end{bmatrix}$$

is

(a) singular
(b) non-singular
(c) skew-symmetric
(d) orthogonal

63. Suppose matrix A satisfies the equation $A^2 - 5A + 7I = O$. If $A^8 = aA + bI$, then value of a is

(a) 1265 (b) 2599
(c) –2599 (d) 0

64. Let α, β, γ be three real numbers and

$$A = \begin{bmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{bmatrix}$$

then
(a) A is singular
(b) A is non-singular
(c) A is orthogonal
(d) none of these

65. Let $A(\theta) = \begin{pmatrix} \sin\theta & i\cos\theta \\ i\cos\theta & \sin\theta \end{pmatrix}$, then

(a) $A(\theta)^{-1} = A(-\theta)$
(b) $A(\theta)^{-1} = A(\pi - \theta)$
(c) $A(\theta)^{-1}$ does not exist
(d) $A(\theta)^2 = A(2\theta)$

66. Let A and B be two matrices such that $AB = BA$, then which one of the following **does not** hold for each $n \in \mathbf{N}$.

(a) $A^n B = BA^n$
(b) $(AB)^n = A^nB^n$
(c) $A^nB = B^nA$
(d) $A^nB^n = B^nA^n$

67. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$. The sum of all values of λ for which there exists a column vector $X \neq O$ such that $AX = \lambda X$, is

(a) 0 (b) 1
(c) 2 (d) 3

68. Let $a, b, c \in \mathbf{R}$ be such that $a + b + c > 0$ and $abc = 2$. Let

$$A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

If $A^2 = I$, then value of $a^3 + b^3 + c^3$ is

(a) 7 (b) 2
(c) 0 (d) –1

69. If A is a 3×3 skew-symmetric matrix with real entries and trace of A^2 equals zero, then

(a) $A = O$ (b) $2A = I$
(c) A is orthogonal (d) none of these

70 Suppose A and B are two 3×3 non-singular matrices such that

$$(AB)^k = A^kB^k$$

for $k = 2015, 2016, 2017$, then
(a) $AB = O$
(b) $BA = O$
(c) $AB = BA$
(d) $AB + BA = O$

71. Let A be a square matrix of order 3 such that $|\text{Adj } A| = 100$, then $|A|$ equals
(a) ±10 (b) –100
(c) 100 (d) 25

72. Let M be a 3×3 matrix satisfying

$$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \quad M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$$

then sum of the diagonal entries of M is

(a) 0 (b) –3
(c) 6 (d) 9

73. If A, B and $A + B$ are non-singular matrices, then

$$(A^{-1} + B^{-1})\,[A - A\,(A + B)^{-1}\,A]$$

equals

(a) O (b) I
(c) A (d) B

74. If $A + B$ is a non-singular matrix, then

$$A - B - A\,(A + B)^{-1}A + B\,(A + B)^{-1}B$$

equals

(a) O (b) I
(c) A (d) B

75. If A and B are two matrices such that

$A + B = AB$, then

(a) $A = I$ (b) $B = I$
(c) $AB = BA$ (d) $AB = I$

76. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A^3 = O$, then $a + d$ equals

(a) ad (b) bc
(c) 1 (d) 0

77. Let $A = \begin{pmatrix} 0 & x & 0 \\ y & 0 & -x \\ 0 & y & 0 \end{pmatrix}$, then A^3 equals

(a) O (b) x^2I
(c) $(x^2 + y^2)\,I$ (d) none of these

Previous Years' AIEEE/JEE Main Questions

1. If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then

(a) $\alpha = a^2 + b^2, \quad \beta = 2ab$

(b) $\alpha = a^2 + b^2, \quad \beta = a^2 - b^2$

(c) $\alpha = 2ab, \quad \beta = a^2 + b^2$

(d) $\alpha = a^2 + b^2, \beta = 2ab$ **[2003]**

2. Let $A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$. The only correct statement about matrix A is

(a) A^{-1} does not exist

(b) $A = (-1)I$, where I is a unit matrix

(c) A is a zero matrix

(d) $A^2 = I$ **[2004]**

3. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$ and $(10)B = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix}$.

If B is the inverse of matrix A, then α is

(a) 2 (b) –1

(c) –2 (d) 5 **[2004]**

4. If $A^2 - A + I = O$, then the inverse of A is

(a) $A - I$ (b) $I - A$

(c) $A + I$ (d) A **[2005]**

5. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then which one of the following holds for all $n \geq 1$, by the principle of mathematical induction

(a) $A^n = nA + (n - 1)I$

(b) $A^n = (2n - 1)A + (n - 1)I$

(c) $A^n = nA - (n - 1)I$

(d) $A^n = (2n - 1)A - (n - 1)I$ **[2005]**

6. If A and B are square matrices of size $n \times n$ such that

$A^2 - B^2 = (A - B)(A + B)$, then which of the following will be always true?

(a) either A or B is an identity matrix

(b) $A = B$

(c) $AB = BA$

(d) either A or B is a zero matrix **[2006]**

7. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ $a, b \in \mathbf{N}$. Then

(a) there exist infinitely many B's such that $AB = BA$

(b) there cannot exist B such that $AB = BA$

(c) there exist more than one but finite number of B's such that $AB = BA$

(d) there exists exactly one B such that $AB = BA$ **[2006]**

8. Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $|A^2| = 25$ then $|\alpha|$ equals

(a) 5^2 (b) 1

(c) 1/5 (d) 5 **[2007]**

9. Let A be a 2×2 matrix with real entries. Let I be the 2×2 identity matrix. Denote by tr(A), the sum of diagonal entries of A. Assume that $A^2 = I$.

Statement-1: If $A \neq I$ and $A \neq -I$, then det $(A) = -1$.

Statement-2: If $A \neq I$ and $A \neq -I$, then tr$(A) \neq O$. **[2008]**

10. Let A be a 2×2 matrix.

Statement-1: $adj\,(adj\,A) = A$

Statement-2: $|adj\,A| = |A|$ **[2009]**

11. Let A be a 2×2 matrix with non-zero entries and let $A^2 = I$, where I is 2×2 identity matrix. Define

$Tr(A)$ = Sum of diagonal elements of A,

$|A|$ = determinant of matrix A.

Statement-1: $Tr\,(A) = 0$

Statement-2: $|A| = 1$ **[2010]**

12. The number of 3×3 non-singular matrices with four entries as 1 and all other entries 0, is

(a) 6 (b) at least 7

(c) less than 4 (d) 5 **[2010]**

13. Consider the system of linear equations

$$x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + x_3 = 3$$
$$3x_1 + 5x_2 + 2x_3 = 1$$

The system has

(a) a unique solution
(b) no solution
(c) infinite number of solutions
(d) exactly 3 solutions **[2010]**

14. The number of values of k for which the linear equations

$$4x + ky + 2z = 0$$
$$kx + 4y + z = 0$$
$$2x + 2y + z = 0$$

possess a non zero solution is

(a) 0 (b) 3
(c) 2 (d) 1 **[2011]**

15. If $\omega \neq 1$ is cube root of unity and $H = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$ then H^{70} equals to

(a) O (b) $-H$
(c) H^2 (d) H **[2011]**

16. If trivial solution is the only solution of the system of linear equations:

$$x - ky + z = 0$$
$$kx + 3y - kz = 0$$
$$3x + y - z = 0$$

then set of all values of k is

(a) $\mathbf{R} - \{2, -3\}$ (b) $\mathbf{R} - \{2\}$
(c) $\mathbf{R} - \{-3\}$ (d) $\{2, -3\}$ **[2011]**

17. Let A and B be two symmetric matrices of order 3.

Statement-1: $A(BA)$ and $(AB)A$ are symmetric matrices

Statement-2: AB is symmetric if matrix multiplication of A with B is commutative **[2011]**

18. Let P and Q be 3×3 matrices such that $P \neq Q$. If $P^3 = Q^3$ and $P^2Q = Q^2P$, then det $(P^2 + Q^2)$ is equal to

(a) 1 (b) 0
(c) -1 (d) -2 **[2012]**

19. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to

(a) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (b) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ (d) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ **[2012]**

20. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to

(a) 11 (b) 5
(c) 0 (d) 4 **[2013]**

21. The number of values of k for which the system of equations

$$(k + 1)x + 8y = 4k$$
$$kx + (k + 3)y = 3k - 1$$

has no solution, is

(a) 1 (b) 2
(c) 3 (d) infinite **[2013]**

22. If the system of linear equations

$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_1 + 3x_2 + 5x_3 = 9$$
$$2x_1 + 5x_2 + ax_3 = b$$

is consistent and has infinite number of solutions, then

(a) $a = 8$, b can be any real number
(b) $b = 15$, a can be any real number
(c) $a \in \mathbf{R} - \{8\}$ and $b \in \mathbf{R} - \{15\}$
(d) $a = 8, b = 15$ **[2013, online]**

23. If p, q, r are 3 real numbers satisfying the matrix equation,

$$[p\ \ q\ \ r]\begin{bmatrix} 3 & 4 & 1 \\ 3 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix} = [3\ \ 0\ \ 1]$$

then $2p + q - r$ equal

(a) -3 (b) -1
(c) 4 (d) 2 **[2013, online]**

24. Consider the system of equations:
$x + ay = 0$, $y + az = 0$, and $z + ax = 0$.
Then the set of all values of a for which the system has a unique solution is

(a) $\mathbf{R} - \{1\}$ (b) $\mathbf{R} - \{-1\}$
(c) $\{1, -1\}$ (d) $\{1, 0, -1\}$
[2013, online]

25. Let $S = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| a_{ij} \in \{0, 1, 2\},\ a_{11} = a_{22} \right\}$

Then the number of non-singular matrices in the set S is

(a) 27 (b) 24
(c) 10 (d) 20 **[2013, online]**

26. The matrix $A^2 + 4A - 5I$, where I is the identity matrix and $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$, equals:

(a) $4\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ (b) $4\begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$

(c) $32\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ (d) $32\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

[2013, online]

27. Let A, other than I or $-I$, be a 2×2 real matrix such that $A^2 = I$, I being the unit matrix. Let Tr(A) be the sum of diagonal elements of A.

Statement-1: Tr$(A) = 0$

Statement-2: det$(A) = -1$ **[2013, online]**

28. If A is an 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then BB' equals:

(a) I (b) B^{-1}

(c) $(B^{-1})'$ (d) $I + B$ **[2014]**

29. If B is a 3×3 matrix such that $B^2 = O$, then det $[(I + B)^{50} - 50B]$ is equal to:

(a) 1 (b) 2

(c) 3 (d) 50 **[2014, online]**

30. Let A be a 3×3 matrix such that

$$A\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

then A^{-1} is :

(a) $\begin{bmatrix} 3 & 1 & 2 \\ 3 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$

[2014, online]

31. If $A = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$ be such that $AB = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, then:

(a) $y = 2x$ (b) $y = -2x$

(c) $y = x$ (d) $y = -x$ **[2014, online]**

32. Let A and B be any two 3×3 matrices. If A is symmetric and B is skew-symmetric, then the matrix $AB - BA$ is:

(a) skew-symmetric

(b) symmetric

(c) neither symmetric nor skew-symmetric

(d) I or $-I$, where I is an identity matrix.

[2014, online]

33. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to:

(a) $(2, -1)$ (b) $(-2, 1)$

(c) $(2, 1)$ (d) $(-2, -1)$ **[2015]**

34. If A is 3×3 matrix such that $|5\,\text{adj}\, A| = 5$, then $|A|$ is equal to:

(a) $\pm\frac{1}{5}$ (b) ± 5

(c) ± 1 (d) $\pm\frac{1}{25}$ **[2015, online]**

35. If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then which one of the following statement is **not** correct?

(a) $A^4 - I = A^2 + I$ (b) $A^3 - I = A(A - I)$

(c) $A^2 + I = A(A^2 - I)$ (d) $A^3 + I = A(A^3 - I)$

[2015, online]

36. If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A\,\text{adj}\, A = AA'$, then $5a + b$ is equal to

(a) -1 (b) 5

(c) 4 (d) 13 **[2016]**

37. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP'$, then $P'Q^{2015}P$ is:

(a) $\begin{bmatrix} 0 & 2015 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2015 & 0 \\ 1 & 2015 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2015 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 2015 & 1 \\ 0 & 2015 \end{bmatrix}$

[2016, online]

38. If $A = \begin{bmatrix} -4 & -1 \\ 3 & 1 \end{bmatrix}$, then the determinant of the matrix $(A^{2016} - 2A^{2015} - A^{2014})$ is:

(a) -175 (b) 2014

(c) 2016 (d) -25 **[2016, online]**

39. Let A be a 3×3 matrix such that $A^2 - 5A + 7I = 0$.

Statement-1: $A^{-1} = \frac{1}{7}(5I - A)$

Statement-2: The polynomial $A^3 - 2A^2 - 3A + I$ can be reduced to $5(A + 4I)$. Then

(a) Both the statements are true.

(b) Both the statements are false.

(c) Statement-1 is true, but Statement-2 is false.

(d) Statement-1 is false, but Statement-2 is true.

[2016, online]

Previous Years' B-Architecture Entrance Examination Questions

1. If A and B are square matrices of the same order then which one of the following is always true?
   (a) $(A+B)^{-1} = A^{-1} + B^{-1}$
   (b) adj (AB) = (adj B) (adj A)
   (c) A and B are non-zero and $|AB| = 0 \Leftrightarrow |A| = 0$ and $|B| = 0$
   (d) $(AB)^{-1} = A^{-1}B^{-1}$ **[2006]**

2. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and let I denote the 3×3 identity matrix. Then $2A^2 - A^3 =$
   (a) $A + I$ (b) $A - I$
   (c) $I - A$ (d) A **[2008]**

3. Let A and B be 2×2 matrices with real entries. Let I be the 2×2 identity matrix. Denote by tr(A) the sum of diagonal entries of A.
   Statement-1: $AB - BA \neq I$
   Statement-2: $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$ and $\text{tr}(AB) = \text{tr}(BA)$ **[2009]**

4. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 real matrix. If $A - \alpha I$ is invertible for every real number α, then
   (a) $bc > 0$ (b) $bc = 0$
   (c) $bc < \min\left(0, \frac{1}{2}ad\right)$ (d) $a = 0$ **[2010]**

5. Let A and B be two 2×2 matrices.
   Statement-1: $A\,(\text{adj}A) = |A|\,I_2$
   Statement-2: $\text{adj}(AB) = (\text{adj}A)(\text{adj}B)$
   [2011]

6. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, be a 2×2 matrix, where a, b, c, d take values 0 or 1 only. The number of such matrices which have inverses is:
   (a) 8 (c) 7
   (c) 6 (d) 5 **[2012]**

7. Let S be the set of all real matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + d = 2$ and $A' = A^2 - 2A$. Then S:
   (a) is an empty set
   (b) has exactly one element
   (c) has exactly two elements
   (d) has exactly four elements **[2013]**

8. Let $S_k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ $k \in \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers, then $(S_2)^n\,(S_k)^{-1}$, for $n \in \mathbf{N}$ is:
   (a) S_{2n+k-1} (b) S_{2^n+k-1}
   (c) S_{2^n-k} (d) S_{2n-k} **[2014]**

9. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. If $10A^{10} + \text{Adj}(A^{10}) = B$, then $b_1 + b_2 + b_3 + b_4$ is equal to:
   (a) 91 (b) 92
   (c) 111 (d) 112 **[2015]**

10. If for a matrix A, $|A| = 6$ and adj $A = \begin{bmatrix} 1 & -2 & 4 \\ 4 & 1 & 1 \\ -1 & k & 0 \end{bmatrix}$, then k is equal to
    (a) 0 (b) 1
    (c) 2 (d) -1 **[2016]**

Answers

Concept-based

1. (c)	**2.** (a)	**3.** (d)	**4.** (c)
5. (b)	**6.** (c)	**7.** (a)	**8.** (a)
9. (c)	**10.** (b)		

Level 1

11. (b)	**12.** (b)	**13.** (c)	**14.** (b)
15. (a)	**16.** (b)	**17.** (b)	**18.** (b)
19. (d)	**20.** (a)	**21.** (c)	**22.** (a)
23. (b)	**24.** (a)	**25.** (a)	**26.** (d)
27. (a)	**28.** (d)	**29.** (c)	**30.** (a)
31. (a)	**32.** (d)	**33.** (d)	**34.** (c)
35. (c)	**36.** (a)	**37.** (a)	**38.** (c)
39. (d)	**40.** (c)	**41.** (d)	**42.** (a)
43. (a)	**44.** (a)	**45.** (a)	**46.** (a)
47. (d)	**48.** (b)	**49.** (c)	**50.** (c)
51. (d)	**52.** (a)	**53.** (b)	**54.** (d)
55. (a)	**56.** (a)	**57.** (b)	**58.** (b)
59. (a)	**60.** (b)		

Level 2

61. (b) **62.** (b) **63.** (a) **64.** (a)
65. (b) **66.** (c) **67.** (d) **68.** (a)
69. (a) **70.** (c) **71.** (a) **72.** (d)
73. (b) **74.** (a) **75.** (c) **76.** (d)
77. (a)

Previous Years' AIEEE/JEE Main Questions

1. (a) **2.** (d) **3.** (d) **4.** (b)
5. (c) **6.** (c) **7.** (a) **8.** (c)
9. (c) **10.** (b) **11.** (c) **12.** (b)
13. (b) **14.** (c) **15.** (d) **16.** (a)
17. (b) **18.** (b) **19.** (c) **20.** (a)
21. (a) **22.** (d) **23.** (a) **24.** (b)
25. (d) **26.** (a) **27.** (a) **28.** (a)
29. (a) **30.** (a) **31.** (a) **32.** (b)
33. (d) **34.** (a) **35.** (c) **36.** (b)
37. (c) **38.** (d) **39.** (a)

Previous Years' B-Architecture Entrance Examination Questions

1. (b) **2.** (d) **3.** (a) **4.** (c)
5. (c) **6.** (c) **7.** (a) **8.** (d)
9. (d) **10.** (c)

Hints and Solutions

Concept-based

1. If $f(x) = 1 + x$, then det $f(A) = 1 + \det(A)$ may not hold. Take $A = -I$.

2. $|\det(A)| = |ad - bc| \le |a||d| + |b||c|$

$$\le (|a| + |b|)(|c| + |d|)$$

Also, $|\det(A)| \le (|a| + |c|)(|b| + |d|)$

For truth of (c), see Example 6

$\left[\text{(a) does not hold in general. Take } A = \begin{bmatrix} 5 & -1 \\ 10 & 2 \end{bmatrix}\right]$

3. (a), (b), (c) cannot hold in general unless $AB = BA$.

4. $\det(ABA) = 1$

$\Rightarrow \det(A)\det(B)\det(A) = 1$

$\Rightarrow \det(A) \ne 0, \quad \det(B) \ne 0$

$\Rightarrow$ Both A^{-1} and B^{-1} exist

Also $ABA = I \Rightarrow BA = A^{-1} \Rightarrow B = A^{-1}A^{-1} = A^{-2}$

$\Rightarrow B^2 = (A^{-2})^2 = A^{-4} \Rightarrow A^4B^2 = I$

5. Let X_1, X_2 X_3 be solutions of $AX = B$, when $B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ respectively. Let C be the 3×3 matrix $[X_1\ X_2\ X_3]$, then $AC = I \Rightarrow A$ is invertible.

$\Rightarrow$ A' is invertible.

Thus $A'Y = O \Rightarrow Y = O$.

6. $\det(A) = \cos^2\theta + \sin^2\theta = 1 \ne 0$

$\therefore$ A^{-1} exist.

Now, $AB = A(C - D)$

$\Rightarrow$ $A^{-1}(AB) = A^{-1}(A(C - D))$

$B = C - D \Rightarrow C = B + D$

$\Rightarrow$ $C' = (B + D)' = B' + D'$

7. $(A - B)A^{-1}(A + B)$

$$\begin{aligned}
&= (I - BA^{-1})(A + B) \\
&= A + B - BA^{-1}A - BA^{-1}B \\
&= A - BA^{-1}B \\
&= A + B - BA^{-1}A - BA^{-1}B \\
&= (A + B) - BA^{-1}(A + B) \\
&= (A + B)(I - BA^{-1}) \\
&= (A + B)A^{-1}(A - B)
\end{aligned}$$

8. $AA' = I$ and $A^2 = I$

$\therefore$ $AA' = A^2 \Rightarrow A^{-1}(AA') = A^{-1}A^2$

$\Rightarrow$ $A' = A$

9. As A, B are orthogonal matrices, $AA' = BB' = I$, $\det(A) \ne 0$, $\det(B) \ne 0$. Let $C = A + B$.

$$C' = A' + B' \Rightarrow AC' = AA' + AB' = I + AB'$$

$\Rightarrow AC'B = B + AB'B = B + A$

Now,

$$\begin{aligned}
\det(A + B) &= \det(AC'B) = \det(A)\det(C')\det(B) \\
&= \det(A)\det(C)(-\det(A))
\end{aligned}$$

$\Rightarrow$ $\det(C) = -(\det(A))^2\det(C) = -\det(C)$

$\Rightarrow$ $\det(C) = 0$

10. $A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & b(a + d) \\ c(a + d) & bc + d^2 \end{pmatrix}$

$\therefore$ $(a + d)A - A^2 = A'$

$\Rightarrow$ $\begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow ad = a, d$

As $ad \ne 0$, we get $a = d = 1$.

Level 1

11. $S_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = I + B$ where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{and}\, B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Note that $B^2 = O$, therefore, $B^r = O \quad \forall\ r \geq 2$.

Thus, $\quad (S_2)^n = (I + B)^n = I + nB$

$$= \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

Also, $\quad S_k^{-1} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$

$$\therefore\ (S_2)^n (S_k)^{-1} = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2n-k \\ 0 & 1 \end{pmatrix} = S_{2n-k}$$

12. $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & 3b \\ 3c & bc + d^2 \end{bmatrix}$$

$$\therefore \quad A^2 - 3A = \begin{bmatrix} a^2 - 3a + bc & 0 \\ 0 & d^2 - 3d + bc \end{bmatrix}$$

Since $\quad A' = A^2 - 3A$, we get

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2 - 3a + bc & 0 \\ 0 & d^2 - 3d + bc \end{bmatrix}$$

$\Rightarrow \qquad b = 0, c = 0, \quad a = a^2 - 3a, d = d^2 - 3d$

$\Rightarrow\ a\,(a - 4) = 0 \quad$ and $d(d - 4) = 0$

$\Rightarrow \qquad a = 0$ or $a = 4$ and $d = 0$ or $d = 4$

But then $a + d$ cannot be 3.

$\therefore \qquad S = \phi$

13. Let

$$A = \begin{pmatrix} a & p \\ b & q \\ c & r \end{pmatrix}, \text{ then}$$

$$B = AA' = \begin{pmatrix} a^2 + p^2 & ab + pq & ac + pr \\ ab + pq & b^2 + q^2 & bc + qr \\ ac + pr & bc + qr & c^2 + r^2 \end{pmatrix}$$

$$\det(B) = \begin{vmatrix} a & p & 0 \\ b & q & 0 \\ c & r & 0 \end{vmatrix}\begin{vmatrix} a & b & c \\ p & q & r \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$\therefore \quad B^{-1}$ does not exist.

14. In $A + B$ two rows are identical.

15. $B = (\cos\,\theta)I + (\sin\,\theta)J$.

16. A is a diagonal matrix, $a_{ij} = 0\ \forall\ i \neq j$

As A is a skew-symmetric matrix,

$$a_{ii} = 0\ \forall\ i$$

$$\therefore \qquad A = O$$

17. $A^2 - 3A + 2I \Rightarrow 2I = A(3I - A)$

$$\Rightarrow \qquad I = A\, \left\{\frac{1}{2}(3I - A)\right\}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{2}\ (3I - A)$$

18. $|\ A^2\ | = |\ 2A\ | = 2^3\ |\ A\ | \quad \Rightarrow \quad |\ A\ |^2 = 8\ |\ A\ |$

19. $(A - A')' = A' - (A')' = A' - A = -\,(A - A')$

$\therefore\ A - A'$ is skew-symmetric matrix.

20. $A = \begin{pmatrix} \cos 2 & \cos 3 & \cos 4 \\ \cos 3 & \cos 4 & \cos 5 \\ \cos 4 & \cos 5 & \cos 6 \end{pmatrix} = A'$

21. $A^3 - 3A + I = 0 \quad \Rightarrow \quad I = A(3I - A^2)$

$\Rightarrow \quad A$ is invertible and hence non-singular.

22. $(A + B)^2 = (A + B)\ (A + B) = A^2 + BA + AB + B^2$

$\therefore\ (A + B)^2 = A^2 + 2AB + B^2$ if and only if $AB = BA$

23. $2X = \begin{bmatrix} i & 2 \\ 3 & 4+i \end{bmatrix} - \begin{bmatrix} i & 0 \\ 3 & -i \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4+2i \end{bmatrix}$

$$\Rightarrow \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 2+i \end{bmatrix}$$

24. $AB = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

and $BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$

Thus, $AB + BA = O$

25. $A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore\ A$ is a nilpotent matrix of index 2

26. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^* = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix}$

$AA^* = I \quad \Rightarrow \quad |\ A\ |\ |\ A^*| = 1$

$\Rightarrow (ad - bc)\ (\bar{a}\,\bar{d} - \bar{b}\,\bar{c}) = 1$

$\Rightarrow |\ ad - bc\ |^2 = 1$

$\Rightarrow ad - bc = e^{i\theta}$ for some $\theta \in \mathbf{R}$.

27. $A = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} = \begin{pmatrix} \cos(4\pi/3) & \sin(4\pi/3) \\ -\sin(4\pi/3) & \cos(4\pi/3) \end{pmatrix}$

Note that $A^3 = I \Rightarrow A^2 - A^{-1} = O$

28. $C^2 + C = I \Rightarrow C(C + I) = I$

$\Rightarrow \quad C^{-1} = C + I$

Thus, $\quad C^{-2} = (C + I)^2 = C^2 + 2C + I$

$= I - C + 2C + I = 2I + C$

29. $CA^3C^{-1} = (CAC^{-1})\ (CAC^{-1})\ (CAC^{-1}) = B^3$

30. $A^2 = (I - X(X'X)^{-1} X')^2$

$= I - 2X\ (X'X)^{-1} X' + X(X'X)^{-1} X'\ X(X'X)^{-1} X'$

$= I - 2X(X'X)^{-1} X' + X(X'X)^{-1} X'$

$= I - X(X'X)^{-1} X' = A$

31. $AA' = I$

$$\Rightarrow \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \begin{pmatrix} p & q \\ -q & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} p^2+q^2 & 0 \\ 0 & p^2+q^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\therefore\ p^2 + q^2 = 1$

32. $AA' = I$

$$\Rightarrow \begin{pmatrix} \lambda & 0 & \lambda \\ \lambda & 0 & -\lambda \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & \lambda & 0 \\ 0 & 0 & 1 \\ \lambda & -\lambda & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2\lambda^2 & 0 & 0 \\ 0 & 2\lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore\ 2\lambda^2 = 1 \quad \Rightarrow \quad \lambda = \pm\, 1/\sqrt{2}$

33. $A' = A$

$$\Rightarrow \begin{pmatrix} a & a+1 & -3 \\ a^2-1 & 2 & 4a \\ -3 & a^2+4 & -1 \end{pmatrix} = \begin{pmatrix} a & a^2-1 & -3 \\ a+1 & 2 & a^2+4 \\ -3 & 4a & -1 \end{pmatrix}$$

$\Rightarrow a + 1 = a^2 - 1 \quad \text{and} \quad 4a = a^2 + 4$

$\Rightarrow a + 1 = 4a - 4 - 1 \quad \Rightarrow \quad 6 - 3a \quad \Rightarrow \quad a = 2.$

34. Since A_t^{-1} does not exist, $|\ A_t\ | = 0$

$\Rightarrow (-30 - t(7 - t)) - 3(-12 - 4t) + 2(14 - 2t - 20) = 0$

$\Rightarrow t^2 + t - 6 = 0 \quad \Rightarrow \quad t = -3,\ 2$

35. $A^{-1} = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$

36. $|\ A\ | = \dfrac{1}{4}\ [(e^{ix} + e^{-ix})^2 - (e^{ix} - e^{-ix})^2]$

$= \dfrac{1}{4}\ (4e^{ix}\ e^{-ix}) = 1 \neq 0$

Thus, A^{-1} exists for all $x \in \mathbf{R}$.

37. $A^2 = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$

$$= \begin{bmatrix} a^2b^2 - a^2b^2 & ab^3 - ab^3 \\ -a^3b + a^3b & -a^2b^2 + a^2b^2 \end{bmatrix} = O$$

38. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

Now, $A^2 = O$

$\Rightarrow a^2 + bc = 0, \quad (a + d)\ b = 0$

$(a + d)\ c = 0, \quad bc + d^2 = 0$

Suppose $a + d \neq 0$, then $b = 0$, $c = 0$.

$\therefore\ a^2 = 0,\ d^2 = 0 \quad \Rightarrow \quad a = 0,\ d = 0$

Contradiction.

Thus, $a + d = 0$ we get $\text{tr}(A) = 0$

39. Suppose A^{-1} exists, then

$A^{-1}\ A^2 - (a + d)A^{-1}\ A = O$

$\Rightarrow A - (a + d)I = O$

$$\Rightarrow \begin{bmatrix} -d & b \\ c & -a \end{bmatrix} = O$$

$\Rightarrow a = b = c = d = 0 \quad \Rightarrow |A| = 0$

A contradiction. Thus, A^{-1} does not exist.

40. We have $A^{-1} = \dfrac{1}{3}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$

$$\Rightarrow A^{-3} = \frac{1}{27}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$= \frac{1}{27}\begin{bmatrix} 1 & -8 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} = \frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$$

41. $A^* = -A = \Rightarrow (\bar{A})' = -A \Rightarrow \overline{a_{ii}} = -a_{ii}\ \forall\ i$

$\Rightarrow 2\text{Re}(a_{ii}) = 0 \Rightarrow a_{ii}$ is purely imaginary.

42. $A^2 = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{bmatrix}$$

Now, $A^3 - 3A^2 - A + 9I$

$$= \begin{bmatrix} 4 & 11 & 1 \\ -9 & -2 & -7 \\ 21 & 11 & 7 \end{bmatrix} - 3\begin{bmatrix} 4 & 3 & 0 \\ -3 & 2 & -2 \\ 6 & 4 & 5 \end{bmatrix}$$

$$- \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = O$$

43. $(3A)' = \begin{bmatrix}1 & 2 & x\\ 2 & 1 & 2\\ 2 & -2 & y\end{bmatrix}$

Now, $9I = (3A)\,(3A)'$

$$= \begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ x & 2 & y\end{bmatrix}\begin{bmatrix}1 & 2 & x\\ 2 & 1 & 2\\ 2 & -2 & y\end{bmatrix}$$

$$= \begin{bmatrix}9 & 0 & x+4+2y\\ 0 & 9 & 2x+2-2y\\ x+4+2y & 2x+2-2y & x^2+4+y^2\end{bmatrix}$$

$\Rightarrow\ x + 2y + 4 = 0$

$2x - 2y + 2 = 0$

$x^2 + y^2 + 4 = 9$

$\Rightarrow\ x = -2,\quad y = -1$

44. Solving $x + 2y = 3$, $3x + 4y = 7$, we get
$x = 1,\quad y = 1$
$\therefore\quad a + 1 = 3 \Rightarrow a = 2.$

45. If $p \neq -2,\ \dfrac{-1}{2}$ then $z = \dfrac{3}{p+2}$

and $y = \dfrac{1}{p+2}$ corresponding to this value we can get x.

If $p = -1/2$, we get $z = 2$.

Corresponding to which we get infinite number of value of x and y.

For $p = -2$, the system of equations is inconsistent.

46. The system of equations will have a unique solution if

$$\begin{vmatrix}1 & 1 & 1\\ 2 & 1 & -1\\ 3 & 2 & k\end{vmatrix} \neq 0 \Rightarrow \begin{vmatrix}1 & 0 & 0\\ 2 & -1 & -3\\ 3 & -1 & k-3\end{vmatrix} \neq 0$$

[using $C_1 \to C_1 - C_1$, $C_3 \to C_3 - C_1$]

$\Rightarrow (k - 3)(-1) -3 \neq 0 \Rightarrow k \neq 0.$

47. $|\,A\,| \neq 0 \Rightarrow 4(x + 1) - (2x - 3)(x + 2) \neq 0.$

$\Rightarrow 2x^2 - 3x - 10 \neq 0$

$\Rightarrow x \neq \dfrac{3 \pm \sqrt{89}}{4}$

48. See Theory.

49. Subtracting first equation from the second and third, we get

$y + 3z = \alpha - 1$

$3y + 9z = \alpha^2 - 1$

$\therefore\quad \alpha^2 - 1 = 3(\alpha - 1) \quad\Rightarrow\quad \alpha = 1 \quad\text{or}\quad \alpha = 2$

50. The system of equations will have a non-trivial solution if $\Delta = 0$ where

$$\Delta = \begin{vmatrix}t & t+1 & t-1\\ t+1 & t & t+2\\ t-1 & t+2 & t\end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix}t & t+1 & t-1\\ 1 & -1 & 3\\ -2 & 2 & -2\end{vmatrix} = \begin{vmatrix}2t+1 & t+1 & t-1\\ 0 & -1 & 3\\ 0 & 2 & -2\end{vmatrix}$$

[using $C_1 \to C_1 + C_2$]

$= (2t + 1)(2 - 6) = -4(2t - 1)$

Note that $\Delta = 0$ for $t = -1/2$ i.e., for just for one value of t.

51. Choice (d) is not true as $|\,A + A\,| = |\,2A\,| = 8|\,A\,|$

52. $A = iB$

$\Rightarrow A^2 = i^2B^2 = (-1)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = -2B$

$\Rightarrow A^4 = (-2B)^2 = 4B^2 = 4(2B) = 8B$

$\Rightarrow A^8 = (8B)^2 = 64B^2 = 64(2B) = 128B$

53. We have

$$A' = \begin{pmatrix}2 & 3+i & i\\ 3-i & \pi & 7-i\\ -i & 7+i & e\end{pmatrix}$$

$$\Rightarrow A^* = (\overline{A'}) = \begin{pmatrix}2 & 3-i & -i\\ 3+i & \pi & 7-i\\ i & 7+i & e\end{pmatrix} = A$$

Thus A is a Hermitian matrix.

54. Statement-2 is true since $|A| \neq 0$, implies A^{-1} exists.

$$\begin{aligned}
\therefore\quad && AX &= B\\
\Rightarrow\quad && A^{-1}(AX) &= A^{-1}B\\
\Rightarrow\quad && (A^{-1}A)X &= A^{-1}B\\
\Rightarrow\quad && IX &= A^{-1}B\\
\Rightarrow\quad && X &= A^{-1}B
\end{aligned}$$

That statement-1 is false can be seen by the following example.

Let $A = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ then

$AB = O$ but neither $A = O$ nor $B = O$

55. Since B commutes with I, we can use binomial theorem to obtain

$$(I + B)^n = I + \binom{n}{1}B + \binom{n}{2}B^2 + \ldots + \binom{n}{n}B^n$$

Since $B^2 = O$, we get $B^r = O\ \forall\ r \geq 2$.

Thus,

$(I + B)^n = I + nB$

Now,

$$A = I + B \text{ where } B = \begin{bmatrix} 0 & \pi \\ 0 & 0 \end{bmatrix}$$

Since $B^2 = O$, we get

$$A^{100} = I + 100B = \begin{bmatrix} 1 & 100\pi \\ 0 & 1 \end{bmatrix}$$

56. Since $a_0 A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = O$, and $a_n \neq 0$, we get $AB = I$ where

$$B = -\frac{a_0}{a_n} A^{n-1} - \frac{a_1}{a_n} A^{n-2} - \cdots - \frac{a_{n-1}}{a_n} I$$

$\Rightarrow \quad B = A^{-1}$.

57. If $|A| \neq 0$, A is invertible and we can write $AX = B$ as $X = A^{-1} B$.

$\therefore$ $AX = B$ has a unique solution and hence is consistent.

Subtracting (2) from (3) and (1) and (2), we get the system of equations as

$$3x + 4y + 5z = a \qquad (4)$$
$$x + y + z = b - a \qquad (5)$$
$$x + y + z = c - b \qquad (6)$$

As a, b, c are in A.P. $b - a = c - b$

$\therefore$ the last two equations are identical.

From (4) and (5) we obtain

$$x = 4b - 5a + k$$
$$y = 4a - 3b - 2k$$
$$z = k$$

where k is an arbitrary complex number. Thus, the system of equations in statement-1 is consistent.

58. Suppose A is symmetric, then $A' = A$

Since $X'AY$ is a 1×1, matrix,

$$X'AY = (X'AY)' = Y'A'(X')' = Y'AX$$

Next, suppose $X'AY = Y'AX$ for each pair of X and Y.

Let $E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Taking $X = E_1$ and $Y = E_2$, then

$$E_1'AE_2 = E_2'AE_1 \quad \Rightarrow \quad a_{12} = a_{21}$$

Thus, A is symmetric.

59. For truth of statement-2, see theory

As det (Adj A) = (det $A)^{n-1} = O$ if det $A = O$

Thus, statement-1 is true and statement-2 is correct reason for it.

60. We have

$$(I + B + B^2 + \ldots + B^k)(I - B)$$
$$= (I - B) + (B - B^2) + (B^2 - B^3) + \ldots + (B^k - B^{k+1})$$
$$= I - B^{k+1}$$

$\Rightarrow \quad I + B + B^2 + \ldots + B^k = (I - B^{k+1})(I - B)^{-1}$

$\therefore$ Statement-2 is true

Also, $BX = X \quad \Rightarrow \quad (I - B)X = O$

$\Rightarrow \quad X = (I - B)^{-1} O = O$

Thus, both the statements are true but statement-2 is not a correct reason for it.

Level 2

61. $$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{pmatrix} \begin{pmatrix} \cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

$\Rightarrow a = \cos 2\theta$, $b = \sin 2\theta$

62. $$\begin{vmatrix} 1 + 1/a & 1 & 1 \\ 1 & 1 + 1/b & 1 \\ 1 & 1 & 1 + 1/c \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} a+1 & b & c \\ a & b+1 & c \\ a & b & c+1 \end{vmatrix}$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$|A| = \frac{1}{abc}(a+b+c+1) \begin{vmatrix} 1 & b & c \\ 1 & b+1 & c \\ 1 & b & c+1 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$|A| = \frac{1}{abc}(a+b+c+1) \begin{vmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{abc}(a+b+c+1) \neq 0$$

63. $A^2 = 5A - 7I$

$$\Rightarrow A^4 = (A^2)^2 = (5A - 7I)(5A - 7I)$$
$$= 25A^2 - 70A + 49I$$
$$= 25(5A - 7I) - 70A + 49I$$
$$= 55A - 126I$$

$$\Rightarrow A^8 = (A^4)^2 = (55A - 126I)(55A - 126I)$$
$$= 3025A^2 - 13860A + 15876I$$
$$= 3025(5A - 7I) - 13860A + 158766I$$
$$= 1265A - 5299I$$

Thus, $a = 1265$.

64. $$|A| = \begin{vmatrix} 1 & \cos(\beta - \alpha) & \cos(\gamma - \alpha) \\ \cos(\alpha - \beta) & 1 & \cos(\gamma - \beta) \\ \cos(\alpha - \gamma) & \cos(\beta - \gamma) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \cos\gamma & 0 \end{vmatrix} \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} = 0$$

$\Rightarrow A$ is singular.

65. $|A(\theta)| = \sin^2\theta - i^2 \cos^2\theta = 1.$

$$\therefore A(\theta)^{-1} = \begin{pmatrix} \sin\theta & -i\cos\theta \\ -i\cos\theta & \sin\theta \end{pmatrix}$$

$$= A(\pi - \theta)$$

66. $A^n B = A^{n-1}(AB) = A^{n-1}(BA)$

$$= A^{n-2}(AB)A$$

$$= A^{n-2}(BA)A = A^{n-2}BA^2$$

Continuing in this way, we get

$A^nB = BA^n$

Similarly, $AB^n = B^nA$, $(AB)^n = A^nB^n$

and $A^nB^n = B^nA^n$

However, $A^nB = B^nA$ does not hold.

67. $AX = \lambda X$

$\Rightarrow (A - \lambda I)X = 0$

As $X \neq 0$, $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 1 \\ 0 & 1-\lambda & -1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1-\lambda & -1 \end{vmatrix} = 0$$

$= (1 - \lambda)[(1 - \lambda)^2 + 1] + 3[-2 - (1 - \lambda)] = 0$

$= (1 - \lambda)^3 - 2(1 - \lambda) - 6 = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 + 5\lambda + 7 = 0$ (1)

If $\lambda_1, \lambda_2, \lambda_3$ are roots of (1), then

$$\lambda_1 + \lambda_2 + \lambda_3 = 3.$$

68. $$A^2 = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{bmatrix}$$

where $\alpha = a^2 + b^2 + c^2$,

$\beta = bc + ca + ab.$

As $A^2 = I$, we get

$a^2 + b^2 + c^2 = \alpha = 1$

$bc + ca + ab = 0$

Now, $(a + b + c)^2 = \alpha + 2\beta = 1$

$\Rightarrow a + b + c = 1$

We have

$a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$

$= 1$

$\Rightarrow a^3 + b^3 + c^3 = 7$

69. Let $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Trace of $A^2 = (a^2 + b^2 + c^2) + (b^2 + d^2 + e^2)$

$+ (c^2 + e^2 + f^2)$

$\Rightarrow a^2 + 2b^2 + 2c^2 + d^2 + 2e^2 + f^2 = 0$

As a, b, c, d, e, f are real numbers, we get

$a = b = c = d = e = f = 0.$

$\therefore A = 0$

70. Let $k = 2015$. We are given

$(AB)^k = A^k B^k$ and $(AB)^{k+1} = A^{k+1}B^{k+1}$

We have

$A^{k+1}B^{k+1} = (AB)^{k+1} = (AB)^k(AB) = (A^kB^k)(AB)$

$\Rightarrow A^{k+1}B^{k+1} = A^k(B^kA)B$

As A, B are non-singular matrices, we get

$$AB^k = B^kA.$$

Similarly, using $(AB)^{k+1} = A^{k+1}B^{k+1}$

and $(AB)^{k+2} = A^{k+2}B^{k+2}$

we get $AB^{k+1} = B^{k+1}A$

Now, $AB^{k+1} = B^{k+1}A = B(B^kA)$

$= B(AB^k)$

$\Rightarrow \quad (AB)B^k = (BA)B^k$

As B is invertible, we get $AB = BA$.

71. $|\text{adj } A| = |A|^2 = 100 = |A|^2 \Rightarrow |A| = \pm 10$

72. $$M\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = M\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

and $$M\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = M\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$$

$$M\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\3\\0\end{bmatrix}\Rightarrow m_{11}=0$$

Similarly, $M\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}\Rightarrow m_{22}=2$

and $m_{33} = 7$

Thus, $m_{11} + m_{22} + m_{33} = 9$

73. $(A^{-1} + B^{-1})\,[A - A\,(A + B)^{-1} A]$

$= B^{-1}\,(B + A)A^{-1}\,\mathrm{A}[I - (A + B)^{-1} A]$

$= B^{-1}(A + B)\,(A + B)^{-1}[A + B - A]$

$= B^{-1}\,I\,B = I$

74. $A - B - A\,(A + B)^{-1} A + B\,(A + B)^{-1} B$

$= A[I - (A + B)^{-1} A] - [I - B\,(A + B)^{-1}]\,B$

$= A(A + B)^{-1}[A + B - A]$

$- [A + B - B]\,(A + B)^{-1} B$

$= A\,(A + B)^{-1} B - A\,(A + B)^{-1} B$

$= O$

75. $A + B = AB$

$\Rightarrow I - (A + B) + AB = I$

$\Rightarrow (I - A)\,(I - B) = I$

$\Rightarrow (I - A)^{-1} = I - B$

$\Rightarrow (I - B)\,(I - A) = I$

$\Rightarrow I - (B + A) + BA = I$

$\Rightarrow A + B = BA$

Thus, $AB = BA$

76. Use

$O = A^3 = (a + d)^2 A$

77. $A^2 = \begin{pmatrix}xy & 0 & -x^2\\ 0 & 0 & 0\\ y^2 & 0 & -xy\end{pmatrix}$

$A^3 = O$

Previous Years' AIEEE/JEE Main Questions

1. $A^2=\begin{bmatrix}a & b\\ b & a\end{bmatrix}\begin{bmatrix}a & b\\ b & a\end{bmatrix}=\begin{bmatrix}a^2+b^2 & 2ab\\ 2ab & a^2+b^2\end{bmatrix}$

$\therefore\ \alpha = a^2 + b^2,\ \beta = 2ab$

2. We have $A^2=\begin{pmatrix}0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0 & -1\\ 0 & -1 & 0\\ -1 & 0 & 0\end{pmatrix}=I$

3. As B is inverse of A, $AB = I$

$$\therefore\quad \begin{pmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}4 & 2 & 2\\ -5 & 0 & \alpha\\ 1 & -2 & 3\end{pmatrix}=10I$$

$$\Rightarrow\quad \begin{pmatrix}10 & 0 & 5-\alpha\\ 0 & 10 & \alpha-5\\ 0 & 0 & 5+\alpha\end{pmatrix}=\begin{pmatrix}10 & 0 & 0\\ 0 & 10 & 0\\ 0 & 0 & 10\end{pmatrix}$$

$\Rightarrow\quad \alpha = 5$

4. $A^2 - A + I = O \Rightarrow I = A - A^2 = A(I - A)$

$\Rightarrow A^{-1} = I - A$

5. For $n = 1$, all (a), (b), (c) and (d) hold good. We have

$$A^2=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$$

If (a) holds for some $n \geq 1$, then

$A^{n+1} = A^nA = [nA + (n - 1)I]\,A$

$= nA^2 + (n - 1)A$

$= n\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}-(n-1)\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$

$= \begin{bmatrix}2n-1 & 0\\ 3n-1 & 2n-1\end{bmatrix}$

$\neq (n+1)\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}+n\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=(n+1)A+nI$

Similarly (b) does not hold for $n + 1$.

For (c),

$A^{n+1} = A^nA = [nA - (n - 1)I]A$

$= nA^2 - (n - 1)A$

$= n\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}-(n-1)\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$

$= \begin{bmatrix}1 & 0\\ n+1 & 1\end{bmatrix}=(n+1)A-nI$

Thus, (c) holds.

Alternatively

Put $B=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$ and note that $A = I + B$ and $B^2 = O$. As I and B commute.

$$A^n = (I + B)^n = I + nB \quad [\because\ B^2 = O]$$
$$= n(B + I) - (n - I)\,I = nA - (n - I)I$$

6. $$A^2 - B^2 = (A - B)(A + B)$$
$$= A^2 - BA + AB - B^2$$
$$\Rightarrow \quad BA = AB$$

7. Each matrix of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ commutes with A.

8. $$|A| = 25\alpha \Rightarrow |A^2| = |A|^2 = 25$$
$\Rightarrow (25\alpha)^2 = 25$
$$\Rightarrow \quad |\alpha| = \frac{1}{5}$$

9. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now, $A^2 = I \Rightarrow \det(A^2) = 1$

$\Rightarrow (\det A)^2 = 1 \Rightarrow \det A = \pm 1.$

Also, $A^2 = I \Rightarrow A = A^{-1}$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det A \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

If det A = 1, then

$a = d,\ b = -b,\ c = -c \Rightarrow a = d,\ b = c = 0.$

In this case $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$|A| = 1 \Rightarrow a^2 = 1 \Rightarrow a = \pm 1.$

$\therefore A = I$ or $A = -I$. A contradiction.

Thus, det $(A) = -1$.

$$\therefore \begin{pmatrix} a & b \\ c & d \end{pmatrix} = -1\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$$

$\therefore a = -d \Rightarrow \text{Tr}(A) = a + d = 0$

$\therefore$ Statement-1 is true and Statement-2 is false.

10. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

adj $A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$|\text{adj } A| = ad - bc = |A|$

$\therefore$ Statement-2 is true.

adj (adj A) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$

Statement-1 is also true but Statement-2 is not a correct reason for it.

11. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Now, $A^2 = I \Rightarrow |A^2| = |I|$

$\Rightarrow \quad |A|^2 = 1 \Rightarrow |A| = \pm 1.$

Suppose $|A| = 1$.

In this case, $A^2 = I \Rightarrow A = A^{-1}$

$$\Rightarrow \begin{pmatrix} a & b \\ c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\Rightarrow \quad a = d,\ b = 0,\ c = 0$

$\text{Tr}(A) = 0 \Rightarrow a + d = 0$

$\Rightarrow \quad 2a = 0 \Rightarrow a = 0$

In this case $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Thus, in this case, if we assume statement-2 is true then we get statement-2 is false.

In case $|A| = -1$, then

$$A = -A^{-1}$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Rightarrow a + d = 0$$

$\therefore |A| = -1 \Rightarrow \text{Tr}(A) = 0.$

Therefore Statement-1 is true and Statement-2 is false.

12. The matrix

$$\begin{bmatrix} 1 & a & b \\ c & 1 & d \\ e & f & 1 \end{bmatrix}$$

where exactly one of a, b, c, d, e, f is 1 and rest of them are zeros, is invertible.

There are six such matrices.

Also, the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is invertible.

Thus, there are at least 7 such matrices which are invertible.

13. Adding the first two equations and subtracting the third from the sum, we obtain

$(x_1 + 2x_2 + x_3) + (2x_1 + 3x_2 + x_3) - (3x_1 + 5x_2 + 2x_3) = 3 + 3 - 1$

$\Rightarrow 0 = 5$

Thus, the system of equation has no solution.

14. The system of equations will have a non-zero solution if and only if

$$\begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} - k\begin{vmatrix} k & 1 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} k & 4 \\ 2 & 2 \end{vmatrix} = 0$$

$\Rightarrow 4(2) - k(k - 2) + 2(2k - 8) = 0$

$\Rightarrow k^2 - 6k + 8 = 0 \Rightarrow k = 2, 4$

15. $H = \omega I \Rightarrow H^{70} = \omega^{70} I$

But $\omega^{70} = (\omega^3)^{23}\omega = \omega$

$\therefore H^{70} = \omega I = H$

16. The system of equations will have only trivial solution if and only if

$$\begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} \neq 0$$

$$\Leftrightarrow (1)\begin{vmatrix} 3 & -k \\ 1 & -1 \end{vmatrix} + k\begin{vmatrix} k & -k \\ 3 & -1 \end{vmatrix} + (1)\begin{vmatrix} k & 3 \\ 3 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow 2(k^2 + k - 6) \neq 0 \Rightarrow k \neq -3, 2$

17. We have $A' = A$ and $B' = B$

Now, $(A(BA))' = (BA)'A' = (A'B')A'$

$= (AB)A = A(BA)$

[$\because$ matrix multiplication is associative]

$\Rightarrow A(BA)$ is symmetric.

Similarly, $(AB)\,A$ is a symmetric matrix

Also, $(AB)' = B'A' = BA$

Note that AB is symmetric if and only if $AB = BA$. Thus, both Statement-1 and Statement-2 are true but Statement-2 is not a correct explanation for the statement-1.

18. $P^3 = Q^3$ and $P^2Q = Q^2P$ gives

$$P^3 - P^2Q = Q^3 - Q^2P$$

$\Rightarrow P^2(P - Q) = -\,Q^2(P - Q)$

$\Rightarrow (P^2 + Q^2)\,(P - Q) = 0$

If det $(P^2 + Q^2) \neq 0$, then $P^2 + Q^2$ is invertible and hence $P = Q$. Therefore, det $(P^2 + Q^2) = 0$.

19. We have

$$A(u_1 + u_2) = Au_1 + Au_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We solve the above equation for $u_1 + u_2$.

We consider the augmented matrix

$$(A/B) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{array}\right)$$

Applying $R_3 \to R_3 - 2R_2 + R_1$ and $R_2 \to R_2 - 2R_1$, we get

$$(A/B) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow u_1 + u_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

20. $|P| = |A|^2 = 16$

$\Rightarrow (1)\,(0) - \alpha\,(4 - 6) + (3)(4 - 6) = 16$

$\Rightarrow 2\alpha - 6 = 16 \Rightarrow \alpha = 11$

21. As the system of equations has no solution

$$\frac{k+1}{k} = \frac{8}{k+3} \neq \frac{4k}{3k-1}$$

But $\dfrac{k+1}{k} = \dfrac{8}{k+3} \Rightarrow k^2 + 4k + 3 = 8k$

$\Rightarrow (k - 1)\,(k - 3) = 0 \Rightarrow k = 1, 3$

For $k = 1$, $\dfrac{8}{k+3} = \dfrac{4k}{3k-1}$

and for $k = 3$, $\dfrac{8}{k+3} = \dfrac{8}{6} = \dfrac{4}{3}$ and $\dfrac{4k}{3k-1} = \dfrac{3}{2}$

Thus, there is exactly one such value of k.

22. Adding first two equations and subtracting from third, we get

$(a - 8)x_3 = b - 15$

If $a \neq 8$, we get value of x_3, and substituting in first two we get values of x_1 and x_2. When $a \neq 8$, we get the system has a unique solution.

Thus, $a = 8$ and hence $b = 15$.

When $a = 8$ and $b = 15$, we solve first two equations in terms of x_3 to obtain infinite number of solutions.

23. $3p + 3q + 2r = 3$ (i)

$4p + 2q = 0$ (ii)

$p + 3q + 2r = 1$ (iii)

From (i) and (iii) $2p = 2$ or $p = 1$

When $p = 1$, $q = -2$ [from (ii)]

Now, $2p + q - r$

$$= 2p + q - \frac{1}{2}[1 - p - 3q]$$

$$= 2(1) + (-2) - \frac{1}{2}[1 - 1 - 6] = -3$$

24. As the system of equation has a unique solution.

$$\Delta = \begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} \neq 0$$

Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (1 + a)\begin{vmatrix} 1 & a & 0 \\ 1 & 1 & a \\ 1 & 0 & 1 \end{vmatrix}$$

Using $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = (1 + a)\begin{vmatrix} 1 & a & 0 \\ 0 & 1 - a & a \\ 0 & -a & 1 \end{vmatrix}$$

$= (1 + a)(1 - a + a^2) = 1 + a^3$

$\Delta \neq 0 \Rightarrow a \neq -1$ [$\because a \in \mathbf{R}$]

25. Let $A = \begin{pmatrix} a & b \\ c & a \end{pmatrix}$; $a, b, c \in \{0, 1, 2\}$.

$|A| = a^2 - bc$

When $a = 0$, $bc \neq 0 \Rightarrow b, c \in \{1, 2\}$

There are four such values of b, c, viz. $(b, c) = (1, 1), (1, 2), (2, 1)$ and $(2, 2)$

When $a = 1$, both b and c cannot be 1. In this case there are 8 such pairs of (b, c).

When $a = 2$, both b and c cannot be 2. In this case also, there are 8 such pairs of (b, c).

26. $$A^2 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$$

$$A^2 + 4A - 5I$$

$$= \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ 16 & -12 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 \\ 8 & 0 \end{bmatrix} = 4\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$$

27. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^2 = I \Rightarrow \det(A)^2 = 1$

$\Rightarrow \det(A) = \pm 1.$

Now, $A^2 = I \Rightarrow A^{-1} = A$

$$\Rightarrow \det(A)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (1)$$

If $\det(A) = 1$, then

$a = d$, $-b = b$, $-c = c$

$\Rightarrow a = d$, $b = 0$, $c = 0$.

$\therefore \det(A) = 1 \Rightarrow a^2 = 1 \Rightarrow a = \pm 1.$

Thus, in this case $A = I$ or $A = -I$.

But $A \neq I, -I$.

Therefore $\det(A) = -1$.

$\Rightarrow$ Statement-2 is true.

Now, from (1)

$$(-1)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Rightarrow a + d = 0 \Rightarrow \operatorname{tr}(A) = 0$

Thus, Statement-1 is true and Statement-2 is correct reason for it.

28. $$\begin{aligned} BB' &= (A^{-1}A')(A^{-1}A')' \\ &= (A^{-1}A')[(A')'(A^{-1})'] \\ &= A^{-1}(A'A)(A^{-1})' \\ &= A^{-1}(AA')(A')^{-1} \\ &= (A^{-1}A)[A'(A')^{-1}] \\ &= (I)(I) = I \end{aligned}$$

29. As $B^2 = 0$, $B^r = 0 \ \forall \ r \geq 2$.

Now, $(I + B)^{50} = I + {}^{50}C_1B + {}^{50}C_2B^2 + \cdots + {}^{50}C_{50}\,B^{50}$

$= I + 50B$ $[\because B^r = 0 \ \forall \ r \geq 2]$

$\Rightarrow (I + B)^{50} - 50B = I$

$\Rightarrow \det(I + B)^{50} - 50B = 1$

30. Applying $C_2 \leftrightarrow C_3$, we get

$$A\begin{bmatrix} 1 & 3 & 2 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $C_1 \leftrightarrow C_2$, we get

$$A\begin{bmatrix}3&1&2\\3&0&2\\1&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\Rightarrow \quad A^{-1}=\begin{bmatrix}3&1&2\\3&0&2\\1&0&1\end{bmatrix}$$

31. $AB=\begin{bmatrix}6\\8\end{bmatrix}$

$\Rightarrow \quad y + 2x + x = 6$

$3y - x + 2 = 8$

$\Rightarrow \quad 3x + y = 6$

$-x + 3y = 6$

$\Rightarrow \quad 3x + y = -x + 3y$

$\Rightarrow \quad y = 2x$

32. Let $C = AB - BA$, then

$C' = (AB - BA)' = (AB)' - (BA)'$

$= B'A' - A'B'$

$= (-B)A - A(-B)$

$= -BA + AB = C$

$\Rightarrow AB - BA$ is symmetric.

33. $AA^T = 9I$

$$\Rightarrow\begin{bmatrix}1&2&2\\2&1&-2\\a&2&b\end{bmatrix}\begin{bmatrix}1&2&a\\2&1&2\\2&-2&b\end{bmatrix}=9\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}9&0&a+4+2b\\0&9&2a+2-2b\\a+4+2b&2a+2-2b&a^2+4+b^2\end{bmatrix}=\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}$$

$\Rightarrow a + 2b = -4,\ a - b = -1,\ a^2 + b^2 = 5$

$\Rightarrow (a, b) = (-2, -1)$

34. $|5 \text{ adj } A| = 5$

$\Rightarrow 5^3\ |\text{adj } A| = 5$

$\Rightarrow |\text{adj } A| = 1/25$

$\Rightarrow |A|^2 = 1/25 \Rightarrow |A| = \pm\ 1/5$

35. $A^2=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}=-I$

$\Rightarrow A^3 = -A$ and $A^4 = I$

Now, $A^4 - I = O = A^2 + I$

and $A^3 - I = -A - I = -A + A^2 = A(A - I)$

and $A^3 + I = -A + I = A^4 - A = A(A^3 - I)$

But $A^2 + I = O$ and $A(A^2 - I) = -\ 2A$.

$\therefore\ A^2 + I \neq A(A^2 - I)$.

36. We know $A(\text{Adj } A) = |A|I_2$

$\therefore\ AA' = |A|I_2$

$$\Rightarrow \begin{bmatrix}5a&-b\\3&2\end{bmatrix}\begin{bmatrix}5a&3\\-b&2\end{bmatrix} = (10a + 3b)\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}25a^2+b^2&15a-2b\\15a-2b&13\end{bmatrix} = (10a + 3b)\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$\Rightarrow 25a^2 + b^2 = 10a + 3b,\ 15a - 2b = 0,\ 10a + 3b = 13$

Now, $b = (15/2)a\ 100 + \dfrac{45}{2}a = 13$

$\Rightarrow a = 2/5,\ b = 3$

$\therefore\ 5a + b = 2 + 3 = 5$

37. $P = \begin{pmatrix}\cos\alpha&\sin\alpha\\-\sin\alpha&\cos\alpha\end{pmatrix}$ where $\alpha = \pi/6$

$P' = \begin{pmatrix}\cos\alpha&-\sin\alpha\\\sin\alpha&\cos\alpha\end{pmatrix}$

We have $PP' = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\Rightarrow P' = P^{-1}$

Thus, $Q = PAP^{-1} \Rightarrow P^{-1}QP = A$

$\Rightarrow A^2 = (P^{-1}QP)^2 = P^{-1}Q^2P$

$\Rightarrow A^3 = P^{-1}Q^3P$

Continuing like this we get,

$A^{2015} = P^{-1}Q^{2015}P = P'Q^{2015}P$

We have $A = I + B$ where $B = \begin{pmatrix}0&1\\0&0\end{pmatrix}$

Also, $B^2 = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$

$\Rightarrow B^r = \begin{pmatrix}0&0\\0&0\end{pmatrix}\ \forall\ r \geq 2$

Thus, $A^{2015} = I + 2015B = \begin{pmatrix}1&2015\\0&1\end{pmatrix}$

38. Let $B = A^{2016} - 2A^{2015} - A^{2014}$

$= A^{2014}(A^2 - 2A - I)$

$= A^{2014}(A - (\sqrt{2} + 1)I)(A + (\sqrt{2} - 1)I)$

We have $|A| = \begin{vmatrix} -4 & -1 \\ 3 & 1 \end{vmatrix} = -1$

$$|A - (\sqrt{2} + 1)I| = \begin{vmatrix} -4-(\sqrt{2}+1) & -1 \\ 3 & 1-(\sqrt{2}+1) \end{vmatrix}$$

$= (-4 - (\sqrt{2} + 1)(1 - (\sqrt{2} + 1) + 3$

$= (\sqrt{2} + 1)^2 + 3(\sqrt{2} + 1) - 1 = 5 + 5\sqrt{2}$

$$|A + (\sqrt{2} - 1)I| = \begin{vmatrix} -4+(\sqrt{2}-1) & -1 \\ 3 & 1+(\sqrt{2}-1) \end{vmatrix}$$

$= (-4 + (\sqrt{2} - 1))(1 + (\sqrt{2} - 1)) + 3$

$= (\sqrt{2} - 1)^2 - 3(\sqrt{2} - 1) - 1$

$= 5 - 5\sqrt{2}$

Thus $\det(B) = (-1)^{2014}(5 + 5\sqrt{2})(5 - 5\sqrt{2})$
$= -25$

39. $A^2 - 5A = 7I = O$

$\Rightarrow 7I = A(5I - A) = (5I - A)A$

$I = AB = BA$

where $B = \frac{1}{7}(5I - A)$

$\therefore A^{-1} = B = \frac{1}{7}(5I - A)$

$\Rightarrow$ Statement-1 is true.

We have

$A^3 - 2A^2 - 3A + \mathrm{I}$

$= A(5A - 7I) - 2(5A - 7I) - 3A + I$

$= 5A^2 - 7A - 10A + 14I - 3A + I$

$= 5(5A - 7I) - 20A + 15I$

$= 25I - 35I - 20A + 15I$

$= 5(A - 4I)$

Thus, Statement-2 is also true.

Previous Years' B-Architecture Entrance Examination Questions

1. $\mathrm{adj}(AB) = (\mathrm{adj}\ B)(\mathrm{adj}\ A)$ [See Theory]

2. $A = I + B$

where $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Note that $B^2 = O$.

$A^2 = (I + B)^2 = I^2 + 2IB + B^2$

$= I + 2B$

$A^3 = AA^2 = (I + B)(I + 2B)$

$= I^2 + BI + 2(IB) + 2B^2$

$= I + 3B$ $[\because B^2 = O]$

Thus, $2A^2 - A^3 = 2(I + 2B) - (I + 3B)$

$= I + B = A.$

3. Statement-2 is true. [See Theory]

$\mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0$

But $\mathrm{tr}(I) = 2$.

$\therefore AB - BA \neq I.$

Thus, both statements are true and Statement-2 is a correct explanation for Statement-1.

4. As $A - \alpha I$ is invertible for all $\alpha \in \mathbf{R}$.

$\det(A - \alpha I) \neq 0 \quad \forall \alpha \in \mathbf{R}.$

$\Rightarrow (a - \alpha)(d - \alpha) - bc \neq 0 \quad \forall \alpha \in \mathbf{R}.$

$\Rightarrow \alpha^2 - (a + d)\alpha + ad - bc \neq 0 \quad \forall \alpha \in \mathbf{R}.$

Therefore

$(a + d)^2 - 4(ad - bc) < 0$

$\Rightarrow (a - d)^2 + 4bc < 0$

Therefore, $bc < 0$.

Also, $a^2 + d^2 - 2ad + 4bc < 0$

$\Rightarrow 0 \leq a^2 + d^2 < 2ad - 4bc$

$\Rightarrow bc < \frac{1}{2}ad.$

Thus, $bc < \min\left(0, \frac{1}{2}ad\right)$

5. Statement-2 is false as

$\mathrm{adj}(AB) = (\mathrm{adj}\ B)(\mathrm{adj}\ A)$

$\neq (\mathrm{adj}\ A)(\mathrm{adj}\ B)$

But Statement-1 is true as

$A(\mathrm{adj}\ A) = |A|I_2.$ [See Theory]

6. $\det(A) = ad - bc$

Note that $\det(A)$ can take value -1, 0 or 1. We have

$\det(A) = 1 \Leftrightarrow ad = 1, bc = 0$

$\Leftrightarrow a = 1,\ d = 1$ or $(b = 0,\ c = 0,\ b = 0;\ c = 1;\ b = 1,\ c = 0)$

and $\det(A) = -1 \Leftrightarrow ad = 0$ or $bc = 1$

This is also possible in 3 cases.

$\therefore A^{-1}$ exists in 6 cases.

7. $$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc & 2b \\ 2c & bc+d^2 \end{bmatrix}$$

$$\therefore A^2 - 2A = \begin{bmatrix} a^2-2a+bc & 0 \\ 0 & a^2-2d+bc \end{bmatrix}$$

Since $A' = A^2 - 2A$, we get

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2-2a+bc & 0 \\ 0 & a^2-2d+bc \end{bmatrix}$$

$\Rightarrow b = 0,\ c = 0,\ a = a^2 - 2a,\ d = d^2 - 2d$

$\Rightarrow a(a - 3) = 0$ and $d(d - 3) = 0$

$\Rightarrow a = 0$ or $a = 3$ and $d = 0$ or $d = 3$

But then $a + d$ cannot be 2.

$\therefore\ \ S = \phi$

8. $$S_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = I + B \text{ where}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Note that $B^2 = O$, therefore, $B^r = O\ \forall\ r \geq 2$.

Thus, $(S_2)^n = (I + B)^n = I + nB$

$$= \begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$$

Also, $S_k^{-1} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$

$$\therefore (S_2)^n (S_k)^{-1} = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2n-k \\ 0 & 1 \end{pmatrix} = S_{2n-k}$$

9. Let $A = I + C$ where $C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

AS $C^2 = O$, we get $C^r = 0\ \forall\ r > 2$.

Now,

$$A^{10} = (I + C)^{10}$$

$$= I + 10C + {}^{10}C_2\,(C^2) + \ldots + {}^{10}C_{10}(C^{10})$$

$$= I + 10C + O = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \operatorname{Adj}(A^{10}) = \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix}$$

As $\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = 10A^{10} + \operatorname{adj}(A^{10})$, we get

$$\Rightarrow \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} 10 & 100 \\ 0 & 10 \end{bmatrix} + \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 90 \\ 0 & 11 \end{bmatrix}$$

$\Rightarrow\ \ b_1 = b_4 = 11,\ b_2 = 90,\ b_3 = 0,$

$\therefore\ b_1 + b_2 + b_3 + b_4 = 112$

10. $|\operatorname{adj} A| = |A|^2 = 36$

$$\Rightarrow \begin{vmatrix} 1 & -2 & 4 \\ 4 & 1 & 1 \\ -1 & k & 0 \end{vmatrix} = 36$$

$$\Rightarrow (-1)\begin{vmatrix} -2 & 4 \\ 1 & 1 \end{vmatrix} - k\begin{vmatrix} 1 & 4 \\ 4 & 1 \end{vmatrix} = 36$$

[Expanding along R_3]

$\Rightarrow (-1)(-2 - 4) - k(1 - 16) = 36$

$\Rightarrow 6 + 15k = 36 \Rightarrow k = 2$

CHAPTER SIX

Permutations and Combinations

FUNDAMENTAL PRINCIPLES OF COUNTING

The Sum Rule

Suppose that A and B are two disjoint events (mutually exclusive); that is, they never occur together. Further suppose that A occurs in m ways and B in n ways. Then A or B can occur in $m + n$ ways. This rule can also be applied to more than two mutually exclusive events.

The Product Rule

Suppose that an event X can be decomposed into two stages, A and B. Let stage A occur in m ways and suppose that these stages are unrelated, in the sense that stage B occurs in n ways regardless of the outcome of stage A. Then event X occurs in mn ways. This rule is applicable even if event X can be decomposed in more than two stages.

PERMUTATIONS AND COMBINATIONS

Suppose we have a set of n distinct objects $\{x_1, x_2, \cdots, x_n\}$ from which we have to take a sample of r $(1 \leq r \leq n)$ objects. In how many ways can this be done? However, the problem is not completely specified; in particular, the phrase 'take a sample' is somewhat ambiguous.

1. Is the order in which we take the sample important? For example, if $r = 2$, do we consider first picking x_1 and then x_2 to be different from picking x_2 and then x_1?
2. Do we allow repetition of objects? For example, do we allow a sample to consists of two x_1?

Since each question has two answers, the product rule says that our original problem is really made up of four separate problems.

1. How many such samples are there if the order is important and we allow repetitions?
2. How many such samples are there if the order is important and we do not allow repetitions?
3. How many such samples are there if the order is not important and we allow repetitions?
4. How many such samples are there if the order is not important and we do not allow repetitions?

We now introduce some terminology. If the order is important, then we say that our sample consists of *arrangements*. If the order is not important then we have *selections*. If repetition is allowed, then we are using *replacement*. An arrangement of r objects with replacement is called an *r-sequence* (or simply sequence). Vehicle numbers are examples of sequences.

An *r-permutation* (or simply *permutation*) is an arrangement without replacement. The different orderings of a deck of cards are examples of permutations. A selection of r objects with replacement is called an *r-multiset*. The letters used to form the word MATHEMATICS can be thought of as a multiset with two Ms, two As, two Ts, one H, one E, one I, one C and one S. An *r-combination* (or simply *combination*) is a selection of r objects without replacement.

In Table 6.1 we illustrate the four different problems and their solutions. We take two objects from $\{x_1, x_2, x_3\}$.

Table 6.1

Repetition		*No repetition*	
$x_1\,x_1$ $x_1\,x_2$ $x_1\,x_3$ $x_2\,x_1$ $x_2\,x_2$ $x_2\,x_3$ $x_3\,x_1$ $x_3\,x_2$ $x_3\,x_3$	r-sequence	$x_1\,x_2$ $x_1\,x_3$ $x_2\,x_1$ $x_2\,x_3$ $x_3\,x_1$ $x_3\,x_2$	r-permutation
$\{x_1, x_1\}$ $\{x_1, x_2\}$ $\{x_1, x_3\}$ $\{x_2, x_2\}$ $\{x_2, x_3\}$ $\{x_3, x_3\}$	r-multiset	$\{x_1, x_2\}$ $\{x_1, x_3\}$ $\{x_2, x_3\}$	r-combination

What happens when $r = 0$? We may view the r-combination as a subset of r elements, and thus a 0-combination may be viewed as the empty set. Since there is exactly one empty set, we may say that there is exactly one 0-combination. Mathematicians also say that there is exactly one 0-sequence, one 0-permutation, and one 0-multiset. By relaxing the definitions we may use the terms sequence and permutation interchangeably and call a multiset, a combination.

SOME IMPORTANT RESULTS

1. The number of sequences (that is, permutations) of n distinct objects taken r at a time, when repetition of objects is allowed, is n^r $(r > 0)$.

2. The number of permutations of n distinct objects taken r $(0 \le r \le n)$ at a time, when repetition is not allowed, is given by

$$^nP_r = \frac{n!}{(n-r)!}$$

Conventionally, $^nP_r = 0$ for $r > n$.

3. The number of permutations of n objects taken all together, when p_1 of the objects are alike and of one kind, p_2 of them are alike and of the second kind, $\cdots$, p_r of them are alike and of the rth kind, where $p_1 + p_2 + \cdots + p_r = n$ is given by

$$\frac{n!}{p_1!\, p_2! \cdots p_r!}$$

4. The number of combinations of n distinct objects taken r $(0 \le r \le n)$ at a time is given by

$$\binom{n}{r} = {}^nC_r = \frac{n!}{(n-r)!r!}$$

By definition $^nC_r = 0$ if $r > n$.

5. The number of ways of selecting zero or more objects out of p_1 identical objects of first kind, p_2 identical objects of second kind, …, p_r identical objects of rth kind is $(p_1 + 1)(p_2 + 1) \ldots (p_r + 1)$.

6. The number of combinations of n distinct objects taken r $(\le n)$ at a time, when k $(0 \le k \le r)$ particular objects always occur, is $^{n-k}C_{r-k}$.

7. The number of combinations of n distinct objects taken r at a time, when $k\,(1 \le k \le n)$ never occur, is $^{n-k}C_r$.

8. If $n > 1$, $^nP_0 < {}^nP_1 < {}^nP_2 < \ldots < {}^nP_{n-1} = {}^nP_r$

9. If $n = 2m + 1$, then

$$^nC_0 < {}^nC_1 < {}^nC_2 \ldots < {}^nC_m = {}^nC_{m+1}$$
$$^nC_{m+1} > {}^nC_{m+2} > {}^nC_{m+3} > \ldots > {}^nC_n$$

10. If $n = 2m$, then

$$^nC_0 < {}^nC_1 < {}^nC_2 \ldots < {}^nC_m$$
$$^nC_m > {}^nC_{m+1} > {}^nC_{m+2} \ldots > {}^nC_n$$

CIRCULAR PERMUTATION

The number of ways of arranging n distinct objects around a circle is $(n-1)!$.

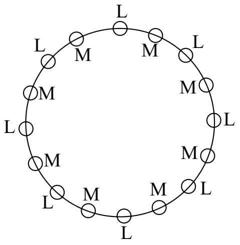
Remark

1. Note that even if one object is lying on the circle, then the circle has to be treated as a row, so far as arranging the objects around the circle is concerned.
2. Note that if no distinction is to be made between clockwise and counter-clockwise arrangements, then the number of arrangements is equal to $(n-1)!/2$.
3. The number of ways of arranging r identical objects and $(n-r)$ distinct objects along a circle is $\dfrac{(n-1)!}{r!}$

Illustration 1

In how many ways can 8 ladies and 5 men can be seated around a round table so that no two men are together.

Solution: 8 ladies can be arranged arround a round table in $(8-1)!$ ways. After the ladies have been arranged, there are 8 places for men. We can place 5 men at these places in $^8P_5$ ways. Thus, required number of ways is $(7!)\,(^8P_5)$.



SOME IMPORTANT IDENTITIES

(i) $^nC_0 = 1 = {}^nC_n$.

(ii) $^nC_r = {}^nC_{n-r}$ $\quad (0 \le r \le n)$.

(iii) $^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$.

(iv) $^nC_r = {}^nC_s$ implies $r = s$ or $r + s = n$.

(v) $^nC_r = \dfrac{n-r+1}{r}\, {}^nC_{r-1}$ $\quad (1 \le r \le n)$.

(vi) If n is even, then the greatest value of ^nC_r is ^nC_m, where $m = n/2$. If n is odd, then the greatest value of ^nC_r is ^nC_m, where $m = (n-1)/2$ or $(n+1)/2$.

(vii) $^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$.

(viii) $^nC_0 + {}^nC_2 + \cdots = {}^nC_1 + {}^nC_3 + \cdots = 2^{n-1}$.

(ix) $^{2n+1}C_0 + {}^{2n+1}C_1 + \cdots + {}^{2n+1}C_n = 2^{2n}$
$^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \ldots + {}^{2n+1}C_{2n+1} = 2^{2n}$

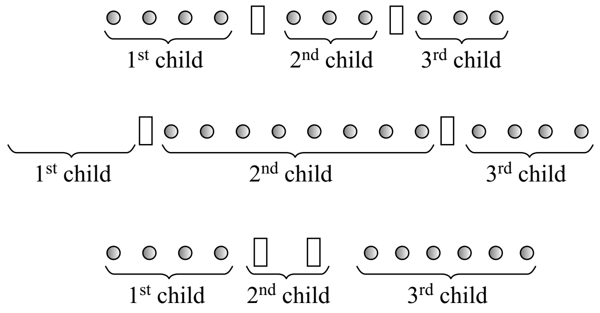
DIVISION OF IDENTICAL OBJECTS

The number of ways of distributing n identical objects among r persons giving zero or more to each is equal to the number of ways of arranging n identical object of one kind and $(r-1)$ identical objects of second kind (separators) in a row

$$= \frac{(n+r-1)!}{n!(r-1)!} = {}^{n+r-1}C_{r-1}$$

Illustration 2

To distribute 10 identical toys among 3 children, we may use two separators as follows:



etc.
The required number of ways

$$= \frac{(10+2)!}{10!2!} = {}^{12}C_2 = 66.$$

Alternative method

1. The number of distributing n identical objects among r persions giving zero or more to each

$= \text{coefficient of } x^n \text{ in } (1-x)^{-r}$

$= \text{coefficient of } x^n \text{ in } (1+x+x^2+\ldots)^r$

2. The number of ways of distributing n identical object among r persons giving at least one to each.

= the number of ways of distributing remaining $(n-r)$ identical objects (after giving one to each of r persons) among r persons giving zero or more to each

$= {}^{(n-r)+(r-1)}C_{r-1} = {}^{n-1}C_{r-1}$

Illustration 3

(i). To distribute 10 toys among 3 children giving at least one to each, we first give one toy to each of the child and distribute the remaining 7 toys among 3 children giving zero or more to each in ${}^{7+3-1}C_{3-1} = {}^9C_2 = 36$ ways.

(ii). However if we wish to give at least two to first, at least three to second and at least one to the third, then, we first give $(2+3+1)$ toys to the three children as desired and distribute the remaining 4 toys among three children giving zero or more to each. This can be done in ${}^{4+3-1}C_{3-1} = {}^6C_2 = 15$ ways.

(iii). If wish to distribute n identical object among r persons so that kth person $(1 \le k \le r)$ does not get more than a_k, then the desired number of ways

$= \text{Coeffcient of } x^n \text{ in } (1+x+\ldots+x^{a_1})(1+x+\ldots+x^{a_2}) \ldots (1+x+\ldots+x^{a_r})$

Illustration 4

Find the number of ways of distributing 10 identical toys among 3 children so as give a_i $(1 \le i \le 3)$ to the ith children, where $2 \le a_1 \le 4$, $3 \le a_2 \le 5$, $1 \le a_3 \le 7$.

The required number of ways

$= \text{coefficient of } x^{10} \text{ in } (x^2+x^3+x^4)(x^3+x^4+x^5)(x+\ldots+x^7)$

$= \text{coefficient of } x^{10} \text{ in } x^6(1+x+x^2)^2(1+\ldots+x^6)$

$$= \text{coefficient of } x^4 \text{ in } \left(\frac{1-x^3}{1-x}\right)^2 \left(\frac{1-x^7}{1-x}\right)$$

$= \text{coefficient of } x^4 \text{ in } (1-2x^3)(1-x)^{-3}$ [ignore any power that is more than 4]

$= \text{coeffcient of } x^4 \text{ in } (1-2x^3)(1+{}^3C_1 x+{}^4C_2 x^2+{}^5C_3 x^3+{}^6C_4 x^4+\ldots)$

$= {}^6C_4 - 2({}^3C_1) = 15 - 6 = 9$

Finding Number Integral Solutions of a Linear Equation

The number of non-negative integral solution of

$$x_1 + x_2 + \ldots + x_r = n,$$

where $n \in \mathbf{N} \cup \{0\}$, and $r \in \mathbf{N}$

= the number of ways of distributing n identical objects among r persons giving zero or more to each

$= {}^{n+r-1}C_{r-1}$.

MULTIPLICATION OF TWO INFINITE SERIES

The following informal method of multiplying two infinite power series is also useful:

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots)(b_0 + b_1x + b_2x^2 + \ldots)$$
$$= c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$$

where $c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \ldots a_nb_0$ for all $n \ge 0$.

DISTRIBUTION INTO GROUPS

1. The number of ways in which n distinct objects can be split into three groups containing respectively r, s and t objects (where r, s and t are distinct and $r+s+t=n$) is

$$({}^nC_r)({}^{n-r}C_s)({}^{n-r-s}C_t)$$

$$= \frac{n!}{r!(n-r)!}\,\frac{(n-r)!}{(n-r-s)!\,s!}\,\frac{(n-r-s)!}{(n-r-s-t)!\,t!}$$

$$= \frac{n!}{r!\,s!\,t!}$$

2. Suppose mn distinct objects are to be divided into m groups, each containing n objects and the order of the groups is not important. Then the number of ways of doing this is given by

$$\frac{(mn)!}{m!(n!)^m}$$

If, however, the order of groups is important, then the number of ways is given by

$$\frac{(mn)!}{(n!)^m}$$

SELECTION

1. The number of ways of selecting one or more objects out of n distinct objects is $2^n - 1$.
2. Suppose we have n distinct objects, and p like objects of one kind, q like objects of a second kind, r like objects of a third kind, etc. Then the number of ways of selecting one or more objects from these objects is
$2^n (p + 1)(q + 1)(r + 1) \cdots - 1$.

EXPONENT OF PRIME *p* IN *n*!

Let $E_p(m)$ denote the exponent of the prime p in the prime factorization of positive integer m. We have

$$E_p(n!) = E_p(1\cdot2\cdot3\cdot4 \cdots (n-1)\cdot n)$$

The last integer amongst $1, 2, 3, \cdots, (n-1), n$ which is divisible by p is $[n/p]p$, where $[x]$ denotes greatest integer $\le x$.

Therefore,

$$E_p(n!) = E_p\left(p\cdot2p\cdot3p\cdots\left[\frac{n}{p}\right]p\right)$$

because the remaining integers from the set $\{1, 2, 3, \cdots, (n-1), n\}$ are not divisible by p.

$$E_p(n!) = E_p\left(p\cdot2p\cdot3p\cdots\left[\frac{n}{p}\right]p\right)$$

$$= \left[\frac{n}{p}\right] + E_p\left(1\cdot2\cdot3\cdots\left[\frac{n}{p}\right]\right)$$

The last integer amongst $1, 2, \cdots, [n/p]$ which is divisible by p is

$$\left[\frac{[n/p]}{p}\right]p = \left[\frac{n}{p^2}\right]p$$

Since the remaining integers are not divisible by p, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1\cdot2\cdots\left[\frac{n}{p^2}\right]\right)$$

Continuing in this way, we get

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots + \left[\frac{n}{p^s}\right]$$

where $p^s \le n < p^{s+1}$.

Illustration 5

Find exponent of 3 in 32!

As 3 divides 3, 6, 9...., 30 amongst 1, 2, . . . 31, 32,

$E_3(32!) = E_3(3 \times 6 \times 9 \times \ldots \times 30)$

$= E_3(3^{10} \times 1 \times 2 \times \ldots \times 10)$

$= 10 + E_3(3 \times 6 \times 9)$

[Ignore 1, 2, 4, 5, 7, 8, 10 as 3does not divide these.]

$= 10 + E_3(3^3 \times 1 \times 2 \times 3)$

$= 10 + 3 + E_3(3)$

$= 10 + 3 + 1 = 14.$

Note, $10 = \left[\frac{31}{3}\right]$, $3 = \left[\frac{31}{3^2}\right]$ and $1 = \left[\frac{31}{3^3}\right]$

SOLVED EXAMPLES

Concept-based Straight Objective Type Questions

Example 1: If ${}^{n+5}p_{n+1} = \frac{1}{2}(11)(n-1)\left({}^{n+3}p_n\right)$ then n is equal to:

(a) 10, 11 (b) 9, 10
(c) 8, 7 (d) 6, 7

Ans: (d).

Solution: $\frac{1}{2}(11)(n-1) = \frac{{}^{n+5}P_{n+1}}{{}^{n+3}P_n}$

$$= \frac{(n+5)!}{4!} \times \frac{3!}{(n+3)!}$$

$\Rightarrow \quad \frac{1}{2}(11)(n-1) = \frac{1}{4}(n+5)(n+4)$

$\Rightarrow \quad 22n - 22 = n^2 + 9n + 20$

$\Rightarrow \quad n^2 - 13n + 42 = 0 \Rightarrow n = 6, 7$

Example 2: If ${}^{n+2}C_3 = {}^{n+3}P_2 - 20$, then n is equal to:

(a) 6 (b) 5
(c) 4 (d) 3

Ans (d).

Solution: $\frac{1}{6}(n+2)(n+1)n = (n+3)(n+2) - 20$

$\Rightarrow \quad n^3 + 3n^2 + 2n = 6(n^2 + 5n + 6) - 120$

$\Rightarrow \quad n^3 - 3n^2 - 28n + 84 = 0$

$\Rightarrow \quad (n - 3)(n^2 - 28) = 0$

As n is a positive integer, we get $n = 3$

Example 3: If ${}^{10}C_{r-1} > 2({}^{10}C_r)$ then r is equal to

(a) 6, 7, 8, 9 (b) 7, 8, 9
(c) 8, 9, 10 (d) 5, 6, 9, 10

Ans: (c).

Solution: Note that $0 \le r \le 10$. Now $\dfrac{{}^{10}C_{r-1}}{{}^{10}C_r} > 2$

$$\Rightarrow \quad \frac{10!}{(r-1)!\,(10-r+1)!}\cdot\frac{r!(10-r)!}{10!} > 2$$

$$\Rightarrow \quad \frac{r}{11-r} > 2 \Rightarrow r > 22 - 2r$$

$$\Rightarrow \quad 3r > 22 \Rightarrow r \ge 8 \Rightarrow r = 8, 9, 10.$$

Example 4: If $\frac{1}{a}({}^nP_{r+1}) = \frac{1}{b}({}^nP_r) = \frac{1}{c}({}^nP_{r-1})$, then $b^2 - (a + b)c$ is equal to.

(a) 0 (b) 1
(c) – 1 (d) – 2

Ans: (a).

Solution: $\dfrac{{}^nP_{r+1}}{{}^nP_r} = \dfrac{a}{b} \Rightarrow \dfrac{(n-r)!}{(n-r-1)!} = \dfrac{a}{b}$

$$\Rightarrow \quad n - r = \frac{a}{b}$$

Similarly $\quad n - r + 1 = \dfrac{b}{c}$

$$\therefore \quad \frac{b}{c} - \frac{a}{b} = 1$$

$$\Rightarrow \quad b^2 - ac = bc \Rightarrow b^2 - (a + b)c = 0.$$

Exemple 5: Let Q_n be the number of possible quadrilaterals formed by joining vertices of an n sided regular polygon. If $Q_{n+1} - Q_n = 20$, then value of n is:

(a) 8 (b) 7
(c) 6 (d) 5

Ans: (c).

Solution: Q_n = Number of ways of choosing four vertices out of n

$$= {}^nC_4$$

$$\therefore \quad 20 = Q_{n+1} - Q_n = {}^{n+1}C_4 - {}^nC_4$$

$$= {}^nC_4 + {}^nC_3 - {}^nC_4 = {}^nC_3$$

$$[\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$$

$$\Rightarrow \quad \frac{1}{6}n\,(n-1)(n-2) = 20 \qquad (1)$$

$$\Rightarrow \quad n^3 - 3n^2 + 2n - 120 = 0$$

$$\Rightarrow \quad n^3 - 6n^2 + 3n^2 - 18n + 20n - 120 = 0$$

$$\Rightarrow \quad (n - 6)(n^2 + 3n + 20) = 0$$

As $n \in \mathbf{N}$, $n = 6$.

TIP

You can directly reach the answer by substituting the given choices in (1).

Example 6: If there are 62 onto mapping from a set X containing n elements to the set $Y = \{-1, 1\}$, then n is equal to:

(a) 4 (b) 5
(c) 6 (d) 7

Ans: (c)

Solution: Let $X = \{x_1, x_2, \ldots, x_n\}$.

Each x_i can have two images *viz* – 1 and 1. Thus, there are 2^n mappings from X to Y. But there are exactly two mapping which are not onto. These are when all the elements are mapped to – 1 or when all the elements are mapped to 1. $\therefore$ there are $2^n - 2$ onto mapping from X to Y.

Set $\quad 2^n - 2 = 62 = \Rightarrow 2^n = 64 = 2^6$

$$\Rightarrow \quad n = 6$$

Example 7: The maximum possible number of points of intersection of 7 straight lines and 5 circles is:

(a) 111 (b) 109
(c) 107 (d) 105

Ans: (a)

Solution: Maximum possible number of points of intersection of 7 lines is ${}^7C_2 = 21$.

Maximum possible number of points of intersection of 5 circles in $2\,({}^5C_2) = 20$

[$\because$ two circles can intersect in two distinct points.]

Maximum possible number of points of intersection of 5 circles and 7 lines is $2(5)(7) = 70$

Thus, maximum possible number of points of intersection is

$$21 + 20 + 70 = 111.$$

Example 8: The number of injective functions from a set X containing m elements to a set Y containing n elements, for $m > n$ is:

(a) ${}^mP_n$ (b) $(m - n)!$
(c) ${}^mC_n$ (d) 0

Ans. (d)

Solution:

TIP

If X and Y are two finite sets and $f : X \to Y$ is an injective function from X to Y then $n \ge m$ where $m = n(X)$ and $n = n(Y)$. The number of injective functions from X to Y is

$$n(n-1)(n-2)\ldots(n-m+1) = {}^nP_m$$

As $m > n$, there does not exist any injective function from X to Y.

Example 9: If ${}^nP_r = 2520$ and ${}^nC_r = 21$ then n is equal to:

(a) 6 (b) 7
(c) 8 (d) 10

Ans: (b)

Solution: $(r!)({}^nC_r) = {}^nP_r \Rightarrow (r!)(21) = 2520$

$$\Rightarrow \quad r! = 120 \Rightarrow r = 5$$

$$\text{Now} \quad {}^nC_5 = 21 = \frac{1}{2!}(7 \times 6) = \frac{7!}{5!2!}$$

$$\Rightarrow \quad n = 7$$

Example 10: If ${}^{15}C_{r+1} : {}^{15}C_{3r} = 3 : 11$, then value of r is:

(a) 2 (b) 3
(c) 4 (d) 5

Ans: (b)

Solution: Note that $0 \le 3r \le 15$

$$\Rightarrow 0 \le r \le 5. \text{ Also, } \frac{15!}{(14-r)!(r+1)!} \times \frac{(3r)!(15-3r)!}{15!} = \frac{3}{11}$$

$$\Rightarrow \quad \frac{(3r)!(15-3r)!}{(14-r)!(r+1)!} = \frac{3}{11} \qquad (1)$$

For the denominator, to be divisible by 11, $11|(14-r)!$. Setting $14 - r = 11$.

$$\Rightarrow \quad r = 3$$

Putting $r = 3$, in LHS of (1) we get

$$\text{LHS of (1)} \; \frac{9!6!}{11!4!} = \frac{6 \times 5}{11 \times 10} = \frac{3}{11} = \text{RHS of (1)}$$

$$\therefore \quad r = 3$$

LEVEL 1

Straight Objective Type Questions

Example 11: If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then a value of n can be

(a) 6 (b) 7
(c) 8 (d) 9

Ans. (b)

Solution: As ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P.,

$$2({}^nC_5) = {}^nC_4 + {}^nC_6$$

$$\Rightarrow \quad 2 = \frac{{}^nC_4}{{}^nC_5} + \frac{{}^nC_6}{{}^nC_5}$$

$$= \frac{n!}{4!(n-4)!}\,\frac{5!(n-5)!}{n!} + \frac{n!}{6!(n-6)!}\,\frac{5!(n-5)!}{n!}$$

$$\Rightarrow \quad 2 = \frac{5}{n-4} + \frac{n-5}{6} \quad \Rightarrow \quad n^2 - 21n + 98 = 0$$

$$\Rightarrow \quad n = 7, 14.$$

Example 12: The least positive integer n for which ${}^{n-1}C_5 + {}^{n-1}C_6 < {}^nC_7$ is

(a) 14 (b) 15
(c) 16 (d) 28

Ans. (a)

Solution: Since ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$ we can write the given inequality as

$${}^nC_6 < {}^nC_7$$

$$\Rightarrow \quad \frac{{}^nC_6}{{}^nC_7} < 1 \quad \Rightarrow \quad \frac{7}{n-6} < 1 \quad \Rightarrow \quad n > 13$$

$\therefore$ the least value of n is 14.

Example 13: If ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$, then the value of r is equal to

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (c)

Solution: We have

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{36}{84} \Rightarrow \frac{n!}{(r-1)!(n-r+1)!}\,\frac{r!(n-r)!}{n!} = \frac{3}{7}$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{3}{7} \quad \Rightarrow \quad 10r = 3n + 3$$

$$\text{Similarly,} \quad \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{84}{126} \quad \Rightarrow \quad \frac{r+1}{n-r} = \frac{2}{3}$$

$$\Rightarrow \quad 5r + 3 = 2n \quad \Rightarrow \quad 5r + 3 = 2n$$

Solving we obtain $r = 3$.

Example 14: In a group of 8 girls, two girls are sisters. The number of ways in which the girls can sit in a row so that two sisters are not sitting together is

(a) 4820 (b) 1410
(c) 2830 (d) 30240

Ans. (d)

◎ **Solution:** The required number of ways
= the number of ways in which 8 girls can sit in a row – the number of ways in which two sisters sit together
= 8! – (2) (7!) = 30240.

◉ **Example 15:** The number of words that can be formed by using the letters of the word MATHEMATICS that start as well as end with T is:

(a) 80720 (b) 90720
(c) 20860 (d) 37528

Ans. (b)

◎ **Solution:** The word MATHEMATICS contains 11 letters viz. M, M, A, A T, T, H, E, I, C, S. The number of words that begin with T and end with T is

$$\frac{9!}{2!\,2!} = 90720.$$

◉ **Example 16:** If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ equals:

(a) $\frac{n}{2}a_n$ (b) $\frac{n}{4}a_n$
(c) na_n (d) $(n–1)a_n$

Ans. (a)

◎ **Solution:** Let $b_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-r}{{}^nC_{n-r}} = \sum_{r=0}^{n} \frac{n-r}{{}^nC_r}$

$$\Rightarrow \quad 2b_n = \sum_{r=0}^{n} \frac{r+(n-r)}{{}^nC_r} = n\sum_{r=0}^{n} \frac{1}{{}^nC_r} = na_n$$

$$\Rightarrow \quad b_n = \frac{n}{2}a_n\,.$$

◉ **Example 17:** *m* men and *w* women are to be seated in a row so that no two women sit together. If *m* > *w*, then the number of ways in which they can be seated is:

(a) $\frac{m!\,(m+1)!}{(m-w+1)!}$ (b) ${}^mC_{m-w}\,(m-w)!$
(c) ${}^{m+w}C_m\,(m-w)!$ (d) none of these

Ans. (a)

◎ **Solution:** We first arrange the *m* men. This can be done in *m*! ways. After *m* men have taken their seats, the women must choose *w* seats out of (*m* + 1) seats marked with *X* below.

X M X M X M X …X M X
1st 2nd 3rd *m*th (*m* + 1)th

They can choose *w* seats in ${}^{m+1}C_w$ ways and arrange *w* women in *w*! ways.
Thus, the required number of arrangements is

$$m!\,({}^{m+1}C_w)\,(w!) = \frac{m!\,(m+1)!\,w!}{w!\,(m+1-w)!} = \frac{m!\,(m+1)!}{(m+1-w)!}$$

◉ **Example 18:** The number of subsets of the set $A = \{a_1, a_2, \ldots, a_n\}$ which contain even number of elements is

(a) 2^{n-1} (b) $2^n - 1$
(c) 2^{n-2} (d) 2^n

Ans. (a)

◎ **Solution:** For each of the first (*n* – 1) elements $a_1, a_2, \ldots, a_{n-1}$ we have two choices: either a_i ($1 \le i \le n–1$) lies in the subset or a_i doesn't lie in the subset. For the last element we have just one choice. If even number of elements have already been selected, we do not include a_n in the subset, otherwise (when odd number of elements have been selected), we include it in the subset.
Thus, the number of subsets of $A = \{a_1, a_2,\ldots, a_n\}$ which contain even number of elements is equal to $2^{n–1}$.

◉ **Example 19:** The number of positive integers < 1,00,000 which contain exactly one 2, one 5 and one 7 in its decimal representation is

(a) 2940 (b) 7350
(c) 2157 (d) 1582

Ans. (a)

◎ **Solution:** We may consider a number of up to 5-digits to be a number of the form

X X X X X

where *X* is a digit from 0 to 9. The digit 2 can occupy any of the five places, 3 can occupy any of the remaining 4 places and 7 any of the 3-remaining places. The remaining 2 places can be filled up by 7 digits. Thus, there are (5) (4) (3) (7) (7) = 2940 positive integers in the desired category.

◉ **Example 20:** The number of ways of factoring 91,000 into two factors, *m* and *n*, such that *m* > 1, *n* > 1 and *gcd* (*m*, *n*) = 1 is

(a) 7 (b) 15
(c) 32 (d) 37

Ans. (a)

◎ **Solution:** We have $91{,}000 = 2^3 \times 5^3 \times 7 \times 13$
Let $A = \{2^3, 5^3, 7, 13\}$ be the set associated with the prime factorization of 91,000. For *m*, *n* to be relatively prime, each element of *A* must appear either in the prime factorization of *m* or in the prime factorization of *n* but not in both. Moreover, the 2 prime factorizations must be composed exclusively from the elements of *A*. Therefore, the number of relatively prime pairs *m*, *n* is equal to the number of ways of partitioning *A* into 2 unordered non-empty subsets. We can partition *A* as follows:

$\{2^3\}\cup\{5^3, 7, 13\}$, $\{5^3\} \cup \{2^3, 7, 13\}$
$\{7\}\cup\{2^3, 5^3, 13\}$, $\{13\} \cup \{2^3, 5^3, 7\}$
and $\{2^3, 5^3\}\cup\{7, 13\}$, $\{2^3, 7\} \cup \{5^3, 13\}$,
$\{2^3, 13\}\cup \{5^3, 7\}$

Therefore, the required number of ways = 4 + 3 = 7.

Example 21: The sum $S = \dfrac{1}{9!}+\dfrac{1}{3!7!}+\dfrac{1}{5!5!}+\dfrac{1}{7!3!}+\dfrac{1}{9!}$ equals

(a) $\dfrac{2^9}{10!}$ (b) $\dfrac{2^{10}}{8!}$

(c) $\dfrac{2^{11}}{9!}$ (d) $\dfrac{2^{10}}{7!}$

Ans. (a)

Solution: We can write S as follows

$$S = \frac{1}{10!}\,[{}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9]$$

$$= \frac{1}{10!}\,(2^9).$$

Example 22: Let $n = 2015$. The least positive integer k for which

$$k(n^2)\,(n^2 - 1^2)\,(n^2 - 2^2)\,(n^3 - 3^2)\,\ldots\,(n^2 - (n-1)^2) = r!$$

for some positive integer r is

(a) 2014 (b) 2013

(c) 1 (d) 2

Ans. (d)

Solution: We can rewrite the given expression as

$$k(n^2)\,(n-1)\,(n+1)\,(n-2)\,(n+2)\,(n-3)\,(n+3)\ldots\,(n+n-1)\;\;(n-n+1) = r!$$

$\Rightarrow\quad kn\,(1)\,(2)\ldots\,(n-1)\,n(n+1)\,(n+2)\ldots\,(2n-1) = r!$

$\Rightarrow\quad kn\,(2n-1)! = r!$

$\therefore$ To convert L.H.S. to a factorial, we shall require, $k = 2$ which will convert it into $(2n)!$

Example 23: If $0 < r < s \le n$ and ${}^nP_r = {}^nP_s$, then value of $r + s$ is

(a) $2n - 2$ (b) $2n - 1$

(c) 2 (d) 1

Ans. (b)

Solution: ${}^nP_r = {}^nP_s \;\Rightarrow\; \dfrac{n!}{(n-r)!} = \dfrac{n!}{(n-s)!}$

$\Rightarrow\quad (n-r)! = (n-s)!$

As $r < s$, $n - r > n - s$. But the only two different factorials which are equal are 0! and 1!. Thus $n - r = 1$ and $n - s = 0$

$\Rightarrow r = n-1$ and $s = n$.

$\Rightarrow\quad r + s = 2n - 1.$

Example 24: If $E = \dfrac{1}{4}.\dfrac{2}{6}.\dfrac{3}{8}.\dfrac{4}{10}\cdots\dfrac{30}{62}.\dfrac{31}{64} = 8^x$, then value of x is

(a) -7 (b) -9

(c) -10 (d) -12

Ans. (d)

Solution: We have

$$E = \frac{31!}{2^{31}(32!)} = \frac{1}{2^{31}(32)} = \frac{1}{2^{36}} = 2^{-36} = (2^3)^{-12} = 8^{-12}$$

Thus, $x = -12$.

Example 25: The value of

$$E = \frac{(1+17)\left(1+\frac{17}{2}\right)\left(1+\frac{17}{3}\right)\ldots\left(1+\frac{17}{19}\right)}{(1+19)\left(1+\frac{19}{2}\right)\left(1+\frac{19}{3}\right)\ldots\left(1+\frac{19}{17}\right)} \text{ is}$$

(a) 1 (b) ${}^{36}C_{17}$

(c) 2/19 (d) ${}^{36}C_{18}$

Ans. (a)

Solution: We have

$$(1+k)\left(1+\frac{k}{2}\right)\left(1+\frac{k}{3}\right)\ldots\left(1+\frac{k}{n}\right)$$

$$= \frac{(1+k)(2+k)(3+k)\ldots(n+k)}{(2)(3)\ldots(n)} = \frac{(n+k)!}{k!\,n!}$$

$$= {}^{n+k}C_k$$

Thus, both the numerator and the denominator of E equals ${}^{36}C_{17} = {}^{36}C_{19}$.

$\therefore\quad E = 1.$

Example 26: Kunal Gaba has n objects, each of weight w. He weighs them in pairs and finds the sum of the weights of all possible pairs is 120g. When his friend Rakshit weighs them in triplets, the sum of all possible weights is 240g. The value of n is

(a) 7 (b) 6

(c) 5 (d) 10

Ans. (b)

Solution: According to the given condition

$$({}^nC_2)\,(2w) = 120 \quad\Rightarrow\quad n(n-1)w = 120$$

and $\quad ({}^nC_3)\,(3w) = 240 \quad\Rightarrow\quad n(n-1)\,(n-2)w = 480$

Thus, $\quad \dfrac{n(n-1)(n-2)w}{n(n-1)w} = \dfrac{480}{120}$

$\Rightarrow\quad n - 2 = 4 \quad\Rightarrow\quad n = 6$

Example 27: If $[y]$ denote the greatest integer $\le y$, and $2\left[\dfrac{x}{8}\right]^2 + 3\left[\dfrac{x}{8}\right] = 20$, then x lies in the smallest interval $[a, b)$ where $b - a$ is equal to

(a) 6 (b) 5

(c) 4 (d) 8

Ans. (d)

Solution: $2\left[\dfrac{x}{8}\right]^2 + 3\left[\dfrac{x}{8}\right] - 20 = 0$

$$\Rightarrow \quad \left[\frac{x}{8}\right] = \frac{5}{2} \quad \text{or} \quad -4$$

As $\left[\frac{x}{8}\right]$ is an integer, we take

$$\left[\frac{x}{8}\right] = -4 \qquad \Rightarrow \qquad -4 \le x/8 < -3$$

$\Rightarrow -32 \le x < -24$. Thus, $a = -32$, and $b = -24$. Therefore, $b - a = 8$

Example 28: The sum $\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$ (where $\binom{p}{q} = 0$ if $p < q$) is maximum where m is

(a) 5 (b) 10

(c) 15 (d) 20

Ans. (c)

Solution: We have $\sum_{i=0}^{m} \binom{10}{i}\binom{20}{m-i}$

= the number of ways of choosing m persons out of 10 men and 20 women

= the number of ways of choosing m persons out of 30 persons

= ${}^{30}C_m$

But ${}^{30}C_m$ is maximum for $m = 15$.

Example 29: Let T_n denote the number of triangles which can be formed by using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals

(a) 5 (b) 7

(c) 6 (d) 4

Ans. (b)

Solution: The number of triangles that can be formed by using the vertices of a regular polygon is ${}^nC_3$. That is, $T_n = {}^nC_3$

$$\begin{aligned}
\text{Now,} \quad T_{n+1} - T_n &= 21 \\
\Rightarrow \quad {}^{n+1}C_3 - {}^nC_3 &= 21 \\
\Rightarrow \quad {}^nC_2 + {}^nC_3 - {}^nC_3 &= 21 \qquad [\because\ {}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r] \\
\Rightarrow \quad \frac{1}{2}\, n(n-1) &= 21 \quad \Rightarrow \quad n = -6 \text{ or } 7.
\end{aligned}$$

As n is a positive integer, $n = 7$.

Example 30: An eight digit number divisible by 9 is to be formed by using 8 digits out of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 without replacement. The number of ways in which this can be done is

(a) 9! (b) 2(7!)

(c) 4(7!) (d) (36) (7!)

Ans. (d)

Solution: We have $0 + 1 + 2 + 3 \ldots + 8 + 9 = 45$

To obtain an eight digit number exactly divisible by 9, we must not use either (0, 9) or (1, 8) or (2, 7) or (3, 6) or (4, 5). [Sum of the remaining eight digits is 36 which is exactly divisible by 9.]

When, we do not use (0, 9), then the number of required 8 digit numbers is 8!.

When one of (1, 8) or (2, 7) or (3, 6) or (4, 5) is not used, the remaining digits can be arranged in 8!– 7! ways.{0 cannot be at extreme left.}

Hence, there are 8! + 4(8! – 7!) = (36) (7!) numbers in the desired category.

Example 31: The number of rational numbers lying in the interval (2015, 2016) all whose digits after the decimal point are non-zero and are in decreasing order is

(a) $\sum_{i=1}^{9} {}^9P_i$ (b) $\sum_{i=1}^{10} {}^9P_i$

(c) $2^9 - 1$ (d) $2^{10} - 1$

Ans. (c)

Solution: A rational number of the desired category is of the form 2015. $x_1\, x_2 \ldots x_k$ where $1 \le k \le 9$ and $9 \ge x_1 > x_2 > \ldots > x_k \ge 1$. We can choose k digits out 9 in ${}^9C_k$ ways and arrange them in decreasing order in just one way. Thus, the desired number of rational numbers is ${}^9C_1 + {}^9C_2 + \ldots + {}^9C_9 = 2^9 - 1$.

Example 32: The number of functions f from the set $A = \{0, 1, 2\}$ in to the set $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) \le f(j)$ for $i < j$ and $i, j \in A$ is

(a) ${}^8C_3$ (b) ${}^8C_3 + 2({}^8C_2)$

(c) ${}^{10}C_3$ (d) ${}^{10}C_4$

Ans. (c)

Solution: A function $f : A \to B$ such that $f(0) \le f(1) \le f(2)$ falls in one of the following four categories.

Case 1 $f(0) < f(1) < f(2)$

There are ${}^8C_3$ functions in this category.

Case 2 $f(0) = f(1) < f(2)$

There are ${}^8C_2$ functions in this category.

Case 3 $f(0) < f(1) = f(2)$

There are again ${}^8C_2$ functions in this category.

Case 4 $f(0) = f(1) = f(2)$

There are ${}^8C_1$ functions in this category.

Thus, the number of desired functions is

$${}^8C_3 + {}^8C_2 + {}^8C_2 + {}^8C_1 = {}^9C_3 + {}^9C_2 = {}^{10}C_3.$$

Example 33: The number of positive integral solution of the equation $x_1\, x_2\, x_3\, x_4\, x_5 = 1050$ is

(a) 1800 (b) 1600
(c) 1400 (d) none of these

Ans. (d)

◎ **Solution:** Using prime factorization of 1050, we can write the given equation as

$$x_1\, x_2\, x_3\, x_4\, x_5 = 2 \times 3 \times 5^2 \times 7$$

We can assign 2, 3 or 7 to any of 5 variables. We can assign entire 5^2 to just one variable in 5 ways or can assign $5^2 = 5 \times 5$ to two variables in ${}^5C_2$ ways. Thus, 5^2 can be assigned in

$$ {}^5C_1 + {}^5C_2 = 5 + 10 = 15 \text{ ways}$$

Thus, the required number of solutions is $5 \times 5 \times 5 \times 15 = 1875$.

◉ **Example 34:** The number of ways in which we can arrange the digits 1, 2, 3, …, 9 such that the product of five digits at any of the five consecutive positions is divisible by 7 is

(a) 7! (b) ${}^9P_7$
(c) 8! (d) 5(7!)

Ans. (c)

◎ **Solution:** Let an arrangement of 9 digit number be $x_1\, x_2\, x_3\, x_4\, x_5\, x_6\, x_7\, x_8\, x_9$. Note that we require product of each of $(x_1, x_2, x_3, x_4, x_5)$; $(x_2, x_3, x_4, x_5, x_6)$; …; $(x_5, x_6, x_7, x_8, x_9)$ is divisible by 7.

This is possible if the 5th digit is 7.

Therefore, we can arrange the 9 digits in desired number of ways in 8! ways.

◉ **Example 35:** Let X be a set containing n elements. The number of reflexive relations that can be defined on X is

(a) 2^{n^2} (b) $2^{n^2 - n}$
(c) $n(2^{n^2 - n})$ (d) $n(2^{n^2})$

Ans. (b)

◎ **Solution:** Let $X = \{x_1, x_2, \ldots x_n\}$. The set $X \times X$ contains n^2 elements. A reflexive relation R on X must contain (x_i, x_i) for $1 \le i \le n$, i.e. R must contain the n elements (x_1, x_1), (x_2, x_2) . . . (x_n, x_n) and any subset of the set containing remaining $n^2 - n$ elements. Therefore, the number of reflexive relations that can be defined on X is $2^{n^2 - n}$

◉ **Example 36:** Suppose X contains m elements and Y contain n elements. The number of functions from X to Y is

(a) nm (b) n^m
(c) ${}^nP_m$ (d) ${}^nC_m$

Ans. (b)

◎ **Solution:** Let $X = \{x_1, x_2, \ldots x_m\}$ and

$$Y = \{y_1, y_2, \ldots y_n\}$$

For each x_i $(1 \le i \le m)$ we have n possible images. Therefore, the number of functions from X to Y is n^m.

◉ **Example 37:** Three boys and three girls are to be seated around a circular table. Among them the boy X does not want any girl as a neighbour and girl Y does not want any boy as a neighbour. The number of possible arrangement is:

(a) 9 (b) 4
(c) 8 (d) 6

Ans. (b)

◎ **Solution:** Clearly X should sit between the two remaining boys and Y should sit between the remaining two girls. We have to arrange two groups $\boxed{B_1\ X\ B_2}$ and $\boxed{G_1\ Y\ G_2}$ along a circle. This can be done is (2 – 1)! ways. But group of boys (girls) can be arranged in 2! ways. Thus, the required number of ways is (2) (2) (1) = 4 ways.

◉ **Example 38:** If ${}^nC_r : {}^nC_{r+1} : {}^nC_{r+2} = 1 : 2 : 3$, then r is equal to

(a) 5 (b) 4
(c) 3 (d) 0

Ans. (b)

◎ **Solution:** $\dfrac{2}{1} = \dfrac{{}^nC_{r+1}}{{}^nC_r} = \dfrac{n-r}{r+1}$

$$\text{and} \qquad \frac{3}{2} = \frac{n-r-1}{r+2}$$

$$\Rightarrow \qquad \frac{3}{2} = \frac{2(r+1)-1}{r+2} \qquad [\because\ n - r = 2(r+1)]$$

$$\Rightarrow \qquad 3r + 6 = 4r + 2 \Rightarrow r = 4.$$

◉ **Example 39:** The number of ways selecting three numbers from 1 to 30, so as to exclude every selection of three consecutive number is:

(a) 4030 (b) 4031
(c) 4032 (d) 4035

Ans. (c)

◎ **Solution:** The numbers of ways of selecting 3 number out of 30 is ${}^{30}C_3$. But this include 28 ways of selecting 3 consecutive numbers. [viz (1, 2, 3), (2, 3, 4), 3, 4, 5), ..., (28, 29, 30)]

Thus, required number of ways is ${}^{30}C_3 - 28 = 4032$.

◉ **Example:40:** If all permutations of the letters of the word P E N C I L are arranged as in a dictionary, then 413th word is

(a) L I C N E P (b) L I C N P E
(c) L I C P N E (d) L I C P E N

Ans. (d)

◎ **Solution:** Letters in the word PENCIL are C, E, I, L, N, P.

There are 5! words each beginning with C, E and I. That is, there are 3(5!) = 360 words beginning with C, E, I

Words beginning with L have serial numbers 361 to 480. 4! = 24 words beginning with LC have serial numbers 361 to 384. Words beginning with LE are numbered from 385 to 408. Words beginning with LIC are numbered from 409 to 414. Now 414th word is L I C P N E and 413th word is L I C P E N.

Example: 41: Out of 10 consonants and 4 vowels, words with 6 consonants and 3 vowels are formed. The number of such words is

(a) $\frac{7}{11}(12!)$ (b) $\frac{9}{10}(12!)$

(c) $\frac{7}{10}(11!)$ (d) $\frac{4}{5}(12!)$

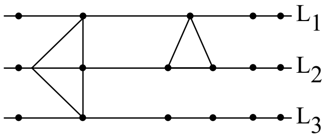
Ans. (a)

Solution: The number of ways of choosing 6 consonants out of 10 is ${}^{10}C_6$ and 3 vowels out of 4 in ${}^{4}C_3$. Thus number of such words is $({}^{10}C_4)\,({}^{4}C_3)\,({}^{9}P_9) = \frac{7}{11}(12!)$

Example: 42: Let L_1, L_2, L_3 be three distinct parrallel lines in the *XY*-plane, *p* distinct the points are taken on each of the three lines. The maximum number of triangles than can be formed by these $3p$ points is:

(a) $p^2(4p-3)$ (b) $p^3 + 3({}^{p}C_2)$
(c) $p^2(3p-4)$ (d) $(p+1)^3 - 1$

Ans. (a)



Solution: To obtain a triangle we may take one point from each of the three lines. This will give us p^3 triangles. Alternatively, we may choose two lines and take one point on one of them and two points on the other line. This will give us

$({}^{3}C_2)\,(2!)\,(p)\,({}^{p}C_2) = 3p^2\,(p-1)$ triangles

$\therefore$ required number of triangles is $p^3 + 3p^2\,(p-1) = p^2(4p-3)$

Alternative Solution

Choose three points not on the same lines. Number of ways $= {}^{3p}C_3 - 3({}^{p}C_3) = p^2(4p-3)$

Example : 43: The number of ways of selecting 4 letters out of the letters of the word MINIMAL is

(a) 16 (b) 17
(c) 18 (d) 20

Ans. (b)

Solution: Letters of the word , M I N I M A L are (*M*, *M*) (*I*, *I*), *A*, *N*, *L*

We can select 4 letters from letters of the word MINIMAL as follows:

Case 1 All letters are distinct.
This can be done in ${}^{5}C_4 = 5$ ways

Case 2 Exactly two letters are identical.
We can choose two identical letters in ${}^{2}C_1$ ways and two remaining distinct letters in ${}^{4}C_2 = 6$ ways
$\therefore$ In this case the selection can be made in (2) (6) = 12 ways.

Case 3 Two pairs of identical letters are selected.
This can be done in ${}^{2}C_2 = 1$ way
$\therefore$ two number of ways
$= 5 + 12 + 1 = 17$.

Example 44: Let $S = \left\{\begin{bmatrix} 18 & a \\ b & 25 \end{bmatrix} : a, b \in \mathbf{N}\right\}$

The number of singular matrices in *S* is

(a) 18 (b) 25
(c) 450 (d) infinite

Ans: (a)

Solution: det $\begin{bmatrix} 18 & a \\ b & 25 \end{bmatrix} = 2 \times 3^2 \times 5^2 - ab = 0$

$\Leftrightarrow \quad ab = 2 \times 3^2 \times 5^2$

$\Leftrightarrow \quad a \mid 2 \times 3^2 \times 5^2$

That is *a* is a divisor of $2 \times 3^2 \times 5^2$

Therefore, *a* is of the form $2^a\, 3^b\, 5^g$ where $\alpha \in \{0, 1\}$, $\beta, \gamma \in \{0, 1, 2\}$.

Thus number of elements in *S* is $2 \times 3 \times 3 = 18$.

Example 45: The number of positive integers *n* such that 2^n divides $n!$ is

(a) exactly 1 (b) exactly 2
(c) infinite (d) none of these

Ans. (d)

Solution: The exponent of 2 in $n!$ is given by

$$E = \left[\frac{n}{2}\right] + \left[\frac{n}{2^2}\right] + \left[\frac{n}{2^3}\right] + \dots$$

where $[x]$ denotes greatest integer $\le x$, As $[x] \le x \;\forall\; x$, $\left[\frac{n}{2^m}\right] = 0$ after finite number of terms. Thus we get

$$E < \frac{n}{2} + \frac{n}{2^2} + \frac{n}{2^3} + \dots = \frac{n/2}{1 - 1/2} = n$$

Thus, there is no positive integer for which 2^n divides $n!$

Example 46: The range of the function ${}^{7-x}\mathrm{P}_{x-3}$ is

(a) $\{1, 2, 3, 4\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
(c) $\{1, 2, 3\}$ (d) $\{1, 2, 3, 4, 5\}$

Ans. (c)

Solution: We must have $7 - x \ge 1$, $x - 3 \ge 0$ and $x - 3 \le 7 - x$

$\Rightarrow \quad x \le 6$, $x \ge 3$ and $x \le 5$

Thus, $\quad 3 \le x \le 5$.

$\therefore \quad$ Range of ${}^{7-x}\mathrm{P}_{x-3}$ is $\{{}^{4}P_0, {}^{3}P_1, {}^{2}P_2\} = \{1, 3, 2\}$

Example 47: The value of $^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is

(a) $^{56}C_3$ (b) $^{56}C_4$
(c) $^{55}C_4$ (d) $^{55}C_3$

Ans. (b)

Solution: $^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$

$$
\begin{aligned}
&= {}^{50}C_4 + ({}^{55}C_3 + {}^{54}C_3 + \ldots + {}^{50}C_3)\\
&= ({}^{50}C_4 + {}^{50}C_3) + ({}^{51}C_3 + \ldots + {}^{55}C_3)\\
&= ({}^{51}C_4 + {}^{51}C_3) + ({}^{52}C_3 + \ldots + {}^{55}C_3)\\
&= ({}^{52}C_4 + {}^{52}C_3) + ({}^{53}C_3 + \ldots + {}^{55}C_3)\\
&= \ldots\ldots\\
&= {}^{56}C_4
\end{aligned}
$$

Example 48: A class consists of 4 boys and g girls. Every Sunday five students, including at least three boys go for a picnic to Appu Ghar, a different group being sent every week. During, the picnic, the class teacher gives each girl in the group a doll. If the total number of dolls distributed was 85, then value of g is

(a) 15 (b) 12
(c) 8 (d) 5

Ans. (d)

Solution: Number of groups having 4 boys and 1 girl

$$= ({}^4C_4)\,({}^gC_1) = g$$

and number of groups having 3 boys and 2 girls

$$= ({}^4C_3)\,({}^gC_2) = 2g(g-1)$$

Thus, the number of dolls distributed

$$= g(1) + (2)\,[2g\,(g-1)] = 4g^2 - 3g$$

We are given $4g^2 - 3g = 85 \quad \Rightarrow \quad g = 5$.

Example 49: Sum of the factors of 9! which are odd and are of the form $3m + 2$, where m is a natural number is

(a) 40 (b) 45
(c) 51 (d) 54

Ans. (a)

Solution: We have $9! = 2^7 \times 3^4 \times 5 \times 7$

Odd factors of the form $3m + 2$ are neither multiples of 2 nor multiples of 3. So the factors may be 1, 5, 7, 35 of which just 5 and 35 are of the form $3m + 2$. Their sum is 40.

Example 50: Sum of all three digit numbers (no digit being zero) having the property that all digits are perfect squares, is

(a) 3108 (b) 6216
(c) 13986 (d) none of these

Ans. (c)

Solution: The non-zero perfect square digits are 1, 4 and 9.
1 can occur at units place in $3 \times 3 = 9$ ways.

$\therefore$ Sum due to 1 at units place is 1×9. Similarly, sum due to 1 at tens place is $1 \times 10 \times 9$ and sum due to 1 at hundreds place is $1 \times 100 \times 9$. We can deal with the digits 4 and 9 in a similar way.

Thus, sum of the desired number is

$$(1 + 4 + 9)\,(1 + 10 + 100)\,(9) = 13986$$

Example 51: The number of ordered pairs (m, n), $m, n \in \{1, 2, \ldots, 100\}$ such that $7^m + 7^n$ is divisible by 5 is

(a) 1250 (b) 2000
(c) 2500 (d) 5000

Ans. (c)

Solution: Note that 7^r $(r \in \boldsymbol{N})$ ends in 7, 9, 3 or 1 (corresponding to $r = 1, 2, 3$ and 4 respectively.)
Thus, $7^m + 7^n$ cannot end in 5 for any values of $m, n \in \boldsymbol{N}$. In other words, for $7^m + 7^n$ to be divisible by 5, it should end in 0.
For $7^m + 7^n$ to end in 0, the forms of m and n should be as follows:

	m	n
1	$4r$	$4s + 2$
2	$4r + 1$	$4s + 3$
3	$4r + 2$	$4s$
4	$4r + 3$	$4s + 1$

Thus, for a given value of m there are just 25 values of n for which $7^m + 7^n$ ends in 0. [For instance, if $m = 4r$, then $n = 2, 6, 10, \ldots, 98$]
$\therefore$ there are $100 \times 25 = 2500$ ordered pairs (m, n) for which $7^m + 7^n$ is divisible by 5.

Example 52: An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only three digits 2, 5 and 7. The smallest value of n for which this is possible is

(a) 6 (b) 7
(c) 8 (d) 9

Ans. (b)

Solution: The number of n-digit distinct numbers that can be formed using digits 2, 5 and 7 is 3^n. We have to find n, so that $3^n \geq 900$

$\Rightarrow \quad 3^{n-2} \geq 100 \quad \Rightarrow \quad n - 2 \geq 5 \quad \Rightarrow \quad n \geq 7.$

Thus, the smallest value of n is 7.

Example 53: The number of ways of arranging p numbers out of 1, 2, 3, …, q so that maximum is q–2 and minimum is 2 (repetition of number is allowed) such that maximum and minimum both occur exactly once, $(p > 5, q > 3)$ is

(a) $^{p-3}C_{q-2}$ (b) $^pC_2\,(q-3)^{q-1}$
(c) $^pC_2 \times {}^qC_3$ (d) $p(p-1)\,(q-5)^{p-2}$

Ans. (d)

◎ **Solution:** First we take one of the numbers as 2 and one another as $q-2$. We can arrange these two numbers in $p(p-1)$ ways. We have to choose remaining $p-2$ numbers from the numbers 3, 4, …, $q-4$, $q-3$. This can be done in $(q-5)^{p-2}$ ways.

Thus, the total number of ways of arranging the numbers in desired way is

$$p(p-1)\,(q-5)^{p-2}.$$

● **Example 54:** The number of ways of selecting 4 cards of an ordinary pack of playing cards so that exactly 3 of them are of the same denomination is

(a) 2496 (b) ${}^{13}C_3 \times {}^{4}C_3 \times 48$
(c) ${}^{52}C_3 \times 48$ (d) none of these

Ans. (a)

◎ **Solution:** We can choose one denomination in ${}^{13}C_1$ ways, then 3 cards of this denomination can be chosen in ${}^{4}C_3$ ways and one remaining card can be chosen in ${}^{48}C_1$ ways. Thus, the total number of choices is $({}^{13}C_1)\,({}^{4}C_3)\,({}^{48}C_1) = 13 \times 4 \times 48 = 2496$.

● **Example 55:** The number of integers x, y, z, w such that $x + y + z + w = 20$ and $x, y, z, w \geq -1$, is

(a) ${}^{24}C_3$ (b) ${}^{25}C_3$
(c) ${}^{26}C_3$ (d) ${}^{27}C_3$

Ans. (d)

◎ **Solution:** Put $x = a-1$, $y = b-1$, $z = c-1$, $w = d-1$, then $a, b, c, d \geq 0$ and $(a-1) + (b-1) + (c-1) + (d-1) = 20$

$$\Rightarrow \qquad a + b + c + d = 24$$

The number of non-negative integral solutions of this equation is

$$^{24+4-1}C_{4-1} = {}^{27}C_3$$

● **Example 56:** If 20% of three subsets (i.e., subsets containing exactly three elements) of the set $A = \{a_1, a_2, \ldots, a_n\}$ contain a_1, then value of n is

(a) 15 (b) 16
(c) 17 (d) 18

Ans. (a)

◎ **Solution:** The number of subsets of A containing exactly three elements is ${}^{n}C_3$ whereas the number of three subsets of A that contain a_1, is ${}^{n-1}C_2$. We are given,

$$^{n-1}C_2 = \frac{20}{100}\left({}^{n}C_3\right)$$

$$\Rightarrow \frac{(n-1)(n-2)}{2} = \frac{1}{5}\,\frac{n(n-1)\,(n-2)}{(6)} \Rightarrow n = 15.$$

● **Example 57:** There are three piles of identical yellow, black and green balls and each pile contains at least 20 balls. The number of ways of selecting 20 balls if the number of black balls to be selected is twice the number of yellow balls, is

(a) 6 (b) 7
(c) 8 (d) 9

Ans. (b)

◎ **Solution:** Let the number of yellow balls be x, that of black be $2x$ and that of green be y. Then

$$x + 2x + y = 20 \text{ or } 3x + y = 20$$

$$\Rightarrow \qquad y = 20 - 3x.$$

As $0 \leq y \leq 20$, we get $0 \leq 20 - 3x \leq 20$

$$\Rightarrow \quad 0 \leq 3x \leq 20 \quad \text{or} \quad 0 \leq x \leq 6$$

$\therefore$ The number of ways of selecting the balls is 7.

● **Example 58:** The total number of permutations of n (>1) different things taken not more that r at a time, when a thing may be repeated any number of times, is

(a) $\dfrac{n}{n-1}(n^r - 1)$ (b) $\dfrac{n^{r-1}}{n-1}$

(c) $\dfrac{n^r + 1}{n+1}$ (d) $\dfrac{n^r + n}{n-1}$

Ans. (a)

◎ **Solution:** When k $(1 \leq k \leq r)$ things are arranged, the number of possible arrangements, when repetition is allowed is

$$\underbrace{n \times n \times \ldots\ldots \times n}_{k \text{ times}} = n^k$$

Thus, the total possible arrangements is

$$n + n^2 + n^3 + \ldots + n^r = \frac{n\,(n^r - 1)}{n-1}.$$

● **Example 59:** The exponent of 7 in the prime factorization of ${}^{100}C_{50}$ is

(a) 0 (b) 2
(c) 4 (d) none of these

Ans. (a)

◎ **Solution:** We have ${}^{100}C_{50} = \dfrac{100!}{50!50!}$.

The exponent of 7 in 50! is $\left[\dfrac{50}{7}\right] + \left[\dfrac{50}{7^2}\right] = 7 + 1 = 8$,

and the exponent of 7 in 100! is $\left[\dfrac{100}{7}\right] + \left[\dfrac{100}{7^2}\right]$

$= 14 + 2 = 16$

Thus, exponent of 7 in the prime factorization of ${}^{100}C_{50}$ is $16 - 2(8) = 0$.

● **Example 60:** In a certain test there are n questions. In this test 2^k students gave wrong answers to at least $(n-k)$ questions, where $k = 0, 1, 2, \ldots, n$. If the total number of wrong answers is 4095, then value of n is

(a) 11 (b) 12
(c) 13 (d) 15

Ans. (b)

Solution: The number of students answering at least r questions incorrectly is 2^{n-r}.

$\therefore$ the number of students answering exactly r $(1 \le r \le n-1)$ questions incorrectly is $2^{n-r} - 2^{n-(r+1)}$.

Also, the number of students answering all questions wrongly is $2^0 = 1$.

Thus, the total number of wrong answers is

$$1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3}) + 3(2^{n-3} - 2^{n-4}) + \dots + (n-1)(2^1 - 2^0) + n(2^0)$$

$$= 2^{n-1} + 2^{n-2} + \dots + 2^0 = 2^n - 1.$$

Now, $2^n - 1 = 4095 \Rightarrow 2^n = 4096 = 2^{12} \Rightarrow n = 12$.

Example 61: The number of ways of permuting letters of the word ENDEANOEL so that none of the letters D, L, N occurs in the last five positions is

(a) 5! (b) 2(5!)
(c) 7(5!) (d) 21(5!)

Ans. (b)

Solution: Letters of the word ENDEANOEL are

$$3E\text{'s}, 2N\text{'s}, 1D, 1O, 1L, 1A$$

Letters D, L, N can be permuted at first 4 places in $\frac{4!}{2!}$ ways and the remaining letters can be permuted $\frac{5!}{3!}$ ways.

$\therefore$ required number of ways $\frac{4!}{2!} \times \frac{5!}{3!} = 2(5!)$

Example 62: Sum of the series $\sum_{r=1}^{n} (r^2 + 1)(r!)$ is

(a) $(n + 1)!$ (b) $(n + 2)! - 1$
(c) $n(n + 1)!$ (d) none of these

Ans. (c)

Solution: We can write

$$r^2 + 1 = (r + 2)(r + 1) - 3(r + 1) + 2$$

Thus, $$\sum_{r=1}^{n} (r^2 + 1)(r!) = \sum_{r=1}^{n} [(r + 2)(r + 1) - (r + 1) - 2\{(r + 1) - 1\}]\, r!$$

$$= \sum_{r=1}^{n} [(r + 2)! - (r + 1)!] - 2\sum_{r=1}^{n} \{(r + 1)! - r!\}$$

$$= (n + 2)! - 2! - 2\{(n + 1)! - 1\} = n(n + 1)!$$

Example 63: If letters of the word SACHIN are arranged in all possible ways and are written out as in a dictionary, then the word SACHIN appears at serial number

(a) 603 (b) 602
(c) 601 (d) 600

Ans. (c)

Solution: Letters appearing in the word SACHIN are

$$A, C, H, I, N, S$$

The words beginning with letters A, C, H, I and N appear before the word SACHIN.

There are 5(5!) = 600 words beginning with A, C, H, I and N.

Word SACHIN appears is the first word beginning with S. Therefore, SACHIN appears at serial number 601.

Example 64: The set $S = \{1, 2, 3, \dots 12\}$ is to be partitioned into three sets A, B, C of equal size. Thus $A \cup B \cup C = S$, $A \cap B = B \cap C = C \cap A = \phi$. The number of ways to partition S is

(a) $\frac{12!}{3!(4!)^3}$ (b) $\frac{12!}{3!(3!)^3}$

(c) $\frac{12!}{(4!)^3}$ (d) $\frac{12!}{(3!)^4}$

Ans. (a)

Solution: Each of the three sets A, B, C contains exactly 4 elements.

Thus, the number of ways of partitioning the set S is

$$\frac{1}{3!}\left({}^{12}C_4\right)\left({}^{8}C_4\right)\left({}^{4}C_4\right) = \frac{1}{3!}\,\frac{12!}{4!\,8!} \times \frac{8!}{4!\,4!}(1) = \frac{12!}{3!\,(4!)^3}$$

Example 65: Ten letters of an alphabet are given. Words with five letters are formed by these given letters. Then the number of words which have at least one letter repeated is

(a) 69760 (b) 30240
(c) 99748 (d) none of these

Ans. (a)

Solution: Number of five letter words that can be formed from 10 letters

$$= 10 \times 10 \times 10 \times 10 \times 10 = 10^5$$

Number of five letter words that have none of their letter repeated

$$= {}^{10}P_5 = 10 \times 9 \times 8 \times 7 \times 6 = 30240$$

$\therefore$ Number of words which have at least one letter repeated

$$= 10^5 - 30240 = 69760$$

Example 66: The number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together is

(a) 30 (b) 35
(c) 6!5! (d) 10!

Ans. (b)

◎ **Solution:** Six '+' signs can be arranged in just one way. There are seven places for '–' signs as shown in the following figure marked with X.

$$X + X + X + X + X + X + X$$

We can choose 4 places out of 7 in ${}^7C_4 = 35$ ways.

◉ **Example 67:** The remainder when $x = 1! + 2! + 3! + 4! + \ldots + 100!$ is divided by 240, is

(a) 153 (b) 33
(c) 73 (d) 187

Ans. (a)

◎ **Solution:** For $r \geq 6$; $r!$ is divisible by 240. Thus, when x is divided by 240, the remainder is $1! + 2! + \ldots + 5! = 153$.

◉ **Example 68:** If $x \in \boldsymbol{N}$ and ${}^{x-1}C_4 - {}^{x-1}C_3 < \frac{5}{4}\,({}^{x-2}P_2)$, then x can take

(a) 8 values (b) 7 values
(c) 6 values (d) none of these

Ans. (b)

◎ **Solution:** For $1 \leq x \leq 3$, each of ${}^{x-1}C_4$, ${}^{x-1}C_3$ and ${}^{x-2}P_2$ is zero. For $x = 4$, ${}^{x-1}C_4 = 0$, ${}^{x-1}C_3 = 1$ and ${}^{x-2}P_2 = 2$ and the inequality is valid trivally. For $x \geq 5$, we have

$$\frac{{}^{x-1}C_4}{{}^{x-2}P_2} - \frac{{}^{x-1}C_3}{{}^{x-2}P_2} < \frac{5}{4}$$

$$\Rightarrow \quad \frac{(x-1)(x-4)}{4!} - \frac{x-1}{3!} < \frac{5}{4}$$

$$\Rightarrow \quad x^2 - 9x - 22 < 0 \Rightarrow (x - 11)\,(x + 2) < 0$$

$$\Rightarrow \quad x - 11 < 0 \;[\because x + 2 > 0]$$

Thus, $5 \leq x < 11$. Hence, x can take 7 values viz. 4, 5, 6, 7, 8, 9 and 10.

◉ **Example 69:** The total number of ways in which 5 balls of different colours can be distributed among three persons so that each person gets at least one ball is:

(a) 75 (b) 150
(c) 210 (d) 243

Ans. (b)

◎ **Solution:** 5 balls can be distributed to 3 persons by giving (2, 2, 1) balls or by giving (3, 1, 1) balls. Each of the above distribution has three such ways. Thus, the required number of ways

$$= (3)\,({}^5C_2)\,({}^3C_2)\,({}^1C_1) + 3({}^5C_3)\,({}^2C_1)\,({}^1C_1)$$
$$= (3)\,(10)\,(3)\,(1) + (3)\,(10)\,(2)\,(1) = 150$$

◉ **Example 70:** Let $n \geq 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and rest by red. If the number of red and blue line segments are equal than value of n is

(a) 4 (b) 5
(c) 6 (d) 7

Ans. (b)

◎ **Solution:** Total number of line segments is ${}^nC_2$. The number of line segments joining adjacent points is n and number of lines joining non-adjacent points in ${}^nC_2 - n$. We are given

$${}^nC_2 - n = n \Rightarrow \frac{1}{2}\,n(n - 1) = 2n$$

$$\Rightarrow \qquad n = 5.$$

◉ **Example 71:** The number of seven digit integers, with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only, is

(a) 55 (b) 66
(c) 77 (d) 88

Ans. (c)

◎ **Solution:** Let x_i $(1 \leq i \leq 7)$ be the digit at the ith place. As $1 \leq x_i \leq 3$, and $x_1 + x_2 + \ldots + x_7 = 10$, at most one x_i can be 3.

Two cases arise.

Case 1.

Exactly one of x_i's is 3. In this case exactly one of the remaining x_i's is 2.

In this case, the number of seven digit numbers is

$$\frac{7!}{5!} = 7 \times 6 = 42$$

Case 2.

None of x_i's is 3.

In this case exactly three of x_i's is 2 and the remaining four x_i's are 1.

In this case, the number of seven digit number is

$$\frac{7!}{3!4!} = 35$$

Hence, the required seven digit numbers is 42 + 35 = 77.

◉ **Example 72:** Let $S = \{1, 2, 3, 4\}$. The total number of unordered pairs of disjoint subsets of S is equal to

(a) 25 (b) 34
(c) 42 (d) 41

Ans. (d)

◎ **Solution:** Let A and B be two subsets of S. If $x \in S$, then x will not belong to $A \cap B$ if x belongs to at most one of A, B. This can happen in 3 ways.

Thus, there are $3^4 = 81$ subsets of S for which $A \cap B = \phi$.

Out of these there is just one way for which $A = B = \phi$.

As, we, are interested in unordered pairs of disjoint sets, the number of such subsets is

$$\frac{1}{2}\left(3^4 - 1\right) + 1 = 41.$$

Example 73: How many different words can be formed by jumbling the letters in the word MISSISSIPPI in which no two S are adjacent?

(a) $(8)\left({}^6C_4\right)\left({}^7C_4\right)$ (b) $(6)(7)\left({}^8C_4\right)$

(c) $(6)(8)\left({}^7C_4\right)$ (d) $(7)\left({}^6C_4\right)\left({}^8C_4\right)$

Ans. (d)

Solution: We can permute M, I, I, I, I, P, P in $\frac{7!}{4!2!}$ ways. Corresponding to each arrangement of these seven letters, we have 8 places where S can be arranged as shown below with X.

$$X \square X \square X \square X \square X \square X \square X \square$$

We can choose 4 places out of 8 in ${}^8C_4$ ways.
Thus, the required number of ways

$$= ({}^8C_4)\left(\frac{7!}{4!2!}\right) = (7)\left({}^8C_4\right)\left({}^6C_4\right)$$

Example 74: From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then the number of such arrangement is

(a) at least 750 but less than 1000
(b) at least 1000
(c) less than 500
(d) at least 500 but less than 750

Ans. (b)

Solution: We can choose 4 novels out of 6 in ${}^6C_4$ ways and 1 dictionary out of 3 in $3C_1$ ways. We can arrange 4 novels and 1 dictionary in the middle in 4! ways. Thus, required number of ways

$$= ({}^6C_4)\,({}^3C_1)\,(4!) = 1080 > 1000$$

Example 75: There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out and then transferred to the other. The number of ways in which this can be done is

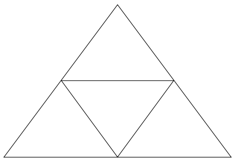
(a) 66 (b) 108
(c) 3 (d) 36

Ans. (b)

Solution: The number of ways of transferring the balls

$$= ({}^3C_2)\,({}^9C_2) = (3)\,(36) = 108$$

Example 76: The number of ways in which the diagram in Figure can be coloured so that each of the smaller triangle is painted with one of the three colours yellow, pink or green and no two adjacent regions are painted with the same colour, is



(a) 24 (b) 12
(c) 36 (d) 16

Ans. (a)

Solution: This can be done in the following ways:

(i) paint the central triangle with one of the three colours;
(ii) paint each of the remaining triangles with any one of the two remaining colours.

Thus, the required number of ways

$$= 3 \times 2 \times 2 \times 2 = 24.$$

Example 77: Four married couples are to be seated in a row having 8 chairs. The number of ways so that spouses are seated next to each other, is

(a) 72 (b) 186
(c) 384 (d) 516

Ans. (c)

Solution: Let us denote the four married couples of C_1, C_2, C_3 and C_4. We consider each couple as one unit. We can permute four units in 4! ways. Each couple can be seated in 2! ways. Thus, the required number of ways is

$$(4!)\,(2)\,(2)\,(2)\,(2) = 384.$$

Example 78: 25 lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. The number of points in which these lines intersect, is

(a) 300 (b) 315
(c) 325 (d) 450

Ans. (a)

Solution: The number of points in which the lines can intersect = the number of ways of choosing two lines out of 25

$${}^{25}C_2 = \frac{25 \times 24}{2} = 300.$$

Example 79: Letter of the word INDIANOIL are arranged in all possible ways. The number of permutations in which A, I, O occur only at odd places, is

(a) 720 (b) 360
(c) 240 (d) 120

Ans. (c)

Solution: A, I, I, I, O can occur at odd places in $\frac{5!}{3!}$ ways, and the remaining letters N, N, D, L can be arranged at the remaining places in $\frac{4!}{2!}$ ways.

Thus, the required number of ways $= \left(\frac{5!}{3!}\right)\left(\frac{4!}{2!}\right) = 240$ ways

Example 80: Let $a_n = \sum_{r=0}^{n} \frac{1}{\left({}^nC_r\right)^2}$ and $b_n = \sum_{r=0}^{n} \frac{(n-r)}{\left({}^nC_r\right)^2}$, then b_n equals

(a) a_n (b) na_n

(c) $\frac{n}{2}a_n$ (d) 0

Ans. (c)

◎ **Solution:** We have

$$b_n = \sum_{r=0}^{n} \frac{n-r}{\left({}^nC_r\right)^2} = \sum_{r=0}^{n} \frac{n-r}{\left({}^nC_{n-r}\right)^2} = \sum_{r=0}^{n} \frac{r}{\left({}^nC_r\right)^2}$$

Thus,

$$b_n + b_n = \sum_{r=0}^{n} \frac{n-r}{\left({}^nC_r\right)^2} + \sum_{r=0}^{n} \frac{r}{\left({}^nC_r\right)^2} = n \sum_{r=0}^{n} \frac{1}{\left({}^nC_r\right)^2} = n\,a_n$$

$$\Rightarrow \qquad b_n = \frac{n}{2} a_n .$$

◉ **Example 81:** A committee of at least three members is to be formed from a group of 6 boys and 6 girls such that it always has a boy and a girl.The number of ways to form such a committee is

(a) $2^{11} - 2^6 - 13$ (b) $2^{12} - 2^7 - 35$
(c) $2^{11} - 2^7 - 35$ (d) $2^{12} - 2^7 - 13$.

Ans. (b)

◎ **Solution:** Required number of ways

$$= ({}^6C_1 + {}^6C_2 + \cdots + {}^6C_6)\,({}^6C_1 + {}^6C_2 + \cdots {}^6C_6) - ({}^6C_1)\,({}^6C_1)$$
$$= (2^6-1)\,(2^6-1) - 36 = 2^{12} - 2^7 - 35$$

◉ **Example 82:** There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then

(a) $N \le 100$ (b) $100 < N \le 140$
(c) $140 < N \le 190$ (d) $N > 190$.

Ans. (a)

◎ **Solution:** Number of triangles

$$= {}^{10}C_3 - {}^6C_3 = 120 - 20 = 100$$

◉ **Example 83:** Assuming the balls to be identical except for difference of colours the number of ways in which one or more ball can be selected from 10 white, 9 green and 7 black balls is:

(a) 629 (b) 630
(c) 879 (d) 880

Ans. (c)

◎ **Solution:** We can select 0 or more balls out of 10 white balls in 11 ways. [select 0 or 1 or 2 ⋯ or 10 white balls.] Thus, the number of ways of making a selection of 0 or more balls

$$= (1+10)\,(1+9)(1+7) = 880$$

Therefore, number of ways of selecting at least one ball = 880 – 1 = 879.

◉ **Example 84:** Let a_n = number of n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them is 0.

Let b_n = number of such n digit integers ending with digit 1, and c_n = the numner of such n digit integers ending with digit 0, then for $n \ge 3$,

(a) $a_n = b_{n-1} + c_{n-2}$ (b) $a_n = b_{n-1} + c_{n-1}$
(c) $a_n = b_{n-2} + c_{n-1}$ (d) $a_n = b_n + c_n$

Ans. (d)

◎ **Solution:** Note that all such numbers begin with 1.

If number ends in 1, then there $b_n = a_{n-1}$ such numbers. If number ends in 0, then $(n-1)$th digit cannot be 0 and thus, there are $c_n = a_{n-2}$ such numbers.

$$\therefore \qquad a_n = b_n + c_n = a_{n-1} + a_{n-2}$$

◉ **Example 85:** In Example 84, a_8 equals

(a) 21 (b) 34
(c) 38 (d) 41

Ans. (b)

◎ **Solution:** If # (S) = number of elements of S, then

$a_1 = 1,\ a_2 = \#\,\{10, 11\} = 2,$
$a_3 = \#\,\{101, 110, 111\} = 3$

Next, $a_n = b_n + c_n = a_{n-1} + a_{n-2}\ \forall\ n \ge 3,$
we get $a_4 = a_3 + a_2 = 5,$
$a_5 = a_4 + a_3 = 8,\ a_6 = 13,\ a_7 = 21$
and $a_8 = 34$

◉ **Example 86:** The number of natural numbers less than one million that can be formed by using the digits, 0, 2, and 3 is

(a) 728 (b) 726
(c) 730 (d) 732

Ans. (a)

◎ **Solution:** The number of k digit natural numbers formed by using the digit 0,2 and 3 is $2(3^{k-1})$.

∴ Number of natural numbers less than one million formed by 0,2 and 3 is

$$= 2 + (2)\,(3) + (2)\,(3^2) + 2(3^3) + 2(3^4) + 2(3^5)$$
$$= 728$$

Assertion-Reason Type Questions

◉ **Example 87:**

Statement-1:

The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$

Statement-2: The number of ways of choosing 3 places from 9 different place is ${}^9C_3$.

Ans. (b)

Solution: Let x_i = number of balls put in ith box, then $x_1 + x_2 + x_3 + x_4 = 10$ where $x_i \geq 1$.

Put $x_i = y_i + 1$, so that equation becomes

$y_1 + y_2 + y_3 + y_4 = 6$. where $y_i \geq 0$

Number of non-negative integral solution of the above equation = Number of ways of arranging 6 identical balls and 3 identical separators

$= \dfrac{(6+3)!}{6!3!} = {}^9C_3$ number of ways of choosing 3 places out of 9 different places.

Example 88: Statement-1: If n is a natural number then $\dfrac{(n^2)!}{(n!)^{n+1}}$ is a natural number.

Statement-2: The number of ways of dividing mn students into m groups each containing n students is $\dfrac{(mn)!}{m!(n!)^m}$

Ans. (a)

Solution: The number of ways of selecting students for the first group is ${}^{mn}C_n$; for the second group is ${}^{mn-n}C_n$ and so on.

$\therefore$ the number of ways of dividing (mn) students into m numbered groups is

$$({}^{mn}C_n)\,({}^{mn-n}C_n)\cdots({}^{n}C_n)$$

$$= \frac{(mn)!}{n!(mn-n)!}\,\frac{(mn-n)!}{n!(mn-2n)!}\cdots\frac{n!}{n!\,0!}$$

$$= \frac{(mn)!}{(n!)^m}$$

As groups are not to be numbered, the desired number of ways is $\dfrac{(mn)!}{m!(n!)^m}$

$\therefore$ Statement-2 is true.

For statement-1, put $m = n$.

Example 89: Let $n \in \boldsymbol{N}$, and

$$f(n) = \begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^nC_n & {}^{n+1}C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$$

Statement-1: $f(n)$ is an integer for all $n \in \mathbf{N}$.

Statement-2: If elements of a determinant are integers, then determinant itself is an integer.

Ans. (a)

Solution: If each element of a determinant is an integer, then its each cofactor is an integer, and hence determinant itself is an integer.

Example 90: Statement-1: $\displaystyle\sum_{j=1}^{n}\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j} = 3^n - 2^n$

Statement-2: $\displaystyle\sum_{k=1}^{n}\left({}^nC_k\right)^2 = {}^{2n}C_n$

Ans. (c)

Solution: $\displaystyle\binom{n}{k}\binom{k}{j} = \frac{n!}{k!(n-k)!}\,\frac{k!}{j!(k-j)!}$

$$= \frac{n!}{j!(n-j)!}\,\frac{(n-j)!}{(k-j)!\,(n-k)!}$$

$$= \binom{n}{j}\binom{n-j}{k-j}$$

Thus, $\displaystyle\sum_{j=1}^{n}\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j} = \sum_{j=1}^{n}\binom{n}{j}\sum_{k=j}^{n}\binom{n-j}{k-j}$

But $\displaystyle\sum_{k=j}^{n}\binom{n-j}{k-j} = \sum_{l=0}^{n-j}\binom{n-j}{l} = 2^{n-j}$

$\therefore$ $\displaystyle\sum_{j=1}^{n}\sum_{k=j}^{n}\binom{n}{k}\binom{k}{j} = \sum_{j=1}^{n}\binom{n}{j}2^{n-j} = (2+1)^n - 2^n$

$$= 3^{\mathrm{n}} - 2^{\mathrm{n}}$$

Statement-2 is false as $\displaystyle\sum_{k=1}^{n}\left({}^nC_k\right)^2 = {}^{2n}C_n - 1$

Example 91: Statement-1: $T_n = \displaystyle\sum_{k=1}^{n} k\left({}^{2n}C_k\right) = 2^{2n-2}$

Statement-2: $\displaystyle\sum_{k=0}^{n} {}^{2n+1}C_k = 2^{2n}$

Ans. (d)

Solution: $\displaystyle 2\sum_{k=0}^{n} {}^{2n+1}C_k = \sum_{k=0}^{n} {}^{2n+1}C_k + \sum_{k=0}^{n} {}^{2n+1}C_k$

$$= \sum_{k=0}^{n} {}^{2n+1}C_k + \sum_{k=0}^{n} {}^{2n+1}C_{2n+1-k}$$

$$= \sum_{k=0}^{2n+1} {}^{2n+1}C_k = 2^{2n+1}$$

$\Rightarrow$ $\displaystyle\sum_{k=0}^{n} {}^{2n+1}C_k = 2^{2n}$

$\therefore$ Statement-2 is true.

Using $k({}^{2n}C_k) = 2n({}^{2n-1}C_{k-1})$, for $k \geq 1$, we get

$$T_n = 2n\sum_{k=1}^{n}\left({}^{2n-1}C_{k-1}\right) = 2n\sum_{j=0}^{n-1} {}^{2n-1}C_j$$

$$= 2n(2^{2n-2}) = n(2^{2n-1})$$

Example 92: Statement-1: The number of ways of distributing at most 12 toys to three children A_1, A_2 and A_3 so that A_1 gets at least one, A_2 at least three and A_3 at most five, is 145.

Statement-2: The number of non-negative integral solutions of $x_1 + x_2 + x_3 \le m$ is ${}^{m-1}P_2$.
Ans. (d)

Solution: Suppose A_i gets x_i toys, then

$$x_1 + x_2 + x_3 \le 12.$$

Let $\quad x_4 = 12 - (x_1 + x_2 + x_3)$, then

$$x_1 + x_2 + x_3 + x_4 = 12 \qquad (1)$$

The number of non-negative integral solutions of (1)

$= \text{coefficient of } t^{12} \text{ in } (t + t^2 + \cdots)(t^3 + t^4 + t^5 + \cdots)(1 + t + \cdots + t^5) \times (1 + t + t^2 + \cdots)$

$= \text{coefficient of } t^{12} \text{ in } t^4(1 - t^6)(1 - t)^{-4}$

$= \text{coefficient of } t^8 \text{ in } (1 - t^6)(1 + {}^4C_1 t + {}^5C_2 t^2 + \cdots)$

$= {}^{11}C_8 - {}^5C_2 = 165 - 10 = 155$

Statement-2 is false as the number of non-negative integral solutions of

$$x_1 + x_2 + x_3 \le m$$

equals the number of non-negative integral solution of

$$x_1 + x_2 + x_3 + x_4 = m,$$

which equals ${}^{m+3}C_m$.

LEVEL 2

Straight Objective Type Questions

Example 93: The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in an English dictionary. The number of words that appear before COCHIN is

(a) 360 (b) 192
(c) 96 (d) 48

Ans. (c)

Solution: Letters appearing in the word COCHIN are

$$C, C, H, I, N, O$$

Words appearing before COCHIN are of the form

$$C\,X\text{ - - - -}$$

where X is one of the letters C, H, I, N and the four remaining places can be filled by the remaining four letters.
Thus, the number of words before COCHIN is (4) (4!) = 96

Example 94: If r, s, t are prime numbers and p, q are natural numbers such that LCM of p, q is $r^2 s^4 t^2$, then the number of ordered pairs (p, q) is

(a) 252 (b) 254
(c) 225 (d) 224

Ans. (c)

Solution: Numbers p and q must be of the form

$$p = r^a s^b t^c, \quad q = r^\alpha s^\beta t^\gamma$$

where $\quad 0 \le a, \alpha \le 2$ and at least one of a, α is 2
$\quad 0 \le b, \beta \le 4$ and at least one of b, β is 4
$\quad 0 \le c, \gamma \le 2$ and at least one of c, γ is 2

Possible values of (a, α), and (c, γ) are

(0, 2), (1, 2), (2, 2), (2, 0), (2, 1).

Possible values of (b, β) are

(0, 4), (1, 4), (2, 4), (3, 4), (4, 4), (4, 0), (4, 1), (4, 2), (4, 3).

Thus, number of possible order pairs (p, q) is $5 \times 9 \times 5 = 225$.

Example 95: If $n > 1$ and n divides $(n - 1)! + 1$, then

(a) n must be prime
(b) n must be divisible by exactly two primes
(c) n must be a composite number
(d) none of these.

Ans. (a)

Solution: If n is not prime, then there exists $r \in \mathbf{N}$ such that $2 \le r \le n - 1$ and $r \mid n$.
As $r \mid n$ and $n \mid [(n - 1)! + 1]$, we get
$\quad r \mid [(n - 1)! + 1]$
As $2 \le r \le n - 1$, $r \mid (n - 1)!$, therefore $r \mid 1$. A contradiction.

Example 96: Which of the following statement is false?

(a) There exist 100 consecutive natural numbers none of which is prime
(b) There exist 1000 consecutive natural number none of which is prime.
(c) Given $n \in \mathbf{N}$, there exist n consecutive natural numbers none of which is prime.
(d) none of these.

Ans. (d).

Solution: Given any $n \in \mathbf{N}$, then

$\quad (n + 1)! + 2, (n + 1)! + 3, \ldots\ldots, (n + 1)! + (n + 1)$

are n consecutive natural numbers none of which is prime.

Example 97: The results of 21 football matches (win, lose or draw) are to be predicted. The number of forecasts that contain exactly 18 correct results is

(a) ${}^{21}C_3\,2^{18}$ (b) ${}^{21}C_{18}\,2^3$
(c) $3^{21} - 2^{18}$ (d) ${}^{21}C_3\,3^{21} - 2^{18}$

Ans. (b)

Solution: 18 correct results can be predicted in ${}^{21}C_{18}$ ways and 3 wrong results in 2^3 ways. Thus, required number of ways is ${}^{21}C_{18}\,2^3$.

Example 98: There are 15 points in a plane, no two of which are in a straight line except 4, all of which are in a straight line. The number of triangles that can be formed by using these 15 points is

(a) 404 (b) 415
(c) 451 (d) 490

Ans. (c)

Solution: The required number of triangle is

$$ {}^{15}C_3 - {}^4C_3 = 455 - 4 = 451 $$

Example 99: If $x, y \in (0, 30)$ such that

$$\left[\frac{x}{3}\right]+\left[\frac{3x}{2}\right]+\left[\frac{y}{2}\right]+\left[\frac{3y}{4}\right]=\frac{11}{6}x+\frac{5}{4}y$$

(where $[x]$ denotes greatest integer $\le x$), then number of ordered pairs (x, y) is

(a) 0 (b) 2
(c) 4 (d) none of these

Ans. (d)

Solution: Let $\{x\} = x - [x]$ denote the fractional part of x. Note that $0 \le \{x\} < 1$. We can write the given equation as

$$\frac{x}{3}-\left\{\frac{x}{3}\right\}+\frac{3x}{2}-\left\{\frac{3x}{2}\right\}+\frac{y}{2}-\left\{\frac{y}{2}\right\}+\frac{3y}{4}-\left\{\frac{3y}{4}\right\}=\frac{11}{6}x+\frac{5}{4}y$$

$$\Rightarrow \quad \left\{\frac{x}{3}\right\}+\left\{\frac{3x}{2}\right\}+\left\{\frac{y}{2}\right\}+\left\{\frac{3y}{4}\right\}=0$$

As each number on the L.H.S. lies in the interval $0 \le x < 1$, we must have

$$\left\{\frac{x}{3}\right\}=\left\{\frac{3x}{2}\right\}=\left\{\frac{y}{2}\right\}=\left\{\frac{3y}{4}\right\}=0$$

$$\Rightarrow \quad \frac{x}{3},\frac{3x}{2},\frac{y}{2} \text{ and } \frac{3y}{4} \text{ must be integers.}$$

$\therefore$ $x = 6, 12, 18, 24,$ $y = 4, 8, 12, 16, 20, 24, 28$

$\Rightarrow$ Number of ordered pairs (x, y) equals $4 \times 7 = 28$.

Example 100: The sum $\sum\limits_{0\le i\le j\le 10}\sum ({}^{10}C_j)({}^jC_i)$ is equal to

(a) $2^{10} - 1$ (b) 2^{10}
(c) $3^{10} - 1$ (d) 3^{10}

Ans. (c)

Solution: We have $\sum\limits_{0\le\ i<j\le 10}\sum {}^{10}C_j({}^jC_i)$

$$= {}^{10}C_1\,({}^1C_0 + {}^1C_1) + {}^{10}C_2\,({}^2C_0 + {}^2C_1 + {}^2C_2) + \ldots + {}^{10}C_{10}\,({}^{10}C_0 + {}^{10}C_1 + \ldots + {}^{10}C_{10})$$
$$= {}^{10}C_1(2) + {}^{10}C_2(2^2) + {}^{10}C_3(2^3) + \ldots + {}^{10}C_{10}(2^{10})$$
$$= (2 + 1)^{10} - 1 = 3^{10} - 1.$$

Example 101: There are p letters a, q letters b, r letters c. The number of ways of selecting k letters out of these if $p < k < q < r$ is

(a) $\frac{1}{2}\,(p + 1)^2 - k$

(b) $\frac{1}{2}\,(p + 1)\,(2k - p)$

(c) $\frac{1}{3}\,(p+1)\,(q+1)\,(r+1)-k$

(d) none of these

Ans. (b)

Solution: Let x *a′s*, y *b′s* and z *c′s* be selected. Then number of selections is equal to the number of non-negative integral solutions of

$$x + y + z = k \tag{1}$$

If we take $x = l$, $0 \le x \le p$ then $y + z = k - l$ and its number of solutions is $k - l + 1$.

Thus, the desired number of selections

$$\sum_{l=0}^{p}(k-l+1) = (k + 1)\,(p + 1) - \frac{1}{2}\,(p + 1)\,(p + 2)$$
$$= \frac{1}{2}\,(p + 1)\,(2k - p).$$

Example 102: The number of ways of choosing n objects out of $(3n + 1)$ objects of which n are identical and $(2n + 1)$ are distinct, is

(a) 2^{2n} (b) 2^{2n+1}
(c) $2^{2n} - 1$ (d) none of these

Ans. (a)

Solution: If we choose k $(0 \le k \le n)$ identical objects, then we must choose $(n - k)$ distinct objects. This can be done in ${}^{2n+1}C_{n-k}$ ways.Thus, the required number of ways

$$= \sum_{k=0}^{n} {}^{2n+1}C_{n-k} = {}^{2n+1}C_n + {}^{2n+1}C_{n-1} + \ldots + {}^{2n+1}C_0 = 2^{2n}.$$

Example 103: The greatest common divisor ${}^{31}C_3$, ${}^{31}C_5$, …, ${}^{31}C_{29}$ is

(a) 31 (b) 2
(c) 17 (d) none of these

Ans. (a)

Solution: We have, for $3 \le r \le 29$

$$r!\,(31 - r)!\,({}^{31}C_r) = 31!$$

As the prime 31 divides R.H.S. and 31 does not divide $r!$ and $(31 - r)!$ for $3 \le r \le 29$, we get $31 \mid ({}^{31}C_r)$.

Also, since ${}^{31}C_{29} = (31)(3)(5)$, ${}^{31}C_3 = (31)(29)(5)$ and ${}^{31}C_5 = (31)(29)(7)$

No prime other than 31 can divide all the numbers.

Thus greatest common divisors of the given numbers is 31.

Example 104: For $x \in \boldsymbol{R}$, let $[x]$ denote the greatest integer $\le x$, then value of $\left[-\frac{1}{3}\right]+\left[-\frac{1}{3}-\frac{1}{100}\right]+\left[-\frac{1}{3}-\frac{2}{100}\right]+\ldots+\left[-\frac{1}{3}-\frac{99}{100}\right]$ is

(a) -100 (b) -123
(c) -135 (d) -153

Ans. (c)

Solution: For $0 \le r \le 66, 0 \le \frac{r}{100} < \frac{2}{3} \Rightarrow -\frac{2}{3} < -\frac{r}{100} \le 0$

$$\Rightarrow \quad -\frac{1}{3}-\frac{2}{3} < -\frac{1}{3}-\frac{r}{100} \le -\frac{1}{3}$$

$$\therefore \quad \left[-\frac{1}{3}-\frac{r}{100}\right] = -1 \qquad \text{for } 0 \le r \le 66$$

Also, for $67 \le r \le 100, \frac{67}{100} \le \frac{r}{100} \le 1 \Rightarrow -1 \le -\frac{r}{100} \le -\frac{67}{100}$

$$\Rightarrow \quad -\frac{1}{3}-1 \le -\frac{1}{3}-\frac{r}{100} \le -\frac{1}{3}-\frac{67}{100}$$

$$\therefore \quad \left[-\frac{1}{3}-\frac{r}{100}\right] = -2 \text{ for } 67 \le r \le 100$$

Hence, $\sum_{r=0}^{100}\left[-\frac{1}{3}-\frac{r}{100}\right] = 67(-1) + 2(-34) = -135.$

Example 105: If $n \le 8$ and ${}^{n}C_{r+1} = 20$ and ${}^{n-1}C_{r} = 10$, then n is equal to

(a) 5 (b) 6
(c) 7 (d) 8

Ans. (b)

Solution $\frac{{}^{n}C_{r+1}}{{}^{n-1}C_{r}} = \frac{20}{10}$

$$\Rightarrow \quad \frac{n!}{(r+1)!\,(n-r-1)!}\;\frac{r!(n-r-1)!}{(n-1)!} = 2$$

$$\Rightarrow \quad \frac{n}{r+1} = 2$$

This shows that n is even and $r + 1 = n/2$ Now. ${}^{n}C_{n/2} = 20$ and $n \le 8$

$\Rightarrow \qquad n = 6.$

EXERCISE

Concept-based Straight Objective Type Questions

1. If $x \in \mathbf{N}$ and $\frac{(2x)!}{3!(2x-3)!} : \frac{x!}{2!(x-2)} = 44:3,$ then x is equal to
   (a) 6 (b) 7
   (c) 11 (d) 12
2. If $x \in \mathbf{N}$ and $\frac{(x+2)!}{(2x-1)!} \cdot \frac{(2x+1)!}{(x+3)!} = \frac{72}{7}$, then x is equal to
   (a) 3 (b) 4
   (c) 5 (d) 6
3. If ${}^{m+n}P_2 = 90$ and ${}^{m-n}P_2 = 30$, then (m, n) is equal to
   (a) (12, 8) (b) (8, 2)
   (c) (6, 4) (d) (7, 4)
4. The number of ways in which you can put five beads of five different colours to form a necklace is:
   (a) 12 (b) 24
   (c) 60 (d) 120
5. 15 lines are drawn in a plane in such a way that no two of them are parallel and no three are concurrent. The number of points of intersections of these lines is
   (a) 455 (b) 465
   (c) 475 (d) 485
6. Suppose $p \in \mathbf{N}$, $n = {}^{p}C_2$ and $m = {}^{n}C_2$. If $4m : n = 18 : 1$, then p is equal to
   (a) 11 (b) 9
   (c) 7 (d) 5
7. If ${}^{2n}C_4$, ${}^{2n}C_5$ and ${}^{2n}C_6$ are in A.P, then n is equal to
   (a) 14 (b) 12
   (c) 7 (d) 6
8. If ${}^{n}P_{r} = 1680$ and ${}^{n}C_{r} = 70$, then n is equal to
   (a) 5 (b) 7
   (c) 8 (d) 10
9. If ${}^{n+1}C_{r+1} : {}^{n}C_{r} : {}^{n-1}C_{r-1} = 11 : 6 : 3$, then value of n is
   (a) 12 (b) 10
   (c) 18 (d) 21
10. Suppose P is a set containing n distinct elements. Let $S = \{(x, y, z.) \mid x, y, z, \in, P$ and at least two of x, y, z are equal$\}$.

The number of elements in S is

(a) $n(3n - 2)$ (b) $n^2 (n - 2)$
(c) $n^3 - {}^{n}C_3$ (d) $n(5n - 4)$

LEVEL 1

Straight Objective Type Questions

11. A man invites 6 non-vegeterian and 5 vegeterian friends for a dinner party. He arrange 6 non-vegeterian friends on one round table and 5 vegeterian friends along another round table. The number of ways this can be done is:

(a) 11! (b) 9!
(c) 2880 (d) 8280

12. In an examination, there are 11 papers. A candidate has to pass in at least 6 papers to pass the examination. The number of ways in which the candidate can pass the examination is:

(a) 2^9 (b) 2^{10}
(c) 2^{11} (d) $2^{11} - {}^{11}C_5$

13. At an election, a voter may vote for any number of candidates not greater than the number to be chosen. There are 10 candidates and 5 members to be selected. The number of ways of in which a voter can vote is:

(a) 630 (b) 635
(c) 637 (d) 639

14. In a class of 10 students there are 3 girls students. The number of ways in which they can be arranged in a row, so that no two girls are consecutive is $k(8!)$, then k is equal to

(a) 12 (b) 24
(c) 36 (d) 42

15. The number of ways of putting 5 identical balls in 10 identical boxes, if not more than one can go in a box is

(a) ${}^{10}P_5$ (b) ${}^{10}C_5$
(c) 10^5 (d) 1

16. The number of four digit numbers that do not contain 4 different digits is:

(a) 4464 (b) 3680
(c) 4120 (d) 7208

17. The number of ways of dividing 10 girls into two groups of 5 each so that two shortest girls are in the different groups is:

(a) 70 (b) 252
(c) 140 (d) 282

18. The number of solutions of $\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$ where $x, y \in \mathbf{N}$ is

(a) 9 (b) 18
(c) 21 (d) 28

19. Let m and n be two digit natural numbers. The number of pairs (m, n) such that n can be subtracted from m without borrowing is:

(a) 2475 (b) 2550
(c) 2675 (d) 2875

20. 5-digit numbers are formed using 2, 3, 5, 7, 9 without repeating the digits. If p is the number of such numbers that exceeds 20000 and q be the number of those that lie between 30000 and 90000 then $p : q$ is:

(a) 5 : 4 (b) 5 : 3
(c) 8 : 3 (d) 15 : 16

21. The number of diagonals of a polygon of 15 sides is

(a) 105 (b) 90
(c) 75 (d) 60

22. A five digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways, this can be done is

(a) 240 (b) 3125
(c) 600 (d) 216

23. The number of subsets of a set containing n distinct object is

(a) ${}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-1}$
(b) $2^n - 1$
(c) ${}^nC_0 + {}^nC_1 + \dots + {}^nC_n$
(d) $2^n + 1$

24. If a, b and r are positive integers, then value of

$${}^aC_r + {}^aC_{r-1}\,{}^bC_1 + {}^aC_{r-2}\,{}^bC_2 + \dots + {}^bC_r \text{ is}$$

(a) $\frac{a!b!}{r!}$ (b) $\frac{(a+b)!}{r!}$
(c) ${}^{a+b}C_r$ (d) ${}^{ab}C_r$

25. The number of 10 digit numbers that can be written by using the digits 0 and 1 is

(a) ${}^{10}C_2 + {}^9C_2$ (b) 2^{10}
(c) $2^{10}–2$ (d) 10!

26. If P_r stands for ${}^rP_r$, then sum of the series $1 + P_1 + 2P_2 + 3P_3 + \dots + nP_n$ is

(a) $(n + 1)!$ (b) $P_{n+1} - 1$
(c) $P_{n-1} + 1$ (d) none of these

27. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then value of $\sum_{r=0}^{n} \frac{n-2r}{{}^nC_r}$ is

(a) $\frac{n}{2}a_n$ (b) $\frac{1}{4}a_n$
(c) na_n (d) 0

28. m men and w women are to be seated in a row so that all women sit together. The number of ways in which they can be seated is
(a) $(m+1)!\,w!$ (b) $m!\,w!$
(c) $m!\,(w-1)!$ (d) ${}^{m+w}C_w$

29. A five digit number divisible by 6 is to be formed by using the digits 0, 1, 2, 3, 4 and 8 without repetition. The total number of ways in which this can be done is
(a) 216 (b) 150
(c) 116 (d) 98

30. Rakshit is allowed to select $(n+1)$ or more books out of $(2n+1)$ distinct books. If the number of ways in which he may not select all of them is 255, then value of n is
(a) 3 (b) 4
(c) 5 (d) 11

31. The number of ways in which a committee of 3 women and 4 men be chosen from 8 women and 7 men if Mr. X refuses to serve on the committee if Mr. Y is a member of the committee is
(a) 420 (b) 840
(c) 1540 (d) 1400

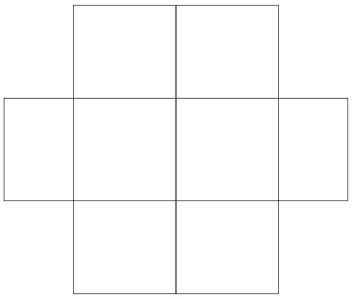
32. The product of first n odd natural numbers equals
(a) $({}^{2n}C_n)({}^nP_n)$ (b) $\left(\frac{1}{2}\right)^n({}^{2n}C_n)({}^nP_n)$
(c) $\left(\frac{1}{4}\right)^n({}^{2n}C_n)({}^{2n}P_n)$ (d) none of these

33. The number of ways in which two teams A and B of 11 players each can be made up from 22 players so that two particular players are on the opposite sides is
(a) 369512 (b) 184755
(c) 184756 (d) 369514

34. The number of ways in which 20 letters $x_1, x_2, \ldots x_{10}, y_1, y_2, \ldots, y_{10}$ be arranged in a line so that suffixes of the letters x and also those of y are respectively in ascending order of magnitude is
(a) 126 (b) 64
(c) 2^{20} (d) 184756

35. The product of n consecutive natural number is always divisible by
(a) ${}^nP_n$ (b) ${}^{2n}C_n$
(c) ${}^{2n}P_n$ (d) ${}^{n+1}P_n$

36. Six X have to be placed in the squares of Figure such that each row contain at least one X. The number of ways in which this can be done is
(a) 25 (b) 26
(c) 27 (d) 30



37. There are three pigeon holes marked M, P, C. The number of ways in which we can put 12 letters so that 6 of them are in M, 4 are in P and 2 are in C is
(a) 2520 (b) 13860
(c) 12530 (d) 25220

38. The greatest number of points of intersection of n circles and m straight lines is
(a) $2mn + {}^mC_2$
(b) $\frac{1}{2}\,m(m-1) + n(2m+n-1)$
(c) ${}^mC_2 + 2({}^nC_2)$
(d) none of these

39. The number of binary sequences of length n that contain even number of 1's is
(a) 2^{n-1} (b) $2^n - 1$
(c) $2^n - 2$ (d) none of these

40. The number of natural numbers with distinct digits is
(a) $9^{10}-1$ (b) $10^{10} - 9^{10}$
(c) $9^{10}\times 1$ (d) none of these

41. The number of five digit numbers that contain 7 exactly once is
(a) $(41)(9^3)$ (b) $(37)(9^3)$
(c) $(7)(9^4)$ (d) $(41)(9^4)$

42. The units digit of $17^{2009} + 11^{2009} - 7^{2009}$ is
(a) 1 (b) 8
(c) 2 (d) 9

43. If $\frac{1}{{}^4C_n} = \frac{1}{{}^5C_n} + \frac{1}{{}^6C_n}$, then value of n is
(a) 3 (b) 4
(c) 1 (d) 2

44. At an election there are five candidates and three members are to be elected, and a voter may vote for any number of candidates not greater than the number

to be elected. The number of ways in which the person can vote is

(a) 25 (b) 30
(c) 35 (d) $2^5 – 2^3$

45. If n is odd and ${}^nC_0 < {}^nC_1 < {}^nC_2 < \ldots < {}^nC_r$, then maximum possible value of r is

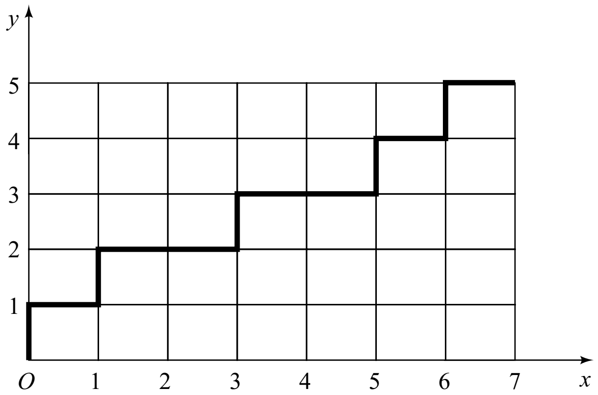
(a) $\frac{n-1}{2}$ (b) $\frac{n+1}{2}$
(c) n (d) none of these

46. The number of arrangements of the letters of the word BANANA in which two N's do not appear adjacently is

(a) 40 (b) 60
(c) 80 (d) 100

47. The number of (staircase) paths in the xy-plane from (0, 0) to (7, 5) where each such path is made up of individual steps going one unit upward (U) or one unit to the right (R). One such path is shown in figure

(a) ${}^{12}C_5$ (b) 12!
(c) 5! 7! (d) ${}^{12}P_5$



48. In Question 47, how many such paths are there if each path must pass through the point (3, 4)?

(a) 175 (b) 145
(c) 95 (d) 78

49. The number of ways we can put 5 different balls in 5 different boxes such that at most three boxes is empty, is equal to

(a) $5^5 – 5$ (b) $5^5 – 10$
(c) $2^5 – 2$ (d) none of these

50. The number of ordered pairs of integers (x, y) satisfying the equation $x^2 + 6x + y^2 = 4$ is

(a) 2 (b) 4
(c) 6 (d) 8

51. The number of the factors of 20! is

(a) 4140 (b) 41040
(c) 4204 (d) 81650

52. The number of five digit numbers that can be formed by using digits 1, 2, 3 only, such that three digits of the formed number are identical, is

(a) 30 (b) 60
(c) 120 (d) 90

53. Let $S_n = \sum_{r=1}^{n} r!$. If $T_n = S_n - 7\left[\frac{S_n}{7}\right]$, where $[x]$ denotes greatest integer $\le x$, then T_{100} is equal to

(a) 7 (b) 5
(c) 11 (d) none of these

54. The number of ordered pairs (m, n), m, $n \in \{1, 2, \ldots, 50\}$ such that $6^m + 9^n$ is a multiple of 5 is

(a) 1250 (b) 2500
(c) 500 (d) 625

55. Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. The number of words which have at least one of their letter repeated is

(a) 69760 (b) 30240
(c) 99748 (d) none of these

56. The number of squares that we can find on a chess board is

(a) 64 (b) 160
(c) 224 (d) 204

57. In a football championship, 153 matches were played. Every team played one match with each other, the number of teams participating in the championship is

(a) 18 (b) 16
(c) 9 (d) none of these

58. The total number of ways of dividing mn distinct objects into n equal groups is

(a) $\frac{(mn)!}{(m!)^n (n!)}$ (b) $\frac{(mn)!}{(n!)^m (m!)}$
(c) $\frac{(mn)!}{m!\,n!}$ (d) none of these

59. The number of ways in which we can get at least 6 successive tails when a coin is tossed 10 times in a row, is

(a) ${}^{10}C_6$ (b) $2^7 - 2^6$
(c) 40 (d) 48

60. Let $R = \{a, b, c, d\}$ and $S = \{1, 2, 3\}$, then the number of functions f, from R to S, which are onto is

(a) 80 (b) 16
(c) 24 (d) 36

61. If ${}^{n+1}C_2 + 2({}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots + {}^nC_2) = x$, then x equals

(a) $1 + 2 + 3 \ldots + n$ (b) $1^2 + 2^2 + \ldots + n^2$
(c) $\frac{1}{2}\, n(n + 1)(n + 3)$ (d) none of these

62. The number of ways in which three girls and nine boys can be seated in two cars, each having numbered seats, 3 in the front and 4 at the back, is

(a) ${}^{14}P_{12}$ (b) ${}^{14}P_{12} \times 14 \times 3$
(c) ${}^{11}P_9$ (d) none of these

63. The number of integral solutions of $x + y + z = 0$ with $x \geq -5$, $y \geq -5$, $z \geq -5$ is

(a) 134 (b) 136
(c) 138 (d) 140

64. A library has n different books and has p copies of each of the book. The number of ways of selecting one or more books from the library is

(a) n^p (b) p^n
(c) $(n + 1)^p - 1$ (d) $(p + 1)^n - 1$

65. The number of ways in which we can get a score of 11 by throwing three dice is

(a) 18 (b) 27
(c) 45 (d) 56

66. Three straight lines l_1, l_2 and l_3 are parallel and lie in the same plane. Five points are taken on each of l_1, l_2 and l_3. The maximum number of triangles which can be obtained with vertices at these points, is

(a) 425 (b) 405
(c) 415 (d) 505

67. The number of 9 digit numbers which have all distinct digits, is

(a) 10! (b) 9!
(c) 8(9!) (d) 10! – 9!

68. A four digit number of distinct digits is formed by using the digits 2, 3, 4, 5, 6, 7, 8. The number of such numbers which are divisible by 25, is

(a) 60 (b) 40
(c) 20 (d) 15

69. The range of the function $f(x) = {}^{15-x}P_{x-8}$ is

(a) $\{1, 6, 20, 24\}$ (b) $\{1, 6, 10, 15\}$
(c) $\{1, 4, 11, 15\}$ (d) $\{1, 8, 11, 19\}$

70. The number of ways of arranging 20 boys so that 3 particular boys are separated is

(a) 9(16!) (b) 15(16!)
(c) 15(17!)/2 (d) none of these

71. A library has n different books and 3 copies of each of the n books. The number of ways of selecting one or more books from the library is

(a) $4^n - 1$ (b) $5^n - 1$
(c) $2^n - 1$ (d) $n^3 - 1$.

72. Let $a_n = 10^n/n!$ for $n \geq 1$. Then a_n take the greatest value when n equals

(a) 20 (b) 18
(c) 6 (d) 9

Assertion-Reason Type Questions

73. **Statement-1:** The expression
$$\binom{40}{r}\binom{60}{0}+\binom{40}{r-1}\binom{60}{1}+\cdots$$ attains maximum value when $r = 50$.

Statement-2: $\binom{2n}{r}$ is maximum when $r = n$.

74. **Statement-1:** The number of non-negative integral solution of
$$x_1 + x_2 + \ldots + x_{20} = 100$$
is $\binom{120}{20}$.

Statement-2: The number of ways of distributing n identical objects among r persons giving zero or more objects to a person is $\binom{n+r-1}{r-1}$.

75. **Statement-1:** The sum of divisors of $n = 2^{10}\ 3^2\ 5^3\ 7^2\ 11^2$ is
$$\frac{1}{480}\ (2^{11} - 1)\ (3^3 - 1)\ (5^4 - 1)\ (7^3 - 1)\ (11^3 - 1)$$

Statement-2: The number of divisors of $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ where $p_1, p_2, \cdots p_r$ are distinct primes and $\alpha_1, \alpha_2, \cdots \alpha_r$ are natural numbers is $(\alpha_1 + 1)\ (\alpha_2 + 1) \cdots (\alpha_r + 1)$.

76. **Statement-1:** The number $\binom{1000}{500}$ is not divisible by 11.

Statement-2: If p is a prime, the exponent of p in $n!$ is
$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\cdots$$
where $[x]$ denotes the greatest integer $\leq x$.

77. **Statement-1:** A student is allowed to select at most n books from a collection of $(2n + 1)$ books. If the total number of ways in which he can select at least one book is 255, then $n = 3$.

Statement-2: $$\binom{2n+1}{0}+\binom{2n+1}{1}+\cdots+\binom{2n+1}{n}=4^n$$

LEVEL 2

Straight Objective Type Questions

78. $2m$ white counters and $2n$ red counters are arranged in a straight line with $(m + n)$ counters on each side of a central mark. The number of ways of arranging the counters, so that the arrangements are symmetrical with respect to the central mark, is

(a) $\dfrac{(m+n)!}{m!n!}$ (b) ${}^{2m+2n}C_{2m}$

(c) $\dfrac{1}{2}\dfrac{(m+n)!}{m!n!}$ (d) none of these

79. Let $x = (2n + 1)(2n + 3)(2n + 5) \ldots (4n - 3)(4n - 1)$ and $y = \left(\dfrac{1}{2}\right)^n \dfrac{(4n)!\,n!}{(2n)!\,(2n)!}$, then $x - y + 2^n$ is equal to

(a) 0 (b) $(2n)!/2^n$

(c) ${}^{2n}C_n$ (d) 2^n

80. There are 3 set of parallel lines containing respectively p lines, q lines and r lines respectively. The greatest number of parallelograms that can be formed by the system

(a) $pqr + (p - 1)(q - 1)(r - 1)$

(b) $\dfrac{1}{4}\{pqr + (p - 1)(q - 1)(r - 1)\}$

(c) $\dfrac{1}{4}pqr(p + 1)(q + 1)(r + 1)$

(d) none of these

81. The number of six digit numbers which have sum of their digits as an odd integer, is

(a) 45000 (b) 450000

(c) 97000 (d) 970000

82. In the identity $\dfrac{m!}{x(x+1)(x+2)\ldots(x+m)} = \sum_{i=0}^{m}\dfrac{A_i}{x+i}$, then value of A_k is

(a) ${}^nC_k$ (b) ${}^nC_{k+1}$

(c) $(-1)^k \cdot {}^nC_k$ (d) none of these

83. The number of integral solutions of $x^2 - y^2 = 352706$, is

(a) 276 (b) 0

(c) 720 (d) infinite

84. Let A be a set containing n elements. The number of ways of choosing two subsets P and Q of A such that $P \cap Q = \phi$ is

(a) 2^n (b) 3^n

(d) $4^n - 2^n$ (d) $4^n - 3^n$.

85. The last digit of $(1! + 2! + \ldots\ldots + 2009!)^{500}$ is

(a) 1 (b) 2

(c) 7 (d) 9

86. If $n = 2^{100}3^2$ and $\{d_1, d_2, \ldots d_k\}$ is the set of all divisors of n, then $\sum_{j=1}^{k}\dfrac{1}{d_j}$ equals

(a) 2 (b) N

(c) $2N$ (d) none of these

87. If ${}^nC_{r-1} = (k^2 - 8)({}^{n+1}C_r)$, then k belongs to

(a) $[-3, -2\sqrt{2})$

(b) $(-3, 3)$

(c) $[-3, -2\sqrt{2}) \cup (2\sqrt{2}, 3]$

(d) $[-2\sqrt{2}, 2\sqrt{2}]$

Previous Years' AIEEE/JEE Main Questions

1. The number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together is given by

(a) 30 (b) 5! ´ 4!

(c) $7! \times 5!$ (d) $6! \times 5!$ **[2003]**

2. A student is to answer 10 questions out of 13 questions in an examination such that he must choose at least 4 from the first five questions. The number of choices available to him is

(a) 196 (b) 280

(c) 346 (d) 140 **[2003]**

3. If ^nC_r denotes the number of combinations of n things taken r at a time, then the expression

$$^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$$

equals

(a) $^{n+2}C_{r+1}$ (b) $^{n+1}C_r$
(c) $^{n+1}C_{r+1}$ (d) $^{n+2}C_r$ **[2003]**

4. The range of the function $^{7-x}P_{x-3}$ is

(a) $\{1, 2, 3, 4\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
(c) $\{1, 2, 3\}$ (d) $\{1, 2, 3, 4, 5\}$ **[2004]**

5. How many ways are there to arrange the letters in the word GARDEN with the vowels in alphabetical order?

(a) 360 (b) 240
(c) 120 (d) 480 **[2004]**

6. The number of ways distributing 8 identical balls in 3 distinct boxes so that none of the boxes is empty is

(a) 3^8 (b) 21
(c) 5 (d) $^8C_3$ **[2004]**

7. If $S_n = \sum_{r=0}^{n} \frac{1}{^nC_r}$ and $t_n = \sum_{r=0}^{n} \frac{r}{^nC_r}$, then $\frac{t_n}{S_n}$ is equal to

(a) $n - 1$ (b) $\frac{1}{2}n - 1$
(c) $\frac{1}{2}n$ (d) $\frac{2n-1}{2}$ **[2004]**

8. The value of $^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$ is

(a) $^{56}C_3$ (b) $^{56}C_4$
(c) $^{55}C_4$ (d) $^{55}C_3$ **[2005]**

9. If the letters of the word SACHIN are arranged in all possible ways and these are written out as in a dictionary, then the word SACHIN appears at serial number

(a) 603 (b) 602
(c) 601 (d) 600 **[2005]**

10. At an election, a voter may vote for any number of candidates, not greater than the number to be elected. There are 10 candidates and 4 are to be elected. If a voter votes for at least one candidate, then the number of ways in which he can vote is

(a) 1110 (b) 5040
(c) 62010 (d) 385 **[2006]**

11. The set $S = \{1, 2, 3 \ldots 12\}$ is to be partitioned into three sets A, B, C of equal size such that, $A \cup B \cup C = S$, $A \cap B = B \cap C = C \cap A = \phi$. The number of ways to partition S is.

(a) $\frac{12!}{3!(4!)^3}$ (b) $\frac{12!}{3!(3!)^3}$
(c) $\frac{12!}{(4!)^3}$ (d) $\frac{12!}{(3!)^4}$ **[2007]**

12. In a shop there are five types of ice-creams available. A child buys six ice-creams

Statement-1: The number of different ways the child can buy the six ice-creams is $^{10}C_5$.

Statement-2: The number of different ways the child can buy the six ice-creams is equal to the number of different ways of arranging 6 A's and 4B's in a row. **[2008]**

13. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. Then the number of such arrangements is

(a) at least 750 but less than 1000
(b) at least 1000
(c) less than 500
(d) at least 500 but less than 750. **[2009]**

14. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is

(a) 66 (b) 108
(c) 3 (d) 36 **[2010]**

15. There are 10 points in a place, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then

(a) $N \leq 100$ (b) $100 < N \leq 140$
(c) $140 < N \leq 190$ (d) $N > 190$ **[2011]**

16. **Statement-1**: The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is $^9C_3$.

Statement-2: The number of choosing 3 places from 9 different places is $^9C_3$. **[2011]**

17. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is:

(a) 629 (b) 630
(c) 879 (d) 880 **[2012]**

18. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then value of n is

(a) 5 (b) 10
(c) 8 (d) 7 **[2013]**

19. The number of ways in which an examiner can assign 30 marks to 8 question, giving no less then 2 marks to any question is:
(a) ${}^{30}C_7$ (b) ${}^{21}C_8$
(c) ${}^{21}C_7$ (d) ${}^{30}C_8$ **[2013, online]**

20. 5-digit numbers are formed using 2, 3, 5, 7, 9 without repeating the digits. If p is the number of such numbers that exceeds 20000 and q be the number of those that lie between 30000 and 90000, then $p : q$ is:
(a) 6 : 5 (b) 3 : 2
(c) 4 : 3 (d) 5 : 3 **[2013, online]**

21. A committee of 4 persons is to be formed from 2 ladies, 2 old men and 4 youngmen such that it includes at least 1 lady, at least one old man and at most 2 youngmen. Then the total number of ways in which this committee can be formed is
(a) 40 (b) 41
(c) 16 (d) 32 **[2013, online]**

22. On the sides AB, BC, CA of a triangle ABC, 3, 4, 5 distinct points (excluding vertices A, B, C) are respectively chosen. The number of triangles that can be constructed using these chosen points as vertices are:
(a) 210 (b) 205
(c) 215 (d) 220 **[2013, online]**

23. The sum of the digits in the unit's place of all the 4-digit numbers formed by using the numbers 3, 4, 5 and 6, without repetition, is:
(a) 432 (b) 108
(c) 36 (d) 18 **[2014, online]**

24. An eight digit number divisible by 9 is to be formed using digits from 0 to 9 without repeating the digits. The number of ways in which this can be done is:
(a) 72(7!) (b) 18(7!)
(c) 40(7!) (d) 36(7!) **[2014, online]**

25. 8-digit numbers are formed using the digits 1, 1, 2, 2, 2, 3, 4, 4. The number of such numbers in which the odd digits do not occupy odd places, is:
(a) 160 (b) 120
(c) 60 (d) 48 **[2014, online]**

26. Two women and some men participated in a chess tournament in which every participant played two games with each of the other participants. If the number of games that the men played betweeen themselves exceeds the number of games that the men played with the women by 66, then the number of men who participated in the tournament lies in the interval:
(a) [8, 9] (b) [10, 12)
(c) (11, 13] (d) (14, 1) **[2014, online]**

* None of the given options is correct

27. Let A and B be two sets containing four and two elements respectively. Then the number of subsets of the set $A \times B$, each having at least three elements is:
(a) 219 (b) 256
(c) 275 (d) 510 **[2015]**

28. The number of integers greater than 6,000 that can be formed, using the digits 3, 5, 6, 7 and 8, without repetition, is:
(a) 216 (b) 192
(c) 120 (d) 72 **[2015]**

29. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0), is:
(a) 901 (b) 861
(c) 820 (d) 780 **[2015]**

30*. The number of ways of selecting 15 teams from 15 men and 15 women, such that team consists of a man and a woman, is:
(a) 1120 (b) 1240
(c) 1880 (d) 1960 **[2015 online]**

31. If in a regular polygon the number of diagonals is 54, then the number of sides of this polygon is:
(a) 10 (b) 12
(c) 9 (d) 16 **[2015 online]**

32. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary, then the position of the word SMALL is:
(a) 46th (b) 59th
(c) 52nh (d) 58th **[2016]**

33. The value of $\sum_{r=1}^{15} r^2\left(\frac{{}^{15}C_r}{{}^{15}C_{r-1}}\right)$ is equal to:
(a) 1240 (b) 560
(c) 1085 (d) 680 **[2016 online]**

34. The sum $\sum_{r=1}^{15}(r^2+1)\times(r!)$ is equal to:
(a) $11\times(11!)$ (b) $10\times(11!)$
(c) (11)! (d) $101\times(10!)$
[2016, online]

35. If the four letter words (need not be meaningful) are to be formed using the letters from the word "MEDITERRANEAN" such that the first letter is A and the fourth letter is E, then the total number of all such words is:
(a) 110 (b) 59
(c) $\frac{11!}{(2!)^3}$ (d) 56 **[2016, online]**

36. If $\frac{^{n+2}C_6}{^{n-2}P_2} = 11$, then n satisfies the equation:

(a) $n^2 + n - 110 = 0$ (b) $n^2 + 2n - 80 = 0$
(c) $n^2 + 3n - 108 = 0$ (d) $4n^2 + 5n - 84 = 0$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. A set B contain 2007 elements. Let C be the set consisting of subsets of B which contain at most 1003 elements. The number of elements in C is:
(a) 2^{2005} (b) 2^{2006}
(c) 2^{1003} (d) 2^{2007} **[2006]**

2. A parallelogram is cut by two set of n parallel lines, parallel to the sides of the parallelogram. The number of parallelograms formed is:
(a) $(^{n+2}C_2)\,(^{n+2}C_2)$ (b) $(^nC_2)\,(^{n+1}C_2)$
(c) $(^nC_2)\,(^{n+2}C_2)$ (d) $(^nC_2)\,(^nC_2)$ **[2007]**

3. If there are 30 onto, mappings from a set containing n elements to the set $\{0, 1\}$, then n equals:
(a) 3 (b) 5
(c) 7 (d) 2 **[2007]**

4. Let A be a set containing ten elements. Then the number of subsets of A containing at least four elements is
(a) 845 (b) 848
(c) 850 (d) 854 **[2008]**

5. A group of $2n$ students consisting of n boys and n girls are to be arranged in a row such that adjacent members are of opposite sex. The number of ways in which this can be done is:
(a) $2(n!)$ (b) $(n!)^2$
(c) $2(n!)^2$ (d) $n!$ **[2009]**

6. The maximum possible number of points of intersection of 8 straight lines and 4 circles is
(a) 164 (b) 76
(c) 104 (d) 32 **[2010]**

7. A committee consisting of at least three members is to be formed from a group of 6 boys and 6 girls such that it always has a boy and a girl. The number of ways to form such committee is:
(a) $2^{12} - 2^7 - 13$ (b) $2^{11} - 2^6 - 13$
(c) $2^{12} - 2^7 - 35$ (d) $2^{11} - 2^7 - 35$ **[2011]**

8. The total number of injective mappings from a set with m elements to a set with n elements for, $m > n$, is:
(a) $\frac{m!}{n!(m-n)!}$ (b) $\frac{m!}{(m-n)!}$
(c) n^m (d) zero **[2012]**

9. The least positive integral value of x for which $^{10}C_{x-1} > 2(^{10}C_x)$ is:
(a) 5 (b) 8
(c) 9 (d) 6 **[2013]**

10. Suppose that six students, including Madhu and Puja, are having six beds arranged in a row. Further, suppose that Madhu does not want a bed adjacent to Puja. Then the number of ways, the beds can be allotted to students is:
(a) 264 (b) 480
(c) 600 (d) 384 **[2014]**

11. In a car with seating capacity of exactly five persons, two persons can occupy the front seat and three persons can occupy the back seat. If amongst the seven persons, who wish to travel by this car, only two of them know driving then number of ways in which the car can be fully occupied and driven by them, is:
(a) 360 (b) 60
(c) 240 (d) 720 **[2015]**

12. If $S = \sum_{i=1}^{n}\left(\frac{^nC_{i-1}}{^nC_i + {^nC_{i-1}}}\right)^3 = \frac{36}{13}$, then n is equal to:
(a) 11 (b) 12
(c) 13 (d) 10 **[2016]**

13. A code word of length 4 consists two distinct consonants in the English alphabet followed by two digits from 1 to 9, with repetition allowed in digits. If the number of code words so formed ending with an even digit is 432 k, then k is equal to:
(a) 5 (b) 49
(c) 35 (d) 7 **[2016]**

Answers

Concept-based

1. (a) **2.** (b) **3.** (b) **4.** (a)
5. (a) **6.** (d) **7.** (c) **8.** (c)
9. (b) **10.** (a)

Level 1

11. (c)	**12.** (b)	**13.** (c)	**14.** (d)
15. (d)	**16.** (a)	**17.** (a)	**18.** (a)
19. (a)	**20.** (b)	**21.** (b)	**22.** (d)
23. (c)	**24.** (c)	**25.** (b)	**26.** (a)
27. (d)	**28.** (a)	**29.** (b)	**30.** (b)
31. (d)	**32.** (b)	**33.** (a)	**34.** (d)
35. (a)	**36.** (b)	**37.** (b)	**38.** (b)
39. (a)	**40.** (d)	**41.** (a)	**42.** (a)
43. (d)	**44.** (a)	**45.** (a)	**46.** (a)
47. (a)	**48.** (a)	**49.** (a)	**50.** (d)
51. (b)	**52.** (c)	**53.** (b)	**54.** (a)
55. (a)	**56.** (d)	**57.** (a)	**58.** (a)
59. (d)	**60.** (d)	**61.** (b)	**62.** (a)
63. (b)	**64.** (d)	**65.** (b)	**66.** (a)
67. (d)	**68.** (b)	**69.** (a)	**70.** (d)
71. (a)	**72.** (d)	**73.** (a)	**74.** (d)
75. (b)	**76.** (a)	**77.** (d)	

Level 2

78. (a)	**79.** (d)	**80.** (d)	**81.** (b)
82. (c)	**83.** (b)	**84.** (b)	**85.** (a)
86. (d)	**87.** (c)		

Previous Years' AIEEE/JEE Main Questions

1. (d)	**2.** (a)	**3.** (a)	**4.** (c)
5. (a)	**6.** (b)	**7.** (c)	**8.** (b)
9. (c)	**10.** (d)	**11.** (a)	**12.** (d)
13. (b)	**14.** (b)	**15.** (a)	**16.** (b)
17. (c)	**18.** (a)	**19.** (c)	**20.** (d)
21. (b)	**22.** (b)	**23.** (b)	**24.** (d)
25. (b)	**26.** (b)	**27.** (a)	**28.** (b)
29. (d)	**30.** (**)	**31.** (b)	**32.** (d)
33. (d)	**34.** (b)	**35.** (b)	**36.** (c)

Previous Years' B-Architecture Entrance Examination Questions

1. (b)	**2.** (a)	**3.** (b)	**4.** (b)
5. (c)	**6.** (c)	**7.** (a)	**8.** (d)
9. (b)	**10.** (b)	**11.** (d)	**12.** (b)
13. (c)			

Hints and Solutions

Concept-based

1. $$\frac{(2x)!}{3!(2x-3)!}\cdot\frac{2!(x-2)!}{x!}=\frac{44}{3}$$

$$\Rightarrow \frac{(2x)(2x-1)(2x-2)}{3x(x-1)}=\frac{44}{3}$$

$\Rightarrow 2x - 1 = 11 \Rightarrow x = 6$

2. $$\frac{(2x+1)(2x)}{x+3}=\frac{72}{7}$$

$\Rightarrow 7(2x^2 + x) = 36(x + 3)$

$\Rightarrow 14x^2 - 29x - 108 = 0$

$\Rightarrow (14x + 27)(x - 4) = 0 \Rightarrow x = 4$

3. $(m + n)(m + n - 1) = 90$ and $(m - n)(m - n - 1) = 30$

$\Rightarrow m + n = 10,\ m - n = 6$

$\therefore\ m = 8,\ n = 2$

4. Number of ways of putting beads in a necklace is 4! But clockwise arrangement abcde and anticlockwise arrangement aedcb are not really different, for, the arrangement abcde gives rise to arrangement aedcb if the necklace is turned over.

$\therefore$ required number of ways $= \frac{1}{4}(4!) = 12$

5. For a point of intersection, we require two lines, therefor, the number of points of intersection is ${}^{15}C_3$.

6. $m = \frac{1}{2}n(n - 1) \Rightarrow \frac{4m}{n} = 2(n - 1) = p(p - 1) - 2$

$\therefore\ 18 = p(p - 1) - 2 \Rightarrow p = 5$

7. $2({}^{2n}C_5) = {}^{2n}C_4 + {}^{2n}C_6$

$$\Rightarrow\ 2=\frac{{}^{2n}C_4}{{}^{2n}C_5}+\frac{{}^{2n}C_6}{{}^{2n}C_5}=\frac{5}{2n-4}+\frac{2n-5}{6}$$

$\Rightarrow 12(2n - 4) = 30 + (2n - 4)(2n - 5)$

$\Rightarrow 2n^2 - 21n + 49 = 0 \Rightarrow n = 7$

8. ${}^nP_r = (r!)({}^nC_r) \Rightarrow 1680 = (r!)(70)$

$\Rightarrow r! = 24 \Rightarrow r = 4$

Also, $1680 = {}^nP_4 = n(n - 1)(n - 2)(n - 3)$

$\Rightarrow (n^2 - 3n)(n^2 - 3n + 2) = 1680$

$\Rightarrow (n^2 - 3n + 1)^2 = 41^2 \Rightarrow n^2 - 3n + 1 = 41$

$\Rightarrow n^2 - 3n - 40 = 0 \Rightarrow (n + 5)(n - 8) = 0$

$\Rightarrow n = 8.$

9. As $(r + 1)({}^{n+1}C_{r+1}) = (n + 1)({}^nC_r)$,

we get, $\frac{n+1}{r+1}=\frac{11}{6}$.

Also, $\frac{n}{r}=\frac{6}{3}\Rightarrow r=\frac{1}{2}n$

Thus, $\frac{n+1}{n/2+1}=\frac{11}{6}\Rightarrow n=10$

10. We can choose x, y, z in n^3 ways out of which ${}^nP_3$ have distinct elements.

Level 1

11. Arrange non-vegeterians in $(6 - 1)! = 5!$ ways and vegeterian in $(5 - 1)! = 4!$ways

    $\therefore$ required number of ways

    $= (5!)\ (4!) = (120)\ (24) = 2880$

12. Number of ways passing the examination is

    $S = {}^{11}C_6 + {}^{11}C_7 + {}^{11}C_8 + {}^{11}C_9 + {}^{11}C_{10} + {}^{11}C_{11} \quad (1)$

    Using ${}^{n}C_{r} = {}^{n}C_{n-r}$, we get

    $S = {}^{11}C_5 + {}^{11}C_4 + {}^{11}C_3 + {}^{11}C_2 + {}^{11}C_1 + {}^{11}C_0 \quad (2)$

    Adding (1) and (2) we get

    $2S = 2^{11} \Rightarrow S = 2^{10}$

13. The voter has to vote for at least one and at most 5 candidates. The number of ways is:

    ${}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5 = 637$

14. Seven boys can be arranged in 7! ways. If $XB_1\ XB_2\ XB_3\ XB_4\ XB_5\ XB_6\ XB_7\ X$ is one such arrangement, then 3 girls can be arranged at any of the 3 places marked with a X. Thus, the number of arrangements is $({}^{8}P_3)\ (7!) = (6 \times 7)\ (8!) = (42)\ (8!)$. Thus, $k = 42$.

15. There is only one way in which five identical balls can be put in 10 boxes.

16. The number of 4 digit numbers is (9) (10) (10) (10), out of these (9) (9) (8) (7) have distinct digits.

17. Excluding two shortest girls, the remaining eight girls can be divided into two groups in $\frac{1}{2!}({}^{8}C_4)({}^{4}C_4) = 35$ ways. The two shortest girls can be put in two different groups in 2 ways. Therefore, the required number of ways is 70.

18. $\frac{1}{x} + \frac{1}{y} = \frac{1}{6} \Rightarrow 6\,(x + y) = xy$

    $\Rightarrow (x - 6)\ (y - 6) = 36$

    $\Rightarrow (x - 6)|36.$

    $\Rightarrow x - 6 = 1, 2, 3, 4, 6, 9, 12, 18, 36$

    $\Rightarrow (x, y) = (7, 42), (8, 24), (9, 18), (10, 15), (12, 12), (15, 10), (18, 9), (24, 8), (42, 7)$

19. Suppose $m = 10a + b$ and $n = 10c + d$ where $1 \le a, c \le 9$ and $0 \le b, d \le 9$. Note that $0 \le d \le b$. Thus d can take $(b + 1)$ values. Similarly , $1 \le c \le a$, therefore, c can take a values. Hence, the required number of pairs (m, n) is $(1 + 2 + \ldots + 10)\ (1 + 2 + .. + 9) = 2475$

20. Any number 5-digit formed by 2, 3, 5, 7, 9, exceeds 20000. Therefore, $p = 5! = 120$

    A number will lie between 30000 and 90000 if it begins with 3, 5 or 7

    $\therefore q = (3)\ (4!)$. Thus $p : q = 5 : 3$.

21. Number of diagonals = ${}^{15}C_2 - 15 = 90$.

22. As $0 + 1 + 2 + 3 + 4 + 5 = 15$, for a five digit number to be divisible by 3 either do not use 0 or 3.

    When 0 is not used, the number of such numbers is 5! and when 3 is not used, the number of such numbers is $5! - 4!$

    $\therefore$ Required number of ways $= 5! + (5! - 4!) = 216$

23. Use: The number of subsets containing exactly r elements is ${}^{n}C_{r}$.

24. The given expression = number of ways of selecting r persons out of a men and b women $= {}^{a+b}C_r$

25. For each place we have two choices: either use 0 or use 1.

26. $1 + P_1 + 2P_2 + 3P_3 + \ldots + nP_n$

    $= 1 + 1! + 2(2!) + 3(3!) + \ldots + n(n!)$

    $$= 1 + \sum_{r=1}^{n} [(r+1) - 1]r! = 1 + \sum_{r=1}^{n} \{(r+1)! - r!\}$$

    $= 1 + (n + 1)! - 1! = (n + 1)!$

27. $$\sum_{r=0}^{n} \frac{n-2r}{{}^{n}C_r} = \sum_{r=0}^{n} \frac{n-r}{{}^{n}C_r} - \sum_{r=0}^{n} \frac{r}{{}^{n}C_r}$$

    $$= \sum_{r=0}^{n} \frac{n-r}{{}^{n}C_{n-r}} - \sum_{r=0}^{n} \frac{r}{{}^{n}C_r} = 0$$

28. Treating w women as one block, we can permute m men and one block in $(m + 1)!$ ways and women in the block in $w!$ ways. Thus, the required number of ways is $(m + 1)!\ w!$

29. Since $0 + 1 + 2 + 3 + 4 + 8 = 18$, the 5 digit number will be divisible by 3 if either 0 or 3 is not used.

    When 0 is not used. For the unit place we have 3 choices (2, 4 or 8) and for the remaining place we have 4! Choices.

    When 3 is not used

    In this case, 0 is used at the unit's place, the number of choices is 4! If 0 is not used at the unit's place, then unit's place can be filled up in 3 ways and the remaining places in $(4! - 3!)$ ways.

    Thus, required number of numbers

    $= 3(4!) + 4! + 3(4! - 3!) = 150$

30. Use ${}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \dots + {}^{2n+1}C_{2n} = 2^{2n} - 1$
31. Women can be chosen in ${}^{8}C_3$ ways and men in ${}^{7}C_4 - {}^{5}C_2$
32. $1.3.5 \dots (2n-1) = \dfrac{(2n)!}{2^n\,(n!)} = \left(\dfrac{1}{2}\right)^n \left({}^{2n}C_n\right)\left({}^{n}P_n\right)$
33. We can choose 10 players from (22 – 2) players in ${}^{20}C_{10}$ ways and one player from 2 players in ${}^{2}C_1$ ways.

   ∴ required number of ways

   $$= \frac{20!}{10!\,10!} \times 2 = 369512$$

34. Choose 10 places (out of 20) for $x_1, x_2, \dots, x_{10}$ in ${}^{20}C_{10}$ ways.
35. Let n consecutive natural numbers be $m+1, m+2, \dots, m+n$ where $m \ge 0$.

   $$P = (m+1)(m+2)\dots(m+n)$$
   $$= \left(\frac{m!\,(m+1)\cdots(m+n)}{m!\,n!}\right) n!$$
   $$= \left({}^{m+n}C_n\right)\left({}^{n}P_n\right)$$

   ∴ P is divisible by ${}^{n}P_n$

36. This can be done in four mutually exclusive ways as follows:

	Row R_1	*Row R_2*	*Row R_3*	*Number of ways*
I	1	3	2	$({}^2C_1)({}^4C_3)({}^2C_2) = 8$
II	1	4	1	$({}^2C_1)({}^4C_4)({}^2C_1) = 4$
III	2	2	2	$({}^2C_2)({}^4C_2)({}^2C_2) = 6$
IV	2	3	1	$({}^2C_2)({}^4C_3)({}^2C_1) = 8$
			Total	26

37. Number of ways is ${}^{12}C_6$ for M, ${}^{6}C_4$ for P and ${}^{2}C_2$ for C.

   Thus, the required number of ways

   $$= ({}^{12}C_6)({}^{6}C_4)({}^{2}C_2)$$

38. The number of points of intersections

   $$= ({}^{n}C_2)(2) + ({}^{m}C_2)(1) + ({}^{n}C_1)({}^{m}C_1)(2)$$
   $$= n(n-1) + \frac{1}{2}\,m(m-1) + 2mn$$

39. Imitate Example 18.
40. Required number of natural numbers

   $$= {}^{9}P_1 + 9 \times {}^{9}P_1 + 9 \times {}^{9}P_2 + \dots + 9 \times {}^{9}P_9$$

41. When 7 is right in the beginning, the number of numbers = 9^4 and when 7 is not in the beginning, the number of numbers is $({}^{4}C_1)(8)(9^3)$

   ∴ Required number of numbers

   $$= 9^4 + ({}^{4}C_1)(8)\,9^3 = (41)(9^3)$$

42. Use unit's digit of 17^{2009} is same as that of the unit's digit of 7^{2009}.
43. $\dfrac{{}^{5}C_n}{{}^{4}C_n} = 1 + \dfrac{{}^{5}C_n}{{}^{6}C_n}$

   $$\Rightarrow \quad \frac{5}{5-n} = 1 + \frac{6-n}{6}$$
   $$\Rightarrow n^2 - 17n + 30 = 0 \quad \Rightarrow \quad n = 2.$$

44. Number of ways = ${}^{5}C_1 + {}^{5}C_2 + {}^{5}C_3 = 25$
45. Use ${}^{2m+1}C_r$ is maximum when $r = m$ or $r = m + 1$.
46. Required number of ways

   = Total number of ways – Number of ways when two N's are together

   $$= \frac{6!}{3!\,2!} - \frac{5!}{3!} = 40$$

47. The number of paths

   = the number of ways of arranging 5 $U'\,S$ and 7 R's.

   $$= \frac{12!}{5!\,7!} = {}^{12}C_5$$

48. Required number of paths

   = (Number of paths upto (3, 4)) × (Number of paths from (3, 4) to (5, 7))

   $$= \frac{7!}{3!\,4!} \times \frac{5!}{4!} = 175$$

49. The number of ways of putting all the balls in exactly one box = 5

   As for each ball there are 5 choices for the box, the number of ways of putting balls in the boxes = 5^5

   Thus, the required number of ways of putting the balls in the boxes = $5^5 - 5$

50. $x^2 + 6x + y^2 = 4 \;\Rightarrow\; (x+3)^2 + y^2 = 13$

   $\Rightarrow\; (x+3, y) = (\pm 3, \pm 2)$ or $(\pm 2, \pm 3)$

   $\Rightarrow\; x = 0, y = \pm 2, x = -6, y = \pm 2,$

   $x = -1, y = \pm 3, y = -5, y = \pm 3$

51. Use $20! = 2^{18}\, 3^8\, 5^4\, 7^2\, (11)(13)(17)(19)$
52. When exactly three digits are identical and the remaining two are different, then the number of such numbers

   $$\left({}^{3}C_1\right)\left(\frac{5!}{3!}\right) = 60$$

   When three digits are identical and the remaining two are also identical, then the number of ways

   $$= \left({}^{3}C_2\right)(2)\left(\frac{5!}{3!\,2!}\right) = 60$$

   Thus, the number of such numbers = 120.

53. Use the fact that $r!$ is divisible by 7 whenever $r \geq 7$.

54. As the last digit of 6^m, $m \in \mathbf{N}$ is 6, $6^m + 9^n$ will be divisible by 5 if the unit's digit of 9^n is 4 or 9. This is possible when n is odd.

    $\therefore$ required number of ordered pairs = 50×25 = 1250.

55. The required number of words = $10^5 - {}^{10}P_5$

56. To get a square of size $r \times r$, we must choose $(r + 1)$ consecutive horizontal and $(r + 1)$ consecutive vertical lines. This can be done in $(9 - r)(9 - r) = (9 - r)^2$ ways.

57. ${}^nC_2 = 153 \Rightarrow n = 18$

58. As groups are indistinguishable, the number of ways

$$= \frac{1}{n!}({}^{mn}C_m)({}^{mn-m}C_m)\ldots({}^mC_m)$$

$$= \frac{1}{n!}\frac{(mn)!}{m!(mn-m)!}\frac{(mn-m)!}{m!(mn-2m)!}\cdots\frac{m!}{m!0!}$$

$$= \frac{1}{n!}\frac{(mn)!}{(m!)^n}$$

59. The sequence of six tails may begin at with the ith toss where $i = 1, 2, 3, 4, 5$. Thus, the number of ways is

$$2^4 + 1(2^3) + 2(1)(2^2) + 2^2(1)2 + 2^3(1) = 48$$

60. Total number of functions = 3^4.

    All the four elements can be mapped to exactly one element in 3 ways, and exactly two elements in $3(2^4 - 2)$.

    Thus, the number of onto functions = $3^4 - 3 - 3(2^4 - 2) = 36$.

61. We have

$$\begin{aligned}&{}^2C_2 + {}^3C_2 + {}^4C_2 + \ldots + {}^nC_2\\ &= ({}^3C_3 + {}^3C_2) + {}^4C_2 + \ldots + {}^nC_2\\ &= {}^4C_3 + {}^4C_2 + \ldots + {}^nC_2\\ &= {}^5C_3 + \ldots + {}^nC_2 = \ldots = {}^{n+1}C_3\end{aligned}$$

Thus, $$\begin{aligned}x &= {}^{n+1}C_2 + 2({}^{n+1}C_3)\\ &= {}^{n+1}C_2 + {}^{n+1}C_3 + {}^{n+1}C_3 = {}^{n+2}C_3 + {}^{n+1}C_3\\ &= \frac{1}{6}n(n+1)(2n+1)\end{aligned}$$

62. We have to arrange 12 persons at 14 seats.

63. Let $x_1 = x + 5$, $y_1 = y + 5$, $z_1 = z + 5$ and find the number of non-negative integral solutions of

$$x_1 + y_1 + z_1 = 15$$

64. For each book we have $p + 1$ choices.

65. Find the number of positive integral solutions of

$$x + y + z = 11$$

    where $1 \leq x, y, z \leq 6$.

66. The total number of points is 15. From these 15 points we can obtain ${}^{15}C_3$ triangles. However, if all the 3 points are chosen on the same straight line, we do not get a triangle. Therefore, the required number of triangles

$$= {}^{15}C_3 - 3({}^5C_3) = 425.$$

67. The required number of ways

    = The number of ways of permuting nine digits out of ten digits 0, 1, 2, …, 9

    – The number of ways of permuting nine digits out of nine digits 1, 2, …, 9

$$= {}^{10}P_9 - {}^9P_9 = \frac{10!}{1!} - 9! = 10! - 9!$$

68. The number will be divisible by 25 if it ends in 25 or 75. Therefore, the required number of numbers is

$$\left({}^5P_2\right)(2) = 40$$

69. We must have $15 - x \geq 1$ and $x - 8 \geq 0$ and $15 - x \geq x - 8 \Rightarrow 8 \leq x \leq 11$. Now,

$$f(8) = {}^7P_0 = 1,\quad f(9) = {}^6P_1 = 6,$$

$$f(10) = {}^5P_2 = 20,\quad f(11) = {}^4P_3 = 24,$$

    Thus, range of f is $\{1, 6, 20, 24\}$.

70. We can arrange remaining 17 boys in 17! ways. For 3 particular boys we have 18 positions. We can arrange 3 particulars boys at these places in ${}^{18}P_3$ ways. Thus, the required number of ways

$$= (17!)\left({}^{18}P_3\right) = 17!\,\frac{18!}{15!}$$

71. For each book we have four choices. We can choose 0, 1, 2 or 3 volumes of the book.

72. We have $\dfrac{a_n}{a_{n+1}} = \dfrac{10^n}{n!} \times \dfrac{(n+1)!}{10^{n+1}} = \dfrac{n+1}{10}$.

    Note that $\dfrac{a_1}{a_2} < 1$, $\dfrac{a_2}{a_3} < 1$, $\dfrac{a_3}{a_4} < 1$, $\dfrac{a_4}{a_5} < 1$,

$$\ldots\ \frac{a_8}{a_9} < 1, \frac{a_9}{a_{10}} = 1, \frac{a_{10}}{a_{11}} > 1, \frac{a_{11}}{a_{12}} > 1, \ldots$$

    Thus, $a_1 < a_2 < a_3 < \ldots < a_9 = a_{10} > a_{11} > a_{12}$

    $\therefore$ a_n is greatest if $n = 9$ or 10.

73. We have $\dfrac{\binom{2n}{r}}{\binom{2n}{r+1}} = \dfrac{(2n)!}{r!(2n-r)!}\,\dfrac{(r+1)!(2n-r-1)!}{(2n)!}$

$$= \frac{r+1}{2n-r}$$

Since for $0 \le r \le n-1$, $\dfrac{r+1}{2n-r} < 1$, we get

$$\binom{2n}{0} < \binom{2n}{1} < \cdots < \binom{2n}{n-1} < \binom{2n}{n}$$

Also, as $\binom{2n}{r} = \binom{2n}{2n-r}$

$$\binom{2n}{2n} < \binom{2n}{2n-1} \cdots < \binom{2n}{2n+1} < \binom{2n}{n}$$

Thus, $\binom{2n}{r}$ is maximum when $r = n$.

Next,

$$\binom{40}{r}\binom{60}{0} + \binom{40}{r-1}\binom{60}{1} + \cdots$$

= the number ways of selecting r persons out of 40 men and 60 women

$$= \binom{100}{r}$$

which is maximum when $r = 50$.

74. The number of ways of distributing n identical objects among r persons giving zero of more objects to a person is equivalent to arranging n identical objects of one kind and $(r-1)$ identical objects of second kind in a row, which is equal to $\dfrac{(n+r-1)!}{n!(r-1)!} = \binom{n+r-1}{r-1}$.

Next, the number of non-negative integral solutions of $x_1 + x_2 + \cdots + x_{20} = 100$ equals the number of ways of distributing 100 identical objects among 20 persons giving zero or more objects to a person, which equal $\binom{100+20-1}{20-1} = \binom{109}{19}$.

75. Sum of the divisors of n

$= (1 + 2 + \cdots - 2^{10})\,(1 + 3 + 3^2)\,(1 + 5 + 5^2 + 5^3)\,(1 + 7 + 7^2)\,(1 + 11 + 11^2)$

$$= (2^{11} - 1)\left(\frac{3^3-1}{3-1}\right)\left(\frac{5^4-1}{5-1}\right)\left(\frac{7^3-1}{7-1}\right)\left(\frac{11^3-1}{11-1}\right)$$

$$= \frac{1}{480}(2^{11} - 1)\,(3^3 - 1)\,(5^4 - 1)\,(7^3 - 1)\,(11^3 - 1)$$

A divisors of m is of the form $p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$ where $0 \le \beta_i \le \alpha_i$ for $i = 1, 2, \ldots, r$.

That is, β_i can take $\alpha_i + 1$ values. Thus, the number of divisors of m is $(\alpha_1 + 1)\,(\alpha_2 + 1)\,\ldots\,(\alpha_r + 1)$

76. For the truth of statement-2, see theory of this chapter.

We have

$$\binom{1000}{500} = \frac{1000!}{500!500!}$$

Exponent of 11 in the prime factorization of 1000! is

$$\left[\frac{1000}{11}\right] + \left[\frac{1000}{11^2}\right] = 90 + 8 = 98$$

Exponent of 11 in the prime factorization of 500! is

$$\left[\frac{500}{11}\right] + \left[\frac{500}{11^2}\right] = 45 + 4 = 49.$$

Thus, exponent of 11 in the prime factorization of $\binom{1000}{500}$ is 0.

$\Rightarrow$ $\binom{1000}{500}$ is not divisible by 11.

77. Let

$$S = \binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{n} \qquad (1)$$

Using $\binom{n}{r} = \binom{n}{n-r}$, we can write (1) as

$$S = \binom{2n+1}{2n+1} + \binom{2n+1}{2n} + \cdots + \binom{2n+1}{n+1} \qquad (2)$$

Adding (1) and (2), we get

$$2S = \binom{2n+1}{0} + \cdots + \binom{2n+1}{2n+1} = 2^{2n+1}$$

$\Rightarrow \quad S = 2^{2n} = 4^n$

The number of ways of choosing r books out of $2n + 1$ is $\binom{2n+1}{r}$. We are given

$$\Rightarrow \quad \binom{2n+1}{1} + \binom{2n+1}{2} + \cdots + \binom{2n+1}{n} = 255$$

$$\Rightarrow \quad 4^n - 1 = 255 \quad \Rightarrow \quad 4^n = 256 = 4^4 \quad \Rightarrow \quad n = 4.$$

Level 2

78. Arrange m white and n red counters on one side of the central mark. This can be done in $\dfrac{(m+n)!}{m!n!}$

79. $x = \dfrac{(2n+1)(2n+2)\cdots(4n-1)(4n)}{(2n+2)(2n+4)\cdots(4n)}$

$= \dfrac{1}{(2n)!}\dfrac{(4n)!n!}{2^n(2n)!} = y$

Thus, $x - y + 2^n = 2^n$

80. Number of parallelograms

$= ({}^pC_2)\,({}^qC_2) + ({}^qC_2)\,({}^rC_2) + ({}^rC_2)\,({}^pC_2)$

81. For the first place we have nine choices. For each of the next four places, we have 10 choices. At this stage we add the numbers already selected and choose the digit at unit's place in 5 ways.

82. A_k is obtained by putting $x = -k$ everywhere in the expression except in $x + k$. Therfore,

$A_k = \dfrac{m!}{(-k)\cdots(-k+k-1)(-k+k+1)\cdots(-k+m)}$

$= \dfrac{(-1)^k m!}{k!(m-k)!} = (-1)^k\left({}^mC_k\right)$

83. As 352706 is even, either both x and y are even or both x and y are odd.

If x, y are both even, $x^2 - y^2$ is divisble by 4, but 352706 is not divisible by 4.

If x, y are both odd, $x^2 - y^2$ is divisible by 4, but 352706 is not divisible by 4.

84. For each $x \in A$, we have four choices;

(i) $x \in P$, $x \in Q$

(ii) $x \in P$, $x \notin Q$

(iii) $x \notin P$, $x \in Q$

(iv) $x \notin P$, $x \notin Q$

Out of these four choices, last 3 choices imply $x \notin P \cap Q$.

Thus, P and Q can be chosen in 3^n ways, so that $P \cap Q = \phi$.

85. As $1! + 2! + 3! + 4! = 33$ and last digit of $n!$, for $n \geq 5$ is 0, last digit of $(1! + 2! + \cdots + 2009!)^{500}$ is same as that of last digit of 3^{500}

But $3^{500} = 9^{250} = (10 - 1)^{250}$, and last digit of $(10 - 1)^{250}$ is 1.

86. Note that $d\,\big|\,n \iff \dfrac{n}{d}\,\Big|\,n$

Now, $\displaystyle\sum_{j=1}^{k}\frac{1}{d_j} = \sum_{j=1}^{k}\frac{d_j}{n} = \frac{1}{n}\sum_{j=1}^{k} d_j$

But divisors of $n = 2^{100}\,3^2$ are

$1, 2, \cdots, 2^{100}, 3, (3)\,(2), \cdots, 3(2^{100}), 3^2\,(3^2)(2), \cdots (3^2)\,(2^{100})$

Thus, $\displaystyle\sum_{j=1}^{k} d_j = (1 + 2 + \cdots + 2^{100})\,(1 + 3 + 3^2)$

$= \left(\dfrac{2^{101}-1}{2-1}\right)(13) = 13(2^{101}-1)$

87. We have, $r - 1 \geq 0$, $r \leq n + 1$

$\Rightarrow 1 \leq r \leq n + 1 \Rightarrow \dfrac{1}{n+1} \leq \dfrac{r}{n+1} \leq 1$

Also, $k^2 - 8 = \dfrac{n!}{(r-1)!\,(n-r+1)!}\cdot\dfrac{r!(n+1-r)!}{(n+1)!}$

$= \dfrac{r}{n+1}$

Thus, $\dfrac{1}{n+1} \leq k^2 - 8 \leq 1$

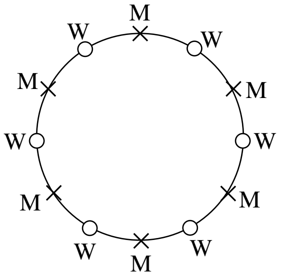
$\Rightarrow 8 < \dfrac{1}{n+1} + 8 \leq k^2 \leq 9$

$\Rightarrow 8 < k^2 \leq 9 \Rightarrow -3 \leq k < -2\sqrt{2}$ or $2\sqrt{2} < k \leq 3$

Hence, $k \in [-3, -2\sqrt{2})$

Previous Years' AIEEE/JEE Main Questions

1. We can arrange 6 men around the round table in 5! ways. There are 6 places for women to sit. Out of these we choose 5 and arrange women there. This can be done in $(5!)\,({}^6C_5)\,(5!) = (5!)\,(6!)$ ways.



2. Required number of ways

$= ({}^5C_4)\,({}^8C_6) + ({}^5C_5)\,({}^8C_5) = 196$

3. ${}^nC_{r+1} + {}^nC_{r-1} + 2({}^nC_r)$

$= ({}^nC_{r-1} + {}^nC_r) + ({}^nC_r + {}^nC_{r+1})$

$= {}^{n+1}C_r + {}^{n+1}C_{r+1} = {}^{n+2}C_{r+1}$

4. We must have

$7 - x \geq 1$, $x - 3 \geq 0$ and $7 - x \geq x - 3$

$\Rightarrow x \leq 6$, $x \geq 3$ and $x \leq 5$

Thus, $3 \leq x \leq 5$

$\therefore$ Range $= \{{}^4P_0, {}^3P_1, {}^2P_2\}$

$= \{1, 3, 2\}$

5. We can arrange the letters of the word GARDEN in 6! ways. Exactly half of these will be with vowels in alphabetical order.

So the required number of ways is (1/2) 6! = 360.

6. If x_1 is the number of balls in the ith box, then we must have

$$x_1 + x_2 + x_3 = 8 \qquad (1)$$

where $x_i \geq 1$.

The number of solutions of (1) is ${}^7C_2 = 21$.

7. We have $t_n = \sum_{r=0}^{n} \frac{r}{{}^nC_r} = \sum_{r=0}^{n} \frac{n-r}{{}^nC_{n-r}}$

$$= \sum_{r=0}^{n} \frac{n-r}{{}^nC_r} = n\, s_n - t_n$$

$$\Rightarrow 2t_n = n\, s_n \Rightarrow \frac{t_n}{s_n} = \frac{n}{2}.$$

8. ${}^{50}C_4 + \sum_{r=1}^{6} {}^{56-r}C_3$

$= {}^{50}C_4 + ({}^{50}C_3 + {}^{51}C_3 + \ldots + {}^{55}C_3)$

$= ({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3 + \ldots + {}^{55}C_3$

$= ({}^{51}C_4 + {}^{51}C_3) + {}^{52}C_3 + \ldots + {}^{55}C_3$

[using ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$]

$= {}^{52}C_4 + {}^{52}C_3 + \ldots + {}^{55}C_3$

$= \ldots = {}^{56}C_4$

9. Number of words which begin with A, C, H, I or N is 5(5!) = 600. These have ranks from 1 to 600. Then words beginning S are listed and SACHIN is the first word beginning with S, therefore, its rank is 601.

10. Required number of ways

$= {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4$

$= 10 + 45 + 120 + 210 = 385$

11. Number of ways

$= ({}^{12}C_4)\,({}^8C_4)\,({}^4C_4)$

$$= \frac{12!}{(41)^3}$$

12. Let x_i = number of ice creams of ith type bought by the child.

Then, $x_1 + x_2 + x_3 + x_4 + x_5 = 6 \qquad (1)$

The number of non-negative integral solution of (1) ${}^{6+5-1}C_{5-1} = {}^{10}C_4 = N$ (say)

$\therefore$ Statement-1 is false.

The number of ways of arranging 6 A's and 4 B's in a row

$$= \frac{10!}{4!6!} = {}^{10}C_4 = N.$$

Thus, statement-2 is true.

13. We can choose 4 novels out of 6 in ${}^6C_4$ ways and 1 dictionary out of 3 in ${}^3C_1$ ways. We can arrange 4 novels and 1 dictionary in the middle in 4! Ways. Thus, required number of ways

$= ({}^6C_4)\,({}^3C_1)\,(4!) = 1080 > 1000$

14. Number of ways

$= ({}^3C_2)\,({}^9C_2) = (3)\,(36) = 108$

15. Number of triangles

$N = {}^{10}C_3 - {}^6C_3$

$$= \frac{1}{6}(10 \times 9 \times 8) - \frac{1}{6}(6 \times 5 \times 4)$$

$= 100$

16. Let x_i = number of balls put in the ith box, then

$x_1 + x_2 + x_3 + x_4 = 10 \qquad (1)$

where $x_i \geq 1$.

Let $x_i = y_i + 1$ where $y_i \geq 0$

Thus, (1) becomes

$y_1 + y_2 + y_3 + y_4 = 6 \qquad (2)$

Number of non-negative integral solutions of (2) = number of ways of arranging 6 identical balls and 3 identical separators in a row.

$$= \frac{(6+3)!}{6!3!} = {}^9C_3$$

= number of ways of choosing 3 places out of 9 different places.

17. We can choose 0 or more balls out of 10 white balls in 11 ways.

[Select 0 or 1 or 2, … or 10 balls.]

Thus, the number ways of selecting 0 or more balls is (1 + 10) (1 + 9) (1 + 7) = 880.

Therefore, the number of ways selecting one or more balls is 880 – 1 = 879.

18. As $T_n = {}^nC_3$,

$T_{n+1} - T_n = 10$

$\Rightarrow {}^{n+1}C_3 - {}^nC_3 = 10$

$\Rightarrow {}^nC_2 + {}^nC_3 - {}^nC_3 = 10$

$\Rightarrow \frac{1}{2}\, n(n-1) = 10 \Rightarrow n = 5$

19. Let x_i = marks assigned to ith question, then

$x_1 + x_2 + \ldots + x_8 = 30,$

$x_i \geq 2$ for $i = 1, 2, \ldots, 8.$

Put $x_i = y_i + 2$, then

$$y_1 + y_2 + \ldots + y_8 = 14 \qquad (1)$$

The number of solutions of (1) is

${}^{14+8-1}C_{8-1} = {}^{21}C_7$

20. Any number that is formed by 2, 3, 5, 7, 9 exceeds 20000.

$\therefore\ p = 5!$

A number that begins with 3, 5 or 7 lies between 3000 and 90000. Therefore,

$q = (3)\ (4!)$

Thus, $p : q = 5 : 3$

21. Different number of ways of forming committee are as under.

	Ladies	Old men	Young men
OPTION 1	1	1	2
OPTION 2	1	2	1
OPTION 3	2	1	1

Thus, the required number of ways

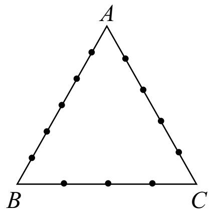
$= ({}^2C_1)\ ({}^2C_1)\ ({}^4C_2) + ({}^2C_1)\ ({}^2C_2)\ ({}^4C_1) + ({}^2C_2)\ ({}^2C_1)\ ({}^4C_1)$

$= 24 + 8 + 8 = 40.$

22. Take one vertex on each side.

This can be done in $({}^3C_1)\ ({}^4C_1)\ ({}^5C_1)$

= 60 ways.



Choose two vertices on one side and one from the remaining two sides. This can be done in

$({}^3C_2)\ ({}^9C_1) + ({}^4C_2)\ ({}^8C_1) + ({}^5C_2)\ ({}^7C_1)$

= 27 + 48 + 70 = 145.

Thus, number of required triangle is 60 + 145 = 205.

23. Each digit occurs at the unit's place exactly 3! = 6 times. This, sum of the digits at the units place

= 6[3 + 4 + 5 + 6] = 108

24. As 0 + 1 + 2 + … + 9 = 45, a number will be divisible by 9, if we do not use (0, 9), (1, 8), (2, 7), (3, 6), (4, 5).

Thus, the number of 8 digit number that are divisible by 9 is 8! + (8! – 7!) (4) = (36) (7!)

25. We can choose places for 1, 1, 3 in ${}^4C_3 = 4$ ways.

$\therefore$ We can arrange the digits

1, 1, 2, 2, 2, 3, 4, 4 is

$(4)\ \frac{3!}{2!}\frac{5!}{3!2!} = 120$

26. Number of games played by men among themselves $= 2({}^nC_2) = n(n-1)$

Number of games played by men with women = 2(2n) = 4n

$\therefore\ n(n-1) - 4n = 66 \Rightarrow n(n-5) = 66$

$\Rightarrow (n-11)\ (n+6) = 0 \Rightarrow n = 11 \in [10, 12)$

27. $n(A \times B) = n(A)\ n(B) = (4)\ (2) = 8$

$\therefore$ number of subset of $A \times B$ which contain at least three elements

$= {}^8C_3 + {}^8C_4 + {}^8C_5 + {}^8C_6 + {}^8C_7$

$= 2^8 - ({}^8C_0 + {}^8C_1 + {}^8C_2)$

= 256 – (1 + 8 + 28) = 219

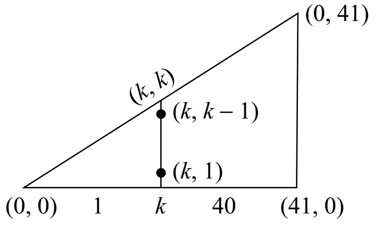
28. Any four digit number beginning with 6, 7 or 8 is greater than 6000. Also, any five digit number is greater than 6000. Thus, required number of numbers is

$({}^3P_1)\ ({}^4P_3) + {}^5P_5 = 192$

29. For $1 \le k \le 40$, the number of integral points on the line $x = k$ and lying in the interior of triangles is $k - 1$.

    Thus, required number of points = $\sum_{k=1}^{40}(k-1)$

    $\frac{1}{2}$ (39) (40) = 780



30. If we allow same man (woman) to be in two distinct teams (for instance if (M_1, W_1) and (M_1, W_2) are two distinct teams), then the answer is $^{225}C_{15}$.

    If we do not allow same man (woman) to be in two distinct teams, then first team can be chosen in (15) (15) ways, second team can be chosen in (14) (14) ways and so on.

    As all the teams 15 are to be chosen, the number of ways

    $= (15)^2\ (14)^2\ (13)^2 \ldots (2^2)\ (1^2)$

    $= (15!)^2$

    It seems that answer 1240 is arrived by the following way

    $15^2 + 14^2 + \ldots + 2^2 + 1^2$

    $= \frac{1}{6}\ (15)\ (16)\ (2 \times 15 + 1) = 120$

    This is an incorrect way as we have to select all the 15 teams.

31. Let number of sides of the regular polygon be n. The number of diagonals of the polygon

    $= {^nC_2} - n = \frac{1}{2}n(n-1) - n = \frac{1}{2}n(n-3)$

    Thus, $\frac{1}{2}n(n-3) = 54$

    $\Rightarrow n^2 - 3n - 108 = 0$

    $\Rightarrow (n-12)(n+9) = 0$

    As $n \ge 3$, $n = 12$.

32. Number of 5 letter words that can be formed by using letters of word SMALL is $\frac{5!}{2!} = 60$.

    Last word (60th word) in the list is SMLLA, 59th word is SMLAL, and 58th word is SMALL.

    Alternate solution

    Number of words beginning with

    (i) A is $\frac{4!}{2!} = 12$ (Ranked 1 to 12)
    (ii) L is $4! = 24$ (Ranked 13 to 36)
    (iii) M is $\frac{4!}{2!} = 12$ (Ranked 37 to 48)
    (iv) SA is $\frac{3!}{2!} = 3$ (Ranked 49 to 51)
    (v) SL is $3! = 6$ (Ranked 52 to 57)
    (vi) SM (ALL) is 1 (Ranked 58th)

33. $\frac{^{15}C_r}{^{15}C_{r-1}} = \frac{15!}{r!(15-r)!}\cdot\frac{(r-1)!(16-r)!}{15!}$

    $= \frac{15-r}{r}$

    $\therefore\ a_r = r^2\left(\frac{^{15}C_r}{^{15}C_{r-1}}\right) = r(15-r)$

    $\Rightarrow \sum_{r=1}^{15} a_r = \sum_{r=1}^{15} r(16-r)$

    $= 16\left\{\frac{1}{2}(15)(16)\right\} - \frac{1}{6}\ (15)(16)(31)$

    $= \frac{1}{2}(15)(16)\left[16 - \frac{31}{3}\right]$

    $= \frac{1}{2}(15)(16)\left(\frac{17}{3}\right) = 680$

34. As $r^2 + 1 = (r+2)(r+1) - 3(r+1) + 2$, we get

    $(r^2 + 1)r! = (r+2)! - 3(r+1)! + 2(r!)$

    $= [(r+2)! - (r+1)!] - 2[(r+1)! - r!]$

    $\Rightarrow \sum_{r=1}^{10}(r^2+1)r! = (12! - 2!) - 2(11! - 1!)$

    $= 11!(12 - 2) = (10)(11!)$

35. Letters in the word M E D I T E R R A N E A N are A, A, D, E, E, E, I, M, N, N, R, R, T. Two middle letters must be chosen from

    A, A, D, E, E, I, M, N, N, R, T. If the chosen letters are distinct, then these can be arranged in $^8P_2 = 56$ ways.

    If the chosen letters are indentical, then these can be chosen and arranged in $^3C_2 = 3$ ways.

    Thus, total number of such letters is 59.

36. $\dfrac{(n+2)!}{6!(n-4)!}\cdot\dfrac{(n-4)!}{(n-2)!} = 11$

$\Rightarrow \dfrac{(n+2)(n+1)(n)(n-1)}{(6)(5)(4)(3)(2)} = 11$

$\Rightarrow (n + 2)\ (n + 1)\ (n)\ (n - 1) = (11)\ (10)\ (9)\ (8)$

$\Rightarrow n + 2 = 11 \Rightarrow n = 9$

$\Rightarrow n^2 + 3n - 108 = 0$

Previous Years' B-Architecture Entrance Examination Questions

1. The number of required subsets is

$N = {}^{2007}C_0 + {}^{2007}C_1 + {}^{2007}C_2 + \dots + {}^{2007}C_{1003}$

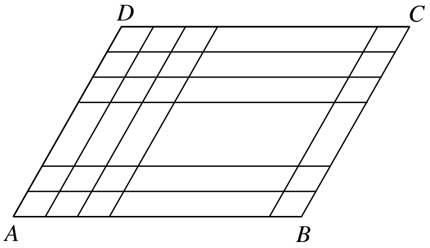
$N = {}^{2007}C_{2007} + {}^{2007}C_{2006} + \dots + {}^{2007}C_{1004}$

Adding above two equations we get

$2N = {}^{2007}C_0 + {}^{2007}C_1 + \dots + {}^{2007}C_{2007}$

$\Rightarrow N = 2^{2006}$

2. There are $(n + 2)$ line segments from A to B and $(n + 2)$ line segments from A to D.



To get a parallelogram, choose two line segments from A to B and two line segments from A to D.

$\therefore$ number of required ways

$= ({}^{n+2}C_2)\ ({}^{n+2}C_2)$

3. Suppose the set contains n elements.

Then number of onto mappings

$= 2^n - 1 - 1 = 2^n - 2$

Thus, $2^n - 2 = 30 \Rightarrow n = 5$.

4. Required number of subsets

$= {}^{10}C_4 + {}^{10}C_5 + \dots + {}^{10}C_{10}$

$= 2^{10} - ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3)$

$= 1024 - (1 + 10 + 45 + 120) = 848$

5. There are following ways for the photograph.

$M_1\ W_1\ M_2\ W_2\ \dots\ M_n\ W_n$

or

$W_1\ M_1\ W_2\ M_2\ \dots\ W_n\ M_n$

This can be done in $(2)\ (n!)\ (n!) = 2(n!)^2$ ways.

6. Number of points of intersections

$= ({}^8C_2) + 2({}^4C_2) + 2({}^8C_1)\ ({}^4C_1)$

$= 104$

7. Required number of ways

$= ({}^6C_1 + {}^6C_2 + \dots + {}^6C_6)\ ({}^6C_1 + {}^6C_2 + \dots + {}^6C_6) - ({}^6C_1)\ ({}^6C_1)$

$= (2^6 - 1)\ (2^6 - 1) - 36 = 2^{12} - 2^7 - 35$

8. If A and B are two finite sets, and $f : A \to B$ is an injective mapping, then $n(A) \le n(B)$.

As $m > n$, there does not exist any mapping from set with m elements to a set with n elements.

9. ${}^{10}C_{x-1} > 2({}^{10}C_x)$

$\Rightarrow \dfrac{10!}{(x-1)!(11-x)!} > 2\dfrac{10!}{x!(10-x)!}$

$\Rightarrow x > 2(11 - x) \Rightarrow 3x > 22$

$\Rightarrow x > 7 \Rightarrow x \ge 8.$

Thus, least positive integral value of x is 8.

10. Number of ways of arranging beds

$= {}^6P_6 = 6! = 720$

Number of ways in which Madhu and Puja have their beds together is $(2)\ ({}^5P_5) = 2(5!) = 240$.

Thus, required number of ways

$= 720 - 240 = 480.$

11. Two cases arise:

Case 1: Both the drivers are occupying the car. In this case we have to choose 3 persons out of 5 for the remaining seats and car can be driven by any of the two drivers. Thus, in this case car can be occupied and driven in $({}^5C_3)\ ({}^2C_1)\ (4!) = 480$ ways.

Case 2: Exactly one drive is occupying the car. In this case we have to choose 4 persons out of the remaining 5 people. In this case car can be occupied and driven in $({}^5C_4)\ ({}^2C_1)\ (4!) = 240$ ways.

Thus, the required number of ways

$= 480 + 240 = 720$

12. ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_n$

$\therefore \dfrac{{}^nC_{r-1}}{{}^nC_{r-1} + {}^nC_r} = \dfrac{n!}{(r-1)!(n-r+1)!}\cdot\dfrac{r!(n+1-r)!}{(n+1)!}$

$$= \frac{r}{n+1}$$

$$\Rightarrow S = \sum_{r=1}^{n}\left(\frac{r}{n+1}\right)^3 = \frac{1}{(n+1)^3}\cdot\frac{n^2(n+1)^2}{4}$$

$$= \frac{n^2}{4(n+1)} = \frac{36}{13}$$

$\Rightarrow 13n^2 = 144(n+1) \Rightarrow n = 12, -12/13$

Thus, $n = 12$

13. Number of code words ending with even digit

= (21) (20) (9) (4)

$\therefore$ $432k$ = (21) (20) (9) (4)

$\Rightarrow k = 35$.

CHAPTER SEVEN

Mathematical Induction and Binomial Theorem

PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n. To prove that $P(n)$ is true for all natural numbers n, we proceed as follows:

1. Prove that $P(1)$ is true.
2. Assume that $P(k)$ is true.
3. Using steps 1 and 2, prove that $P(k + 1)$ is true.

Note

Sometimes step 2 is replaced by step 2′.

2′. Assume that $P(n)$ is true for all $n \in \mathbf{N}$, where $1 \le n \le k$.

The *First Principle of Mathematical Induction* comprises steps 1, 2 and 3. Steps 1, 2′ and 3 constitute another principle of mathematical induction, usually called the *Second Principle of Mathematical Induction*.

BINOMIAL THEOREM (FOR A POSITIVE INTEGRAL INDEX)

If n is a positive integer and x, y are two complex numbers, then

$$(x+y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \dots + {}^nC_rx^{n-r}y^r + \dots + {}^nC_{n-1}xy^{n-1} + {}^nC_ny^n \qquad (1)$$

The coefficients ${}^nC_0$, ${}^nC_1$, $\cdots$, ${}^nC_n$ are called *binomial coefficients*.

PROPERTIES OF THE BINOMIAL EXPANSION

1. There are $(n + 1)$ terms in the expansion (1).
2. In any term of (1), the sum of the exponents of x and y is always n.
3. Binomial coefficients of terms equidistant from the beginning and the end are equal, since ${}^nC_r = {}^nC_{n-r}$ $(0 \le r \le n)$.
4. The term ${}^nC_rx^{n-r}y^r$ is the $(r + 1)$th term from the beginning of the expansion. It is usually denoted by T_{r+1} and is called the *general term* of the expansion.

MIDDLE TERM(S)

1. ***n* is even**

If n is *even*, then the expansion (1) has just one middle term, *viz.* $\left(\frac{n}{2}+1\right)$ th term. It is given by

$${}^nC_{n/2}\,x^{n/2}\,y^{n/2}.$$

2. ***n* is odd**

If n is *odd*, then the expansion (1) has two middle terms, *viz.* $\left(\frac{n+1}{2}\right)$ th term and $\left(\frac{n+1}{2}+1\right)$ th term. These are given by

$${}^nC_{(n-1)/2}\;x^{(n-1)/2}\;y^{(n+1)/2}$$

and $${}^nC_{(n+1)/2}\;x^{(n+1)/2}\;y^{(n-1)/2}$$

Illustration 1

Find the middle term of $\left(\frac{3}{2}x-\frac{2}{5}y\right)^{10}$

As $n = 10$ is even, middle term of $\left(\frac{3}{2}x-\frac{2}{5}y\right)^{10}$ is $\left(\frac{10}{2}+1\right)$th or 6th term. It is given by

$$T_6 = T_{5+1} = {}^{10}C_5\left(\frac{3}{2}x\right)^{10-5}\left(-\frac{2}{5}y\right)^5$$

$$= -252\left(\frac{3}{5}\right)^5 x^5y^5$$

Illustration 2

Find the middle terms of $\left(\frac{1}{2}x+\frac{2}{3}\sqrt{x}\right)^{11}$

As $n = 11$, is odd, the binomial expansion of $\left(\frac{1}{2}x+\frac{2}{3}\sqrt{x}\right)^{11}$ has two middle terms viz. $\left(\frac{11+1}{2}\right)$th and $\left(\frac{11+1}{2}+1\right)$ th terms, that is, 6th and 7th terms.

$$T_6 = T_{5+1} = {}^{11}C_5\left(\frac{1}{2}x\right)^{11-5}\left(\frac{2}{3}\sqrt{x}\right)^5$$

$$= \frac{11!}{5!6!}\cdot\frac{1}{2^6}\cdot\frac{2^5}{3^5}x^{6+5/2}$$

$$= \frac{77}{81}x^{17/2}$$

and $T_7 = T_{6+1} = {}^{11}C_6\left(\frac{1}{2}x\right)^{11-6}\left(\frac{2}{3}\sqrt{x}\right)^6$

$$= \frac{11!}{5!6!}\cdot\frac{1}{2^5}\cdot\frac{2^6}{3^6}x^{5+3} = \frac{308}{243}x^8$$

The Greatest Coefficient

If n is even, the greatest coefficient in the expansion of $(x + y)^n$ is ${}^nC_{n/2}$.

If n is *odd*, there are two greatest coefficients in the expansion of $(x + y)^n$. These are ${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$.

The Greatest Term

To find the greatest term in the expansion of $(x + y)^n$, $x, y > 0$, $n \in \boldsymbol{N}$ put

$$\alpha = \frac{(n+1)\,y}{x+y}$$

If α is not an integer, then the expansion of $(x + y)^n$ has just one greatest term and it is given by $k = [\alpha]$ where $[\alpha]$ denotes the greatest integer $\le \alpha$.

If α is an integer, then expansion of $(x + y)^n$ has two greatest terms *viz.* αth and $(\alpha + 1)$th terms.

TIP

If we are given that t_r is numerically the greatest term, we use

$|t_{r-1}| < |t_r|$

and $|t_{r+1}| < |t_r|$

to obtain two inequalities from which some other results can be obtained.

Illustration 3

Find numerically greatest term in the expansion of $(5 - 3x)^7$ when $x = 2/3$.

We have

$T_{r+1} = {}^7C_r\, 5^{7-r}\,(-3x)^r$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{7!\,(r-1)!\,(8-r)!}{r!\,(7-r)!\,7!}\,\frac{5^{7-r}}{5^{7-r+1}}\,\frac{(-3x)^r}{(-3x)^{r-1}}$$

$$= \frac{8-r}{r}\,\frac{1}{5}\,(-3x)$$

For $x = 2/3$,

$$\left|\frac{T_{r+1}}{T_r}\right| = \frac{(8-r)(2)}{5r}$$

Similarly $\frac{T_{r+2}}{T_{r+1}} = \frac{(7-r)(2)}{5(r+1)}$

For $(r + 1)$th term to be numerically greatest,

$|T_r| < |T_{r+1}|$ and $|T_{r+2}| < |T_{r+1}|$

$\Rightarrow 5r < 16 - 2r$ and $14 - 2r < 5r + 5$

$\Rightarrow 9 < 7r < 16 \Rightarrow r = 2.$

Thus, 3rd term is numerically greatest.

Some Other Useful Expansions

In the following expansions C_r stands for ${}^nC_r$

1. $(x + y)^n + (x - y)^n$
   $= 2\,[C_0\,x^n + C_2\,x^{n-2}\,y^2 + C_4\,x^{n-4}\,y^4 + \ldots]$
2. $(x + y)^n - (x - y)^n$
   $= 2\,[C_1\,x^{n-1}\,y + C_3\,x^{n-3}\,y^3 + C_5\,x^{n-5}\,y^5 + \ldots]$
3. $n\,(n-1)\,(n-2)\ldots(n-k+1)\,(x+y)^{n-k}$
   $= \sum_{r=k}^{n} r(r-1)(r-2)\ldots(r-k+1)C_r\; x^{r-k}y^{n-r}$
4. $(1 + x)^n + (1 - x)^n = 2\,[C_0 + C_2\,x^2 + C_4\,x^4 + \ldots]$
5. $(1 + x)^n - (1 - x)^n = 2\,[C_1\,x + C_3\,x^3 + \ldots]$
6. $C_1 + 2C_2\,x + 3C_3\,x^2 + \ldots + nC_n\,x^{n-1}$
   $= n(1 + x)^{n-1}$
7. $C_0\,x + C_1\frac{x^2}{2} + C_2\frac{x^3}{3} + \ldots + C_n\frac{x^{n+1}}{n+1}$
   $= \int_0^x (1+t)^n\,dt$

SOME PROPERTIES OF THE BINOMIAL COEFFICIENTS

1. $C_0 + C_1 + C_2 + \ldots + C_n = 2^n$
2. $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$
3. $C_0 - C_1 + C_2 - \ldots + (-1)^n\,C_n = 0$
4. For $n > 1$,
   $C_1 - 2C_2 + 3C_3 - \ldots + (-1)^{n-1}\,nC_n = 0$
5. If $P_n = C_0\,C_1\,C_2 \ldots C_n$, then $\frac{P_{n+1}}{P_n} = \frac{(n+1)^n}{n!}$

REVERSING TECHNIQUE

If $a_0, a_1, a_2, \ldots, a_n$ are in A.P., then sum of the series

$$a_0\,C_0 + a_1\,C_1 + \ldots + a_n\,C_n$$

can be obtained by the reversing technique explained as follows.

Let

$$S = a_0\, C_0 + a_1\, C_1 + a_2\, C_2 + \ldots + a_{n-1}\, C_{n-1} + a_n\, C_n \quad (1)$$

Using $C_r = C_{n-r}$ and reversing the order in which terms are written above, we obtain

$$S = a_n\, C_0 + a_{n-1}\, C_1 + a_{n-2}\, C_2 + \ldots + a_1\, C_{n-1} + a_0\, C_n \quad (2)$$

Adding (1) and (2) we get

$2S = (a_0 + a_n)\, C_0 + (a_1 + a_{n-1})\, C_1 + (a_2 + a_{n-2})\, C_2 + \ldots + (a_{n-1} + a_1)\, C_{n-1} + (a_n + a_0)\, C_n$

As $a_0, a_1, a_2, \ldots a_n$ are in A.P., we have

$$a_0 + a_n = a_1 + a_{n-1} = a_2 + a_{n-2} = \ldots,$$

so that $2S = (a_0 + a_n)\,(C_0 + C_1 + C_2 + \ldots + C_n)$

$= (a_0 + a_n)2^n.$

$\Rightarrow \quad S = \dfrac{1}{2}\,(a_0 + a_n)2^n = (a_0 + a_n)2^{n-1}.$

e.g. (i) $C_0 + 2C_1 + 3C_2 + \ldots + (n + 1)C_n$

$= (1 + n + 1)2^{n-1} = (n + 2)2^{n-1}$

(ii) $C_1 + 2C_2 + \ldots + nC_n$

$= 0C_0 + 1C_1 + 2C_2 + \ldots + nC_n$

$= (0 + n)2^{n-1} = n(2^{n-1})$

MULTINOMIAL THEOREM

If we wish to expand $(x + y + z)^n$ (where n is a positive integer and x, y, z are complex numbers), we may take $y + z = a$, expand the binomial $(x + a)^n$, and then expand $(y + z)^r$ in each term of the expansion of $(x + a)^n$. In general, we have the following result:

$$(a_1 + a_2 + \ldots + a_m)^n = \sum \frac{n!}{n_1!\, n_2! \cdots n_m!}\, a_1^{n_1}\, a_2^{n_2} \cdots a_m^{n_m} \quad (1)$$

where the summation is taken over all non-negative integers $n_1, n_2, \ldots, n_m$ such that $n_1 + n_2 + \cdots + n_m = n$.

The number of distinct terms in the expansion (1) is ${}^{n+m-1}C_{m-1}$.

SOME USEFUL TIPS AND TRICKS

1. To find remainder when x^n is divided by y, try to express x or some power of x as $ky \pm 1$. For instance, to find the remainder when 7^{200} is divided by 50, we write $7^2 = 50 - 1$ and then use binomial theorem.
2. If $a, b, r \in \mathbf{Q}$ and $\sqrt{r}$ is an irrational number, then sometime it is useful to switch from $a + b\sqrt{r}$ to $a - b\sqrt{r}$.
3. If $p, q \in \mathbf{Q}$ and $\sqrt{p}$, $\sqrt{q}$ are irrational, then often it is preferable to simplify $(\sqrt{p} + \sqrt{q})^{2n}$ by first using $(\sqrt{p} + \sqrt{q})^2 = p + q + 2\sqrt{pq}$.
4. If three consecutive binomial coefficients ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ are in A.P., then $r = \dfrac{1}{2}(n + \sqrt{n+2})$
5. Four consecutive binomial coefficients can never be in A.P.
6. Three consecutive binomial coefficients can never be in G.P. or H.P.

SOME PARTICULAR EXPANSIONS

For $|x| < 1$

1. $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots$
2. $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots$
3. $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots$
4. $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$
5. $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \ldots$
6. $(1 - x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \ldots$
7. $(1 - x)^{-n} = 1 + {}^nC_1\, x + {}^{n+1}C_2\, x^2 + {}^{n+2}C_3\, x^3 + \ldots$

where if n is a natural number.

Expansion when *x* is very small

If x is very small so that x^2 and higher powers of x can be neglected, then

$$(1 + x)^n = 1 + nx.$$

If x is very small so that x^3 and higher powers of x can be neglected, then

$$(1 + x)^n = 1 + nx + \frac{1}{2}\, n\,(n - 1)\, x^2$$

SOLVED EXAMPLES

Concept-based Straight Objective Type Questions

Example 1: Suppose $a \in \mathbf{R}$. If the coefficient of x^5 in the expansion of $\left(ax + \dfrac{1}{x^3}\right)^{17}$ is 680, then a is equal to

(a) ± 2 (b) ± 1

(c) ± 1/2 (d) ± 1/3

Ans. (b)

◎ **Solution:** $T_{r+1} = {}^{17}C_r\,(ax)^{17-r}\left(\frac{1}{x^3}\right)^r$

$$= {}^{17}C_r\,a^{17-r}\,x^{17-4r}$$

For the coefficient of x^5, we set

$17 - 4r = 5$ or $r = 3$.

∴ ${}^{17}C_3\,a^{14} = 680$

⇒ $a^{14} = \dfrac{680 \times 3!14!}{17!} = 1$

⇒ $a = \pm 1$.

◉ **Example 2:** The absolute term (that is, term independent of x) in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is:

(a) $\frac{1}{9}$ (b) $\frac{2}{17}$

(c) $\frac{7}{18}$ (d) $\frac{11}{18}$

Ans. (c)

◎ **Solution:** $T_{r+1} = {}^{9}C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$

$$= {}^{9}C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-3r}$$

For term independent of x, we set $18 - 3r = 0$ or $r = 6$. Thus, term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is

$${}^{9}C_6\left(\frac{3}{2}\right)^{9-6}\left(-\frac{1}{3}\right)^6 = \frac{7}{18}$$

◉ **Example 3:** Let a_r denote the coefficient of y^{r-1} in the expansion of $(1 + 2y)^{10}$. If $\dfrac{a_{r+2}}{a_r} = 4$, then r is equal to

(a) 2 (b) 3

(c) 4 (d) 5

Ans. (c)

◎ **Solution:** $T_{r+1} = {}^{10}C_r\,(2y)^r = ({}^{10}C_r\,2^r)\,y^r$.

∴ $a_r = {}^{10}C_r\,2^r$

Now

$$4 = \frac{a_{r+2}}{a_r} = \frac{{}^{10}C_{r+2}}{{}^{10}C_r}\,\frac{2^{r+2}}{2^r}$$

⇒ ${}^{10}C_{r+2} = {}^{10}C_r \Rightarrow r + 2 = 10 - r \Rightarrow r = 4$

◉ **Example 4:** Suppose $k \in \mathbf{R}$. Let a be the coefficient of the middle term in the expansion of $\left(\frac{k}{x} + \frac{x}{k}\right)^{10}$, and b be the term independent of x in the expansion of $\left(\frac{k^2}{x} + \frac{x}{k}\right)^{10}$. If $\frac{a}{b} = 1$, then k is equal to

(a) 1 (b) 2

(c) –3 (d) any non-zero number

Ans. (a)

◎ **Solution:** Middle term of $\left(\frac{k}{x} + \frac{x}{k}\right)^{10}$ is $\left(\frac{10}{2} + 1\right)$th = 6th term. We have

$$T_6 = {}^{10}C_5\left(\frac{k}{x}\right)^{10-5}\left(\frac{x}{k}\right)^5 = {}^{10}C_5$$

∴ $a = {}^{10}C_5$

Also, $(r + 1)$th term of $\left(\frac{k^2}{x} + \frac{x}{k}\right)^{10}$ is

$$T_{r+1} = {}^{10}C_r\left(\frac{k^2}{x}\right)^{10-r}\left(\frac{x}{k}\right)^r$$

$$= {}^{10}C_r\,k^{20-3r}\,x^{2r-10}$$

For term independent of x, set $2r - 10 = 0$ or $r = 5$. Thus, term independent of x in the expansion of $\left(\frac{k^2}{x} + \frac{x}{k}\right)^{10}$ is

$${}^{10}C_5\,k^5 = b.$$

As $\dfrac{a}{b} = 1$, we get $k^5 = 1$ or $k = 1$

◉ **Example 5:** The remainder when $6^n - 5n$ is divided by 25, is

(a) 1 (b) 24

(c) 0 (d) n

Ans. (a)

◎ **Solution:** $6^n - 5n = (1 + 5)^n - 5n$

$= 1 + 5n + {}^nC_2(5^2) + {}^nC_3(5^3) + \ldots + {}^nC_n(5^n) - 5n$

$= 1 + 25m$

where $m = {}^nC_2 + {}^nC_3(5) + \ldots + {}^nC_n(5^{n-2})$

is a whole number.

Thus, remainder when $6^n - 5n$ is divided by 25 is 1.

◉ **Example 6:** If coefficients of r th and $(r + 1)$th term in the expansion of $(3 + 2x)^{74}$ are equal, then r is equal to:

(a) 28 (b) 29

(c) 30 (d) 31

Ans. (c)

◎ **Solution:** $T_{r+1} = {}^{74}C_r\,3^{74-r}\,(2x)^r$

⇒ coefficient of $(r + 1)$th term is ${}^{74}C_r\,3^{74-r}\,2^r$

We are given

$${}^{74}C_{r-1}\,3^{74-r+1}\,2^{r-1} = {}^{74}C_r\,3^{74-r}\,2^r$$

⇒ $\dfrac{3}{2} = \dfrac{74!}{r!\,(74-r)!}\cdot\dfrac{(r-1)!\,(75-r)!}{74!}$

$= \dfrac{75 - r}{r}$

⇒ $3r = 150 - 2r \Rightarrow r = 30$

Example 7: If the coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 42, then value of n is

(a) 45 (b) 55
(c) 65 (d) 75

Ans. (b)

Solution: Let three consecutive terms be $(r + 1)$th, $(r + 2)$th and $(r + 3)$th. Then

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{1}{7} \Rightarrow \frac{r+1}{n-r} = \frac{1}{7} \Rightarrow n - 8r = 7 \qquad (1)$$

and $$\frac{{}^nC_{r+1}}{{}^nC_{r+2}} = \frac{7}{42} = \frac{1}{6} \Rightarrow \frac{r+2}{n-r-1} = \frac{1}{6}$$

$$\Rightarrow \quad n - 7r = 13 \qquad (2)$$

From (1) and (2), $n = 55$.

Example 8: The coefficient of x^9 in the expansion of $E = (1 + x)^9 + (1 + x)^{10} + \ldots + (1 + x)^{100}$ is

(a) ${}^{101}C_9$ (b) ${}^{101}C_{10}$
(c) ${}^{100}C_9$ (d) ${}^{100}C_{10}$

Ans. (b)

Solution: $$E = \frac{(1+x)^9\left[(1+x)^{92} - 1\right]}{1 + x - 1}$$

$$= \frac{(1+x)^{101} - (1+x)^9}{x}$$

Coefficient of x^9 in E

$= \text{coefficient of } x^{10} \text{ in } [(1 + x)^{101} - (1 + x)^9]$

$= {}^{101}C_{10}$

Example 9: If the 7th term in the binomial expansion of $\left(\frac{3}{84^{1/3}} + \sqrt{3}\ln x\right)^9$, $(x > 0)$ is equal to 729, then x is equal to

(a) e, $1/e$ (b) $2e$, $1/2e$
(c) e^2, e^{-2} (d) $1/e$

Ans. (a)

Solution: $$T_7 = {}^9C_6\left(\frac{3}{84^{1/3}}\right)^{9-6}\left(\sqrt{3}\ln x\right)^6$$

$$\Rightarrow \quad \frac{9 \times 8 \times 7}{3 \times 2} \cdot \frac{3^3}{84}\, 3^3(\ln x)^6 = 729$$

$$\Rightarrow \quad (\ln x)^6 = 1$$

$$\Rightarrow \quad \ln x = \pm 1 \Rightarrow x = e,\ 1/e$$

Example 9: Sum of the series

$$S = \sum_{j=0}^{8} \frac{1}{(j+1)(j+2)}\left({}^8C_j\right) \text{ is}$$

(a) $\frac{1003}{90}$ (b) $\frac{1013}{90}$
(c) $\frac{1023}{90}$ (d) $\frac{1033}{90}$

Ans. (b)

Solution: For $0 \le j \le 8$,

$$\frac{1}{(j+1)(j+2)}\left({}^8C_j\right)$$

$$= \frac{8!}{(j+1)(j+2)\, j!\,(8-j)!}$$

$$= \frac{1}{10 \times 9}\,\frac{10!}{(j+2)!\,(10-(j+2))!}$$

$$= \frac{1}{90}\left({}^{10}C_{j+2}\right)$$

Thus,

$$S = \frac{1}{90}\left(\sum_{j=0}^{10} {}^{10}C_{j+2}\right)$$

$$= \frac{1}{90}\left(\sum_{k=0}^{10} {}^{10}C_k - {}^{10}C_0 - {}^{10}C_1\right)$$

$$= \frac{1}{90}\ (2^{10} - 1 - 10) = \frac{1013}{90}$$

Example 10: If sum of the coefficients in the expansion of $(x + y)^n$ is 2048, then the greatest coefficient in the expansion is:

(a) ${}^{10}C_6$ (b) ${}^{11}C_6$
(c) ${}^{10}C_7$ (d) ${}^{12}C_6$

Ans. (b)

Solution: Sum of the coefficients in the expansion of $(x + y)^n$ is 2^n. Therefore,

$$2^n = 2048 = 2^{11} \Rightarrow n = 11.$$

As n is odd, the greatest value of ${}^nC_r$ is ${}^nC_k$ where

$k = \frac{1}{2}(n - 1)$ or $k = \frac{1}{2}(n + 1)$.

Here $n = 11$, therefore $k = 5$ or 6.

Thus, greatest coefficient is ${}^{11}C_6$.

LEVEL 1

Straight Objective Type Questions

Example 11: 5th term from the end in the expansion of $\left(\frac{x^2}{3}-\frac{3}{x^2}\right)^8$ is:

is:

(a) $25x^2$

(b) –70

(c) 70

(d) $-25x^2$

Ans. (c)

Solution:

TIP

As number of terms in the binomial expansion of $(x + a)^n$ is $(n + 1)$, the rth term from the end is $((n + 1) + (-1)(r - 1))$th $= (n + 2 - r)$th term from the beginning.

Here $n = 8$, $r = 5$, thus, 5th term from the end is $(8 + 2 - 5)$th = 5th term from the beginning, and it is given by

$$t_5 = {}^8C_4\left(\frac{x^2}{3}\right)^4\left(-\frac{3}{x^2}\right)^{8-4} = {}^8C_4 = 70$$

Example 12: If

$$P(x)=\frac{1}{\sqrt{3x+1}}\left[\left(\frac{1+\sqrt{3x+1}}{5}\right)^n-\left(\frac{1-\sqrt{3x+1}}{5}\right)^n\right]$$

is a 5th degree polynomial, then value of n is

(a) 9

(b) 11

(c) 23

(d) 21

Ans. (b)

Solution: $P(x) = \frac{1}{5^n\, a}[(1 + a)^n - (1 - a)^n]$

where $a = \sqrt{3x+1}$

Now,

$$P(x) = \frac{2}{5^n a}\left[{}^nC_1\, a + {}^nC_3\, a^3 + \ldots\right]$$

Note that n must be odd, so that $\sqrt{3x+1}$ does not appear in $P(x)$.

$$\therefore \quad P(x) = \frac{2}{5^n}\left[{}^nC_1 + {}^nC_3\, a^2 + \ldots + {}^nC_n\, a^{n-1}\right]$$

$$= \frac{2}{5^n}\left[{}^nC_1 + {}^nC_3\,(3x + 1) + {}^nC_5\,(3x + 1)^2 + \ldots + {}^nC_n\,(3x + 1)^{(n-1)/2}\right]$$

For this to be a polynomial of degree 5, $\frac{n-1}{2} = 5 \Rightarrow n = 11$.

Example 13: The two consecutive terms whose coefficients in the expansion of $(3 + 2x)^{94}$ are equal, are

(a) 37th, 38th

(b) 38th, 39th

(c) 47th, 48th

(d) 48th, 49th

Ans. (b)

Solution: Suppose coefficients of $(r + 1)$th and $(r + 2)$th terms in the expansion of $(3 + 2x)^{94}$ are equal. Then

$${}^{94}C_r\, 3^{94-r}\, 2^r = {}^{94}C_{r+1}\, 3^{94-(r+1)}\, 2^{r+1}$$

$$\Rightarrow \quad \frac{94!}{r!\,(94-r)!}\cdot\frac{(r+1)!\,(94-r-1)!}{94!} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{r+1}{94-r} = \frac{2}{3} \Rightarrow 3r + 3 = 188 - 2r$$

$$\Rightarrow \quad 5r = 185 \Rightarrow r = 37$$

Example 14: Coefficient of the constant term in the expansion of

$$E = (x^{2/3} + 4x^{1/3} + 4)^5\left(\frac{1}{x^{1/3}-1}+\frac{1}{x^{2/3}+x^{1/3}+1}\right)^{-9}$$

is

(a) 74

(b) 98

(c) 148

(d) 168

Ans. (d)

Solution: $E=((x^{1/3}+2)^2)^5\left(\frac{x^{2/3}+x^{1/3}+1+x^{1/3}-1}{x-1}\right)^{-9}$

$$= (2 + x^{1/3})^{10}\left(\frac{x^{2/3}+2x^{1/3}}{x-1}\right)^{-9}$$

$$= (2 + x^{1/3})^{10}\,\frac{(x-1)^9}{(x^{1/3})^9\,(x^{1/3}+2)^9}$$

$$= (2 + x^{1/3})x^{-3}\,(x - 1)^9$$

Constant term in E

$= 2$[constant term in $x^{-3}\,(x - 1)^9$]

$= 2(-1)^9$ [coefficient of x^3 in $(1 - x)^9$]

$= 2(-1)\,{}^9C_3\,(-1) = 168$

Example 15: Coefficient of x^4 in the expansion of $\left(1+3x+\frac{2}{x^2}\right)^6$ is:

(a) 240

(b) 735

(c) 820

(d) 936

Ans. (b)

Solution: The general term in the expansion of $\left(1+3x+\frac{2}{x^2}\right)^6$ is

$$\frac{6!}{a!\,b!\,c!}\,1^a\,(3x)^b\left(\frac{2}{x^2}\right)^c$$

where $a, b, c \ge 0$ and $a + b + c = 6$

$$= \frac{6!\,3^b\,2^{6-a-b}}{a!\,b!\,(6-a-b)!}\,x^{b+2a+2b-12}$$

for coefficient of x^4, set $2a + 3b - 12 = 4$

Now $2a + 3b = 16$ and $a + b + c = 6 \Rightarrow (a, b, c) = (2, 4, 0)$

Thus, required coefficient is

$$\frac{6!\,(3^4)\,(2^0)}{2!\,4!} = 735$$

◉ **Example 16:** Coefficient of x^{17} in the polynomial $P(x)$ $= \prod_{r=0}^{17}\left(x + {}^{35}C_r\right)$ is

(a) 2^{34} (b) ${}^{36}C_{17}$

(c) $2^{35} - {}^{36}C_{17}$ (d) 0

Ans. (a)

◎ **Solution:** $P(x)$ is a polynomial of degree 18, and coefficient of x^{17} is

$$N = {}^{35}C_0 + {}^{35}C_1 + {}^{35}C_2 + \ldots + {}^{35}C_{17} \qquad (1)$$

Using ${}^nC_r = {}^nC_{n-r}$, we can write N as

$$N = {}^{35}C_{35} + {}^{35}C_{34} + \ldots + {}^{35}C_{18} \qquad (2)$$

Adding (1) and (2), we get

$$2N = {}^{35}C_0 + {}^{35}C_1 + \ldots + {}^{35}C_{35} = 2^{35}$$

$$\Rightarrow \qquad N = 2^{34}$$

◉ **Example 17:** The coefficient of x^{20} in the expansion of

$$\left(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{20!}x^{20}\right)^3$$

is:

(a) 0 (b) $\frac{3}{20!}$

(c) $\frac{3^{20}}{20!}$ (d) ${}^{60}C_{20}$

Ans. (c)

◎ **Solution:** Note that, coefficient of x^{20} in

$$\left(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{20!}x^{20}\right)^3$$

= coefficient of x^{20} in

$$\left(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \ldots + \frac{1}{20!}x^{20} + \frac{1}{21!}x^{21} + \ldots\right)^3$$

= coefficient of x^{20} in $e^{3x} = \frac{3^{20}}{20!}$

◉ **Example 18:** The coefficient of x^7 in the expansion of

$$\left(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5\right)^2$$

is:

(a) $\frac{2}{7!}\left({}^8C_3\right)$ (b) $\frac{2^{10}}{5!}$

(c) $\frac{4}{15}$ (d) $\frac{7}{20!}$

Ans. (a)

◎ **Solution:** Required coefficient is

$$\frac{1}{2!\,5!} + \frac{1}{3!\,4!} + \frac{1}{4!\,3!} + \frac{1}{5!\,2!}$$

$$= \frac{1}{7!}\left({}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5\right)$$

$$= \frac{2}{7!}\left({}^7C_2 + {}^7C_3\right) = \frac{2}{7!}\left({}^8C_3\right)$$

◉ **Example 19:** Sum of the series

$\sum_{k=1}^{\infty}\sum_{r=0}^{k}\frac{2^{2r}}{7^k}\left({}^kC_r\right)$ is:

(a) 1/7 (b) 4/7

(c) 2.5 (d) 5.2

Ans. (c)

◎ **Solution:** $\sum_{r=0}^{k}\frac{2^{2r}}{7^k}\left({}^kC_r\right) = \frac{1}{7^k}\sum_{r=0}^{k}\left({}^kC_r\,4^r\right)$

$$= \frac{1}{7^k}(1 + 4)^k = \left(\frac{5}{7}\right)^k$$

Thus, $\sum_{k=1}^{\infty}\sum_{r=0}^{k}\frac{2^{2r}}{7^k}\left({}^kC_r\right)$

$$= \sum_{k=1}^{\infty}\left(\frac{5}{7}\right)^k = \frac{5/7}{1 - 5/7} = 2.5$$

◉ **Example 20:** If the 6th term from the beginning is equal to the 6th term from the end in the expansion of $\left(2^{1/5} + \frac{1}{3^{1/5}}\right)^n$, then n is equal to:

(a) 7 (b) 9

(c) 10 (d) 12

Ans. (c)

◎ **Solution:** 6th term from the beginning

$$= {}^nC_5\,(2^{1/5})^{n-5}\left(\frac{1}{3^{1/5}}\right)^5 = {}^nC_5\left(\frac{2^{n/5}}{6}\right)$$

Also, 6th term from the end

$= (n + 2 - 6) = (n - 4)$ th from the beginning

$$= {}^nC_{n-5}\,(2^{1/5})^{n-(n-5)}\left(\frac{1}{3^{1/5}}\right)^{n-5} = {}^nC_5\left(\frac{6}{3^{n/5}}\right)$$

We are given

$${}^nC_5\left(\frac{2^{n/5}}{6}\right) = {}^nC_5\left(\frac{6}{3^{n/5}}\right) \Rightarrow 6^{n/5} = 6^2 \Rightarrow n = 10$$

◉ **Example 21:** If the coefficients of x^7 and x^8 in $\left(2+\frac{x}{3}\right)^n$ are equal, then value of n is

(a) 56 (b) 55
(c) 47 (d) 19

Ans. (b)

◎ **Solution:** T_{r+1}, the $(r+1)$th term in the expansion of $\left(2+\frac{x}{3}\right)^n$ is given by

$$T_{r+1} = {}^nC_r\,2^{n-r}\left(\frac{x}{3}\right)^r = {}^nC_r\left(\frac{2^{n-r}}{3^r}\right)x^r$$

According to the given condition

$${}^nC_7\left(\frac{2^{n-7}}{3^7}\right) = {}^nC_8\left(\frac{2^{n-8}}{3^8}\right) \quad\Rightarrow\quad \frac{{}^nC_7}{{}^nC_8} = \frac{2^{n-8}}{3^8}\cdot\frac{3^7}{2^{n-7}}$$

$$\Rightarrow\quad \frac{n!}{7!\,(n-7)!}\,\frac{8!\,(n-8)!}{n!} = \frac{1}{6} \quad\Rightarrow\quad \frac{8}{n-7} = \frac{1}{6}$$

$$\Rightarrow\quad 48 = n-7 \quad\Rightarrow\quad n = 55$$

◉ **Example 22:** In the expansion of $\left(x^3-\frac{1}{x^2}\right)^{15}$, the constant term equals

(a) ${}^{15}C_9$ (b) 0
(c) $-({}^{15}C_9)$ (d) ${}^{15}C_{11}$

Ans. (c)

◎ **Solution:** T_{r+1}, the $(r+1)$th term in the expansion of $\left(x^3-\frac{1}{x^2}\right)^{15}$ is

$$T_{r+1} = {}^{15}C_r\,(x^3)^{15-r}\left(-\frac{1}{x^2}\right)^r = {}^{15}C_r\,(-1)^r\,x^{45-5r}$$

To obtain constant term, we set $45 - 5r = 0 \Rightarrow r = 9$

$\therefore$ Coefficient of constant term is ${}^{15}C_9\,(-1)^9 = -\,{}^{15}C_9$.

◉ **Example 23:** If a is real and the 4th term in the expansion of $\left(ax+\frac{1}{x}\right)^n$ is 5/2, for each $x \in \mathbf{R} - \{0\}$, then values of n and a are respectively

(a) 5, 1/2 (b) 6, – 1/2
(c) 3, 1/3 (d) 6, 1/2

Ans. (d)

◎ **Solution:** We have

$$T_4 = T_{3+1} = {}^nC_3\,(ax)^{n-3}\left(\frac{1}{x}\right)^3$$

$$= {}^nC_3\,a^{n-3}x^{n-6} = 5/2$$

As this is true for each $x \in \mathbf{R} - \{0\}$, we get

$$n - 6 = 0 \text{ and } {}^nC_3\,a^{n-3} = 5/2$$

$$\Rightarrow \quad n = 6 \text{ and } {}^6C_3\,a^3 = 5/2$$

$$\therefore \quad a^3 = \frac{5}{2}\times\frac{3!\,3!}{6!} = \frac{5}{2}\times\frac{1}{20} = \frac{1}{8}$$

$$\Rightarrow \quad a = 1/2.$$

Thus, $n = 6,\ a = 1/2$.

◉ **Example 24:** If the 6th term in the expansion of $\left(\frac{1}{x^{8/3}}+x^2\log_{10}x\right)^8$ is 5600, then value of x is

(a) 2 (b) $\sqrt{5}$
(c) $\sqrt{10}$ (d) 10

Ans. (d)

◎ **Solution:** Note that for $\log_{10}x$ to be defined, $x > 0$.

We have $T_6 = T_{5+1} = {}^8C_5\left(\frac{1}{x^{8/3}}\right)^{8-5}(x^2\log_{10}x)^5$

$$\Rightarrow \quad 5600 = \frac{8!}{5!\,3!}\left(\frac{1}{x^8}\right)x^{10}\,(\log_{10}x)^5$$

$$\Rightarrow \quad 5600 = 56\,x^2\,(\log_{10}x)^5$$

$$\Rightarrow \quad 100 = x^2\,(\log_{10}x)^5$$

$$\Rightarrow \quad 100/x^2 = (\log_{10}x)^5$$

The curves $y = 100/x^2$,
$y = (\log_{10}x)^5$
intersect in just one point.
See Fig. 7.1. This points is (10, 1).
Therefore, $x = 10$.

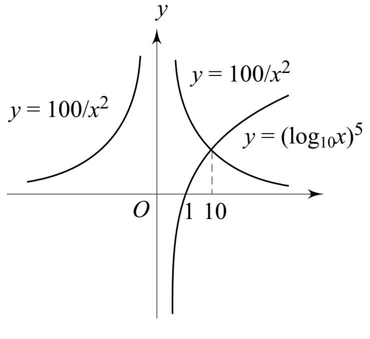


Fig. 7.1

◉ **Example 25:** If the $(r+1)$th term in the expansion of $\left(\frac{a^{1/3}}{b^{1/6}}+\frac{b^{1/2}}{a^{1/6}}\right)^{21}$ has equal exponents of both a and b, then value of r is

(a) 8 (b) 9
(c) 10 (d) 11

Ans. (b)

Solution: We have

$$T_{r+1} = {}^{21}C_r \left(\frac{a^{1/3}}{b^{1/6}}\right)^{21-r} \left(\frac{b^{1/2}}{a^{1/6}}\right)^{r}$$

$$= {}^{21}C_r \frac{a^{7-r/3}}{b^{7/2-r/6}} \cdot \frac{b^{r/2}}{a^{r/6}} = {}^{21}C_r\, a^{7-r/2}\, b^{2r/3-7/2}$$

Since exponents of a and b in the $(r+1)$th term are equal,

$$7-\frac{r}{2} = \frac{2r}{3}-\frac{7}{2} \Rightarrow \frac{21}{2} = \frac{7}{6}r \Rightarrow r = 9.$$

Example 26: If A and B are coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then

(a) $A = B$ (b) $A = 2B$
(c) $2A = B$ (d) none of these

Ans. (b)

Solution: We know that coefficient of x^r in the expansion of $(1+x)^m$ is ${}^mC_r$.

Thus, $A = {}^{2n}C_n$ and $B = {}^{2n-1}C_n$.

We have $$\frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{(2n)!}{n!\,n!}\,\frac{(n!)\,(n-1)!}{(2n-1)!} = \frac{2n}{n} = 2$$

$\Rightarrow$ $A = 2B$

Example 27: If x^{2k} occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$, then

(a) $n-2k$ is a multiple of 2
(b) $n-2k$ is a multiple of 3
(c) $k = 0$
(d) none of these

Ans. (b)

Solution: T_{r+1} the $(r+1)$th term in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$ is given by

$$T_{r+1} = {}^{n-3}C_r\,(x)^{n-3-r}\left(\frac{1}{x^2}\right)^r = {}^{n-3}C_r\, x^{n-3-3r}$$

As x^{2k} occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{n-3}$, we must have $n-3-3r = 2k$ for some non-negative integer r.

$\Rightarrow$ $3(1+r) = n-2k$ $\Rightarrow$ $n-2k$ is a multiple of 3.

Example 28: Coefficient of the term independent of x in the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is

(a) 210 (b) 105
(c) 70 (d) 35

Ans. (a)

Solution: We have

$$x+1 = (x^{1/3})^3+1 = (x^{1/3}+1)\,(x^{2/3}-x^{1/3}+1),$$

$$x-1 = \left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)$$

and $$x-x^{1/2} = \sqrt{x}(\sqrt{x}-1),$$

Thus, $$\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}$$
$$= x^{1/3}+1-(1+x^{-1/2}) = x^{1/3}-x^{-1/2}$$

Now, the $(r+1)$th term in the expansion of $(x^{1/3}-x^{-1/2})^{10}$ is

$$T_{r+1} = {}^{10}C_r\,(x^{1/3})^{10-r}\,(x^{-1/2})^r\,(-1)^r$$
$$= {}^{10}C_r\, x^{10/3-5r/6}\,(-1)^r$$

For this term to be independent of x,

$$\frac{10}{3}-\frac{5r}{6} = 0 \quad\Rightarrow\quad r = \frac{10}{3}\times\frac{6}{5} = 4$$

Thus, coefficient of the term independent of x in the given expression is

$${}^{10}C_4 = \frac{10!}{4!6!} = 210.$$

Example 29: If $(1+x)^n = C_0+C_1x+C_2x^2+\ldots+C_nx^n$, then value of $C_0+2C_1+3C_2+\ldots+(n+1)C_n$ is

(a) 2^{n-1} (b) $n(2^{n-1})$
(c) $n(2^{n-1})+2^n$ (d) $(n+1)2^n$

Ans. (c)

Solution: Let $E = C_0+2C_1+3C_2+\ldots+nC_{n-1}+(n+1)C_n$ (1)

Using $C_r = C_{n-r}$, we can rewrite (1) as

$$E = (n+1)\,C_0+nC_1+(n-1)C_2+\ldots+2C_{n-1}+C_n \qquad (2)$$

Adding (1) and (2), we get

$$2E = (n+2)C_0+(n+2)C_1+(n+2)C_2+\ldots+(n+2)C_n$$
$$= (n+2)\,\{C_0+C_1+\ldots+C_n\} = (n+2)2^n$$

$\Rightarrow$ $E = (n+2)2^{n-1}$

Alternative Solution

We have

$$C_0\,x+C_1\,x^2+C_2\,x^3+\ldots+C_n\,x^{n+1} = x\,(1+x)^n$$

Differentiating both the sides, we get

$$C_0+2C_1\,x+3C_2\,x^2+\ldots+(n+1)C_n\,x^n = (1+x)^n+nx(1+x)^{n-1} \qquad (1)$$

Putting $x = 1$, we get

$$C_0+2C_1+3C_2+\ldots+(n+1)C_n = 2^n+n(1)\,2^{n-1} = (n+2)2^{n-1}$$

Example 30: If $n > 1$, and $(1+x)^n = C_0+C_1\,x+C_2\,x^2+\ldots+C_n\,x^n$ then value of $C_0-2C_1+3C_2-4C_3+\ldots+(-1)^n(n+1)C_n$ is

(a) -1 (b) 0
(c) 1 (d) 2

Ans. (b)

Solution: Putting $x = -1$ in the equation (1) of alternative solution to Example 29, we get

$$C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots + (-1)^n (n+1)C_n = 0 + 0 = 0$$

Example 31: If $n \in \mathbf{N}$, $n > 1$, then value of
$E = a - {}^nC_1 (a-1) + {}^nC_2 (a-2) + \ldots + (-1)^n (a-n)({}^nC_n)$ is

(a) a (b) 0
(c) a^2 (d) 2^n

Ans. (b)

Solution: We can write E as

$$a\,[{}^nC_0 - {}^nC_1 + {}^nC_2 - \ldots + (-1)^n ({}^nC_n)] + [{}^nC_1 - (2)({}^nC_2) + (3)({}^nC_3) - \ldots - (-1)^n (n)({}^nC_n)]$$
$$= 0 + F \text{ where}$$
$$F = {}^nC_1 - (2)({}^nC_2) + (3)({}^nC_3) - \ldots - (-1)^n (n)({}^nC_n)$$

We have $(1+x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + {}^nC_3\,x^3 + \ldots + {}^nC_n\,x^n$

Differentiating, we get

$$n(1+x)^{n-1} = {}^nC_1 + 2({}^nC_2)x + 3({}^nC_3)x^2 + \ldots + n({}^nC_n)x^{n-1}$$

Putting $x = -1$, we get

$$0 = {}^nC_1 - 2({}^nC_2) + 3({}^nC_3) - \ldots - (-1)^{n-1}(n)({}^nC_n)$$

$\Rightarrow$ $0 = F.$

Thus, $E = 0 + 0 = 0.$

Example 32: Suppose ABC is a triangle and n is a natural number, then sum of the series

$$S = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r \cos\,[nA - (n-r)B]$$

is:

(a) $(a+b)^n$ (b) $(a-b)^n$
(c) c^n (d) c^{2n}

Ans. (c)

Solution: $S = \displaystyle\sum_{r=0}^{n} {}^nC_r\; a^{n-r} b^r\, \mathrm{Re}[e^{i(nB-(n-r)A)}]$

$$= \mathrm{Re} \sum_{r=0}^{n} {}^nC_r\, (ae^{-iB})^{n-r} (be^{\,iA})^r$$
$$= \mathrm{Re}\,(ae^{-iB} + be^{iA})^n$$
$$= \mathrm{Re}\,[a(\cos B - i\sin B) + b\,(\cos A + i\sin A)]^n$$
$$= \mathrm{Re}\,[(a\cos B + b\cos A) - i\,(a\sin B - b\sin A)]^n$$

But $a\cos B + b\cos A = c$ [projection formula]

and $a\sin B = b\sin A$ [Law of sines]

$\therefore$ $S = \mathrm{Re}(c^n) = c^n.$

Example 33: The positive integer just greater than $S = (1 + 0.0001)^{10000}$ is

(a) 2 (b) 3
(c) 4 (d) 5

Ans. (b)

Solution: Let $n = 10000$, then

$$S = \left(1 + \frac{1}{n}\right)^n$$
$$= 1 + {}^nC_1\left(\frac{1}{n}\right) + {}^nC_2\left(\frac{1}{n}\right)^2 + \cdots + {}^nC_n\left(\frac{1}{n}\right)^n$$
$$= 1 + 1 + \frac{1}{2!}\frac{n(n-1)}{n^2} + \frac{1}{3!}\frac{n(n-1)(n-2)}{n^3} + \cdots + \frac{1}{n!}\frac{n(n-1)\cdots(n-n+1)}{n^n}$$
$$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$
$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

But $\displaystyle\frac{1}{r!} = \frac{1}{1.2\ldots r} \le \frac{1}{2^{r-1}} \quad \forall\, r \ge 2.$

Thus, $\displaystyle S \le 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$

$$= 1 + \frac{1 - (1/2)^n}{1 - 1/2} = 3 - \left(\frac{1}{2}\right)^{n-1} < 3.$$

Example 34: The coefficient of the middle term in the binomial expansion of $(1 + \alpha x)^4$ and of $(1 - \alpha x)^6$ is the same if α equals

(a) $-\dfrac{3}{10}$ (b) $\dfrac{10}{3}$

(c) $-\dfrac{5}{3}$ (d) $\dfrac{3}{5}$

Ans. (a)

Solution: Middle term in the expansion of $(1 + \alpha x)^4$ is ${}^4C_2(\alpha x)^2 = 6\alpha^2 x^2$ and the middle term in the expansion of $(1 - \alpha x)^6$ is ${}^6C_3(-\alpha x)^3 = -20\,\alpha^3 x^3$.

We are given $6\alpha^2 = -20\,\alpha^3 \Rightarrow \alpha = 0$ or $\alpha = -\dfrac{3}{10}$

Example 35: For natural numbers m and n

$$(1-y)^m (1+y)^n = 1 + a_1 y + a_2 y^2 + \ldots$$

and $a_1 = a_2 = 10$, then (m, n) equals

(a) (35, 45) (b) (20, 45)
(c) (35, 20) (d) (45, 35)

Ans. (a)

◎ **Solution:** $(1-y)^m(1+y)^n = (1 - my + {}^mC_2 y^2 - \ldots)(1 + ny$ $+ {}^nC_2 y^2 + \ldots)$

$$= 1 + (n - m)y + ({}^nC_2 + {}^mC_2 - mn)y^2 + \ldots$$

We are given

$$n - m = a_1 = 10$$

and $\quad {}^nC_2 + {}^mC_2 - mn = a_2 = 10.$

$\Rightarrow \quad n = m + 10$ and

$$\frac{1}{2}n(n-1) + \frac{1}{2}m(m-1) - mn = 10$$

$\Rightarrow \quad (m + 10)(m + 9) + m(m - 1) - 2m(m + 10) = 20$

$\Rightarrow \quad m^2 + 19m + 90 + m^2 - m - 2m^2 - 20m = 20$

$\Rightarrow \quad -2m = -70 \quad \Rightarrow \quad m = 35.$

$\therefore \quad n = 45$

◉ **Example 36:** The sum of the series $S = {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots + {}^{20}C_{10}$ is

(a) $-({}^{20}C_{10})$ (b) $\frac{1}{2}({}^{20}C_{10})$

(c) 0 (d) ${}^{20}C_{10}$

Ans. (b)

◎ **Solution:** Using ${}^nC_r = {}^nC_{n-r}$, we can write

$2S = {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots {}^{20}C_{10} + {}^{20}C_{20} - {}^{20}C_{19} + {}^{20}C_{18} - \ldots + {}^{20}C_{10}$

$\Rightarrow \quad S = 0 + \frac{1}{2}({}^{20}C_{10}) = \frac{1}{2}({}^{20}C_{10})$

◉ **Example 37:** The expansion of

$\left(x + \sqrt{x^3 - 1}\right)^5 + \left(x - \sqrt{x^3 - 1}\right)^5$ is a polynomial of degree

(a) 5 (b) 6

(c) 7 (d) 8

Ans. (c)

◎ **Solution:** Using $(a + b)^5 + (a - b)^5$

$$= 2[a^5 + {}^5C_2\, a^3b^2 + {}^5C_4\, ab^4],$$

we get $\left(x + \sqrt{x^3 - 1}\right)^5 + \left(x - \sqrt{x^3 - 1}\right)^5$

$$= 2[x^5 + 10x^3(x^3 - 1) + 5x(x^3 - 1)^2]$$

which is a polynomial of degree 7.

◉ **Example 38:** The coefficient of x^r in the expansion of

$S = (x + 3)^{n-1} + (x + 3)^{n-2}(x + 2) + (x + 3)^{n-3}(x + 2)^2 + \ldots + (x + 2)^{n-1}$

is

(a) $3^{n-r} - 2^{n-r}$ (b) ${}^nC_r(3^r - 2^r)$

(c) ${}^nC_r(3^{n-r} - 2^{n-r})$ (d) none of these

Ans. (c)

◎ **Solution:** We can write

$$S = \frac{(x+3)^{n-1}\left[1 - \left(\dfrac{x+2}{x+3}\right)^n\right]}{1 - \dfrac{x+2}{x+3}} = (3 + x)^n - (2 + x)^n$$

$\therefore$ Coefficient of $x^r = {}^nC_r(3^{n-r} - 2^{n-r})$

◉ **Example 39:** Number of zeros at then end of $99^{1001} + 1$ is

(a) 2 (b) 4

(c) 1002 (d) 1004

Ans. (a)

◎ **Solution:** Let $n = 1001$. We can write

$$99^{1001} + 1 = (100 - 1)^n + 1$$
$$= {}^nC_0 100^n - {}^nC_1\, 100^{n-1} + {}^nC_2\, 100^{n-2} - \ldots$$
$$= 100\, m + {}^nC_{n-1}(100) - 1 + 1$$

where $\quad m = {}^nC_0 100^{n-1} - {}^nC_1\, 100^{n-2} + {}^nC_2\, 100^{n-3} - \ldots - {}^nC_{n-2}\, 100 + {}^nC_{n-1}$

As ${}^nC_{n-1} = 1001$, the units digit of m is different from 0. Thus, the number of zeros at the end of $99^{1001} + 1$ is two.

◉ **Example 40:** The coefficient of x^k in the expansion of $E = 1 + (1 + x) + (1 + x)^2 + \ldots + (1 + x)^n$ is

(a) ${}^nC_k$ (b) ${}^{n+1}C_k$

(c) ${}^{n+1}C_{k+1}$ (d) none of these

Ans. (c)

◎ **Solution:** We have

$$E = 1 + (1 + x) + (1 + x)^2 + \ldots + (1 + x)^n$$

$$= \frac{1 - (1+x)^{n+1}}{1 - (1+x)} = \frac{1}{x}\left[(1+x)^{n+1} - 1\right]$$

Thus, coefficient of x^k in E

$$= \text{Coefficient of } x^{k+1} \text{ in } [(1 + x)^{n+1} - 1]$$
$$= {}^{n+1}C_{k+1}$$

◉ **Example 41:** The number of irrational terms in the expansion of $(5^{1/6} + 2^{1/8})^{100}$ is

(a) 96 (b) 97

(c) 98 (d) 99

Ans. (b)

◎ **Solution:** T_{r+1}, the $(r + 1)$th in the expansion of $(5^{1/6} + 2^{1/8})^{100}$ is given by

$$T_{r+1} = {}^{100}C_r\,(5^{1/6})^{100-r}\,(2^{1/8})^r$$

As 5 and 2 are relatively prime, T_{r+1} will be rational if $\frac{100 - r}{6}$ and $\frac{r}{8}$ are both integers. i.e. if $100 - r$ is a multiple of 6 and r is a multiple of 8. As $0 \le r \le 100$, multiples of 8 upto 100 and corresponding value of $100 - r$ are

$$r = 0, 8, 16, 24, \ldots, 88, 96$$
$$100 - r = 100, 92, 84, 76, \ldots, 12, 4$$

Out of these values for $100 - r$, multiples of 6 are 84, 60, 36, 12

$\therefore$ there are just four rational terms in the expansion of $\left(5^{1/6}+2^{1/8}\right)^{100}$.

$\Rightarrow$ number of irrational terms is $101 - 4 = 97$

Example 42: The sum of the rational terms of $\left(2^{1/5}+\sqrt{3}\right)^{20}$ is

(a) 71 (b) 85
(c) 97 (d) none of these

Ans. (d)

Solution: T_{r+1}, the $(r + 1)$th term in the expansion of $\left(2^{1/5}+\sqrt{3}\right)^{20}$ is

$$T_{r+1} = {}^{20}C_r\,(2^{1/5})^{20-r}\,(3^{1/2})^r = {}^{20}C_r\,2^{4-r/5}\,(3)^{r/2}$$

As 2 and 3 are relatively prime $2^{4-r/5}$ and $3^{r/2}$ will be rational if both $4 - r/5$ and $r/2$ are integers, i.e. if $r/5$ and $r/2$ are integers. This is possible if

$$r = 0, 10, 20.$$

Thus, the sum of rational terms is

$${}^{20}C_0(2^4)(3^0) + {}^{20}C_{10}(2^2)(3^5) + {}^{20}C_{20}(2^0)(3^{10})$$

As this is more than 71, 85 or 97, answer is (d).

Example 43: If in the expansion of $\left(x^3-\dfrac{1}{x^2}\right)^n$, $n \in \mathbf{N}$, sum of the coefficient of x^5 and x^{10} is 0, then value of n is

(a) 5 (b) 10
(c) 15 (d) 20

Ans. (c)

Solution: T_{r+1}, the $(r + 1)$th term in the expansion of $\left(x^3-\dfrac{1}{x^2}\right)^n$ is

$$T_{r+1} = {}^nC_r(x^3)^{n-r}\left(-\frac{1}{x^2}\right)^r = {}^nC_r(-1)^r\,x^{3n-5r}$$

For the coefficient of x^5, we set $3n - 5r = 5 \Rightarrow r = \dfrac{3n-5}{5} = p$(say) and

for the coefficient of x^{10}, we set $3n - 5r = 10 \Rightarrow r = \dfrac{3n-10}{5} = q$ (say)

Note that $p - q = 1$. We are given

$${}^nC_p(-1)^p + {}^nC_q(-1)^q = 0$$

$$\Rightarrow \quad {}^nC_p(-1)^p + {}^nC_{p-1}(-1)^{p-1} = 0$$

$$\Rightarrow \quad {}^nC_p = {}^nC_{p-1} \Rightarrow n - p = p - 1$$

$$\Rightarrow \quad n = 2p-1 \quad \Rightarrow \quad p = \frac{n+1}{2}$$

$$\text{Thus,} \quad \frac{n+1}{2} = \frac{3n-5}{5} \quad \Rightarrow \quad 5n + 5 = 6n - 10$$

$$\Rightarrow \quad 15 = n$$

Example 44: In the expansion of $(1 + px)^n$, $n \in \mathbf{N}$, the coefficient of x and x^2 are 8 and 24 respectively, then

(a) $n = 3, p = 2$ (b) $n = 4, p = 2$
(c) $n = 4, p = 3$ (d) $n = 5, p = 3$

Ans. (b)

Solution: We have

$$(1 + px)^n = 1 + {}^nC_1(px) + {}^nC_2(px)^2 + \ldots$$
$$= 1 + npx + \frac{1}{2}\,n(n-1)p^2x^2 + \ldots$$

According to the hypothesis,

$$np = 8 \text{ and } \frac{1}{2}\,n(n-1)p^2 = 24$$

Putting $p = 8/n$ in the second expression, we get

$$\frac{1}{2}\,n(n-1)\left(\frac{8}{n}\right)^2 = 24 \Rightarrow \frac{n-1}{n} = \frac{24\times 2}{8\times 8} = \frac{3}{4}$$

$\Rightarrow$ $4n - 4 = 3n$ $\Rightarrow$ $n = 4$

Putting this value in $np = 8$, we get $p = 2$.

Example 45: The greatest value of the term independent of x, as α varies over $\boldsymbol{R}$, in the expansion of $\left(x\cos\alpha+\dfrac{\sin\alpha}{x}\right)^{20}$ is

(a) ${}^{20}C_{10}$ (b) ${}^{20}C_{15}$
(c) ${}^{20}C_{19}$ (d) ${}^{20}C_{10}\left(\dfrac{1}{2}\right)^{10}$

Ans. (d)

Solution: T_{r+1}, the $(r+1)$th term in the expansion of $\left(x\cos\alpha+\dfrac{\sin\alpha}{x}\right)^{20}$ is ${}^{20}C_r\,(x\cos\alpha)^{20-r}\left(\dfrac{\sin\alpha}{x}\right)^r$

$$= {}^{20}C_r\,x^{20-2r}\,(\cos\alpha)^{20-r}\,(\sin\alpha)^r$$

For this term to be independent of x, we set $20 - 2r = 0$

$\Rightarrow$ $r = 10$.

Let β = Term independent of x, then

$$\beta = {}^{20}C_{10}\,(\cos\alpha)^{10}\,(\sin\alpha)^{10}$$
$$= {}^{20}C_{10}\,[\cos\alpha\sin\alpha]^{10}$$

Thus, the greatest possible value of β is ${}^{20}C_{10}\left(\dfrac{1}{2}\right)^{10}$.

Example 46: Sum of the last 20 coefficients in the expansion of $(1 + x)^{39}$, when expanded in ascending powers of x, is

(a) 2^{19} (b) 2^{18}
(c) ${}^{40}C_{20}-2^{19}$ (d) 2^{38}

Ans. (d)

Solution: There are 40 terms in the expansion of $(1 + x)^{39}$. Sum of the last 20 coefficients is

$$S = {}^{39}C_{20} + {}^{39}C_{21} + \ldots + {}^{39}C_{38} + {}^{39}C_{39}$$

$$\Rightarrow \qquad S = {}^{39}C_{19} + {}^{39}C_{18} + \ldots + {}^{39}C_1 + {}^{39}C_0$$
$$[\text{using } {}^nC_r = {}^nC_{n-r}]$$

Adding the above two expressions, we get

$$2S = {}^{39}C_0 + {}^{39}C_1 + \ldots + {}^{39}C_{39} = 2^{39}$$
$$\Rightarrow \qquad S = 2^{38}$$

◉ **Example 47:** If $(1 + x)^{15} = C_0 + C_1x + C_2x^2 + \ldots + C_{15}x^{15}$ then value of the expression $S = C_2 + 2C_3 + 3C_4 + \ldots + 14C_{15}$ is

(a) $13(2^{14}) + 1$ (b) $13(2^{15})$
(c) $13(2^{14})$ (d) none of these

Ans. (a)

◎ **Solution:** We add $-1C_0 + 0C_1$ to both the sides of S and let

$$S_1 = S - 1C_0 + 0C_1 = S - 1.$$

Note that

$$S_1 = -1C_0 + 0C_1 + 1C_2 + 2C_3 + \ldots + 13C_{14} + 14C_{15} \qquad (1)$$

Using $C_r = C_{n-r}$, we rewrite the above expression as

$$S_1 = 14C_0 + 13C_1 + 12C_2 + \ldots + 0C_{14} + (-1)C_{15} \qquad (2)$$

Adding (1) and (2), we get

$$2S_1 = 13\,[C_0 + C_1 + C_2 + \ldots + C_{15}] = 13(2^{15})$$
$$\Rightarrow \quad S_1 = 13(2^{14}) \qquad \Rightarrow \qquad S = 13(2^{14}) + 1$$

◉ **Example 48:** Let $R = (2+\sqrt{3})^{2n}$ and $f = R - [R]$ where [ ] denotes the greatest integer function, then $R(1 - f)$ is equal to

(a) 1 (b) 2^{2n}
(c) $2^{2n} - 1$ (d) ${}^{2n}C_n$

Ans. (a)

◎ **Solution:** As $0 < 2 - \sqrt{3} < 1$ we get $0 < F = (2 - \sqrt{3})^{2n} < 1$

We have

$$R + F = (2 + \sqrt{3})^{2n} + (2 - \sqrt{3})^{2n}$$
$$= 2\,[{}^{2n}C_0\,2^{2n} + {}^{2n}C_2\,2^{2n-2}\left(\sqrt{3}\right)^2 +$$
$${}^{2n}C_4\,(2^{2n-4})(\sqrt{3})^4 \ldots + {}^{2n}C_{2n}\left(\sqrt{3}\right)^{2n}]$$

$\Rightarrow$ $R + F$ is an even integer.

$\Rightarrow$ $[R] + f + F$ is an even integer.

$\Rightarrow$ $f + F$ is an integer.

But $0 \le f < 1$ and $0 < F < 1 \quad \Rightarrow \quad 0 < f + F < 2$

But the only integer between 0 and 2 is 1. Thus,

$$f + F = 1 \qquad \Rightarrow \qquad 1 - f = F$$

Now, $\quad R(1 - f) = RF = \left(2+\sqrt{3}\right)^{2n}\left(2-\sqrt{3}\right)^{2n}$

$$= (4 - 3)^{2n} = 1^{2n} = 1$$

◉ **Example 49:** If [ ] denotes the greatest integer function, then $[(\sqrt{2} + 1)^6]$ is equal

(a) 199 (b) 198
(c) 197 (d) 196

Ans. (c)

◎ **Solution:** Let $(\sqrt{2} + 1)^6 = R + f$ where R is an integer and $0 \le f < 1$.

Note that

$$R = [(\sqrt{2} + 1)^6]$$

Also $0 < \sqrt{2} - 1 < 1 \Rightarrow 0 < F = (\sqrt{2} - 1)^6 < 1$.

We have, $R + f + F = \left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$

$$= 2\,[{}^6C_0\,(\sqrt{2})^6 + {}^6C_2\,(\sqrt{2})^4 + {}^6C_4\,(\sqrt{2})^2 + {}^6C_6]$$
$$= 2\,[2^3 + 15(2^2) + 15(2) + 1] = 198$$
$$\Rightarrow \qquad f + F = 198 - R \text{ is an integer.}$$

As $0 \le f < 1$ and $0 < F < 1$, $0 < f + F < 2$.

But the only integer between 0 and 2 is 1. Therefore

$$f + F = 1$$

Thus, $\qquad 1 = 198 - R \qquad \Rightarrow \qquad R = 197$

◉ **Example 50:** Sum of the coefficients of the terms of degree m in the expansion of $(1 + x)^n\,(1 + y)^n\,(1 + z)^n$ is

(a) $({}^nC_m)^3$ (b) $3({}^nC_m)$
(c) ${}^nC_{3m}$ (d) ${}^{3n}C_m$

Ans. (d)

◎ **Solution:** We have

$$(1 + x)^n\,(1 + y)^n\,(1 + z)^n$$
$$= \left(\sum_{r=0}^{n} {}^nC_r x^r\right)\left(\sum_{s=0}^{n} {}^nC_s y^s\right)\left(\sum_{t=0}^{n} {}^nC_t z^t\right)$$
$$= \sum_{0 \le r,s,t \le n} ({}^nC_r)({}^nC_s)({}^nC_t)\,x^r y^s z^t$$

For sum of the coefficients of degree m, we must have $r + s + t = m$ where r, s, t are integers with $r, s, t \ge 0$.

Sum of such coefficients

$$= \sum_{\substack{r,s,t \ge 0 \\ r+s+t=m}} \left({}^nC_r\right)\left({}^nC_s\right)\left({}^nC_t\right)$$

=the number of ways of choosing a total number of m balls out of n distinct black, n distinct white and n distinct green balls

$= {}^{3n}C_m$

◉ **Example 51:** The remainder when 2^{2000} is divided by 17 is

(a) 1 (b) 2
(c) 8 (d) 12

Ans. (a)

Solution: We have $2^4 = 16 = 17 - 1$

$$\therefore \quad 2^{2000} = (2^4)^{500} = (17-1)^{500}$$
$$= {}^{500}C_0\, 17^{500} - {}^{500}C_1\, 17^{499} + {}^{500}C_2\, 17^{498} - {}^{500}C_3 17^{497} + \ldots - {}^{500}C_{499}\,(17) + (-1)^{500}$$
$$= 17m + 1$$

where m is some positive integer.

Thus, the required remainder is 1.

Example 52: If $(1 + x + x^2)^{48} = a_0 + a_1\, x + a_2\, x^2 + \ldots + a_{96}\, x^{96}$, then value of $a_0 - a_2 + a_4 - a_6 + \ldots + a_{96}$ is

(a) –1	(b) 0
(c) 1	(d) 48

Ans. (c)

Solution: Putting $x = i$, we get

$$(1 + i + i^2)^{48} = a_0 + a_1\, i + a_2\, i^2 + \ldots + a^{98}\, i^{96}$$
$$\Rightarrow \quad i^{48} = (a_0 - a_2 + a_4 - \ldots) + i(a_1 - a_3 + \ldots)$$

Equating the real parts we get

$$a_0 - a_2 + a_4 - a_6 + \ldots + a_{96} = 1$$

Example 53: Sum of the series $S = 3^{-1}\,({}^{10}C_0) - {}^{10}C_1 + (3)\,({}^{10}C_2) - 3^2({}^{10}C_3) + \ldots + 3^9\,({}^{10}C_{10})$ is

(a) 2^9	(b) $2^{10} - 1$
(c) $\frac{1}{3}\,(2^{11} - 2)$	(d) $\frac{1}{3}\,(2^{10})$

Ans. (d)

Solution: We have

$$S = 3^{-1}\,[{}^{10}C_0 - {}^{10}C_1\,(3) + {}^{10}C_2\,(3^2) - {}^{10}C_3\,(3^3) + \ldots + {}^{10}C_{10}(3^{10})]$$
$$= \frac{1}{3}\,(1-3)^{10} = \frac{1}{3}\,(2^{10})$$

Example 54: The coefficient of x^{53} in

$$\sum_{m=0}^{100} {}^{100}C_m (x-5)^{100-m} (4)^m \text{ is}$$

(a) ${}^{100}C_{47}$	(b) ${}^{100}C_{53}$
(c) $-\,{}^{100}C_{53}$	(d) 2^{53}

Ans. (c)

Solution: We have

$$\sum_{m=0}^{100} {}^{100}C_m (x-5)^{100-m} (4)^m = (x - 5 + 4)^{100} = (x-1)^{100}$$

$\therefore$ Coefficient of x^{53} is ${}^{100}C_{53}\,(-1)^{47} = -\,{}^{100}C_{53}$.

Example 55: Let

$$E = 1^{2017} + 2^{2017} + 3^{2017} + \ldots + 2016^{2017},$$

then E is divisible by

(a) 2016	(b) 2017
(c) 4033	(d) 2016^2

Ans. (b)

Solution: For $1 \le x \le 1013$,

$$x^{2017} + (2017 - x)^{2017}$$
$$= x^{2017} + {}^{2017}C_0\,(2017)^{2017} + {}^{2017}C_1\,(2017)^{2016}\,(-x) + {}^{2017}C_2\,(2017)^{2015}\,(-x)^2 + \ldots + {}^{2017}C_{2016}\,(2017)\,(-x)^{2016} + (-x)^{2017}$$
$$= (2017)m$$

where $m = {}^{2017}C_0\,(2017)^{2016} + {}^{2017}C_1\,(2017)^{2015}\,(-x) + \ldots + {}^{2017}C_{2016}\,(-x)^{2016}$,

is a positive integer.

Now, write E as sum of 1013 groups as follows:

$$E = (1^{2017} + 2016^{2017}) + (2^{2017} + 2015^{2017}) + \ldots + (1013^{2017} + 1014^{2017})$$

and note that each group is divisible by 2017.

Example 56: Let $(1 + x + x^2)^{100} = \sum_{r=0}^{200} a_r x^r \qquad (1)$

and $a = \sum_{r=0}^{200} a_r$, then value of $\sum_{r=1}^{200} \frac{ra_r}{25a}$ is

(a) 5	(b) 4
(c) 3	(d) 2

Ans. (b)

Solution: $a = \sum_{r=0}^{200} a_r = \sum_{r=0}^{200} a_r (1)^r = 3^{100}$

Differentiating (1), we get

$$100\,(1 + x + x^2)^{99}\,(1 + 2x) = \sum_{r=1}^{200} ra_r\; x^{r-1}$$

Put $x = 1$, to obtain

$$100(3)^{99}\,(3) = \sum_{r=1}^{200} ra_r$$
$$\Rightarrow \quad \sum_{r=1}^{200} \frac{ra_r}{25a} = \frac{100a}{25a} = 4$$

Example 57: If a_i $(i = 0, 1, 2, \ldots 16)$ are real constants such that for every real value of x

$(1 + x + x^2)^8 = a_0 + a_1x + a_2x^2 + \ldots + a_{16}x^{16}$, then a_5 is equal to

(a) 502	(b) 504
(c) 506	(d) 508

Ans. (b)

Solution: $(1 + x + x^2)^8 = \sum\limits_{\substack{p,q,r \ge 0 \\ P+q+r=8}} \frac{8!}{p!q!r!} 1^p\, x^p\, (x^2)^r$

For a_5, we require, $q + 2r = 5$,

$\Rightarrow$ $q = 5, r = 0, p = 3$ or $q = 3, r = 1, p = 4$,

or $q = 1, r = 2, p = 5$.

Thus, coefficient of x^5 is

$$\frac{8!}{5!\,3!} + \frac{8!}{3!\,1!\,4!} + \frac{8!}{1!\,2!\,5!} = 504$$

Example 58: Sum of the last 12 coefficients of in the binomial expansion of $(1 + x)^{23}$ is:

(a) 2^{22} (b) 2^{23}
(c) $2^{23} - {}^{23}C_{11}$ (d) $2^{23} - 2^{11}$

Ans. (a)

Solution: Sum of last 12 coefficients is

$$S = {}^{23}C_{12} + {}^{23}C_{13} + \ldots + {}^{23}C_{23} \quad (1)$$

Using ${}^nC_r = {}^nC_{n-r}$, we get

$$S = {}^{23}C_{11} + {}^{23}C_{10} + \ldots + {}^{23}C_0 \quad (2)$$

Adding (1) and (2), we get

$$2S = {}^{23}C_0 + {}^{23}C_1 + \ldots {}^{23}C_{23} = 2^{23}$$

$$\Rightarrow \quad S = 2^{22}$$

Example 59: If coefficient of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zeros, then (a, b) is equal to

(a) $\left(14, \frac{251}{3}\right)$ (b) $\left(14, \frac{272}{3}\right)$
(c) $\left(16, \frac{272}{3}\right)$ (d) $\left(16, \frac{251}{3}\right)$

Ans. (c)

Solution: $S = (1 + ax + bx^2)(1 - 2x)^{18}$

$$= (1 + ax + bx^2)\,[1 + {}^{18}C_1(-2x) + {}^{18}C_2(-2x)^2 + {}^{18}C_3(-2x)^3 + {}^{18}C_4(-2x)^4 + \ldots]$$

Coefficient of x^3 in the expansion of S is ${}^{18}C_3(-2)^3 + a({}^{18}C_2(-2)^2) + {}^{18}C_1(-2)b = 0$

Divide by ${}^{18}C_1(-2)$ to obtain

$$\frac{544}{3} - 17a + b = 0 \quad (1)$$

Similarly, coefficient of x^4 is

$${}^{18}C_4(-2)^4 + a({}^{18}C_3(-2)^3) + {}^{18}C_2(-2)^2\,b = 0$$

Divide by ${}^{18}C_2(-2)^2$ to obtain

$$80 - \frac{32}{3}a + b = 0 \quad (2)$$

Subtract (2) from (1) to obtain

$$\frac{304}{3} - \frac{19}{3}a = 0 \Rightarrow a = 16$$

From (1), $b = 17 \times 16 - \frac{544}{3} = \frac{272}{3}$

Example 60: Let a, b, c, d be any four consecutive coefficients in the binomial expansion of $(1 + x)^n$, then

$$\frac{a}{a+b} + \frac{c}{c+d} - 2\left(\frac{b}{b+c}\right) \text{ is equal to}$$

(a) n (b) 1
(c) 0 (d) $2n$

Ans. (c)

Solution: Suppose $a = {}^nC_r$, $b = {}^nC_{r+1}$, $C = {}^nC_{r+2}$, and $d = {}^nC_{r+3}$. Now

$$\frac{a}{a+b} = \frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}} = \frac{{}^nC_r}{{}^{n+1}C_{r+1}} = \frac{r+1}{n+1}$$

Similarly, $\dfrac{b}{b+c} = \dfrac{(r+1)+1}{n+1}$ and $\dfrac{c}{c+d} = \dfrac{(r+2)+1}{n+1}$

Thus, $$\frac{a}{a+b} + \frac{c}{c+d} - 2\left(\frac{b}{b+c}\right) = 0$$

Example 61: Suppose $a_0 = 2017, a_1\,a_2, \ldots, a_{n-1}, 2023 = a_n$ are in A.P. Let

$$S = \frac{1}{2^{n+1}}\sum_{r=0}^{n}\left({}^nC_r\right)a_r - 1000$$

Then S is equal to

(a) 10 (b) 40
(c) 1013 (d) 2021

Ans. (a)

Solution: Put ${}^nC_r = C_r$ and let

$$T = a_0\,C_0 + a_1\,C_1 + a_2\,C_2 + \ldots + a_n\,C_n \quad (1)$$

Using $C_r = C_{n-r}$, we get

$$T = a_n\,C_0 + a_{n-1}\,C_1 + \ldots + a_0\,C_n \quad (2)$$

Adding (1) and (2), we get

$$2T = (a_0 + a_n)C_0 + (a_1 + a_{n-1})C_1 + (a_2 + a_{n-2})C_2 + \ldots + (a_n + a_0)C_n$$

But $a_0 + a_n = a_1 + a_{n-1} = a_2 + a_{n-2}$
$= \ldots = a_n + a_0 = 2017 + 2023$

$\therefore$ $2T = 4040\,(C_0 + C_1 + \ldots + C_n)$

$\Rightarrow$ $T = (2020)2^n = (1010)2^{n+1}$

Thus,

$$S = \frac{1}{2^{n+1}}(T) - 1000 = 10$$

Example 62: The interval in which x (> 0) must lie so that the greatest term in the expansion of $(1 + x)^{2n}$ has the greatest coefficient is

(a) $\left(\frac{n-1}{n}, \frac{n}{n-1}\right)$ (b) $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$

(c) $\left(\frac{n}{n+2}, \frac{n+2}{n}\right)$ (d) none of these

Ans. (b)

◎ **Solution:** The greatest coefficient in the expansion of $(1 + x)^{2n}$ is ${}^{2n}C_n$. We are given ${}^{2n}C_n\, x^n$ is the greatest term.

$$\therefore \qquad {}^{2n}C_{n-1}\, x^{n-1} < {}^{2n}C_n\, x^n \text{ and } {}^{2n}C_{n+1}\, x^{n+1} < {}^{2n}C_n\, x^n$$

$$\Rightarrow \qquad \frac{{}^{2n}C_{n-1}}{{}^{2n}C_n} < x < \frac{{}^{2n}C_n}{{}^{2n}C_{n+1}}$$

$$\Rightarrow \qquad \frac{n}{n+1} < x < \frac{n+1}{n}.$$

◉ **Example 63:** If the coefficients of the rth, $(r + 1)$th and $(r + 2)$th terms in the binomial expansion of $(1 + y)^m$ are in A.P., then m and r satisfy the equation

(a) $m^2 - m(4r + 1) + 4r^2 - 2 = 0$
(b) $m^2 - m(4r - 1) + 4r^2 + 2 = 0$
(c) $m^2 - m(4r - 1) + 4r^2 - 2 = 0$
(d) $m^2 - m(4r + 1) + 4r^2 + 2 = 0$.

Ans. (a)

◎ **Solution:** Coefficient of the rth term is ${}^mC_{r-1}$. According to the given condition ${}^mC_{r-1} + {}^mC_{r+1} = 2({}^mC_r)$

$$\Rightarrow \qquad \frac{{}^mC_{r-1}}{{}^mC_r} + \frac{{}^mC_{r+1}}{{}^mC_r} = 2 \Rightarrow \frac{r}{m+1-r} + \frac{m-r}{r+1} = 2$$

$$\Rightarrow \qquad r^2 + r + (m - r)(m + 1 - r) = 2(m + 1 - r)(r + 1)$$

$$\Rightarrow \qquad r^2 + r + m^2 + (1 - 2r)\, m - r(1 - r) = 2m\,(r + 1) + 2(1 - r^2)$$

$$\Rightarrow \qquad m^2 - m(4r + 1) + 4r^2 - 2 = 0.$$

◉ **Example 64:** Let $x > -1$, then statement

$$P(n) : (1 + x)^n > 1 + nx$$

is true for

(a) all $n \in \mathbf{N}$
(b) all $n > 1$
(c) all $n > 1$ provided $x \neq 0$
(d) none of these.

Ans. (c)

Solution: For $n = 2$,

$P(2)$: $(1 + x)^2 = 1 + 2x + x^2 > 1 + 2x$ as $x \neq 0$.

Assume that

$$P(k): (1 + x)^k > 1 + kx \qquad (1)$$

for some $k \in \mathbf{N}$, $k > 1$.

As $x > -1$, multiplying both the sides of (1) by $1 + x$, we get

$$\begin{aligned}(1 + x)^{k+1} &> (1 + kx)(1 + x)\\ &= 1 + (k + 1)x + kx^2\\ &> 1 + (k + 1)x \qquad [\because \; kx^2 > 0]\end{aligned}$$

Thus, $P(k + 1)$ is true.

By the principle of mathematical induction $P(n)$ is true for all $n > 1$ provided $x \neq 0$.

◉ **Example 65:** For each $n \in \mathbf{N}$, $2^{3n} - 1$ is divisible by

(a) 8 (b) 16
(c) 32 (d) none of these

Ans. (d)

◎ **Solution:** We have

$$\begin{aligned}2^{3n} - 1 &= (2^3)^n - 1 = (1 + 7)^n - 1\\ &= 1 + {}^nC_1\,(7) + {}^nC_2\,(7^2) + \ldots + {}^nC_n\,(7^n) - 1\\ &= 7m\end{aligned}$$

where $m = {}^nC_1 + {}^nC_2\,(7) + \ldots + {}^nC_n\,(7^{n-1})$ is a positive integer.

Thus, 7 divides $2^{3n} - 1$ for all $n \in \mathbf{N}$.

◉ **Example 66:** For each $n \in \mathbf{N}$, $x^{2n-1} + y^{2n-1}$ is divisible by

(a) $x + y$ (b) $(x + y)^2$
(c) $x^3 + y^3$ (d) none of these

Ans. (a)

◎ **Solution:** $x^{2n-1} + y^{2n-1}$

$$= (x + y)(x^{2n-2} - x^{2n-3}y + x^{2n-4}y^2 - x^{2n-5}y^3 + \ldots + y^{2n-2})$$

Thus $x^{2n-1} + y^{2n-1}$ is divisible by $x + y$ for all $n \in \mathbf{N}$.

◉ **Example 67:** For all $n \in \mathbf{N}$, $n(n + 1)(2n + 1)$ is divisible by

(a) 6 (b) 8
(c) 15 (d) 9

Ans. (a)

◎ **Solution:** We have $n(n + 1)(2n + 1) = 6\{1^2 + 2^2 + \ldots + n^2\}$,

$\therefore$ $n(n + 1)(2n + 1)$ is divisible by 6 for all $n \in \mathbf{N}$.

◉ **Example 68:** If $a_n = \sqrt{7 + \sqrt{7 + \sqrt{7 + \cdots}}}$ having n radical signs then by methods of mathematical induction which of the following is true

(a) $a_n > 7 \;\forall\; n \geq 1$ (b) $a_n > 3 \;\forall\; n \geq 1$
(c) $a_n < 4 \;\forall\; n \geq 1$ (d) $a_n < 3 \;\forall\; n \geq 1$

Ans. (c)

◎ **Solution:** We have $a_1 = \sqrt{7} < 4$. Assume $a_k < 4$ for some natural number k.

We have

$$a_{k+1} = \sqrt{7 + a_k} < \sqrt{7 + 4} < 4$$

$\therefore a_n < 4 \;\forall\; n \geq 1$.

◉ **Example 69:** If $a_n = 2^{2^n} + 1$, then for $n > 1$, last digit of a_n is

(a) 5 (b) 7
(c) 3 (d) 4

Ans. (b)

◎ **Solution:** For $n = 2$, $a_n = 2^4 + 1 = 16 + 1 = 17$. Assume that $a_k = 2^{2^k} + 1 = 10m + 7$ where $k > 1$ and m is some positive integer.

Now, $a_{k+1} = 2^{2^{k+1}} + 1 = (2^{2^k})^2 + 1$
$= (10m + 6)^2 + 1 = 10\,(10m^2 + 12m + 3) + 7$

Thus, last digit of a_n is 7 $\forall$ $n > 1$.

◉ **Example 70:** Let $S(K)$: $1 + 3 + 5 + \ldots + (2K - 1) = 3 + K^2$. Then which of the following is true?

(a) $S(K) \nRightarrow S(K + 1)$
(b) $S(K) \Rightarrow S(K + 1)$
(c) $S(1)$ is correct
(d) principle of mathematical induction can be used to prove the formula.

Ans. (b)

◎ **Solution:** $S(1)$: $1 = 3 + 1^2$ which is not true. Suppose S(K) is true, then

$$1 + 3 + 5 + \ldots + (2K - 1) = 3 + K^2$$

Adding $(2K + 1)$ to both the sides, we get

$$1 + 3 + 5 + \ldots + (2K - 1) + (2K + 1) = 3 + K^2 + 2K + 1$$
$$= 3 + (K + 1)^2$$

which is $S(K + 1)$.

Thus, $S(K) \Rightarrow S(K + 1)$.

◉ **Example 71:** Let $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$ and $\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \cdots + n\frac{C_n}{C_{n-1}} = \frac{1}{k} n(n+1)$, then value of k is

(a) 2 (b) 3
(c) 6 (d) 12

Ans. (a)

◎ **Solution:** $r\frac{C_r}{C_{r-1}} = r\frac{n!}{r!\,(n-r)!}\frac{(r-1)!\,(n-r+1)!}{n!}$
$= n - r + 1$

$$\Rightarrow \quad \sum_{r=1}^{n} r\frac{C_r}{C_{r-1}} = \sum_{r=1}^{n}(n - r + 1) = \frac{1}{2}n(n+1)$$

$\therefore$ $k = 2$.

◉ **Example 72:** The last term in the binomial expansion of $\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right)^n$ is $\left(\frac{1}{(3)9^{\frac{1}{3}}}\right)^{\log_3 8}$, then the 5th term from the beginning is

(a) ${}^{10}C_6$ (b) $2({}^{10}C_4)$
(c) $\frac{1}{2}({}^{10}C_4)$ (d) $-\,{}^{10}C_6$

Ans. (b)

◎ **Solution:** ${}^{n}C_n\left(-\frac{1}{\sqrt{2}}\right)^n = (3^{-1-2/3})^{\log_3 8} = 3^{-5\log_3 2} = \frac{1}{2^5}$

$\Rightarrow$ $n = 10$.

$\therefore$ 5th term from the beginning

$$= {}^{10}C_4\,(\sqrt{2})^{10-4}\left(-\frac{1}{\sqrt{2}}\right)^4 = 2({}^{10}C_4)$$

◉ **Example 73:** Degree of the polynomial $\left(x^2 - \sqrt{1 - x^3}\right)^4 + \left(x^2 + \sqrt{1 - x^3}\right)^4$, is

(a) 7 (b) 8
(c) 9 (d) 12

Ans. (b)

◎ **Solution:** We have

$$(x - y)^4 + (x + y)^4 = 2\,[{}^4C_0\,x^4 + {}^4C_2\,x^2y^2 + {}^4C_4\,y^4]$$

Thus,

$$\left(x^2 - \sqrt{1 - x^3}\right)^4 + \left(x^2 + \sqrt{1 - x^3}\right)^4$$
$$= 2[{}^4C_0(x^2)^4 + {}^4C_2(x^2)^2\,(1 - x^3) + {}^4C_4(1 - x^3)^2]$$

which is a polynomial of degree 8.

◉ **Example 74:** Coefficient of $\frac{1}{x}$ in the expansion of $(1 + x)^n\,(1 + 1/x)^n$ is

(a) ${}^{2n}C_{n-1}$ (b) ${}^{2n}C_n$
(c) 1 (d) 0

Ans. (a)

◎ **Solution:** We have

$$(1 + x)^n\left(1 + \frac{1}{x}\right)^n = \frac{(1 + x)^{2n}}{x^n}$$

$\therefore$ Coefficient of x^{-1} in $(1 + x)^n\left(1 + \frac{1}{x}\right)^n$

$= \text{Coefficient of } x^{n-1} \text{ in } (1 + x)^{2n}$
$= {}^{2n}C_{n-1}$

◉ **Example 75:** Coefficient of x^4 in $(1 + x + x^2 + x^3)^{11}$ is

(a) 99 (b) 330
(c) 990 (d) 1050

Ans. (c)

◎ **Solution:** We have

$$1 + x + x^2 + x^3 = 1 + x + x^2(1 + x) = (1 + x)(1 + x^2)$$

$\Rightarrow$ $(1 + x + x^2 + x^3)^{11} = (1 + x)^{11}(1 + x^2)^{11}$
$= (1 + {}^{11}C_1\,x + {}^{11}C_2\,x^2 + {}^{11}C_3\,x^3 + {}^{11}C_4x^4 + \cdots)\,(1 + {}^{11}C_1\,x^2 + {}^{11}C_2\,x^4 + \cdots)$

Thus, coefficient of x^4 in $(1 + x + x^2 + x^3)^{11}$

$= (1)\,({}^{11}C_2) + ({}^{11}C_2)\,({}^{11}C_1) + ({}^{11}C_4)\,(1)$
$= 55 + (55)(11) + 330 = 990$

◉ **Example 76:** The sixth term in the expansion of $(x^{1/3} + y^{1/2})^n$, if the binomial coefficient of the third term form the end is 45, is

(a) $252\,x^{5/3}\,y^{5/2}$ (b) 45
(c) $45\,x^{5/3}\,y^{5/2}$ (d) none of these

Ans. (a)

◎ **Solution:** Binomial coefficients of the third term from the end in the expansion of $(x^{1/3} + y^{1/2})^n$, is

$${}^nC_2 = 45 \Rightarrow n\,(n-1) - 90 = 0 \Rightarrow n = 10.$$

$\therefore$ Sixth term in the expansion of $(x^{1/3} + y^{1/2})^n$ is

$${}^nC_5\,(x^{1/3})^{n-5}\,(y^{1/2})^5 = 252\ x^{5/3}\,y^{5/2}$$

Example 77: If $a > 0$ and coefficient of x^2, x^3, x^4 in the expansion of $(1 + x/a)^6$ are in A.P., then a equals

(a) $\left(4+\sqrt{7}\right)/3$ (b) $\left(4+\sqrt{3}\right)/3$

(c) $2-\sqrt{3}$ (d) $2+\sqrt{3}$

Ans. (a)

Solution: Coefficient of x^r in the expansion of $(1 + x/a)^6$

$$= {}^6C_r\,(1/a)^r$$

According to given condition ${}^6C_2(1/a)^2$, ${}^6C_3(1/a)^3$, ${}^6C_4(1/a)^4$ are in A.P.

$$\therefore\quad 2\left({}^6C_3\right)\left(\frac{1}{a}\right)^3 = {}^6C_2\left(\frac{1}{a}\right)^2 + {}^6C_4\left(\frac{1}{a}\right)^4$$

$$\Rightarrow\quad 2\,(20)\,a = 15(a^2+1)$$

$$\Rightarrow\quad 3a^2 - 8a + 3 = 0$$

$$\Rightarrow\quad a = \left(4+\sqrt{7}\right)/3 \text{ as } a > 0.$$

Example 78: Sum of two middle terms in the expansion of $(1 + x)^{2n-1}$ is

(a) ${}^{2n-1}C_{n-1}$ (b) ${}^{2n-1}C_n$

(c) ${}^{2n}C_n$ (d) ${}^{2n}C_{n+1}$

Ans. (c)

Solution: Two middle terms in the expansion of $(1 + x)^{2n-1}$ are $\left(\dfrac{2n-1+1}{2}\right)$th and $\left(\dfrac{2n-1+1}{2}+1\right)$th terms, that is, nth and $(n+1)$th terms.

Coefficients of the middle terms in the expansion of $(1 + x)^{2n-1}$ are ${}^{2n-1}C_{n-1}$ and ${}^{2n-1}C_n$ and their sum $= {}^{2n-1}C_{n-1} + {}^{2n-1}C_n = {}^{2n}C_n$

Example 79: If a is a real number and if the middle term of $\left(\dfrac{a}{3}+3\right)^8$ is 1120, then value of a is

(a) ± 2 (b) ± 1

(c) $\pm\sqrt{3}$ (d) $\pm\sqrt{2}$

Ans. (a)

Solution: Middle term of $\left(\dfrac{a}{3}+3\right)^8$ is the $\left(\dfrac{8}{2}+1\right)$th or 5th term. We have

$$T_5 = T_{4+1} = {}^8C_4\left(\frac{a}{3}\right)^4 (3)^4 = 70a^4$$

We are given $\quad T_5 = 1120 \Rightarrow 70a^4 = 1120$

$$\Rightarrow\quad a^4 = 16 = 2^4 \Rightarrow a = \pm 2$$

Example 80: Suppose F is the fractional part of $M = \left(\sqrt{13}+\sqrt{11}\right)^6$, then value of $M\,(1-F)$ is

(a) 128 (b) 64

(c) 32 (d) 16

Ans. (b)

Solution: Let $N = \left(\sqrt{13}-\sqrt{11}\right)^6$,

As $\left(\sqrt{13}-\sqrt{11}\right) = \dfrac{2}{\sqrt{13}+\sqrt{11}}$, we get $0 < N < 1$

Also, $M + N = \left(\sqrt{13}+\sqrt{11}\right)^6 + \left(\sqrt{13}-\sqrt{11}\right)^6$,

$$= 2\Big[{}^6C_0\left(\sqrt{13}\right)^6 + {}^6C_2\left(\sqrt{13}\right)^4\left(\sqrt{11}\right)^2 + {}^6C_4\left(\sqrt{13}\right)^2\left(\sqrt{11}\right)^4 + {}^6C_6\left(\sqrt{11}\right)^6\Big]$$

$\Rightarrow M + N$ is an integer, say, J

Let $M = K + F$, where K is the greatest integer contained in M. We have

$$J = M + N = K + N + F$$

As $0 < N < 1$, $0 < F < 1$ we get, $0 < N + F < 2$

Also, $N + F$ is an integer

$\Rightarrow\quad N + F = 1 \Rightarrow 1 - F = N$

Thus, $\quad M\,(1-F) = \left(\sqrt{13}+\sqrt{11}\right)^6\left(\sqrt{13}-\sqrt{11}\right)^6 = 2^6 = 64$

Example 81: The remainder when $8^{2n} - 62^{2n+1}$ is divided by 9 is

(a) 7 (b) 8

(c) 0 (d) 2

Ans. (b)

Solution: $8^{2n} - 62^{2n+1}$

$$= (9-1)^{2n} - (63-1)^{2n+1}$$

$$= ({}^{2n}C_0\,9^{2n} - {}^{2n}C_1\,9^{2n-1} + {}^{2n}C_2\,9^{2n-2} - \cdots - {}^{2n}C_{2n-1}\,(9) + {}^{2n}C_{2n}) - ({}^{2n+1}C_0(63^{2n+1}) - {}^{2n+1}C_1(63^{2n}) + {}^{2n+1}C_2\,(63^{2n-1}) - \cdots - + {}^{2n+1}C_1(63) - {}^{2n+1}C_0)$$

$$= 9m + 1 + 1 = 9m + 2 \text{ for some integer } m$$

Thus, when $8^{2n} - 62^{2n+1}$ is divided by 9, the remainder is 2.

Example 82: The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is

(a) 132 (b) 144

(c) – 132 (d) – 144

Ans. (d)

Solution: $(1 - x - x^2 + x^3)^6 = (1 - x)^6\,(1 - x^2)^6$

coefficient of x^7 in $(1 - x)^6\,(1 - x^2)^6$

$$= \text{coefficient of } x^7 \text{ in } [1 - {}^6C_1\,x + {}^6C_2\,x^2 - \cdots + {}^6C_6\,x^6] \times [1 - {}^6C_1\,x^2 + {}^6C_2\,x^4 - \cdots + {}^6C_6\,x^{12}]$$

$$= (-{}^6C_1)\,(-{}^6C_3) + (-{}^6C_3)\,({}^6C_2) + (-{}^6C_5)\,(-{}^6C_1)$$

$$= -144.$$

Example 83: If $n \in \mathbf{N}$, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3-1}\right)^{2n}$ is

(a) an odd positive integer

(b) an even positive integer

(a) a rational number but not a natural number
(d) an irrational number

Ans. (d)

◎ **Solution:** $\left(\sqrt{3}+1\right)^{2n}-\left(\sqrt{3}-1\right)^{2n}$

$$=2\left[{}^{2n}C_1\left(\sqrt{3}\right)^{2n-1}+{}^{2n}C_3\left(\sqrt{3}\right)^{2n-3}+\cdots+{}^{2n}C_{2n-1}\left(\sqrt{3}\right)\right]$$

$$=2\sqrt{3}\left[{}^{2n}C_1\left(3^{n-1}\right)+{}^{2n}C_3\,3^{n-2}+\cdots+{}^{2n}C_{2n-1}\right]$$

which is an irrational number.

◉ **Example 84:** The coefficient of x^n in the expansion of

$$\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2\,..+\frac{1}{n!}x^n\right)^2$$

(a) $\dfrac{2^n}{n!}$ (b) $\dfrac{2^n}{n}$
(c) $n!$ (d) $\dfrac{1}{n!}$

Ans. (a)

◎ **Solution:** Coefficient of x^n in

$$\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\ldots+\frac{1}{n!}x^n\right)^2$$

= coefficient of x^n is

$$\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\ldots+\frac{1}{n!}x^n+\frac{1}{(n+1)!}x^{n+1}\ldots\right)^2$$

= coefficient of x^n in $e^{2x}=\left(1+2x+\dfrac{2^2\,x^2}{2!}+\dfrac{2^3x^3}{3!}\ldots\right)$

$$=\frac{2^n}{n!}$$

Alternate Solution

Coefficient of x^n in

$$\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+.\cdots+\frac{1}{n!}x^n\right)\left(1+\frac{1}{1!}x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n\right)$$

$$=\frac{1}{n!}+\frac{1}{1!(n-1)!}+\frac{1}{2!(n-2)!}+.\cdots+\frac{1}{(n-1)!1!}+\frac{1}{n!}$$

$$=\frac{1}{n!}\left[{}^nC_0+{}^nC_1+{}^nC_2+\cdots{}^nC_{n-1}+{}^nC_n\right]=\frac{2^n}{n!}$$

◉ **Example 85:** The coefficient of x^n in the expansion of $(1+x)(1-x)^n$ is

(a) $(-1)^n\,n$ (b) $(-1)^n(1-n)$
(c) $(-1)^{n-1}(n-1)$ (d) $(n-1)$

Ans. (b)

◎ **Solution:** $(1+x)(1-x)^n=[2-(1-x)](1-x)^n$

$$=2(1-x)^n-(1-x)^{n+1}$$

$\therefore$ coefficient of x^n in $(1+x)(1-x)^n$

$$=2(-1)^n-{}^{n+1}C_n(-1)^n$$

$$=(-1)^n(2-n-1)=(-1)^n(1-n)$$

◉ **Example 86:** Sum of the coefficients of x^3 and x^6 in the expansion of $\left(x^2-\dfrac{1}{x}\right)^9$ is

Ans. (a)

◎ **Solution:** $T_{r+1}={}^9C_r\,(x^2)^{9-r}\left(-\dfrac{1}{x}\right)^r=(-1)^r\,({}^9C_r)x^{18-3r}$

For coefficient of x^3, x^6, we set $18-3r=3,6$

$\Rightarrow$ $r=5,4.$

$\therefore$ sum of coefficients of x^3 and x^6

$$=(-1)^5\,({}^9C_4)+(-1)^4\,({}^9C_5)=0$$

◉ **Example 87:** For $n\geq 2$, let

$$a_n=\sum_{r=0}^{n}\frac{1}{C_r^2}\text{, then value}$$

of $$b_n=\sum_{r=1}^{n}\frac{1}{r^2\,C_r^2}\text{ equals}$$

(a) $\dfrac{1}{n^2}\,a_n$ (b) $\dfrac{1}{n^2}\,a_{n-1}$
(c) a_n (d) a_n^2

Ans. (b)

◎ **Solution:** For $r\geq 1$, $r\,C_r=r\,({}^nC_r)=n\,({}^{n-1}C_{r-1})$

Thus, $b_n=\displaystyle\sum_{r=1}^{n}\frac{1}{n^2\left({}^{n-1}C_r\right)^2}=\frac{1}{n^2}\,a_{n-1}$

◉ **Example 88:** If in the expansion of $\left(x^3-\dfrac{1}{x^2}\right)^n$ the sum of the coefficients of x^5 and x^{10} is 0, then the coefficient of x^{20} is:

(a) ${}^{20}C_6$ (b) $-{}^{20}C_6$
(c) ${}^{15}C_5$ (d) $-{}^{15}C_5$

Ans. (d)

◎ **Solution:** $(r+1)$th term in the expansion of

$$\left(x^3-\frac{1}{x^2}\right)^n\text{ is}$$

$$T_{r+1}={}^nC_r(x^3)^{n-r}\left(-\frac{1}{x^2}\right)^r$$

$$= {}^nC_r\, x^{3n-5r}\,(-1)^r$$

For coefficient of x^{10}, set $3n - 5r = 5$

$$\Rightarrow \qquad r = \frac{3}{5}n - 1 = r_1 \text{ (say)}$$

For coefficient of x^{10}, set $3n - 5r = 10$

$$\Rightarrow \qquad r = \frac{3}{5}n - 2 = r_2\text{(say)}$$

Note that $r_1 = r_2 + 1$.

We are given

$$ {}^nC_{r_1}(-1)^{r_1} + {}^nC_{r_2}(-1)^{r_2} = 0$$

$$\Rightarrow \qquad {}^nC_{r_2} = {}^nC_{r_2+1}\ [\because\ r_1 = r_2 + 1]$$

$$\Rightarrow\ r_2 + r_2 + 1 = n \Rightarrow r_2 = \frac{1}{2}(n-1)$$

$$\Rightarrow \quad \frac{3}{5}n - 2 = \frac{1}{2}n - \frac{1}{2} \Rightarrow \frac{1}{10}n = \frac{3}{2} \Rightarrow n = 15$$

For coefficient of x^{20}, set $3n - 5r = 20$

$$\Rightarrow \qquad 5r = 45 - 20 = 25 \text{ or } r = 5.$$

Thus, coefficient of x^{20} is $-\,{}^{15}C_5$

Example 89: If $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)\ldots(1 + x + x^2 + \ldots + x^n)$

$= a_0 + a_1x + a_2x^2 + \ldots + a_m x^m$,

then value of a_1 is

(a) $m + 1$ (b) $n + 1$

(c) n (d) m

Ans. (c)

Solution: a_1 = coefficient of x in

$(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3)\ldots(1 + x + \ldots + x^n)$

$= $ coefficient of x in $\underbrace{(1 + x)(1 + x)\ldots(1 + x)}_{n \text{ times}}$

$= $ coefficient of x in $(1 + x)^n$

$= {}^nC_1 = n$

Example 90: If $x^n = a_0 + a_1(1 + x) + a_2(1 + x)^2 + \ldots + a_n(1 + x)^n$

$= b_0 + b_1(1 - x) + b_2(1 - x)^2 + \ldots + b_n(1 - x)^n$

then for $n = 201$, (a_{101}, b_{101}) is equal to:

(a) $(-{}^{201}C_{101}, -{}^{201}C_{101})$ (b) $({}^{201}C_{101}, -{}^{201}C_{101})$

(c) $(-{}^{201}C_{101}, {}^{201}C_{101})$ (d) $({}^{201}C_{101}, {}^{201}C_{101})$

Ans. (b)

Solution: $x^n = [(1 + x) - 1]^n = [1 - (1 - x)]^n$

$$= \sum_{k=0}^{n} {}^nC_k\,(1 + x)^{n-k}(-1)^k$$

$$= \sum_{k=0}^{n} {}^nC_k\,(-1)^k(1 - x)^k$$

$\therefore \qquad a_k$ = coefficient of $(1 + x)^k$ in

$$\sum_{k=0}^{n} {}^nC_k\ (1 + x)^{n-k}(-1)^k = (-1)^{n-k}\,{}^nC_{n-k} = (-1)^{n-k}\,{}^nC_k$$

and $\qquad b_k = {}^nC_k(-1)^k$

For $\qquad n = 201$, $k = 101$, we get

$$(a_{101}, b_{101}) = ({}^{201}C_{101}, -\,{}^{201}C_{101})$$

Assertion-Reason Type Questions

Example 91:

Statement-1: For each natural number n, $(n + 1)^7 - n^7 - 1$ is divisible by 7

Statement-2: For each natural number n, $n^7 - n$ is divisible by 7.

Ans. (a)

Solution: For Statement-2, write $n = 7r + k$, $k = -3, -2, -1, 0, 1, 2, 3$

Now, $\qquad n^7 - n = (7r + k)^7 - (7r + k)$

Using binomial theorem, we get

$\qquad n^7 - n - (k^7 - k)$ is divisible by 7.

$\Leftrightarrow \qquad n^7 - n$ is divisible by 7 if and only if $k^7 - k$ is divisible by 7

For $\qquad k = \pm 3, \pm 2, \pm 1$, $k^7 - k$ is divisible by 7 if and only if $k^6 - 1$ is divisible by 7.

But $\qquad k^6 - 1 = 728, 63, 0$ for $k = \pm 3, \pm 2, \pm 1$.

$\therefore \qquad k^6 - 1$ is divisible by 7 for $k = \pm 3, \pm 2, \pm 1$.

Thus, $n^7 - n$ is divisible by 7 for each $n \in \mathbf{N}$.

$\Rightarrow \qquad (n + 1)^7 - (n + 1) - (n^7 - n)$ is divisible by 7

$\Rightarrow \qquad (n + 1)^7 - n^7 - 1$ is divisible by 7.

Therefore, both the statements are true and statement - 2 is a correct reason for Statement-1.

Example 92: Let $S_1 = \sum_{j=1}^{10} j(j-1)\left({}^{10}C_j\right)$

$$S_2 = \sum_{j=1}^{10} j\left({}^{10}C_j\right) \text{and } S_3 = \sum_{j=1}^{10}\left(j^2\right)\left({}^{10}C_j\right)$$

Statement-1 $\quad S_3 = 55 \times 2^9$

Statement-2 $\quad S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$

Ans. (c)

Solution: We have

$$(1+x)^{10} = \sum_{j=0}^{10} \left({}^{10}C_j\right) x^{j}$$

Differentiating both the sides with respect to x, we get

$$10(1+x)^9 = \sum_{j=1}^{10} j\left({}^{10}C_j\right) x^{j-1} \qquad (1)$$

Again differentiating both the sides with respect to x, we get

$$(10)\,(9)\,(1+x)^8 = \sum_{j=1}^{10} j(j-1)\left({}^{10}C_j\right) x^{j-2} \qquad (2)$$

Putting $x = 1$ in (1) and (2), we get

$$S_2 = \sum_{j=1}^{10} j\left({}^{10}C_j\right) = 10\,(2^9)$$

and $$S_1 = \sum_{j=1}^{10} j(j-1)\left({}^{10}C_j\right) = (10)\,(9)\,(2^8) = (90)\,(2^8)$$

Adding the above two equation, we get

$$S_3 = \sum_{j=1}^{10} [j + j\,(j-1)]({}^{10}C_j)$$
$$= (10)\,(2^8)\,(2+9)$$
$$\Rightarrow \quad S_3 = (55)\,(2^9)$$

Thus, Statement-1 is true but Statement-2 is false.

Example 93: Let $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$.

Statement-1: $5C_0^{\,2} + 7C_1^{\,2} + 9C_2^{\,2} + \cdots + (5+2n)C_n^{\,2} = (5+n)\dfrac{(2n)!}{n!\,n!}$

Statement-2: $C_0^{\,2} + C_1^{\,2} + \cdots + C_n^{\,2} = {}^{2n}C_n$

Ans. (a)

Solution $C_0^{\,2} + C_1^{\,2} + \ldots + C_n^{\,2}$

$= C_0C_n + C_1C_{n-1} + \cdots + C_nC_0$

= number of ways of choosing n persons out of n men and n women

$= {}^{2n}C_n$

$\therefore$ Statement-2 is true

Let

$$S = 5C_0^{\,2} + 7C_1^{\,2} + 9C_2^{\,2} + \cdots + (5+2n)C_n^{\,2} \qquad (1)$$

Using $C_r = C_{n-r}$, we can rewrite (1) as

$$S = (5+2n)\,C_0^{\,2} + (3+2n)\,C_1^{\,2} + \ldots + 5C_n^{\,2} \qquad (2)$$

Adding (1) and (2), we get

$$2S = (10+2n)\,(C_0^{\,2} + C_1^{\,2} + \ldots + C_n^{\,2})$$

$$\Rightarrow \quad S = (5+n)\,({}^{2n}C_n) = (5+n)\frac{(2n)}{n!\,n!}$$

Example 94: Let

$$(1+t)^n = C_0 + C_1t + C_2t^2 + \cdots + C_nt^n \qquad (1)$$

Statement-1: $\dfrac{C_0}{(1)(2)} + \dfrac{C_1}{(2)(3)} + \dfrac{C_2}{(3)(4)} + \cdots + \dfrac{C_n}{(n+1)(n+2)}$

$$= \frac{1}{n+1}\left[\frac{2^{n+2}}{n+2} - 1\right]$$

Statement-2: $\dfrac{C_0}{2} - \dfrac{C_1}{3} + \dfrac{C_2}{4} - \cdots + (-1)^n\dfrac{C_n}{n+2} = 0$

Ans. (b)

Solution: From (1)

$$\int_0^x (1+t)^n\,dt = \int_0^x [C_0 + C_1t + C_2t^2 + \cdots + C_nt^n]\,dt$$

$$\Rightarrow \quad \frac{1}{n+1}\,[(1+x)^{n+1} - 1]$$
$$= \frac{C_0}{1}x + \frac{C_2}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1} \qquad (2)$$

Multiplying (1) by t and integrating, we get

$$\int_0^x t(1+t)^n\,dt = \int_0^x [C_0t + C_1t^2 + \ldots + C_nt^{n+1}]\,dt$$

$$\Rightarrow \quad \frac{x(1+x)^{n+1}}{n+1} - \frac{(1+x)^{n+2}}{(n+1)(n+2)}$$
$$= \frac{C_0}{2}x^2 + \frac{C_1}{3}x^3 + \frac{C_2}{4}x^4 + \ldots + \frac{C_n}{n+2}x^{n+2}. \qquad (3)$$

Putting $x = -1$ in (3), we obtain that the second statement is true.

Putting $x = 1$ in (2) and (3) and subtracting, we obtain that Statement-1 is also true.

Example 95: Let $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$.

Statement-1: For $m \geq 2$, $C_0 - C_1 + C_2 - \cdots + (-1)^{m-1}\,C_{m-1} = (-1)^{m-1}\,({}^{n-1}C_{m-1})$

Statement-2: ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$ for $1 \leq r \leq n-1$

Ans. (a)

Solution: $C_0 = {}^{n-1}C_0$

$$-C_1 = -{}^{n-1}C_0 - {}^{n-1}C_1$$
$$C_2 = {}^{n-1}C_1 + {}^{n-1}C_2$$
$$-C_3 = -{}^{n-1}C_2 - {}^{n-1}C_3$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$(-1)^{m-1}(C_{m-1}) = (-1)^{m-1}({}^{n-1}C_{m-2} + {}^{n-1}C_{m-1})$$

Adding the above equations, we get

$$C_0 - C_1 + C_2 - C_3 + \cdots + (-1)^{m-1}C_{m-1}$$
$$= (-1)^{m-1}\,({}^{n-1}C_{m-1})$$

Example 96: Let $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$.

Statement-1: $S = C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + \cdots + (C_0 + C_1 + \cdots + C_{n-1}) = n(2^{n-1})$

Statement-2: $\displaystyle\sum_{j=1}^{n}\sum_{i<j} (C_i + C_j) = (n+1)2^n.$

Ans. (c)

◎ **Solution:** We can write

$$S = nC_0 + (n-1)C_1 + (n-2)C_2 + \cdots + 1C_{n-1} + 0C_n \quad (1)$$

Using $C_r = C_{n-r}$, we can rewrite (1) as

$$S = 0C_0 + 1C_1 + 2C_2 + \cdots + (n-1)C_{n-1} + nC_n \quad (2)$$

Adding (1) and (2) we obtain

$$2S = n[C_0 + C_1 + C_2 + \cdots + C_n] = n(2^n)$$

$$\Rightarrow \qquad S = n(2^{n-1})$$

In the expression $\sum_{j=1}^{n} \sum_{i<j} (C_i + C_j)$ each C_i $(0 \le i \le n)$ occurs exactly n times. Thus

$$\sum_{j=1}^{n} \sum_{i<j} (C_i + C_j) = n \sum_{k=0}^{n} C_k = n\left(2^n\right)$$

⊛ **Example 97:** Let $(1 + x)^{3n} = C_0 + C_1x + C_2x^2 + \ldots + C_{3n}x^{3n}$, and $\omega \ne 1$ be a cube root of unity

Statement-1: $C_0 + C_1\omega + C_2\omega^2 + C_3 + C_4\omega + C_5\omega^2 + \cdots = (-1)^n$

Statement-2: Cube roots unity form a triangle of area $\sqrt{3}$ square units.

Ans. (c)

◎ **Solution:** $C_0 + C_1\omega + C_2\omega^2 + C_3 + C_4\omega + C_5\omega^2 + \ldots$

$$= \sum_{k=0}^{3n} C_k\,\omega^k = (1 + \omega)^{3n}$$

$$= (-\omega^2)^{3n} = (-1)^{3n}\,\omega^{6n}$$

$$= (-1)^n\,(1) = (-1)^n.$$

Statement-2 is false as area of triangle formed by cube roots of unity is $3\sqrt{3}/4$ square units.

⊛ **Example 98: Statement-1:** Coefficient of x^n in

$$\sum_{j=0}^{n} (-1)^j\,(2x + 3)^{n-j}\,(5 - 2x)^j$$

is $(n + 1)2^n$

Statement-2: $\sum_{j=0}^{n} a^{n-j}b^j = \dfrac{a^{n+1} - b^{n+1}}{a - b}$, $a \ne b$,

and coefficient of x^m in $(\alpha x + \beta)^{m+1}$ is $(m + 1)\alpha^m\beta$

Ans. (a)

◎ **Solution:** $\sum_{j=0}^{n} a^{n-j}b^j = a^n + a^{n-1}b + a^{n-2}b^2 + \ldots + ab^{n-1} + b^n$

$$= \frac{a^n\left[1 - (b/a)^{n+1}\right]}{1 - b/a} \qquad \text{[sum of a G.P.]}$$

$$= \frac{a^{n+1} - b^{n+1}}{a - b}$$

Also, coefficient of x^m in $(\alpha x + \beta)^{m+1}$ is $(m + 1)\alpha^m\beta$

$\therefore$ Statement-2 is true.

Next, $\sum_{j=0}^{n} (-1)^j\,(2x + 3)^{n-j}\,(5 - 2x)^j$

$$= \sum_{j=0}^{n} (2x + 3)^{n-j}\,(2x - 5)^j$$

$$= \frac{(2x + 3)^{n+1} - (2x - 5)^{n+1}}{(2x + 3) - (2x - 5)}$$

$$= \frac{1}{8}\,[(2x + 3)^{n+1} - (2x - 5)^{n+1}]$$

Coefficient of x^n is

$$\frac{1}{8}[{}^{n+1}C_n\,2^n\,(3) - {}^{n+1}C_n\,2^n\,(-5)]$$

$$= (n + 1)\,2^n$$

$\therefore$ Statement-1 is true and Statement-2 is correct explanation for it.

LEVEL 2

Straight Objective Type Questions

⊛ **Example 99:** The number of rational terms in the expansion of $\left(1 + \sqrt{2} + \sqrt[3]{5}\right)^6$ is

(a) 7 (b) 11

(c) 12 (d) 12

Ans. (a)

◎ **Solution:** A term in the expansion of $\left(1 + \sqrt{2} + \sqrt[3]{5}\right)^6$ is of the form

$$\frac{6!}{r!\,s!\,(6 - r - s)!}(1)^{6-r-s}\left(\sqrt{2}\right)^r\left(\sqrt[3]{5}\right)^s$$

$$= \frac{6!}{r!\,s!\,(6 - r - s)!}\left(2^{r/2}\right)(5^{s/3})$$

This term will be rational if $2^{r/2}$ and $5^{s/3}$ are both rational numbers. This is possible if and only if r is a multiple of 2 and s is a multiple of 3. Possible values of r are 0, 2, 4 and 6 whereas the possible values of s are 0, 3 and 6. Also note that $0 \le r + s \le 6$.

For $r = 0$, s can take value 0, 3 or 6.

For $r = 2$, s can take value 0 or 3.

For $r = 4$, s can take value 0.
For $r = 6$, s can take value 0.
Thus, the number of rational terms is $3 + 2 + 1 + 1 = 7$

Example 100: The sum of $(n + 1)$ terms of the series $\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \frac{C_3}{5} + \dots$ is

(a) $\frac{1}{n+1}$ (b) $\frac{1}{n+2}$

(c) $\frac{1}{n(n+1)}$ (d) $\frac{1}{(n+1)(n+2)}$

Ans. (d)

Solution: We have

$$(1 - x)^n = C_0 - C_1 x + C_2 x^2 - C_3 x^3 + \dots + (-1)^n C_n x^n$$

$$\Rightarrow \quad x(1 - x)^n = C_0 x - C_1 x^2 + C_2 x^3 - C_3 x^4 + \dots + (-1)^n C_n x^{n+1}$$

$$\Rightarrow \int_0^1 x(1 - x)^n \, dx = \int_0^1 [C_0 x - C_1 x^2 + C_2 x^3 - C_3 x^4 + \dots + (-1)^n C_n x^{n+1}] dx \qquad (1)$$

L.H.S. of (1)

$$= \int_0^1 x(1-x)^n \, dx = \int_0^1 (1-x)(1-(1-x))^n \, dx$$

$$\left[\text{using } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 (1-x)x^n dx = \left(\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right)\Bigg]_0^1$$

$$= \frac{1}{(n+1)(n+2)}$$

R.H.S. of (1)

$$= \left(C_0 \frac{x^2}{2} - C_1 \frac{x^3}{3} + C_2 \frac{x^4}{4} - \dots + \frac{(-1)^n C_n}{n+1} x^{n+1}\right)\Bigg]_0^1$$

$$= \frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \dots + \frac{(-1)^n C_n}{n+1}$$

Thus,

$$\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \dots + (-1)^n \frac{C_n}{n+1} = \frac{1}{(n+1)(n+2)}$$

Example 101: Two values of x for which the sixth term of the expansion of $E = \left(3^{\log_3 \sqrt{9^{|x-2|}}} + 7^{(1/5)\log_7[(4).3^{|x-2|}-9]}\right)^7$ is 567, is

(a) 3, 4 (b) 3, 1

(c) 2, 7 (d) none of these

Ans. (b)

Solution: Put $y = 3^{\log_3 \sqrt{9^{|x-2|}}} \Rightarrow \log_3 y = \log_3 \sqrt{9^{|x-2|}}$

$$\Rightarrow \qquad y = \sqrt{9^{|x-2|}} = 3^{|x-2|}$$

Next, put $z = \left(7^{(1/5)\log_7[(4).3^{|x-2|}-9]}\right)$

$$\Rightarrow \qquad \log_7 z = \frac{1}{5} \log_7 [(4)3^{|x-2|} - 9] = \log_7 [(4)\, 3^{|x-2|} - 9]^{1/5}$$

$$\Rightarrow \qquad z = [(4)3^{|x-2|} - 9]^{1/5}$$

Now, $E = (y + z)^7$ and the 6th term in its expansion is given by

$$T_6 = {}^7C_5\, y^{7-5} z^5 = 21\left(3^{|x-2|}\right)^2 \left\{(4)\, 3^{|x-2|} - 9\right\}$$

$$\Rightarrow \qquad 567 = 21\{3^{2|x-2|}\}\, \{(4)\, 3^{|x-2|} - 9\}$$

$$\Rightarrow \qquad 27 = \{(4)\, 3^{3|x-2|} - (9)\, (3^{2|x-2|})\}$$

$$\Rightarrow \qquad 27 = (4)\, 3^{3|x-2|} - (9)\, 3^{2|x-2|}$$

$\Rightarrow \quad 4u^3 - 9u^2 - 27 = 0$ where $u = 3^{|x-2|}$

Note that $u = 3$ satisfies this equation*.

$$\therefore \qquad 3^{|x-2|} = 3 \qquad \Rightarrow \qquad |x - 2| = 1 \Rightarrow x - 2 = \pm 1$$

$$x = 2 \pm 1 = 3 \text{ or } 1.$$

Example 102: Let a_n denote the term independent of x in the expansion of $\left[x + \frac{\sin(1/n)}{x^2}\right]^{3n}$, then $\lim_{x\to\infty} \frac{(a_n)n!}{{}^{3n}P_n}$ equals

(a) 0 (b) 1

(c) e (d) $e/\sqrt{3}$

Ans. (a)

Solution: $T_{r+1} = {}^{3n}C_r x^{3n-r} \left[\sin\left(\frac{1}{n}\right)\right]^r x^{-2r}$

For this term to be independent of x, set

$3n - r - 2r = 0 \quad \Rightarrow \quad r = n$

$$\therefore \qquad a_n = {}^{3n}C_n \sin^n\left(\frac{1}{n}\right)$$

Now, $\frac{n!a_n}{{}^{3n}P_n} = \sin^n\left(\frac{1}{n}\right) \to 0$ as $n \to \infty$

Example 103: The largest term in the expansion of $(3 + 2x)^{50}$, where $x = \frac{1}{5}$, is

(a) ${}^{50}C_6 3^{44}\left(\frac{2}{5}\right)^6$ (b) ${}^{50}C_7 3^{43}\left(\frac{2}{5}\right)^7$

(c) ${}^{50}C_{43} 3^7\left(\frac{2}{5}\right)^{43}$ (d) ${}^{50}C_{44} 3^6\left(\frac{2}{5}\right)^{44}$

Ans. (a)

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* From available answer, possible values of u are $3^{|3-2|} = 3$ for the choices (a) and (b) and $3^{|2-2|} = 3^0 = 1$ for the choice (c).

Solution: Let $k = \dfrac{(50+1)(2x)}{3+2x} = \dfrac{(51)(2/5)}{17/5} = 6$.

As k is an integer, there are two greatest term *viz.* 6th and 7th terms

We have

$$T_7 = {}^{50}C_6 3^{44} \left(\frac{2}{5}\right)^6.$$

Remark

In fact $T_6 = T_7$.

Example 104: If n is odd, then sum of the series $C_0^{\,2} - C_1^{\,2} + C_2^{\,2} - C_2^{\,3} + \ldots + (-1)^n C_n^{\,2}$ is

(a) 0 (b) 1

(c) ${}^{2n}C_n$ (d) $-({}^{2n}C_n)$

Ans. (a)

Solution: Let $n = 2m - 1$. Using $C_r = C_{n-r}$, we get sum of the series in 0.

Example 105: If n is even, then sum of the series $C_0^{\,2} - C_1^{\,2} + C_2^{\,2} - C_3^{\,2} + \ldots + (-1)^n C_n^{\,2}$ is

(a) ${}^nC_{\frac{n}{2}} (-1)^{n/2}$ (b) ${}^nC_{\frac{n}{2}}$

(c) ${}^nC_{n-1}$ (d) $(-1)^{n/2}\ {}^nC_2$

Ans. (a)

Solution: Let $n = 2m$. We have

$$\sum_{r=0}^{2m} (-1)^r C_r^2 = \text{Coefficient of the constant term in}$$

$$(1+x)^{2m}\left(1 - \frac{1}{x}\right)^{2m}$$

$$\begin{aligned} &= \text{Coefficient of } x^{2m} \text{ in } (1+x)^{2m}(x-1)^{2m} \\ &= \text{Coefficient of } x^{2m} \text{ in } (1-x^2)^{2m} \\ &= {}^{2m}C_m(-1)^m = {}^nC_{\frac{n}{2}}(-1)^{n/2} \end{aligned}$$

Example 106: If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n$, then value of $C_0^{\,2} + 2C_1^{\,2} + 3C_2^{\,2} + \ldots + (n+1)C_n^2$ is

(a) $(2n+1)({}^{2n}C_n)$ (b) $(2n-1)({}^{2n}C_n)$

(c) $\left(\frac{n}{2}+1\right)({}^{2n}C_n)$ (d) $\left(1+\frac{n}{2}\right)({}^{2n-1}C_n)$

Ans. (c)

Solution: Let

$$S = C_0^{\,2} + 2C_1^{\,2} + 3C_2^{\,2} + \ldots + (n+1)C_n^{\,2} \qquad (1)$$

Using $C_r = C_{n-r}$, we rewrite (1) as

$$S = (n+1)C_0^{\,2} + nC_1^{\,2} + \ldots + C_n^{\,2} \qquad (2)$$

Adding (1) and (2), we get

$$2S = (n+2)\,[C_0^{\,2} + C_1^{\,2} + \ldots + C_n^{\,2}] = (n+2)\,({}^{2n}C_n)$$

$$\Rightarrow \quad S = \left(\frac{n}{2}+1\right)({}^{2n}C_n)$$

Example 107: If C_r stands for ${}^nC_r$, the sum of the series

$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}\,[C_0^{\,2} - 2C_1^{\,2} + 3C_2^{\,2} - \ldots + (-1)^n(n+1)C_n^{\,2}]$$

where n is an even positive integer, is equal to

(a) $(-1)^{n/2}(n+2)$ (b) $(-1)^n(n+1)$

(c) $(-1)^{n/2}(n+1)$ (d) none of these

Ans. (a)

Solution: Let $n = 2m$, and

$$S = C_0^{\,2} - 2C_1^{\,2} + 3C_2^{\,2} - \ldots + (-1)^{2m}(2m+1)C_{2m}^2 \qquad (1)$$

Using $C_r = C_{n-r}$, we rewrite (1) as

$$S = (2m+1)C_0^{\,2} - (2m)C_1^{\,2} + (2m-1)C_2^{\,2} - \ldots + C_{2m}^2 \qquad (2)$$

we get

$$\begin{aligned} 2S &= (2m+2)\,[C_0^{\,2} - C_1^{\,2} + C_2^{\,2} - \ldots + C_{2m}^2] \\ &= (2m+2)\,({}^{2m}C_m)\,(-1)^m \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad S &= (m+1)\,({}^{2m}C_m)\,(-1)^m \\ &= (-1)^{n/2}\left(\frac{n}{2}+1\right)\frac{n!}{\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!} \end{aligned}$$

$$\Rightarrow \quad \frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}\,S = (-1)^{n/2}(n+2)$$

Example 108: Sum of the coefficients of integral powers of x in the expansion of $\left(1 - 2\sqrt{x}\right)^{50}$ is

(a) $\frac{1}{2}(3^{50}+1)$ (b) $\frac{1}{2}(3^{50})$

(c) $\frac{1}{2}(2^{50}-2)$ (d) $\frac{1}{2}(2^{50}+2)$

Ans. (a)

Solution: The coefficients of integral powers of x are

$${}^{50}C_0,\ {}^{50}C_2\, 2^2,\ {}^{50}C_4\, 2^4, \ldots, {}^{50}C_{50}\, 2^{50}$$

Let

$$\begin{aligned} S &= {}^{50}C_0 + {}^{50}C_2\, 2^2 + {}^{50}C_4\, 2^4 + \ldots + {}^{50}C_{50}\, 2^{50} \\ &= \frac{1}{2}\left[(1+2)^{50} + (1-2)^{50}\right] = \frac{1}{2}\left[3^{50}+1\right] \end{aligned}$$

Example 109: Given that the 4th term in the expansion of $\left(2 + \frac{3x}{8}\right)^{10}$ has the maximum numerical value, then x lies in the interval

(a) $\left(-\frac{64}{21}, -2\right) \cup \left(2, \frac{64}{21}\right)$

(b) $\left(-\frac{60}{23},-2\right)\cup\left(2,\frac{64}{21}\right)$

(c) $\left(-\frac{64}{21},-2\right)$

(d) none of these

Ans. (a)

◎ **Solution:** Since T_4 is numerically the greatest term,

$$|T_3| < |T_4| \text{ and } |T_5| < |T_4|$$

$$\Rightarrow \quad \left|\frac{T_3}{T_4}\right| < 1 \text{ and } \left|\frac{T_5}{T_4}\right| < 1.$$

But $$\frac{T_3}{T_4} = \frac{{}^{10}C_2(2^8)(3x/8)^2}{{}^{10}C_3\left(2^7\right)(3x/8)^3} = \frac{2}{x}$$

and $$\frac{T_5}{T_4} = \frac{{}^{10}C_4\left(2^6\right)(3x/8)^4}{{}^{10}C_3\left(2^7\right)(3x/8)^3} = \frac{21x}{64}$$

Now, $$\left|\frac{T_3}{T_4}\right| < 1 \quad \Rightarrow \quad \left|\frac{2}{x}\right| < 1 \quad \Rightarrow \quad x < -2 \text{ or } x > 2.$$

Next, $$\left|\frac{T_5}{T_4}\right| < 1 \quad \Rightarrow \quad \left|\frac{21x}{64}\right| < 1 \quad \Rightarrow \quad -\frac{64}{21} < x < \frac{64}{21}$$

Thus $x \in (-\infty,-2)\cup(2,\infty)$ and $x \in \left(-\frac{64}{21},\frac{64}{21}\right)$

Hence, $x \in \left(-\frac{64}{21},-2\right)\cup\left(2,\frac{64}{21}\right)$.

◉ **Example 110:** Assuming x to be so small that x^3 and higher powers of x can be neglected, then value of

$$E = \left(1-\frac{3}{2}x\right)^5 (2+3x)^6 \text{, is}$$

(a) $64 + 96x + 720\,x^2$ (b) $65 - 97x + 721x^2$

(c) $64 - 96x + 720x^2$ (d) $64 + 96x - 720x^2$

Ans. (d)

◎ **Solution:** We have

$$\left(1-\frac{3}{2}x\right)^5 (2+3x)^6 = 2^6\left(1-\frac{3}{2}x\right)^5\left(1+\frac{3}{2}x\right)^6$$

$$= 2^6\left(1+\frac{3}{2}x\right)\left[\left(1-\frac{3}{2}x\right)\left(1+\frac{3}{2}x\right)\right]^5$$

$$= 2^6\left(1+\frac{3}{2}x\right)\left(1-\frac{9}{4}x^2\right)^5$$

$$= 2^6\left(1+\frac{3}{2}x\right)\left(1-\frac{45}{4}x^2\right)$$

$$= 2^6\left(1+\frac{3}{2}x-\frac{45}{4}x^2\right) = 64 + 96x - 720x^2$$

◉ **Example 111:** If $(1 + x + x^2)^8 = a_0 + a_1x + a_2x^2 + \cdots + a_{16}x^{16}$, then a_5 equals

(a) 504 (b) 506

(c) 508 (d) 502

◎ **Solution:** We have

$$(1 + x + x^2)^8 = \sum_{\substack{p,q\ge 0\\ p+q\le 8}} \frac{8!}{p!q!(8-p-q)!}x^p(x^2)^q$$

For coefficient of x^5, we set $p + 2q = 5$

This is possible if $p = 5$, $q = 0$, $p = 3$, $q = 1$, $p = 1$, $q = 2$

Thus, coefficient of x^5 is

$$\frac{8!}{5!3!} + \frac{8!}{3!4!} + \frac{8!}{2!5!} = 504$$

◉ **Example 112:** Coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is

(a) 1051 (b) 1106

(c) 1113 (d) 1120

Ans. (c)

◎ **Solution:** We have

$$\begin{aligned}(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12} &= (1 + x^2)^4(1 + 7x^3 + 31x^6\\ &\quad + 35x^9 + \ldots)(1 + 12x^4 + 66x^8 + \ldots)\end{aligned}$$

As expansion of $(1 + x^2)^4$ contains only even power, while multiplying last two expansions, we shall only consider odd powers.

$$\begin{aligned}\therefore (1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12} &= (1 + 4x^2 + 6x^4 + 4x^6 + x^8)(7x^3 + 84x^7\\ &\quad + 35x^9 + 462x^{11} + \ldots)\end{aligned}$$

Thus, coefficient of x^{11} is

$$(1)(462) + 4(35) + (6)(84) + (1)(7) = 1113$$

◉ **Example 113:** If the coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5:10:14, then n is equal to

(a) 5 (b) 6

(c) 7 (d) 8

◎ **Solution:** Let three consecutive coefficients of $(1 + x)^{n+5}$ be ${}^{n+5}C_r$, ${}^{n+5}C_{r+1}$, ${}^{n+5}C_{r+2}$

We are given

$$\frac{{}^{n+5}C_r}{{}^{n+5}C_{r+1}} = \frac{5}{10} \text{ and } \frac{{}^{n+5}C_{r+1}}{{}^{n+5}C_{r+2}} = \frac{10}{14}$$

$$\Rightarrow \quad \frac{r+1}{n+5-r} = \frac{1}{2} \text{ and } \frac{r+2}{n+5-r-1} = \frac{5}{7}$$

$$\Rightarrow \quad 3r = n + 3 \text{ and } 12r = 5n + 6$$

$$\therefore \quad 4(n + 3) = 5n + 6 \Rightarrow n = 6$$

◉ **Example 114:** Sum of the series

$$\sum_{k=0}^{n} {}^nC_k(-1)^k\frac{1}{a_k}$$

where $a_k = \sum_{i=0}^{k} {}^kC_i\, b_i$

where $b_i = \sum_{j=0}^{i} {}^iC_j \left(\frac{-2}{3}\right)^j$

is

(a) $\frac{1}{2^n}$ (b) $\frac{1}{3^n}$

(c) $\frac{1}{4^n}$ (d) $\left(\frac{3}{4}\right)^n$

Ans. (c)

Solution:

$$b_i = \sum_{j=0}^{i} {}^iC_j \left(\frac{-2}{3}\right)^j = \left(1-\frac{2}{3}\right)^i = \left(\frac{1}{3}\right)^i$$

$$\therefore \quad a_k = \sum_{i=0}^{k} {}^kC_i \left(\frac{1}{3}\right)^i = \left(1+\frac{1}{3}\right)^k = \left(\frac{4}{3}\right)^k$$

Thus,
$$\sum_{k=0}^{n} {}^nC_k (-1)^k \left(\frac{3}{4}\right)^k = \left(1-\frac{3}{4}\right)^n = \frac{1}{4^n}$$

Example 115: Suppose m and n are positive integers and let

$$S = \sum_{k=0}^{n} (-1)^k \frac{1}{k+m+1} ({}^nC_k)$$

and
$$T = \sum_{k=0}^{m} (-1)^k \frac{1}{k+n+1} ({}^mC_k),$$

then $S - T$ is equal to

(a) 0 (b) $n^m - m^n$

(c) $(n + 1)^m - (m + 1)^n$ (d) $(1 - n)^m - (1 - m)^n$

Ans. (a)

Solution:

$$\begin{aligned} S &= \sum_{k=0}^{n} (-1)^k ({}^nC_k)\left(\int_0^1 x^{k+m} dx\right) \\ &= \int_0^1 \left(\sum_{k=0}^{n} (-1)^k ({}^nC_k) x^{k+m}\right) dx \\ &= \int_0^1 (1-x)^n x^m dx \\ &= \int_0^1 (1-(1-x))^n (1-x)^m dx \\ &= \int_0^1 x^n (1-x)^m dx \\ &= \int_0^1 x^n \left(\sum_{k=0}^{m} (-1)^k ({}^mC_k) x^k\right) dx \\ &= \sum_{k=0}^{m} (-1)^k ({}^mC_k) \int_0^1 x^{n+k} dx \\ &= \sum_{k=0}^{m} (-1)^k ({}^mC_k) \frac{1}{n+k+1} = T \end{aligned}$$

Thus, $S - T = 0$

Example 116: Coefficient of x^m in the expansion of

$$S = (1 + x)^{2m} + x(1 + x)^{2m-1} + x^2(1 + x)^{2m-2} + \ldots + x^{2m}$$

is

(a) ${}^{2m}C_m + {}^{2m-1}C_{m-1}$ (b) ${}^{2m+1}C_m$

(c) 0 (d) $2m^2 + 2m + 1$

Solution: Note S is a G.P. with first term $(1 + x)^{2m}$ and common ratio $\frac{x}{1+x}$ and number of terms $2m + 1$.

Thus,

$$S = \frac{(1+x)^{2m}\left[1-\left(\frac{x}{1+x}\right)^{2m+1}\right]}{1-x/(1+x)} = (1 + x)^{2m+1} - x^{2m+1}$$

$\therefore$ Coefficient of x^m in S is ${}^{2m+1}C_m$.

Example 117: Sum of the first 20 terms of the series

$$\frac{1}{(2)(4)} + \frac{(1)(3)}{(2)(4)(6)} + \frac{(1)(3)(5)}{(2)(4)(6)(8)} + \cdots$$

is

(a) $\frac{1}{2} - \frac{1}{2^{40}}({}^{40}C_{20})$ (b) $\frac{1}{2} - \frac{1}{2^{41}}({}^{42}C_{21})$

(c) $\frac{1}{2} - \frac{1}{2^{42}}({}^{42}C_{21})$ (d) $\frac{1}{2} - \frac{1}{2^{43}}({}^{40}C_{20})$

Ans. (c)

Solution: Let a_k denote the kth term of the series, then

$$\begin{aligned} a_k &= \frac{(1)(3)(5)\cdots(2k-1)}{(2)(4)(6)\cdots(2k)(2k+2)} \\ &= \frac{(1)(3)(5)\cdots(2k-1)(2k+2-2k-1)}{(2)(4)(6)\cdots(2k)(2k+2)} \\ &= b_k - b_{k+1} \end{aligned}$$

where

$$\begin{aligned} b_k &= \frac{(1)(3)(5)\cdots(2k-1)}{(2)(4)(6)\cdots(2k)} \\ &= \frac{(1)(2)(3)(4)\cdots(2k-1)(2k)}{[(2)(4)(6)\cdots(2k)]^2} \\ &= \frac{1}{2^{2k}}({}^{2k}C_k) \end{aligned}$$

Thus,

$$\sum_{k=1}^{20} a_k = \sum_{k=1}^{20} (b_k - b_{k+1}) = b_1 - b_{21}$$

$$= \frac{1}{2} - \frac{1}{2^{42}}\ ({}^{42}C_{21})$$

Example 118: The term independent of x in the expansion of

$$\left(1 - \frac{1}{2x} + 3x^7\right)\left(x^2 + \frac{1}{3x}\right)^{10}$$

is

(a) $10/3^8$
(b) $-2/3$
(c) $10/3^9$
(d) $11/6$

Ans. (a)

Solution: $(r+1)$th term in the expansion of $\left(x^2 + \frac{1}{3x}\right)^{10}$ is

$$t_{r+1} = {}^{10}\mathrm{C}_r\,(x^2)^{10-r}\left(\frac{1}{3x}\right)^r$$

$$= {}^{10}C_r\left(\frac{1}{3}\right)^r x^{20-3r}$$

We require terms containing x^0, x and x^{-7} in the expansion of $\left(x^2 + \frac{1}{3x}\right)^{10}$.

For x^0, set $20 - 3r = 0$. No value of r.
For x^1, set $20 - 3r = 1$. No value of r.
For x^{-7}, set $20 - 3r = -7 \Rightarrow r = 9$.
Thus, required coefficient is

$$3\left[{}^{10}C_9\left(\frac{1}{3}\right)^9\right] = \frac{10}{3^8}$$

Example 119: Let $P(x)$ be a polynomial with real coefficients such that $\int_0^1 x^m\ P(1-x)dx = 0\ \forall\ m \in \mathbf{N} \cup \{0\}$, then

(a) $P(x) = x^n\,(1-x)^n$ for some $n \in \mathbf{N}$
(b) $P(x) = (1-x)^{2n}$ for some $n \in \mathbf{N}$
(c) $P(x) = 1 - x^m\,(1-x)^n$ for some $m, n \in \mathbf{N}$
(d) $P(x) = 0$

Ans. (d)

Solution: $0 = \int_0^1 x^m P(1-x)dx = \int_0^1 (1-x)^m\ P(x)$

Let $\quad P(x) = \sum_{k=0}^{n} a_k x^k$ where $a_k \in \mathbf{R}$

$$= \sum_{k=0}^{n} a_k\ [1 - (1-x)]^k$$

$$= \sum_{k=0}^{n} a_k\left(\sum_{j=0}^{k} (-1)^k({}^kC_j)(1-x)^j\right)$$

$$= \sum_{k=0}^{n} b_k(1-x)^k$$

where $\quad b_k = \sum_{j=k}^{n} (-1)^k({}^jC_k)a_j$

Now $\int_0^1 (P(x))^2 dx = \int_0^1 P(x)\left(\sum_{k=0}^{n} b_k(1-x)^k\right)dx$

$$= \sum_{k=0}^{n} b_k\int_0^1 (1-x)^k P(x)dx = 0$$

As $P(x)$ is a polynomial,

$$P(x) \equiv 0\ \forall\ x \in [0, 1]$$
$$\Rightarrow \quad P(x) \equiv 0$$

Example 120: Suppose $[x]$ denote the greatest integer $\le x$, and $n \in \mathbf{N}$, then

$$\lim_{n\to\infty}\frac{[{}^nC_o x^2] + [{}^nC_1 x^2] + \cdots + [{}^nC_n x^2]}{2^n - 2}$$

is equal to

(a) $\frac{1}{2}\ x^2$
(b) x^2
(c) $2x^2$
(d) $4x^2$

Ans. (b)

Solution: We know $x - 1 < [x] \le x\ \forall\ x \in \mathbf{R}$, therefore ${}^nC_k x^2 - 1 < [{}^nC_k x^2] \le {}^nC_k x^2$

$$\Rightarrow \quad \sum_{k=0}^{n}({}^nC_k x^2 - 1) < \sum_{k=0}^{n}[{}^nC_k x^2] \le \sum_{k=0}^{n}{}^nC_k x^2$$

$$\Rightarrow \quad \frac{x^2(2^n) - (n+1)}{2^n - 2} < \frac{1}{2^n - 2}\sum_{k=0}^{n}[{}^nC_k x^2]$$

$$\le \frac{x^2(2^n)}{2^n - 2}$$

Taking limit as $n \to \infty$, and using sandwich theorem, we get

$$\lim_{n\to\infty}\frac{1}{2^n - 2}\sum_{k=0}^{n}\left[{}^nC_k x^2\right] = x^2$$

EXERCISE

Concept-based Straight Objective Type Questions

1. Suppose $k \in \mathbf{R}$. If the coefficient of x in the expansion of $\left(x^2 + \frac{k}{x}\right)^5$ is 270, then k is equal to:
   (a) 3 (b) – 3
   (c) 9 (d) – 9

2. Suppose $p \in \mathbf{R}$. If the fourth term in the expansion of $\left(px + \frac{1}{x}\right)^n$ is $\frac{5}{2}$, then (n, p) is equal to:
   (a) (5, 1/2) (b) (6, 1/2)
   (c) (8, 1/2) (d) (10, 1/2)

3. If the coefficient of $(3r + 4)$th term and $(2r - 2)$th term in the expansion of $(1 + x)^{20}$ are equal then r is equal to:
   (a) 4 (b) 5
   (c) 7 (d) 8

4. If sum of the coefficient in the binomial expansion of $(1 + 2x)^n$ is 6561, then coefficient of the fourth term is:
   (a) 442 (b) 446
   (c) 448 (d) 472

5. Coefficient of x^{10} in the expansion of $(1 + x^2)^{10}\left(x^2 + 2 + \frac{1}{x^2}\right)^5$ is
   (a) ${}^{10}C_5$ (b) ${}^{15}C_5$
   (c) 0 (d) 2

6. If $(1 + ax)^n = 1 + 10x + 40x^2 + \ldots$, then value of $\frac{a+n}{a-n}$ is
   (a) 5/7 (b) 7/3
   (c) –5/7 (d) – 7/3

7. The ratio of the coefficient of x^{10} in $(1 + x^2)^{10}$ and the term independent of x in $\left(x + \frac{2}{x}\right)^{10}$ is
   (a) 32 : 1 (b) 1 : 32
   (c) 1 : 64 (d) 64 : 1

8. If coefficient of 9th, 10th and 11th terms of $(1 + x)^n$ are in A.P., then value of n can be
   (a) 15 (b) 32
   (c) 23 (d) 17

9. $(p + 2)$th term from the end in the binomial expansion of $\left(x^2 - \frac{2}{x^2}\right)^{2n+1}$ is
   (a) ${}^{2n+1}C_{2n-p}\,(-2)^{2n-p}\,x^{2p+1-2n}$
   (b) ${}^{2n+1}C_{2n-p}\,(-2)^{2n-p}\,x^{2n-2p}$
   (c) ${}^{2n+1}C_{2n-p}\,(-2)^{2n-p}\,x^{2n-2p+1}$
   (d) ${}^{2n+1}C_{2n-p}\,(-2)^{2n-p}\,x^{2p-1+2n}$

10. Let $C_r = {}^{15}C_r$, $0 \le r \le 15$. Sum of the series $S = \sum_{r=1}^{15} r\left(\frac{C_r}{C_{r-1}}\right)$ is
   (a) 40 (b) 60
   (c) 100 (d) 120

LEVEL 1

Straight Objective Type Questions

11. If the middle term of $\left(x^2 + \frac{1}{x}\right)^n$ is $924\,x^6$, then value of n is
   (a) 8 (b) 10
   (c) 12 (d) 20

12. The greatest term in the expansion of $\sqrt{5}\left(1 + \frac{1}{\sqrt{5}}\right)^{20}$ is:
   (a) ${}^{20}C_5\left(\frac{1}{25}\right)$ (b) ${}^{20}C_5\left(\frac{1}{5}\right)$
   (c) ${}^{20}C_5$ (d) ${}^{20}C_5\,(5)$

13. Let $S(\theta)$ denote the sum of coefficients in the expansion of $(\sqrt{2} - x\sin\theta + x^2\cos\theta)^{2n}$. Maximum value of $S(\theta)$ is
   (a) 4^n (b) 8^n
   (c) 2^n (d) 1

14. Suppose
   $(x + a)^n = T_0 + T_1 + T_2 + \ldots + T_n$
   Then $(T_0 - T_2 + T_4 - \ldots)^2 + (T_1 - T_3 + T_5 - \ldots)^2$ is equal to
   (a) $(x^2 + a^2)^n$ (b) $(x^2 - a^2)^n$
   (c) $(x + a)^{2n}$ (d) $(x - a)^{2n}$

15. For $0 \le r < 2n$, $({}^{2n+r}C_n)\,({}^{2n-r}C_n)$ cannot exceed
    (a) ${}^{4n}C_n$ (b) ${}^{4n}C_{2n}$
    (c) ${}^{6n}C_{3n}$ (d) none of these.
16. If sum of the coefficients in the expansion $(2x^2 - 3cx + c^2)^{17}$ is zero, then c is equal to
    (a) 2, 3 (b) 1, 2
    (c) 1, 3 (d) 2, – 1
17. Coefficient of constant term in the expansion of $(x^3 + 5(3)^{-\log\sqrt{3}\;\sqrt{x^3}})^4$ is
    (a) 125 (b) $\log_{\sqrt{3}} 5$
    (c) $\log_3 5$ (d) 150
18. If the number of terms in the expansion of $(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)^{20}$ is m, then unit's place of 2^m is
    (a) 2 (b) 8
    (c) 6 (d) 4
19. Let P_n denote the product of all the coefficients in the expansion of $(1 + x)^n$. If $(20)!\,P_{n+1} = 21^{20}\,P_n$, then n is equal to
    (a) 21 (b) 20
    (c) 19 (d) 18
20. Coefficient of the term independent of x in the expansion of $(1 + x)^{2n}\left(\frac{x}{1-x}\right)^{-2n}$ is:
    (a) ${}^{2n}C_n$ (b) $(-1)^n\,{}^{2n}C_n$
    (c) ${}^{n^2-n}C_n$ (d) 0
21. Coefficient of x^4 in the expansion of $\left(\frac{x}{2} - \frac{3}{x^2}\right)^{10}$ is
    (a) $\frac{504}{256}$ (b) $\frac{405}{256}$
    (c) $\frac{459}{512}$ (d) $-\frac{135}{256}$
22. If α and β are the coefficients of x^r and x^{n-r} respectively in the expansion of $(1 + x)^n$, then
    (a) $\alpha = \beta$
    (b) $\alpha + \beta = n$
    (c) $\alpha = k\beta$ for some constant k
    (d) $\alpha + \beta = 2^n$
23. If the coefficients of $(2r + 4)$th and $(r - 2)$th terms in the expansion of $(1 + x)^{18}$ are equal, then value of r is
    (a) 5 (b) 6
    (c) 7 (d) 9
24. The total number of terms in the expansion of $(x + a)^{200} + (x - a)^{200}$ after simplification is
    (a) 101 (b) 102
    (c) 201 (d) 202
25. The coefficient of x^5 in the expansion of $(1 + x^2)^5\,(1 + x)^4$ is
    (a) 30 (b) 40
    (c) 50 (d) 60
26. If $n > 2$, $r > 1$ and the coefficients of $(3r)$th and $(r + 2)$th terms in the expansion of $(1 + x)^{2n}$ are equal, then
    (a) $n = 2r$ (b) $n = 3r$
    (c) $n = 2r + 1$ (d) $n = 2r + 2$
27. If $(1 + px)^n = 1 + 24x + 2642x^2 + \ldots$ then
    (a) $p = 2, n = 6$ (b) $p = 2, n = 12$
    (c) $p = 3, n = 6$ (d) none of these
28. Sum of the series $\sum_{r=0}^{10} (-1)^r \left({}^{10}C_r\right)\left[\frac{1}{2^r} + \frac{3^r}{2^{2r}}\right]$ is
    (a) $\frac{2^{10}+1}{2^{20}}$ (b) $\frac{2^{10}-1}{2^{20}}$
    (c) $\frac{2^{20}-1}{2^{20}}$ (d) none of these
29. If m and n are positive integers, then value of ${}^mC_0\,{}^nC_k + {}^mC_1\,{}^nC_{k-1} + \ldots + {}^mC_k\,{}^nC_0$
    (a) ${}^{m+n}C_k$ (b) ${}^{m+n}C_{m+k}$
    (c) ${}^{m+n}C_{n+k}$ (d) none of these
30. The expression $\left(x + \sqrt{x^3 - 1}\right)^6 + \left(x - \sqrt{x^3 - 1}\right)^6$ is a polynomial of degree
    (a) 6 (b) 7
    (c) 9 (d) 10
31. The number of distinct terms in the expansion of $(1 + 3x + 3x^2 + x^3)^7$ is
    (a) 18 (b) 19
    (c) 28 (d) 22
32. If $x > 0$, then the number of positive terms in the expansion of $(1 + ix)^{4n}$ is
    (a) $n + 1$ (b) $n - 1$
    (c) n (d) $2n$
33. The number of terms whose values depend on x in the expansion of $\left(x^2 - 2 + \frac{1}{x^2}\right)^n$ is
    (a) $2n$ (b) $2n + 1$
    (c) $2n - 1$ (d) 0
34. If the third term in the expansion of $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$, $x > 1$, is 1000, then value of x is
    (a) $\sqrt{10}$ (b) 10
    (c) 100 (d) $10\sqrt{10}$

35. Let $R = \left(\sqrt{2}+1\right)^{2n+1}$, $n \in \mathbf{N}$, and $f = R - [R]$, where [ ] denote the greatest integer function, Rf is equal to
(a) 1
(b) 2^{2n+1}
(c) $2^{2n} - 1$
(d) none of these

36. The coefficient of x^{60} in $(1 + x)^{51} (1 - x + x^2)^{50}$ is
(a) ${}^{50}C_{20}$
(b) $-({}^{50}C_{20})$
(c) ${}^{51}C_{20}$
(d) $-({}^{51}C_{20})$

37. If $a = 99^{50} + 100^{50}$ and $b = 101^{50}$, then
(a) $a < b$
(b) $a = b$
(c) $a > b$
(d) $a - b = 100^{49}$

38. If $(1 + x - 2x^2)^6 = \sum_{r=0}^{12} A_r x^r$, then value of $A_2 + A_4 + A_6 + \ldots + A_{12}$, is
(a) 30
(b) 31
(c) 32
(d) none of these

39. The sum of the coefficients in the expansion of $(1 + x - 3x^2)^{4321}$ is
(a) 0
(b) 1
(c) –1
(d) 2^{4320}

40. The greatest integer contained in $\left(\sqrt{3}+1\right)^6$ is
(a) 208
(b) 416
(c) 415
(d) 207

41. The number of terms in the expansion of $(x^2 + 6x + 9)^{30}$ is
(a) 31
(b) 61
(c) 91
(d) none of these

42. The expansion of $(x + y + z)^n$ is given by
(a) $\sum \frac{n!}{p!q!r!} x^p y^q z^r$ where $p, q, r \geq 0$, $p + q + r = n$
(b) $\sum \frac{n!}{p!q!r!} x^p y^q z^r$, where $p + q + r = n$
(c) $\sum \frac{(n+1)!}{p!q!r!} x^p y^q z^r$, where $p + q + r = n$
(d) none of these

43. The number of terms in the expansion of the multinomial $(x + y + z)^n$ is
(a) $\frac{1}{2}(n + 1)(n + 2)$
(b) $\frac{1}{2} n(n + 1)$
(c) $\frac{1}{2}(n + 1)(n + 3)$
(d) none of these

44. The number of terms in the expansion of $(x_1 + x_2 + \ldots + x_r)^n$ is
(a) ${}^{n+1}C_r$
(b) ${}^{n+r-1}C_{r-1}$
(c) ${}^{n+r+1}C_{r+1}$
(d) ${}^{n+r-1}C_r$

45. If the coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ and the coefficient of x^{-7} is the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$ are equal, then
(a) $ab = 1$
(b) $ab = 11$
(c) $ab = 5$
(d) $ab = 6$

46. If $x + y = 1$, then value of $\sum_{r=0}^{n} (r)({}^nC_r)\, x^{n-r} y^r$ is
(a) 1
(b) 0
(c) nx
(d) ny

47. If sum of the coefficients in the expansion of $\left(x + \frac{1}{x}\right)^n$ is 128, then coefficient of x in the expansion of $\left(x + \frac{1}{x}\right)^n$ is
(a) 51
(b) 48
(c) 35
(d) 28

48. If, in the binomial expansion of $(a - b)^n$, $n \geq 5$ the sum of 5th and 6th term is zero, then value of $\frac{a}{b}$ is
(a) $\frac{n-5}{6}$
(b) $\frac{6}{n-4}$
(c) $\frac{n-4}{5}$
(d) $\frac{5}{n-4}$

49. The value of $2C_0 + \frac{2^2}{2}C_1 + \frac{2^3}{3}C_2 + \ldots + \frac{2^{11}}{11}C_{10}$, is
(a) $\frac{1}{11}(2^{11} - 1)$
(b) $\frac{1}{11}(3^{11} - 1)$
(c) $\frac{1}{11}(11^3 - 1)$
(d) $\frac{1}{11}(11^2 - 1)$

50. If $C_r = {}^nC_r$, then value of $2\left(\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \cdots n\frac{C_n}{C_{n-1}}\right)$ is
(a) $n(n - 1)$
(b) $n(n + 1)$
(c) $n^2 - 1$
(d) $n^2 + 1$

51. If $C_r = {}^nC_r$, then the value of $\left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right) \cdots \left(1 + \frac{C_n}{C_{n-1}}\right)$
(a) $\frac{(n+1)!}{2^n}$
(b) $\frac{(n+1)^n}{n!}$
(c) $\frac{2^{n-1}}{n!}$
(d) $\frac{2^n - 1}{n!}$

52. If $C_r = {}^nC_r$, then $C_0 - \frac{1}{3}C_1 + \frac{1}{5}C_2 \cdots$ upto $(n + 1)$ terms equal
    (a) $\int_0^1 x(1-x)^n\,dx$ (b) $\int_0^1 (1-x^2)^n\,dx$
    (c) 0 (d) $1 + \int_0^1 x(1+x)^n dx$

53. Let $(1 + x^2)^2 (1 + x)^n = \sum_{r=1}^{n+4} a_r x^r$. If a_1, a_2, a_3 are in *A.P.*, then n is equal to
    (a) 6 (b) 5
    (c) 7 (d) 2, 3, 4

54. The value of $\sum_{k=0}^{n}\left({}^{4n+1}C_k + {}^{4n+1}C_{2n-k}\right)$ is
    (a) $2^{4n} + {}^{4n+1}C_{2n}$ (b) 2^{4n+1}
    (c) 2^{4n} (d) $2^{4n+1} + {}^{4n+1}C_n$

55. Coeffiecient of t^{24} in the expansion of $(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ is
    (a) ${}^{12}C_6 + 2$ (b) ${}^{12}C_6 + 1$
    (c) ${}^{12}C_6$ (d) ${}^{12}C_6 + 3$

56. If the value of the third term in the expansion of $\left(x + x^{\log_{10} x}\right)^5$ is 10^6, then x may have value(s)
    (a) 1, 10 (b) $10, 10^{-5/2}$
    (c) $10, 10^{-3/2}$ (d) $10^2, 10^{-3/2}$

57. The greatest term in the expansion of $(1 + x)^{10}$ when $x = 2/3$ is
    (a) $210(3/2)^6$ (b) $210(2/3)^4$
    (c) $210(3/2)^4$ (d) none of these

58. The greatest term in the expansion of $(3 + 5x)^{15}$, when $x = 1/5$, is
    (a) ${}^{15}C_3\,(3^{13})$ (b) ${}^{15}C_4\,(3^{12})$
    (c) ${}^{15}C_4\,(3^{10})$ (d) none of these

59. If sum of the coefficients of x^7 and x^4 in the expansion of $\left(\frac{x^2}{a} - \frac{b}{x}\right)^{11}$ is zero, then
    (a) $ab = 1$ (b) $a = b$
    (c) $ab = -1$ (d) $a + b = 0$

60. The value of $\dfrac{19^3 + 6^3 + (3)(19)(6)(25)}{3^6 + 6(243)(2) + (15)(81)(4) + (20)(27)(8) + (15)(9)(16) + (6)(3)(32) + 2^6}$ is
    (a) 19 (b) 6
    (c) 25 (d) 1

61. In the expansion of $(x + a)^n$, if the sum of odd terms be A and the sum of even terms be B, then value of $(x^2 - a^2)^n$ is
    (a) AB (b) $2AB$
    (c) $4AB$ (d) $A^2 - B^2$

62. Coefficient of the term independent of x in the expansion of $\left(x + \frac{1}{x}\right)^4 \left(x - \frac{1}{x}\right)^{12}$ is
    (a) 192 (b) 194
    (c) 196 (d) 198

63. For all $n \in \mathbf{N}$, $(3)\,(5^{2n+1}) + 2^{3n+1}$ is divisible by
    (a) 19 (b) 17
    (c) 23 (d) 25

64. 1.2.3 + 2.3.4 + 3.4.5 + … upto n terms is equal to
    (a) $\frac{1}{4}$ $(n + 1)(n + 2)(n + 3)$
    (b) $\frac{1}{4}$ $(n + 2)(n - 2)(n - 3)(n + 3)$
    (c) $\frac{1}{4}$ $n(n + 1)(n + 2)(n + 3)$
    (d) none of these

65. If a, b, c, $\in \mathbf{N}$, $a^n + b^n$ is divisible by c when n is odd but not when n is even, then value of c is
    (a) $a + b$ (b) $a - b$
    (c) $a^3 + b^3$ (d) $a^3 + b^3$

Assertion-Reason Type Questions

66. Let $(1 + x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$

    Statement-1: $\sum_{r=0}^{n} C_r \sin(rx) \cos(n - r)x = 2^n \sin(nx)$

    Statement-2: $\sum_{r=0}^{n} C_r = 2^n$

67. **Statement-1:** The coefficient of the term of independent of x in the expansion of $\left(x + \frac{9}{x} + 6\right)^n$ is $\frac{3^n (2n)!}{n!n!}$.

Statement-2: The coefficient of x^r in the expansion of $(1 + x)^n$ is $\binom{n}{r}$.

68. **Statement-1:** For any positive integers m, n (with $n \geq m$),

$$\binom{n}{m}+\binom{n-1}{m}+\binom{n-2}{m}+\cdots+\binom{m}{m}=\binom{n+1}{m} \qquad (1)$$

Statement-2: Coefficient of x^r in the expansion of $(1 + x)^n$ is $\binom{n}{r}$.

69. **Statement-1:** If n is an odd prime, then greatest integer contained in $\left(2+\sqrt{5}\right)^n - 2^{n+1}$ is divisible by 20 n.

Statement-2: If p is a prime and $1 \leq r \leq p - 1$, then $\binom{p}{r}$ is divisible by p.

70. **Statement-1:** Greatest term in the expansion of $\left(\sqrt{3}+\sqrt{2}\right)^{50}$ is

$\binom{50}{22}\, 3^{14}\, 2^{11}$.

Statement-2: Greatest term in the expansion of $(1 + x)^n$, $x > 0$ is the rth term if $\dfrac{(n+1)x}{x+1}$ is not an integer and $r = \left[\dfrac{(n+1)x}{x+1}\right]$, where $[y]$ denotes the greatest integer $\leq y$.

LEVEL 2

Straight Objective Type Questions

71. Let S denote the set of real numbers x such that

$$\left(x+\sqrt{x^2-1}\right)^3+\left(x-\sqrt{x^2-1}\right)^3 = 2\cos\,(3\cos^{-1} x)$$

Then S contains

(a) one element　　(b) two elements

(c) infinte elements　　(d) no element

72. Value of

$({}^{40}C_0)\ ({}^{40}C_{15}) - ({}^{40}C_1)({}^{40}C_{16}) + ({}^{40}C_2)({}^{40}C_{12}) \cdots - ({}^{40}C_{25})\ ({}^{40}C_{40})$

equals

(a) 0　　(b) ${}^{40}C_{25}$

(c) ${}^{40}C_{20}$　　(d) -1

73. The hundred's digit of 3^{100} is

(a) 0　　(b) 1

(c) 2　　(d) 7

74. If $a,b,\ p \in \mathbf{N}$ and p is prime, then $(a + b)^p - a^p - b^p$ is divisible by

(a) p　　(b) p^2

(c) $\frac{1}{2}p(p - 1)$　　(d) $\frac{1}{2}p(p + 1)$

75. $C_0^2+\frac{1}{2}C_1^2+\frac{1}{3}C_2^2+\cdots+\frac{1}{n+1}C_n^2$ equals

(a) $\frac{1}{n}\left({}^{2n}C_n\right)$　　(b) $\frac{1}{n+1}\left({}^{2n+1}C_n\right)$

(c) $\frac{1}{n}\left({}^{2n}C_{n+1}\right)$　　(d) 2^n

76. Sum of the series $\displaystyle\sum_{k=0}^{n}\frac{(2n)!}{(k!)^2\,(n-k!)^2}$ equals

(a) $n^n + {}^{2n}C_n$　　(b) $({}^{2n}C_n)^2$

(c) ${}^{2n}C_n$　　(d) n^n

77. Sum of the series

$$S = n^2\left(\frac{C_0}{C_1}\right) + (n-1)^2\left(\frac{C_1}{C_2}\right) + \cdots + 1^2\left(\frac{C_{n-1}}{C_n}\right)$$

equals

(a) $\frac{1}{6}\ n(n+1)\ (2n + 1)$　　(b) $\frac{1}{3}\ n\ (n + 1)(n + 2)$

(c) $\frac{1}{6}\ n\ (n + 1)\ (n + 2)$　　(d) $\frac{1}{6}\ n^2(n + 1)\ (n + 2)$

78. If $\displaystyle\sum_{r=0}^{2n} a_r\,(x-2)^r = \sum_{r=0}^{2n} b_r\,(x-3)^r$ and

$a_k = \dfrac{1}{{}^{n+k}C_n}$ for $0 \leq k \leq n$ then b_n equals

(a) $n - 1$　　(b) n

(c) $2n - 1$　　(d) $n + 1$

79. Coefficient of x^{50} in the expansion of $(1 + x)^{41}\ (1 - x + x^2)^{40}$ is

(a) 0　　(b) 1

(c) ${}^{40}C_{19}$　　(d) ${}^{40}C_{29}$

80. If $n \in \mathbf{N}$, then $\displaystyle\sum_{i<j}\left({}^{n}C_i\right)\left({}^{n+1}C_j\right)$ equals

(a) 2^{2n}　　(b) 2^{2n-1}

(c) $2^{n+1}C_n$　　(d) $2n + 1$

Previous Years' AIEEE/JEE Main Questions*

1. If sum of the coefficients in the expansion of $(x + y)^n$ is 4096, then the greatest coefficient in the expansion is
(a) 792 (b) 924
(c) 1594 (d) 2990 **[2002]**

2. Let a_r denote the coefficient of x^r in the expansion of $(1 + x)^{p+q}$, then
(a) $a_p = a_q$ (b) $a_p = -a_q$
(c) $a_p a_q = 1$ (d) none of these **[2002]**

3. The positive integer just greater that $(1 + 0.0001)^{10000}$ is
(a) 2 (b) 3
(c) 4 (d) 5 **[2002]**

4. If $r, n \in \mathbf{N}$ $r > 1$, $n > 2$ and the coefficient of $(r + 2)$th term and (3r)th term in the expansion of $(1 + x)^{2n}$ are equal, then n equals
(a) $3r$ (b) $3r + 1$
(c) $2r$ (d) $2r + 1$ **[2002]**

5. The number of integral terms in the expansion of $(3^{1/2} + 5^{1/8})^{256}$ is
(a) 33 (b) 34
(c) 35 (d) 32 **[2003]**

6. *If x is positive, the first negative term in the expansion of $(1 + x)^{27/5}$ is
(a) 5th term (b) 8th term
(c) 6th term (d) 7th term **[2003]**

7. Let $S(K)$: $1 + 3 + 5 + \ldots + (2K - 1) = 3 + K^2$. Then which of the following is true?
(a) $S(K) \nRightarrow S(K + 1)$
(b) $S(K) \Rightarrow S(K + 1)$
(c) $S(1)$ is correct
(d) Principle of mathematical induction can be used to prove the formula. **[2004]**

8. The coefficients of the middle terms in the binomial expansions in powers of x of $(1 + \alpha x)^4$ and of $(1 - \alpha x)^6$ is the same if α equals
(a) $-\frac{3}{10}$ (b) $\frac{10}{3}$
(c) $-\frac{5}{3}$ (d) $\frac{3}{5}$ **[2004]**

9. The coefficient of x^n in the expansion of $(1 + x)(1 - x)^n$ is
(a) $(-1)^{n-1}(n - 1)^2$ (b) $(-1)^n(1 - n)$
(c) $n - 1$ (d) $(-1)^{n-1}n$ **[2004]**

10. If the coefficients of rth, $(r + 1)$th and $(r + 2)$th terms in the binomial expansion of $(1 + y)^m$ are in A.P., then m and r satisfy the equation
(a) $m^2 - m(4r + 1) + 4r^2 - 2 = 0$
(b) $m^2 - m(4r - 1) + 4r^2 + 2 = 0$
(c) $m^2 - m(4r - 1) + 4r^2 - 2 = 0$
(d) $m^2 - m(4r + 1) + 4r^2 + 2 = 0$ **[2005]**

11. If the coefficient of x^7 in $\left[ax^2 + \frac{1}{bx}\right]^{11}$, equals the coefficients of x^{-7} in $\left[ax - \frac{1}{bx^2}\right]^{11}$, then a and b satisfy the relation
(a) $\frac{a}{b} = 1$ (b) $ab = 1$
(c) $a - b = 1$ (d) $a + b = 1$ **[2005]**

12. *If x is small that x^3 are higher powers of x may be neglected, then

$$\frac{(1+x)^{3/2} - \left(1 + \frac{1}{2}x\right)^3}{(1-x)^{1/2}}$$

may be approximated as
(a) $-\frac{3}{8}x^2$ (b) $\frac{1}{2}x - \frac{3}{8}x^2$
(c) $1 - \frac{3}{8}x^2$ (d) $3x + \frac{3}{8}x^2$ **[2005]**

13. For natural numbers m and n, if $(1 - y)^m (1 + y)^n = 1 + a_1y + a_2y^2 + \ldots$ and $a_1 = a_2 = 10$, then (m, n) equals
(a) (35, 45) (b) (20, 45)
(c) (35, 20) (d) (45, 35) **[2006]**

14. The sum of the series

$${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots + {}^{20}C_{10} \text{ is}$$

(a) $-\left({}^{20}C_{10}\right)$ (b) $\frac{1}{2}\left({}^{20}C_{10}\right)$
(c) 0 (d) ${}^{20}C_{10}$ **[2007]**

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* Questions marked with * may be skipped for the present JEE Main syllabus.

15. In the binomial expansion of $(a - b)^n$, $n \geq 0$, the sum of 5th and 6th terms is zero, then $\frac{a}{b}$ equals

(a) $\frac{5}{n-4}$ (b) $\frac{6}{n-5}$

(c) $\frac{n-5}{6}$ (d) $\frac{n-4}{5}$ **[2007]**

16. **Statement-1:** $\sum_{r=0}^{n} (r+1)\,({}^nC_r) = (n+2)2^{n-1}$

Statement-2: $\sum_{r=0}^{n} (r+1)\,({}^nC_r)\,x^r$

$= (1+x)^n + nx\,(1+x)^{n-1}$ **[2008]**

17. The remainder left out when $8^{2n} - 62^{2n+1}$ is divided by 9 is

(a) 7 (b) 8

(c) 0 (d) 2 **[2009]**

18. Let $S_1 = \sum_{j=1}^{10} j(j-1)({}^{10}C_j)$, $S_2 = \sum_{j=1}^{10} j\,({}^{10}C_j)$ and

$S_3 = \sum_{j=1}^{10} (j^2)({}^{10}C_j)$

Statement-1: $S_3 = 55 \times 2^9$

Statement-2: $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$ **[2010]**

19. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is

(a) 132 (b) 144

(c) – 132 (d) – 144 **[2011]**

20. **Statement-1:** For each natural number n, $(n+1)^7 - n^7 - 1$ is divisible by 7.

Statement-2: For each natural number n, $n^7 - n$ is divisible by 7. **[2011]**

21. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is

(a) an odd positive integer

(b) an even positive integer

(c) a rational number other than a positive integer

(d) an irrational number **[2012]**

22. The term independent of x in the expansion of

$\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10}$ is

(a) 120 (b) 210

(c) 310 (d) 4 **[2013]**

23. If the 7th term in the binomial expansion of $\left(\frac{3}{84^{1/3}} + \sqrt{3}\ln x\right)^9$, $x > 0$ is equal to 729, then x can be:

(a) e^2 (b) e

(c) $\frac{1}{2}e$ (d) $2e$ **[2013, online]**

24. If for positive integers $r > 1$, $n > 2$, the coefficients of the $(3r)$th and $(r+2)$th powers of x in the expansion of $(1+x)^{2n}$ are equal, then n is equal to:

(a) $2r$ (b) $2r - 1$

(c) $3r$ (d) $r + 1$ **[2013, online]**

25. The ratio of the coefficient of x^{15} to the term independent of x in the expansion of $\left(x^2 + \frac{2}{x}\right)^{15}$ is:

(a) 7 : 16 (b) 7 : 64

(c) 1 : 4 (d) 1 : 32 **[2013, online]**

26. The sum of the rational terms in the binomial expansion of $(2^{1/2} + 3^{1/5})^{10}$ is:

(a) 25 (b) 32

(c) 9 (d) 41 **[2013, online]**

27. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to:

(a) $\left(14, \frac{251}{3}\right)$ (b) $\left(14, \frac{272}{3}\right)$

(c) $\left(16, \frac{272}{3}\right)$ (d) $\left(16, \frac{251}{3}\right)$ **[2014]**

28. The number of terms in the expansion of $(1+x)^{101}(1 + x^2 - x)^{100}$ in powers of x is

(a) 302 (b) 301

(c) 202 (d) 101 **[2014, online]**

29. The coefficients of x^{50} in the binomial expansion of $(1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \ldots + x^{1000}$ is:

(a) $\frac{(1000)!}{(50)!(950)!}$ (b) $\frac{(1000)!}{(49)!(951)!}$

(c) $\frac{(1001)!}{(51)!(950)!}$ (d) $\frac{(1001)!}{(50)!(951)!}$

[2014, online]

30. If $\left(2 + \frac{x}{3}\right)^{55}$ is expanded in the ascending powers of x and the coefficients of powers of x in two consecutive terms of the expansion are equal, then these terms are:

(a) 7th and 8th (b) 8th and 9th

(c) 28th and 29th (d) 27th and 28th

[2014, online]

31. If $1 + x^4 + x^5 = \sum_{i=0}^{5} a_i (1+x)^i$, for all x in $\mathbf{R}$, then a_2 is:

(a) – 4 (b) 6

(c) – 8 (d) 10 **[2014, online]**

32. The coefficient of x^{1012} in the expansion of $(1 + x^n + x^{253})^{10}$, (where $n \leq 22$ is a positive integer), is
(a) 1 (b) ${}^{10}C_4$
(c) $4n$ (d) ${}^{253}C_4$ **[2014, online]**

33. The sum of coefficients of integral powers of x in the binomial expansion of $(1-2\sqrt{x})^{50}$ is:
(a) $\frac{1}{2}(3^{50}+1)$ (b) $\frac{1}{2}(3^{50})$
(c) $\frac{1}{2}(3^{50}-1)$ (d) $\frac{1}{2}(2^{50}+1)$ **[2015]**

34. If the coefficients of the three successive terms in the binomial expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 42, then the first of these terms in the expansion is:
(a) 6th (b) 7th
(c) 8th (d) 9th **[2015 online]**

35. The term independent of x in the binomial expansion of $\left(1-\frac{1}{x}+3x^5\right)\left(2x^2-\frac{1}{x}\right)^8$ is:
(a) 400 (b) 496
(c) – 400 (d) – 496 **[2015 online]**

36. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion is:
(a) 64 (b) 2187
(c) 243 (d) 729 **[2016]**

37. For $x \in \mathbf{R}$, $x \neq -1$, if
$(1+x)^{2016} + x(1+x)^{2015} + x^2(1+x)^{2014} + \ldots + x^{2016} = \sum_{i=0}^{2016} a_i x^i$, then a_{17} is equal to:
(a) $\frac{2017!}{17!2000!}$ (b) $\frac{2016!}{17!1999!}$
(c) $\frac{2016!}{16!}$ (d) $\frac{2017!}{2000!}$ **[2016 online]**

38. If the coefficients of x^{-2} and x^{-4} in the expansion of $\left(x^{\frac{1}{3}}+\frac{1}{2x^{\frac{1}{3}}}\right)^{18}$, $(x > 0)$, are m and n respectively, then $\frac{m}{n}$ is equal to:
(a) 27 (b) 182
(c) $\frac{5}{4}$ (d) $\frac{4}{5}$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. If in the expansion of $\left(x^3-\frac{1}{x^2}\right)^n$, the sum of the coefficients of x^5 and x^{10} is 0, then the coefficient of the third term is:
(a) 455 (b) 105
(c) 605 (d) 120 **[2006]**

2. If $(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3) \ldots (1 + x + x^2 + \ldots + x^n) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$, then the value of a_1 is
(a) $m + 1$ (b) $(n + 1)$
(c) n (d) m **[2006]**

3. If the ratio of the 7th term from the beginning to the 7th term from the end in the expansion of $(2^{1/3} + 3^{-1/3})^n$ is 1/6, then the value of n is
(a) 9 (b) 12
(c) 6 (d) 3 **[2007]**

4. The remainder when 7^{128} is divided by 10 is
(a) 1 (b) 3
(c) 7 (d) 9 **[2008]**

5. The value of the sum
$$\sum_{j=0}^{8} \frac{1}{(j+1)(j+2)}\binom{8}{j}$$
is
(a) $\frac{1003}{90}$ (b) $\frac{1013}{90}$
(c) $\frac{1023}{90}$ (d) $\frac{1033}{90}$ **[2008]**

6. **Statement-1:** The coefficients of x^n in
$\sum_{j=0}^{n} (-1)^j (2x+5)^{n-j}(5-2x)^j$ is $(n+1)\,2^n$

Statement-2:
$$\sum_{j=0}^{8} a^{n-j} b^j = \frac{a^{n+1}-b^{n+1}}{a-b}, \; a \neq b$$

and coefficient of $(\alpha x + \beta)^m$ is α^m **[2009]**

7. If $x^n = a_0 + a_1\,(1 + x) + a_2\,(1 + x)^2 + \ldots + a_n\,(1 + x)^n$
$= b_0 + b_1\,(1 - x) + b_2\,(1 - x)^2 + \ldots + b_n\,(1 - x)^n$
then for $n = 101$, (a_{50}, b_{50}) equals:
(a) $(-\,{}^{101}C_{50}, {}^{101}C_{50})$ (b) $({}^{101}C_{50}, -\,{}^{101}C_{50})$
(c) $(-\,{}^{101}C_{50}, -\,{}^{101}C_{50})$ (d) $({}^{101}C_{50}, {}^{101}C_{50})$ **[2009]**

8. If the third term in the expansion of $\left[\frac{1}{x} + x^{\log_{10} x}\right]^5$, $x > 1$, is 1000, then x equals
(a) 10 (b) 1
(c) $\frac{1}{\sqrt{10}}$ (d) 100 **[2010]**

9. If the sum of the coefficients in the expansion of $(x + y)^n$ is 2048, then the greatest coefficient in the expansion is:
(a) ${}^{10}C_6$ (b) ${}^{11}C_6$
(c) ${}^{11}C_7$ (d) ${}^{12}C_6$ **[2011]**

10. If $a_i (i = 0, 1, 2, \ldots, 16)$ be real constants such that for every real value of x, $(1 + x + x^2)^8 = a_0 + a_1x + a_2x^2 + \ldots + a_{16}x^{16}$, then a_5 is equal to:
(a) 502 (b) 504
(c) 506 (d) 508 **[2012]**

11. The coefficient of t^{24} in $(1 + t^2)^{12}\,(1 + t^{12})\,(1 + t^{24})$ is:
(a) ${}^{12}C_6 + 13$ (b) ${}^{12}C_6 + 2$
(c) ${}^{12}C_6 + 1$ (d) ${}^{12}C_6$ **[2013]**

12. Sum of the last 30 coefficients of powers of x in the binomial expansion of $(1 + x)^{59}$ is:
(a) 2^{29} (b) 2^{28}
(c) $2^{59} - 2^{29}$ (d) 2^{58} **[2014]**

13. If in the binomial expansion of $(1 - x)^m\,(1 + x)^n$, the coefficients of x and x^2 are respectively 3 and –4, then the ratio $m : n$ is equal to:
(a) 10 : 7 (b) 8 : 11
(c) 10 : 13 (d) 7 : 10 **[2015]**

14. For a positive integer n, if the mean of the binomial coefficients in the expansion of $(a + b)^{2n - 3}$ is 16, then n is equal to:
(a) 5 (b) 7
(c) 9 (d) 4 **[2016]**

Answers

Concept-based

1. (a) **2.** (b) **3.** (a) **4.** (c)
5. (c) **6.** (d) **7.** (b) **8.** (c)
9. (a) **10.** (d)

Level 1

11. (c) **12.** (a) **13.** (b) **14.** (a)
15. (b) **16.** (b) **17.** (d) **18.** (a)
19. (b) **20.** (b) **21.** (b) **22.** (a)
23. (b) **24.** (a) **25.** (d) **26.** (a)
27. (b) **28.** (a) **29.** (a) **30.** (c)
31. (d) **32.** (a) **33.** (a) **34.** (c)
35. (a) **36.** (a) **37.** (a) **38.** (b)
39. (c) **40.** (c) **41.** (b) **42.** (a)
43. (a) **44.** (b) **45.** (a) **46.** (d)
47. (c) **48.** (c) **49.** (b) **50.** (b)
51. (b) **52.** (b) **53.** (d) **54.** (a)
55. (a) **56.** (b) **57.** (b) **58.** (d)
59. (a) **60.** (d) **61.** (d) **62.** (d)
63. (b) **64.** (c) **65.** (a) **66.** (d)
67. (a) **68.** (d) **69.** (a) **70.** (d)

Level 2

71. (d) **72.** (a) **73.** (a) **74.** (a)
75. (b) **76.** (b) **77.** (c) **78.** (d)
79. (a) **80.** (a)

Previous Years' AIEEE/JEE Main Questions

1. (b) **2.** (a) **3.** (b) **4.** (c)
5. (a) **6.** (b) **7.** (b) **8.** (a)
9. (b) **10.** (a) **11.** (b) **12.** (a)
13. (a) **14.** (b) **15.** (d) **16.** (a)
17. (d) **18.** (c) **19.** (d) **20.** (a)
21. (d) **22.** (b) **23.** (b) **24.** (a)
25. (d) **26.** (d) **27.** (c) **28.** (c)
29. (d) **30.** (b) **31.** (a) **32.** (b)
33. (c) **34.** (b) **35.** (a) **36.** (b)
37. (a) **38.** (b)

Previous Years' B-Architecture Entrance Examination Questions

1. (d) **2.** (c) **3.** (a) **4.** (a)
5. (b) **6.** (a) **7.** (a) **8.** (d)
9. (b) **10.** (b) **11.** (b) **12.** (d)
13. (d) **14.** (a)

Hints and Solutions

Concept-based

1. $T_{r+1} = {}^5C_r\,(x^2)^{5-r}\left(\frac{k}{x}\right)^r = {}^5C_r\,k^r\,x^{10-3r}$

set $10 - 3r = 1$ or $r = 3$. Therefore,

$^5C_3\, k^3 = 270 \Rightarrow k^3 = 27 \Rightarrow k = 3.$

2. $T_4 = {}^nC_3(px)^{n-3}\left(\dfrac{1}{x}\right)^3 = {}^nC_3\, p^{n-3}\, x^{n-6} = \dfrac{5}{2}$

$\Leftrightarrow n - 6 = 0$ and $^6C_3\, p^3 = \dfrac{5}{2} \Leftrightarrow n = 6,\ p = 1/2$

3. $^{20}C_{3r+3} = {}^{20}C_{2r-3} \Rightarrow (3r + 3) + (2r - 3) = 20 \Rightarrow r = 4$

4. Sum of the coefficients

$= (1 + 2)^n = 3^n = 3^8 \Rightarrow n = 8.$

$T_4 = {}^8C_3(2x)^3 = {}^8C_3\, 2^3\, x^3$

Thus, coefficient of fourth term is 448.

5. $(1 + x^2)^{10}\left(x^2 + 2 + \dfrac{1}{x^2}\right)^5$

$= (1 + x^2)^{10}\left(x + \dfrac{1}{x}\right)^5 = \dfrac{1}{x^5}\,(1 + x^2)^{15}$

$\therefore$ coefficient of x^{10} in $(1 + x^2)\left(x^2 + 2 + \dfrac{1}{x^2}\right)^5$

$=$ coefficient of x^{15} in $(1 + x^2)^{15} = 0.$

6. $10 = na,\ 40 = \dfrac{n(n-1)}{2}a^2$

$\dfrac{80}{100} = \dfrac{n(n-1)a^2}{n^2a^2} \Rightarrow \dfrac{4}{5} = \dfrac{n-1}{n} \Rightarrow n = 5$

and $a = 2$

7. $T_{r+1} = (r + 1)$th term in $(1 + x^2)^{10} = {}^{10}C_r\,(x^2)^r$

$\therefore$ coefficient of x^{10} in $(1 + x^2)^{10}$ is $^{10}C_5$.

Also, $t_{r+1} = (r + 1)$th term in $\left(x + \dfrac{2}{x}\right)^{10}$

$= {}^{10}C_r\,(x)^{10-r}\left(\dfrac{2}{x}\right)^r$

$= {}^{10}C_r\, 2^r\, x^{10-2r}$

For this term to be independent of x, set $10 - 2r = 0$ or $r = 5$.

$\therefore$ term independent of x is $^{10}C_5\,(2^5)$

Required ratio is $^{10}C_5 : {}^{10}C_5(2^5) = 1 : 32$

8. $2(^nC_9) = {}^nC_8 + {}^nC_{10}$

$\Rightarrow 2 = \dfrac{9}{n-8} + \dfrac{n-9}{10}$

$\Rightarrow 20(n - 8) = 90 + (n - 8)(n - 9)$

$\Rightarrow n^2 - 37n + 322 = 0$

$\Rightarrow (n - 14)(n - 23) = 0$

Thus, n can be 23

9. $(p + 2)$th term from the end

$= [2n + 2 - (p + 1)]$th $= (2n - p + 1)$th

from the beginning

and $T_{2n-p+1} = {}^{2n+1}C_{2n-p}\,(x^2)^{2n+1-(2n-p)}\left(-\dfrac{2}{x^2}\right)^{2n-p}$

$= {}^{2n+1}C_{2n-p}\,(-2)^{2n-p}\,x^{2p+1-2n}$

10. $\dfrac{C_r}{C_{r-1}} = \dfrac{15!}{r!(15-r)!} \times \dfrac{(r-1)!\,(16-r)!}{15!}$

$= \dfrac{16-r}{r}$

$\Rightarrow S = \displaystyle\sum_{r=1}^{15} r\left(\dfrac{C_r}{C_{r-1}}\right) = \sum_{r=1}^{15}(16-r)$

$= \dfrac{1}{2}\,(15)\,(16) = 120$

Level 1

11. As there is just one middle term, n must be even and middle term is $\left(\dfrac{n}{2} + 1\right)$ th term, and it is given by

$T_{n/2+1} = {}^nC_{n/2}\,(x^2)^{n-n/2}\left(\dfrac{1}{x}\right)^{n/2}$

$= {}^nC_{n/2}\, x^{n/2}$

Now, $^nC_{n/2}\, x^{n/2} = 924x^6$

$\Rightarrow n/2 = 6 \Rightarrow n = 12$

[Also, note that $^{12}C_6 = 924$]

12. We have

$T_{r+1} = {}^{20}C_r\,\dfrac{\sqrt{5}}{(\sqrt{5})^r} = {}^{20}C_r\,\dfrac{1}{(\sqrt{5})^{r-1}}$

Suppose $(r + 1)$th term is the greatest term. Then

$\dfrac{T_{r+1}}{T_r} > 1$ and $\dfrac{T_{r+1}}{T_{r+2}} > 1$

$\Rightarrow \dfrac{20-r}{r+1}\dfrac{1}{\sqrt{5}} > 1$ and $\dfrac{r+2}{19-r}\sqrt{5} > 1$

$\Rightarrow 20 - r > \sqrt{5}\,r + \sqrt{5}$ and $r\sqrt{5} + 2\sqrt{5} > 19 - r$

$\Rightarrow 19 - 2\sqrt{5} < \left(\sqrt{5} + 1\right)r < 20 - \sqrt{5}$

$\Rightarrow \dfrac{1}{4}(19 - 2\sqrt{5})(\sqrt{5} - 1) < r < \dfrac{1}{4}(20 - \sqrt{5})(\sqrt{5} - 1)$

$\Rightarrow \dfrac{1}{4}\,(21\sqrt{5} - 29) < r < \dfrac{1}{4}\,(21\sqrt{5} - 25)$

$\Rightarrow 4.48 < r < 5.49 \Rightarrow r = 5$,

$\therefore$ greatest term $= {}^{20}C_5 \left(\frac{1}{\sqrt{5}}\right)^4 = {}^{20}C_5 \left(\frac{1}{25}\right)$

13. $S(\theta) = (\sqrt{2} - \sin\theta + \cos\theta)^{2n}$

Maximum possible value of

$\sqrt{2} - \sin\theta + \cos\theta = \sqrt{2} - \sqrt{2}\left(\cos\frac{\pi}{4}\sin\theta - \sin\frac{\pi}{4}\cos\theta\right)$

$= \sqrt{2} - \sqrt{2}\sin(\theta - \pi/4)$

is $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$

$\therefore$ Maximum possible value of $S(\theta)$ is $(2^{3/2})^{2n} = 8^n$

14. Note that $T_r = {}^nC_r\, x^{n-r} a^r$.

$(x + ai)^n = {}^nC_0\, x^n + {}^nC_r\, x^{n-1}(ia)$
$\quad + {}^nC_2\, x^{n-2}(ia)^2 + {}^nC_3\, x^{n-3}(ia)^3$
$\quad + \ldots + {}^nC_n\,(ia)^n$

$= (T_0 - T_2 + T_4 - \ldots)$
$+ i(T_1 - T_3 + T_5 - \ldots)$

$\Rightarrow (x^2 + a^2)^n = |(x + ai)^n|^2$
$= (T_0 - T_2 + T_4 - \ldots)^2 + (T_1 - T_3 + T_5 - \ldots)^2$

15. For $0 \le r < 2n$,

$(1 + x)^{4n} = (1 + x)^{2n+r}(1 + x)^{2n-r}$
$= (A_0 + A_1x + A_2x^2 + \ldots + A_{2n+r}\,x^{2n+r})$
$(B_0 + B_1\,x + B_2\,x^2 + \ldots + B_{2n-r}\,x^{2n-r})$

where $A_k = {}^{2n+r}C_k$ $(0 \le k \le 2n + r)$

and $B_k = {}^{2n-r}C_k$ $(0 \le k \le 2n - r)$

Coefficient of x^{2n} on the RHS

$A_{2n}\,B_0 + A_{2n-1}\,B_1 + \ldots + A_n\,B_n$
$\quad + A_{n+1}\,B_{n-1} + \ldots + A_r\,B_{2n-r}$

= coefficient of x^{2n} on LHS

$= {}^{4n}C_{2n}$.

Thus, $A_n\,B_n < {}^{4n}C_{2n}$

$\Rightarrow \left({}^{2n+r}C_n\right)\left({}^{2n-r}C_n\right) < {}^{4n}C_{2n}$

16. Sum of the coefficients $= (2 - 3c + c^2)^{17}$

We are given $(2 - 3c + c^2)^{17} = 0$

$\Rightarrow c^2 - 3c + 2 = 0 \Rightarrow c = 1, 2$

17. Let $y = 3^{-\log_{\sqrt{3}}(x^{3/2})}$

$\Rightarrow \log_3 y = -\frac{\log_3(x^{3/2})}{\log_3\sqrt{3}} = -2\log_3(x^{3/2})$

$\Rightarrow y = x^{-3}$

$(r + 1)$th term in the expansion of

$(x^3 + 5x^{-3})^4$ is

$T_{r+1} = {}^4C_r\,(x^3)^{4-r}(5x^{-3})^r$
$= {}^4C_r\,5^r\,x^{12-3r-3r}$

For $(r + 1)$th term to be independent of x, set $12 - 6r = 0$ or $r = 2$

$\therefore$ required constant term

$= {}^4C_2\,5^2 = (6)(25) = 150$

18. $(1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5)^{20}$

$= ((1 + x)^5)^{20} = (1 + x)^{100}$

m = number of terms = 101.

We have

$2^m = 2^{101} = (2^5)^{20}(2)$

$= (30 + 2)^{20}(2) = (10m_1 + 2^{20})(2)$

Let k = Unit's digit of 2^m is

= units digit of 2^{21}

Now, $2^{21} = (2^5)^4(2) = (30 + 2)^4(2)$

$\therefore$ k = unit's digit of $2^5 = 2$.

19. Suppose

$(1 + x)^{n+1} = \sum_{k=0}^{n+1} B_k\,x^k$, where $B_k = {}^{n+1}C_k$

and $(1 + x)^n = \sum_{k=0}^{n} C_k\,x^k$, where $C_k = {}^nC_k$.

We have

$\frac{P_{n+1}}{P_n} = B_0\left(\frac{B_1}{A_0}\right)\left(\frac{B_2}{A_1}\right)\cdots\left(\frac{B_{n+1}}{A_n}\right)$

Now, $\frac{B_{r+1}}{A_r} = \frac{(n+1)!}{(r+1)!\,(n-r)!}\cdot\frac{r!\,(n-r)!}{n!}$

$= \frac{n+1}{r+1} \qquad (0 \le r \le n)$

Thus, $\frac{P_{n+1}}{P_n} = \frac{(n+1)^n}{r!}$

$\Rightarrow \frac{21^{20}}{20!} = \frac{(n+1)^n}{r!} \Rightarrow n = 20$

20. Coefficient of the term independent of x in the expansion of $(1 + x)^{2n}\left(\frac{x}{1-x}\right)^{-2n}$

or $\dfrac{(1+x)^{2n}\,(1-x)^{2n}}{x^{2n}}$

= coefficient of x^{2n} in $(1 - x^2)^{2n}$

= coefficient of t^n in $(1 - t)^{2n}$

$= {}^{2n}C_n(-1)^n$

21. $T_{r+1} = {}^{10}C_r\left(\dfrac{x}{2}\right)^{10-r}\left(-\dfrac{3}{x^2}\right)^r = {}^{10}C_r\,\dfrac{(-3)^r}{2^{10-r}}\,x^{10-3r}$

For the coefficient of x^4 set $10 - 3r = 4$

$\Rightarrow \quad r = 2.$

22. $\alpha = {}^nC_r$ and $\beta = {}^nC_{n-r}$

$\therefore \quad \alpha = \beta$

23. ${}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$\Rightarrow \; 2r + 3 = r - 3$ or $18 = (2r + 3) + (r - 3)$

$\Rightarrow \qquad r = -6$ or $r = 6$

24. $(x + a)^{200} + (x - a)^{200}$

$= 2[{}^{200}C_0\,x^{200} + {}^{200}C_2\,x^{198}\,a^2 + \ldots + {}^{200}C_{200}\,a^{200}]$

These are 101 terms.

25. $(1 + x^2)^5\,(1 + x)^4$

$= (1 + 5x^2 + 10x^4 + \ldots)\,(1 + 4x + 6x^2 + 4x^3 + \ldots)$

$\therefore$ Coefficient of x^5 is (5)(4) + (4)(10) = 60

26. ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1} \;\Rightarrow\; 3r - 1 + r + 1 = 2n$

$\Rightarrow \quad n = 2r.$

27. $(1 + px)^n = 1 + npx + \dfrac{1}{2}n(n-1)p^2x^2 + \ldots$

$\therefore \quad np = 24,\; \dfrac{1}{2}n(n-1)p^2 = 264$

Solve to obtain $n = 12$, $p = 2$.

28. $\displaystyle\sum_{r=0}^{10}(-1)^r\left({}^{10}C_r\right)\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}\right]$

$= \left(1-\dfrac{1}{2}\right)^{10} + \left(1-\dfrac{3}{4}\right)^{10} = \dfrac{2^{10}+1}{2^{20}}$

29. Given expression

= number of ways of choosing k persons out m men and n women

$= {}^{m+n}C_k$

30. Use $(x + a)^6 + (x - a)^6$

$= \; 2[x^6 + {}^6C_2\,x^4a^2 + {}^6C_4\,x^2\,a^4 + {}^6C_6\,a^6]$

Putting $a = \sqrt{x^3-1}$, we find it to be a polynomial of 9.

31. $(1 + 3x + 3x^2 + x^3)^7 = ((1 + x)^3)^7 = (1 + x)^{21}$

32. Positive terms are

${}^{4n}C_0,\; {}^{4n}C_4x^4,\; {}^{4n}C_8x^8,\; \ldots,\; {}^{4n}C_{4n}\,x^{4n}$

$\therefore$ number of positive terms = $(n + 1)$.

33. $\left(x^2 - 2 + \dfrac{1}{x^2}\right)^n = \left(x - \dfrac{1}{x}\right)^{2n}$

Number of terms in the expansion is $(2n + 1)$ out of which just one is independent of x.

34. $T_3 = {}^5C_2\left(\dfrac{1}{x}\right)^{5-2}\left(x^{\log_{10}x}\right)^2 = 1000$

$\Rightarrow \; 10x^{2\log_{10}x-3} = 1000$

$\Rightarrow \; (2t - 3)t = 2$ where $t = \log_{10}x$

$\Rightarrow \; 2t^2 - 3t - 2 = 0$

$\Rightarrow \; (2t + 1)\,(t - 2) = 0$

$\Rightarrow \; x = \dfrac{1}{\sqrt{10}}$ or $x = 100.$

35. Let $F = \left(\sqrt{2}-1\right)^{2n+1}$. Note that $0 < F < 1$.

Also, $R - F = 2m$, where

$m = 2[{}^{2n+1}C_1(\sqrt{2})^{2n} + {}^{2n+1}C_3\,(\sqrt{2})^{2n-2} + \ldots + {}^{2n+1}C_{2n+1}]$ is an integer

$\Rightarrow \; [R] + f - F = 2m \;\Rightarrow\; f - F = 2m - [R]$ is an integer.

But $-1 < f - F < 1$. Thus, $f - F = 0$

$\therefore \; Rf = RF = 1$

36. The coefficient of x^{60} in $(1 + x)^{51}\,(1 - x + x^2)^{50}$

= the coefficient of x^{60} in $(1 + x)\,(1 - x^3)^{50}$

$= {}^{50}C_{20}$

37. $101^{50} - 99^{50} = (100 + 1)^{50} - (100 - 1)^{50}$

$= 2[{}^{50}C_1(100)^{49} + {}^{50}C_3(100)^{47} + \ldots + {}^{50}C_{49}(100)]$

$= 100^{50}$ + a positive term.

$\Rightarrow \; b = a$ + (a positive term) $\;\Rightarrow\; a < b$.

38. Clearly $A_0 = 1$.

Putting $x = 1$ and $x = -1$, we get

$0 = A_0 + A_1 + A_2 + \ldots + A_{12}$

and $(-2)^6 = A_0 - A_1 + A_2 + \ldots + A_{12}$

Adding we get

$64 = 2[A_0 + A_2 + \ldots + A_{12}]$

$\Rightarrow A_2 + A_4 + \ldots + A_{12} = 31$

39. Put $x = 1$.

40. Let $\quad R = \left(\sqrt{3}+1\right)^6$ and $R - [R] = f$.

Put $\quad F = \left(\sqrt{3}-1\right)^6$, then $0 < F < 1$ and

$$R + F = \left(\sqrt{3}+1\right)^6 + \left(\sqrt{3}-1\right)^6$$
$$= 2[3^3 + {}^6C_2\,(3^2) + {}^6C_4(3) + 1]$$
$$= 2[27 + 135 + 45 + 1] = 416$$

$\Rightarrow \quad f + F = 416 - [R]$ is an integer.

Now, show that $f + F = 1$.

41. Use $(x^2 + 6x + 9)^{30} = (x + 3)^{60}$

42. See Theory.

43. Number of terms in $(x + y + z)^n$

$= $ number of non-negative integral solution of $p + q + r = n$

$$= {}^{n+2}C_2 = \frac{1}{2}\,(n+2)\,(n+1)$$

44. Similar to Question 43.

45. T_{r+1}, the $(r + 1)$th term in the expansion $\left(ax^2 + \dfrac{1}{bx}\right)^{11}$ is

$$T_{r+1} = {}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r\,\frac{a^{11-r}}{b^r}\,x^{22-3r}$$

For the coefficient of x^7, set $22 - 3r = 7 \quad \Rightarrow \quad r = 5$.

$\therefore$ Coefficient of x^7 in $\left(ax^2 + \dfrac{1}{bx}\right)^{11}$ is ${}^{11}C_5\,\dfrac{a^6}{b^5}$.

Next, t_{r+1}, the $(r + 1)$th term in the expansion of $\left(ax - \dfrac{1}{bx^2}\right)^{11}$ is

$$t_{r+1} = {}^{11}C_r\,(ax)^{11-r}\left(-\frac{1}{bx^2}\right)^r$$
$$= {}^{11}C_r\,\frac{a^{11-r}}{b^r}\,(-1)^r\,x^{11-3r}$$

For the coefficient of x^{-7}, set $11 - 3r = -7 \Rightarrow r = 6$.

$\therefore$ Coefficient of x^{-7} in the expansion of $= {}^{11}C_6\,\dfrac{a^5}{b^6}$

We are given $\quad {}^{11}C_5\,\dfrac{a^6}{b^5} = {}^{11}C_6\,\dfrac{a^5}{b^6} \quad \Rightarrow \quad ab = 1.$

46. We have $(a + t)^n = \displaystyle\sum_{r=0}^{n} {}^nC_r a^{n-r} t^r$

Differentiating both the sides with respect to t, we get

$$n(a + t)^{n-1} = \sum_{r=0}^{n} r\left({}^nC_r\right)a^{n-r}t^{r-1}$$

Multiplying both the sides by t and putting $a = x$ and $t = y$, we get

$$n(x + y)^{n-1}\,y = \sum_{r=0}^{n} r\left({}^nC_r\right)x^{n-r}\,y^r$$

$$\Rightarrow \sum_{r=0}^{n} r\left({}^nC_r\right)x^{n-r}\,y^r = ny \qquad [\because \quad x + y = 1]$$

47. Sum of the coefficients in the expansion of $\left(x + \dfrac{1}{x}\right)^n$ is equal to 2^n.

$\therefore \qquad 2^n = 128 = 2^7 \quad \Rightarrow \quad n = 7$

Now, T_{r+1}, the $(r + 1)$th term in the expansion of $\left(x + \dfrac{1}{x}\right)^7$ is

$$T_{r+1} = {}^7C_r\,x^{7-r}\left(\frac{1}{x}\right)^r = {}^7C_r\,x^{7-2r}$$

For the coefficient of x, we set $7 - 2r = 1 \Rightarrow r = 3$.

Thus, coefficient of x in the expansion of $\left(x + \dfrac{1}{x}\right)^7$ is ${}^7C_3 = 35$.

48. $\quad {}^nC_4\,a^{n-4}\,(-b)^4 + {}^nC_5\,a^{n-5}\,(-b)^5 = 0$

$$\Rightarrow {}^nC_4\,a^{n-4}\,b^4 = {}^nC_5\,a^{n-5}\,b^5$$

$$\Rightarrow \frac{a^{n-4}b^4}{a^{n-5}b^5} = \frac{{}^nC_5}{{}^nC_4} \quad \Rightarrow \quad \frac{a}{b} = \frac{n-4}{5}$$

49. Given expression is equal to

$$\int_0^2 \left(C_0 + C_1x + C_2x^2 + \cdots C_{10}x^{10}\right)dx$$
$$= \int_0^2 (1 + x)^{10}\;dx = \frac{1}{11}\left(3^{11} - 1\right)$$

50. $$\frac{C_r}{C_{r-1}} = \frac{n!}{r!(n-r)!}\cdot\frac{(r-1)!\,(n-r+1)!}{n!}$$
$$= \frac{n - r + 1}{r}$$
$$\Rightarrow r\,\frac{C_r}{C_{r-1}} = n - r + 1$$
$$\Rightarrow 2\sum_{r=1}^{n} r\,\frac{C_r}{C_{r-1}} = 2\sum_{r=1}^{n}(n - r + 1) = n(n+1)$$

51. From the previous question, we have

$$\left(1 + \frac{C_r}{C_{r-1}}\right) = \frac{n+1}{r}$$
$$\therefore \quad \left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right)\cdots\left(1 + \frac{C_n}{C_{n-1}}\right) = \frac{(n+1)^n}{n!}$$

52. $C_0 - \dfrac{1}{3}C_1 + \dfrac{1}{5}C_2 - \cdots$

$$= \int_0^1\left[C_0 - C_1x^2 + C_2x^4 - \cdots\right]dx = \int_0^1\left(1 - x^2\right)^n dx$$

53. $(1 + 2x^2 + x^4)\ (1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \ldots)$

$= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$

$\Rightarrow a_1 = {}^nC_1,\ a_2 = {}^nC_2 + 2,\ a_3 = {}^nC_3 + 2({}^nC_1)$

Now, a_1, a_2, a_3 are in A.P. implies

$2({}^nC_2 + 2) = {}^nC_1 + {}^nC_3 + 2({}^nC_1)$

$\Rightarrow n(n - 1) + 4 = 3n + \dfrac{1}{6}n(n - 1)\ (n - 2)$

$\Rightarrow 6(n^2 - n) + 24 = 18n + n^3 - 3n^2 + 2n$

$\Rightarrow n^3 - 9n^2 + 26n - 24 = 0$

$\Rightarrow (n - 2)\ (n - 3)\ (n - 4) = 0 \quad \Rightarrow \quad n = 2, 3, 4$

54. $\displaystyle\sum_{k=0}^{n} \left({}^{4n+1}C_k + {}^{4n+1}C_{2n-k}\right)$

$= \displaystyle\sum_{k=0}^{2n} ({}^{4n+1}C_k) + {}^{4n+1}C_{2n}$

$= 2^{4n} + {}^{4n+1}C_{2n}$ [See Theory]

55. Coefficient of t^{24} in $(1 + t^2)^{12}\ (1 + t^{12})\ (1 + t^{24})$

= Coefficient of t^{24} in $(1 + {}^{12}C_1\, t^2 + {}^{12}C_2\, t^4 + \ldots)$
$(1 + t^{12} + t^{24})$

$= {}^{12}C_{12} + {}^{12}C_6 + 1 = {}^{12}C_6 + 2$

56. $T_3 = {}^5C_2x^{5-2}\left(x^{\log_{10} x}\right)^2 = 10^6 \Rightarrow x^{2\log_{10} x + 3} = 10^5$

$\Rightarrow t(2t + 3) = 5$ where $t = \log_{10}x \Rightarrow t = -5/2,\ 1$

57. $\alpha = \dfrac{(10+1)(2/3)}{1+2/3} = \dfrac{22}{5} \quad \Rightarrow \quad [\alpha] = 4$

Thus, the greatest term is 5th term and it is given by

${}^{10}C_4\left(\dfrac{2}{3}\right)^4 = 210\left(\dfrac{2}{3}\right)^4$

58. $\alpha = \dfrac{(15+1)(5)(1/5)}{3+5(1/5)} = 4$

Thus, 4th and 5th terms are the greatest, and $T_5 = {}^{15}C_4(3)^{11}$

59. $T_{r+1} = {}^{11}C_r\left(\dfrac{x^2}{a}\right)^{11-r}\left(-\dfrac{b}{x}\right)^r$

$= {}^{11}C_r(-1)^r\ a^{r-11}b^rx^{22-3r}$

For coefficient of x^4 and x^7, we put $22 - 3r = 4$ and $22 - 3r = 7$

$\Rightarrow \quad r = 6$ and $r = 5$

We are given

${}^{11}C_6(-1)^6\ a^{-5}\ b^6 + {}^{11}C_5\ (-1)^5\ a^{-6}\ b^5 = 0$

$\Rightarrow \quad \dfrac{b^6}{a^5} = \dfrac{b^5}{a^6} \quad \Rightarrow \quad ab = 1$

60. Numerator $= 19^3 + 6^3 + 3(19)\ (6)\ (19 + 6)$
$= (19 + 6)^3 = 25^3$

and denominator $= (3 + 2)^6 = 5^6$

Thus, given expression = 1.

61. We have $(x + a)^n = A + B$ and $(x - a)^n = A - B$.

$\therefore\ (x^2 - a^2)^n = (x + a)^n\ (x - a)^n = (A + B)\ (A - B)$
$= A^2 - B^2$.

62. $\left(x + \dfrac{1}{x}\right)^4\left(x - \dfrac{1}{x}\right)^{12} = \left(x + \dfrac{1}{x}\right)^4\left(x - \dfrac{1}{x}\right)^4\left(x - \dfrac{1}{x}\right)^8$

$= \left(x^2 - \dfrac{1}{x^2}\right)^4\left(x - \dfrac{1}{x}\right)^8$

$= \left({}^4C_0x^8 - {}^4C_1x^4 + {}^4C_2 - {}^4C_3\dfrac{1}{x^4} + {}^4C_4\dfrac{1}{x^8}\right)$

$\times\left({}^8C_0x^8 - {}^8C_1x^6 + {}^8C_2x^4 - \cdots + {}^8C_8\dfrac{1}{x^8}\right)$

$\therefore$ Coefficient of the term independent of x

$= ({}^4C_0)\ ({}^8C_8) - ({}^4C_1)\ ({}^8C_6) + ({}^4C_2)\ ({}^8C_4) - ({}^4C_3)\ ({}^8C_2)$
$+ ({}^4C_4)\ ({}^8C_0) = 198$

63. For $n = 1$, $3(5^{2n+1}) + 2^{3n+1} = 3(5^3) + 2^4 = (17)\ (23)$

Assume $3(5^{2k+1}) + 2^{3k+1} = 17m$ for some m.

Using this show that $3(5^{2k+3}) + 2^{3k+4}$ is divisible by 17.

64. $t_r = r(r + 1)\ (r + 2) = (r + 1)\ \{(r + 1)^2 - 1\}$

$= (r + 1)^3 - (r + 1)$

$\Rightarrow \displaystyle\sum_{r=1}^{n} t_r = \sum_{r=1}^{n+1} r^3 - \sum_{r=1}^{n+1} r$

$= \dfrac{1}{4}(n+1)^2\ (n+2)^2 - \dfrac{1}{2}(n+2)\ (n+1)$

$= \dfrac{1}{4}(n+1)(n+2)\left(n^2 + 3n + 2 - 2\right)$

$= \dfrac{1}{4}n\ (n+1)(n+2)(n+3)$.

Remark

- ❖ We can also prove this by using the principle of mathematical induction.

65. Use $a^{2m+1} + b^{2m+1}$
$= (a + b)\ (a^{2m} - a^{2m-1}\, b + a^{2m-2}\, b^2 - \ldots$
$- ab^{2m-1} + b^{2m}]$

66. Let $S = \sum_{r=0}^{n} C_r \sin(rx)\cos[(n-r)x]$ $\quad (1)$

Using $C_r = C_{n-r}$, we can write

$$S = \sum_{r=0}^{n} C_{n-r}\sin(rx)\cos[(n-r)x]$$

$$= \sum_{r=0}^{n} C_r \sin[(n-r)x]\cos(rx) \quad (2)$$

Adding (1) and (2), we get

$$2S = \sum_{r=0}^{n} C_r\{\sin(rx)\cos[(n-r)x] + \sin[(n-r)x]\cos(rx)\}$$

$$= \sum_{r=0}^{n} C_r \sin(nx) = \sin(nx)\sum_{r=0}^{n} C_r$$

$$= 2^n \sin(nx)$$

$$\Rightarrow \quad S = 2^{n-1}\sin(nx)$$

67. $\left(x + \dfrac{9}{x} + 6\right)^n = \left(\dfrac{x^2+6x+9}{x}\right)^n = \dfrac{(3+x)^{2n}}{x^n}$

$$= \frac{3^{2n}}{x^n}\left(1+\frac{x}{3}\right)^{2n}$$

$\therefore$ Coefficient of the term independent of $x\left(x+\dfrac{9}{x}+6\right)^n$

$$= 3^{2n}\left[\text{Coefficient of } x^n \text{ in the expansion of} \left(1+\frac{x}{3}\right)^{2n}\right]$$

$$= 3^{2n}\binom{2n}{n}\left(\frac{1}{3}\right)^n = \frac{3^n(2n)!}{n!n!}$$

[using Statement-2]

68. LHS of (1)

$= \text{coefficient of } x^m \text{ in } \{(1+x)^m + (1+x)^{m+1} + \cdots + (1+x)^n\}$ [using Statement-2]

$= \text{coefficient of } x^m$ in

$$(1+x)^m\,\frac{\{(1+x)^{n-m+1}-1\}}{1+x-1}$$

$= \text{coefficient of } x^{m+1} \text{ in } [(1+x)^{n+1} - (1+x)^m]$

$$= \binom{n+1}{m+1}$$

69. We have

$$\binom{p}{r} = \frac{p!}{r!(p-r)!} \quad \Rightarrow \quad r!(p-r)!\binom{p}{r} = p!$$

As $p|p!$, we get $p|r!\,(p-r)!\binom{p}{r}$

But for $1 \le r \le p-1$, neither $r!$ nor $(p-r)!$ is divisible by p.

$$\therefore \quad p\left|\binom{p}{r}\right.$$

We have $\sqrt{5} - 2 = \dfrac{1}{\sqrt{5}+2}$

$\Rightarrow 0 < \sqrt{5} - 2 < 1 \quad \Rightarrow \quad 0 < f = \left(\sqrt{5}-2\right)^n < 1.$

Let $\left(2+\sqrt{5}\right)^n = N + F$ where $0 < F < 1$. Now,

$$N + F - f - 2^{n+1} = \left(\sqrt{5}+2\right)^n - \left(\sqrt{5}-2\right)^n$$

$$= 2\left[\binom{n}{1}5^{(n-1)/2}(2) + \binom{n}{3}5^{(n-3)/2}\left(2^3\right) + \cdots + \binom{n}{n-2}(5)2^{n-1}\right] \quad (1)$$

[$\because$ n is odd]

Since n is an odd prime, each of $\binom{n}{1}, \binom{n}{3}, \ldots \binom{n}{n-2}$ is divisible by n.

Thus RHS of (1) is divisible by $20\,n$

Also, $F - f$ is an integer. Since $0 < F < 1$ and $0 < f < 1$, we get $-1 < F - f < 1$. As $F - f$ is an integer, we get $F - f = 0$ or $F = f$.

$\therefore$ integral part of $\left(2+\sqrt{5}\right)^n - 2^{n+1}$ is $N - 2^{n+1}$, which is divisible by $20\,n$.

70. That Statement-2 is false can be seen from theory.

We can write $\left(\sqrt{3}+\sqrt{2}\right)^{50} = 2^{25}\left(1+\sqrt{\dfrac{3}{2}}\right)^{50}$.

Let $k = \dfrac{(50+1)\sqrt{3/2}}{1+\sqrt{3/2}}$

$$= \frac{51\sqrt{3}\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right)} = 51\left(3-\sqrt{6}\right)$$

$\simeq 51\ (0.5505) \simeq 28.07$

$\Rightarrow [k] = 28.$

Thus, the greatest term in the expansion of $\left(\sqrt{3}+\sqrt{2}\right)^{50}$ is the 29th term and it equals

$$2^{25}\binom{50}{28}\left(\sqrt{\frac{3}{2}}\right)^{28} = \binom{50}{22}3^{14}2^{11}$$

Level 2

71. Note that $\sqrt{x^2 - 1}$ is defined if $x^2 \geq 1$ i.e. if $x \leq -1$ or $x \geq 1$ But $\cos^{-1} x$ is defined for $|x| \leq 1$.

    Thus, only possible values of x are 1 and -1. But these do not satisfy the given equation.

72. $({}^{40}C_0)\ ({}^{40}C_{15}) - ({}^{40}C_1)\ ({}^{40}C_{16}) + ({}^{40}C_2)\ ({}^{40}C_{17})$
    $+ \cdots - ({}^{40}C_{25})\ ({}^{40}C_{40})$
    $= ({}^{40}C_0)\ ({}^{40}C_{25}) - ({}^{40}C_1)\ ({}^{40}C_{24}) + ({}^{40}C_2)\ ({}^{40}C_{23})$
    $+ \cdots - ({}^{40}C_{25})\ ({}^{40}C_0)$
    = Coefficient of x^{25} in $[{}^{40}C_0 - {}^{40}C_1\, x + {}^{40}C_2\, x^2 - \cdots$
    $+ {}^{40}C_{40}\, x^{40}] \times [{}^{40}C_0 + {}^{40}C_1\, x + \cdots + {}^{40}C_{40}\, x^{40}]$
    = Coefficient of x^{25} in $(1 - x)^{40}\ (1 + x)^{40}$
    = Coefficient of x^{25} in $(1 - x^2)^{40} = 0$

73. $3^{100} = 9^{50} = (10 - 1)^{50}$
    $= {}^{50}C_0\, 10^{50} - {}^{50}C_1\, 10^{49} + \cdots\ {}^{50}C_{47}\, 10^3$
    $+ {}^{50}C_{48}\, 10^2 - {}^{50}C_{49}\, (10) + {}^{50}C_{50}$
    $= 1000m + 122000 + 1 = 1000\ (m + 122) + 1$
    where m is a natural number
    Thus, hundred's digit of 3^{100} is 0.

74. $(a + b)^p - a^p - b^p$
    $= {}^pC_1\, a^{p-1}\, b + {}^pC_2\, a^{p-2}\, b^2 + \cdots + {}^pC_{p-1}\, ab^{p-1}$
    For $1 \leq r \leq p - 1$,
    $({}^pC_r)\ (r!)\ (p - r)! = p!$
    As p is prime, $p \mid p!$ but
    $p \nmid (r!)$ and $p \nmid (p - r)!$
    $\therefore\ p \mid {}^pC_r$
    Thus, p divides $(a + b)^p - a^p - b^p$.

75. We have
    $(n + 1)\ ({}^nC_r) = (r + 1)\ ({}^{n+1}C_{r+1})$
    $$\Rightarrow \frac{1}{r+1}\ ({}^nC_r) = \frac{1}{n+1}\ ({}^{n+1}C_{r+1})$$
    $$\text{Thus, } C_0^2 + \frac{1}{2}C_1^2 + \frac{1}{3}C_1^2 + \cdots + \frac{1}{n+1}C_n^2$$
    $$= C_0\left(\frac{1}{1}C_0\right) + C_1\left(\frac{1}{2}C_1\right) + C_2\left(\frac{1}{3}C_2\right)$$
    $$+ \cdots + C_n\left(\frac{1}{n+1}C_n\right)$$
    $$= \frac{1}{n+1}\ [({}^nC_n)\ ({}^{n+1}C_1) + ({}^nC_{n-1})\ ({}^{n+1}C_2)$$
    $$+ ({}^nC_{n-2})\ ({}^{n+1}C_3) + \cdots + ({}^nC_0)\ ({}^{n+1}C_{n+1})]$$
    $$= \frac{1}{n+1}\ ({}^{2n+1}C_{n+1}) = \frac{1}{n+1}\ ({}^{2n+1}C_n)$$

76. $$\sum_{k=0}^{n} \frac{(2n)!}{(k!)^2\ ((n-k)!)^2} = \frac{(2n)!}{n!\,n!}\sum_{k=0}^{n}\left(\frac{n!}{k!\,(n-k)!}\right)^2$$
    $$= ({}^{2n}C_n)\ \sum_{k=0}^{n}\left({}^nC_k\right)^2$$
    $$= ({}^{2n}C_n)\ ({}^{2n}C_n) = ({}^{2n}C_n)^2$$

77. As $\dfrac{C_k}{C_{k+1}} = \dfrac{k+1}{n-k}$,
    $$S = n^2\left(\frac{1}{n}\right) + (n-1)^2\left(\frac{2}{n-1}\right) + \cdots + 1^2\left(\frac{n+0}{1}\right)$$
    $$= \sum_{k=0}^{n}(n-k)(k+1) = n\sum_{k=0}^{n}(k+1) - \sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k$$
    $$= n\left(\frac{1}{2}\right)(n + 1)\ (n + 2) - \left(\frac{1}{6}\right)n\ (n + 1)\ (2n + 1)$$
    $$- \left(\frac{1}{2}\right)\ n\ (n + 1)$$
    $$= \frac{1}{6}\,n\ (n + 1)\ (n + 2)$$

78. Put $x - 3 = t$, so that
    $$\sum_{r=0}^{2n} b_r\, t^r = \sum_{r=0}^{2n} a_r\,(1+t)^r$$
    $\Rightarrow b_n$ = coefficient of t^n of the R.H.S.
    $= a_n\ ({}^nC_n) + (a_{n+1})\ ({}^{n+1}C_n) + \cdots + (a_{2n})\ ({}^{2n}C_n)$
    $$= \sum_{k=0}^{n} a_k\ ({}^{n+k}C_k) = n + 1$$

79. $(1 + x)^{41}\ (1 - x + x^2)^{40}$
    $= (1 + x)\ [(1 + x)\ (1 - x + x^2)]^{40}$
    $= (1 + x)\ (1 - x^3)^{40}$
    Coefficient of x^{50} in $(1 + x)^{41}\ (1 - x + x^2)^{40}$
    = Coefficient of x^{50} in $(1 - x^3)^{40}$
    + Coefficient of x^{49} in $(1 - x^3) = 0 + 0 = 0$

80. $$\sum_{i<j} ({}^nC_i)\ \ ({}^{n+1}C_j)$$
    $$= \sum_{i=0}^{n} ({}^nC_i)\ \ [{}^{n+1}C_{i+1} + {}^{n+1}C_{i+2} + \cdots + {}^{n+1}C_{n+1}]$$

$$= \sum_{i=0}^{n} ({}^nC_i)\,[({}^nC_i + {}^nC_{i+1}) + ({}^nC_{i+1} + {}^nC_{i+2}) + \cdots + ({}^nC_{n-1} + {}^nC_n) + {}^nC_n]$$

$$= \sum_{i=0}^{n} \left({}^nC_i\right)^2 + 2\sum_{i<j} \left({}^nC_i\right)\left({}^nC_j\right)$$

$$= ({}^nC_0 + {}^nC_1 + \cdots + {}^nC_n)^2 = (2^n)^2 = 2^{2n}$$

Previous Years' AIEEE/JEE Main Questions

1. Sum of the coefficients of $(x + y)^n$ is 2^n.

   $\therefore\ 2^n = 4096 = 2^{12} \Rightarrow n = 12$

   Greatest coefficient $= {}^{12}C_6 = 924$

2. $a_r = {}^{p+q}C_r$

   $\Rightarrow a_p = {}^{p+q}C_p = {}^{p+q}C_q = a_q$

3. Let $n = 0.0001$ and

   $$S = \left(1 + \frac{1}{n}\right)^n$$

   $$= 1 + {}^nC_1\left(\frac{1}{n}\right) + {}^nC_2\left(\frac{1}{n}\right)^2 + \ldots + {}^nC_n\left(\frac{1}{n}\right)^n$$

   $$= 1 + 1 + \frac{1}{2!}\frac{n(n-1)}{n^2} + \frac{1}{3!}\frac{n(n-1)(n-2)}{n^3} + \ldots + \frac{1}{n!}\frac{n(n-1)\cdots(n-n+1)}{n^2}$$

   $$= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$

   $$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

   But $\frac{1}{r!} = \frac{1}{1.2\ldots r} \le \frac{1}{2^{r-1}}\ \forall\ r \ge 2.$

   Thus, $S \le 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$

   $$= 1 + \frac{1 - (1/2)^n}{1 - 1/2} = 3 - \left(\frac{1}{2}\right)^{n-1} < 3.$$

4. ${}^{2n}C_{r+1} = {}^{2n}C_{3r-1}$

   $\Rightarrow r + 1 = 3r - 1$ or $(r + 1) + (3r - 1) = 2n$

   $\Rightarrow r = 1$ or $n = 2r$

   As $r > 1$, $n = 2r$.

5. $(r + 1)$th term of $(3^{1/2} + 5^{1/8})^{256}$ is

   $t_{r+1} = {}^{256}C_r\,(3^{1/2})^{256-r}\,(5^{1/8})^r$

   $= {}^{256}C_r\,3^{128 - r/2}\,5^{r/8}$

   This will be an integer if both $r/2$ and $r/8$ are integers, and $0 \le r \le 256$

   $\Rightarrow r = 0, 8, 16, \ldots, 256$

   Thus, there are 33 integral terms.

6. Not relevant to present JEE (Main) syllabus.

7. Note that $S(1)$ is not true

   Assume $S(K)$ is true, then

   $1 + 2 + \ldots + (2K - 1) = 3 + K^2$

   $\Rightarrow 1 + 2 + \ldots + (2K - 1) + (2K + 1)$

   $= 3 + K^2 + (2K + 1)$

   $= 3 + (K + 1)^2$

   Thus, $S(K) \Rightarrow S(K + 1)$

8. Third term of $(1 + \alpha x)^4$ is the middle term and the 4th term of $(1 - \alpha x)^6$ is the middle term. We are given

   ${}^4C_2\,\alpha^2 = {}^6C_3\,(-\alpha)^3 \Rightarrow 6\alpha^2 = -20\alpha^3$

   $\Rightarrow \alpha = -3/10$

9. Coefficient of x^n in $(1 + x)(1 - x)^n$

   $= (1 + x)(-1)^n(x - 1)^n$

   $= (-1)^n(1 + x)\{{}^nC_0\,x^n - {}^nC_1\,x^{n-1} + \ldots\}$

   $= (-1)^n({}^nC_0 - {}^nC_1) = (-1)^n(1 - n)$

10. Coefficient of $(r + 1)$th term in $(1 + y)^m$ is ${}^mC_r$. We are given ${}^mC_{r-1}$, ${}^mC_r$ ${}^mC_{r+1}$ are in A.P.

    $\Rightarrow 2({}^mC_r) = {}^mC_{r-1} + {}^mC_{r+1}$

    $$\Rightarrow 2 = \frac{{}^mC_{r-1}}{{}^mC_r} + \frac{{}^mC_{r+1}}{{}^mC_r}$$

    $$= \frac{m!}{(r-1)!(m-r+1)!}\cdot\frac{r!(m-r)!}{m!} + \frac{m!}{(r+1)!(m-r-1)!}\cdot\frac{r!(m-r)!}{m!}$$

    $$\Rightarrow 2 = \frac{r}{m-r+1} + \frac{m-r}{r+1}$$

    $\Rightarrow 2(m - r + 1)(r + 1) = r(r + 1) + (m - r)(m - r + 1)$

    $\Rightarrow 2(mr - r^2 + r + m - r + 1)$

    $= r^2 + r + m^2 - 2mr + r^2 + m - r$

    $\Rightarrow m^2 - m(4r + 1) + 4r^2 - 2 = 0$

11. $(r + 1)$th term in the expansion of

$\left(ax^2 + \frac{1}{bx}\right)^{11}$ is

$T_{r+1} = {}^{11}C_r\,(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r$

$= {}^{11}C_r\,a^{11-r}\left(\frac{1}{b}\right)^r x^{22-3r}$

For coefficient of x^7, we set

$22 - 3r = 7$ or $r = 5$

$\therefore$ Coefficient of x^7 is ${}^{11}C_5\,\frac{a^6}{b^5}$

Next, $(r + 1)$th term in the expansion of $\left[ax = \frac{1}{bx^2}\right]^{11}$ is

$t_{r+1} = {}^{11}C_r\,(ax)^{11-r}\left(-\frac{1}{bx^2}\right)^r$

$= {}^{11}C_r\,a^{11-r}\left(-\frac{1}{b}\right)^r x^{11-3r}$

For coefficient of x^{-7}, we set $11 - 3r = -7$ or $r = 6$. Thus, coefficient of

x^{-7} is ${}^{11}C_6\,\frac{a^5}{b^6}$. We are given

${}^{11}C_5\,\frac{a^6}{b^5} = {}^{11}C_6\,\frac{a^5}{b^6}$

$\Rightarrow \frac{a^6}{b^5} = \frac{a^5}{b^6}$ $\qquad [\because {}^nC_r = {}^nC_{n-r}]$

$\Rightarrow (ab)^6 = (ab)^5$ or $ab = 1$

12. Not relevant to present JEE (Main) syllabus.

13. $(1 - y)^m (1 + y)^n = 1 + a_1y + a_2y^2 + \ldots$

We have

$(1 - y)^m (1 + y)^n$

$= \left(1 - my + \frac{m(m-1)}{2!}y^2 + \cdots\right)$

$\times \left(1 + ny + \frac{n(n-1)}{2!}y^2 + \cdots\right)$

$= 1 + (n - m)y + \frac{1}{2}(m^2 - m - 2mn + n^2 - n)y^2 + \ldots$

Thus, $10 = a_1 = n - m$

and $10 = a_2 = \frac{1}{2}[(n - m)^2 - (n + m)]$

$\Rightarrow n - m = 10$

and $n + m = 80$

$\Rightarrow n = 45,\ m = 35$

14. ${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - \ldots + {}^{20}C_{10} - {}^{20}C_{11} + \ldots + {}^{20}C_{20} = 0$

Using ${}^nC_r = {}^nC_{n-r}$, we get

$2({}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - \ldots + {}^{20}C_{10}) - {}^{20}C_{10} = 0$

$\Rightarrow {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - \ldots + {}^{20}C_{10} = \frac{1}{2}({}^{20}C_{10})$

15. $T_5 + T_6 = 0$

$\Rightarrow {}^nC_4\,a^{n-4}(-b)^4 + {}^nC_5\,a^{n-5}(-b)^5 = 0$

$\Rightarrow \frac{a^{n-4}b^4}{a^{n-5}b^5} = \frac{{}^nC_5}{{}^nC_4}$

$\Rightarrow \frac{a}{b} = \frac{n!}{5!(n-5)!}\,\frac{4!(n-4)!}{n!} = \frac{n-4}{5}$

16. $\sum_{r=0}^{n}(r+1)\,({}^nC_r)\,x^r$

$= \sum_{r=0}^{n}({}^nC_r)\frac{d}{dx}\{x^{r+1}\}$

$= \frac{d}{dx}\left\{\sum_{r=0}^{n}{}^nC_r x^{r+1}\right\}$

$= \frac{d}{dx}\left\{x\left(\sum_{r=0}^{n}{}^nC_r x^r\right)\right\}$

$= \frac{d}{dx}\{x(1 + x)^n\}$

$= (1 + x)^n + nx(1 + x)^{n-1}$

Thus, Statement-2 is true.

Putting $x = 1$, we get

$\sum_{r=0}^{n}(r+1)\,({}^nC_r) = 2^n + n(1)\,2^{n-1} = (n + 2)\,2^{n-1}$

Thus, statemnen-2 is true and is a correct reason for Statement-1.

17. $8^{2n} - 62^{2n+1}$

$= (9 - 1)^{2n} - (63 - 1)^{2n+1}$

$= ({}^{2n}C_0\,9^{2n} - {}^{2n}C_1\,9^{2n-1} + {}^{2n}C_2\,9^{2n-2} - \ldots - {}^{2n}C_{2n-1}\,(9) + {}^{2n}C_{2n})$

$- ({}^{2n+1}C_0\,(63^{2n+1}) - {}^{2n+1}C_1\,(63^{2n})$

$+ {}^{2n+1}C_2\,(63^{2n-1}) - \ldots + {}^{2n+1}C_1\,(63) - {}^{2n+1}C_0)$

$= 9m + 1 + 1 = 9m + 2$ for some integer m

Thus, when $8^{2n} - 62^{2n+1}$ is divided by 9, the remainder is 2.

18. We have

$$(1 + x)^{10} = \sum_{j=0}^{10} ({}^{10}C_j)x^j$$

Differentiating both the side w.r.t. x, we get

$$10(1 + x)^9 = \sum_{j=1}^{10} j({}^{10}C_j)x^{j-1} \qquad (1)$$

Again differentiating both the sides w.r.t, x, we get

$$(10)\ (9)\ (1 + x)^8 = \sum_{j=1}^{10} j\ \ (j - 1)\ ({}^{10}C_j)x^{j-2} \qquad (2)$$

Putting $x = 1$ in (1) and (2), we get

$$S_2 = \sum_{j=1}^{10} j({}^{10}C_j) = 10(2^9)$$

and $S_1 = \sum_{j=1}^{10} j(j-1)\ ({}^{10}C_j) = (10)\ (9)\ (2^8) = (90)\ (2^8)$

Adding the above two equations, we get

$$S_3 = \sum_{j=1}^{n} [j + j(j-1)]\ ({}^{10}C_j) = (10)\ (2^8)\ (2 + 9)$$

$\Rightarrow S_3 = (55)\ (2^9)$

Thus, Statement-1is true but Statement-2 is false.

19. $1 - x - x^2 + x^3 = (1 - x) - x^2\,(1 - x)$

$= (1 - x)\ (1 - x^2)$

We have $(1 - x - x^2 + x^3)^6 = (1 - x)^6\ (1 - x^2)^6$

Coefficient of x^7 in $(1 - x - x^2 + x^3)^6$

$=$ Coefficient of x^7 in $(1 - {}^6C_1\,x + {}^6C_2\,x^2 - \ldots$

$+ {}^6C_6\,x^6)\ (1 - {}^6C_1\,x^2 + {}^6C_2\,x^4 - {}^6C_3\,x^6 + \ldots)$

$= (-\,{}^6C_5)\ (-\,{}^6C_1) + (-\,{}^6C_3)\ ({}^6C_2)$

$+ (-\,{}^6C_1)\ (-\,{}^6C_3)$

$= 36 - 300 + 120 = -\,144$

20. For Statement-2, write

$n = 7k + r$ where $r = -\,3, -\,2, -\,1, 0, 1, 2, 3.$

Now $n^7 - n = (7k + r)^7 - (7k + r)$

$\Rightarrow n^7 - n - (r^7 - r)$ is a multiple of 7.

Thus, $n^7 - n$ is a divisible of 7 if and only if $r^7 - r$ is a divisible of 7.

For $r = -\,3$, $r^7 - r = (-3)^7 - (-3)$

$= (-3)\ [3^6 - 1]$

$= (-3)\ (128)$

is a divisible of 7.

For $r = -2$, $r^7 - r = (-\,2)^7 - (-\,2)$

$= -\,128 + 2 = -\,126$

is a divisible of 7.

Similarly, for $r = -\,1, 0, 1, 2, 3$, $r^7 - r$ is divisible by 7.

Thus, $n^7 - n$ is divisible by 7.

$\Rightarrow (n + 7)^7 - (n + 1)$ is divisible by 7.

$\therefore\ (n + 1)^7 - (n + 1) - (n^7 - n)$ is divisible by 7

$\Rightarrow (n + 1)^7 - n^7 - 1$ is divisible by 7.

Thus, Statement-1 is true.

Statement-1 and Statement-2 are both true.

Hence Statement-2 is a correct explanation for Statement-1.

21. Using binomial theorem we have

$$(\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n}$$

$$= 2\left[{}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_3(\sqrt{3})^{2n-3} + \cdots + {}^{2n}C_{2n-1}(\sqrt{3})\right]$$

$$= 2\sqrt{3}\ \ [{}^{2n}C_1\ (3^{n-1}) + {}^{2n}C_3\ (3^{n-2}) + \ldots + {}^{2n}C_{2n-1}]$$

This is an irrational number.

22. We have

$$\frac{x+1}{x^{2/3} - x^{1/3} + 1} = x^{1/3} + 1$$

and $\dfrac{x+1}{x - x^{1/2}} = \dfrac{x-1}{\sqrt{x}(\sqrt{x} - 1)} = 1 + x^{-1/2}$

$(r + 1)^{\text{th}}$ term of $\left(\dfrac{x+1}{x^{2/3} - x^{1/3} + 1} + \dfrac{x-1}{x - x^{1/2}}\right)^{10}$

$= (x^{1/3} - x^{-1/2})^{10}$ is

$T_{r+1} = {}^{10}C_r\ (x^{1/3})^{10-r}\ (x^{-1/2})^r$

Note that T_{r+1} will be independent of x if

$\dfrac{10 - r}{3} - \dfrac{r}{2} = 0 \Rightarrow r = 4$. Thus, term independent of x equals ${}^{10}C_4 = 210$.

23. $T_7 = T_{6+1} = {}^9C_6 \left(\dfrac{3}{84^{1/3}}\right)^{9-6} (\sqrt{3}\ln x)^6$

$= (84)\left(\dfrac{3^3}{84}\right)(3^3 (\ln x)^6) = 729$

$\Rightarrow (\ln x)^6 = 1$

$\Rightarrow \ln x = \pm 1 \Rightarrow x = e,\ 1/e$

Thus, $x = e$.

24. See solution to Question 4.

25. $T_{r+1} = {}^{15}C_r (x^2)^{15-r} \left(\dfrac{2}{x}\right)^r$

$= {}^{15}C_r\, 2^r\, x^{30-3r}$.

For coefficient of x^{15}, set

$30 - 3r = 15 \Rightarrow r = 5$

and for term independent of x, set

$30 - 3r = 0 \Rightarrow r = 10$

The required ratio is

${}^{15}C_5\, 2^5 : {}^{15}C_{10}\, 2^{10} = 1 : 32$

26. $T_{r+1} = {}^{10}C_r (2^{1/2})^{10-r} (3^{1/5})^r$

$= {}^{10}C_r\, 2^5\, 2^{-r/2}\, 3^{r/5}\ (0 \le r \le 10)$

For rational terms, both $r/2$ and $r/5$ must be integers.

This is possible if $r = 0$ or $r = 10$

Thus, required sum

$= {}^{10}C_0\, 2^5 + {}^{10}C_{10}\, 2^5\, 2^{-5}\, 3^2 = 41$.

27. $(1 + ax + bx^2)(1 - 2x)^{18}$

$= (1 + ax + bx^2)\,[1 - {}^{18}C_1 (2x) + {}^{18}C_2 (2x)^2 - {}^{18}C_3 (2x)^3 + {}^{18}C_4 (2x)^4 - \ldots]$

Coefficient of x^3

$= -\,{}^{18}C_3 (2^3) + ({}^{18}C_2)2^2\, a - ({}^{18}C_1)(2)\, b = 0$

Coefficient of x^4

$= {}^{18}C_4\, 2^4 - {}^{18}C_3 (2^3)(a) + {}^{18}C_2 (2^2)(b) = 0$

$\Rightarrow b = -\dfrac{544}{3} + 17a$

and $b = -80 + \dfrac{32}{3}a$

$\Rightarrow \dfrac{-544}{3} + 17a = -80 + \dfrac{32}{3}a$

$\Rightarrow a = 16$

$\therefore\ b = 272/3$

28. $(1 + x)^{101} (1 + x^2 - x)^{100}$

$= (1 + x)(1 + x)^{100} (1 - x + x^2)^{100}$

$= (1 + x)(1 - x^3)^{100}$

$= (1 - x^3)^{100} + x(1 - x^3)^{100}$

$= \sum_{r=0}^{100} (-1)^r\, ({}^{100}C_r)\, (x^{3r} + x^{3r+1})$

Thus, number of terms is 202.

29. We have

$E = (1 + x)^{1000} + x(1 + x)^{999} + x^2(1 + x)^{998} + \ldots + x^{1000}$

$$= \frac{(1+x)^{1000}\left[1-\left(\dfrac{x}{1+x}\right)^{1001}\right]}{1-x/(1+x)} = (1 + x)^{1001} - x^{1001}$$

$\therefore$ Coefficient of x^{50} in E

$= {}^{1001}C_{50} = \dfrac{(1001)!}{(50)!(951)!}$

30. $\left(2+\dfrac{x}{3}\right)^{55} = \sum_{r=0}^{55} {}^{55}C_r \left(\dfrac{x}{3}\right)^r (2^{55-r})$

$= \sum_{r=0}^{55} {}^{55}C_r \left(\dfrac{2^{55-r}}{3^r}\right) x^r$

We are given coefficients in two consecutive powers are equal

$\Rightarrow {}^{55}C_r \left(\dfrac{2^{55-r}}{3^r}\right) = {}^{55}C_{r+1} \left(\dfrac{2^{54-r}}{3^{r+1}}\right)$

$\Rightarrow 6 = \dfrac{55!}{(r+1)!(54-r)!} \cdot \dfrac{r!(55-r)!}{55!}$

$\Rightarrow 6 = \dfrac{55-r}{r+1}$

$\Rightarrow 6r + 6 = 55 - 4 \Rightarrow r = 7$

Thus, these terms are 8th and 9th.

31. Replacing x by $x - 1$, we get

$\sum_{i=0}^{5} a_i (1+x-1)^i = 1 + (x - 1)^4 + (x - 1)^5$

$\Rightarrow \sum_{i=0}^{5} a_i\, x^i = 1 + (1 - x)^4 - (1 - x)^5$

a_2 = coefficient of x^2 on RHS

$= {}^4C_2 - {}^5C_2 = 6 - 10 = -4.$

32. $(1 + x^n + x^{253})^{10} = \sum\limits_{\substack{r,s\geq 0\\ r+s\leq 10}} \dfrac{10!\,x^{253r}\,x^{ns}}{r!s!(10-r-s)!}$

To obtain coefficient of 1012, we set

$253r + ns = 1012$

$\Rightarrow r = 4,\ s = 0.$

$\therefore$ required coefficient $= \dfrac{10!}{4!0!6!}$

$= {}^{10}C_4$

33. $(1-2\sqrt{x})^{50} = 1 + {}^{50}C_1\ (-2\sqrt{x}) + {}^{50}C_2\ (-2\sqrt{x})^2$

$+ {}^{50}C_3\ (-2\sqrt{x})^3 + {}^{50}C_4\ (-2\sqrt{x})^4$

$+ \ldots + {}^{50}C_{49}\ (-2\sqrt{x})^{49}$

$+ {}^{50}C_{50}\ \left(-2\sqrt{x}\right)^{50}$

The sum of coefficients of the integral powers of

$= 1 + {}^{50}C_2\ (-2)^2 + {}^{50}C_4\ (-2)^4 + \ldots + {}^{50}C_{50}\ (-2)^{50}$

$= \dfrac{1}{2}\ [(1 + (-2))^{50} + (1 - (-2))^{50}]$

$= \dfrac{1}{2}\ (3^{50} + 1)$

34. Let the first term be the $(r + 1)$th term.

We have

$t_{r+1} = {}^nC_r\, x^r$

${}^nC_r : {}^nC_{r+1} : {}^nC_{r+2} = 1 : 7 : 42$

$\Rightarrow \dfrac{{}^nC_r}{{}^nC_{r+1}} = \dfrac{1}{7}$ and $\dfrac{{}^nC_{r+1}}{{}^nC_{r+2}} = \dfrac{7}{42}$

$\Rightarrow \dfrac{r+1}{n-r} = \dfrac{1}{7}$ and $\dfrac{r+2}{n-r-1} = \dfrac{1}{6}$

$\Rightarrow n = 8r + 7$ and $n = 7r + 13$

$\therefore 8r + 7 = 7r + 13 \Rightarrow r = 6$

Thus, first of these terms is the 7th term.

35. General term, T_{r+1}

of $\left(2x^2 - \dfrac{1}{x}\right)^8$ is

$T_{r+1} = {}^8C_r(2x^2)^{8-r}\left(-\dfrac{1}{x}\right)^r$

$= {}^8C_r\ (-1)^r\ 2^{8-r}\ x^{16-3r}$

To obtain term independent of x in the expansion of

$$\left(1 - \frac{1}{x} + 3x^5\right)\left(2x^2 - \frac{1}{x}\right)^8 \qquad (1)$$

we set $16 - 3r = 0$, $16 - 3r = 1$ and $16 - 3r = -5$

But $16 - 3r = 0$ has no solution;

$16 - 3r = 1 \Rightarrow r = 5$

and $16 - 3r = -5 \Rightarrow r = 7$

Thus, term independent of x in (1)

$= (-1)\ ({}^8C_5)\ (-1)^5\ 2^{8-5} + 3({}^8C_7)\ (-1)^7\ 2^{8-7}$

$= (56)\ (8) - 3(8)\ (2) = 400$

36. Number of terms in the expansion of $\left(1 - \dfrac{2}{x} + \dfrac{4}{x^2}\right)^n$

is ${}^{n+2}C_2 = \dfrac{1}{2}\ (n + 2)\ (n + 1)$

$\Rightarrow \dfrac{1}{2}\ (n + 1)\ (n + 1) = 28 \Rightarrow n = 7$

Sum of the coefficients is obtained by putting $x = 1$ in the given expression. Thus, required sum $= (1 - 2 + 4)^7 = 3^7 = 2187.$

37. $\sum\limits_{i=0}^{2016} a_j x^i = (1 + x)^{2016} + x(1 + x)^{2015} + x^2(1 + x)^{2014}$

$+ \ldots + x^{2016}$

$= \dfrac{(1+x)^{2016}\left[1 - \left(\dfrac{x}{1+x}\right)^{2017}\right]}{1 - \dfrac{x}{1+x}}$

$= (1 + x)^{2017} - x^{2017}$

$\therefore a_{17}$ = coefficient of x^{17}

$= {}^{2017}C_{17} = \dfrac{2017!}{17!2000!}$

38. $T_{r+1} = {}^{18}C_r\ (x^{1/3})^{18-r}\left(\dfrac{1}{2x^{1/3}}\right)^r$

$= {}^{18}C_r\ x^{6-2r/3}\left(\dfrac{1}{2}\right)^r$

For coefficient of x^{-2}, $6 - 2r/3 = -2 \Rightarrow 8 = 2r/3$ $\Rightarrow r = 12$

and for coefficient of x^{-4},

$6 - 2r/3 = -4 \Rightarrow 10 = 2r/3 \Rightarrow r = 15$

$\therefore\ m = {}^{18}C_{12}\left(\frac{1}{2}\right)^{12},\ n = {}^{18}C_{15}\left(\frac{1}{2}\right)^{15}$

Thus, $\frac{m}{n} = \frac{{}^{18}C_{12}}{{}^{18}C_{15}}\ (2^3)$

$= \frac{18!}{12!6!}\cdot\frac{15!3!}{18!}\ (2^3)$

$= \frac{15\times 14\times 13}{6\times 5\times 4}\cdot(8) = 182$

Previous Years' B-Architecture Entrance Examination Questions

1. $T_{r+1} = {}^{n}C_r\ (x^3)^{n-r}\left(-\frac{1}{x^2}\right)^r$

$= {}^{n}C_r\ (-1)^r\ x^{3n-5r}$

For coefficient of x^5, set $3n - 5r = 5$

$\Rightarrow r = (3n - 5)/5 = l$ (say)

For coefficient of x^{10}, set $3n - 5r = 10$

$\Rightarrow r = (3n - 10)/5 = m$ (say)

Note that $l - m = 1$.

We are given

${}^{n}C_l\ (-1)^l + {}^{n}C_m\ (-1)^m = 0$

$\Rightarrow {}^{n}C_l = {}^{n}C_m \Rightarrow l + m = n.$

$\Rightarrow \frac{3n-5}{5} + \frac{3n-10}{5} = n$

$\Rightarrow 6n - 15 = 5n \Rightarrow n = 15.$

Coefficient of third term

$= {}^{15}C_2\ (-1)^2 = 105$

2. a_1 = coefficient of x in

$(1 + x)\ (1 + x + x^2)\ (1 + x + x^2 + x^3)$

$\ldots\ (1 + x + x^2 + \ldots + x^n)$

= coefficient of x in $(1 + x)^n$

$= {}^{n}C_1 = n$

3. Let $a = 7^{\text{th}}$ term from the beginning

$= T_7 = T_{6+1} = {}^{n}C_6\ (2^{1/3})^{n-6}\ (3^{-1/3})^6$

and let $b = 7^{\text{th}}$ term from the end

$= (n + 2 - 7)$th or $(n - 5)$th term from the beginning

$= {}^{n}C_{n-6}\ (2^{1/3})^{n-(n-6)}\ (3^{-1/3})^{n-6}$

$= {}^{n}C_6\ (2^{1/3})^6\ (3^{-1/3})^{n-6}$

We are given

$\frac{a}{b} = \frac{1}{6}$

$\frac{(2^{1/3})^{n-6}(3^{-1/3})^6}{(2^{1/3})^6(3^{-1/3})^{n-6}} = \frac{1}{6}$

$\Rightarrow \left(\frac{(2^{1/3})}{(3^{-1/3})}\right)^{n-12} = \frac{1}{6}$

$\Rightarrow 6^{(n-12)/3} = 6^{-1}$

$\Rightarrow n - 12 = -3 \Rightarrow n = 9.$

4. $7^{128} = (7^2)^{64} = (50 - 1)^{64} = (1 - 50)^{64}$

$= 1 + \sum_{k=1}^{64} {}^{64}C_k(-50)^k$

Thus, remainder when 7^{128} is divided by 10 is 1.

5. For $0 \le j \le 8$,

$\frac{1}{(j+1)(j+2)}\ ({}^{8}C_j)$

$= \frac{8!}{(j+1)(j+2)j!(8-j)!}$

$= \frac{1}{10\times 9}\frac{10!}{(j+2)!(10-(j+2))!}$

$= \frac{1}{90}({}^{10}C_{j+2})$

Thus,

$S = \frac{1}{90}\left(\sum_{j=0}^{10} {}^{10}C_{j+2}\right)$

$= \frac{1}{90}\left(\sum_{k=0}^{10} {}^{10}C_k - {}^{10}C_0 - {}^{10}C_1\right)$

$= \frac{1}{90}\ (2^{10} - 1 - 10) = \frac{1013}{90}$

6. $\sum_{j=0}^{n} a^{n-j}b^j = a^n + a^{n-1}\ b + a^{n-2}\ b^2 + \ldots + ab^{n-1} + b^n$

$= \frac{a^n[1-(b/a)^{n+1}]}{1-b/a}$ [sum of a G.P.]

$= \frac{a^{n+1} - b^{n+1}}{a - b}$

Also, coefficient of x^m in $(\alpha x + \beta)^m$ is $(m + 1)\ \alpha^m\ \beta$

$\therefore$ Statement-2 is true.

Next, $\sum_{j=0}^{n} (-1)^j \; (2x + 3)^{n-j} \; (5 - 2x)^j$

$= \sum_{j=0}^{n} (2x+3)^{n-j} \; (2x - 5)^j$

$= \dfrac{(2x+3)^{n+1} - (2x-5)^{n+1}}{(2x+3)-(2x-5)}$

$= \dfrac{1}{8} \; [(2x + 3)^{n+1} - (2x - 5)^{n+1}]$

Coefficient of x^n is

$\dfrac{1}{8} \; [{}^{n+1}C_n \, 2^n \, (3) - {}^{n+1}C_n \, 2^n \, (-5)]$

$= (n + 1) \, 2^n$

$\therefore$ Statement-1 is true and Statement-2 is correct explanation for it.

7. $x^n = [(1 + x) - 1]^n = [1 - (1 - x)]^n$

$= \sum_{k=0}^{n} {}^nC_k \; (1 + x)^{n-k} \, (-1)^k$

$= \sum_{k=0}^{n} {}^nC_k \; (-1)^k \, (1 - x)^k$

$\therefore$ a_k = coefficient of $(1 + x)^k$ in

$\sum_{k=0}^{n} {}^nC_k \; (1 + x)^{n-k} \, (-1)^k = (-1)^{n-k} \, {}^nC_{n-k} = (-1)^{n-k} \, {}^nC_k$

and $b_k = {}^nC_k \, (-1)^k$

For $n = 101$, $k = 51$, we get

$(a_{51}, b_{51}) = ({}^{101}C_{51}, -{}^{101}C_{51})$

8. $T_3 = {}^5C_2 \left(\dfrac{1}{x}\right)^3 (x^{\log_{10} x})^2$

$= 10 \; x^{2\log_{10} x - 3}$

As $T_3 = 1000$, we get

$x^{2\log_{10} x - 3} = 100$

$(2a - 3)a = 2$ where $a = \log_{10} x$

$\Rightarrow 2a^2 - 3a - 2 = 0$

$\Rightarrow (2a + 1) \, (a - 2) = 0$

As $x > 1$, $a = \log_{10} x > 0$, thus,

$a = 2 \Rightarrow \log_{10} x = 2 \Rightarrow x = 100$

9. Sum of the coefficients in the expansion of $(x + y)^n$ is 2^n. Therefore,

$2^n = 2048 = 2^{11} \Rightarrow n = 11.$

As n is odd, the greatest value of ${}^nC_r$ is ${}^nC_k$ where

$k = \dfrac{1}{2} \; (n - 1)$ or $k = \dfrac{1}{2} \; (n + 1).$

Here $n = 11$, therefore $k = 5$ or 6.

Thus, greatest coefficient is ${}^{11}C_6$.

10. $(1 + x + x^2)^8 = \sum_{\substack{p,q,r \ge 0 \\ P+q+r=8}} \dfrac{8!}{p!q!r!} \; 1^p \, x^p (x^2)^r$

For a_5, we require, $q + 2r = 5$,

$\Rightarrow q = 5, r = 0, p = 3$ or $q = 3, r = 1, p = 4,$

or $q = 1, r = 2, p = 5.$

Thus, coefficient of x^5 is

$\dfrac{8!}{5!3!} + \dfrac{8!}{3!1!4!} + \dfrac{8!}{1!2!5!} = 504$

11. Coefficient to t^{24} in $(1 + t^2)^{12} \, (1 + t^{12}) \, (1 + t^{24})$

= Coefficients of t^{24} in $(1 + {}^{12}C_1 \, t^2 + {}^{12}C_2 \, t^4 + \ldots)$
$(1 + t^{12} + t^{24})$

$= {}^{12}C_{12} + {}^{12}C_6 + 1 = {}^{12}C_6 + 2$

12. Let

S = sum of the last 30 coefficients in $(1 + x)^{59}$

$= {}^{59}C_{30} + {}^{59}C_{31} + \ldots + {}^{59}C_{59}$ (1)

Using ${}^nC_r = {}^nC_{n-r}$, we get

$S = {}^{59}C_{29} + {}^{59}C_{28} + \ldots + {}^{59}C_0$ (2)

Adding (1) and (2), we get

$2S = {}^{59}C_0 + {}^{59}C_1 + \ldots + {}^{59}C_{59} = 2^{59}$

$\Rightarrow S = 2^{58}.$

13. We have

$(1 - x)^m \, (1 + x)^n$

$= (1 - mx + {}^mC_2 \, x^2 - \ldots) \, (1 + nx + {}^nC_2 \, x^2 + \ldots)$

$= 1 + (n + m)x + ({}^mC_2 - mn + {}^nC_2)x^2 + \ldots$

We are given

$n - m = 3, \; {}^mC_2 - mn + {}^nC_2 = -4$

$\Rightarrow n - m = 3, \; \dfrac{1}{2} \; (m^2 - 2mn + n^2 - (m + n)) = -4$

$\Rightarrow n - m = 3, \; (m - n)^2 - (m + n) = -8$

$\Rightarrow n - m = 3,\ m + n = 17$

$\therefore\ n = 10,\ m = 7$

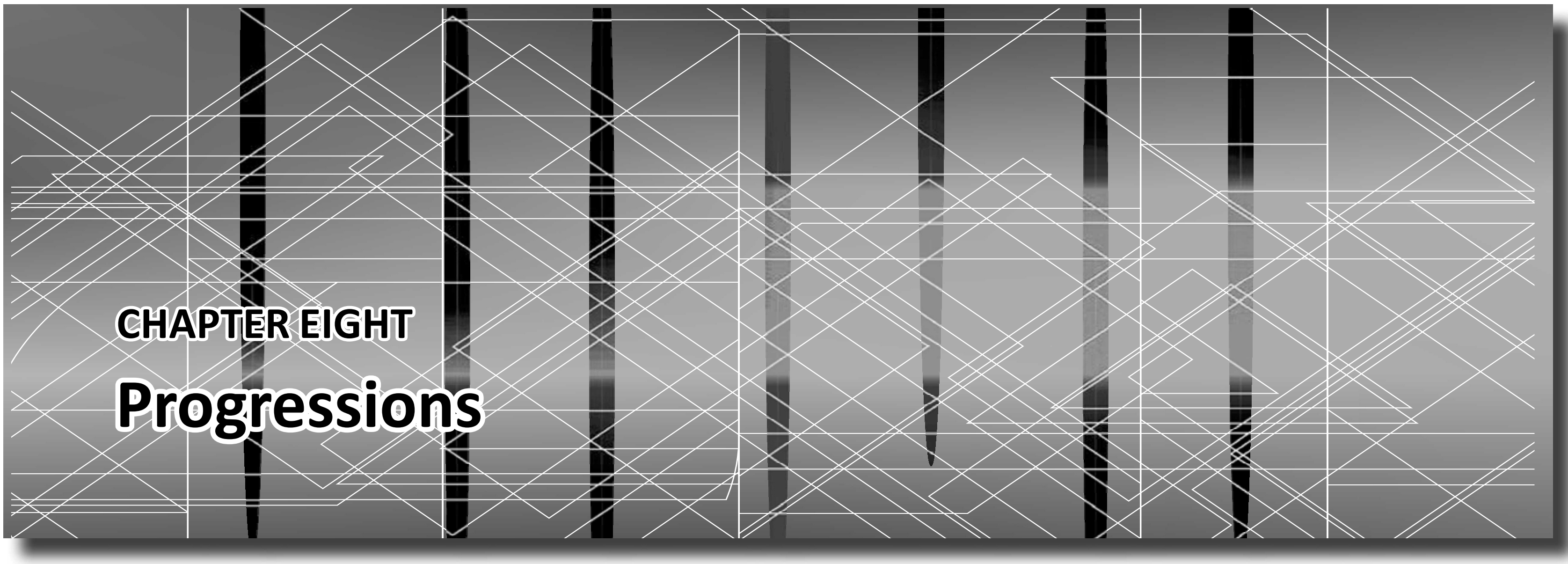
$\Rightarrow m : n = 7 : 10$

14. Mean of binomial coefficients in the expansion of $(a + b)^{2n-3}$ is

$$\frac{1}{2n-3+1}\left[{}^{2n-3}C_0 + {}^{2n-3}C_1 + \dots + {}^{2n-3}C_{2n-3}\right]$$

$$= \frac{1}{2(n-1)}\,2^{2n-3} = \frac{2^{2(n-2)}}{n-1} = 16 = 2^4$$

$\Rightarrow n = 5$



CHAPTER EIGHT

Progressions

SEQUENCES

A sequence is a function whose domain is the set of natural numbers. If the range of a sequence is a subset of real numbers (complex numbers), then it is called a *real sequence* (*complex sequence*).

If $f: \mathbf{N} \to \mathbf{C}$, is a sequence, we usually denote it by

$$\langle f(1), f(2), f(3), \cdots\rangle = \langle f(n)\rangle$$

If $t_n = f(n)$, then the sequence is written as $\langle t_1, t_2, t_3, \cdots\rangle$

ARITHMETIC PROGRESSION

A sequence of numbers $\langle t_n\rangle$ is said to be in arithmetic progression (A.P.) when the difference $t_n - t_{n-1}$ is a constant for all $n \in \mathbf{N}$, $n > 1$. This constant is called the common difference of the A.P., and it is usually denoted by the letter d.

If a is the first term and d the common difference of the A.P., then its nth term t_n is given by

$$t_n = a + (n - 1)d$$
$$t_n = t_m + (n - m)d$$

The sum S_n of the first n terms of such an A.P. is given by

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

where l is the last term, that is, the nth term of the A.P.

Some Facts About A.P.

1. If a fixed number is added to (subtracted from) each term of a given A.P., then the resulting sequence is also an A.P., and it has the same common difference as that of the given A.P.
2. If each term of an A.P. is multiplied by a fixed constant (or divided by a non-zero fixed constant), then the resulting sequence is also an A.P.
3. If $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ are two arithmetic progressions, then $a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots$ is also an A.P.
4. If $a_1, a_2, a_3, \ldots$ are in A.P., then

   $$a_m - a_n = (m - n)d$$

   where d is the common difference of the A.P.

Illustration 1

If 20th term of an A.P. is 25 and 25th term is 15, find 17th term of the A.P.

If d is the common difference of the A.P., then

$(25 - 20)d = a_{25} - a_{20} = 15 - 25$

$\Rightarrow d = -2.$

Also $a_{17} - a_{20} = (17 - 20)d = 6$

$\Rightarrow a_{17} = 25 + 6 = 31$

5. If we have to take three terms in A.P. whose sum is known, it is convenient to take them as $a - d$, a, $a + d$. In general, if we have to take $(2r + 1)$ terms in A.P., we take them as $a - rd$, $a - (r - 1)d$, $\cdots$, $a - d$, a, $a + d$, $\cdots$, $a + rd$.
6. If we have to take four terms in A.P., whose sum is known, we take them as $a - 3d$, $a - d$, $a + d$, $a + 3d$.
7. The *arithmetic mean A* of any two numbers a and b is given by $(a + b)/2$. It is to be noted that the sequence a, A, b is in A.P.

   If $a_1, a_2, \cdots, a_n$ are n numbers, then the arithmetic mean A, of these numbers is given by

   $$A = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

8. The n numbers $A_1, A_2, \cdots, A_n$ are said to be *n arithmetic means between a and b* if $a, A_1, A_2, \cdots, A_n, b$ is an A.P. Here a is the first term and b is the $(n + 2)$th term of the A.P. If d is the common difference of this A.P., then

   $$b = a + (n + 2 - 1)d = a + (n + 1)d \Rightarrow d = \frac{b - a}{n + 1}$$

   Thus,

   $$A_1 = a + \frac{b - a}{n + 1}, A_2 = a + \frac{2(b - a)}{n + 1},$$
   $$\cdots A_n = a + \frac{n(b - a)}{n + 1}$$

TIP

Sometimes, it is useful to use $A_n = b - d$ etc.

Sum of n Arithmetic means $A_1 + A_2 + \ldots + A_n = \frac{n}{2}(a + b)$

GEOMETRIC PROGRESSION

A sequence is said to be in geometric progression (G.P.) when its first term is non-zero and each of its succeeding terms is r times the preceding term, where r is some fixed non-zero number. The fixed number r is known as the *common ratio* of the G.P. It is clear from the definition that no term of a G.P. is zero. If a is the first term of a G.P. and r its common ratio, then its nth term, t_n, is given by

$$t_n = ar^{n-1}$$

The sum S_n of the first n terms of such a G.P. is given by

$$S_n = \begin{cases} \dfrac{a(r^n - 1)}{r - 1} & \text{if } r \neq 1 \\ na & \text{if } r = 1 \end{cases}.$$

If $-1 < x < 1$, then

$$\lim_{n \to \infty} x^n = 0.$$

Therefore, the sum of the infinite G.P.

$$1 + x + x^2 + \cdots = \frac{1}{1 - x}.$$

Thus $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots \; (-1 < x < 1) \qquad (1)$

Differentiating both the sides w.r.t. x, we get

$$(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \; (-1 < x < 1) \qquad (2)$$

Differentiating (1) $(m - 1)$ times w.r.t. x, we get for $-1 < x < 1$

$$(m - 1)!\,(1 - x)^{-m} = (m - 1)! + m!x + \frac{(m+1)!}{2!}x^2 + \frac{(m+2)!}{3!}x^3 + \cdots$$

$$\Rightarrow \quad (1 - x)^{-m} = 1 + {}^{m}C_1\,x + {}^{m+1}C_2\,x^2 + {}^{m+2}C_3\,x^3 + \cdots \qquad (3)$$

for $-1 < x < 1$

Also, if n is any real number, then

$$(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \; (-1 < x < 1)$$

If $-1 < r < 1$, then the sum of the infinite G.P.

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r}$$

Some Facts About G.P.

1. If each term of a G.P. is multiplied (divided) by a fixed non-zero constant, then the resulting sequence is also a G.P.
2. If $a_1, a_2, a_3, \cdots$ and $b_1, b_2, b_3, \cdots$ are two geometric progressions, then the sequence $a_1b_1, a_2b_2, a_3b_3, \cdots$ is also a G.P.
3. The *geometric mean* G of two positive numbers a and b is given by $\sqrt{ab}$. It is to be noted that a, G and b are in G.P. If $a_1, a_2, \cdots, a_n$ are n positive numbers, then their geometric mean is given by

   $$G = (a_1\, a_2 \cdots a_n)^{1/n}$$
4. The n numbers $G_1, G_2, \ldots G_n$ are said to be n geometric means between two positive real numbers a and b if $a, G_1, G_2, \ldots, G_n, b$ forms a G.P. Let r be the common ratio of this G.P., then

   $b = (n + 2)$th term of the G.P.

   $= ar^{n+2-1} = ar^{n+1}$

   $\Rightarrow r = \left(\frac{b}{a}\right)^{1/(n+1)}$

   Thus, $G_1 = a\left(\frac{b}{a}\right)^{1/(n+1)}$, $G_2 = a\left(\frac{b}{a}\right)^{2/(n+1)}$

   $\ldots, G_n = a\left(\frac{b}{a}\right)^{n/(n+1)}$

   Also, note that

   $G_1\, G_2 \ldots G_n = (ab)^{n/2}$
5. If we have to take three terms in G.P., we take them as a/r, a, ar and four terms as a/r^3, a/r, ar, ar^3.
6. If $a_1, a_2, a_3, \cdots$ is a G.P. ($a_i > 0\ \forall i$), then $\log a_1, \log a_2, \log a_3, \cdots$ is an A.P. In this case the converse also holds.

   Suppose $a_1, a_2, a_3, \cdots$ is a G.P. Let $a_i = AR^{i-1}$, where A is the first term and R is the common ratio of the G.P. Then

   $$\log a_i = \log A + (i - 1)\log R$$

   This shows that $\log a_1, \log a_2, \log a_3, \cdots$ is an A.P. whose first term is $\log A$ and the common difference is $\log R$. Conversely, assume that $\log a_1, \log a_2, \log a_3, \cdots$ is an A.P. Suppose that $\log a_i = a + (i - 1)d$, where a is the first term and d is the common difference of the A.P. Then

   $$a_i = \alpha^{a + (i-1)d} = \alpha^a (\alpha^d)^{i-1}$$

   where α is the base of the logarithm. This shows that $a_1, a_2, a_3, \cdots$ is a G.P., with the first term α^a and common ratio α^d.

ARITHMETICO-GEOMETRIC SEQUENCE

Suppose $a_1, a_2, \cdots, a_n, \cdots$ is an A.P., and $b_1, b_2, \cdots, b_n, \cdots$ is a G.P. Then the sequence $a_1 b_1, a_2 b_2, \cdots, a_n b_n, \cdots$ is said to be an arithmetico-geometric sequence. An arithmetico-geometric sequence is of the form

$$ab, (a + d)br, (a + 2d)br^2, (a + 3d)br^3, \cdots$$

Sum to n term of an Arithemtico-Geometric series.

Let $S_n = ab + (a + d)br + (a + 2d)br^2 + \cdots +$

$[a + (n - 2)d]br^{n-2} + [a + (n - 1)d]br^{n-1} \cdots \qquad (1)$

We multiply each term by r and write the first term below the second, the second below the third and so on.

$$rS_n = abr + (a + d)br^2 + \cdots + [a + (n - 3)d]br^{n-2} + [a + (n - 2)d]br^{n-1} + [a + (n - 1)d]br^n$$

Subtracting, we get

$$(1 - r)S_n = ab + dbr + dbr^2 + \cdots + dbr^{n-2} + dbr^{n-1} - [a + (n - 1)d]br^n$$

$$= ab + \frac{dbr(1 - r^{n-1})}{1 - r} - [a + (n - 1)d]br^n$$

$$\Rightarrow S_n = \frac{ab}{1 - r} + \frac{dbr(1 - r^{n-1})}{(1 - r)^2} - \frac{[a + (n - 1)d]br^n}{1 - r}, \; r \neq 1$$

If $-1 < r < 1$, then

$$\lim_{n \to \infty} r^n = 0 \quad \text{and} \quad \lim_{n \to \infty} nr^n = 0$$

Therefore, the sum of an infinite number of terms of the series in (1) is

$$S = \frac{ab}{1 - r} + \frac{dbr}{(1 - r)^2} \qquad (-1 < r < 1)$$

HARMONIC PROGRESSION

The sequence $a_1, a_2, \cdots, a_n, \cdots$ where $a_i \neq 0$ for each i is said to be in harmonic progression (H.P.) if the sequence $1/a_1, 1/a_2, \cdots, 1/a_n, \cdots$ is in A.P. Note that a_n, the nth term of the H.P., is given by

$$a_n = \frac{1}{a + (n - 1)d} \quad \text{where } a = \frac{1}{a_1} \text{ and } d = \frac{1}{a_2} - \frac{1}{a_1}$$

Some Facts about H.P.

1. If a and b are two non-zero numbers, then the *harmonic mean* of a and b is a number H such that the sequence a, H, b is a H.P. We have

   $$\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \quad \text{or} \quad H = \frac{2ab}{a + b}$$

   If $a_1, a_2, \cdots, a_n$ are n non-zero numbers, then the harmonic mean H of these numbers is given by

   $$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

2. The n numbers $H_1, H_2, \cdots, H_n$ are said to be *harmonic means between a and b* if $a, H_1, H_2, \cdots, H_n, b$ are in H.P., that is, if $1/a, 1/H_1, 1/H_2, \cdots, 1/H_n, 1/b$ are in A.P. Let d be the common difference of this A.P. Then

   $$\frac{1}{b} = \frac{1}{a} + (n + 2 - 1)d$$

   $$\Rightarrow d = \frac{1}{n + 1}\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{a - b}{(n + 1)\, ab}$$

   Thus

   $$\frac{1}{H_1} = \frac{1}{a} + \frac{a - b}{(n + 1)ab},$$

   $$\frac{1}{H_2} = \frac{1}{a} + \frac{2(a - b)}{(n + 1)ab}, \cdots, \frac{1}{H_n} = \frac{1}{a} + \frac{n(a - b)}{(n + 1)ab}$$

   From here, we can get the values of $H_1, H_2, \cdots, H_n$.

SUMMATION OF SOME SERIES OF NATURAL NUMBERS

1. $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{1}{2}\, n(n + 1)$

2. $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}\, n(n + 1)\, (2n + 1)$

3. $\displaystyle\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\sum_{k=1}^{n} k\right)^2$

   $$= (1 + 2 + 3 + \ldots + n)^2$$

   $$= \left[\frac{1}{2} n(n+1)\right]^2 = \frac{1}{4}\, n^2(n + 1)^2$$

4. To find $\displaystyle\sum_{k=1}^{n} k^4$ and $\displaystyle\sum_{k=1}^{n} k^5$ we can use the identities:

   $$k^5 - (k - 1)^5 = 5k^4 - 10k^3 + 10k^2 - 5k + 1$$

   $$\text{and } k^6 - (k - 1)^6 = 6k^5 - 15k^4 + 20k^3 - 15k^2 + 6k - 1$$

The first one gives

$$\sum_{k=1}^{n} \{k^5 - (k - 1)^5\} = 5 \sum_{k=1}^{n} k^4 - 10 \sum_{k=1}^{n} k^3 + 10 \sum_{k=1}^{n} k^2 - 5 \sum_{k=1}^{n} k + n$$

$$\Rightarrow \quad n^5 + 10 \left\{\frac{1}{4} n^2 (n+1)^2\right\} - 10 \left\{\frac{1}{6} n(n+1)(2n+1)\right\} + 5\left\{\frac{1}{2} n\,(n+1)\right\} - n = 5\sum_{k=1}^{n} k^4$$

Similarly we can use 2nd identity to obtain sum of $\displaystyle\sum_{k=1}^{n} k^5$.

SUM OF THE PRODUCTS OF TWO TERMS OF A SEQUENCE

To obtain the sum $\displaystyle\sum_{i<j} a_i a_j$, we use the identity

$$2 \sum_{i<j} a_i a_j = (a_1 + a_2 + \ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots + a_n^2)$$

METHOD OF DIFFERENCE FOR SUMMATION OF SERIES

If possible express *r*th term as difference of two terms as follows:

$$t_r = f(r) - f(r \pm 1).$$

Illustration 2

If the given series is 1.3.5 + 3.5.7 + 5.7.9 + …

then $t_r = (2r - 1)(2r + 1)(2r + 3)$

and $t_{r+1} = (2r + 1)(2r + 3)(2r + 5)$

$\Rightarrow \quad (2r + 5)t_r = (2r - 1)t_{r+1}$

$\Rightarrow \quad (2r - 3)t_r + 8t_r = (2r - 1)t_{r+1}$

$\Rightarrow \quad 8t_r = f(r + 1) - f(r)$

where
$$\begin{aligned} f(r) &= (2r - 3)t_r \\ &= (2r - 3)(2r - 1)(2r + 1)(2r + 3) \\ &= (4r^2 - 9)(4r^2 - 1) \end{aligned}$$

This gives

$$8\sum_{r=1}^{n} t_r = \sum_{r=1}^{n} \{f(r+1) - f(r)\} = f(n+1) - f(1)$$

$$\begin{aligned} \Rightarrow \quad 8\sum_{r=1}^{n} t_r &= [4(n+1)^2 - 9][4(n+1)^2 - 1] \\ &\quad - (4 - 9)(4 - 1) \\ &= 16(n+1)^4 - 40(n+1)^2 + 9 + 15 \end{aligned}$$

$$\Rightarrow \quad \sum_{r=1}^{n} t_r = 2(n+1)^4 - 5(n+1)^2 + 3$$

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: Suppose $a_1, a_2, \ldots$ are in A.P. If $a_8 : a_5 = 3 : 2$, then $a_{17} : a_{23}$ is:

(a) 1 : 2 (b) 3 : 4

(c) 4 : 11 (d) 8 : 11

Ans. (b)

Solution: Let *a* be the first term and *d* be the common difference of the A.P. $a_1, a_2, a_3, \ldots$ Then

$$\frac{3}{2} = \frac{a_8}{a_5} = \frac{a + 7d}{a + 4d}$$

$\Rightarrow \quad 3a + 12d = 2a + 14d \Rightarrow a = 2d.$

Now,
$$\frac{a_{17}}{a_{23}} = \frac{a + 16d}{a + 22d} = \frac{2d + 16d}{2d + 22d} = \frac{18}{24} = \frac{3}{4}$$

Example 2: If 7th term of an A.P. is 9 and 9th term of the A.P. is 7, then 20th term of the A.P. is

(a) – 2 (b) – 3

(c) – 4 (d) – 6

Ans. (c)

Solution: If *d* is common difference of the A.P., then

$$2d = a_9 - a_7 = 7 - 9 = -2$$

$\Rightarrow \quad d = -1.$

$\therefore \quad a_{20} = a_9 + (20 - 9)d = 7 - 11 = -4$

Example 3: Suppose *m*th term of an A.P. is 1/*n* and *n*th term of the A.P. is 1/*m*. If *r*th term of the A.P. is 1, then *r* is equal to

(a) *mn* (b) *m* + *n*

(c) *m* – *n* (d) *m* + *n* – 1

Ans. (a)

Solution: Let *d* be the common difference of the A.P., then

$$(m - n)d = a_m - a_n = \frac{1}{n} - \frac{1}{m}$$

$\Rightarrow \quad d = \dfrac{1}{mn}$

Now $\quad a_r - a_m = (r - m)d$

$\Rightarrow \quad 1 = a_r = \dfrac{1}{n} + (r - m)\dfrac{1}{mn} = \dfrac{r}{mn}$

$\Rightarrow \quad r = mn.$

Example 4: The number of terms of the A.P. 1, 4, 7, . . . that must be taken to obtain a sum of 715 is

(a) 24 (b) 23

(c) 22 (d) 21

Ans. (c)

Solution: Here $a = 1$, $d = 3$ and $S_n = 715$. We have

$$715 = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}[2(1) + (n - 1)(3)]$$

$\Rightarrow \quad 1430 = 3n^2 - n$ or $3n^2 - n - 1430 = 0$

$$\Rightarrow \quad n = \frac{1}{6}\left(1 \pm \sqrt{1 + 3 \times 4 \times 1430}\right) = \frac{1}{6}(1 \pm 131)$$

$$= 22, -65/3$$

As *n* must be a natural number $n = 22$

Example 5: If sum of first 20 terms of an A.P. is equal to sum of first 30 terms of the A.P. then sum of the first 50 terms of the A.P. is

(a) – 1 (b) 0
(c) 10 (d) 25

Ans. (b)

◎ **Solution:** Let a be the first term and d be the common difference of the A.P., then

$$S_{20} = S_{30}$$

$$\Rightarrow \quad \frac{20}{2}[2a + (20 - 1)d] = \frac{30}{2}[2a + (30 - 1)d]$$

$$\Rightarrow \quad 2a + 49d = 0$$

Now, $S_{50} = \frac{50}{2}[2a + (50 - 1)d] = 0.$

◉ **Example 6:** If for every $n \in \mathbf{N}$, sum to n terms of an A.P. is $5n^2 + 7n$ then its 10th term is

(a) 7/2 (b) 570
(c) 102 (d) 52

Ans. (c)

◎ **Solution:** We have

$$a_r = S_r - S_{r-1}$$

$$\Rightarrow \quad a_{10} = S_{10} - S_9 = 5(10^2 - 9^2) + 7(10 - 9) = 102$$

◉ **Example 7:** If the sum of three numbers in A.P. is 24 and their product is 440, then common difference of the A.P. can be

(a) 3 (b) 2
(c) 5 (d) – 5

Ans. (a)

◎ **Solution:** Let the three number in A.P. be $a - d, a, a + d$. Then

$$24 = (a - d) + a + (a + d) = 3a$$

$$\Rightarrow \quad a = 8$$

Also, $440 = (a - d)\,a(a + d) = 8(64 - d^2)$

$$\Rightarrow \quad d^2 = 64 - 55 = 9 \Rightarrow d = \pm 3.$$

$\therefore$ d can be 3.

◉ **Example 8:** The digits of a three digit number N are in A.P. If sum of the digits is 15 and the number obtaind by reversing the digits of the number is 594 less than the original number, then $\frac{1000}{N - 252}$ is equal to

(a) 5/6 (b) 5/3
(c) 0.06 (d) 0.03

Ans. (b)

◎ **Solution:** Let three digits of N be:

Hundred's digit = $a - d$
ten's digit = a,
and unit's digit = $a + d$

$$(a - d) + a + (a + d) = 15 \Rightarrow a = 5$$

and

$$N = 100(5 - d) + 50 + (5 + d) \qquad (1)$$

Number obtained by reversing digits of N is

$$N_1 = 100(5 + d) + 50 + (5 - d) \qquad (2)$$

Now,

$$594 = N - N_1 = 100(-2d) + 2d$$

$$\Rightarrow \quad 2d = -6 \Rightarrow d = -3$$

Thus, $N = 852$ and

$$\frac{1000}{N - 252} = \frac{1000}{600} = \frac{5}{3}$$

◉ **Example 9:** If sum of four numbers in A.P. is 28 and product of two middle terms is 45, then product of the first and last terms is

(a) 11 (b) 13
(c) 15 (d) 17

Ans. (b)

◎ **Solution:** Let four numbers in A.P. be

$$a - 3d,\ a - d,\ a + d,\ a + 3d$$

We are given

$$28 = (a - 3d) + (a - d) + (a + d) + (a + 3d) = 4a$$

$$\Rightarrow \quad a = 7$$

Also, $(a - d)(a + d) = 45$

$$\Rightarrow \quad 49 - d^2 = 45 \Rightarrow d = \pm 2.$$

Now, $(a - 3d)(a + 3d) = a^2 - 9d^2 = 13$

◉ **Example 10:** n arithmetic means are inserted between 3 and 17. If the ratio of first to last mean is 1 : 3, then n is equal to:

(a) 4 (b) 5
(c) 6 (d) 8

Ans. (c)

◎ **Solution:** Let the n arithmetic means between 3 and 17 be $A_1, A_2, \ldots, A_n$. If d is the common difference, then,

$$d = \frac{17 - 3}{n + 1} = \frac{14}{n + 1}$$

Also, $\frac{A_1}{A_n} = \frac{1}{3} \Rightarrow \frac{3 + d}{17 - d} = \frac{1}{3}$

$$\Rightarrow \quad \frac{3 + 14/(n+1)}{17 - 14/(n+1)} = \frac{1}{3} \Rightarrow \frac{3n + 17}{17n + 3} = \frac{1}{3} \Rightarrow n = 6$$

◉ **Example 11:** Suppose $a, b, c > 0$ and $p \in \mathbf{R}$. If

$$(a^2 + b^2)p^2 - 2(ab + bc)p + (b^2 + c^2) = 0 \qquad (1)$$

then a, b, c are in

(a) A.P (b) G.P
(c) H.P. (d) A.G.P.

Ans. (b)

◎ **Solution:** Write (1) as

$$(a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) = 0$$

$$\Rightarrow \quad (ap - b)^2 + (bp - c)^2 = 0$$

As $ap - b,\ bp - c \in \mathbf{R}$.

from (2), we get

$$ap - b = 0,\ bp - c = 0$$

$$\Rightarrow \quad p = \frac{b}{a} = \frac{c}{b}$$

$\Rightarrow$ a, b, c are in G.P.

Example 12: Suppose $(m + n)$th term of a G.P. is p and $(m - n)$th term is q, then its nth is

(a) $\sqrt{pq}$ (b) $p\left(\frac{q}{p}\right)^{m/n}$

(c) $p\left(\frac{q}{p}\right)^{m/2n}$ (d) $\sqrt{\frac{p}{q}}$

Ans. (c)

Solution: Let a be the first term and r be the common ratio. We have

$$p = a_{m+n} = ar^{m+n-1}$$

$$q = a_{m-n} = ar^{m-n-1}$$

$$\Rightarrow \quad \frac{p}{q} = \frac{a_{m+n}}{a_{m-n}} = r^{2n} \Rightarrow r = \left(\frac{p}{q}\right)^{1/2n}$$

Now, $a_n = ar^{n-1} = \dfrac{ar^{m+n-1}}{r^m}$

$$= p\left(\frac{p}{q}\right)^{-m/2n} = p\left(\frac{q}{p}\right)^{m/2n}$$

Example 13: The number of terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}, \ldots$ needed to obtain a sum of $\frac{3069}{512}$ is:

(a) 9 (b) 10

(c) 11 (d) 12

Ans. (b)

Solution: Here $a = 3$, $r = 1/2$.

Let the number of terms needed be n, then

$$\frac{3069}{512} = S_n = \frac{3\left(1 - (1/2)^n\right)}{1 - 1/2}$$

$$\Rightarrow \quad \frac{1023}{1024} = 1 - \left(\frac{1}{2}\right)^n$$

$$\Rightarrow \quad \left(\frac{1}{2}\right)^n = \frac{1}{1024} \Rightarrow n = 10$$

Example 14: Sum to 25 terms of the series 0.5 + 0.55 + 0.555 + . . . is:

(a) $\frac{5}{81}(224 - 10^{-25})$ (b) $\frac{5}{9}(224 - 10^{-25})$

(c) $\frac{5}{81}(224 - 10^{-24})$ (d) none of these

Ans. (a)

Solution: Let

$$S = 0.5 + 0.55 + 0.555 + \ldots + \text{upto 25 terms}$$

$$= \frac{5}{9}[0.9 + 0.99 + 0.999 + \ldots + \text{upto 25 terms}]$$

$$= \frac{5}{9}\left[(1 - 0.1) + (1 - (0.1)^2) + (1 - (0.1)^3) + \ldots + (1 - (0.1)^{25})\right]$$

$$= \frac{5}{9}\left[25 - \frac{(0.1)\left(1 - (0.1)^{25}\right)}{1 - 0.1}\right]$$

$$= \frac{5}{81}\ [224 - 10^{-25}]$$

Example 15: The sum of three numbers in A.P. is 21. If the second number is reduced by 1 while third is increased by 1, three resulting numbers form an increasing G.P., the common ratio of the G.P. is

(a) 2 (b) 3

(c) 4 (d) 7

Ans. (a)

Solution: Let three numbers in A.P. be $a - d, a, a + d$.

As $21 = (a - d) + a + (a + d) = 3a$

$\Rightarrow$ $a = 7$

Also, $7 - d, 7 - 1, 7 + d + 1$ are in G.P.

$\Rightarrow$ $6^2 = (7 - d)(8 + d)$

$\Rightarrow$ $36 = 56 - d - d^2$

$\Rightarrow$ $d^2 + d - 20 = 0$

$\Rightarrow$ $(d + 5)(d - 4) = 0 \Rightarrow d = -5, 4$

As, $7 - d, 6, 8 + d$ is an increasing G.P., $d = 4$. The common ratio of the G.P. is $\dfrac{6}{7 - d} = \dfrac{6}{7 - 4} = 2$.

LEVEL 1

Straight Objective Type Questions

Example 16: If one geometric G and two arithmetic means A_1 and A_2 are inserted between two distinct positive numbers, then

$$\left(\frac{2A_1 - A_2}{G}\right)\left(\frac{2A_2 - A_1}{G}\right)$$

is equal to

(a) 0 (b) 1

(c) – 1.5 (d) – 2.5

Ans. (b)

◎ **Solution:** Let two distinct positive numbers be a and b. Then $G^2 = ab$

Let common difference of the A.P. a, A_1, A_2, b. be d.

We have

$$2A_1 - A_2 = 2(a + d) - (a + 2d) = a$$

and $2A_2 - A_1 = 2(b - d) - (b - 2d) = b$

Thus, $\left(\dfrac{2A_1 - A_2}{G}\right)\left(\dfrac{2A_2 - A_1}{G}\right) = \dfrac{ab}{G^2} = 1$

● **Example 17:** If sum of the infinite G.P. $p + 1 + \dfrac{1}{p} + \dfrac{1}{p^2} + \ldots$, $(p > 2)$ is 49/6, then sum of the 3rd term and the 4th term of the G.P. is

(a) $\dfrac{8}{49}$ (b) $\dfrac{1}{7}$

(c) $\dfrac{6}{7}$ (d) $\dfrac{23}{49}$

Ans. (a)

◎ **Solution:** For $p > 1$,

$$\frac{49}{6} = p + 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots = \frac{p}{1 - 1/p}$$

$\Rightarrow$ $\dfrac{49}{6} = \dfrac{p^2}{p - 1}$

$\Rightarrow$ $6p^2 - 49p + 49 = 0$

$\Rightarrow$ $6p^2 - 42p - 7p + 49 = 0$

$\Rightarrow$ $(6p - 7)(p - 7) = 0$

As $p > 2, p = 7$

Thus, $a_3 + a_4 = p\left(\dfrac{1}{p}\right)^2 + p\left(\dfrac{1}{p}\right)^3$

$$= \frac{1}{7} + \frac{1}{49} = \frac{8}{49}$$

● **Example 18:** If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in A.P. such that $a_4 - a_7 + a_{10} = m$, then sum of the first 13 terms of the A.P. is

(a) 10 m (b) 12 m

(c) 13 m (d) 15 m

Ans. (c)

◎ **Solution:** Let d be the common difference of the A.P., then

$$m = a_4 - a_7 + a_{10} = a_4 + (a_{10} - a_7)$$

$$= a_1 + 3d + 3d = a_1 + 6d$$

Now,

$$S_{13} = \frac{13}{2}\,[2a_1 + (13 - 1)d] = 13m.$$

● **Example 19:** If $(20)^{19} + 2(21)(20)^{18} + 3(21)^2(20)^{17} + \ldots + 20(21)^{19} = k(20)^{19}$ then k is equal to

(a) 400 (b) 100

(c) 441 (d) 420

Ans. (a)

◎ **Solution:** Dividing by $(20)^{19}$, we can write

$k = 1 + 2x + 3x^2 + \ldots + 20x^{19}$

where $x = 21/20$.

$$k = \frac{d}{dx}[x + x^2 + \ldots + x^{20}]$$

$$= \frac{d}{dx}\left[\frac{x(x^{20} - 1)}{x - 1}\right]$$

$$= \frac{(x - 1)(21x^{20} - 1) - x(x^{20} - 1)}{(x - 1)^2}$$

$$= \frac{20x^{21} - 21x^{20} + 1}{(x - 1)^2}$$

But $20x^{21} - 21x^{20}$

$$= 20\left(\frac{21}{20}\right)^{21} - 21\left(\frac{21}{20}\right)^{21} = 0$$

$\therefore$ $k = \dfrac{1}{(21/20 - 1)^2} = 400$

● **Example 20:** Three positive numbers form an increasing G.P. If the middle term in this G.P. is tripled, the new numbers are in A.P. Then the common ratio of G.P.is:

(a) $3 + 2\sqrt{2}$ (b) $2\sqrt{2} - \sqrt{3}$

(c) $2 + 2\sqrt{3}$ (d) $2\sqrt{2} + \sqrt{3}$

Ans. (a)

◎ **Solution:** Let the three positive numbers in the increasing G.P. be

$$\frac{a}{r}, a, ar \text{ where } r > 1.$$

We are given, $\dfrac{a}{r}$, $3a$, ar are in A.P.

$\therefore$ $\dfrac{a}{r} + ar = 2(3a)$

$\Rightarrow$ $r^2 - 6r + 1 = 0$

$\Rightarrow$ $r = 3 \pm 2\sqrt{2}$

As $r > 1, r = 3 + 2\sqrt{2}$

● **Example 21:** Suppose m arithmetic means are inserted between 1 and 31. If the ratio of the second mean to the mth mean is 1 : 4, then m is equal to

(a) 7 (b) 9

(c) 11 (d) 15

Ans. (b)

◎ **Solution:** Let m arithmetic means between 1 and 31 be $A_1, A_2, \ldots, A_m$, then $1, A_1, A_2 \ldots A_m, 31$ is an A.P. Let d be the common difference of this A.P. Now,

$$31 = (m+2)\text{th term of the A.P.}$$

$$= 1 + (m + 2 - 1)d$$

$$\Rightarrow \qquad d = \frac{30}{m+1}.$$

Also, $A_2 = 1 + 2d,\ A_m = 1 + md = 31 - d.$

Now, $\dfrac{A_2}{A_m} = \dfrac{1}{4} \Rightarrow 4A_2 = A_m$

$\Rightarrow \qquad 4(1 + 2d) = 31 - d \Rightarrow d = 3$

Thus $\dfrac{30}{m+1} = 3 \Rightarrow m = 9.$

◉ **Example 22:** In a geometric progression the ratio of the sum of the first 5 terms to the sum of their reciprocals is 49 and sum of the first and the third term is 35. The fifth term of the G.P. is

(a) 7 (b) 7/2
(c) 7/4 (d) 7/8

Ans. (c)

◎ **Solution:** Let five terms in G.P. be $\dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2$

Then

$$\frac{a\left(r^{-2}+r^{-1}+1+r+r^2\right)}{(1/a)\left(r^2+r+1+r^{-1}+r^{-2}\right)} = 49$$

$\Rightarrow \qquad a^2 = 49 \Rightarrow a = \pm 7.$

Also, $\dfrac{a}{r^2} + a = 35$

Therefore $\quad a = -7$ is not possible

Thus, $\quad a = 7$ and $a/r^2 = 28.$

Now, fifth term $= ar^2 = a\left(\dfrac{a}{28}\right) = \dfrac{7}{4}$

◉ **Example 23:** If $|x| < 1$, and rth term of a series is $1 + x + x^2 + \ldots + x^{r-1}$, then sum to n terms of the series is

(a) $\dfrac{n+(n+1)x - x^{n+1}}{(1-x)^2}$ (b) $\dfrac{n-(n+1)x + x^{n+1}}{(1-x)^2}$

(c) $\dfrac{(n+1)x - x^{n+1} - n}{(1-x)^2}$ (d) $\dfrac{n-(n+1)x - x^{n+1}}{(1-x)^2}$

Ans. (b)

◎ **Solution:** $t_r = 1 + x + x^2 + \ldots x^{r-1} = \dfrac{1-x^r}{1-x}$

Now,
$$\sum_{r=1}^{n} t_r = \sum_{r=1}^{n}\left(\frac{1-x^r}{1-x}\right)$$

$$= \frac{n}{1-x} - \frac{1}{1-x}\sum_{r=1}^{n} x^r$$

$$= \frac{n}{1-x} - \frac{1}{1-x}\,\frac{x(1-x^n)}{1-x}$$

$$= \frac{1}{(1-x)^2}\left[n - nx - x + x^{n+1}\right]$$

$$= \frac{1}{(1-x)^2}\left[n - (n+1)x + x^{n+1}\right]$$

◉ **Example 24:** Let m be a positive integer, then

$$S = \sum_{k=1}^{m} k\left(\frac{1}{k} + \frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{m}\right)$$

is equal to:

(a) $\dfrac{1}{4}\ m(m+2)$ (b) $\dfrac{1}{4}\ m(m+3)$

(c) $\dfrac{1}{4}\ m(m+4)$ (d) $\dfrac{1}{4}\ m(m+6)$

Ans. (b)

◎ **Solution:**

$$S = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}\right)$$

$$+\ 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}\right)$$

$$+\ 3\left(\frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{m}\right)$$

$$+ \ldots$$

$$+\ (m-1)\left(\frac{1}{m-1} + \frac{1}{m}\right) + m\left(\frac{1}{m}\right)$$

$$= (1)\,(1) + \frac{1}{2}(1+2) + \frac{1}{3}(1+2+3)$$

$$+\ \frac{1}{4}(1+2+3+4)$$

$$+ \ldots + \frac{1}{m}(1+2+3+\ldots+m)$$

$$= \sum_{k=1}^{m}\frac{1}{k}(1+2+\ldots+k)$$

$$= \sum_{k=1}^{m}\frac{1}{k}\cdot\frac{k(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{m}(k+1)$$

$$= \frac{m}{4}\left[2 + (m+1)\right]$$

$$= \frac{1}{4}\ m(m+3)$$

◉ **Example 25:** Let $f(n) = \left[\dfrac{1}{5} + \dfrac{3n}{100}\right]n$, where $[x]$ denotes the greatest integer less than or equal to x. Then $\displaystyle\sum_{n=1}^{61} f(n)$

(a) 2013 (b) 1869
(c) 1947 (d) 1661

Ans. (d)

Solution: If $\frac{3n}{100} < 0.8$ or $n \le 26$, then

$$\left[\frac{1}{5}+\frac{3n}{100}\right] = 0.$$

If $0.8 \le \frac{3n}{100} < 1.8$, or $27 \le n \le 59$

then $\left[\frac{1}{5}+\frac{3n}{100}\right] = 1$

For $n = 60, 61$ $\left[\frac{1}{5}+\frac{3n}{100}\right] = 2.$

$$\text{Thus, } \sum_{n=1}^{61} f(n) = \sum_{n=27}^{59} n + \sum_{n=60}^{61} 2n$$

$$= \frac{1}{2}(33)(27+59) + 2(60+61)$$

$$= 1661$$

Example 26: Sum of the series $S = 1^2 - 2^2 + 3^2 - 4^2 + \ldots - 2008^2 + 2009^2$ is

(a) 2019045 (b) 1005004
(c) 2000506 (d) none of these

Ans. (a)

Solution: We can write S as

$$S = (1-2)(1+2) + (3-4)(3+4) + \ldots + (2007 - 2008)(2007+2008) + 2009^2$$

$$= -[1+2+3+4+\ldots+2008] + 2009^2$$

$$= -\frac{1}{2}(2008)(2009) + 2009^2 = 2019045$$

Example 27: The value of n for which $704 + \frac{1}{2}(704) + \frac{1}{4}(704) + \ldots$ upto n terms $= 1984 - \frac{1}{2}(1984) + \frac{1}{4}(1984) - \ldots$ upto n terms is

(a) 5 (b) 3
(c) 4 (d) 10

Ans. (a)

Solution: According to the given condition

$$(704)(2)\left\{1-\frac{1}{2^n}\right\} = \frac{(1984)(2)}{3}\left\{1-\frac{(-1)^n}{2^n}\right\}$$

$$\Rightarrow \quad 128 = \frac{2112}{2^n} - 1984\frac{(-1)^n}{2^n}$$

If n is odd, we get $2^n = 32$ or $n = 5$
If n is even, we get $128 = 128/2^n \Rightarrow n = 0$.

Example 28: The positive integer n for which $2{\times}2^2 + 3{\times}2^3 + 4{\times}2^4 + \ldots + n{\times}2^n = 2^{n+10}$ is

(a) 510 (b) 511
(c) 512 (d) 513

Ans. (d)

Solution: We have

$$2^{n+10} = 2\times 2^2 + 3\times 2^3 + 4\times 2^4 + \ldots + n\times 2^n$$

$$\Rightarrow \quad 2(2^{n+10}) = 2\times 2^3 + 3\times 2^4 + \ldots + (n-1)\times 2^n + n\times 2^{n+1}$$

Subtracting, we get

$$-2^{n+10} = 2{\cdot}2^2 + 2^3 + 2^4 + \ldots + 2^n - n{\cdot}2^{n+1}$$

$$= 8 + \frac{8\left(2^{n-2}-1\right)}{2-1} - n\times 2^{n+1}$$

$$= 2^{n+1} - (n)2^{n+1}$$

$$\Rightarrow \quad 2^{10} = 2n - 2 \quad \Rightarrow \quad n = 513$$

Example 29: Sum of the series $S = 1 + \frac{1}{2}(1+2) + \frac{1}{3}(1+2+3) + \frac{1}{4}(1+2+3+4) + \ldots$ upto 20 terms is

(a) 110 (b) 111
(c) 115 (d) 116

Ans. (c)

Solution: Let t_k denote the kth term of the series. Then

$$t_k = \frac{1}{k}(1+2+3+\ldots+k) = \frac{1}{k}\frac{k(k+1)}{2} = \frac{k+1}{2}.$$

$$\text{Thus,} \quad S = \frac{1}{2}[2+3+4+\ldots+21] = \frac{10}{2}(2+21) = 115$$

Example 30: If $1^2 + 2^2 + 3^2 + \ldots + 2009^2 = (2009)(4019)(335)$ and $(1)(2009) + (2)(2008) + (3)(2007) + \ldots + (2009)(1) = (2009)(335)(x)$, then x equals

(a) 2011 (b) 2009
(c) 2008 (d) 2007

Ans. (a)

Solution: Let

$$S = (1)(2009) + (2)(2008) + (3)(2007) + \ldots + (2009)(1)$$

and $K = 1^2 + 2^2 + 3^2 + \ldots + 2009^2$

Adding, we get

$$S + K = (2010)[1+2+3+\ldots+2009]$$

$$\Rightarrow (2009)(335)(x) + (2009)(4019)(335) = (2010)(1005)(2009)$$

$$\Rightarrow \quad x = 2011$$

Example 31: If $x > 0$, and $\log_2 x + \log_2\left(\sqrt{x}\right) + \log_2\left(\sqrt[4]{x}\right) + \log_2\left(\sqrt[8]{x}\right) + \log_2\left(\sqrt[16]{x}\right) + \ldots = 4$

then x equals

(a) 2 (b) 3
(c) 4 (d) 5

Ans. (c)

Solution: We can write the given equation as

$$\log_2\left(x^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots}\right)=4$$

$$\Rightarrow \quad \log_2(x^2)=4 \quad \Rightarrow \quad x^2=2^4 \quad \Rightarrow \quad x=4$$

Example 32: If $(1 + 3 + 5 + \ldots + p) + (1 + 3 + 5 + \ldots + q) = (1 + 3 + 5 + \ldots + r)$ where each set of parentheses contains the sum of consecutive odd integers as shown, the smallest possible value of $p + q + r$, (where $p > 6$) is

(a) 12 (b) 21
(c) 45 (d) 54

Ans. (b)

Solution: We know that

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2$$

Thus, the given equation can be written as

$$\left(\frac{p+1}{2}\right)^2+\left(\frac{q+1}{2}\right)^2=\left(\frac{r+1}{2}\right)^2$$

$$\Rightarrow \quad (p + 1)^2 + (q + 1)^2 = (r + 1)^2$$

Therefore, $(p + 1, q + 1, r + 1)$ forms a Pythagorean triplet. As $p > 6, p + 1 > 7$.

The first Pythagorean triplet containing a number > 7 is $(6, 8, 10)$.

$\therefore$ We may take $p + 1 = 8, q + 1 = 6, r + 1 = 10$

$$\Rightarrow \quad p + q + r = 21.$$

Example 33: Let $a_1, a_2, \ldots, a_{10}$ be in A.P. and $h_1, h_2, \ldots, h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is

(a) 2 (b) 3
(c) 5 (d) 6

Ans. (d)

Solution: Let d be the common difference of the A.P., then

$$3 = a_{10} = 2 + 9d \quad \Rightarrow \quad d = 1/9$$

$$\therefore \quad a_4 = 2 + 3d = 7/3$$

Next, let D be the common difference of the A.P. $\frac{1}{h_1}, \frac{1}{h_2}, \ldots, \frac{1}{h_{10}}$ Then

$$\frac{1}{3}=\frac{1}{h_{10}}=\frac{1}{2}+9D \quad \Rightarrow \quad D=-\frac{1}{54}$$

$$\therefore \quad \frac{1}{h_7}=\frac{1}{h_1}+6D=\frac{7}{18} \quad \Rightarrow \quad h_7=\frac{18}{7}$$

Hence, $a_4 h_7 = (7/3)(18/7) = 6$.

Example 34: Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. If $a < b < c$ and $a + b + c = 3/2$, then the value of a is

(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$
(c) $\frac{1}{2}-\frac{1}{\sqrt{3}}$ (d) $\frac{1}{2}-\frac{1}{\sqrt{2}}$

Ans. (d)

Solution: We have $a + c = 2b \Rightarrow 3b = a + b + c = 3/2 \Rightarrow b = 1/2$

Thus, $a + c = 1$ and $a^2 c^2 = b^4 = 1/16$

$\Rightarrow \quad a^2(1 - a)^2 = 1/16 \quad \Rightarrow \quad a(1 - a) = \pm 1/4$

$\Rightarrow 4a^2 - 4a \pm 1 = 0 \quad \Rightarrow \quad (2a - 1)^2 = 0$ or $(2a - 1)^2 = 2$

$$\Rightarrow \quad a=\frac{1}{2} \text{ or } a=\frac{1}{2}\pm\frac{1}{\sqrt{2}}$$

As $a < b < c$ and $b = \frac{1}{2}$, we get $a = \frac{1}{2}-\frac{1}{\sqrt{2}}$

Example 35: Let $S_1, S_2, \ldots$ be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of a diagonal of S_{n+1}. If the length of a side of S_1 is 10 cm, then the smallest value of n for which Area $(S_n) < 1$ is

(a) 7 (b) 8
(c) 9 (d) 10

Ans. (b)

Solution: Let a_n denote the length of a side of S_n. We are given

Length of a side of S_n = Length of a diagonal of S_{n+1}

$$\Rightarrow \quad a_n=\sqrt{2}\;a_{n+1} \quad \Rightarrow \quad \frac{a_{n+1}}{a_n}=\frac{1}{\sqrt{2}}.$$

Thus, $a_1, a_2, a_3, \ldots$ is a G.P. with first term 10 and common ratio $1/\sqrt{2}$. Therefore,

$$a_n = 10\left(1/\sqrt{2}\right)^{n-1}$$

Also, Area $(S_n) = a_n^2 < 1$

$\Rightarrow \quad [10\,(1/\sqrt{2}\,)^{n-1}]^2 < 1 \quad \Rightarrow \quad 100 < 2^{n-1}$

$\Rightarrow$ smallest possible value of n is 8.

Example 36: Let T_r be the rth term of an AP, for $r = 1, 2, \ldots$. If for some positive integers m and n, we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, the T_{m+n} equals

(a) $\frac{1}{mn}$ (b) $\frac{1}{m}+\frac{1}{n}$
(c) $\frac{1}{m}$ (d) 0

Ans. (c)

◎ **Solution:** Let a be the first term and d be the common difference of the given A.P. Then according to the hypothesis,

$$T_m - T_n = \frac{1}{n} - \frac{1}{m}$$

$$\Rightarrow \quad (m-n)d = \frac{m-n}{mn} \quad \Rightarrow \quad d = \frac{1}{mn}.$$

$$\Rightarrow \quad T_{m+n} - T_m = (m+n-m)\,\frac{1}{mn} = \frac{1}{m}$$

◉ **Example 37:** Consider an infinite geometric series with the first term a and common ratio r. If its sum is 4 and the second term is 3/4, then

(a) $a = 4/7,\ r = 3/7$ (b) $a = 2,\ r = 3/8$
(c) $a = 3/2,\ r = 1/2$ (d) $a = 3,\ r = 1/4$

Ans. (d)

◎ **Solution:** We have

$$\frac{a}{1-r} = 4 \text{ and } ar = \frac{3}{4} \quad \Rightarrow \quad r = \frac{3}{4a}.$$

Thus, $$\frac{a}{1-3/4a} = 4 \quad \Rightarrow \quad a^2 - 4a + 3 = 0 \Rightarrow a = 1, 3$$

When $a = 1$, $r = 3/4$ and when $a = 3$, $r = 1/4$.

◉ **Example 38:** If the sum of first $2n$ terms of the A.P. 2, 5, 8, …, is equal to the sum of first n terms of the A.P. 57, 59, 61, …, then n equals

(a) 10 (b) 12
(c) 11 (d) 13

Ans. (c)

◎ **Solution:** Sum to $2n$ terms of 2, 5, 8, … is

$$\frac{2n}{2}\,[2(2) + (2n-1)(3)] = n(6n+1)$$

and sum to n terms of 57, 59, 61, …, is

$$\frac{n}{2}\,[2(57) + (n-1)(2)] = n(56+n)$$

According to given condition

$$n(6n+1) = n(56+n) \Rightarrow 6n+1 = 56+n \Rightarrow n = 11$$

◉ **Example 39:** For a positive integer n, let

$$a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{(2^n)-1}. \text{ Then}$$

(a) $a(100) < 100$ (b) $a(100) > 100$
(c) $a(200) < 100$ (d) none of these

Ans. (a)

◎ **Solution:** We have

$$a(n) = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \cdots + \frac{1}{7}\right)$$

$$+ \left(\frac{1}{8} + \cdots + \frac{1}{15}\right) + \cdots + \left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-1}+1} + \cdots + \frac{1}{(2^n)-1}\right)$$

$$< 1 + \frac{2}{2} + \frac{4}{4} + \frac{8}{8} + \cdots + \frac{2^{n-1}}{2^{n-1}} = n$$

Thus, $a(100) < 100$.

◉ **Example 40:** If $x > 1$, $y > 1$, $z > 1$ are in G.P., then $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Ans. (c)

◎ **Solution:** As x, y, z are in G.P. $\Rightarrow y^2 = xz$

$$\Rightarrow \quad \ln(y^2) = \ln(xz) \quad \Rightarrow \quad 2\ln(y) = \ln(x) + \ln(z)$$

$$\Rightarrow \quad 2(1 + \ln(y)) = (1 + \ln(x)) + (1 + \ln(z))$$

$$\Rightarrow \quad 1 + \ln(x),\ 1 + \ln(y),\ 1 + \ln(z) \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{1}{1+\ln(x)}, \frac{1}{1+\ln(y)}, \frac{1}{1+\ln(z)} \text{ are in H.P.}$$

◉ **Example 41:** Let x be the arithmetic mean and y, z be the two geometric means between any two positive numbers. Then value of $\frac{y^3+z^3}{xyz}$ is

(a) 2 (b) 3
(c) $\frac{1}{2}$ (d) $\frac{3}{2}$

Ans. (a)

◎ **Solution:** Let two positive numbers be a and b. Then $x = (a+b)/2$. Also, a, y, z, b are in G.P. If r is the common ratio of this G.P., then $b = ar^3 \Rightarrow r = (b/a)^{1/3}$.
We have

$$\frac{y^3+z^3}{xyz} = \frac{a^3r^3 + a^3r^6}{x(ar)(ar^2)} = \frac{a(1+r^3)}{x} = \frac{a+b}{(a+b)/2} = 2$$

◉ **Example 42:** The sum to n terms of the series $\frac{1}{1^3} + \frac{1+2}{1^3+2^3} + \frac{1+2+3}{1^3+2^3+3^3} + \ldots$ is

(a) $\frac{n}{n+1}$ (b) $\frac{n}{2(n+1)}$

(c) $\frac{2n}{n+1}$ (d) $\frac{2}{n(n+1)}$

Ans. (c)

◎ **Solution:** If t_n denotes the nth term of the series, then

$$t_n = \frac{1+2+3+\cdots+n}{1^3+2^3+3^3+\cdots+n^3} = \frac{\frac{1}{2}n(n+1)}{\frac{1}{4}n^2(n+1)^2}$$

$$= \frac{2}{n(n+1)} = 2\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$\Rightarrow \quad \sum_{k=1}^{n} t_k = 2\left\{\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right\}$$

$$= 2\left(1-\frac{1}{n+1}\right) = \frac{2n}{n+1}$$

Example 43: Let $\sum_{r=1}^{n} r^4 = f(n)$, then $\sum_{r=1}^{n} (2r-1)^4$ is equal to

(a) $f(2n) - 16f(n)$ (b) $f(2n) - 7f(n)$
(c) $f(2n-1) - 8f(n)$ (d) none of these

Ans. (a)

Solution: We have

$$\sum_{r=1}^{n} (2r-1)^4 = \sum_{r=1}^{2n} r^4 - \sum_{r=1}^{n} (2r)^4 = f(2n) - 16f(n)$$

Example 44: The sum of first n terms of the series $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \ldots$ is $n(n+1)^2/2$ when n is even. When n is odd the sum of the series is

(a) $n^2(3n+1)/4$ (b) $n^2(n+1)/2$
(c) $n^3(n-1)/2$ (d) none of these

Ans. (b)

Solution: Let $n = 2m$, then

$$S_{2m} = 1^2 + 2\times2^2 + 3^2 + 2\times4^2 + \ldots + (2m-1)^2 + 2(2m)^2$$
$$= 2m(2m+1)^2/2 = m(2m+1)^2$$

When $n = 2m - 1$,

$$S_{2m-1} = 1^2 + 2\times2^2 + 3^2 + 2\times4^2 + \ldots + (2m-1)^2$$
$$= S_{2m} - 2(2m)^2 = m(2m+1)^2 - 2(2m)^2$$
$$= m\,[4m^2 + 4m + 1 - 8m] = m(2m-1)^2$$
$$= n^2\,(n+1)/2$$

Example 45: If the first and $(2n+1)$th terms of an A.P.; G.P. and H.P. are equal and their $(n+1)$th terms are a, b and c respectively, then

(a) $a > b > c$ (b) $ac = b^2$
(c) $a + b = c$ (d) none of these

Ans. (b)

Solution: Let A be the first term, D be the common difference and B be the $(2n+1)$th term of the A.P., then

$$B = A + 2nD \Rightarrow D = \frac{B-A}{2n}.$$

Also, $\quad a = A + (n+1-1)D = \dfrac{A+B}{2}.$

Similarly, $\quad b = \sqrt{AB}$ and $c = \dfrac{2AB}{A+B}.$

Clearly, we have $\quad b^2 = ac$

Example 46: If $a_{n+1} = \dfrac{1}{1-a_n}$ for $n \geq 1$ and $a_3 = a_1$, then $(a_{2022})^{2022}$ equals

(a) -1 (b) 1
(c) 0 (d) none of these

Ans. (b)

Solution: We have $a_2 = \dfrac{1}{1-a_1}$ and

$$a_3 = \frac{1}{1-a_2} = \frac{1}{1-1/(1-a_1)} = \frac{1-a_1}{-a_1}$$

Since $a_3 = a_1$, we get $-a_1^2 = 1 - a_1$
$\Rightarrow a_1^2 - a_1 + 1 = 0 \Rightarrow a_1 = -\omega$ or $-\omega^2$

We have $\quad a_5 = \dfrac{1}{1-a_4} = \dfrac{1}{1-1/(1-a_3)} = \dfrac{1-a_3}{-a_3} = \dfrac{1-a_1}{-a_1}$

$$= a_3 = a_1$$

Continuing in this way we obtain
$a_1 = a_3 = a_5 = a_7 = a_9 = a_{11} = \ldots = a_{2022}$
Thus, $(a_{2022})^{2022} = (-\omega)^{2022}$ or $(-\omega^2)^{2022} = 1$

Example 47: If $f(x+y) = f(xy)\ \forall\ x, y\ \varepsilon\ \mathbf{R}$, and $f(2013) = 2013$, then $f(-2013)$ equals

(a) 2013 (b) 0
(c) -2013 (d) none of these

Ans. (a)

Solution: $f(2013) = f(2013 + 0) = f[(2013\,(0)] = f(0)$
$\Rightarrow \quad f(0) = 2013$
Also, $\quad f(-2013) = f(-2013 + 0) = f(0) = 2013.$
It can also be seen that f is a constant function.

Example 48: If $\sum_{r=1}^{n} t_r = \dfrac{1}{12}\,n(n+1)(n+2)$, the value $\sum_{r=1}^{n} \dfrac{1}{t_r}$ is

(a) $\dfrac{2n}{n+1}$ (b) $\dfrac{n-1}{(n+1)!}$

(c) $\dfrac{4n}{n+1}$ (d) $\dfrac{3n}{n+2}$

Ans. (c)

Solution: We have $t_r = \sum_{k=1}^{r} t_k - \sum_{k=1}^{r-1} t_k$

$$= \frac{1}{12}\,r(r+1)(r+2) - \frac{1}{12}\,(r-1)(r)(r+1)$$

$$= \frac{1}{4}\,r(r+1)$$

Now, $\dfrac{1}{t_r} = \dfrac{4}{r(r+1)} = 4\left(\dfrac{1}{r} - \dfrac{1}{r+1}\right)$

$$\Rightarrow \quad \sum_{r=1}^{n} \frac{1}{t_r} = 4\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) = 4\left(1 - \frac{1}{n+1}\right) = \frac{4n}{n+1}$$

Example 49: Sum to n term of the series $\dfrac{1}{2} + \dfrac{3}{4} + \dfrac{7}{8} + \dfrac{15}{16} + \cdots$ is

(a) $\dfrac{n}{2^n}$ (b) $\dfrac{n+1}{2^{n+1}}$

(c) $n + 1 - \dfrac{1}{2^n}$ (d) $n - 1 + \dfrac{1}{2^n}$

Ans. (d)

Solution: We have $t_r = 1 - \dfrac{1}{2^r}$ for $r \geq 1$.

$$\text{Thus,} \quad \sum_{r=1}^{n} t_r = \sum_{r=1}^{n}\left(1 - \frac{1}{2^r}\right) = n - \frac{(1/2)\left(1 - 1/2^n\right)}{1 - 1/2}$$

$$= n - 1 + \frac{1}{2^n}$$

Example 50: If $0 < \phi < \dfrac{\pi}{2}$, and $x = \displaystyle\sum_{n=0}^{\infty} \cos^{2n}\phi$, $y = \displaystyle\sum_{n=0}^{\infty} \sin^{2n}\phi$ and $z = \displaystyle\sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$, then

(a) $xyz = xz + y$ (b) $xyz = xz + z$

(c) $xyz = x + y + z$ (d) $xyz = yz + z$

Ans. (c)

Solution: We have $x = \dfrac{1}{1 - \cos^2\phi} = \dfrac{1}{\sin^2\phi}$

and $y = \dfrac{1}{\cos^2\phi}$

$$\text{Also,} \quad z = \frac{1}{1 - \cos^2\phi \sin^2\phi} = \frac{1}{1 - \dfrac{1}{xy}} = \frac{xy}{xy - 1}$$

$$\Rightarrow \quad xyz - z = xy \quad \text{or} \quad xyz = xy + z$$

$$\text{But} \quad \frac{1}{x} + \frac{1}{y} = 1 \quad \Rightarrow \quad x + y = xy$$

$$\therefore \quad xyz = x + y + z$$

Example 51: The sum to n terms of the series $\dfrac{1}{\sqrt{7}+\sqrt{10}} + \dfrac{1}{\sqrt{10}+\sqrt{13}} + \dfrac{1}{\sqrt{13}+\sqrt{16}} + \cdots$ is

(a) $\dfrac{1}{3}\left(\sqrt{7+3n} - \sqrt{7}\right)$ (b) $\dfrac{\sqrt{4+3n} - 2}{3}$

(c) $\dfrac{1}{3}\left(\sqrt{10+3n} - \sqrt{10}\right)$ (d) none of these

Ans. (a)

Solution: We have

$$t_r = \frac{1}{\sqrt{4+3r} + \sqrt{7+3r}} = \frac{\sqrt{7+3r} - \sqrt{4+3r}}{3}$$

Thus, sum to n terms is equal to

$$\frac{1}{3}\left[\left(\sqrt{10} - \sqrt{7}\right) + \left(\sqrt{13} - \sqrt{10}\right) + \cdots + \left(\sqrt{7+3n} - \sqrt{4+3n}\right)\right]$$

$$= \frac{1}{3}\left(\sqrt{7+3n} - \sqrt{7}\right)$$

Example 52: If $S_n = \displaystyle\sum_{r=1}^{n} t_r = \dfrac{1}{6}\, n(2n^2 + 9n + 13)$, then $\displaystyle\sum_{r=1}^{n} \sqrt{t_r}$ equals

(a) $\dfrac{1}{2}\, n(n+1)$ (b) $\dfrac{1}{2}\, n(n+2)$

(c) $\dfrac{1}{2}\, n(n+3)$ (d) $\dfrac{1}{2}\, n(n+5)$

Ans. (c)

Solution: We have $t_n = S_n - S_{n-1}\ \forall\ n \geq 2$

$$\therefore \quad t_n = \frac{1}{6}\, n(2n^2 + 9n + 13) - \frac{1}{6}\,(n-1)\,\{(2(n-1)^2 + 9(n-1) + 13\}$$

$$= \frac{1}{6}\,[2(n^3 - (n-1)^3) + 9(n^2 - (n-1)^2) + 13(n - n + 1)]$$

$$= \frac{1}{6}\,[6n^2 - 6n + 2 + 9(2n - 1) + 13]$$

$$= \frac{1}{6}\,(6n^2 + 12n + 6)$$

$$= (n+1)^2$$

Also, $t_1 = S_1 = 4 = (1+1)^2$

$$\therefore \sum_{r=1}^{n} \sqrt{t_r} = \sum_{r=1}^{n} (r+1) = \frac{1}{2}\,(n+1)(n+2) - 1 = \frac{1}{2}\, n(n+3)$$

Example 53: The interior angles of a convex polygon are in A.P. If the smallest angle is 100° and the common difference is 4°, then the number of sides is

(a) 5 (b) 7

(c) 36 (d) 44

Ans. (a)

Solution: We know that the sum of the interior angles of a convex polygon of n sides is $(n-2)\,(180°)$. We are given $a = 100$ and $d = 4$.

$$\therefore \quad \frac{n}{2}\,[2(100) + (n-1)(4)] = (n-2)(180)$$

$$\Rightarrow \quad n^2 - 41n + 180 = 0 \quad \Rightarrow \quad n = 5, 36$$

If $n = 36$, then interior angles of the polygon are 100°, 104°, …, 176°, 180°, 184°, …, 244°.

But no interior angle can be 180°. Therefore, $n = 5$

Example 54: If a and x are positive integers such that $x < a$ and $\sqrt{a-x}$, $\sqrt{x}$, $\sqrt{a+x}$ are in A.P., then least possible value of a is

(a) 5 (b) 7
(c) 11 (d) none of these

Ans. (a)

Solution: We have $2\sqrt{x} = \sqrt{a+x} + \sqrt{a-x}$

$$\Rightarrow 2\sqrt{x} = \frac{2x}{\sqrt{a+x}-\sqrt{a-x}} \Rightarrow \sqrt{a+x}-\sqrt{a-x} = \sqrt{x}$$

$$\therefore\ 3\sqrt{x} = 2\sqrt{a+x} \Rightarrow 4(a+x) = 9x \Rightarrow a = \frac{5}{4}x$$

As a and x are positive integers, least possible value of x is 4 and that of a is 5.

Example 55: If $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$ upto $\infty = \frac{\pi^2}{6}$, then value of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ up to ∞ is

(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{6}$
(c) $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{12}$

Ans. (c)

Solution: We have $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$ upto ∞

$$= \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}\cdots \text{ upto } \infty$$

$$-\frac{1}{2^2}\left[1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$= \frac{\pi^2}{6}-\frac{1}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{8}$$

Example 56: If the length of sides of a right triangle are in A.P., then the sines of the acute angle are

(a) 3/5, 4/5 (b) $\sqrt{2/3}, \sqrt{1/3}$
(c) $\sqrt{\frac{\sqrt{5}-1}{2}}, \sqrt{\frac{\sqrt{5}+1}{2}}$ (d) $\sqrt{\frac{\sqrt{3}-1}{2}}, \sqrt{\frac{\sqrt{3}+1}{2}}$

Ans. (a)

Solution: Let the sides of the triangle be $a - d$, a, $a + d$ where $a > d > 0$.

By the Pythagorean theorem, we have

$$(a + d)^2 = (a - d)^2 + a^2$$

$$\Rightarrow \quad 4ad = a^2 \quad \Rightarrow \quad d = a/4$$

We have $\sin\theta = \frac{a}{a+d} = \frac{4}{5}$

and $\sin(90° - \theta) = 3/5$

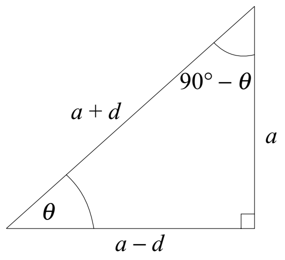


Fig. 8.1

Example 57: If exp $\{(\sin^2 x + \sin^4 x + \sin^6 x + \ldots$ upto $\infty) \ln 2\}$ satisfies the equation $x^2 - 17x + 16 = 0$ then value of $\frac{2\cos x}{\sin x + 2\cos x}$ $(0 < x < \pi/2)$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$
(c) $\frac{5}{2}$ (d) none of these

Ans. (a)

Solution: We have $\sin^2 x + \sin^4 x + \sin^6 x + \ldots$

$$= \frac{\sin^2 x}{1-\sin^2 x} = \tan^2 x$$

Therefore, $\alpha = \exp\{(\sin^2 x + \sin^4 x + \sin^6 x + \ldots \text{ upto } \infty) \ln 2\}$

$= \exp\{\tan^2 x \ln 2\} = \exp\{\ln 2^{\tan^2 x}\}$

$= 2^{\tan^2 x}$

As α satisfies the equation $x^2 - 17x + 16 = 0$ we get

$$\alpha = 1 \text{ or } \alpha = 16$$

Since $0 < x < \pi/2$, $\tan^2 x > 0 \Rightarrow \alpha = 2^{\tan^2 x} > 1$. Therefore,

$$2^{\tan^2 x} = 16 = 2^4 \Rightarrow \tan^2 x = 4 \Rightarrow \tan x = 2 \quad [\because \tan x > 0]$$

Thus, $\frac{2\cos x}{\sin x + 2\cos x} = \frac{2}{\tan x + 2} = \frac{2}{2+2} = \frac{1}{2}$

Example 58: Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in A.P., then $f'(a)$, $f'(b)$, $f'(c)$ are in

(a) G.P. (b) H.P.
(c) A.G.P. (d) A.P.

Ans. (d)

Solution: Let $f(x) = Ax^2 + Bx + C$.

$$f(1) = f(-1) \Rightarrow A + B + C \Rightarrow A - B + C$$

$$\Rightarrow \quad 2B = 0 \Rightarrow B = 0.$$

$$\therefore \quad f(x) = Ax^2 + C \Rightarrow f'(x) = 2Ax$$

As a, b, c are in A.P., $2Aa$, $2Ab$, $2Ac$ are in A.P., i.e., $f'(a)$, $f'(b), f'(c)$ are in A.P.

◉ **Example 59:** If $\exp\{(\tan^2 x - \tan^4 x + \tan^6 x - \tan^8 x + \ldots)\log_e 16\}$, $0 < x < \pi/4$, satisfies the quadratic equation $x^2 - 3x + 2 = 0$, then value of $\cos^2 x + \cos^4 x$ is

(a) 4/5 (b) 21/16
(c) 17/11 (d) 19/31

Ans. (b)

◎ **Solution:** We have

$$\tan^2 x - \tan^4 x + \tan^6 x - \tan^8 x + \ldots$$

$$= \frac{\tan^2 x}{1-\left(-\tan^2 x\right)} = \frac{\tan^2 x}{\sec^2 x} = \sin^2 x$$

Therefore

$$y = \exp\{(\tan^2 x - \tan^4 x + \tan^6 x - \tan^8 x + \ldots)\log_e 16\}$$
$$= \exp\{(\sin^2 x)\log_e 16\}$$
$$= \exp\{\log_e(16^{\sin^2 x})\} = 16^{\sin^2 x}$$

As y satisfies $x^2 - 3x + 2 = 0$, we get $y = 1$ or $y = 2$

$\Rightarrow$ $16^{\sin^2 x} = 1$ or $16^{\sin^2 x} = 2$

Since $0 < x < \pi/4$, $0 < \sin x < 1/\sqrt{2}$ $\Rightarrow$ $0 < \sin^2 x < 1/2$

$\therefore$ $16^{\sin^2 x} = 1$ is not possible.

Thus, $16^{\sin^2 x} = 2 \Rightarrow \sin^2 x = 1/4$

Thus, $\cos^2 x + \cos^4 x = (1 - \sin^2 x) + (1 - \sin^2 x)^2 = 21/16$

◉ **Example 60:** If $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$, then value of $\frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \frac{1}{9\cdot 11} + \cdots$ is

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{34}$

Ans. (a)

◎ **Solution:** We have

$$\frac{\pi}{4} = \left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\cdots$$

$$= \frac{2}{1\cdot 3}+\frac{2}{5\cdot 7}+\frac{2}{9.11}+\cdots$$

$$\Rightarrow \quad \frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\frac{1}{9\cdot 11}+\cdots = \frac{\pi}{8}$$

◉ **Example 61:** For x ε **R**, let $[x]$ denote the greatest integer $\le x$. Largest natural number n for which

$$E = \left[\frac{\pi}{2}\right]+\left[\frac{1}{100}+\frac{\pi}{2}\right]+\left[\frac{2}{100}+\frac{\pi}{2}\right]\ldots+\left[\frac{n}{100}+\frac{\pi}{2}\right] < 43,$$

is

(a) 41 (b) 42
(c) 43 (d) 97

Ans. (a)

◎ **Solution:** Since $3.14 < \pi < 3.142$, $1.57 < \pi/2 < 1.571$

Thus, $\left[\frac{\pi}{2}+\frac{n}{100}\right] = 1$ for $n = 0, 1, 2, \ldots, 42$

Thus, largest natural number n for which $E < 43$ is 41.

◉ **Example 62:** If the ratio of sums to n terms of two A.P.'s is $(5n + 7) : (3n + 2)$, then the ratio of their 17th terms is

(a) 172 : 99 (b) 172 : 101
(c) 175 : 99 (d) 175 : 101

Ans. (b)

◎ **Solution:** Let two A.P.'s be

$$a, a + d, a + 2d, a + 3d, \ldots$$

and $A, A + D, A + 2D, A + 3D, \ldots$

According to the given condition,

$$\frac{\frac{n}{2}[2a+(n-1)d]}{\frac{n}{2}[2A+(n-1)D]} = \frac{5n+7}{3n+2} \quad \Rightarrow \quad \frac{a+\frac{n-1}{2}d}{A+\frac{n-1}{2}D} = \frac{5n+7}{3n+2}$$

Putting $\frac{n-1}{2} = 16$ or $n = 33$, we get

$$\frac{a+16d}{A+16D} = \frac{5(33)+7}{3(33)+2} = \frac{172}{101}$$

◉ **Example 63:** If $H_1, H_2, \ldots, H_n$ are n harmonic means between a and $b(\ne a)$, then value of $\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}$ is equal to

(a) $n + 1$ (b) $n - 1$
(c) $2n$ (d) $2n + 3$

Ans. (c)

◎ **Solution:** As $a, H_1, H_2, \ldots, H_n$ b are in H.P.,

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2} \cdots \frac{1}{H_n}, \frac{1}{b} \text{ are in A.P.}$$

Let d be the common difference of this A.P., then

$$\frac{1}{b} = \frac{1}{a} + (n+1)d$$

and $$\frac{1}{H_n} - \frac{1}{H_1} = (n-1)d$$

Now, $$\frac{H_1+a}{H_1-a} = \frac{1/a+1/H_1}{1/a-1/H_1} = \frac{1/a+1/H_1}{-d}$$

and $$\frac{H_n+b}{H_n-b} = \frac{1/b+1/H_n}{1/b-1/H_n} = \frac{1/b+1/H_n}{d}$$

$$\therefore \frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b} = \frac{1/a+1/H_1}{-d}+\frac{1/b+1/H_n}{d}$$

$$= \frac{1}{d}\left[\left(\frac{1}{b}-\frac{1}{a}\right)+\left(\frac{1}{H_n}-\frac{1}{H_1}\right)\right] = 2n$$

Example 64: If three positive real numbers a, b, c are in A.P. such that $abc = 4$, then the minimum possible value of b is

(a) $2^{3/2}$ (b) $2^{2/3}$

(c) $2^{1/3}$ (d) $2^{5/2}$

Ans. (b)

Solution: Let d be the common difference of the A.P., then

$$4 = abc = (b - d)b\,(b + d) = b(b^2 - d^2)$$

$$\Rightarrow \quad b^3 = 4 + bd^2 \ge 4 \qquad [\because\ b > 0,\ d^2 \ge 0]$$

$$\Rightarrow \quad b \ge 2^{2/3}$$

Thus, minimum possible value of b is $2^{2/3}$, that is the case when $d = 0$.

Example 65: For $0 < x < \pi$ the values of x which satisfy the relation $9^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\cdots \text{ upto } \infty} = 3^4$ are given by

(a) $\frac{\pi}{3}, \frac{2\pi}{3}$ (b) $\frac{\pi}{3}, \frac{3\pi}{4}$

(c) $\frac{\pi}{4}, \frac{3\pi}{4}$ (d) $\frac{\pi}{3}, \frac{\pi}{4}$

Ans. (a)

Solution: Since $0 < x < \pi$, $-1 < \cos x < 1 \Rightarrow$ $0 \le |\cos x| < 1$. We can write the given expression as $9^{1/(1-|\cos x|)} = 81 = 9^2$

$$\Rightarrow \quad \frac{1}{1-|\cos x|} = 2 \quad \Rightarrow \quad 1 - |\cos x| = \frac{1}{2}$$

$$\Rightarrow \quad |\cos x| = 1/2 \quad \Rightarrow \quad \cos x = \pm 1/2$$

$$\Rightarrow \quad x = \pi/3 \text{ or } 2\pi/3$$

Example 66: The sum upto $(2n + 1)$ terms of the series $a^2 - (a + d)^2 + (a + 2d)^2 - (a + 3d)^2 + \ldots$ is

(a) $a^2 + 3nd^2$

(b) $a^2 + 2nad + n(n - 1)d^2$

(c) $a^2 + 3nad + n(n - 1)d^2$

(d) $(a + nd)^2 + n(n + 1)d^2$

Ans. (d)

Solution: We can write the sum upto $(2n + 1)$ terms as

$[a + (a + d)]\,(-d) + [(a + 2d) + (a + 3d)]\,(-d) + \ldots$

$[(a + (2n - 2)\,d) + (a + (2n - 1)d]\,(-d) + (a + 2nd)^2$

$$= (-d)\,[a + (a + d) + (a + 2d) + \ldots$$
$$+ a + (2n - 1)d] + (a + 2nd)^2$$
$$= (-d)\,\frac{2n}{2}\,\{a + a + (2n - 1)\,d\} + (a + 2nd)^2$$
$$= -2nad - n(2n - 1)d^2 + a^2 + 4\,n(ad) + 4n^2d^2$$
$$= a^2 + 2nad + n(2n + 1)d^2 = (a + nd)^2 + n(n + 1)d^2$$

Example 67: Value of $y = (0.36)^{\log_{0.25}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots \text{ upto } \infty\right)}$ is

(a) 0.9 (b) 0.8

(c) 0.6 (d) 0.25

Ans. (c)

Solution: We have

$$S = \frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots = \frac{1/3}{1-1/3} = \frac{1}{2}$$

$$\therefore \quad y = (0.36)^{\log_{0.25}(1/2)}$$

$$\Rightarrow \quad \log y = \frac{\log(1/2)}{\log(0.25)}\,\log(0.36)$$

$$= \frac{-\log 2}{-\log 4}\log(0.36) = \frac{1}{2}\log(0.36) = \log 0.6$$

$$\Rightarrow \quad y = 0.6$$

Example 68: Sum to n terms of the series

$S = 1 + 2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2 + \cdots$ is given by

(a) n^2 (b) $(n + 1)^2$

(c) $n(n + 1)$ (d) none of these

Ans. (a)

Solution: Let $x = 1 + 1/n$. Then

$$S = 1 + 2x + 3x^2 \ldots + nx^{n-1}$$

$$\Rightarrow \quad xS = x + 2x^2 + \ldots + (n - 1)x^{n-1} + nx^n$$

Subtracting, we get

$$(1 - x)S = 1 + x + x^2 + \ldots + x^{n-1} - nx^n$$
$$= \frac{1-x^n}{1-x} - nx^n$$

$$\Rightarrow \quad \left(-\frac{1}{n}\right)S = (-n)\left[1-\left(1+\frac{1}{n}\right)^n\right] - n\left(1+\frac{1}{n}\right)^n = -n$$

$$\Rightarrow \quad S = n^2$$

Example 69: Suppose a, b, c are positive real numbers. If a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then

(a) $a = b = c$ (b) $2b = 3a + c$

(c) $b^2 = \sqrt{ac/8}$ (d) none of these

Ans. (a)

Solution: We have $2b = a + c$ and $b^2 = \frac{2a^2c^2}{a^2+c^2}$

$$\therefore \quad \left(\frac{a+c}{2}\right)^2 = \frac{2a^2c^2}{a^2+c^2} \Rightarrow (a + c)^2\,(a^2 + c^2) = 8a^2c^2$$

$$\Rightarrow \quad (a^2 + c^2)^2 + 2ac(a^2 + c^2) - 8a^2c^2 = 0$$

$$\Rightarrow \quad (a^2 + c^2 + 4ac)(a^2 + c^2 - 2ac) = 0$$

$$\Rightarrow \quad [(a + c)^2 + 2ac](a - c)^2 = 0$$

$$\Rightarrow \quad a - c = 0 \text{ or } (a + c)^2 + 2ac = 0$$

As $(a + c)^2 + 2ac > 0$, we get $a = c \Rightarrow a = b = c$

Example 70: If $0 < \theta, \phi < \pi/2$ and $x = \sum_{n=0}^{\infty} \sin^{2n}\phi$, $y = \sum_{n=0}^{\infty} \cos^{2n}\theta$ and $z = \sum_{n=0}^{\infty} \cos^n(\theta + \phi)\cos^n(\theta - \phi)$, then

(a) $xyz + 1 = yz - zx$ (b) $xyz - 1 = yz + zx$
(c) $xyz - xy = yz - zx$ (d) $xyz + 1 = yz + zx$

Ans. (c)

Solution: We have $x = \dfrac{1}{1 - \sin^2\phi} = \dfrac{1}{\cos^2\phi}$,

$$y = \frac{1}{1 - \cos^2\theta} = \frac{1}{\sin^2\theta},$$

Also, $\cos(\theta + \phi)\cos(\theta - \phi) = \cos^2\phi - \sin^2\theta = \dfrac{1}{x} - \dfrac{1}{y}$

As, $0 < \cos^2\phi < 1$, $0 < \sin^2\theta < 1$, $-1 < \cos^2\phi - \sin^2\theta < 1$

$$\Rightarrow \quad -1 < \frac{1}{x} - \frac{1}{y} < 1$$

Thus, $$z = \sum_{n=0}^{\infty}\left(\frac{1}{x} - \frac{1}{y}\right)^n = \frac{1}{1 - \left(\dfrac{1}{x} - \dfrac{1}{y}\right)} = \frac{xy}{xy - y + x}$$

$$\Rightarrow \quad z(xy - y + x) = xy \quad \text{or} \quad xyz - xy = yz - xz$$

Example 71: If a, b, c are in H.P. then $a - \dfrac{b}{2}, \dfrac{b}{2}, c - \dfrac{b}{2}$ are in

(a) A.P. (b) G.P.
(c) A.G.P. (d) H.P.

Ans. (b)

Solution: $b = \dfrac{2ac}{a + c}$, and

$$\left(a - \frac{b}{2}\right)\left(c - \frac{b}{2}\right) = ac - \frac{b}{2}(a + c) + \frac{b^2}{4}$$

$$= ac - ac + \frac{b^2}{4} = \frac{b^2}{4}$$

Example 72: Sum to n terms of the series $\dfrac{1}{(1 + x)(1 + 2x)} + \dfrac{1}{(1 + 2x)(1 + 3x)} + \dfrac{1}{(1 + 3x)(1 + 4x)} + \dots$ is

(a) $\dfrac{nx}{(1 + x)(1 + nx)}$ (b) $\dfrac{n}{(1 + x)[1 + (n + 1)x]}$

(c) $\dfrac{x}{(1 + x)(1 + (n - 1)x)}$ (d) none of these

Ans. (b)

Solution: If t_r denotes the rth term of the series, then

$$x\,t_r = \frac{x}{(1 + rx)(1 + (r + 1)x)} = \frac{1}{1 + rx} - \frac{1}{1 + (r + 1)x}$$

$$\Rightarrow \quad x\sum_{r=1}^{n} t_r = \sum_{r=1}^{n}\left[\frac{1}{1 + rx} - \frac{1}{1 + (r + 1)x}\right]$$

$$= \frac{1}{1 + x} - \frac{1}{1 + (n + 1)x}$$

$$= \frac{nx}{(1 + x)(1 + (n + 1)x)}$$

$$\Rightarrow \quad \sum_{r=1}^{n} t_r = \frac{n}{(1 + x)[1 + (n + 1)x]}$$

Example 73: If $a_1, a_2, \dots a_n$ are in H.P. then the expression $a_1a_2 + a_2a_3 + \dots + a_{n-1}a_n$ is equal to

(a) $(n - 1)a_1 a_n$ (b) $n(a_1 - a_n)$
(c) $(n - 1)(a_1 - a_n)$ (d) $na_1 a_n$

Ans. (a)

Solution: $a_1, a_2, \dots a_n$ are in H.P.

$$\Rightarrow \quad \frac{1}{a_1}, \frac{1}{a_2}, \dots \frac{1}{a_n} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \cdots \frac{1}{a_n} - \frac{1}{a_{n-1}} = d \text{ (say)}$$

$$\Rightarrow \quad a_1a_2 = \frac{1}{d}(a_1 - a_2),\ a_2a_3 = \frac{1}{d}(a_2 - a_3), \dots$$

$$a_{n-1}a_n = \frac{1}{d}(a_{n-1} - a_n)$$

Thus, $$a_1a_2 + a_2a_3 + \dots + a_{n-1}a_n$$

$$= \frac{1}{d}[a_1 - a_2 + a_2 - a_3 + \dots + a_{n-1} - a_n]$$

$$= \frac{1}{d}(a_1 - a_n)$$

But $$\frac{1}{a_n} = \frac{1}{a_1} + (n - 1)d \ \Rightarrow\ \frac{a_1 - a_n}{a_n a_1} = (n - 1)d$$

$$\therefore \quad a_1a_2 + a_2a_3 + \dots + a_{n-1}a_n = (n - 1)a_1 a_n$$

Example 74: If the sum of the series $2 + \dfrac{5}{x} + \dfrac{25}{x^2} + \dfrac{125}{x^3} + \dots$ is finite, then

(a) $|x| > 5$ (b) $-5 < x < 5$
(c) $|x| < 5/2$ (d) $|x| > 5/2$

Ans. (a)

Solution: We can rewrite the series as

$$1 + 1 + \frac{5}{x} + \left(\frac{5}{x}\right)^2 + \left(\frac{5}{x}\right)^3 + \dots$$

We can sum up this series if $\left|\frac{5}{x}\right| < 1 \quad \Leftrightarrow \quad |x| > 5$

Example 75: In a G.P. consisting of positive terms, each term equals the sum of the next two terms. Then common ratio of the G.P. is

(a) $\frac{1}{2}(1-\sqrt{5})$ (b) $\frac{1}{2}\sqrt{5}$

(c) $\sqrt{5}$ (d) $\frac{1}{2}(\sqrt{5}-1)$

Ans. (d)

Solution: We are given

$$ar^n = ar^{n+1} + ar^{n+2} \ \forall \ n \geq 1$$

$$\Rightarrow \quad 1 = r + r^2 \Rightarrow r^2 + r - 1 = 0 \Rightarrow r = \frac{1}{2}(-1 \pm \sqrt{5})$$

As G.P. is of positive terms, $r > 0$. Thus, $r = \frac{1}{2}(\sqrt{5}-1)$

Example 76: Suppose a, b, c, are in A.P. and $-1 < a, b, c < 1$. Let $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$, then x, y, z are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

Ans. (c)

Solution: We have $\quad x = \frac{1}{1-a},\ y = \frac{1}{1-b},\ z = \frac{1}{1-c}.$

As a, b, c are in A.P., $-a, -b, -c$ are in A.P. $\Rightarrow 1-a, 1-b, 1-c$ are in A.P.

$\Rightarrow \frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c}$ are in H.P.

Example 77: If $\log_5 2$, $\log_5(2^x - 5)$ and $\log_5(2^x - 7/2)$ are in A.P., then x is equal to

(a) 7 (b) 3

(c) 4, 5 (d) 8

Ans. (b)

Solution: As $\log_5 2$, $\log_5(2^x - 5)$, $\log_5(2^x - 7/2)$ are in A.P., we get

$$2\log_5(2^x - 5) = \log_5 2 + \log_5(2^x - 7/2)$$

$$\Rightarrow \quad (2^x - 5)^2 = 2(2^x - 7/2)$$

$$\Rightarrow (2^x)^2 - 12\,(2^x) + 32 = 0 \quad \Rightarrow \quad (2^x - 4)\,(2^x - 8) = 0$$

$$\Rightarrow \qquad 2^x = 2^2, 2^3 \qquad \Rightarrow \qquad x = 2, 3.$$

Clearly, $x \neq 2$. Therefore $x = 3$

Example 78: Sum of the series $\sum_{r=1}^{n} r \log \frac{r+1}{r}$ is

(a) $\log \frac{(n+1)^n}{n!}$ (b) $\frac{\log(n+1)}{n!}$

(c) $n! \log(n+1)$ (d) none of these

Ans. (a)

Solution: Let $t_r = r \log \frac{r+1}{r} = r\,[\log(r+1) - \log r]$

$$= (r+1)\log(r+1) - r\log r - \log(r+1)$$

$$\Rightarrow \quad \sum_{r=1}^{n} t_r = \sum_{r=1}^{n} \{(r+1)\log(r+1) - r\log r\} - \sum_{r=1}^{n} \log(r+1)$$

$$= (n+1)\log(n+1) - 1\log 1 - \log(n+1)!$$

$$= \log \frac{(n+1)^{n+1}}{(n+1)!} = \log \frac{(n+1)^n}{n!}$$

Example 79: Let A_1, A_2 be two arithmetic means, G_1, G_2 be two geometric means, and H_1, H_2 be two harmonic means between two positive numbers a and b, then values $\frac{G_1 G_2}{H_1 H_2} \cdot \frac{H_1 + H_2}{A_1 + A_2}$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$

(c) 1 (d) 2

Ans. (c)

Solution: We have $A_1 + A_2 = a + b$,

$$G_1 G_2 = ab \text{ and } \frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b}.$$

$$\therefore \quad \frac{G_1 G_2}{H_1 H_2} \cdot \frac{H_1 + H_2}{A_1 + A_2} = \frac{G_1 G_2}{A_1 + A_2}\left(\frac{1}{H_1} + \frac{1}{H_2}\right)$$

$$= \frac{ab}{a+b}\left(\frac{1}{a} + \frac{1}{b}\right) = 1.$$

Example 80: Sum of the series $S_n = (n)\,(n) + (n-1)\,(n+1) + (n-2)\,(n+2) + \ldots + 1(2n-1)$ is

(a) $\frac{1}{6}\,n\,(n+1)\,(n+2)$ (b) $\frac{1}{3}n^3 - n^2$

(c) n^3 (d) none of these.

Ans. (d)

Solution: $S_n = \sum_{r=0}^{n-1} (n-r)\,(n+r) = \sum_{r=0}^{n-1} (n^2 - r^2)$

$$= n^3 - \frac{1}{6}(n-1)\,n(2n-1)$$

Example 81: Suppose for each $n \in \mathbf{N}$, $1^4 + 2^4 + 3^4 + \ldots + n^4 = an^5 + bn^4 + cn^3 + dn^2 + en + f$, then value of b is

(a) 1/5 (b) 1/2

(c) 1/3 (d) 1

Ans. (b)

Solution: We have

$$1^4 + 2^4 + \ldots + n^4 = an^5 + bn^4 + cn^3 + dn^2 + en + f \qquad (1)$$

and $\qquad 1^4 + 2^4 + \ldots + n^4 + (n+1)^4$

$$= a(n+1)^5 + b(n+1)^4 + c(n+1)^3 + d(n+1)^2 + e(n+1) + f \qquad (2)$$

Subtracting (1) from (2), we get

$$(n+1)^4 = a[(n+1)^5 - n^5] + b[(n+1)^4 - n^4] + c\,[(n+1)^3 - n^3] + d[(n+1)^2 - n^2] + e\,[(n+1) - n]$$

Comparing coefficients fo n^4, we get

$$1 = 5a \quad \Rightarrow \quad a = 1/5$$

Comparing coefficient fo n^3, we get

$$4 = a\,({}^5C_2) + 4b \Rightarrow b = 1/2.$$

◉ **Example 82:** Value of $S = 2012 + \frac{1}{3}\left(2011 + \frac{1}{3}\left(2010 + \frac{1}{3}(2009 + \dots + \frac{1}{3}\left(2 + \frac{1}{3}(1)\right)\dots\right)\right)$ is

(a) 3018 (b) 3017.5

(c) $3017.5 - \frac{1}{4}\left(\frac{1}{3}\right)^{2011}$ (d) $3018 - \frac{1}{4}\left(\frac{1}{3}\right)^{2011}$

Ans. (c)

◎ **Solution:** Note that

$$S = 2012 + \frac{1}{3}(2011) + \frac{1}{3^2}(2010) + \frac{1}{3^3}(2009) + \dots + \frac{1}{3^{2011}}$$

This is an A.G.P. with $a = 2012$, $b = 1$, $d = -1$, $r = 1/3$, $n = 2012$

Thus,
$$S = \frac{2012}{1 - 1/3} + \frac{(-1)(1)(1 - (1/3)^{2011})}{1 - 1/3} - \frac{[2012 + (2012 - 1)(-1)](1)(1/3)^{2012}}{(1 - 1/3)^2}$$

$$= 3018 - \frac{1}{2}\left[1 - \left(\frac{1}{3}\right)^{2011}\right] - \frac{1}{4}\left(\frac{1}{3}\right)^{2010}$$

$$= 3017.5 - \frac{1}{4}\left(\frac{1}{3}\right)^{2011}$$

◉ **Example 83:** Suppose a, b, c are positive real numbers satisfying the system of equations

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) = abc \qquad (1)$$

and
$$(a^4 + a^2b^2 + b^4)(b^4 + b^2c^2 + c^4)(c^4 + c^2a^2 + a^4) = a^3b^3c^3 \qquad (2)$$

then

(a) $a = b = c = 1/3$

(b) $abc = 1$

(c) $\frac{1}{3}(a + b + c) = (abc)^{1/2}$

(d) a, b, c are in G.P.

Ans. (a)

◎ **Solution:** We have $a^4 + a^2b^2 + b^4 = (a^2 + b^2)^2 - a^2b^2 = (a^2 + ab + b^2)(a^2 - ab + b^2)$ etc.

Thus, dividing (2) by (1), we get

$$(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) = a^2b^2c^2 \qquad (3)$$

But A.M. ≥ G.M.

$$\Rightarrow \quad \frac{1}{2}(x^2 + y^2) \ge x^2y^2$$

and the equality holds if and only if $x = y$.

$$\Rightarrow \quad x^2 - xy + y^2 \ge xy \ \forall\ x, y > 0$$

and the equality holds if and only if $x = y$.

Thus, equality in (3) can hold if and only if $a = b = c$.

Putting this in (1), we obtain

$$(3a^2)^3 = a^3 \quad \Rightarrow \quad a^3 = (1/3)^3 \text{ or } a = 1/3.$$

$$\therefore \quad a = b = c = 1/3.$$

◉ **Example 84:** Suppose a, b, c are distinct positive real numbers such that a, $2b$, $3c$ are in A.P. and a, b, c are in G.P. The common ratio of G.P. is

(a) 2 (b) 1/2

(c) 1/3 (d) 3

Ans. (c)

◎ **Solution:** We have $2(2b) = a + 3c$ and $b^2 = ac$

$$\therefore \quad \frac{1}{16}(a + 3c)^2 = ac$$

$$\Rightarrow \quad a^2 + 9c^2 + 6ac = 16ac$$

$$\Rightarrow \quad a^2 - 10ac + 9c^2 = 0$$

$$\Rightarrow \quad (a - c)(a - 9c) = 0 \Rightarrow a = 9c \ [\because \ a \ne c]$$

$$\Rightarrow \quad \frac{c}{a} = \frac{1}{9} \text{ or } r^2 = \frac{1}{9} \Rightarrow r = \frac{1}{3}$$

$[\because a, b, c > 0]$

◉ **Example 85:** Let S_n denote the sum of the cubes of first n natural numbers and s_n denote the sum of the first n natural numbers. Then $\sum_{r=1}^{n} \frac{S_r}{s_r}$ equals

(a) $\frac{1}{2}n(n + 1)$ (b) $\frac{1}{2}(n + 1)(n + 2)$

(c) $\frac{1}{6}n(n + 1)(n + 2)$ (d) $\frac{1}{4}n^2(n + 1)^2$

Ans. (c)

◎ **Solution:** We have

$$S_r = 1^3 + 2^3 + \dots r^3 = \frac{1}{4}r^2(r + 1)^2$$

$$s_r = 1 + 2 + \dots + r = \frac{1}{2}r(r + 1)$$

$$\Rightarrow \quad \frac{S_r}{s_r} = \frac{1}{2}r(r + 1) = \frac{1}{2}(r^2 + r)$$

$$\Rightarrow \quad \sum_{r=1}^{n}\frac{S_r}{s_r} = \frac{1}{2}\sum_{r=1}^{n}(r^2+r)$$

$$= \frac{1}{2}\left[\frac{1}{6}n(n+1)(2n+1)+\frac{1}{2}n(n+1)\right]$$

$$= \frac{1}{12}n(n+1)(2n+1+3) = \frac{1}{6}n(n+1)(n+2)$$

◉ **Example 86:** The sum to infinity of the series

$S = 1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\ldots$ is

(a) 4 (b) 6

(c) 2 (d) 3

Ans. (d)

◎ **Solution:** Let $S = 1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\ldots$

$$\frac{1}{3}S = \frac{1}{3}+\frac{2}{3^2}+\frac{6}{3^3}+\frac{10}{3^4}+\ldots$$

Subtracting, we get

$$\frac{2}{3}S = 1+\frac{1}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\frac{4}{3^4}+\ldots$$

$$= 1+\frac{1}{3}+\frac{4/3^2}{1-1/3} = \frac{4}{3}+\frac{2}{3} = 2$$

$$\Rightarrow \quad S = 3$$

◉ **Example 87:** Sum of the series $P = \frac{1}{2\sqrt{1}+\sqrt{2}}$

$+\frac{1}{3\sqrt{2}+2\sqrt{3}}+\ldots+\frac{1}{100\sqrt{99}+99\sqrt{100}}$ is

(a) 1/10 (b) 3/10

(c) 9/10 (d) 1/2

Ans. (c)

◎ **Solution:** Let t_r denote the rth term of the series, then

$$t_r = \frac{1}{(r+1)\sqrt{r}+r\sqrt{r+1}} = \frac{1}{\sqrt{r(r+1)}}\ \frac{1}{\sqrt{r+1}+\sqrt{r}}$$

$$= \frac{\sqrt{r+1}-\sqrt{r}}{\sqrt{r(r+1)}} = \frac{1}{\sqrt{r}}-\frac{1}{\sqrt{r+1}}$$

Thus, $$P = \sum_{r=1}^{99}\left[\frac{1}{\sqrt{r}}-\frac{1}{\sqrt{r+1}}\right] = 1-\frac{1}{\sqrt{100}} = \frac{9}{10}$$

◉ **Example 88:** Let $x = \sqrt{2\sqrt{2\sqrt{2\sqrt{2\ldots \text{upto } \infty}}}}$ and

$y = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots \text{upto } \infty}}}}$, then $x - y$ equals

(a) 0 (b) 1

(c) 2 (d) none of these

Ans. (a)

◎ **Solution:** We have

$$x^2 = 2x \text{ and } y^2 = 2 + y$$

$$\Rightarrow \quad x(x-2) = 0 \text{ and } (y-2)(y+1) = 0$$

$$\Rightarrow \quad x = 2,\ y = 2 \qquad [\because \quad x, y > 0]$$

$$\Rightarrow \quad x - y = 0$$

◉ **Example 89:** If $[x]$ denotes the greatest integer $\le x$, then value of $S = \left[\sqrt{1}\right]+\left[\sqrt{2}\right]+\left[\sqrt{3}\right]+\ldots+\left[\sqrt{2024}\right]$ is

(a) 59000 (b) 58750

(c) 59730 (d) 65138

Ans. (c)

◎ **Solution:** For any positive integer k and x, we have

$$[\sqrt{x}] = k \Leftrightarrow k^2 \le x < (k+1)^2 \Leftrightarrow x \in [k^2, k^2+2k)$$

$$\therefore \quad [\sqrt{x}] = 1 \quad \text{for } x = 1, 2, 3,$$

$$[\sqrt{x}] = 2 \quad \text{for } x = 4, 5, 6, 7, 8,$$

$$[\sqrt{x}] = 3 \quad \text{for } x = 9, 10, 11, 12, 13, 14, 15,$$

$$\vdots \quad \vdots \quad \vdots$$

$$[\sqrt{x}] = 44 \quad \text{for } x = 1936, \ldots, 2024$$

Thus,

$$S = (1)(3) + (2)(5) + 3(7) + \ldots + 44(89)$$

$$= \sum_{k=1}^{44}(k)(2k+1) = 2\sum_{k=1}^{44}(k)^2+\sum_{k=1}^{44}(k)$$

$$= \frac{2}{6}(44)(45)(89)+\frac{1}{2}(44)(45)$$

$$= 59730$$

◉ **Example 90:** A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute. If $a_1 = a_2 = \ldots = a_{10} = 150$ and $a_{10}, a_{11}, \ldots$ are in an A.P. with common difference –2, then the time taken by him to count all the notes is

(a) 125 minutes (b) 135 minutes

(c) 24 minutes (d) 34 minutes

Ans. (d)

◎ **Solution:** Suppose he takes n minutes to count 4500 notes. We have $a_1 + a_2 + \ldots + a_{10} = 10(150)$, $a_{11} = 148$ and $a_{11}, a_{12}, \ldots, a_n$ is an A.P. with common difference $d = -2$. We are given

$$a_1 + a_2 + \ldots + a_{10} + a_{11} + \ldots + a_n = 4500$$

$$\Rightarrow \quad a_{11} + a_{12} + \ldots + a_n = 3000$$

$$\Rightarrow \quad \frac{n-10}{2}[a_{11} + a_n] = 3000$$

$$\Rightarrow \quad \frac{1}{2}(n-10)[148 + 148 + (n-11)(-2)] = 3000$$

$$\Rightarrow \quad (n-10)(148 - n + 11) = 3000$$

$$\Rightarrow \quad (n-10)(159 - n) = 3000$$

$$\Rightarrow \quad n^2 - 169n + 4590 = 0$$

$$\Rightarrow \quad n^2 - 135n - 34n + 4590 = 0$$

$$\Rightarrow \quad (n-135)(n-34) = 0 \quad \Rightarrow \quad n = 135, 34$$

All the notes get counted in 34 minutes.

Example 91: A man saves ₹ 200 in each of the first three months of his service. In each of the subsequent months his savings was ₹ 40 more than the savings of the immediately previous month. His total savings from the start of service will be ₹ 11040 after

(a) 21 months (b) 18 months
(c) 19 months (d) 20 months

Ans. (a)

Solution: Suppose his total savings is ₹ 11040 after n months. His savings are

$$200, 200, 200, 240, 280, \dots \text{ upto } n \text{ terms}$$

Note that

$$200, 240, 280, \dots \text{ upto } (n-2) \text{ terms}$$

is an A.P. whose sum is 10640.

Thus,

$$\frac{1}{2}(n-2)[2(200)+(n-3)(40)] = 10640$$

$$\Rightarrow (n-2)(n+7) = 532$$

$$\Rightarrow n^2 + 5n - 546 = 0 \Rightarrow n = -26, 21$$

As $n > 0$, $n = 21$.

Example 92: Let a_n be the nth term of an A.P. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, then the common difference of the A.P. is:

(a) $\alpha - \beta$ (b) $\frac{1}{100}(\alpha - \beta)$
(c) $\beta - \alpha$ (d) $\frac{1}{200}(\alpha - \beta)$

Ans. (b)

Solution: Let d the common difference of the A.P., then

$$\alpha - \beta = \sum_{r=1}^{100}\left(a_{2r} - a_{2r-1}\right) = \sum_{r=1}^{100} d = 100d$$

$$\Rightarrow d = \frac{1}{100}(\alpha - \beta).$$

Example 93: If m times the mth term of an A.P. with non-zero common difference equals n times the nth term of the A.P., where $m \neq n$, then $(m + n)$th term of this A.P. is

(a) $(m + n)$ times mth term
(b) zero
(c) $m + n$
(d) $-(m + n)$

Ans. (b)

Solution: Let a be the first term and d be the common difference of the A.P. We thve

$$ma_m = na_n$$

$$\Rightarrow \quad m[a + (m-1)d] = n[a + (n-1)d]$$

$$\Rightarrow \quad (m-n)(a-d) + (m^2 - n^2)d = 0$$

$$\Rightarrow \quad (m-n)[a - d + (m+n)d] = 0$$

$$\Rightarrow \quad a + (m + n - 1)d = 0 \qquad [\because m \neq n]$$

$$\Rightarrow \quad a_{m+n} = 0$$

Example 94: Let $a_1, a_2, \dots$ be in H.P. with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is

(a) 22 (b) 23
(c) 24 (d) 25

Ans. (d)

Solution: $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots$ are in A.P.

Let d be the common difference of this A.P., then

$$\frac{1}{a_{20}} - \frac{1}{a_1} = 19d$$

$$\Rightarrow \quad \frac{1}{25} - \frac{1}{5} = 19d \Rightarrow d = -\frac{4}{475}$$

Now, $$a_n < 0 \Rightarrow \frac{1}{5} - \frac{(n-1)4}{475} < 0$$

$$\Rightarrow \quad n - 1 > 95/4 \Rightarrow n \geq 25.$$

Thus, least value of n is 25.

Example 95: A species has an initial population 4^{10}. At the end of first day, the population increases by 50%. At the end of second day, it decreases by the same percentage. If the process continues in the same pattern, the number of days for the population to reach 3^{10} is

(a) 10 (b) 20
(c) 50 (d) 100

Ans. (b)

Solution: Let a_n be the population at the end of nth day, then

$$a_n = \begin{cases} 4^{10}\left(\frac{3}{2}\right)^{n/2}\left(\frac{1}{2}\right)^{n/2} & \text{if } n \text{ is even} \\ 4^{10}\left(\frac{3}{2}\right)^{(n+1)/2}\left(\frac{1}{2}\right)^{(n-1)/2} & \text{is } n \text{ is odd} \end{cases}$$

$$= \begin{cases} 4^{10}(3/4)^{n/2} & \text{if } n \text{ is even} \\ 4^{10}\left(\frac{3}{4}\right)^{(n-1)/2}\left(\frac{3}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$

As all answers are even natural numbers we set $a_n = 3^{10}$ assuming n to be even.

$$\Rightarrow \quad 4^{10}\left(\frac{3}{4}\right)^{n/2} = 3^{10} \Rightarrow n = 20$$

Remark

- ❖ $a_n = 3^{10}$ is not possible for any odd n.

Example 96: The sum of first three terms of an increasing A.P. is 9 and sum of their squares is 35. The sum to n terms of the series is

(a) $3n^2$ (b) $2n^2$
(c) $6n - 2n^2$ (d) n^2

Ans.(d)

Solution: Let d be common difference of the A.P., then

$$(a_2 - d) + a_2 + (a_2 + d) = 9$$

and $$(a_2 - d)^2 + a_2^2 + (a_2 + d)^2 = 35$$

$$\Rightarrow \quad a_2 = 3 \text{ and } 3a_2^2 + 2d^2 = 35$$

$$\Rightarrow \quad d = 2.$$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}\,[2a_2 + (n-3)d]$$

$$= \frac{n}{2}\,[2(3) + (n-3)\,(2)] = n^2$$

Example 97: The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7, 8, 9), and so on. The sum of first and last term of the nth group is

(a) $2\,(n^2 - n + 1)$ (b) $n^2 - 2n + 2$
(c) $n^2 + n$ (d) $n^2 - n + 2$

Ans. (a)

Solution: Note that the nth group contain $(2n - 1)$ natural number and the last term of the nth group is

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

$$\therefore \text{ First terms of } n\text{th group is } (n-1)^2 + 1$$

Thus, sum of first and last term of nth group is

$$n^2 + (n-1)^2 + 1 = 2(n^2 - n + 1)$$

Example 98: In an arithmetic progression of 16 distinct terms with $a_1 = 16$, the sum is equal to square of the last term. The common difference of the A.P. is

(a) 8/15 (b) – 4/5
(c) – 8/5 (d) – 8/15

Ans. (c)

Solution: We are given

$$\frac{16}{2}\,[a_1 + a_{16}] = a_{16}^2$$

$$\Rightarrow \quad a_{16}^2 - 8a_{16} - (8)\,(16) = 0$$

$$\Rightarrow \quad (a_{16} - 16)\,(a_{16} + 8) = 0$$

As $a_{16} \neq 16 = a_1$, $a_{16} = -8$

Now, $a_{16} - a_1 = -24 \Rightarrow 15d = -24$

$$\Rightarrow \quad d = -8/5$$

Assertion-Reason Type Questions

Example 99: Suppose $\dfrac{a-x}{px} = \dfrac{a-y}{qy} = \dfrac{a-z}{rz}$.

Statement-1: If p, q, r are in A.P., then x, y, z are in H.P.
Statement-2: If p, q, r are in H.P., then x, y, z are in G.P.

Ans. (c)

Solution: Suppose

$$\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz} = k \text{ (say)}$$

$$\Rightarrow \quad p = \frac{a-x}{kx},\ q = \frac{a-y}{ky},\ r = \frac{a-z}{kz}$$

Now, p, q, r are in A.P.

$$\Rightarrow \quad \frac{a-x}{kx}, \frac{a-y}{ky}, \frac{a-z}{kz} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{a}{x} - 1, \frac{a}{y} - 1, \frac{a}{z} - 1 \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{a}{x}, \frac{a}{y}, \frac{a}{z} \text{ are in A.P.} \quad \Rightarrow \quad x, y, z \text{ are in H.P.}$$

Next, p, q, r are in H.P.

$$\Rightarrow \quad \frac{kx}{a-x}, \frac{ky}{a-y}, \frac{kz}{a-z} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{x}{a-x}, \frac{y}{a-y}, \frac{z}{a-z} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{x}{a-x} + 1, \frac{y}{a-y} + 1, \frac{z}{a-z} + 1 \text{ are in A.P.}$$

$\Rightarrow$ $a - x, a - y, a - z$ are in H.P.

Nothing can be said about x, y, z.

Example 100: Let a, b, c, d, e be five non-zero numbers such that a, b, c are in A.P., b, c, d are in G.P, and c, d, e are in H.P.

Statement-1: If $a = 2$, $e = 18$, $b > 0$, then $b = 4$ and $c = 6$.
Statement-2: If $a = 2$, $e = 18$, $c = -6$, then $b < 0$.

Ans. (b)

Solution: $a + c = 2b$, $c^2 = bd$ and $d = \dfrac{2ce}{c+e}$.

Now, $$a = 2, e = 18 \quad \Rightarrow \quad b = \frac{2+c}{2},\ d = \frac{36c}{c+18}.$$

$$\therefore \quad c^2 = \frac{1}{2}(2+c)\left(\frac{36c}{c+18}\right)$$

$$\Rightarrow \quad c(c + 18) = 18(c + 2)$$

$$\Rightarrow \quad c^2 = 36 \quad \Rightarrow \quad c = \pm 6.$$

If $c = 6$, then $b = \dfrac{2+6}{2} = 4$

If $c = -6$, then $b = \dfrac{2-6}{2} = -2 < 0$.

◉ **Example 101:** Suppose a, b, c, d are four real numbers and

$$\Delta(x) = \begin{vmatrix} x+a & x+b & x+a-c \\ x+b & x+c & x-1 \\ x+c & x+d & x-b+d \end{vmatrix},$$

Statement-1: If a, b, c, d are in A.P. and $\int_0^2 \Delta(x)\,dx = -4$, then common difference of the A.P. is ± 1

Statement-2: If a, b, c, d are in A.P., then $\Delta(x)$ is independent of x.

Ans. (b)

◎ **Solution:** Let D be the common difference of the A.P. Using $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get

$$\Delta(x) = \begin{vmatrix} x+a & x+b & x+a-c \\ D & D & 2D-1 \\ D & D & 2D+1 \end{vmatrix}$$

Again using $C_1 \to C_1 - C_2$, we get

$$\Delta(x) = \begin{vmatrix} -D & x+b & x+a-d \\ 0 & D & 2D-1 \\ 0 & D & 2D+1 \end{vmatrix} = -2D^2$$

Now, $\int_0^2 (-2D^2)dx = -4 \;\Rightarrow\; 4D^2 = 4 \;\Rightarrow\; D = \pm 1$

Also, $\Delta(x)$ is independent of x.

◉ **Example 102:** Suppose a^2, b^2, c^2 are in A.P.

Statement-1: $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in H.P.

Statement-2: $b + c$, $c + a$, $a + b$ are in H.P.

Ans. (d)

◎ **Solution:** a^2, b^2, c^2 are in A.P.

$\Rightarrow$ $a^2 + bc + ca + ab,\ b^2 + bc + ca + ab,\ c^2 + ab + bc + ca$ are in A.P.

$\Rightarrow$ $(a + b)(a + c),\ (b + a)(b + c),\ (c + a)(c + b)$ are in A.P.

Dividing by $(b + c)(c + a)(a + b)$, we get

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A.P.} \qquad (1)$$

$\Rightarrow$ $b + c$, $c + a$, $a + b$ are in H.P.

and $\dfrac{a+b+c}{b+c}, \dfrac{a+b+c}{c+a}, \dfrac{a+b+c}{a+b}$ are in A.P.

$\Rightarrow$ $\dfrac{a}{b+c}+1, \dfrac{b}{c+a}+1, \dfrac{c}{a+b}+1$ are in A.P.

$\Rightarrow$ $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in A.P.

◉ **Example 103:** The natural numbers are divided into rows as follows:

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & 3 & 4 & \\ 5 & 6 & 7 & 8 & 9 \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots} \end{array}$$

Statement-1: Sum of the numbers in the 10th row is a number which can be written as sum of two cubes in two different ways.

Statement-2: Sum of the numbers in the rth row is $\dfrac{1}{3}(r^3 + (r + 1)^3)$ if $r \ge 2$.

Ans. (c)

◎ **Solution:** The rth row contains $(2r - 1)$ numbers and its last term is r^2.

When read from right to left the rth row forms an A.P. with first term r^2 and common difference – 1.

$\therefore$ sum of the terms in the rth row

$$= \frac{2r-1}{2}[2(r^2) + (2r - 1 - 1)(-1)]$$

$$= (2r - 1)(r^2 - r + 1) = r^3 + (r - 1)^3$$

Sum of the terms in the 10th row

$$= 10^3 + 9^3 = 1729 = 12^3 + 1^3$$

◉ **Example 104:** **Statement-1:** The sum of the series

$$1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + \ldots + (361 + 380 + 400) \text{ is } 8000$$

Statement-2: $\displaystyle\sum_{k=1}^{n}\left(k^3 - (k-1)^3\right) = n^3$

for each natural number n.

Ans. (a)

◎ **Solution:** Statement-2 is true since

$$\sum_{k=1}^{n}\left(k^2 - (k-1)^3\right) = (1^3 - 0^3) + (2^3 - 1^3) + \ldots + (n^3 - (n-1)^3) = n^3$$

$$\Rightarrow \quad \sum_{k=1}^{n}(k-(k-1))\left((k^2 + k(k-1) + (k-1)^2\right) = n^3$$

$$\Rightarrow \quad \sum_{k=1}^{n}\left(k^2 + k(k-1) + (k-1)^2\right) = n^3$$

Putting $n = 20$, we get

$$\sum_{k=1}^{20}\left((k-1)^2 + k(k-1) + k^2\right) = 20^3$$

$\Rightarrow$ $1 + (1 + 2 + 4) + (4 + 6 + 9) + \ldots + (361 + 380 + 400) = 8000$

Thus, Statement-1 is also true and Statement-2 is a correct reason for it.

Example 105: Statement-1: For every natural number $n \geq 2$,

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}} > \sqrt{n}$$

Statement-2: For every natural number $n \geq 2$,

$\sqrt{n(n+1)} < n + 1$

Ans. (b)

Solution: As $n < n + 1 \; \forall \; n \geq 2$

$\Rightarrow \quad n(n + 1) < (n + 1)^2 \; \forall \; n \geq 2$

$\Rightarrow \quad \sqrt{n(n+1)} < n + 1 \; \forall \; n \geq 2$

Thus, Statement-2 is true.

Also, for $n \geq 2$

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}}$$

$$> \underbrace{\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+\ldots+\frac{1}{\sqrt{n}}}_{n \text{ times}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\ldots+\frac{1}{\sqrt{n}} > \sqrt{n}$$

$\therefore$ Statement-1 is true.

Thus, both the statements are correct but Statement-2 is not a correct explanation of Statement-1.

LEVEL 2

Straight Objective Type Questions

Example 106: If $(1- y)\,(1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5) = 1 - y^6$, $(y \neq 1)$, then a value y/x is

(a) 1/2 (b) 2

(c) 1/4 (d) 4

Ans. (b)

Solution: We can rewrite the given expression as

$$(1 - y)\,\frac{\left[1-(2x)^6\right]}{1-2x} = 1 - y^6,$$

one of the possible values of y is clearly $2x$. Therefore, one of the possible values of y/x is 2.

Example 107: If $\frac{22}{7}$ and π appear as two distinct terms of an A.P., then common difference of the A.P. must be

(a) an integer

(b) a rational number

(c) an irrational number

(d) 0

Ans. (c)

Solution: Let the A.P. be $a_1, a_2, \ldots$ suppose

$$a_m = \frac{22}{7} \text{ and } a_n = \pi$$

If d is the common difference of the A.P., then

$$(m - n)d = a_m - a_n = \frac{22}{7} - \pi$$

$$\Rightarrow \quad d = \frac{1}{m-n}\left(\frac{22}{7} - \pi\right)$$

which is a irrational number as π is an irrational number.

Example 108: Let $a_m = \underbrace{111\ldots1}_{m \text{ times}}$, then which of the following is a prime number?

(a) a_4 (b) a_5

(c) a_{91} (d) a_{100}

Ans. (b)

Solution: Show that a_5 is not divisible by any prime upto 105.

Example 109: Let $I_n = \int\limits_0^{\pi/4} \tan^n x \, dx$. Then $I_2 + I_4$, $I_3 + I_5$, $I_4 + I_6$, $I_5 + I_7$, …are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

Ans. (c)

Solution: We have for $r \geq 2$,

$$I_r + I_{r+2} = \int_0^{\pi/4} \tan^r x\,(1+\tan^2 x)\,dx$$

$$= \int_0^{\pi/4} \tan^r x \sec^2 x \, dx$$

$$= \frac{1}{r+1}\tan^{r+1} x\Big]_0^{\pi/4} = \frac{1}{r+1}$$

Thus, the given sequence becomes,

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

This is clearly an H.P.

Example 110: If a, b, c are in A.P. p, q, r are in H.P. and ap, bq, cr are in G.P., then $\frac{p}{r}+\frac{r}{p}$ is equal to

(a) $\frac{a}{c}+\frac{c}{a}$ (b) $\frac{a}{c}-\frac{c}{a}$

(c) $\frac{b}{q}+\frac{q}{b}$ (d) $\frac{b}{q}-\frac{a}{p}$

Ans. (a)

◎ **Solution:** According to the given condition

$$2b = a + c,\ q = \frac{2pr}{p+r} \text{ and } b^2q^2 = (ap)\,(cr)$$

Substituting values of b and q from first two expressions in the last expression, we get

$$\frac{(a+c)^2}{4}\left(\frac{2pr}{p+r}\right)^2 = (ac)\,(pr)$$

$$\Rightarrow \quad \frac{(a+c)^2}{ac} = \frac{(p+r)^2}{pr} \quad \Rightarrow \quad \frac{a}{c}+\frac{c}{a} = \frac{p}{r}+\frac{r}{p}$$

● **Example 111:** Sum to n terms of the series $\frac{1}{1\cdot2\cdot3\cdot4}+\frac{1}{2\cdot3\cdot4\cdot5}+\frac{1}{3\cdot4\cdot5\cdot6}+\cdots$ is

(a) $\frac{1}{24}-\frac{1}{n}$ (b) $\frac{1}{18}-\frac{1}{3(n+1)(n+2)(n+3)}$

(c) $\frac{1}{2^n}-\frac{1}{n^2}$ (d) $\frac{1}{18}-\frac{1}{3(n+3)(n+4)}$

Ans. (b)

◎ **Solution:** Let t_r denote the rth term of the given series, then

$$t_r = \frac{1}{r(r+1)(r+2)(r+3)} \text{ and}$$

$$t_{r+1} = \frac{1}{(r+1)(r+2)(r+3)(r+4)}$$

$$\Rightarrow \quad rt_r = \frac{1}{(r+1)(r+2)(r+3)} \text{ and}$$

$$(r+4)t_{r+1} = \frac{1}{(r+1)(r+2)(r+3)}$$

Thus, $rt_r - (r+4)t_{r+1} = 0$

$\Rightarrow$ $rt_r - (r+1)t_{r+1} = 3t_{r+1}$

Putting $r = 1, 2, \ldots, n-2, n-1$, we get

$1t_1 - 2t_2 = 3t_2$

$2t_2 - 3t_3 = 3t_3$

$3t_3 - 4t_4 = 3t_4$

$\ldots$

$(n-2)t_{n-2} - (n-1)t_{n-1} = 3t_{n-1}$

$(n-1)t_{n-1} - nt_n = 3t_n$

Adding above expressions, we get

$$t_1 - nt_n = 3(t_2 + t_3 + \ldots + t_n)$$

$$\Rightarrow \quad 4t_1 - nt_n = 3(t_1 + t_2 + \ldots + t_n)$$

$$\Rightarrow \quad \frac{1}{1\cdot2\cdot3}-\frac{1}{(n+1)(n+2)(n+3)} = 3S_n$$

$$\Rightarrow \quad S_n = \frac{1}{3}\left[\frac{1}{6}-\frac{1}{(n+1)(n+2)(n+3)}\right]$$

● **Example 112:** Sum to n terms of the series $\frac{1}{5!}+\frac{1!}{6!}+\frac{2!}{7!}+\frac{3!}{8!}+\ldots$ is

(a) $\frac{2}{5!}-\frac{1}{(n+1)!}$ (b) $\frac{1}{4}\left(\frac{1}{4!}-\frac{n!}{(n+4)!}\right)$

(c) $\frac{1}{4}\left(\frac{1}{3!}-\frac{3!}{(n+2)!}\right)$ (d) none of these

Ans. (b)

◎ **Solution:** We have $t_r = \frac{(r-1)!}{(r+4)!}$ and $t_{r+1} = \frac{r!}{(r+5)!}$

Now, $rt_r - (r+5)t_{r+1} = \frac{r!}{(r+4)!}-\frac{r!}{(r+4)!} = 0$

$\Rightarrow \quad rt_r - (r+1)t_{r+1} = 4t_{r+1}$

$\Rightarrow \quad 4\sum_{r=1}^{n-1} t_{r+1} = \sum_{r=1}^{n-1}[rt_r - (r+1)\,t_{r+1}]$

$\Rightarrow \quad 4(t_2 + t_3 + \ldots + t_n) = 1t_1 - nt_n$

$\Rightarrow \quad 4(t_1 + t_2 + \ldots t_n) = 5t_1 - nt_n$

$$= 5\left(\frac{0!}{5!}\right)-\frac{n(n-1)!}{(n+4)!} = \frac{1}{4!}-\frac{n!}{(n+4)!}$$

$$\Rightarrow t_1 + t_2 + \ldots + t_n = \frac{1}{4}\left[\frac{1}{4!}-\frac{n!}{(n+4)!}\right]$$

● **Example 113:** Sum to n terms of the series $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$ is

(a) $\frac{n}{24}\,(n^2 + 9n + 13)$ (b) $\frac{n}{24}\,(2n^2 + 7n + 15)$

(c) $\frac{n}{24}\,(2n^2 + 9n + 13)$ (d) $\frac{n}{24}\,(n^2 + 11n + 11)$

Ans. (c)

◎ **Solution:** Let t_r denote the rth term of the series, then

$$t_r = \frac{1^3+2^3+\ldots+r^3}{1+3+\ldots+(2r-1)} = \frac{\frac{1}{4}r^2(r+1)^2}{r^2}$$

$$= \frac{1}{4}(r+1)^2$$

$$\Rightarrow \quad \sum_{r=1}^{n} t_r = \frac{1}{4}\sum_{r=1}^{n}(r+1)^2 = \frac{1}{4}\left[\sum_{r=1}^{n+1} r^2 - 1\right]$$

$$= \frac{1}{4}\left[\frac{(n+1)\,(n+2)\,(2n+3)}{6} - 1\right]$$

$$= \frac{1}{24}\ [2n^3 + 9n^2 + 13n + 6 - 6]$$

$$= \frac{n}{24}(2n^2 + 9n + 13)$$

◉ **Example 114:** Sum of the series $\sum_{r=1}^{n} \frac{r}{(r+1)!}$ is

(a) $1 - \frac{1}{n!}$ (b) $1 - \frac{1}{(n+1)!}$

(c) $2 - \frac{1}{(n+1)!}$ (d) none of these

Ans. (b)

◎ **Solution:** Let t_r denote the rth term of the series, then

$$t_r = \frac{r}{(r+1)!} = \frac{r+1-1}{(r+1)!} = \frac{1}{r!} - \frac{1}{(r+1)!}$$

$$\Rightarrow \quad \sum_{r=1}^{n} t_r = \sum_{r=1}^{n}\left(\frac{1}{r!} - \frac{1}{(r+1)!}\right) = 1 - \frac{1}{(n+1)!}$$

◉ **Example 115:** Sum to n terms of the series $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \frac{7}{4.5.6} + \ldots$ is

(a) $\frac{n(n+1)}{2(n+2)(n+3)}$ (b) $\frac{n(3n+1)}{4(n+1)(n+2)}$

(c) $\frac{1}{6} - \frac{5}{(n+1)(n+4)}$ (d) none of these

Ans. (b)

◎ **Solution:** We have $t_r = \frac{2r-1}{r(r+1)(r+2)}$ and

$$t_{r+1} = \frac{2r+1}{(r+1)(r+2)(r+3)}$$

$$\Rightarrow \quad rt_r = \frac{2r-1}{(r+1)(r+2)},\ (r+3)t_{r+1} = \frac{2r+1}{(r+1)(r+2)}$$

$$\Rightarrow \quad rt_r - (r+3)t_{r+1} = \frac{-2}{(r+1)(r+2)}$$

$$\Rightarrow \quad rt_r - (r+1)t_{r+1} + 2\left(\frac{1}{r+1} - \frac{1}{r+2}\right) = 2t_{r+1}$$

$$\Rightarrow \quad \sum_{r=1}^{n-1}[rt_r - (r+1)t_{r+1}] + 2\sum_{r=1}^{n-1}\left(\frac{1}{r+1} - \frac{1}{r+2}\right) = 2\sum_{r=1}^{n-1} t_{r+1}$$

$$\Rightarrow \quad t_1 - nt_n + 1 - \frac{2}{n+1} = 2(t_2 + t_3 + \ldots + t_n)$$

$$\Rightarrow \quad 3t_1 - nt_n + 1 - \frac{2}{n+1} = 2\sum_{r=1}^{n} t_r$$

$$\frac{1}{2} - \frac{n(2n-1)}{n(n+1)(n+2)} + \frac{n-1}{n+1} = 2\sum_{r=1}^{n} t_r$$

$$\therefore \quad 2\sum_{r=1}^{n} t_r = \frac{(n+1)(n+2) - 2(2n-1) + 2(n-1)(n+2)}{2\,(n+1)(n+2)}$$

$$\Rightarrow \quad \sum_{r=1}^{n} t_r = \frac{n(3n+1)}{4(n+1)(n+2)}$$

◉ **Example116:** If $\frac{48}{(2)(3)} + \frac{47}{(3)(4)} + \frac{46}{(4)(5)} + \ldots + \frac{2}{(48)(49)} + \frac{1}{(49)(50)}$

$$= \frac{51}{k} + k\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{50}\right)$$

then k is equal to

(a) 2 (b) – 1

(c) – 1/2 (d) 1

Ans. (b)

◎ **Solution:** Let $t_r = \frac{50-r}{r(r+1)}$, $r \geq 2$.

We have $\sum_{r=2}^{49} \frac{50-r}{r(r+1)}$

$$= 50\sum_{r=2}^{49}\left(\frac{1}{r} - \frac{1}{r+1}\right) - \sum_{r=2}^{49} \frac{1}{r+1}$$

$$= 50\left(\frac{1}{2} - \frac{1}{50}\right) - \sum_{r=2}^{49} \frac{1}{r+1}$$

$$= 24 - \sum_{r=1}^{50} \frac{1}{r} + \left(1 + \frac{1}{2}\right)$$

$$= \frac{51}{2} - \sum_{r=1}^{50} \frac{1}{r}$$

$$\therefore \quad k = -1$$

EXERCISES

Concept-based Straight Objective Type Questions

1. Let $a_n = \left[\log\left(\frac{7}{5}\right)\right]^n$ and $b_n = \log_5 (a_n)\ \forall\ n \in \mathbf{N}$. Then $b_1, b_2, b_3, \ldots$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

2. Suppose a, b, c are positive and form an increasing G.P. If $a + b + c = xb$, then

(a) $x > 3$ (b) $x > 4$
(c) $x > \sqrt{5}$ (d) $x > \sqrt{3} + \sqrt{5}$

3. If for each $n \in \mathbf{N}$, $S_n = nA + \frac{1}{2}n(n + 1)B$ is sum of the first n terms of an A.P., the common difference of the A.P. is

(a) $A + B$ (b) $A + 2B$
(c) B (d) $B/2$

4. An A.P. consists of $2n$ terms. If two middle terms are $\alpha - \beta$ and $\alpha + \beta$, then sum of the $2n$ terms of the A.P. is

(a) $2n\alpha$ (b) $2n\beta$
(c) $2n(\alpha + \beta)$ (d) $2n(\alpha - \beta)$

5. If nth terms of the arithmetic progressions
3, 10, 17, 24, . . .
and 63, 65, 67, 69, . . .
are equal, then n is equal to

(a) 71 (b) 37
(c) 11 (d) 13

6. Suppose $a, c \in \mathbf{Q}$ and $b \in \mathbf{R} - \mathbf{Q}$, then which one of the following can be true?

(a) $a, b + 2, c$ are in A.P. (b) $a, b + 1, c$ are in G.P.
(c) $a, b - 3, c$ are in H.P. (d) $a, b - 2, c$ are in A.P.

7. Suppose $a, b, c > 0$, $x, y, z > 1$ and a, b, c are in G.P. If $a^{\log x} = b^{\log y} = c^{\log z}$, then

(a) x, y, z are in A.P. (b) x, y, z are in G.P.
(c) x, y, z are in H.P. (d) none of these

8. The ratio of the sum of first three terms of a G.P. to the sum of first 6 terms of the G.P. is 125 : 152, then common ratio of the G.P. is

(a) $\frac{2}{3}$ (b) $\frac{3}{4}$
(c) $\frac{3}{5}$ (d) $\frac{2}{5}$

9. There are four numbers of which the first three are in G.P. and the last three are in A.P. with common difference 9. If the first and last number are equal, the sum of the four numbers is

(a) 18 (b) 21
(c) 24 (d) 27

10. Suppose A.M. of two positive numbers be 7 and G.M. between them be 5. The A.M. between their squares is

(a) 57 (b) 73
(c) 78 (d) 37

11. Sum of an infinite G.P. is 2 and sum of their cubes is 24, then 5th term of the G.P. is

(a) 3/16 (b) 3/8
(c) – 3/8 (d) – 3/16

12. Suppose a, b are two positive numbers. The product of three arithmetic means inserted between a, b is 15/2 and product of three arithmetic means inserted between $1/a$ and $1/b$ is 5/18. Then ab is equal to:

(a) 9 (b) 6
(c) 3 (d) 1

13. If $\frac{a+b}{1-ab}, b, \frac{b+c}{1-bc}$ are in A.P., then $\frac{1}{a}, b, \frac{1}{c}$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

14. The sum of the infinite series

$$\frac{5}{17} + \frac{55}{(17)^2} + \frac{555}{(17)^3} + \cdots$$

is

(a) $\frac{1}{17}$ (b) $\frac{85}{112}$
(c) $\frac{85}{324}$ (d) $\frac{19}{289}$

15. Sum of the series

$$\sum_{r=1}^{2016} (-1)^r (a + rd)$$

is

(a) $a + 2016d$ (b) $2015d$
(c) 0 (d) $1008d$

LEVEL 1

Straight Objective Type Questions

16. If $5a - b$, $2a + b$, $a + 2b$ are in A.P. and $(a - 1)^2$, $(ab + 1)$, $(b + 1)^2$ are in G.P., $a \neq 0$, then a is equal to

(a) 2, – 1/4 (b) –2, – 1/4
(c) – 2, 1/4 (d) 2, 1/4

17. If $a + c \neq b$ and $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$, then a, b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

18. In an A.P., the first term is 1 and sum of the first p terms is 0, then sum of the first $(p + q)$ terms is

(a) $\frac{q(p+q)}{1-p}$ (b) $\frac{q}{1-p}$
(c) $\frac{(p+q)p}{1-p}$ (d) $\frac{2(p+q)}{1-p}$

19. If both the A.M. between m and n and G.M. between two distinct positive numbers a and b are equal to $\frac{ma+nb}{m+n}$, then n is equal to

(a) $\frac{a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ (b) $\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$
(c) $\frac{\sqrt{ab}}{\sqrt{a}+\sqrt{b}}$ (d) $\frac{1}{\sqrt{a}+\sqrt{b}}$

20. Product of two positive integers a and b is 192. Let $g = hcf\,(a, b)$ and $l = lcm\,(a, b)$. If the ratio of A.M. between g and l to the H.M. between g and l is $\frac{169}{48}$, then smaller of a, b is

(a) 8, 24 (b) 12, 16
(c) 4, 12 (d) 6, 24

21. The least positive integer n such that $1 - \frac{2}{3} - \frac{2}{3^2} - \dots - \frac{2}{3^{n-1}} < \frac{1}{100}$ is

(a) 4 (b) 5
(c) 6 (d) 7

22. Let the positive numbers a, b, c, d be in A.P. Then abc, abd, acd, bcd are

(a) Not in A.P./G.P./H.P. (b) in A.P.
(c) in G.P. (d) in H.P.

23. The harmonic mean of the roots of the equation $(5 + \sqrt{2})x^2 - (4 + \sqrt{2})x + 8 + 2\sqrt{2} = 0$ is

(a) 2 (b) 4
(c) 6 (d) 8

24. If $\cos (x - y)$, $\cos x$ and $\cos (x + y)$ are in H.P. then value of $\cos x \sec (y/2)$ is

(a) $\pm\sqrt{2}$ (b) $\pm\sqrt{3}$
(c) ± 2 (d) ± 1

25. For a positive integer n, let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{(2^n) - 1}$. Then

(a) $a(200) > 100$ (b) $a(100) > 100$
(c) $a(50) < 25$ (d) none of these

26. Let $S = \frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + \dots$ upto ∞. Then S is equal to

(a) 38/81 (b) 4/19
(c) 36/171 (d) none of these

27. If $a, b, c, \dots$ are in G.P., and $a^{1/x} = b^{1/y} = c^{1/z} = \dots$, then $x, y, z, \dots$ are in

(a) H.P. (b) G.P.
(c) A.P. (d) none of these

28. If $a^x = b^y = c^z = d^u$ and a, b, c, d are in G.P., then x, y, z, u are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

29. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0$, then a, b, c, d are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

30. If a, b, c are in A.P., then $2^{ax+1}, 2^{bx+1}, 2^{cx+1}$, $x \in \mathbf{R}$, are in

(a) A.P. (b) G.P. and when $x > 0$
(c) G.P. only when $x < 0$ (d) G.P. for all x

31. The rational number which is equal to the number $2.\overline{357}$ with recurring decimal is

(a) $\frac{2355}{1001}$ (b) $\frac{2370}{999}$
(c) $\frac{2355}{999}$ (d) $\frac{2359}{991}$

32. The sum to infinity of the series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\dots$ is
    (a) $\frac{16}{35}$ (b) $\frac{11}{8}$
    (c) $\frac{35}{16}$ (d) $\frac{17}{6}$

33. If the sum to n terms of an A.P. is $3n^2 + 5n$, while $T_m = 164$, then value of m is
    (a) 25 (b) 26
    (c) 27 (d) 28

34. If G_1 and G_2 are two geometric means and A is the arithmetic mean inserted between two positive numbers then the value of $\frac{G_1^2}{G_2}+\frac{G_2^2}{G_1}$ is
    (a) $A/2$ (b) A
    (c) $2A$ (d) none of these

35. If A_1, A_2 be two arithmetic means and G_1, G_2 be two geometric means between two positive numbers a and b, then $\frac{A_1+A_2}{G_1G_2}$ is equal to
    (a) $\frac{ab}{a+b}$ (b) $\frac{2ab}{a+b}$
    (c) $\frac{a+b}{2ab}$ (d) $\frac{a+b}{ab}$

36. The first and last terms of an A.P. are a and l respectively. If s is the sum of all the terms of the A.P., then the common difference of the A.P. is
    (a) $\frac{l^2-a^2}{s-(l+a)}$ (b) $\frac{l^2-a^2}{2s-(l+a)}$
    (c) $\frac{l^2+a^2}{2s-(l+a)}$ (d) $\frac{l^2+a^2}{s-(l-a)}$

37. The value of $x = 2^{1/4}\, 4^{1/8}\, 8^{1/16}\, 16^{1/32} \dots$ is
    (a) 2 (b) 3/2
    (c) 1 (d) 1/2

38. The sum of an infinite number of terms of a G.P. is 20, and the sum of their squares is 100, then the first term of the G.P. is
    (a) 5 (b) 8/5
    (c) 3/5 (d) 8

39. The eighth term of a geometric progression is 128 and common ratio is 2. The product of the first five terms is
    (a) 4^6 (b) 4^5
    (c) 4^3 (d) 4^8

40. If l, m, n are positive and are respectively the pth, qth and rth terms of a G.P., then
    $$\Delta = \begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} \text{ is equal to}$$
    (a) pqr (b) $p+q+r$
    (c) $p+q+r+pqr$ (d) 0

41. Let a_1, a_2, a_3 … be terms of an A.P. If $\frac{a_1+a_2+\dots+a_p}{a_1+a_2+\dots+a_q} = \frac{p^2}{q^2}$, $p \neq q$ then $\frac{a_6}{a_{21}}$ equals
    (a) $\frac{11}{41}$ (b) $\frac{41}{11}$
    (c) $\frac{7}{2}$ (d) $\frac{2}{7}$

42. The sum to 10 terms of the series $\sqrt{2}+\sqrt{6}+\sqrt{18}+\sqrt{54}+\dots$ is
    (a) $121\left(\sqrt{6}+\sqrt{2}\right)$ (b) $\frac{121}{2}\left(\sqrt{3}+1\right)$
    (c) $243\left(\sqrt{3}+1\right)$ (d) $243\left(\sqrt{3}-1\right)$

43. If $1^2 + 2^2 + \dots + n^2 = 1015$, then value of n is
    (a) 15 (b) 14
    (c) 13 (d) 16

44. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$ upto 22nd terms is
    (a) 23276 (b) 22736
    (c) 22738 (d) 227152

45. Suppose a, b, c are distinct real numbers. If a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then
    (a) $-\frac{a}{2}$, b, c are in G.P.
    (b) $a + b = c$
    (c) $a = b + c$
    (d) a, b, c are in G.P.

46. If mth term of an A.P. is n and nth term in m, then its rth term is
    (a) $m + n - r$ (b) $m + n + r$
    (c) $m - n + r$ (d) $r - (m + n)$

47. If a, b, c are positive real numbers that are in G.P., then the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root if a/d, b/e, c/f are in
    (a) A.P. (b) G.P.
    (c) H.P. (d) none of these

48. If $\log_{10}2$, $\log_{10}(2^x + 1)$ and $\log_{10}(2^x + 3)$ are in A.P., then
(a) $x = 0$ (b) $x = 1$
(c) $x = \frac{1}{2}\log_{10}2$ (d) $x = \frac{1}{2}\log_{2}5$

49. Suppose $\frac{1}{2}x$, $|x + 1|$, $|x - 1|$ are in A.P., then sum to 10 terms of the A.P. is
(a) 54 (b) 36
(c) 28 (d) none of these

50. The harmonic mean and geometric mean of two positive numbers are in the ratio 4 : 5, then two numbers are in the ratio
(a) 4 : 1 (b) 3 : 1
(c) 2 : 1 (d) 1 : 3

51. The sum to 16 terms of the series
$$1^3 + \frac{1+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \frac{1^3+2^3+3^3+4^3}{1+3+5+7} + \ldots \text{ is}$$
(a) 445 (b) 446
(c) 447 (d) 448

52. If $\log(a + c) + \log(a + c - 2b) = 2\log(a - c)$, then
(a) a, b, c are in A.P.
(b) a, b, c are in G.P.
(c) a, b, c are in H.P.
(d) $b + c$, $c + a$, $a + b$ are in H.P.

53. The sum of integers from 1 to 100 that are divisible by 2 or 5 is
(a) 3050 (b) 3150
(c) 3250 (d) 3350

54. The common difference d of the A.P. in which $T_7 = 9$ and $T_1T_2T_7$ is least is
(a) 33/2 (b) 5/4
(c) 33/20 (d) 7/3

55. If $3^{2\sin 2\theta - 1}$, 14, $3^{4-2\sin 2\theta}$ are first three terms of an A.P., then its fifth term is given by
(a) 25 (b) 40
(c) 53 (d) – 12

56. The sum to n terms of the series $\frac{3}{2} + \frac{5}{4} + \frac{9}{8} + \frac{17}{16} + \ldots$ is equal to
(a) $n + 1 - 2^{-n}$ (b) $1 + 2^{-n}$
(c) $2^n - 1$ (d) $2^n - n + 1$

57. The sum of first n terms of the series
$1^2 + 2(2^2) + 3^2 + 2(4^2) + 5^2 + 2(6^2) + \ldots$ is $n(n + 1)^2/2$ when n is even. When n is odd, the sum is
(a) $n(n - 1)^2/2$ (b) $n^2(n + 1)/2$
(c) $n(n + 2)/3$ (d) $n(n + 3)/2$

58. Sum to n terms of the series $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \ldots$ is
(a) $\tan^{-1}\left(\frac{n}{n+2}\right)$ (b) $\tan^{-1}\left(\frac{2n-1}{2n+1}\right)$
(c) $\tan^{-1}\left(\frac{1}{3n}\right)$ (d) $\tan^{-1}\left(\frac{1}{2n}\right)$

59. Sum to 30 terms of the series $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots$ is
(a) $\frac{116}{465}$ (b) $\frac{495}{1984}$
(c) $\frac{435}{791}$ (d) $\frac{485}{791}$

60. If $n!$, $3(n!)$ and $(n + 1)!$ are in G.P., then $n!$, $5(n!)$ and $(n + 1)!$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) none of these

61. If $S_n = 81 + 54 + 36 + 24 + \ldots$ upto n terms, then value of
$$x = \frac{S_n - 4S_{n-1} + 6S_{n-2} - 4S_{n-3} + S_{n-4}}{S_{n-1} - 4S_{n-2} + 6S_{n-3} - 4S_{n-4} + S_{n-5}} \text{ is equal}$$
(a) $\frac{2}{3}$ (b) $\frac{3}{2}$
(c) $\frac{1}{2}$ (d) 2

62. The sum to n terms of the series $1^2 + (1)(2) + 3^2 + (3)(4) + 5^2 + (5)(6) + 7^2 + \ldots$ upto n terms when n is odd is
(a) $\frac{1}{12}(n + 1)(4n^2 - n + 3)$
(b) $\frac{1}{12}n(4n^2 + 3n - 4)$
(c) $\frac{1}{6}(n + 1)(4n^2 - n + 5)$
(d) none of these

63. If $L = \lim\limits_{n\to\infty} (1 + 3^{-1})(1 + 3^{-2})(1 + 3^{-4})(1 + 3^{-8}) \ldots (1 + 3^{-2^n})$, then
(a) $L = \frac{2}{3}$ (b) $L = \frac{3}{2}$
(c) $L = 1$ (d) $L = 2$

64. Suppose $a_1, a_2, \ldots, a_n$ are n distinct odd natural numbers not divisible by any prime greater than 5, then value of the expression $\frac{1}{a_1}+\frac{1}{a_2}+\ldots\frac{1}{a_n}$ cannot exceed
    (a) 2 (b) 1
    (c) 1/2 (d) none of these

65. Coefficient of x^{99} in the expansion of $(x-1)(x-3)(x-5)\ldots(x-199)$ is
    (a) – 100 (b) – 1000
    (c) –10000 (d) – 100000

66. Sum to n terms of the series $1+(1+2)+(1+2+3)+\ldots$ is
    (a) $\frac{1}{2}n(n+1)(n+2)$ (b) $\frac{1}{3}n(n+1)(n+2)$
    (c) $\frac{1}{4}n(n+1)(2n+1)$ (d) $\frac{1}{6}n(n+1)(n+2)$

67. If $x=\sum_{n=0}^{\infty}a^n$, $y=\sum_{n=0}^{\infty}b^n$, $0<a<b<1$, and $z=\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^n$, then
    (a) $x+yz=x(y+z)$ (b) $xyz=x+y+z$
    (c) $xy+z=y(x+z)$ (d) none of these

68. If the sum of an infinite decreasing G.P. is 3 and sum of the cubes of its terms is 108/13, then common ratio is given by
    (a) $\frac{1}{5}$ (b) $\frac{1}{4}$
    (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

69. The sum of the series $S=\sum_{r=1}^{n}\log\left(\frac{a^{r+1}}{b^{r-1}}\right)$ is
    (a) $\frac{n}{2}\log\left(\frac{a^{n-1}}{b^n}\right)$ (b) $\frac{n}{2}\log\left(\frac{a^{n+3}}{b^{n-1}}\right)$
    (c) $\frac{n}{2}\log\left(\frac{a^{n+2}}{b^{n-1}}\right)$ (d) none of these

70. Fifth term of a G.P. is 2, then the product of its 9 terms is
    (a) 256 (b) 512
    (c) 1024 (d) 204

71. If s is the sum to infinity of a G.P., whose first term is a, then the sum of the first n terms of the G.P. is
    (a) $s\left[1-\left(1-\frac{a}{s}\right)^n\right]$ (b) $s\left(1-\frac{a}{s}\right)^n$
    (c) $a\left[1-\left(1-\frac{a}{s}\right)^n\right]$ (d) $\frac{a}{s}\left[1-\left(1-\frac{a}{s}\right)^n\right]$

72. For a sequence $<a_n>$, $a_1=5$, and $\frac{a_{r+1}}{a_r}=\frac{1}{2}\forall\ r$, then $\sum_{n=1}^{\infty}a_{2n-1}$ is
    (a) 20/7 (b) 20/5
    (c) 20/3 (d) 20

73. If H.M. and G.M. of two positive numbers are in the ratio 12 : 13, then the numbers are in the ratio
    (a) 1 : 2 (b) 2 : 3
    (c) 3 : 5 (d) 4 : 9

74. The H.M. between two numbers is 16/5, their A.M. is A and G.M. is G. If $2A+G^2=26$ then the numbers are
    (a) 6, 8 (b) 4, 8
    (c) 2, 8 (d) 1, 8

75. If $a_r=a+(r-1)d$ is rth term of an A.P., then $a_n^2-2a_{n-1}^2+a_{n-2}^2$ is independent of
    (a) n (b) d
    (c) a (d) a and n both

76. If $A=1+r^a+r^{2a}+r^{3a}+\ldots$ upto ∞ and $B=1+r^b+r^{2b}+r^{3b}+\ldots$upto ∞, then $\frac{a}{b}$ is equal to
    (a) $\frac{\log(1+A)}{\log(1+B)}$ (b) $\frac{\log(A-1)}{\log(B-1)}$
    (c) $\log_B A$ (d) none of these

77. Suppose $a_1=45$, $a_2=41$ and $a_k=2a_{k-1}-a_{k-2}\ \forall\ k\geq 3$, then, value of
    $$S=\frac{1}{12}\left[a_1^2-a_2^2+a_3^2-a_4^2+\ldots+a_{11}^2-a_{12}^2\right]$$ is
    (a) 89 (b) 90
    (c) 91 (d) 92

78. Suppose $P(x)=1-x+x^2-x^3+\ldots+x^{2012}$ is expressed as a polynomial in y, as $Q(y)=a_0+a_1y+\ldots+a_{2012}y^{2012}$ where $y=x-2$, then $\sum_{i=0}^{2012}a_i$ equals
    (a) $\frac{1}{4}(3^{2013}+1)$ (b) $\frac{1}{3}(3^{2013}-1)$
    (c) 0 (d) 1

79. The sum of first 26 terms of an A.P. $a_1, a_2, a_3, \ldots$. if $a_2 + a_6 + a_9 + a_{18} + a_{21} + a_{25} = 165$, is
   (a) 705 (b) 715
   (c) 725 (d) 735

80. If the sum of first n terms of an A.P. is cn^2, then the sum of squares of these n terms is
   (a) $\dfrac{n(4n^2-1)c^2}{6}$ (b) $\dfrac{n(4n^2+1)c^2}{3}$
   (c) $\dfrac{n(4n^2-1)c^2}{3}$ (d) $\dfrac{n(4n^2+1)c^2}{6}$

Assertion-Reason Type Questions

81. **Statement-1 :** If $S_n = \sum_{k=1}^{n} a_k = 3n^2 + 2n - 7$ for each n, then, $a_1, a_2, a_3, \ldots$ are in A.P.

   Statement-2 : Sum to n terms of an A.P. is always of the form $an^2 + bn$.

82. **Statement-1 :** The largest interval for which we can find the sum of the series

   $1 + (2x - 1) + (2x - 1)^2 + (2x - 1)^3 + \ldots$ is (0, 1).

   Statement-2 : An infinite geometric series

   $$a + ar + ar^2 + \ldots$$

   can be summed up if $|r| < 1$.

83. **Statement-1 :** If $a, b, c > 0$ and the expression

   $(a + c)x^2 + \left(\sqrt{(a+c)^2 + 4b^2}\right) x + b > 0$ $\forall\ x \in \mathbf{R}$, then a, b, c are in G.P.

   Statement-2 : $ax^2 + bx + c > 0\ \forall\ x \in \mathbf{R}$ if and only if $a > 0$ and $b^2 - 4ac < 0$.

84. Let α, β be roots of $ax^2 + 2bx + c = 0$ and $\Delta = b^2 - ac$.

   Statement-1 : If $\alpha + \beta, \alpha^2 + \beta^2, \alpha^3 + \beta^3$ are in G.P. then $c\Delta = 0$

   Statement-2 : If $\alpha, \alpha\beta, \beta$ are in A.P. then $b + c = 0$

85. Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P.

   Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.
   Statement-1: The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.
   Statement-2: The numbers b_1, b_2, b_3, b_4 are in H.P.

LEVEL 2

Straight Objective Type Questions

86. The nth term of the sequence $2\frac{1}{2}, 1\frac{7}{13}, 1\frac{1}{9}, \frac{20}{23} \ldots$ is
   (a) $\dfrac{15}{5n+3}$ (b) $\dfrac{20}{5n+4}$
   (c) $\dfrac{20}{5n+3}$ (d) $\dfrac{20}{5n+2}$

87. H.M. between $\dfrac{1}{28}$ and $\dfrac{1}{10}$ is
   (a) $\dfrac{1}{19}$ (b) $\dfrac{1}{18}$
   (c) $\dfrac{1}{17}$ (d) $\dfrac{1}{16}$

88. If $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in A.P., then
   (a) a, b, c are in A.P. (b) a, b, c are in G.P.
   (c) a, b, c are in H.P. (d) none of these

89. If $a_1, a_2, \ldots, a_n$ are in A.P., with common difference $d \neq 0$, then sum of the series $\sin d\,[\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n]$ is
   (a) $\sec a_1 - \sec a_n$ (b) $\operatorname{cosec} a_1 - \operatorname{cosec} a_n$
   (c) $\cot a_1 - \cot a_n$ (d) $\tan a_1 - \tan a_n$

90. If $H_n = 1+\dfrac{1}{2}+\dfrac{1}{3}+\ldots+\dfrac{1}{n}$, then value of

   $1+\dfrac{3}{2}+\dfrac{5}{3}+\ldots+\dfrac{2n-1}{n}$ is

(a) $H_n + n$ (b) $2n - H_n$
(c) $n - 1 + H_n$ (d) $H_n + 2n$

91. If a, b, c are in H.P., then $\dfrac{a}{b+c-a}, \dfrac{b}{c+a-b}, \dfrac{c}{a+b-c}$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) A.G.P.

92. Sum of all the terms in the nth row of the triangle

1
2 3
4 5 6
7 8 9 10
. . . .
. . . .

is
(a) $\dfrac{1}{2}n^2(n+1)$ (b) $\dfrac{1}{2}n(n^2+1)$
(c) $n(n^2+1)$ (d) $n^2(n+1)$

93. Let a and b be two positive real numbers. Suppose A_1, A_2 are two arithmetic means; G_1, G_2 are two geometric means and H_1, H_2 are two harmonic means between a and b, then $\dfrac{A_1 + A_2}{H_1 + H_2}$ is equal to
(a) $\dfrac{2(a^2+b^2)+5ab}{9\,ab}$ (b) $\dfrac{a^2+b^2}{9\,ab}+5$
(c) $\dfrac{a^2+b^2+5(a+b)}{9\,ab}$ (d) $\dfrac{a^2+b^2+7(a+b)}{3\,(a+b)\,ab}$

94. Let $a\,(n) = 1 - \dfrac{1}{2} + \dfrac{1}{3} + \ldots + (-1)^{n-1}\dfrac{1}{n}$, then $\dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{2n}$ is equal to
(a) $a\,(2n)$ (b) $a\,(2n) - a\,(n)$
(c) $a\,(3n) - a\,(2n)$ (d) none of these

95. In Fig. 8.2 each square is filled with the arithmetic mean of the numbers in the boxes sharing a side with it. If the first square is filled with 5, then value of x is
(a) 5 (b) 0
(c) – 5 (d) cannot be determined.

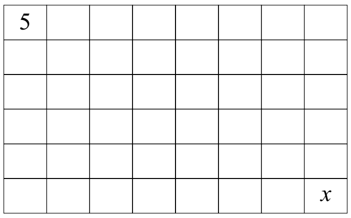


Fig. 8.2

96. Let a, b and c be three real numbers satisfying

$$[a\quad b\quad c]\begin{bmatrix}1 & 9 & 5\\ 8 & 2 & 5\\ 7 & 3 & 5\end{bmatrix} = [0\quad 0\quad 0] \qquad (1)$$

and let $b = 6$. If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)^n$ is
(a) 6 (b) 7
(c) 6/7 (d) ∞

Previous Years' AIEEE/JEE Main Questions

1. Fifth term of a G.P. is 2, then the product of its 9 terms is
(a) 256 (b) 512
(c) 1024 (d) 2048 **[2002]**

2. The value of $2^{1/4}\,.\,4^{1/8}\,.\,8^{1/16}\ldots$ upto ∞ is
(a) 1 (b) 2
(c) $\dfrac{3}{2}$ (d) 4 **[2002]**

3. If 1, $\log_9(3^{1-x}+2)$, $\log_3(4(3^x)-1)$ are in A.P., then x equals
(a) $\log_3 4$ (b) $1 - \log_3 4$
(c) $1 - \log_4 3$ (d) $\log_4 3$ **[2002]**

4. $1^3 - 2^3 + 3^3 - 4^3 + \ldots + 9^3$ equals
(a) 425 (b) – 425
(c) 475 (d) – 475 **[2002]**

5. The sum of integers from 1 to 100 that are divisible by 2 or 5 is
(a) 3000 (b) 3050
(c) 3600 (d) 3250 **[2002]**

6. Sum of an infinite G.P. is 20 and sum of their squares is 100. The common ratio of the G.P. is
(a) $\dfrac{1}{5}$ (b) $\dfrac{2}{5}$
(c) $\dfrac{3}{5}$ (d) $\dfrac{4}{5}$ **[2002]**

7. Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in A.P., then $f'(a), f'(b)$, and $f'(c)$ are in

   (a) G.P. (b) H.P.
   (c) A.G.P. (d) A.P. **[2003]**

8. Let T_r be the rth term of an A.P. whose first term is a and common difference is d. If for some positive integers $m, n, m \neq n$, $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then $a - d$ equals

   (a) $\frac{1}{mn}$ (b) 1
   (c) 0 (d) $\frac{1}{m} + \frac{1}{n}$ **[2004]**

9. The sum of the first n terms of the series

   $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \dots$ is $\frac{1}{2}n(n + 1)^2$

   when n is even. When n is odd the sum is

   (a) $\frac{1}{4}n(n + 1)^2$ (b) $\frac{1}{2}n^2(n + 1)$
   (c) $\frac{3}{2}n(n + 1)$ (d) $\frac{1}{4}n^2(n + 1)^2$ **[2004]**

10. If a, b, c are in A.P. are $|a| < 1$, $|b| < 1$, $|c| < 1$, and $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$, then x, y, z are in

    (a) A.P. (b) G.P.
    (c) H.P. (d) A.G.P. **[2005]**

11. If the expression in powers of x of the function $\frac{1}{(1-ax)(1-bx)}$ is $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$, then a_n is

    (a) $\frac{b^{n+1} - a^{n+1}}{b - a}$ (b) $\frac{b^n - a^n}{b - a}$
    (c) $\frac{a^n - b^n}{b - a}$ (d) $\frac{a^{n+1} - b^{n+1}}{b - a}$ **[2006]**

12. Let $a_1, a_2, a_3, \dots$ be terms of an A.P. if

    $$\frac{a_1 + a_2 + \dots + a_p}{a_1 + a_2 + \dots + a_q} = \frac{p^2}{q^2},\ p \neq q,$$

    then $\frac{a_6}{a_{21}}$ equals

    (a) $\frac{11}{41}$ (b) $\frac{41}{11}$
    (c) $\frac{7}{2}$ (d) $\frac{2}{7}$ **[2006]**

13. If $a_1, a_2, \dots a_n$ are in H.P., then the expression $a_1a_2 + a_2a_3 + \dots + a_{n-1}a_n$ is equal to

    (a) $(n - 1)\,a_1 a_n$ (b) $n(a_1 - a_n)$
    (c) $(n - 1)(a_1 - a_n)$ (d) $n\,a_1 a_n$ **[2006]**

14. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then common ratio of the G.P. is

    (a) $\frac{1}{2}(1 - \sqrt{5})$ (b) $\frac{1}{2}\sqrt{5}$
    (c) $\sqrt{5}$ (d) $\frac{1}{2}(\sqrt{5} + 1)$ **[2007]**

15. **Statement-1:** For every natural number $n \geq 2$,

    $$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots \frac{1}{\sqrt{n}} > \sqrt{n}$$

    Statement-2: For every natural number $n \geq 2$, $\sqrt{n(n+1)} < n + 1$. **[2008]**

16. The sum to infinity of the series $S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \dots$, is

    (a) 4 (b) 6
    (c) 2 (d) 3 **[2009]**

17. A person is to count 4500 currency notes. Let a_n denote the number of notes he counts in the n^{th} minute. If $a_1 = a_2 = \dots a_{10} = 150$ and $a_{10}, a_{11}, \dots$ are in A.P. with common difference -2, then the time taken by him to count all the notes is

    (a) 125 minutes (b) 135 minutes
    (c) 24 minutes (d) 34 minutes **[2010]**

18. A man saves ₹ 200 in each of the first three months of his service. In each of the subsequent months his saving is ₹ 40 more than the saving of the immediately previous month. His total saving from the start of the service will be ₹ 11040 after.

    (a) 21 months (b) 18 months
    (c) 19 months (d) 20 months **[2011]**

19. Let a_n be the nth term of a_n A.P.

    If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, then

    common difference of the A.P. is

    (a) $\alpha - \beta$ (b) $\frac{1}{100}(\alpha - \beta)$
    (c) $\beta - \alpha$ (d) $\frac{1}{200}(\alpha - \beta)$ **[2011]**

20. If 100 times the 100th term of an A.P. with non zero common difference equals the 50 times its 50th term, then the 150 term of this A.P. is

(a) 150 times its 50^{th} term
(b) 150
(c) zero
(d) –150 **[2012]**

21. **Statement-1 :** The sum of the series 1 + (1+2 +4) + (4 + 6 + 9) + (9 + 12 + 16) + ... + (361 + 380 + 400) is 800.

Statement-2 : $\sum_{k=1}^{n}\left(k^3-(k-1)^3\right)=n^3$, for any natural number n **[2012]**

22. If x, y, z are in A.P. and $\tan^{-1} x$, $\tan^{-1} y$ and $\tan^{-1} z$ are also in A.P., then

(a) $2x = 3y = 6z$ (b) $6x = 3y = 2z$
(c) $6x = 4y = 3z$ (d) $x = y = z$ **[2013]**

23. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, . . . is

(a) $\frac{7}{9}(99 - 10^{-20})$ (b) $\frac{7}{81}(179 + 10^{-20})$

(c) $\frac{7}{9}(99 + 10^{-20})$ (d) $\frac{7}{81}(179 - 10^{-20})$ **[2013]**

24. Given sum of the first n terms of an A.P. is $2n + 3n^2$. Another A.P. is formed with the same first term and is double the common difference, the sum of n terms of the new A.P. is

(a) $n + 4n^2$ (b) $6n^2 - n$
(c) $n^2 + 4n$ (d) $3n + 2n^2$ **[2013, online]**

25. The sum

$$\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\ldots \text{ upto 11 terms}$$

is

(a) $\frac{7}{2}$ (b) $\frac{11}{4}$

(c) $\frac{11}{2}$ (d) $\frac{60}{11}$ **[2013, online]**

26. Given a sequence of 4 numbers, first three of which are in G.P., and the last three are in A.P. with common difference six. If first and last term of the sequence are equal, then the last term is

(a) 16 (b) 8
(c) 4 (d) 2 **[2013, online]**

27. The value of

$1^2 + 3^2 + 5^2 + \ldots + 25^2$

is

(a) 2925 (b) 1469
(c) 1728 (d) 1456 **[2013, online]**

28. Let $a_1, a_2, a_3, \ldots$ be an A.P. such that

$$\frac{a_1+a_2+\ldots+a_p}{a_1+a_2+\ldots+a_q}=\frac{p^2}{q^2};\ p \neq q$$

Then $\frac{a_6}{a_{21}}$ is equal to

(a) $\frac{41}{11}$ (b) $\frac{121}{1681}$

(c) $\frac{11}{41}$ (d) $\frac{121}{1861}$ **[2013, online]**

29. The sum of the series

$2^2 + 2(4^2) + 3(6^2) + \ldots$ upto 10 terms is

(a) 11300 (b) 11200
(c) 12100 (d) 12300 **[2013, online]**

30. If $S=\tan^{-1}\left(\frac{1}{n^2+n+1}\right)+\tan^{-1}\left(\frac{1}{n^2+3n+3}\right)$

$+\ldots+\tan^{-1}\left(\frac{1}{1+(n+19)(n+20)}\right)$, then

$\tan(S)$ is equal to

(a) $\frac{20}{401+20n}$ (b) $\frac{n}{n^2+20n+1}$

(c) $\frac{20}{n^2+20n+1}$ (d) $\frac{n}{401+20n}$

[2013, online]

31. If $a_1, a_2, a_3, \ldots, a_n \ldots$ are in A.P. such that

$a_4 - a_7 + a_{10} = m$, then sum of the first 13 terms of the A.P. is

(a) $10\,m$ (b) $12\,m$
(c) $13\,m$ (d) $15\,m$ **[2013, online]**

32. Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. Then the common ratio of the G.P. is

(a) $3+\sqrt{2}$ (b) $2-\sqrt{3}$

(c) $2+\sqrt{3}$ (d) $\sqrt{2}+\sqrt{3}$ **[2014]**

33. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \ldots + 10(11)^9 = k(10)^9$, then k is equal to

(a) $\frac{441}{100}$ (b) 100

(c) 110 (d) $\frac{221}{10}$ **[2014]**

34. Given an A.P. whose terms are all positive integers. The sum of its nine terms is greater than 200 and less than 220. If the second term in it is 12, then its 4th term is

(a) 8 (b) 16
(c) 20 (d) 24 **[2014, online]**

35. If the sum

$$\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\ldots+\text{ upto}$$

20 terms is equal to $\frac{k}{21}$, then k is equal to

(a) 120 (b) 180
(c) 240 (d) 60 **[2014, online]**

36. In a geometric progression, if the ratio of the sum of first 5 terms to the sum of their reciprocals is 49, and the sum of the first and the third term is 35. Then the first term of this geometric progression is

(a) 7 (b) 21
(c) 28 (d) 42 **[2014, online]**

37. The sum of the first 20 terms common between the series 3 + 7 + 11 + 15 + . . . and 1 + 6 + 11 + 16 + . . ., is

(a) 4000 (b) 4020
(c) 4200 (d) 4220 **[2014, online]**

38. Let G be the geometric mean of two positive numbers a and b, and M be the arithmetic mean of $\frac{1}{a}$ and $\frac{1}{b}$. If $\frac{1}{M}$: G is 4 : 5, then a : b can be

(a) 1 : 4 (b) 1 : 2
(c) 2 : 3 (d) 3 : 4 **[2014, online]**

39. The least positive integer n such that $1-\frac{2}{3}-\frac{2}{3^2}-\ldots-\frac{2}{3^{n-1}}<\frac{1}{100}$, is

(a) 4 (b) 5
(c) 6 (d) 7 **[2014, online]**

40. The number of terms in an A.P. is even: the sum of the odd terms in it is 24 and that the even terms is 30. If the last terms exceeds the first term by $10\frac{1}{2}$, then the number of terms in the A.P. is

(a) 4 (b) 8
(c) 12 (d) 16 **[2014, online]**

41. Let $f(n)=\left[\frac{1}{3}+\frac{3n}{100}\right]n$, where [$n$] denotes the greatest integer less than or equal to n. then $\sum_{n=1}^{56} f(n)$ is equal to

(a) 56 (b) 689
(c) 1287 (d) 1399 **[2014, online]**

42. If m is the A.M. of two distinct real number l and n (l, $n > 1$) and G_1, G_2 and G_3 are three geometric between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals,

(a) $4l^2 mn$ (b) $4lm^2n$
(c) $4lmn^2$ (d) $4l^2m^2n^2$ **[2015]**

43. The sum of first 9 terms of the series

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots\text{ is}$$

(a) 71 (b) 96
(c) 142 (d) 192 **[2015]**

4. The value of $\sum_{r=16}^{30}(r+2)(r-3)$ is equal to

(a) 7785 (b) 7780
(c) 7775 (d) 7770 **[2015, online]**

45. The sum of the 3rd and the 4th terms of a G.P. is 60 and the product of its first three terms is 1000. If the first term of this G.P. is positive, then its 7th term is:

(a) 7290 (b) 320
(c) 640 (d) 2430 **[2015, online]**

46. If $\sum_{n=1}^{5}\frac{1}{n(n+1)(n+2)(n+3)}=\frac{k}{3}$, then k is equal to

(a) $\frac{55}{336}$ (b) $\frac{17}{105}$
(c) $\frac{1}{6}$ (d) $\frac{19}{112}$ **[2015, online]**

47. If the 2nd, 5th and 9th terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is

(a) $\frac{8}{5}$ (b) $\frac{4}{3}$
(c) 1 (d) $\frac{7}{4}$ **[2016]**

48. If the sum of the first ten terms of the series

$$\left(1\frac{3}{5}\right)^2+\left(2\frac{2}{5}\right)^2+\left(3\frac{1}{5}\right)^2+4^2+\left(4\frac{4}{5}\right)^2+\cdots,\text{is}\frac{16}{5}m,$$

then m is equal to

(a) 102 (b) 101
(c) 100 (d) 99 **[2016, online]**

49. Let a_1, a_2, a_3, …, a_n, … be in A.P. If $a_3 + a_7 + a_{11} + a_{15} = 72$, then the sum of its first 17 terms is equal to

(a) 306 (b) 204
(c) 153 (d) 612 **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. If the first three terms of a sequence $\frac{1}{16}$, a, b, $\frac{1}{6}$ are in G.P. and the last three are in H.P., then the values of a and b respectively are

   (a) $\frac{1}{12}, \frac{4}{9}$ (b) $\frac{4}{7}, \frac{3}{4}$

   (c) $\frac{1}{9}, \frac{1}{12}$ (d) $-\frac{1}{4}, 1$ **[2006]**

2. If a, x, b are in H.P. and a, y, z, b are in G.P., then the value of $\frac{yz}{x\left(y^3+z^3\right)}$ is

   (a) ab (b) $\frac{1}{2ab}$

   (c) $\frac{1}{2}ab$ (d) $2ab$ **[2006]**

3. If $x = 1 + a + a^2 + \ldots$ ($|a| < 1$) and $y = 1 + b + b^2 + \ldots$ ($|b| < 1$), then sum of the series $1 + ab + a^2b^2 + \ldots$ is

   (a) $\frac{xy}{x-y-1}$ (b) $\frac{xy}{x-y+1}$

   (c) $\frac{xy}{x+y+1}$ (d) $\frac{xy}{x+y-1}$ **[2007]**

4. If three distinct positive numbers a, b, c are in A.P. such that $abc = 4$, then value of b is always

   (a) greater than $(2)^{2/3}$ (b) less than $(2)^{2/3}$

   (c) equal to $(2)^{2/3}$ (d) equal to $(2)^{3/2}$ **[2007]**

5. The value of the sum

   $$\sum_{i=1}^{20} i\left[\frac{1}{i}+\frac{1}{i+1}+\frac{1}{i+2}+\ldots+\frac{1}{20}\right]$$

   is

   (a) 100 (b) 105

   (c) 110 (d) 115 **[2008]**

6. The sum of the numbers between 200 and 400 that are divisible by 7 is

   (a) 8729 (b) 7511

   (c) 6328 (d) 5712 **[2009]**

7. Let a, b and c be distinct real numbers. If a, b, c are in geometric progression and $a + b + c = xb$, then x lies in the set

   (a) $(1, 3)$ (b) $(-1, 0) \cup (1, 2)$

   (c) $(-\infty, -1) \cup (3, \infty)$ (d) $(0, 1)$ **[2010]**

8. If the sum of first n terms of two A.P.'s are in the ratio $3n + 8 : 7n + 15$, then the ratio of 12th term is

   (a) 8 : 7 (b) 7 : 16

   (c) 74 : 169 (d) 13 : 47 **[2012]**

9. A tree, in each year grows 5 cm less than it grew in the previous year. If it grew half a metre in the first year, then the height of the tree (in metres) when it ceases to grow, is

   (a) 3.00 (b) 2.75

   (c) 2.50 (d) 2.00 **[2013]**

10. If $\frac{48}{(2)(3)} + \frac{47}{(3)(4)} + \frac{46}{(4)(5)} + \ldots + \frac{2}{(48)(49)} + \frac{1}{(49)(50)} = \frac{51}{2} + K\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{50}\right)$, then K equals

    (a) -1 (b) $-\frac{1}{2}$

    (c) 1 (d) 2 **[2014]**

11. If $\log_{10} 2$, $\log_{10}(2^x - 1)$ and $\log_{10}(2^x + 3)$ are three consecutive terms of an A.P. for

    (a) no real x (b) exactly one real x

    (c) exactly two real x (d) more than two real x **[2014]**

12. Let a, b, c, d and and e be real numbers such that $a > b > 0$ and $c > 0$. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., b, c, d are in G.P. and c, d, e are in A. P., then $\frac{ab^2}{(2a-b)^2}$ is equal to

    (a) c (b) $\sqrt{de}$

    (c) e (d) d **[2015]**

13. If $\sum_{k=1}^{n} \phi(k) = \frac{2n}{n+1}$, then $\sum_{k=1}^{10} \frac{1}{\phi(k)}$ is equal to

    (a) $\frac{11}{20}$ (b) 220

    (c) $\frac{55}{18}$ (d) 110 **[2015]**

14. Let a, b, c, d and e be distinct positive number. If a, b, c and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ both are in A.P. and b, c, d are in G.P. then

(a) a, b, c are in G.P (b) a, b, c are in A.P.
(c) a, c, e are in A.P. (d) a, c, e are in G.P.

Answers

Concept-based

1. (a)	**2.** (a)	**3.** (c)	**4.** (a)
5. (d)	**6.** (b)	**7.** (d)	**8.** (c)
9. (b)	**10.** (b)	**11.** (a)	**12.** (c)
13. (a)	**14.** (b)	**15.** (d)	

Level 1

16. (c)	**17.** (c)	**18.** (a)	**19.** (b)
20. (c)	**21.** (d)	**22.** (d)	**23.** (b)
24. (a)	**25.** (a)	**26.** (a)	**27.** (c)
28. (c)	**29.** (b)	**30.** (d)	**31.** (c)
32. (c)	**33.** (c)	**34.** (c)	**35.** (d)
36. (b)	**37.** (a)	**38.** (d)	**39.** (b)
40. (d)	**41.** (a)	**42.** (a)	**43.** (b)
44. (a)	**45.** (a)	**46**. (a)	**47.** (c)
48. (d)	**49.** (d)	**50.** (a)	**51.** (b)
52. (c)	**53.** (a)	**54.** (c)	**55.** (c)
56. (a)	**57.** (b)	**58.** (a)	**59.** (b)
60. (a)	**61.** (a)	**62.** (a)	**63.** (b)
64. (a)	**65.** (c)	**66.** (d)	**67.** (a)
68. (c)	**69.** (b)	**70.** (b)	**71.** (a)
72. (c)	**73.** (d)	**74.** (c)	**75.** (d)
76. (d)	**77.** (d)	**78.** (a)	**79.** (b)
80. (c)	**81.** (d)	**82.** (a)	**83.** (d)
84. (b)	**85.** (c)		

Level 2

86. (c)	**87.** (a)	**88.** (a)	**89.** (c)
90. (b)	**91.** (c)	**92.** (b)	**93.** (a)
94. (a)	**95.** (a)	**96.** (b)	

Previous Years' AIEEE/JEE Main Questions

1. (b)	**2.** (b)	**3.** (b)	**4.** (a)
5. (b)	**6.** (c)	**7.** (d)	**8.** (c)
9. (b)	**10.** (c)	**11.** (a)	**12.** (a)
13. (a)	**14.** (d)	**15.** (b)	**16.** (d)
17. (d)	**18.** (a)	**19.** (b)	**20.** (c)
21. (a)	**22.** (d)	**23.** (b)	**24.** (b)
25. (c)	**26.** (b)	**27.** (a)	**28.** (c)
29. (c)	**30.** (c)	**31.** (c)	**32.** (c)
33. (b)	**34.** (c)	**35.** (a)	**36.** (c)
37. (b)	**38.** (a)	**39.** (d)	**40.** (b)
41. (d)	**42.** (b)	**43.** (b)	**44.** (b)
45. (b)	**46.** (a)	**47.** (b)	**48.** (b)
49. (a)			

Previous Years' B-Architecture Entrance Examination Questions

1. (d)	**2.** (b)	**3.** (d)	**4.** (a)
5. (d)	**6.** (a)	**7.** (c)	**8.** (b)
9. (b)	**10.** (a)	**11.** (b)	12. (c)
13. (b)	**14.** (d)		

Hints and Solutions

Concept-based

1. $a_1, a_2, a_3, \ldots$ is a G.P.

   $\Rightarrow \log_5 a_1, \log_5 a_2, \log_5 a_3, \ldots$ are in A.P.

2. If $r > 1$ is the common ratio of the G.P., then $\frac{b}{r} + b + br = xb \Rightarrow x - 1 = r + \frac{1}{r} > 2$.

3. Use : $a_n = S_n - S_{n-1}$ and $d = a_n - a_{n-1}$.

4. Middle terms are nth and $(n + 1)$th terms.

   $\alpha - \beta = t_n = a + (n - 1)d$

   and $\alpha + \beta = t_{n+1} = a + nd$

   $S_{2n} = \frac{2n}{2}[2a + (2n - 1)d] = 2n\alpha$

5. Use : $3 + 7(n - 1) = 63 + 2(n - 1)$

6. For example, if $a = 2$, $b = \sqrt{2} - 1$, $c = 1$, then a, $b + 1$, c forms a G.P.

7. $a^{\log x} = b^{\log y} = c^{\log z} = k$ (say)

   $\Rightarrow a = k^{1/\log x}, b = k^{1/\log y}, c = k^{1/\log z}$

   Now, a, b, c are in G.P.

   $\Rightarrow \frac{1}{\log x}, \frac{1}{\log y}, \frac{1}{\log z}$ are in A.P.

   $\Rightarrow \log x, \log y, \log z$ are in H.P.

   However, nothing can be said about x, y, z.

8. $S_3 = \frac{a(1-r^3)}{1-r}$, $S_6 = \frac{a(1-r^6)}{1-r}$

   $\frac{S_3}{S_6} = \frac{125}{152} \Rightarrow \frac{1-r^3}{1-r^6} = \frac{125}{152}$

$\Rightarrow 1 + r^3 = \dfrac{152}{125} \Rightarrow r^3 = \dfrac{27}{125} = \left(\dfrac{3}{5}\right)^3$

$\Rightarrow r = 3/5$

9. As the common difference is 9, we let the four numbers be

$a, a - 18, a - 9, a$

As $a, a - 18, a - 9$ are in G.P.

$(a - 18)^2 = a(a - 9)$

$\Rightarrow a^2 - 36a + 324 = a^2 - 9a$

$\Rightarrow 27a = 324 \Rightarrow a = 12$

Thus, sum of the four numbers is 21

10. Let two positive numbers be a, b. Then

$a + b = 2(7) = 14$ and $\sqrt{ab} = 5$ or $ab = 25$

Now, $a^2 + b^2 = (a + b)^2 - 2ab = 196 - 50 = 146$

$\Rightarrow \dfrac{1}{2}(a^2 + b^2) = 73.$

11. $\dfrac{a}{1-r} = 2, \dfrac{a^3}{1-r^3} = 24$

$\therefore \dfrac{8(1-r)^3}{1-r^3} = 24 \Rightarrow \dfrac{1-2r+r^2}{1+r+r^2} = 3$

$\Rightarrow 2r^2 + 5r + 2 = 0 \Rightarrow r = -2, -1/2$

As $|r| < 1$, $r = -1/2$, $a = 3$.

Now, $t_5 = ar^4 = 3/16$

12. Let three arithmetic means between a and b be a_1, a_2, a_3. Then a, a_1, a_2, a_3, b are in A.P. Let common difference of this A.P. be d. Then

$d = \dfrac{1}{4}(b - a)$

$a_1 = a + b = \dfrac{1}{4}(3a + b)$

$a_2 = a + 2d = \dfrac{1}{2}(a + b)$

$a_3 = a + 3d = \dfrac{1}{4}(a + 3b)$

If A_1, A_2, A_3 are arithmetic means between $1/a$ and $1/b$, then

$A_1 = \dfrac{1}{4}\left(\dfrac{3}{a}+\dfrac{1}{b}\right) = \dfrac{a+3b}{4ab}$

$A_2 = \dfrac{1}{2}\left(\dfrac{1}{a}+\dfrac{1}{b}\right) = \dfrac{a+b}{2ab}$

$A_3 = \dfrac{1}{4}\left(\dfrac{1}{a}+\dfrac{3}{b}\right) = \dfrac{3a+b}{4ab}$

Now, $\dfrac{1}{64}(3a + b)(a + b)(a + 3b) = \dfrac{15}{2}$

and $\dfrac{1}{64(ab)^3}(a + 3b)(a + b)(3a + b) = \dfrac{5}{18}$

Thus, $(ab)^3 = \dfrac{15/2}{5/18} = 27$

$\Rightarrow ab = 3.$

13. $b - \dfrac{a+b}{1-ab} = \dfrac{b+c}{1-bc} - b$

$\Rightarrow \dfrac{b-ab^2-a-b}{1-ab} = \dfrac{b+c-b+b^2c}{1-bc}$

$\Rightarrow -\dfrac{a(b^2+1)}{1-ab} = \dfrac{c(1+b^2)}{1-bc}$

$\Rightarrow -a + abc = c - abc$

$\Rightarrow 2abc = a + c$

$2b = \dfrac{1}{a}+\dfrac{1}{c}$

$\therefore \dfrac{1}{a}, b, \dfrac{1}{c}$ are in A.P.

14. Let

$S = \dfrac{5}{17}+\dfrac{55}{(17)^2}+\dfrac{555}{(17)^3}+\ldots$

$\dfrac{1}{17}S = \dfrac{5}{(17)^2}+\dfrac{55}{(17)^3}+\ldots$

Subtracting, we get

$\left(1-\dfrac{1}{17}\right)S = \dfrac{5}{17}+\dfrac{50}{(17)^2}+\dfrac{500}{(17)^3}+\dfrac{5000}{(17)^4}+\ldots$

$\dfrac{16}{17}S = \dfrac{5/17}{1-10/17} = \dfrac{5}{7} \Rightarrow S = \dfrac{85}{112}$

15. $\sum_{r=1}^{2016}(-1)^r(a + rd) = a\sum_{r=1}^{2016}(-1)^r + d\sum_{r=1}^{2016}(-1)^r(r)$

But $\sum_{r=1}^{2016}(-1)^r = 0$ and $\sum_{r=1}^{2016}(-1)^r(r) = 1008$

Level 1

16. $2(2a + b) = (5a - b) + (a + 2b) \Rightarrow b = 2a.$

Also, $(a - 1)^2(b + 1)^2 = (ab + 1)^2$

$\Rightarrow (a - 1)(b + 1) = \pm(ab + 1)$

$\Rightarrow (a - 1)(2a + 1) = \pm(2a^2 + 1)$

$2a^2 - a - 1 = 2a^2 + 1$

or $2a^2 - a - 1 = -2a^2 - 1$

$\Rightarrow a = -2,\ a = 1/4$

17. $\dfrac{1}{a} + \dfrac{1}{c-b} + \dfrac{1}{c} + \dfrac{1}{a-b} = 0$

$\Rightarrow \dfrac{a+c-b}{a(c-b)} + \dfrac{a+c-b}{c(a-b)} = 0$

As $a + c - b \neq 0,\ a(c - b) + c(a - b) = 0$

$\Rightarrow b = \dfrac{2ac}{a+c} \Rightarrow a, b, c$ are in H.P.

18. $0 = S_p = \dfrac{p}{2}[2a + (p - 1)d] \Rightarrow d = \dfrac{2}{1-p}$

$S_{p+q} = \dfrac{p+q}{2}[2a + (p + q - 1)d]$

$= \dfrac{1}{2}(p + q)\,[2a + (p - 1)d + qd]$

$= \dfrac{(p+q)q}{1-p}$

19. $\dfrac{ma+nb}{m+n} = \dfrac{1}{2}(m + n) = \sqrt{ab}$ (1)

$\Rightarrow ma + nb = \sqrt{ab}\ (m + n)$

$\Rightarrow m\sqrt{a}\left(\sqrt{a}-\sqrt{b}\right) = n\sqrt{b}\left(\sqrt{a}-\sqrt{b}\right)$

$\Rightarrow \dfrac{m}{\sqrt{b}} = \dfrac{n}{\sqrt{a}} = k\text{(say)}$ (2)

From (1) and (2)

$\dfrac{1}{2}k\left(\sqrt{a}+\sqrt{b}\right) = \sqrt{ab} \Rightarrow k = \dfrac{2\sqrt{ab}}{\sqrt{a}+\sqrt{b}}$

Thus, $n = k\sqrt{a} = \dfrac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

20. We have $gl = ab = 192$.

Also, $\dfrac{1}{2}(g + l) : \dfrac{2gl}{g+l} = 169 : 48$

$\Rightarrow \dfrac{(g+l)^2}{4gl} = \dfrac{169}{48}$

$\Rightarrow (g + l)^2 = \dfrac{169}{48}(4)\,(192) = 13^2\,(4)^2$

$\Rightarrow g + l = 52$

Note that g, l are roots of

$t^2 - (g + l)t + gl = 0$

or $t^2 - 52t + 192 = 0$

$\Rightarrow t = 4, 48.$

$\therefore\ g = 4,\ l = 48$

Let $a = 4a_1$, $b = 4b_1$, where $hcf\,(a_1, b_1) = 1$.

Also, $4a_1\,b_1 = 48 \Rightarrow a_1\,b_1 = 12$.

$\Rightarrow a_1 = 1,\ b_1 = 12$ or $a_1 = 3,\ b_1 = 4$

Thus, smaller of a, b, is 4 or 12.

21. $\dfrac{99}{100} < \dfrac{2}{3} + \dfrac{2}{3^2} + \ldots + \dfrac{2}{3^{n-1}}$

$\Rightarrow \dfrac{(2/3)\left[1-(1/3)^{n-1}\right]}{1-1/3} > \dfrac{99}{100}$

$\Rightarrow \dfrac{1}{100} > \dfrac{1}{3^{n-1}} \Rightarrow 3^{n-1} > 100$

$\Rightarrow n - 1 > 5 \Rightarrow n > 6$

Thus, least value of n is 7.

22. a, b, c, d are in A.P.

$\Rightarrow \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}, \dfrac{1}{d}$ are in H.P.

$\Rightarrow \dfrac{1}{d}, \dfrac{1}{c}, \dfrac{1}{b}, \dfrac{1}{a}$ are in H.P.

Now multiply each term by $abcd$

23. $\alpha + \beta = \dfrac{4+\sqrt{2}}{5+\sqrt{2}},\ \alpha\beta = \dfrac{8+2\sqrt{2}}{5+\sqrt{2}}$

H.M. $= \dfrac{2\,\alpha\,\beta}{\alpha+\beta} = \dfrac{16+4\sqrt{2}}{4+\sqrt{2}} = 4$

24. $\dfrac{2}{\cos x} = \dfrac{1}{\cos(x-y)} + \dfrac{1}{\cos(x+y)}$

$\Rightarrow 2(\cos^2 x - \sin^2 y) = 2\cos x\,\cos x\,\cos y.$

$\Rightarrow \cos^2 x(1 - \cos y) = \sin^2 y = 1 - \cos^2 y$

$\Rightarrow \cos^2 x = 1 + \cos y = 2\cos^2\left(\dfrac{y}{2}\right).$

$\Rightarrow \cos x\,\sec\left(\dfrac{y}{2}\right) = \pm\sqrt{2}.$

25. $a(2n) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \ldots + \dfrac{1}{2^{2n}-1}$

$> 1 + \dfrac{1}{2} + \left(\dfrac{1}{3}+\dfrac{1}{4}\right) + \left(\dfrac{1}{5}+\dfrac{1}{6}+\dfrac{1}{7}+\dfrac{1}{8}\right) + \ldots$

$+ \left(\dfrac{1}{2^{2n-1}+1} + \ldots + \dfrac{1}{2^{2n}}\right) - \dfrac{1}{2^{2n}}$

$> \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{2}{4} + \dfrac{4}{8} + \ldots + \dfrac{2^{2n-1}}{2^{2n}} + \left(\dfrac{1}{2} - \dfrac{1}{2^{2n}}\right) > \dfrac{n}{2}$ if $n > 1$

$\therefore\ a(200) > 100.$

26. $$S = \frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + \ldots \qquad (1)$$

$$\frac{1}{19}S = \frac{4}{19^2} + \frac{44}{19^3} + \ldots \qquad (2)$$

Subtracting (2) from (1), we get

$$\left(\frac{18}{19}\right)S = \frac{4}{19} + \frac{40}{19^2} + \frac{400}{19^3} + \ldots$$

$$= \frac{4/19}{1 - 10/19} = \frac{4}{9}$$

$$\Rightarrow \quad S = \frac{19}{18} \times \frac{4}{9} = \frac{38}{81}$$

27. $a^{1/x} = b^{1/y} = c^{1/z} = \ldots = k(\text{say})$

$$\Rightarrow \quad \frac{\log a}{x} = \frac{\log b}{y} = \frac{\log c}{z} = \ldots \log k,$$

Now, use $\log a, \log b, \log c \ldots$ are in A.P.

28. $a^x = b^y = c^z = d^u$

$\Rightarrow x \log a = y \log b = z \log c = u \log d$

$$\Rightarrow \quad \frac{\log a}{1/x} = \frac{\log b}{1/y} = \frac{\log c}{1/z} = \frac{\log d}{1/u}$$

Now, use $\log a, \log b, \log c, \log u$ are in A.P.

29. $(a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2) \le 0$

$\Rightarrow (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \le 0$

As a, b, c, d and p are real this is possible if and only if

$ap - b = 0,\ bp - c = 0,\ cp - d = 0$

$$\Rightarrow \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c} \quad \Rightarrow \quad a, b, c, d \text{ are in G.P.}$$

30. a, b, c are in A.P. $\Rightarrow$ $2^{ax}, 2^{bx}, 2^{cx}$ are in G.P.

$\Rightarrow 2^{ax+1}, 2^{bx+1}, 2^{cx+1}$ are in G.P.

31. $$2\cdot\overline{357} = 2 + \frac{357}{1000} + \frac{357}{(1000)^2} + \ldots$$

$$= 2 + \frac{357/1000}{1 - 1/1000} = 2 + \frac{357}{999} = \frac{2355}{999}$$

32. $$S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \ldots \qquad (1)$$

$$\frac{1}{5}S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \ldots \qquad (2)$$

Subtracting (2) from (1), we get

$$\frac{4}{5}\ S = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \ldots = 1 + \frac{3/5}{1 - 1/5}$$

$$= 1 + \frac{3}{4} = \frac{7}{4}$$

$$\Rightarrow \quad S = \frac{35}{16}.$$

33. $T_m = S_m - S_{m-1}$

$\Rightarrow 164 = 3m^2 + 5m - [3(m-1)^2 + 5(m-1)$

$= 6m + 2$

$\Rightarrow \quad m = 27.$

34. $G_1 = ar,\ G_2 = ar^2$ where $r = \left(\frac{b}{a}\right)^{1/3}$.

$$\frac{G_1^{\,2}}{G_2} + \frac{G_2^{\,2}}{G_1} = \frac{G_1^{\,3} + G_2^{\,3}}{G_1\,G_2} = \frac{a^3r^3 + a^3r^6}{a^2\,r^3}$$

$$= a + ar^3 = a + b = 2A$$

35. Use $A_1 + A_2 = a + b$ and $G_1\,G_2 = ab$.

36. Let n be the number of terms. Then

$$s = \frac{n}{2}(a + l) \Rightarrow n = \frac{2s}{a + l}.$$

Also,

$$d = \frac{l - a}{n - 1} = \frac{(l - a)(l + a)}{2s - (l + a)} = \frac{l^2 - a^2}{2s - (l + a)}$$

37. $x = 2^S$

where $$S = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \ldots$$

$$\Rightarrow \frac{1}{2}S = \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \ldots$$

Subtracting, we get

$$\frac{1}{2}S = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots = \frac{1/4}{1 - 1/2}$$

$$= \frac{1}{4} \times 2 = \frac{1}{2}$$

$\Rightarrow \qquad S = 1.$

$\therefore \qquad x = 2$

38. $$\frac{a}{1 - r} = 20 \text{ and } \frac{a^2}{1 - r^2} = 100.$$

$$\Rightarrow \quad \frac{400\,(1 - r)^2}{1 - r^2} = 100$$

$$\Rightarrow \quad \frac{1 - r}{1 + r} = \frac{1}{4} \Rightarrow 4 - 4r = 1 + r$$

$\Rightarrow r = 3/5$

Thus, $$\frac{a}{2/5} = 20 \Rightarrow a = 8$$

39. $ar^7 = 128 = 2^7$ and $r = 2 \Rightarrow a = 1.$

Product of first five terms = $a^5 r^{10} = 1024$.

40. $l = AR^{p-1}$, $m = AR^{q-1}$, $n = AR^{r-1}$

$$\Delta = \begin{vmatrix} a+(p-1)d & p & 1 \\ a+(q-1)d & q & 1 \\ a+(r-1)d & r & 1 \end{vmatrix}$$

where $a = \log A$, $d = \log R$.

Use $C_1 \to C_1 - (a-d)C_3 - dC_3$ to show $\Delta = 0$.

41. $$\frac{\frac{p}{2}[2a+(p-1)d]}{\frac{q}{2}[2a+(q-1)d]} = \frac{p^2}{q^2} \Rightarrow \frac{a+\frac{p-1}{2}d}{a+\frac{q-1}{2}d} = \frac{p}{q}$$

Now, put $p = 11$ and $q = 41$.

42. $a = \sqrt{2}$, $r = \sqrt{3}$

$$S_{10} = \frac{a(r^{10}-1)}{r-1} = \frac{\sqrt{2}(3^5-1)}{\sqrt{3}-1}$$

$$= \frac{\sqrt{2}(242)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = 121(\sqrt{6}+\sqrt{2})$$

43. $\frac{1}{6}n(n+1)(2n+1) = 1015$

$\Rightarrow (2n)(2n+1)(2n+2) = (28)(29)(30)$

$\Rightarrow 2n = 28$ or $n = 14$.

44. $$t_r = 1^2 + 2^2 + \ldots + r^2 = \frac{1}{6}r(r+1)(2r+1)$$

$$= \frac{1}{6}(2r^3 + 3r^2 + r)$$

$$\therefore \sum_{r=1}^{n} t_r = \frac{1}{3}\cdot\frac{1}{4}n^2(n+1)^2$$

$$+ \frac{1}{2}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{1}{12}n(n+1)$$

$$= \frac{1}{12}n(n+1)[n^2 + n + 2n + 1 + 1]$$

$$= \frac{1}{12}n(n+1)(n^2 + 3n + 2)$$

$$= \frac{1}{12}n(n+1)^2(n+2)$$

$$\therefore \sum_{r=1}^{22} t_r = \frac{1}{12}(22)(23)^2(24) = 23276$$

45. $2b = a + c$, $b^2 = \dfrac{2a^2c^2}{a^2+c^2}$

$$\Rightarrow \frac{(a+c)^2}{4} = \frac{2a^2c^2}{a^2+c^2}$$

$\Rightarrow (a^2 + c^2)^2 + 2ac(a^2 + c^2) - 8a^2c^2 = 0$

$\Rightarrow (a^2 + c^2 + 4ac)(a^2 + c^2 - 2ac) = 0$

$\Rightarrow [(a+c)^2 + 2ac](a-c)^2 = 0$

$\Rightarrow (a+c)^2 + 2ac = 0$ $\quad [\because \ a \neq c]$

$\Rightarrow 4b^2 + 2ac = 0$

$\Rightarrow -\dfrac{a}{2}$, b, c are in G.P.

46. $a_m = n$, $a_n = m$

$\Rightarrow (m-n)d = a_n - a_n = -(m-n)$

$\Rightarrow d = -1$.

Now, $a_r - a_m = (r-m)(-1)$

$\Rightarrow a_r = a_m - r + m = m + n - r$.

47. As $b^2 = ac$, $ax^2 + 2bx + c = 0$ can be written as

$ax^2 + 2\sqrt{ac}\,x + c = 0 \Rightarrow (\sqrt{a}\,x + \sqrt{c})^2 = 0$

$\Rightarrow x = -\sqrt{\dfrac{c}{a}}, -\sqrt{\dfrac{c}{a}}$.

$\therefore -\sqrt{\dfrac{c}{a}}$ is a root of $dx^2 + 2ex + f = 0$

$$\Rightarrow d\left(\frac{c}{a}\right) - 2e\sqrt{\frac{c}{a}} + f = 0.$$

$$\Rightarrow \frac{d}{a} - 2e\sqrt{\frac{1}{ac}} + \frac{f}{c} = 0.$$

$$\Rightarrow 2\frac{e}{b} = \frac{d}{a} + \frac{f}{c} \Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in A.P.}$$

48. 2, $2^x + 1$, $2^x + 3$ are in G.P.

$\Rightarrow (2^x + 1)^2 = 2(2^x + 3) \Rightarrow 2^{2x} = 5$

$\Rightarrow x = \dfrac{1}{2}\log_2 5$

49. As $\frac{1}{2}x$, $|x+1|$, $|x-1|$ are in A.P.

$$2|x+1| = \frac{1}{2}x + |x-1| \qquad (1)$$

If $x < -1$, then (1) give us

$$-2(x+1) = \frac{1}{2}x - (x-1)$$

$\Rightarrow x = -2$

In this case common difference is 2 and sum to 10 terms is

$$\frac{10}{2}[2(-1) + (9)(2)] = 85$$

If $-1 \le x < 1$,

$$2(x+1) = \frac{1}{2}\, x - (x-1) \Rightarrow x = -2/5$$

In this case common difference is 4/5, and sum to 10 terms is

$$\frac{10}{2}\left[2\left(-\frac{1}{5}\right)+(9)\left(\frac{4}{5}\right)\right] = 34$$

If $x \ge 1$, then (1) becomes

$$2x + 2 = \frac{1}{2}\, x + x - 1$$

$$\Rightarrow \frac{1}{2}\, x = -3$$

Not possible as $x \ge 1$.

50. $\dfrac{2ab}{a+b} : \sqrt{ab} = 4 : 5 \Rightarrow \dfrac{2\sqrt{ab}}{a+b} = \dfrac{4}{5}$

$$\Rightarrow \frac{a+b}{\sqrt{ab}} = \frac{5}{2}$$

$$\Rightarrow \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \frac{5}{2} \quad \Rightarrow \quad \sqrt{\frac{a}{b}} = 2, \frac{1}{2}$$

$$\Rightarrow \quad a : b = 4 : 1 \text{ or } 1 : 4$$

51. $t_r = \dfrac{1^3+2^3+\ldots+r^3}{1+3+\ldots(2r-1)} = \dfrac{r^2\,(r+1)^2}{4r^2} = \dfrac{(r+1)^2}{4}$

Thus, $\displaystyle\sum_{r=1}^{16} t_r = \frac{1}{4}\left(\sum_{r=1}^{17} r^2 - 1\right)$

$$= \frac{1}{4}\left(\frac{1}{6}(17)\,(18)\,(35) - 1\right) = 446$$

52. Given equation implies

$$(a+c)\,(a+c-2b) = (a-c)^2$$

$$\Rightarrow (a+c)^2 - (a-c)^2 = 2b(a+c)$$

$$\Rightarrow \quad 4ac = 2b(a+c)$$

$$\Rightarrow b = \frac{2ac}{a+c} \Rightarrow a, b, c \text{ are in H.P.}$$

53. Required sum

$$= (2+4+\ldots+100) + (5+10+\ldots+100) - (10+20+\ldots+100)$$

$$= \frac{50}{2}(2+100) + \frac{20}{2}(5+100) - \frac{10}{2}(10+100)$$

$$= (50)\,(51) + (10)\,(105) - 550$$

$$= 2550 + 1050 - 550 = 3050$$

54. $T_1\, T_2\, T_7 = (T_7 - 6d)\,(T_7 - 5d)T_7$

$$= 9(9-6d)\,(9-5d)$$

$$= 729\left(d - \frac{3}{2}\right)\left(d - \frac{9}{5}\right)$$

But $(x-a)\,(x-b)$ is least when $x = \dfrac{1}{2}\,(a+b)$.

$\therefore\ T_1\, T_2\, T_7$ is least when $d = \dfrac{1}{2}\left(\dfrac{3}{2}+\dfrac{9}{5}\right) = \dfrac{33}{20}$

55. We have $\dfrac{t}{3} + \dfrac{81}{t} = 28$ where $t = 3^{2\sin 2\theta}$

$$\Rightarrow t^2 - 84t + 243 = 0 \Rightarrow t = 3, 81$$

But $1/9 \le t \le 9$. Thus, $t = 3$.

$\therefore\ a_1 = 1,\ d = 13 \quad \Rightarrow \quad a_5 = a_1 + 4d = 53.$

56. Write the given series as

$$\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{2^2}\right)+\left(1+\frac{1}{2^3}\right)\ldots+\left(1+\frac{1}{2^n}\right)$$

$$= n + \frac{\frac{1}{2}\left(1-\frac{1}{2^n}\right)}{1-1/2} = n + 1 - \frac{1}{2^n}$$

57. We have $S_{2m} = \dfrac{2m\,(2m+1)^2}{2} = m(2m+1)^2$

$$\Rightarrow S_{2m-1} = m(2m+1)^2 - 2(2m)^2 = m(2m-1)^2$$

Now, put $n = 2m - 1$.

58. $a_r = \tan^{-1}\left(\dfrac{1}{1+r(r+1)}\right) = \tan^{-1}\left(\dfrac{r+1-r}{1+(r+1)r}\right)$

$$= \tan^{-1}(r+1) - \tan^{-1}(r)$$

Thus, $S_n = \displaystyle\sum_{r=1}^{n} a_r = \tan^{-1}(n+1) - \tan^{-1}(1)$

$$= \tan^{-1}\left(\frac{n+1-1}{1+(n+1)\,(1)}\right) = \tan^{-1}\left(\frac{n}{n+2}\right)$$

59. $t_r = \dfrac{1}{r\,(r+1)\,(r+2)} = \dfrac{1}{2r} - \dfrac{1}{r+1} + \dfrac{1}{2(r+2)}$

[split into partial fractions]

$$= \frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+1}\right) - \frac{1}{2}\left(\frac{1}{r+1} - \frac{1}{r+2}\right)$$

$$\Rightarrow \quad S_n = \sum_{r=1}^{n} t_r = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\,\frac{n}{n+1} - \frac{n}{4\,(n+2)}$$

Thus, $S_{30} = \dfrac{1}{2}\,\dfrac{30}{31} - \dfrac{30}{4(32)} = \dfrac{30}{4}\left[\dfrac{64-31}{(31)\,(32)}\right] = \dfrac{495}{1984}$

60. $9(n!)^2 = n!(n+1)! \quad \Rightarrow \quad 9 = n + 1 \Rightarrow n = 8$

This shows that 8!, 5(8!), 9(8!) are in A.P.

61. Here $a = 81$, $r = \dfrac{2}{3}$.

Also, $S_n - S_{n-1} = ar^{n-1}$.

Let y = numerator of x, then

$$y = (S_n - S_{n-1}) - 3(S_{n-1} - S_{n-2})$$
$$+ 3(S_{n-2} - S_{n-3}) - (S_{n-3} - S_{n-4})$$
$$= ar^{n-1} - 3ar^{n-2} + 3ar^{n-3} - ar^{n-4}$$
$$= ar^{n-4}(r^3 - 3r^2 + 3r - 1)$$

Thus, $x = r = \dfrac{2}{3}$.

62. Let $n = 2m - 1$, then

$$S_{2m-1} = 1^2 + (1)(2) + 3^2 + (3)(4) + \ldots (2m-3)(2m-2) + (2m-1)^2$$
$$= \sum_{r=1}^{2m-1} r^2 + \sum_{r=1}^{m-1}\left[(2r-1)(2r) - (2r)^2\right]$$
$$= \frac{1}{6}(2m-1)(2m)(4m-1) - 2\sum_{r=1}^{m-1} r$$
$$= \frac{1}{3}m(2m-1)(4m-1) - (m-1)m$$
$$= \frac{1}{3}\left(\frac{n+1}{2}\right)n(2n+1) - \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)$$
$$= \frac{1}{12}(n+1)[2n(2n+1) - 3(n-1)]$$
$$= \frac{1}{12}(n+1)(4n^2 - n + 3)$$

63. $(1 - 3^{-1})L = \lim\limits_{n\to\infty} (1 - 3^{-1})(1 + 3^{-1})(1 + 3^{-2})(1 + 3^{-4}) \ldots (1 + 3^{-2^n})$

$$\frac{2}{3}L = \lim_{n\to\infty}(1 - 3^{-2})(1 + 3^{-2})(1 + 3^{-4}) \ldots (1 + 3^{-2^n})$$
$$= \lim_{n\to\infty}(1 - 3^{-2^2})(1 + 3^{-4}) \ldots (1 + 3^{-2^n})$$
$$= \ldots = \lim_{n\to\infty}(1 - 3^{-2^{n+1}}) = 1$$

$\Rightarrow \qquad L = 3/2$

64. Note that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$$
$$< \left(1 + \frac{1}{3} + \frac{1}{3^2} + \ldots\right)\left(1 + \frac{1}{5} + \frac{1}{5^2} + \ldots\right)$$
$$= \left(\frac{1}{1 - 1/3}\right)\left(\frac{1}{1 - 1/5}\right) = \frac{15}{8} < 2$$

65. Coefficient of x^{99}

$= -(1 + 3 + 5 + \ldots 199) = -100^2 = -10000$

66. $t_r = 1 + 2 + \ldots + r = \dfrac{1}{2}r(r+1) = \dfrac{1}{2}(r^2 + r)$

$$\sum_{r=1}^{n} t_r = \frac{1}{2}\left[\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)\right]$$
$$= \frac{1}{6}n(n+1)(n+2)$$

67. $x = \dfrac{1}{1-a}$, $y = \dfrac{1}{1-b}$, $z = \dfrac{1}{1 - a/b} = \dfrac{b}{b-a}$

$$z = \frac{y-1}{y} \cdot \frac{yx}{y-x} = \frac{xy - x}{y - x}$$

$\Rightarrow yz - xz = xy - x \;\Rightarrow\; x + yz = x(y + z).$

68. $\dfrac{a}{1-r} = 3$, $\dfrac{a^3}{1-r^3} = \dfrac{108}{13}$ where $|r| < 1$.

$$\therefore \frac{27(1-r)^3}{1-r^3} = \frac{108}{13}$$
$$\Rightarrow \frac{1 - 2r + r^2}{1 + r + r^2} = \frac{4}{13}$$

$\Rightarrow 13 - 26r + 13r^2 = 4 + 4r + 4r^2$

$\Rightarrow 9r^2 - 30r + 9 = 0 \Rightarrow 3r^2 - 10r + 3 = 0.$

$\Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = 1/3 \qquad [\because |r| < 1]$

69. $S = \displaystyle\sum_{r=1}^{n} [(r+1)\log a - (r-1)\log b]$

$$= \frac{n}{2}[(2 + n + 1)\log a - (0 + n - 1)\log b]$$
$$= \frac{n}{2}\log\left(\frac{a^{n+3}}{b^{n-1}}\right)$$

70. $t_5 = ar^4 = 2$.

We have $t_1 t_2 \ldots t_9 = a^9 r^{1+2+\ldots+8}$

$= a^9 r^{36} = (ar^4)^9 = 2^9 = 512$

71. $s = \dfrac{a}{1-r} \Rightarrow 1 - r = \dfrac{a}{s} \Rightarrow r = 1 - \dfrac{a}{s}$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{a/s}\left[1 - \left(1 - \frac{a}{s}\right)^n\right]$$
$$= s\left[1 - \left(1 - \frac{a}{s}\right)^n\right]$$

72. Note that $< a_n >$ is a G.P. with first term equal to 5 and common ratio 1/2.

$$\therefore \quad \sum_{n=1}^{\infty} a_{2n-1} = \frac{a_1}{1-r^2} = \frac{5}{1-1/4} = 20/3.$$

73. $\frac{2ab}{a+b} : \sqrt{ab} = 12 : 13 \quad \Rightarrow \quad \frac{2\sqrt{ab}}{a+b} = \frac{12}{13}$

$$\Rightarrow \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \frac{13}{6}$$

$$\Rightarrow \sqrt{\frac{a}{b}} = \frac{2}{3} \text{ or } \frac{3}{2} \Rightarrow a : b = 4 : 9 \text{ or } 9 : 4$$

74. $\frac{2ab}{a+b} = \frac{16}{5}$, $a + b + ab = 26$

$$\Rightarrow \quad a + b = \frac{5}{8} ab, \; a + b + ab = 26$$

$$\therefore \; \frac{13}{8} ab = 26 \Rightarrow ab = 16 \text{ and } a + b = 10$$

Thus, numbers are 2 and 8.

75. $a_n^2 - 2a_{n-1}^2 + a_{n-2}^2$

$$= (a_n - a_{n-1})(a_n + a_{n-1}) - (a_{n-1} - a_{n-2})(a_{n-1} + a_{n-2})$$
$$= d(a_n + a_{n-1} - a_{n-1} - a_{n-2}) = d(2d) = 2d^2$$

76. $A = \frac{1}{1-r^a}$, $B = \frac{1}{1-r^b}$

$$\Rightarrow \quad 1 - r^a = \frac{1}{A}, \; 1 - r^b = \frac{1}{B}$$

$\Rightarrow a \log r = \log(1 - 1/A)$ and

$b \log r = \log(1 - 1/B)$

$$\therefore \; \frac{a}{b} = \frac{\log(1 - A^{-1})}{\log(1 - B^{-1})}$$

77. Note that $a_k - a_{k-1} = a_{k-1} - a_{k-2} \; \forall \; k \geq 3$ implies $a_1, a_2, a_3 \ldots, a_{12}$ are in A.P.

Also, $d = -4$. We have

$$12S = (a_1 - a_2)(a_1 + a_2) + (a_3 - a_4)(a_3 + a_4) + \ldots + (a_{11} - a_{12})(a_{11} + a_{12})$$
$$= 4(a_1 + a_2 + \ldots + a_{12})$$
$$= 4 \times \frac{12}{2}[2a_1 + (12 - 1)(-4)]$$

$$\Rightarrow \quad S = 2[2(45) - 44] = 92$$

78. We have

$$P(x) = \frac{1 - (-x)^{2013}}{1 - (-x)} = \frac{1 + x^{2013}}{1 + x}$$

Also $\sum_{i=0}^{2012} a_i = Q(1) = P(3)$ $\quad [\because y = 1 = x - 2]$

$$= \frac{1}{4}(3^{2013} + 1)$$

79. We have

$a_1 + a_{26} = a_2 + a_{25} = a_3 + a_{24} = \ldots = a_9 + a_{18}$.

Thus,

$a_2 + a_6 + a_9 + a_{18} + a_{21} + a_{25} = 165$

$\Rightarrow \; (a_2 + a_{25}) + (a_6 + a_{21}) + (a_9 + a_{18}) = 165$

$\Rightarrow \; 3(a_1 + a_{26}) = 165 \quad \Rightarrow \quad a_1 + a_{26} = 55$

$\therefore \; a_1 + a_2 + \ldots + a_{26} = \frac{26}{2}(a_1 + a_{26})$

$= 13 \times 55 = 715$

80. We are given $S_n = cn^2$. Now,

$t_n = S_n - S_{n-1} = cn^2 - c(n-1)^2 = (2n-1)c$

We have

$$t_1^2 + t_2^2 + \ldots + t_n^2 = \sum_{k=1}^{n} c^2 (2k-1)^2$$
$$= c^2 \sum_{k=1}^{n} \left(4k^2 - 4k + 1\right)$$
$$= c^2 \left[\frac{4}{6} n(n+1)(2n+1) - \frac{4}{2} n(n+1) + n\right]$$
$$= \frac{c^2}{3} n\left[2\left(2n^2 + 3n + 1\right) - 6n - 6 + 3\right]$$
$$= \frac{1}{3} n\left(4n^2 - 1\right)c^2$$

81. Sum to n terms of an A.P. is of the form $an + bn^2$ and not of the form $c + bn + an^2$ where $c \neq 0$.

82. $|2x - 1| < 1 \quad \Rightarrow \quad -1 < 2x - 1 < 1$

$\Rightarrow \quad 0 < x < 1$

83. As $a + c > 0$, we must have

$(a + c)^2 + 4b^2 - 4(a + c)b < 0$

$\Rightarrow (2b - a - c)^2 < 0$

This is not possible as $2b - a - c \in \mathbf{R}$.

84. $(\alpha^2 + \beta^2)^2 = (\alpha + \beta)(\alpha^3 + \beta^3)$

$\Rightarrow \alpha^4 + \beta^4 + 2\alpha^2\beta^2 = \alpha^4 + \alpha\beta^3 + \alpha^3\beta + \beta^4$

$\Rightarrow \alpha\beta(\alpha - \beta)^2 = 0$

$\Rightarrow \alpha\beta[(\alpha + \beta)^2 - 4\alpha\beta] = 0$

$$\Rightarrow \frac{c}{a}\left[\frac{4b^2}{a^2}-\frac{4c}{a}\right] = 0 \quad \Rightarrow \quad c\Delta = 0$$

Next, α, $\alpha\beta$, β are in A.P. $\Rightarrow$ $\alpha + \beta = 2\alpha\beta$

$$\Rightarrow -\frac{2b}{a}=\frac{2c}{a} \quad \Rightarrow \quad b + c = 0$$

85. Let $a_1 = a$, $a_2 = ar$, $a_3 = ar^2$ and $a_4 = ar^3$.

As a_1, a_2, a_3, a_4 are distinct, $r \neq 1$

$b_1 = a$, $b_2 = a + ar = a(1 + r)$,

$b_3 = a(1 + r + r^2)$

and $b_4 = a(1 + r + r^2 + r^3)$.

As $b_2 - b_1 = ar \neq ar^2 = b_3 - b_2$.

$\Rightarrow b_1, b_2, b_3, b_4$ are not in A.P.

$$\frac{b_2}{b_1} = 1 + r \neq \frac{1+r+r^2}{1+r}=\frac{b_3}{b_2}$$

$\Rightarrow b_1, b_2, b_3, b_4$ are not in G.P.

Next, $\dfrac{1}{b_2}-\dfrac{1}{b_1}=\dfrac{-r}{a(1+r)}$

$$\frac{1}{b_3}-\frac{1}{b_2}=\frac{-r^2}{a(1+r)\left(1+r+r^2\right)}$$

Since $\dfrac{1}{b_2}-\dfrac{1}{b_1}\neq\dfrac{1}{b_3}-\dfrac{1}{b_2}$ we get b_1, b_2, b_3, b_4 cannot be in H.P.

Level 2

86. The given sequence can be written as

$$\frac{5}{2},\frac{20}{13},\frac{10}{9},\frac{20}{23},\cdots$$

The reciprocal of the numbers in the sequence are

$$\frac{8}{20},\frac{13}{20},\frac{18}{20},\frac{23}{20},\cdots$$

This is an A.P.

nth term of this A.P. is

$$a_n = \frac{8+5(n-1)}{20} = \frac{3+5n}{20}$$

Thus, nth term of the given sequence is $\dfrac{20}{5n+3}$.

87. H.M. between a and b is

$$\frac{2ab}{a+b} = \frac{2(1/28)(1/10)}{1/28+1/10} = \frac{1}{19}$$

88. $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right)$, $b\left(\dfrac{1}{c}+\dfrac{1}{a}\right)$, $c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in A.P.

$\Rightarrow a\left(\dfrac{1}{b}+\dfrac{1}{c}\right)+1$, $b\left(\dfrac{1}{c}+\dfrac{1}{a}\right)+1$, $c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)+1$ are in A.P.

$\Rightarrow a\left(\dfrac{bc+ca+ab}{abc}\right)$, $b\left(\dfrac{bc+ca+ab}{abc}\right)$,

$c\left(\dfrac{bc+ca+ab}{abc}\right)$ are in A.P.

$\Rightarrow a, b, c$ are in A.P.

89. $\sin d\,[\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \cdots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n]$

$$= \frac{\sin(a_2-a_1)}{\sin a_1 \sin a_2}+\frac{\sin(a_3-a_2)}{\sin a_2 \sin a_3}+\cdots+\frac{\sin(a_n-a_{n-1})}{\sin a_{n-1}\sin a_n}$$

$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3)$

$+ \cdots + (\cot a_{n-1} - \cot a_n)$

$= \cot a_1 - \cot a_n$

90. $1 + \dfrac{3}{2} + \dfrac{5}{3} + \cdots + \dfrac{2n-1}{n}$

$$= (2-1) + \left(2-\frac{1}{2}\right)+\left(2-\frac{1}{3}\right)+\cdots+\left(2-\frac{1}{n}\right)$$

$= 2n - H_n$

91. a, b, c are in H.P.

$\Rightarrow \dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c}$ are in A.P.

$\Rightarrow \dfrac{a+b+c}{a}-2$, $\dfrac{a+b+c}{b}-2$, $\dfrac{a+b+c}{c}-2$ are in A.P.

$\Rightarrow \dfrac{b+c-a}{a}$, $\dfrac{c+a-b}{b}$, $\dfrac{a+b-c}{c}$ are in A.P.

$\Rightarrow \dfrac{a}{b+c-a}$, $\dfrac{b}{c+a-b}$, $\dfrac{c}{a+b-c}$ are in H.P.

92. Last term of the nth row

$= 1 + 2 + 3 + \cdots + n = \dfrac{1}{2}n(n+1)$

Note that sum of the terms in the nth row

= sum to n terms of an A.P. whose first term in $\dfrac{1}{2}n(n+1)$ and common difference is (-1).

$$= \frac{1}{2}n\left[2\left(\frac{1}{2}\right)n(n+1)+(n-1)(-1)\right]=\frac{1}{2}n(n^2+1)$$

93. We have

$$A_1 = a + \frac{1}{3}(b - a) = \frac{2a+b}{3}$$

and $A_2 = \dfrac{a+2\,b}{3}$

$\therefore\ A_1 + A_2 = a + b$

Also, $\dfrac{1}{H_1} = \dfrac{1}{3}\left(\dfrac{2}{a}+\dfrac{1}{b}\right) = \dfrac{1}{3}\left(\dfrac{2b+a}{ab}\right)$

and $\dfrac{1}{H_2} = \dfrac{1}{3}\left(\dfrac{2a+b}{ab}\right)$

Also, $\dfrac{1}{H_1}+\dfrac{1}{H_2} = \dfrac{1}{a}+\dfrac{1}{b}$

$\Rightarrow H_1 + H_2 = \dfrac{a+b}{ab} H_1\, H_2$

Therefore,

$$\frac{A_1 + A_2}{H_1 + H_2} = \frac{ab}{H_1\,H_2} = \frac{ab}{9}\,\frac{(a+2b)\,(2a+b)}{(ab)^2}$$

$$= \frac{2\,(a^2+b^2)+5\,ab}{9\,ab}$$

94. $\dfrac{1}{n+1}+\dfrac{1}{n+2}+\cdots+\dfrac{1}{2n}$

$$= \left(1+\frac{1}{2}+\cdots+\frac{1}{2n}\right) - \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$$= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\right)$$

$$+\ 2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n}\right) - \left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$$

$= a\ (2n)$

95. Let m be the minimum number filled in the boxes. Let a, b, c, d be the numbers in the boxes sharing a side with the box containing m, then

$m = \dfrac{1}{4}\ (a + b + c + d)$

$\Rightarrow (a - m) + (b - m) + (c - m) + (d - m) = 0 \qquad (1)$

As $a - m \ge 0,\ b - m \ge 0\ ,\ c - m \ge 0,\ d - m \ge 0,$

(1) is possible if and only if

$a - m = 0,\ b - m = 0,\ c - m = 0\ d - m = 0$

Thus, each of the box contains m.

As the first box contains 5, $m = 5$.

Therefore, $x = 5$

96. $a + 8b + 7c = 0$

$9a + 2b + 3c = 0$

$5a + 5b + 5c = 0$

as $b = 6$, we get $a = 1$, $c = -7$

The roots of quadratic equation

$x^2 + 6x - 7 = 0 \Rightarrow x = 1, -7$

Now, $\dfrac{1}{\alpha}+\dfrac{1}{\beta} = 1-\dfrac{1}{7} = \dfrac{6}{7}$

Thus, $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n = \frac{1}{1-6/7} = 7$

Previous Year's AIEEE/JEE Main Questions

1. $t_5 = 2$. Let r be the common ratio of the G.P. $t_1t_2t_3t_4t_5t_6t_7t_8t_9$

$= \left(\dfrac{2}{r^4}\right)\left(\dfrac{2}{r^3}\right)\left(\dfrac{2}{r^2}\right)\left(\dfrac{2}{r}\right)(2)(2r)(2r^2)(2r^3)(2r^4)$

$= 2^9 = 512.$

2. Let $E = 2^{1/4}4^{1/8}8^{1/16}\ \ldots = 2^5$

where

$S = \dfrac{1}{4}+\dfrac{2}{8}+\dfrac{3}{16}+\dfrac{4}{32}+\cdots$

$\Rightarrow\quad \dfrac{1}{2}S = \dfrac{1}{8}+\dfrac{2}{16}+\dfrac{3}{32}+\cdots$

Subtracting, we get

$\dfrac{1}{2}S = \dfrac{1}{4}+\dfrac{1}{8}+\dfrac{1}{16}+\dfrac{1}{32}+\cdots$

$= \dfrac{1/4}{1-1/2} = \dfrac{1}{2}$

$\Rightarrow S = 1$

$\therefore\ E = 2^1 = 2$

3. 1, $\log_9\ (3^{1-x} + 2)$, $\log_3\ (4(3^x) - 1)$ are in A.P.

$\Rightarrow 1,\ \dfrac{\log_3(3^{1-x}+2)}{\log_3 9}$, $\log_3\ (4(3^x) - 1)$ are in A.P.

$\Rightarrow 2,\ \log_3\ (3^{1-x} + 2),\ 2\log_3\ (4(3^x) - 1)$ are in A.P.

$\Rightarrow 3^2,\ 3^{1-x} + 2,\ (4(3^x) - 1)^2$ are in G.P.

$\Rightarrow (3^{1-x} + 2)^2 = 3^2\ (4(3^x) - 1)^2$

$\Rightarrow \dfrac{3}{y}+2 = 3(4(y) - 1)$ where $y = 3^x$

$\Rightarrow 12y^2 - 5y - 3 = 0$

$\Rightarrow (4y - 3)\ (3y + 1) = 0$

$\Rightarrow y = 3/4 \qquad [\because\ 3y + 1 > 0]$

$\Rightarrow 3^x = 3/4 \Rightarrow x = 1 - \log_3 4$

4. $1^3 - 2^3 + 3^3 - 4^3 + \cdots - 8^3 + 9^3$

$= (1^3 - 2^3) + (3^3 - 4^3) + \cdots + (7^3 - 8^3) + 9^3$

$= (1 - 2)\,(1^2 + 2^2 + 2) + (3 - 4)\,(3^2 + 4^2 + (3)(4))$

$+ (5 - 6)\,(5^2 + 6^2 + (5)\,(6))$

$+ (7 - 8)\,(7^2 + 8^2 + (7)(8)) + 9^3$

$= -\,(1^2 + 2^2 + 3^2 + \cdots 8^2)$

$- (2 + 12 + 30 + 56) + 9^3$

$= 9^3 - \frac{1}{6}(8)(9)(17) - 100$

$= 729 - 204 - 100 = 425$

5. Let $S_1 = \{2x \mid 1 \le x \le 50\}$

$S_2 = \{5x \mid 1 \le x \le 20\}$

$S_1 \cap S_2 = \{10x \mid 1 \le x \le 10\}$

Required sum

$= \sum_{x\in S_1} x + \sum_{x\in S_2} x - \sum_{x\in S_1 n S_2} x$

$= 2\left(\frac{50}{2}\right)(51) + 5\left(\frac{20}{2}\right)(21) - 10\left(\frac{10}{2}\right)(11)$

$= (50)\,(51) + 5(10)\,(21) - 5(10)\,(11)$

$= (50)\,[51 + 21 - 11] = 3050$

6. $\frac{a}{1-r} = 20,\ \frac{a^2}{1-r^2} = 100$

$\Rightarrow 20\left(\frac{a}{1+r}\right) = 100 \quad \Rightarrow \frac{a}{1+r} = 5$

$\therefore \frac{a/(1-r)}{a/(1+r)} = \frac{20}{5} \Rightarrow \frac{1+r}{1-r} = 4$

$\Rightarrow 1 + r = 4 - 4r$

$\Rightarrow 5r = 3 \Rightarrow r = 3/5.$

7. Let $f(x) = Ax^2 + Bx + C$

As $f(1) = f(-1)$, $B = 0$

$\therefore f(x) = Ax^2 + C$

$\Rightarrow f'(x) = 2Ax$

Now, a, b, c are in A.P.

$\Rightarrow 2Aa$, $2Ab$, $2Ac$ are in A.P.

$\Rightarrow f'(a), f'(b), f'(c)$ are in A.P.

8. $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$

$\Rightarrow a + (m-1)d = \frac{1}{n}$ and $a + (n-1)d = \frac{1}{m}$

Subtracting we get

$(m-n)d = \frac{1}{n} - \frac{1}{m} \quad \Rightarrow d = \frac{1}{mn}.$

Thus, $a - d = \frac{1}{n} - md$

$= \frac{1}{n} - \frac{1}{n} = 0$

9. Let $n = 2m$, then

$1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + 5^2 + 2\cdot 6^2 + \ldots + (2m-1)^2 + 2(2m)^2$

$= \frac{1}{2}(2m)(2m+1)^2$

$\Rightarrow 1^2 + 2\cdot 2^2 + 3^2 + 2\cdot 4^2 + \cdots + (2m-1)^2$

$= m(4m^2 + 4m + 1) - 8m^2$

$= m(2m - 1)^2$

Putting $n = 2m - 1$, we get desired sum.

$= \frac{1}{2}n^2(n+1)$

10. We have $x = \frac{1}{1-a}$, $y = \frac{1}{1-b}$, $z = \frac{1}{1-c}$

a, b, c are in A.P.

$\Rightarrow 1 - a$, $1 - b$, $1 - c$ are in A.P.

$\Rightarrow \frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c}$ are in H.P.

11. $\frac{1}{(1-ax)(1-bx)} = \frac{1}{(b-a)x}\left[\frac{1}{1-b} - \frac{1}{1-ax}\right]$

$= \frac{1}{(b-a)x}\left[(1-bx)^{-1} - (1-ax)^{-1}\right]$

Thus, coefficient of x^n in $\frac{1}{(1-ax)(1-bx)}$ is $\frac{1}{(b-a)x}$ × [coefficient of x^{n+1} in $(1 - bx)^n$ – coefficient of x^{n+1} in $(1 - ax)^n$]

$= \frac{1}{b-a}\{b^{n+1} - a^{n+1}\}$

12. Let d be the common difference of the A.P., then

$$\frac{\frac{p}{2}[2a_1 + (p-1)d]}{\frac{q}{2}[2a_1 + (q-1)d]} = \frac{p^2}{q^2}$$

$$\Rightarrow \frac{a_1 + \frac{1}{2}(p-1)d}{a_1 + \frac{1}{2}(q-1)d} = \frac{p}{q}$$

Putting $p = 11$ and $q = 41$, we get

$$\frac{a_6}{a_{21}} = \frac{11}{41}$$

13. $a_1, a_2, \ldots a_n$ are in H.P.

$\Rightarrow \quad \frac{1}{a_1}, \frac{1}{a_2}, \ldots \frac{1}{a_n}$ are in A.P.

$$\Rightarrow \quad \frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \cdots \frac{1}{a_n} - \frac{1}{a_{n-1}} = d \text{ (say)}$$

$$\Rightarrow \quad a_1a_2 = \frac{1}{d}(a_1 - a_2), a_2a_3 = \frac{1}{d}(a_2 - a_3), \cdots$$

$$\Rightarrow a_{n-1}a_n = \frac{1}{d}(a_1 - a_n)$$

Thus, $\quad a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n$

$$= \frac{1}{d}[a_1 - a_2 + a_2 - a_3 + \cdots + a_{n-1} - a_n]$$

$$= \frac{1}{d}(a_1 - a_n)$$

But $\frac{1}{a_n} = \frac{1}{a_1} + (n-1)d \Rightarrow \frac{a_1 - a_n}{a_na_1} = (n-1)d$

$\therefore \quad a_1a_2 + a_2a_3 + \cdots a_{n-1}a_n = (n-1)a_1a_n$

14. $ar^n = ar^{n+1} + ar^{n+2}$, $a > 0$, $r > 0$

$\Rightarrow \quad 1 = r + r^2 \Rightarrow r^2 + r - 1 = 0$

$\Rightarrow \quad r = \frac{1}{2}(\sqrt{5} + 1)$ as $r > 0$

15. As $n < n + 1 \ \forall \ n \geq 2$

$\Rightarrow \quad n(n+1) < (n+1)^2 \ \forall \ n \geq 2$

$\Rightarrow \quad \sqrt{n(n+1)} < n + 1 \ \forall \ n \geq 2$

Thus, Statement-2 is true.

Also, for $n \geq 2$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$$

$$> \underbrace{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}}}_{n \text{ times}}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

$\therefore$ Statement-1 is true.

Thus, both the statements are correct but Statement-2 is not a correct explanation of Statement-1.

16. Let $S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} +$

$$\frac{1}{3}S = \frac{1}{3} + \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + \cdots$$

Subtracting, we get

$$\frac{2}{3}S = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \cdots$$

$$= 1 + \frac{1}{3} + \frac{4/3^2}{1 - 1/3} = \frac{4}{3} + \frac{2}{3} = 2$$

$\Rightarrow \quad S = 3$

17. Suppose he takes n minutes to count 4500 notes. We have $a_1 + a_2 + \cdots + a_{10} = 10(150)$, $a_{11} = 148$ and $a_{11}, a_{12}, \ldots, a_n$ is an A.P. with common difference $d = -2$.

We are given.

$a_1 + a_2 + \ldots + a_{10} + a_{11} + \ldots + a_n = 4500$

$\Rightarrow \quad a_{11} + a_{12} + \ldots + a_n = 3000$

$$\Rightarrow \quad \frac{n-10}{2}[a_{11} + a_n] = 3000$$

$$\Rightarrow \quad \frac{1}{2}(n-10)[148 + 148 + (n-11)(-2)] = 3000$$

$\Rightarrow \quad (n - 10)(148 - n + 11) = 3000$

$\Rightarrow \quad (n - 10)(159 - n) = 3000$

$\Rightarrow \quad n^2 - 169n + 4590 = 0$

$\Rightarrow \quad n^2 - 135n - 34n + 4590 = 0$

$\Rightarrow \quad (n - 135)(n - 34) = 0 \quad \Rightarrow \quad n = 135, 34$

All the notes get counted in 34 minutes.

18. Suppose his total saving is ₹ 11040 after n months. His savings are

200, 200, 200, 240, 280, ... upto n terms

Note that

200, 240, 280, ... upto $(n - 2)$ terms is an A.P. whose sum is 10640.

Thus,

$$\frac{1}{2}(n-2)[2(200) + (n-3)(40)] = 10640$$

$\Rightarrow (n - 2)(n + 7) = 532$

$\Rightarrow \quad n^2 + 5n - 546 = 0$

$\Rightarrow n = -26, 21$

As $n > 0$, $n = 21$.

19. Let d the common difference of the A.P. then

$$\alpha - \beta = \sum_{r=1}^{100}(a_{2r} - a_{2r-1}) = \sum_{r=1}^{100} d = 100d$$

$$\Rightarrow d = \frac{1}{100}(a - \beta).$$

20. We have

$100a_{100} = 500a_{50}$

$\Rightarrow \quad 100\,[a + 99d] = 50[a + 49d]$

$\Rightarrow \quad (100 - 50)a = 50(49 - 2 \times 99)d$

$\Rightarrow \quad a = -149d \quad \Rightarrow \quad a + 149d = 0$

or $\quad a_{150} = 0$

21. Statement-2 is true since

$$\sum_{k=1}^{n}\left(k^2 - (k-1)^3\right) = (1^3 - 0^3) + (2^3 - 1^3) + \cdots + (n^3 - (n-1)^3) = n^3$$

$$\Rightarrow \quad \sum_{k=1}^{n}(k-(k-1))\left(k^2 + k(k-1) + (k-1)^2\right) = n^3$$

$$\Rightarrow \quad \sum_{k=1}^{n}\left(k^2 + k(k-1) + (k-1)^2\right) = n^3$$

Putting $n = 20$, we get

$$\sum_{k=1}^{20}\left((k-1)^2 + k(k-1) + k^2\right) = 20^3$$

$$\Rightarrow \quad 1 + (1 + 2 + 4) + (4 + 6 + 9) + \cdots + (361 + 380 + 400) = 8000$$

22. $2y = x + z$

and $2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$$\Rightarrow \quad \frac{2y}{1-y^2} = \frac{x+z}{1-xz}$$

$\Rightarrow \quad y^2 = xz \quad [\because 2y = x + z]$

$\Rightarrow \quad \frac{1}{4}(x+z)^2 = xz \;\Rightarrow\; (x - z)^2 = 0$

$\Rightarrow \quad x = z$

$\therefore\; 2y = 2x \Rightarrow \; y = x$

$\Rightarrow x = y = z$

23. $S = 0.7 + 0.77 + 0.777 + \cdots +$ upto 20 terms

$= \frac{7}{9}[0.9 + 0.99 + 0.999 + \cdots$ upto 20 terms$]$

$= \frac{7}{9}[20 - (0.1 + 0.01 + 0.001 + \cdots$ upto 20 terms$)]$

$$= \frac{7}{9}\left[20 - \frac{(1/10)\left(1-(1/10)^{20}\right)}{1 - 1/10}\right]$$

$$= \frac{7}{81}\left[179 + 10^{-20}\right]$$

24. $S_n = 2n + 3n^2$

$a = S_1 = 2 + 3 = 5$

$a + (a + d) = S_2 = 2(2) + 3(2^2) = 16$

$\therefore\; d = 16 - 10 = 6$

Sum of required A.P.

$$= \frac{n}{2}[2a + (n-1)(2d)]$$

$$= \frac{n}{2}[2(5) + (n-1)(2)(6)]$$

$= 6n^2 - n$

25. $$a_r = \frac{2r+1}{1^2 + 2^2 + \ldots + r^2}$$

$$= \frac{(2r+1)}{(1/6)r(r+1)(2r+1)}$$

$$= \frac{6}{r(r+1)} = 6\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$\therefore\; \sum_{r=1}^{11} a_r = 6\left(1 - \frac{1}{12}\right) = \frac{11}{2}$$

26. Let first three numbers be

$a/r,\ a,\ ar$

and last three numbers be

$a,\ ar,\ 2ar - a$

We are given

$ar - a = 6$

and $\quad 2ar - a = a/r$

$\Rightarrow 2r^2 - r = 1$

$\Rightarrow (2r + 1)\,(r - 1) = 0$

$\Rightarrow r = -1/2,\ 1$

$r \neq 1$ is not possible, for then $a\,(1) - a = 6$ or $0 = 16$

Let $r = -1/2$.

Thus, $a(-1/2) - a = 6 \quad \Rightarrow a = -4$

$\therefore$ last term = first term = $a/r = 8$

27. $1^2 + 3^2 + 5^2 + \cdots + 25^2$

$$= \sum_{r=1}^{13}(2r-1)^2 = \sum_{r=1}^{13}(4r^2 - 4r + 1)$$

$$= 4\left(\frac{1}{6}\right)(13)(14)(27) - \frac{4}{2}(13)(14) + (13)$$

$= 2925$

28. See Solution to Question 12.

29. $\sum_{k=1}^{10} k(2k)^2 = 4\sum_{k=1}^{10} k^3$

$\frac{4}{4}(10)^2(10+1)^2 = 12100$

30. $t_k = \tan^{-1}\left(\frac{1}{1+(n+k-1)(n+k)}\right)$

$= \tan^{-1}\left(\frac{(n+k)-(n+k-1)}{1+(n+k)(n+k-1)}\right)$

$= \tan^{-1}(n+k) - \tan^{-1}(n+k-1)$

$\therefore\ S = \sum_{k=1}^{20} t_k = \tan^{-1}(n+20) - \tan^{-1}(n)$

$= \tan^{-1}\left(\frac{20}{1+n(n+20)}\right)$

$\Rightarrow \tan(S) = \frac{20}{n^2+20n+1}$

31. Let d be the common difference of the A.P., then

$m = a_4 - a_7 + a_{10} = a_4 + (a_{10} - a_7)$

$= a_1 + 3d + 3d = a_1 + 6d$

Now,

$S_{13} = \frac{13}{2}[2a_1 + (13-1)d] = 13m$

32. Let three positive numbers in G.P. be

$\frac{a}{r}$, a, ar, $a > 0$, $r > 1$

Now, $2(2a) = \frac{a}{r} + ar$

$\Rightarrow r^2 - 4r + 1 = 0$

$\Rightarrow r = 2+\sqrt{3}$ as $r > 1$.

33. $k = 1 + 2\left(\frac{11}{10}\right) + 3\left(\frac{11}{10}\right)^2 + \cdots + 10\left(\frac{11}{10}\right)$

$\frac{11}{10}k = \frac{11}{10} + 2\left(\frac{11}{10}\right)^2 + \cdots + 9\left(\frac{11}{10}\right)^9 + 10\left(\frac{11}{10}\right)^{10}$

Subtracting, we get

$-\frac{1}{10}k = 1 + \frac{11}{10} + \left(\frac{11}{10}\right)^2 + \cdots + \left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10}$

$= \frac{(11/10)^{10} - 1}{11/10 - 1} - 10\left(\frac{11}{10}\right)^{10}$

$= 10\left(\frac{11}{10}\right)^{10} - 10 - 10\left(\frac{11}{10}\right)^{10}$

$k = 100$

34. $S_9 = \frac{9}{2}[2a + (9-1)d] = 9(a + 4d)$

$200 < 9(a + 4d) < 220$

As $a + 4d$ is a positive integer

$a + 4d = 23, 24$

Also, $a + d = 12$

$\therefore\ 3d = 11, 12$

As d is an integer, $d = 4$

Thus, $t_4 = a + 3d = (a + 4d) - d = 20$

[$a + 4d = 23$ is not possible]

35. From Question 25,

$a_r = 6\left(\frac{1}{r} - \frac{1}{r+1}\right)$

$\Rightarrow \sum_{r=1}^{20} a_r = 6\left(1 - \frac{1}{21}\right) = \frac{120}{21} = \frac{k}{21}$

$\therefore\ k = 120$

36. Let first 5 terms of the G.P. be

$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Reciprocals are

$\frac{r^2}{a}, \frac{r}{a}, \frac{1}{a}, \frac{1}{ar}, \frac{1}{ar^2}$

We are given $\frac{a/r^2 + a/r + a + ar + ar^2}{r^2/a + r/a + 1/a + 1/ar + 1/ar^2} = 49$

$\Rightarrow a^2 = 49 \Rightarrow a = \pm 7$

Also, $\frac{a}{r^2} + a = 35$

As $r^2 > 0$, $a = 7$

and $\frac{1}{r^2} + 1 = 5 \Rightarrow r^2 = \frac{1}{4}$

$\therefore$ first term $= a/r^2 = 28$

37. Two A.P. are

3, 7, 11, 15, 19, 23, 27, 31, …

and 1, 6, 11, 16, 21, 26, 31, …

Common terms are

11, 31, 51, …

Its an A.P. with first term 11, and common ratio 20.

Thus, sum to 20 common terms is

$$\frac{20}{2}[2(11)+(20-1)(20)]=4020$$

38. $G = \sqrt{ab}$,

$$M = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{a+b}{2ab}$$

$$\frac{1/M}{G}=\frac{4}{5} \quad \Rightarrow MG = \frac{5}{4}$$

$$\Rightarrow \frac{a+b}{2\sqrt{ab}}=\frac{5}{4} \quad \Rightarrow \sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}=\frac{5}{2}$$

$$\Rightarrow \sqrt{\frac{a}{b}}=2, \frac{1}{2}$$

$$\Rightarrow \frac{a}{b}=4, \frac{1}{4}$$

$\therefore$ $a : b = 1 : 4$

39. Let S_n denote the sum to n terms of the given geometric series, that is,

$$S_n = 1-\frac{2}{3}-\frac{2}{3^2}-\frac{2}{3^3}\cdots-\frac{2}{3^{n-1}}$$

$$= 1-\frac{(2/3)\left[1-(1/3)^{n-1}\right]}{1-1/3}$$

$$= 1-1+\left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^{n-1}$$

Now, $\left(\frac{1}{3}\right)^{n-1} < \frac{1}{100}$

$\Rightarrow 3^{n-1} > 100 \;\Rightarrow n-1 \geq 5 \quad \Rightarrow n \geq 6$

Thus, least value of n is 6.

40. Let the A.P. be a, $a + d$, ..., $a + (2n - 1)d$

It is given

$$a + (a+2d) + \cdots + (a + (2n-2)d) = 24 \quad (1)$$

and $(a + d) + (a + 3d) + ...+ (a + (2n - 1)d) = 30$ (2)

From (1) and (2), we get

$$\underbrace{d+d+\cdots+d}_{n\text{ times}}=6 \;\Rightarrow nd = 6$$

Also,

$$a + (2n-1)d - a = 10\frac{1}{2}=\frac{21}{2} = \frac{21}{2}$$

$$(2n-1)d = \frac{21}{2}$$

$$\Rightarrow \frac{6(2n-1)}{2n}=\frac{21}{2} \quad \Rightarrow \frac{2n-1}{n}=\frac{7}{4}$$

$\therefore$ number of terms in the A.P. is 8.

41. $\frac{3n}{100}<\frac{2}{3} \quad \Rightarrow 9n < 200$

$\Rightarrow n < 200/9$

$\Rightarrow n \leq 22$

Next,

$\frac{2}{3}\leq\frac{3n}{100}<\frac{5}{3} \quad \Rightarrow 200 \leq 9n < 500$

$\Rightarrow \; 23 \leq n \leq 55.$

$$\therefore \sum_{n=1}^{56} f(n) = \sum_{n=1}^{22} f(n)+\sum_{n=23}^{55} f(n)+f(56)$$

$$= 0+\sum_{n=23}^{55} n+2(56)$$

$$= \frac{33}{2}(23+55)+112=1399$$

42. $m = (l + n)/2$

Also, l, G_1, G_2, G_3, n are in G.P.

Let its common ratio be r, then

$n = lr^4 \Rightarrow r^4 = n/l.$

Now, $G_1^4 + 2G_2^4 + G_3^4$

$= l^4\, [r^4 + 2(r^2)^4 + (r^3)^4]$

$$= l^4\left[\frac{n}{l}+\frac{2n^2}{l^2}+\frac{n^3}{l^3}\right]$$

$= ln\, (l^2 + 2nl + n^2)$

$= ln\, (l + n)^2 = 4lm^2n$

43. $$t_r = \frac{1^3+2^3+\cdots+r^3}{1+3+\cdots+(2r-1)}$$

$$= \frac{r^2(r+1)^2/4}{r^2}$$

$$= \frac{1}{4}(r+1)^2$$

$$\Rightarrow \sum_{r=1}^{9} t_r = \frac{1}{4}\sum_{r=1}^{9}(r+1)^2$$

$$= \frac{1}{4}\left[\sum_{r=1}^{10} r^2-1\right]$$

$$= \frac{1}{4}\left[\frac{1}{6}(10)(10+1)(20+1)-1\right]$$

$= 96$

44. $\sum_{r=16}^{30}(r+2)(r-3)$

$= \sum_{r=1}^{15}(r+15+2)(r+15-3)$

$= \sum_{r=1}^{15}(r+17)(r+12)$

$= \sum_{r=1}^{15}(r^2+29r+204)$

$= \frac{1}{6}(15)(16)(31)+\frac{29}{2}(15)(16)+(204)(15)$

$= 7780$

45. $ar^2 + ar^3 = 60$ (1)

and $(a)(ar)(ar^2) = 1000$ (2)

(2) $\Rightarrow (ar)^3 = 10^3 \Rightarrow ar = 10$

$\therefore$ (1) gives us

$10r + 10r^2 = 60 \Rightarrow r^2 + r - 6 = 0$

$\Rightarrow (r + 3)(r - 2) = 0 \Rightarrow r = -3, 2.$

As $a > 0$, $ar = 10$, therefore, $r = 2$.

Thus, $t_7 = ar^6 = (ar)r^5 = 10(2^5) = 320$

46. $\frac{k}{3} = \sum_{n=1}^{5}\frac{1}{n(n+1)(n+2)(n+3)}$

$= \sum_{n=1}^{5}\left[\frac{1}{6n}-\frac{1}{2(n+1)}+\frac{1}{2(n+2)}-\frac{1}{6(n+3)}\right]$

$= \sum_{n=1}^{5}\left[\frac{1}{6}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{3}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\right.$

$\left.+\frac{1}{6}\left(\frac{1}{n+2}-\frac{1}{n+3}\right)\right]$

$\Rightarrow \frac{k}{3} = \frac{1}{6}\left(1-\frac{1}{6}\right)-\frac{1}{3}\left(\frac{1}{2}-\frac{1}{7}\right)+\frac{1}{6}\left(\frac{1}{3}-\frac{1}{8}\right)$

$\Rightarrow k = \frac{1}{2}\left(\frac{5}{6}\right)-\frac{5}{14}+\frac{1}{2}\left(\frac{5}{24}\right)$

$= \frac{5}{2}\left[\frac{1}{6}-\frac{1}{7}+\frac{1}{24}\right] = \frac{5}{2}\left(\frac{11}{168}\right) = \frac{55}{336}$

47. Let a_n denotes the nth term of the A.P., then

$\frac{a_5}{a_2} = \frac{a_9}{a_5} = r$ (say)

$\Rightarrow \frac{a_5 - a_2}{a_2} = \frac{a_9 - a_5}{a_5}$

$\Rightarrow \frac{3d}{a_2} = \frac{4d}{a_5}$ [d = common difference of the A.P.]

$\Rightarrow r = \frac{a_5}{a_2} = \frac{4}{3}$

48. Note that

$1\frac{3}{5}, 2\frac{2}{5}, 3\frac{1}{5}, 4, 4\frac{4}{5}, \ldots$

forms an A.P. with first term $a = 1\frac{3}{5}$ and common difference $d = \frac{4}{5}$

Sum to 10 terms

$= \sum_{k=1}^{10}[a+(k-1)d]^2$

$= 10a^2 + 2ad(1 + 2 + \ldots + 9) + d^2(1^2 + 2^2 + \ldots + 9^2)$

$= 10a^2 + (ad)(9)(10) + d^2\left(\frac{1}{6}\right)(9)(10)(19)$

$= 10\left(\frac{8}{5}\right)^2 + \left(\frac{8}{5}\right)\left(\frac{4}{5}\right)(9)(10) + \left(\frac{4}{5}\right)^2(15)(19)$

$= \frac{16}{5}[8 + 36 + 57] = \frac{16}{5}(101)$

$\therefore m = 101$

49. For A.P.

$a_r + a_s = 2a_1 + (r + s - 2)d$

$\therefore a_3 + a_{15} + 2a_1 + 16d = 2a_9$

Similarly $a_7 + a_{11} = 2a_9$.

Now, $72 = 4a_9 \Rightarrow a_9 = 18$

Sum to first 17 terms

$= \frac{17}{2}[2a_1 + (17 - 1)d]$

$= 17a_9 = (17)(18) = 306$

Previous Years' B-Architecture Entrance Examination Questions

1. $\frac{1}{16}$, a, b are in G.P.

$\Rightarrow a^2 = b/16$ (1)

Also, as a, b, 1/6 are in H.P.

$$b = \frac{2(a)(1/6)}{a+1/6} = \frac{2a}{6a+1} \qquad (2)$$

From (1) and (2)

$16a^2 = \frac{2a}{6a+1}$

$\Rightarrow 8a\ (6a + 1) = 1$ $\quad [\because\ a \neq 0]$

$\Rightarrow 48a^2 + 8a - 1 = 0$

$\Rightarrow (12a - 1)\ (4a + 1) = 0$

$\Rightarrow a = 1/12,\ a = -1/4$

When $a = 1/12,\ b = 1/9$

When $a = -1/4,\ b = 1$

2. $x = \dfrac{2ab}{a+b}$

Let common ratio of G.P. a, y, z, b be r, then $r^3 = b/a$ and

$y = ar,\ z = b/r$

$\therefore\ yz = ab$

and $y^3 + z^3 = a^3r^3 + \dfrac{b^3}{r^3}$

$= a^3\left(\dfrac{b}{a}\right) + b^3\left(\dfrac{a}{b}\right)$

$= ab\ (a + b)$

$\Rightarrow x(y^3 + z^3) = \dfrac{2ab}{a+b}(ab)(a+b) = 2a^2b^2$

$\therefore\ \dfrac{yz}{x(y^3+z^3)} = \dfrac{ab}{2(ab)^2} = \dfrac{1}{2ab}$

3. $x = 1 + a + a^2 + \cdots = \dfrac{1}{1-a}$

$y = \dfrac{1}{1-b}$

$\Rightarrow a = 1 - 1/x,\ b = 1 - 1/y$

Now, $1 + ab + a^2b^2 + \ldots$

$= \dfrac{1}{1-ab} = \dfrac{1}{1-(1-1/x)(1-1/y)}$

$= \dfrac{xy}{x+y-1}$

4. $3b = a + b + c \geq 3(abc)^{1/3}$

$\Rightarrow b \geq 4^{1/3} = 2^{2/3}$

5. $\displaystyle\sum_{i=1}^{20} i\left(\frac{1}{i}+\frac{1}{i+1}+\frac{1}{i+2}+\ldots+\frac{1}{20}\right)$

$= \left(\dfrac{1}{1}+\dfrac{1}{2}+\cdots+\dfrac{1}{20}\right)$

$+2\left(\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{20}\right)+3\left(\dfrac{1}{3}+\cdots+\dfrac{1}{20}\right) +20\left(\dfrac{1}{20}\right)$

$= (1)(1)+\dfrac{1}{2}(1+2)+\dfrac{1}{3}(1+2+3)+\cdots$

$+\dfrac{1}{20}(1+2+\cdots+20)$

$= \displaystyle\sum_{k=1}^{20}\frac{1}{k}(1+2+\cdots+k) = \frac{1}{2}\sum_{k=1}^{20}(k+1)$

$= \dfrac{1}{2}\cdot\dfrac{1}{2}(20)(2 + 21) = 115$

6. Required numbers are

203, 210, …, 399

Now, $399 = 203 + 7\ (n - 1)$

$\Rightarrow 57 = 29 + (n - 1) \Rightarrow n = 29$

Thus, required sum is

$= \dfrac{29}{2}[203+399] = 8729$

7. $a + b + c = xb$

$\Rightarrow \dfrac{b}{r}+b+br = xb$, where r is common ratio of the G.P and $r \neq 1$

$\Rightarrow r+\dfrac{1}{r} = (x - 1)$

But $r+\dfrac{1}{r} < -2$ or $r+\dfrac{1}{r} > 2$

$\therefore\ x - 1 < -2$ or $x - 1 > 2$

$\Rightarrow x < -1$ or $x > 3$

$\Rightarrow x \in (-\infty, -1) \cup (3, \infty)$

8. Let two A.P.'s be

$a,\ a + d,\ a + 2d, \ldots$

and $A,\ A + D,\ A + 2D, \ldots$

We are given

$$\frac{\frac{n}{2}[2a+(n-1)d]}{\frac{n}{2}[2A+(n-1)D]} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \frac{a+\frac{n-1}{2}d}{A+\frac{n-1}{2}D} = \frac{3n+8}{7n+15}$$

Put $(n - 1)/2 = 11$ or $n = 23$.

$$\therefore\ \frac{a+11d}{A+11D} = \frac{3(23)+8}{7(23)+15} = \frac{77}{176} = \frac{7}{16}$$

9. Let a_n denote the growth of height in the nth year. We have

$a_1 = 50$ cm, $a_r = 50 - 5(r - 1) = 55 - 5r$

The tree ceases to grow in the rth year where $a_r = 0$
$\Rightarrow r = 11$

So height of tree is

$a_1 + a_2 + \cdots + a_9$

$= \frac{10}{2}[50+5]\text{cm} = 275 \text{ cm} = 2.75 \text{ m}$

10. rth term of the series is

$$t_r = \frac{49-r}{(r+1)(r+2)}$$

$$= \frac{50}{r+1} - \frac{51}{r+2}$$

$$= 50\left(\frac{1}{r+1} - \frac{1}{r+2}\right) - \frac{1}{r+2}$$

$$\Rightarrow \sum_{r=1}^{48} t_r = 50\sum_{r=1}^{48}\left(\frac{1}{r+1} - \frac{1}{r+2}\right) - \sum_{r=1}^{48}\left(\frac{1}{r+2}\right)$$

$$= 50\left(\frac{1}{2} - \frac{1}{50}\right) - \sum_{r=1}^{48}\frac{1}{r+2}$$

$$= \frac{51}{2} - \sum_{r=1}^{50}\left(\frac{1}{r}\right)$$

$\therefore K = -1$

11. $2, 2^x - 1, 2^x + 3$ are in G.P.

$\Rightarrow (2^x - 1)^2 = 2(2^x + 3)$

$\Rightarrow 2^{2x} - 2(2^x) + 1 = 2(2^x) + 6$

$\Rightarrow 2^{2x} - 4(2^x) - 5 = 0$

$\Rightarrow (2^x - 5)\ (2x + 1) = 0$

$\Rightarrow 2^x = 5 \quad [\because 2^x + 1 > 0]$

Thus, there is exactly one value of x.

12. Let r be the common ratio of the G.P. b, c, d, then

$b = \frac{c}{r}$, $c = c$, $d = cr$

As c, d, e, are in A.P.

$e = 2d - c = 2cr - c = c(2r - 1)$

Also, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$$\Rightarrow \frac{1}{a} = \frac{2}{b} - \frac{1}{c} = \frac{2r}{c} - \frac{1}{c} = \frac{2r-1}{c}$$

Also, $\frac{2}{b} - \frac{1}{a} = \frac{1}{c}$

$$\Rightarrow \frac{2a-b}{ab} = \frac{1}{c}$$

$$\therefore \frac{ab^2}{(2a-b)^2} = \frac{ab^2}{(ab/c)^2} = \frac{c^2}{a}$$

$= c(2r - 1) = e$

13. Let $S_n = \sum_{k=1}^{n}\phi(k) = \frac{2n}{n+1}$

$$\Rightarrow \phi(n) = S_n - S_{n-1} = \frac{2n}{n+1} - \frac{2(n-1)}{n}$$

$$= \frac{2[n^2 - (n^2 - 1)]}{(n+1)n} = \frac{2}{n(n+1)}$$

$$\Rightarrow \frac{1}{\phi(n)} = \frac{n(n+1)}{2}$$

$$\Rightarrow \sum_{k=1}^{10}\frac{1}{\phi(n)} = \frac{1}{2}\sum_{k=1}^{10}(k^2 + k)$$

$$= \frac{1}{2}\cdot\frac{1}{6}(10)(11)(21) + \frac{1}{2}\cdot\frac{1}{2}(10)(11)$$

$= 220$

14. $2b = a + c$, $\frac{2}{d} = \frac{1}{c} + \frac{1}{e}$, $c^2 = bd$

Now, $\frac{2ce}{c+e} = d = \frac{c^2}{b}$

$$\Rightarrow \frac{e}{c+e} = \frac{c}{2b} = \frac{c}{a+c}$$

$\Rightarrow ae + ce = c^2 + ce$

$\Rightarrow c^2 = ae \Rightarrow a, c, e$ are in G.P.

CHAPTER NINE

Limits and Continuity

LIMITS

Let f be a function defined in some neighbourhood of c, but not necessarily at the point c. We are interested in the behaviour of $f(x)$ when the distance between x and c is small, i.e., $|x - c|$ is small. Specifically, we would like to know whether there is some real number l such that $f(x)$ approaches l as x approaches c. We write

$$\lim_{x \to c} f(x) = l$$

to indicate that the difference between $f(x)$ and l can be made arbitrarily small (smaller than any pre-assigned positive number) simply by requiring that x be sufficiently close to c but different from c.

Take a positive number ε. $\lim_{x \to c} f(x) = l$, if $|f(x) - l| < \varepsilon$ for all x sufficiently close to c but different from it. That is, there exists a positive number δ such that if $0 < |x - c| < \delta$, then $|f(x) - l| < \varepsilon$.

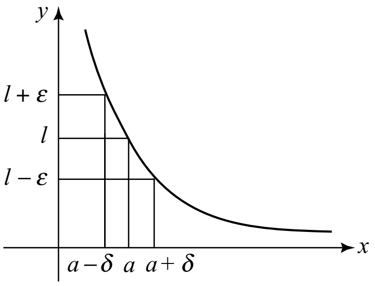


Fig. 9.1

ONE-SIDED LIMITS

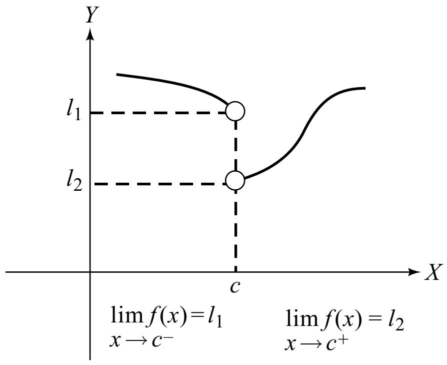
The *left-hand limit* is written as

$$\lim_{x \to c-} f(x) = l$$

If x approaches c from its left, then it crosses over values of the form $c - h$, $h > 0$, the difference between $f(x)$ and l can be made arbitrarily small. More precisely, $\lim\limits_{x \to c-} f(x) = l$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $c - \delta < x < c$, then $|f(x) - l| < \varepsilon$.

Similarly we define, the *right-hand limit* written as

$$\lim_{x \to c+} f(x) = l$$



Y
l_1
l_2
X
c
$\lim\limits_{x \to c^-} f(x) = l_1$
$\lim\limits_{x \to c^+} f(x) = l_2$

Fig. 9.2

If $\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x)$, then the limit of $f(x)$ as $x \to c$ exists and is written as $\lim_{x \to c} f(x)$. The limit of a function $f(x)$ may not exists in any of the following situations:

1. $\lim_{x \to c+} f(x)$ does not exist, i.e., $f(x)$ becomes infinitely small or large as $x \to c+$. For example, as $x \to 0+$, the function $f(x) = 1/x$ becomes infinitely large, as in this case x crosses over values which are positive but sufficiently small.
2. $\lim_{x \to c-} f(x)$ does not exist as for example, in $f(x) = 1/x$ as $x \to 0-$, where x crosses over values which are negative, so that $f(x)$ becomes infinitely small.
3. Both $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist but are unequal. for example, in $f(x) = \begin{cases} 1, & x > c \\ -1, & x < c \end{cases}$ In this case, $\lim_{x \to c+} f(x) = 1$ and $\lim_{x \to c-} f(x) = -1$.

x	$c + .001$	$c + .01$	$c + .1$	$c - .001$	$c - .01$	$c - .1$
$f(x)$	1	1	1	–1	–1	–1

FREQUENTLY USED LIMITS

1. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1 = \lim\limits_{x\to 0} \cos x = \lim\limits_{x\to 0} \dfrac{\tan x}{x}$

$$= \lim_{x\to 0} \frac{\sin^{-1} x}{x} = \lim_{x\to 0} \frac{\tan^{-1} x}{x}.$$

2. $\lim\limits_{x\to 0} (1 + x)^{1/x} = e = \lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$.
3. $\lim\limits_{x\to 0} \dfrac{\log_a (1+x)}{x} = \log_a e \quad (a > 0,\ a \neq 1)$.

   In particular taking $a = e$, $\lim\limits_{x\to 0} \dfrac{\log (1+x)}{x} = 1$.
4. $\lim\limits_{x\to 0} \dfrac{a^x - 1}{x} = \log a \quad (a > 0)$.

   In particular, $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = \log e = 1$.
5. $\lim\limits_{x\to 0} \dfrac{(1+x)^m - 1}{x} = m,\ m \in \mathbf{R}$.
6. $\lim\limits_{x\to\infty} \dfrac{\log x}{x^m} = 0 \quad (m > 0)$.
7. $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$.

SOME THEOREMS ON LIMITS

The calculation of limits is based on the following theorems. If $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ exist, then

1. $\lim\limits_{x\to c} (f(x) \pm g(x)) = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$.
2. $\lim\limits_{x\to c} [kf(x)] = k \lim\limits_{x\to c} f(x)$ where k is a scalar.
3. $\lim\limits_{x\to c} (f(x) \cdot g(x)) = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x)$.
4. $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)}$ (provided $\lim\limits_{x\to c} g(x) \neq 0$).
5. If $f(x) > 0$ and $\lim\limits_{x\to a} f(x) = L$ then $L \geq 0$

For many elementary functions (e.g., polynomial, trigonometric, logarithmic and exponential), we have

$$\lim_{x\to c} f(x) = f\left(\lim_{x\to c} x\right) = f(c),$$

in general, for continuous functions.

COMPUTATION OF LIMITS

Suppose we have find $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$

1. If f, g are Polynomial, Trigonometric, Logarithmic or Exponential function such that $g(a) \neq 0$. Then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)}$ provided $f(a)$ and $g(a)$ are finite.

Illustration 1

Evaluate $\lim\limits_{x\to 2} \dfrac{x^2 + x + 2}{x^3 + 1}$

Since the numerator and denominator are polynomials and if $f(x) = x^2 + x + 2$ and $g(x) = x^3 + 1$ then $f(2) = 4 + 2 + 2 = 8$ and $g(2) = 8 + 1 = 9$, thus

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{f(2)}{g(2)} = \frac{8}{9}.$$

Illustration 2

Evaluate $\lim\limits_{x\to 0} \dfrac{3 + \sin x}{\cos x}$

Since the numerator and denominator are trigonometric function and if $f(x) = 3 + \sin x$ and $g(x) = \cos x$ then $f(0) = 3$ and $g(0) = 1$, thus

$$\lim_{x\to 0} \frac{3 + \sin x}{\cos x} = \frac{3 + 0}{1} = 3$$

2. If $g(a) = 0$ and $f(a) = 0$ and f, g are polynomials then $(x - a)$ is a factor of both f and g. In such a case we factorise the numerator and denominator and cancel out the common factor and then the situation is as above.

Illustration 3

$$\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}$$

If $f(x) = x^3 - 1$ and $g(x) = x^2 - 1$ then $f(1) = g(1) = 0$. Therefore $x - 1$ is a factor of $x^3 - 1$ and $x^2 - 1$, hence

$$\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1} \frac{(x-1)\left(x^2 + x + 1\right)}{(x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{x^2 + x + 1}{x + 1}$$

$$= \frac{1+1+1}{1+1} = \frac{3}{2}.$$

3. If f and g are of the form $\sqrt{p(x)} - \sqrt{q(x)}$ and $\sqrt{r(x)} - \sqrt{s(x)}$ where $p(x)$, $q(x)$, $r(x)$, $s(x)$ are polynomials such that $f(a) = g(a) = 0$ then we use rationalisation of the denominator and numerator, e.g.

Illustration 4

$$\lim_{x\to a}\frac{\sqrt{4a+3x}-\sqrt{x+6a}}{\sqrt{2a+5x}-\sqrt{3a+4x}}$$

$$= \lim_{x\to a}\frac{(2x-2a)\left(\sqrt{2a+5x}+\sqrt{3a+4x}\right)}{(x-a)\left(\sqrt{4a+3x}+\sqrt{x+6a}\right)}$$

(Rationalizing the numerator and denominator)

$$= 2\lim_{x\to a}\frac{\sqrt{2a+5x}+\sqrt{3a+4x}}{\sqrt{4a+3x}+\sqrt{x+6a}} = \frac{2\,(2\sqrt{7a})}{2\sqrt{7a}} = 2.$$

4. $\lim\limits_{x\to 0}\dfrac{a^{f(x)}-1}{g(x)}$, where $f(x)\to 0$ as $x\to 0$

and $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$ exist

then $\lim\limits_{x\to 0}\dfrac{a^{f(x)}-1}{g(x)} = \lim\limits_{x\to 0}\dfrac{a^{f(x)}-1}{f(x)}\cdot\dfrac{f(x)}{g(x)}$

$$= \log a \lim_{x\to 0}\frac{f(x)}{g(x)}$$

Illustration 5

$$\lim_{x\to 0}\frac{a^{\sin x}-1}{x\cos x}$$

$$= \lim_{x\to 0}\frac{a^{\sin x}-1}{\sin x}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = \log a.1.1 = \log a$$

5. $\lim\limits_{x\to 0}(1+f(x))^{1/g(x)}$, where $f(x)\to 0$ as $x\to 0$ and

$\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = l$ exist

then

$$\lim_{x\to 0}(1+f(x))^{1/g(x)} = \lim_{x\to 0}(1+f(x))^{1/f(x).(f(x)/g(x))}$$

$$= e^{l}$$

Illustration 6

To find $\lim\limits_{x\to 0}(1+(x^2+3x))^{1/\sin x}$, write the required limit as

$$\lim_{x\to 0}(1+(x^2+3x)^{[1/(x^2+3x)]\,(x(x+3)/\sin x)}$$

$$= \lim_{x\to 0}(1+(x^2+3x))^{[(1/(x^2+3x)]\,(x(x+3)/\sin x)}$$

$$= \lim_{x\to 0}(1+(x^2+3x))^{(1/(x^2+3x))\,(x/\sin x)\,(x+3)}$$

$$= e^{1.3} = e^{3}$$

L'HÔPITAL'S RULES FOR CALCULATING LIMITS

1. (0/0 *form*) If f and g are differentiable functions on (0, δ) (see the definition of differentiability in the next chapter) such that
   (i) $g'(x)\neq 0$ for any $x\in(0,\delta)$
   (ii) $\lim\limits_{x\to 0+} f(x) = 0 = \lim\limits_{x\to 0+} g(x)$
   (iii) $\lim\limits_{x\to 0+}\dfrac{f'(x)}{g'(x)} = L$, then
   $\lim\limits_{x\to 0+}\dfrac{f(x)}{g(x)} = L.$
2. (L′ Hôpital's rule when $x\to\infty$). If f and g are differentiable functions on $[a,\infty)$ such that
   (i) $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} g(x) = 0$
   (ii) $\lim\limits_{x\to\infty}\left(\dfrac{f'(x)}{g'(x)}\right) = L$, then $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = L.$
3. (∞/∞ *form*). If f and g are differentiable functions on (0, δ) such that
   (i) $g'(x)\neq 0$ for any $x\in(0,\delta)$
   (ii) $f(x)\to\infty$, $g(x)\to\infty$ as $x\to 0+$
   (iii) $\lim\limits_{x\to 0+}\dfrac{f'(x)}{g'(x)} = L$, then $\lim\limits_{x\to 0+}\dfrac{f(x)}{g(x)} = L$

Illustration 7

What is wrong with the following application of L'Hôpital's rule?

$$\lim_{x\to 1}\frac{x^3+3x-4}{2x^2+x-3} = \lim_{x\to 1}\frac{3x^2+3}{4x+1} = \lim_{x\to 1}\frac{6x}{4} = \frac{3}{2}.$$

The expression $\dfrac{3x^2+3}{4x+1}$ is not of $\dfrac{0}{0}$ form as $x\to 1$. Therefore, it is not correct to apply L'Hôpital rule. In fact

$$\lim_{x\to 1}\frac{3x^2+3}{4x+1} = \frac{\lim\limits_{x\to 1}\left(3x^2+3\right)}{\lim\limits_{x\to 1}(4x+1)} = \frac{6}{5}.$$

4. The following illustrations will show the methods of converting 0°, 1^∞, 0^∞, ∞^∞, $\infty-\infty$ etc., to the forms $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty}$.

Illustration 8

$$\lim_{x\to 0+}(\sin x)^{x}$$

It is of the form 0^0. If L is the limit sought, then

$$\log L = \lim_{x\to 0+} x \log \sin x = \lim_{x\to 0+} \frac{\log \sin x}{1/x}\left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0+} \frac{-\cos x}{\sin x\left(\frac{-1}{x^2}\right)} = \lim_{x\to 0} \frac{x\cos x}{(\sin x)/x} = \frac{0}{1} = 0$$

Thus $L = 1$

Illustration 9

$$\lim_{x\to 0} (1 - 3x)^{4/x} \quad (1^{\infty} \text{ form})$$

If L is the required limit then

$$\log L = \lim_{x\to 0} \frac{4\log(1-3x)}{x}\left(\frac{0}{0}\right) = \lim_{x\to 0} \frac{4(-3)}{1-3x} = -12$$

Thus $L = e^{-12}$

USE OF SERIES EXPANSION IN FINDING LIMITS

The following series expansions are useful in evaluating limits

1. $(1 + x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$ (for $|x| < 1$ and $p \in \mathbf{Q}$, the set of rational numbers)
2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
3. $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots$
4. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (|x| < 1)$
5. $\log(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) \quad (|x| < 1)$
6. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
7. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$
8. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$
9. $\sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{x^5}{5} + \cdots \quad (-1 < x < 1)$
10. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad (-1 < x < 1)$

Illustration 10

$$\lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x}$$

Let $y = (1 + x)^{1/x}$

$$\Rightarrow \log y = \frac{1}{x}\log(1 + x) = \frac{1}{x}\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]$$

$$= \left[1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right]$$

$$\Rightarrow y = e^{\left[1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right]} = e\cdot e^{-\frac{x}{2} + \frac{x^2}{3} - \cdots}$$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \cdots\right)^2 + \cdots\right]$$

Thus $\frac{y-e}{x}$

$$= e\left[-\frac{1}{2} + \frac{x}{3} + \text{terms containing } x^2 \text{ and higher powers of } x\right]$$

$$\therefore \lim_{x\to 0} \frac{(1+x)^{1/x} - e}{x} = -\frac{e}{2}.$$

SEQUENTIAL LIMITS

A sequence is a function whose domain is the set of natural numbers. If f: $\mathbf{N} \to \boldsymbol{R}$ then we write $f(1) = f_1, f(2) = f_2 \cdots$, $f(n) = f_n, \cdots$ A sequence which has finite limit is said to be convergent. Thus $\lim_{n\to\infty} a_n = a$ if for any $\in\, > 0$ there exists a number N(depending on $\in$) > 0 such that $|a_n - a| < \,\in$ holds true for all $n > N$.

Illustration 11

$\lim_{n\to\infty} \frac{1}{n} = 0$, $\lim_{n\to\infty} \frac{1}{n^2} = 0$ so $\left\langle\frac{1}{n}\right\rangle, \left\langle\frac{1}{n^2}\right\rangle$ are convergent sequences. If $x_n = \frac{2n-1}{2n+1}$ then $\lim_{n\to\infty} x_n = 1$.

$\lim_{x\to a} f(x) = l$ if and only if for every sequence x_n (of values of x) such that $\lim_{n\to\infty} x_n = a$ we have $\lim_{n\to\infty} f(x_n) = l$.

The above criterion is very useful in showing $\lim_{x\to a} f(x)$ (for a given f) does not exist. It is enough to give a sequence x_n with $\lim\limits_{n\to\infty} x_n = a$ such that $\lim\limits_{n\to\infty} f(x_n)$ does not exist. Alternatively, give two sequences x_n and x'_n such that $\lim\limits_{n\to\infty} x_n = a = \lim\limits_{n\to\infty} x'_n$ but $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(x'_n)$.

Illustration 12

To show that $\lim\limits_{x\to 1} \sin\dfrac{1}{x-1}$ does not exist let $f(x) = \sin\dfrac{1}{(x-1)}$ and choose $x_n = 1+\dfrac{1}{n\pi}$, $x'_n = 1+\dfrac{2}{(4n+1)\pi}$, so $\lim\limits_{n\to\infty} x_n = 1 = \lim\limits_{n\to\infty} x'_n$. Now $f(x_n) = \sin n\pi = 0$ so $\lim\limits_{n\to\infty} f(x_n) = 0$. $f(x_n) = \sin\dfrac{4n+1}{2}\pi = 1$. Hence $\lim\limits_{n\to\infty} f(x'_n) = 1$. Thus $\lim\limits_{n\to\infty} f(x)$ does not exist.

SOME USEFUL RESULTS ON LIMITS

1. If $\lim\limits_{x\to c} f(x) = l$ with $l \neq 0$ and $\lim\limits_{x\to c} g(x) = 0$ then $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$ does not exist.
2. (Sandwich theorem) If f, g and h are functions having a common domain D and defined near a. Suppose that $h(x) \leq f(x) \leq g(x)$ for $x \in D$ and $\lim\limits_{x\to a} h(x) = \lim\limits_{x\to a} g(x) = l$ then $\lim\limits_{x\to a} f(x) = l$.
3. $\lim\limits_{x\to 0} e^x = 1$, $\lim\limits_{x\to\infty} e^x = \infty$, $\lim\limits_{x\to\infty} e^{-x} = 0$,

   $\lim\limits_{x\to-\infty} e^x = 0$, $\lim\limits_{x\to a}\dfrac{e^x - e^a}{x-a} = e^a$.

Illustration 13

To show $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$.

For values of x near 0, $0 \leq x\sin\dfrac{1}{x} \leq x$ and $\lim\limits_{x\to 0} x = 0$ so by Sandwich theorem $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$.

Illustration 14

Find $\lim\limits_{x\to 0+}\dfrac{[x]}{x}$

For $x \in (0, 1)$, $[x] = 0$ so $\dfrac{[x]}{x} = 0$

Hence $\lim\limits_{x\to 0+}\dfrac{[x]}{x} = 0$.

CONTINUITY

In ordinary language, to say that a certain process is continuous implies that it goes on without interruption and without abrupt changes. In mathematics, the word continuous has much the same meaning. We shall first introduce the idea of continuity at a point c, and then that of continuity on an interval.

The function f is said to *continuous* at $x = c$ if $\lim_{x\to c} f(x) = f(c)$ or, equivalently, $\lim_{h\to 0} f(c+h) = f(c)$.

The function f is said to be *continuous on an open interval* (a, b) if it is continuous at each point of (a, b).

The function f is said to be *continuous on a closed interval* $[a, b]$ if

(i) f is continuous at each point of (a, b),

(ii) $\lim\limits_{x\to a+} f(x) = f(a)$, (i.e., right hand side continuity at a)

(iii) $\lim\limits_{x\to b-} f(x) = f(b)$. (i.e., left hand side continuity at b)

Graphically, $f(x)$ is continuous if its graph can be drawn on paper without raising the pencil. A function f is said to be *discontinuous* at c if it is not continuous at that point. A function f is said to be *discontinuous on an interval* if it is discontinuous at least at one point of the interval. A function f is discontinuous at c under any of the following circumstances:

(i) f is not defined at c (e.g., $f(x) = 1/x$ at $x = 0$).
(ii) $\lim_{x\to c} f(x)$ does not exist.
(iii) f is defined at c and $\lim_{x\to c} f(x)$ exists but $\lim_{x\to c} f(x) \neq f(c)$. For example.

$$f(x) = \begin{cases} \dfrac{x^3-1}{x-1} & x \neq 1 \\ 4 & x = 1 \end{cases}$$

But

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x^2+x+1) = 3 \neq f(1)$$

One can prove that if f and g are continuous at $x = c$ and α is a scalar, then $f + g$, αf, fg, max (f, g) and min (f, g) are continuous at c. Moreover, if $g(c) \neq 0$, then the quotient f/g is also continuous at c.

If g is continuous at c and f is continuous at $g(c)$, then the composition $f\,o\,g$ is continuous at c.

All elementary functions (e.g., polynomial, trigonometric, logarithmic, exponential and inverse trigonometric functions) are continuous at each point of their domains.

Illustration 15

Examine the continuity at origin

$$f(x) = \begin{cases} \dfrac{|x|}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

$$\lim_{x\to 0+} \frac{|x|}{x} = \lim_{x\to 0+} \frac{x}{x} = 1 \text{ but } \lim_{x\to 0-} \frac{|x|}{x}$$

$$= -\lim_{x\to 0-} \frac{x}{x} = -1. \text{ So } f \text{ is not continuous at } x = 0.$$

FUNCTIONS CONTINUOUS ON A CLOSED INTERVAL

A function $f(x)$, continuous on the interval $[a, b]$, possesses the following properties:

1. $f(x)$ is bounded on $[a, b]$, i.e., there exist $k, K \in \boldsymbol{R}$ such that $k \le f(x) \le K$ for all $x \in [a, b]$.
2. $f(x)$ has minimum and maximum values on $[a, b]$.
3. *Intermediate Value Theorem* If $m = \min f(x)$ and $M = \max f(x)$ for $a \le x \le b$, then for any A satisfying the inequalities $m \le A \le M$, there exists a point $x_0 \in [a, b]$ for which $f(x_0) = A$, i.e., *a* continuous function assumes every value between any two of its values.
4. If f is monotonically increasing on an open interval I and $x_0 \in I$ then

   $\lim\limits_{x\to x_0+} f(x)$ and $\lim\limits_{x\to x_0-} f(x)$ are finite and

   $\lim\limits_{x\to x_0-} f(x) \le f(x_0) \le \lim\limits_{x\to x_0+} f(x)$.
5. If f is monotonically increasing and continuous on an interval I then f^{-1} exist and is continuous and monotonically increasing.
6. Let f be a function which is continuous at c and $f(x) \neq 0$. Then there exists an interval $(c - \delta, c + \delta)$ throughout which $f(x)$ is of the same sign.
7. Let f be a continuous function on $[a, b]$. If $f(a)$ and $f(b)$ are of opposite sign, then there exists a point $c \in (a, b)$ such that $f(c) = 0$

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: Let $f(x) = \begin{cases} \dfrac{x - |x|}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$ then

(a) $\lim\limits_{x\to 0+} f(x) = 1$

(b) $\lim\limits_{x\to 0-} f(x) = 0$

(c) $\lim\limits_{x\to 0+} f(x) \neq \lim\limits_{x\to 0-} f(x)$

(d) $\lim\limits_{x\to 0+} f(x)$ does not exist

Ans. (c)

Solution: If $x > 0$, $|x| = x$, so

$$\lim_{x\to 0+} \frac{x - |x|}{x} = \lim_{x\to 0+} \frac{x - x}{x} = 0$$

for $x < 0$, $|x| = -x$, so

$$\lim_{x\to 0-} \frac{x - |x|}{x} = \lim_{x\to 0-} \frac{x + x}{x} = 2$$

Thus $\lim\limits_{x\to 0+} f(x) \neq \lim\limits_{x\to 0-} f(x)$

Example 2: Let $f(x) = \left[\dfrac{\sin x}{x}\right]$, $x \neq 0$, where [.] denotes the greatest integer function then $\lim\limits_{x\to 0} f(x)$

(a) does not exist
(b) is equal to 1
(c) is equal to 0
(d) $\lim\limits_{x\to 0+} f(x) = 1$

Ans. (c)

Solution: For $x \neq 0$ $\left|\dfrac{\sin x}{x}\right| < 1$. For $-1 < x < 0$, $\dfrac{\sin x}{x} > 0$, so $\dfrac{\sin x}{x}$ tends to 1 as $x \to 0$ but through values which are less than 1. Hence $\left[\dfrac{\sin x}{x}\right] = 0$ as x tends to 0.

Thus $\lim\limits_{x\to 0} f(x) = 0$.

Example 3: $\lim\limits_{x\to\infty} \dfrac{(x+1)^{10} + (x+2)^{10} + \cdots + (x+100)^{10}}{x^{10} + 10^{10}}$ is equal to

(a) 100
(b) 0
(c) 1
(d) 10

Ans. (a)

Solution: $f(x) = \dfrac{(x+1)^{10} + (x+2)^{10} + \cdots + (x+100)^{10}}{x^{10} + 10^{10}}$

$$= \frac{\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+\cdots+\left(1+\frac{100}{x}\right)^{10}}{1+\left(\frac{10}{x}\right)^{10}}$$

As $\frac{1}{x} \to 0, \text{for } x \to \infty$ so $\lim\limits_{x\to\infty} f(x) = \frac{1+1\cdots+1-(100 \text{ times})}{1}$

$$= 100.$$

Example 4: $\lim\limits_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$ is equal

(a) 1 (b) –1

(c) 2 (d) $-\frac{1}{2}$

Ans. (b)

Solution: For $x \neq 1, f(x) = \frac{1}{1-x} - \frac{3}{1-x^3}$

$$= \frac{1}{(1-x)}\left[1-\frac{3}{1+x+x^2}\right]$$

$$= \frac{1}{(1-x)} \cdot \frac{x^2+x-2}{\left(1+x+x^2\right)} = \frac{(x+2)(x-1)}{(1-x)\left(1+x+x^2\right)}$$

$$= -\frac{x+2}{1+x+x^2}$$

so $\lim\limits_{x\to 1} f(x) = -\lim\limits_{x\to 1}\frac{x+2}{1+x+x^2} = -1$.

Example 5: $\lim\limits_{x\to\pi}\frac{\sin 3x}{\sin 2x}$ is equal to

(a) $\frac{3}{2}$ (b) 0

(c) 1 (d) $-\frac{3}{2}$

Ans. (d)

Solution: $\lim\limits_{x\to\pi}\frac{\sin 3x}{\sin 2x} = \lim\limits_{u\to 0}\frac{\sin 3(\pi-u)}{\sin 2(\pi-u)}$ $(u = x - \pi)$

$$= \lim_{u\to 0}\frac{\sin 3u}{-\sin 2u} = -\frac{3}{2}\lim \frac{\sin 3u}{3u} \times \frac{2u}{\sin 2u}$$

$$= -\frac{3}{2}.$$

Example 6: If $f(x) = \begin{cases} mx^2+x+n & , \quad x<0 \\ nx+m & , \quad 0\le x\le 1 \\ 2nx^3+x^2-2x+m & , \quad x>1 \end{cases}$

and $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 1} f(x)$ exist then

(a) $m = 2, n = 1$ (b) $m = 0, n = 1$

(c) $m = 1, n = 2$ (d) $m = 1, n = 1$

Ans. (d)

Solution: $\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0+}(nx+m) = m$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-}\left(mx^2+x+n\right) = n$$

Hence $m = n$.

Also $\lim\limits_{x\to 1-} f(x) = \lim\limits_{x\to 1-}(nx+m) = n+m$

and $\lim\limits_{x\to 1+} f(x) = \lim\limits_{x\to 1+}\left(2nx^3+x^2-2x+m\right)$

$$= 2n+1-2+m = 2n+m-1$$

Hence $n+m = 2n+m-1 \Rightarrow n = 1$ so $m = 1$

Example 7: If $f(x) = \begin{cases} -2\sin x & \text{if} \quad x \le -\frac{\pi}{2} \\ A\sin x + B & \text{if} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x & \text{if} \quad x \ge \frac{\pi}{2} \end{cases}$

is a continuous function then

(a) $A = 1, B = -1$ (b) $A = -1, B = 1$

(c) $A = 0, B = 2$ (d) $A = -2, B = 2$

Ans. (b)

Solution: $f\left(-\frac{\pi}{2}\right) = -2\sin\left(-\frac{\pi}{2}\right) = 2$

$$\lim_{x\to -\pi/2+} f(x) = \lim_{x\to -\pi/2}(A\sin x + B) = -A+B$$

so $2 = -A + B$ … (i)

Also $f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$$\lim_{x\to\frac{\pi}{2}-} f(x) = \lim_{x\to\pi/2-}(A\sin x + B) = A+B$$

Hence A + B = 0 (ii)

Solving (i) and (ii), we get $A = -1, B = 1$.

Example 8: The function $f(x) = \frac{x^2-1}{x^3-1}$ is not defined for $x = 1$. The value of $f(1)$ so that the function extended by this value is continuous is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$

(c) 1 (d) 0

Ans. (a)

Solution: For f to be continuous at $x = 1$,

$$f(1) = \lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{x^2-1}{x^3-1}$$

$$= \lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)\left(x^2+x+1\right)}$$

$$= \lim_{x\to 1}\frac{x+1}{x^2+x+1} = \frac{2}{3}.$$

Example 9: Let $y = -\dfrac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1}$, then

(a) $\lim\limits_{x\to 0} y = -1$ (b) $\lim\limits_{x\to 0} y = 1$

(c) $\lim\limits_{x\to 0+} y = -1$ (d) $\lim\limits_{x\to 0+} y = \lim\limits_{x\to 0-} y$

Ans. (c)

Solution: As $x \to 0+$, $\dfrac{1}{x} \to \infty$ so $2^{-\frac{1}{x}} \to 0$ Hence

$$\lim_{x\to 0+} f(x) = -\lim_{x\to 0+} \frac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1} = -\lim_{x\to 0+} \frac{1-2^{-\frac{1}{x}}}{1+2^{-\frac{1}{x}}} = -1$$

As $x \to 0-, \dfrac{1}{x} \to -\infty$ so $2^{\frac{1}{x}} \to 0$

$$\lim_{x\to 0-} f(x) = -\lim_{x\to 0-} \frac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1} = -(-1) = 1.$$

Example 10: The value of $\lim\limits_{x\to\infty} \dfrac{x^6}{6^x}$ is

(a) 1 (b) 0

(c) – 1 (d) not a finite number

Ans. (b)

Solution: $\lim\limits_{x\to\infty} \dfrac{x^6}{6^x}\left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x\to\infty} \frac{6x^5}{6^x(\log 6)}\left(\frac{\infty}{\infty}\right) \text{ (L' Hopital Rule)}$$

$$= \lim_{x\to\infty} \frac{6.5.4.3.2.1}{6^x(\log 6)^6} = 0.$$

LEVEL 1

Straight Objective Type Questions

Example 11: $\lim\limits_{x\to\infty}\left(\dfrac{x+1}{x+2}\right)^{2x+1}$ is

(a) e (b) e^{-2}

(c) e^{-1} (d) 1

Ans. (b)

Solution:

$$\lim_{x\to\infty}\left(\frac{x+1}{x+2}\right)^{2x+1} = \left[\lim_{x\to\infty}\left(1-\frac{1}{x+2}\right)^{-(x+2)}\right]^{-\frac{2x+1}{x+2}}$$

$$= \left[\lim_{x\to\infty}\left(1-\frac{1}{x+2}\right)^{-(x+2)}\right]^{-\frac{2+1/x}{1+2/x}} = e^{-2}$$

Example 12: $\lim\limits_{x\to 0} \dfrac{(\cos x)^{1/2}-(\cos x)^{1/3}}{\sin^2 x}$ is

(a) 1/6 (b) – 1/12

(c) 2/3 (d) 1/3

Ans. (b)

Solution: $\lim\limits_{x\to 0} \dfrac{(\cos x)^{1/2}-(\cos x)^{1/3}}{\sin^2 x}\left(\dfrac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{-\frac{1}{2}(\cos x)^{-1/2}\sin x + \frac{1}{3}(\cos x)^{-2/3}\sin x}{2\sin x\cos x}$$

(using L'Hôpital Rule)

$$= \lim_{x\to 0} \frac{-\frac{1}{2}(\cos x)^{-1/2} + \frac{1}{3}(\cos x)^{-2/3}}{2\cos x}$$

$$= -\frac{1}{2}\left[\frac{1}{2}-\frac{1}{3}\right] = -\frac{1}{12}.$$

Example 13: $\lim\limits_{x\to 1} \dfrac{1+\log x - x}{1-2x+x^2}$ equals

(a) 1 (b) 0

(c) – 1 (d) – 1/2

Ans. (d)

Solution: $\lim\limits_{x\to 1} \dfrac{1+\log x - x}{1-2x+x^2} = \lim\limits_{x\to 1} \dfrac{1/x-1}{-2+2x}$

(using L'Hôpital Rule)

$$= \lim_{x\to 1} -\frac{1}{2}\frac{(1-x)}{x(1-x)} = -\frac{1}{2}\lim_{x\to 1}\frac{1}{x} = -\frac{1}{2}$$

Example 14: $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^2 \tan x}$ equals to

(a) 1/3 (b) 2/3

(c) 1/2 (d) 1

Ans. (a)

Solution: $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^2\tan x} = \lim\limits_{x\to 0} \dfrac{\sin x - x\cos x}{x^2 \sin x}\left(\dfrac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{\cos x - \cos x + x\sin x}{2x\sin x + x^2\cos x}$$

(using L'Hôpital Rule)

$$= \lim_{x\to 0}\frac{\sin x}{2\sin x + x\cos x} = \lim_{x\to 0}\frac{(\sin x)/x}{(2(\sin x)/x) + \cos x} = \frac{1}{3}$$

Example 15: $\lim_{n\to\infty}\left[\frac{1}{1-n^4} + \frac{8}{1-n^4} + \cdots + \frac{n^3}{1-n^4}\right]$ is

(a) 1/4 (b) 1/8
(c) 1/2 (d) none of these

Ans. (d)

Solution: $\lim_{n\to\infty}\left[\frac{1}{1-n^4} + \frac{8}{1-n^4} + \cdots + \frac{n^3}{1-n^4}\right]$

$$= \lim_{n\to\infty}\frac{1}{1-n^4}\,[1^3 + 2^3 + \cdots + n^3]$$

$$= \lim_{n\to\infty}\frac{1}{4}\frac{(n(n+1))^2}{1-n^4} = \frac{1}{4}\lim_{n\to\infty}\frac{(1+1/n)^2}{1/n^4 - 1} = -\frac{1}{4}$$

Example 16: $\lim_{x\to\pi/3}\frac{2\sin(x-\pi/3)}{1-2\cos x}$ is

(a) $1/\sqrt{2}$ (b) $2/\sqrt{3}$
(c) 2/3 (d) 1/3

Ans. (b)

Solution: $\lim_{x\to\pi/3}\frac{2\sin(x-\pi/3)}{1-2\cos x}\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to\pi/3}\frac{2\cos(x-\pi/3)}{2\sin x}$$

[using L'Hôpital Rule]

$= \cos 0/\sin(\pi/3) = 2/\sqrt{3}$

Example 17: $\lim_{x\to\pi/4}\frac{1-\cot^3 x}{2-\cot x-\cot^3 x}$ is

(a) 11/4 (b) 3/4
(c) 1/2 (d) none of these

Ans. (b)

Solution: $\lim_{x\to\pi/4}\frac{1-\cot^3 x}{2-\cot x-\cot^3 x}$

$$= \lim_{x\to\pi/4}\frac{(1-\cot x)\left(1+\cot x+\cot^2 x\right)}{(1-\cot x)\left(\cot^2 x+\cot x+2\right)}$$

$$= \lim_{x\to\pi/4}\frac{1+\cot x+\cot^2 x}{\cot^2 x+\cot x+2} = \frac{3}{4}$$

Example 18:

$$\lim_{x\to 1}\left[\left(\frac{4}{x^2-x^{-1}} - \frac{1-3x+x^2}{1-x^3}\right)^{-1} + 3\,\frac{x^4-1}{x^3-x^{-1}}\right] \text{ is}$$

(a) 3 (b) 2
(c) 4 (d) 28/3

Ans. (a)

Solution:

$$\lim_{x\to 1}\left[\left(\frac{4}{x^2-x^{-1}} - \frac{1-3x+x^2}{1-x^3}\right)^{-1} + 3\,\frac{x^4-1}{x^3-x^{-1}}\right]$$

$$= \lim_{x\to 1}\left[\left(\frac{4x}{x^3-1} - \frac{1-3x+x^2}{1-x^3}\right)^{-1} + \frac{3x\left(x^4-1\right)}{x^4-1}\right]$$

$$= \lim_{x\to 1}\left[\left(\frac{4x+1-3x+x^2}{x^3-1}\right)^{-1} + 3x\right]$$

$$= \lim_{x\to 1}[x - 1 + 3x] = 3$$

Example 19: $\lim_{x\to 3}\frac{\sqrt{1-\cos 2(x-3)}}{x-3}$

(a) $= \sqrt{2}$ (b) does not exist
(c) = 1 (d) $= -\sqrt{2}$

Ans. (b)

Solution: $1 - \cos 2(x-3) = 2\sin^2(x-3)$

ϕι $\sqrt{1-\cos 2(x-3)} = \sqrt{2}\,|\sin(x-3)|$

So, $\lim_{x\to 3+}\frac{\sqrt{1-\cos 2(x-3)}}{x-3} = \sqrt{2}\lim_{x\to 3+}\frac{\sin(x-3)}{x-3} = \sqrt{2}$

whereas

$$\lim_{x\to 3-}\frac{\sqrt{1-\cos 2(x-3)}}{x-3} = \sqrt{2}\lim_{x\to 3-}\frac{-\sin(x-3)}{x-3} = -\sqrt{2}$$

Example 20: Let $f : \mathbf{R} \to [0, \infty)$ be such that $\lim_{x\to 5} f(x)$ exists and $\lim_{x\to 5}\frac{(f(x))^2-9}{\sqrt{|x-5|}} = 0$

Then $\lim_{x\to 5} f(x)$ equals

(a) 0 (b) 1
(c) 2 (d) 3

Ans. (d)

Solution: As $\lim_{x\to 5}\frac{(f(x))^2-9}{\sqrt{|x-5|}} = 0$, so

$(f(x))^2 - 9 = K|x-5|^{\alpha}P(x)$, where $\alpha > 1/2$ and $P(x)$ is a function of x such that $\lim_{x\to 5} P(x)$ exist

ϕι $\lim_{x\to 5}(f(x))^2 = \lim_{x\to 5}\left[K|x-5|^{\alpha}P(x) + 9\right] = 9$

The only possible answer is $\lim_{x\to 5} f(x) = 3$

Example 21: If $f(x) = \begin{cases} \dfrac{\sin(1+[x])}{[x]} & \text{for } [x] \neq 0 \\ 0 & \text{for } [x] = 0 \end{cases}$ where $[x]$ denotes the greatest integer $\leq x$, then $\lim\limits_{x \to 0-} f(x)$ equals

(a) 1 (b) 0
(c) – 1 (d) none of these

Ans. (b)

Solution: For $-1 < x < 0$, $[x] = -1$, so

$$\lim_{x \to 0-} \frac{\sin(1+[x])}{[x]} = \frac{\sin 0}{-1} = 0.$$

Example 22: Let $f(x) = \begin{cases} x\left(\sin(1/x) + \sin\left(1/x^2\right)\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

then $\lim\limits_{x \to \infty} f(x)$ equals

(a) 0 (b) – 1/2
(c) 1 (d) none of these

Ans. (c)

Solution: $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \left[\dfrac{\sin(1/x)}{1/x} + \dfrac{1}{x} \cdot \dfrac{\sin\left(1/x^2\right)}{1/x^2}\right]$

$$= \left(\lim_{u \to 0} \frac{\sin u}{u} + \lim_{u \to 0} u \frac{\sin u^2}{u^2}\right)\left(u = \frac{1}{x}\right)$$

$$= 1 + 0.1 = 1.$$

Example 23: The value of

$$\lim_{n \to \infty} \left(\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \text{up to } n \text{ terms}\right) \text{ is}$$

(a) 1/4 (b) 1/2
(c) 1 (d) none of these

Ans. (b)

Solution:

$$L = \lim_{n \to \infty} \frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right]$$

$$= \lim_{n \to \infty} \frac{1}{2}\left[1 - \frac{1}{2n+1}\right] = \frac{1}{2}$$

Example 24: If $\lim\limits_{x \to 0} (1 + ax)^{b/x} = e^4$, where a and b are natural numbers then

(a) $a = 4, b = 2$ (b) $a = 8, b = 4$
(c) $a = 16, b = 8$ (d) none of these

Ans. (d)

Solution: $e^4 = \lim\limits_{x \to 0} (1 + ax)^{b/x}$

$$= \lim_{x \to 0} \left[(1 + ax)^{(1/ax)}\right]^{ab} = e^{ab}$$

$\Rightarrow$ $ab = 4$. But none of the values in (a), (b), (c) satisfy $ab = 4$.

Example 25: $\lim\limits_{x \to 0} \dfrac{a^x - 1}{\sqrt{a+x} - \sqrt{a}}$ is

(a) $2\sqrt{a}\ \log a$ (b) $\sqrt{a}\ \log a$
(c) $\log a$ (d) none of these

Ans. (a)

Solution: $= \lim\limits_{x \to 0} \dfrac{a^x - 1}{\sqrt{a+x} - \sqrt{a}}$

$$= \lim_{x \to 0} \frac{a^x \log a}{\frac{1}{2} \cdot \frac{1}{\sqrt{a+x}}} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} 2\, a^x \log a \left(\sqrt{a+x}\right) = 2\sqrt{a}\ \log a.$$

Example 26: Let $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$.

The value of $\lim\limits_{h \to 0} \dfrac{f(1-h) - f(1)}{h^3 + 3h}$ is

(a) 50/3 (b) 22/3
(c) 13 (d) none of these

Ans. (d)

Solution: $= \lim\limits_{h \to 0} \dfrac{f(1-h) - f(1)}{h^3 + 3h}$

$$= \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} \cdot \frac{-1}{h^2 + 3}$$

$$= f'(1)\,(-1/3) = 53/3.$$

Example 27: $\lim\limits_{x \to 0} \dfrac{e^{x^2} - \cos x}{x^2}$ is equal to

(a) 1/2 (b) 3/2
(c) 2/3 (d) 2

Ans. (b)

Solution:

$$\lim_{x \to 0} \frac{e^{x^2} - 1 + 1 - \cos x}{x^2} = \lim_{x \to 0}\left(\frac{e^{x^2} - 1}{x^2} + \frac{2\sin^2 x/2}{4(x/2)^2}\right)$$

$$= 1 + \frac{1}{2} = \frac{3}{2}$$

Example 28: Let $f(x) = \langle x \rangle^*$, where $\langle x \rangle^*$ is the distance from x to the integer nearest to x then $\lim\limits_{x \to 2} f(x)$ is

(a) 2 (b) 1
(c) 0 (d) none of these

Ans. (c)

◎ **Solution:** $\lim\limits_{x\to 2+} f(x) = \lim\limits_{h\to 0+} <2+h>^{*} = \lim\limits_{h\to 0+} h = 0$

and $\lim\limits_{x\to 2-} f(x) = \lim\limits_{h\to 0+} <2-h>^{*} = \lim\limits_{h\to 0+} h = 0.$

Hence $\lim\limits_{x\to 2} f(x) = 0.$

◉ **Example 29:** $\lim\limits_{x\to 0} \dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$ is equal to

(a) 1/2 (b) 1
(c) 2 (d) – 1/4

Ans. (c)

◎ **Solution:** For $x \neq 0$, $\dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$

$$= \frac{2\dfrac{\sin^2 x}{x^2}\times(3+\cos x)}{4\dfrac{\tan 4x}{4x}}$$

Therefore, the required limit $= \dfrac{2\times 1\times 4}{4\times 1} = 2$

◉ **Example 30:** $\lim\limits_{x\to 0} \dfrac{\sin(\pi\cos^2 x)}{x^2}$ is equal to

(a) $-\pi$ (b) π
(c) $\pi/2$ (d) 1

Ans. (b)

◎ **Solution:** $\lim\limits_{x\to 0} \dfrac{\sin(\pi\cos^2 x)}{x^2}\left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{-\cos(\pi\cos^2 x)\,2\pi\cos x\sin x}{2x}$$

$$= -\pi \lim_{x\to 0}\cos(\pi\cos^2 x)\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\cos x$$

$$= \pi$$

◉ **Example 31:** Let $f(x) = \text{sgn}(\text{sgn}(\text{sgn}\, x))$. Then $\lim\limits_{x\to 0} f(x)$ is

(a) 1 (b) 2
(c) 0 (d) none of these

Ans. (d)

◎ **Solution:** By definition we have for $x \neq 0$, sgn (sgn x) = sgn $\left(\dfrac{x}{|x|}\right) = \dfrac{x/|x|}{|x|/|x|} = \dfrac{x}{|x|}$ = sgn x. Thus, sgn [sgn) (sgn x)] =

$$\text{sgn}\, x = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$

Therefore, $\lim\limits_{x\to 0+} f(x) = 1$ but $\lim\limits_{x\to 0-} f(x) = -1$.

◉ **Example 32:** $\lim\limits_{x\to a-}\left(\dfrac{|x|^3}{a} - \left[\dfrac{x}{a}\right]^3\right)$ $(a > 0)$, where $[x]$ denotes the greatest integer less than or equal to x is

(a) $a^2 - 3$ (b) $a^2 - 1$
(c) a^2 (d) none of these

Ans. (c)

◎ **Solution:** For, $0 < x < a$, $\left[\dfrac{x}{a}\right] = 0$,

So $\lim\limits_{x\to a-}\left(\dfrac{|x|^3}{a} - \left[\dfrac{x}{a}\right]^3\right) = \dfrac{a^3}{a} = a^2$.

◉ **Example 33:** $\lim\limits_{x\to 0}\left\{\dfrac{1+\tan x}{1+\sin x}\right\}^{\text{cosec}\, x}$ is equal to

(a) e (b) e^{-1}
(c) 1 (d) none of these

Ans. (c)

◎ **Solution:** $\lim\limits_{x\to 0}\left\{\dfrac{1+\tan x}{1+\sin x}\right\}^{\text{cosec}\, x}$

$$= \lim_{x\to 0}\{1+\tan x\}^{\cot x\sec x}\left\{\frac{1}{1+\sin x}\right\}^{\text{cosec}\, x}.$$

$$= \lim_{x\to 0}\{1+\tan x\}^{\cot x\cdot\sec x}\cdot\frac{1}{(1+\sin x)^{\text{cosec}\, x}}$$

$$= e^1\cdot\frac{1}{e} = 1.$$

◉ **Example 34:** $\lim\limits_{n\to\infty}(6^n + 5^n)^{1/n}$ is equal to

(a) 6 (b) 5
(c) 5/6 (d) e

Ans. (a)

◎ **Solution:** $\lim\limits_{n\to\infty}(6^n+5^n)^{1/n} = 6\lim\limits_{n\to\infty}(1+(5/6)^n)^{1/n}$

$$= 6\lim_{n\to\infty}\left(1+(5/6)^n\right)^{\left[(6/5)^n.(1/n)(5/6)^n\right]}$$

$$= 6e^0 = 6.$$

◉ **Example 35:** If $f(x) = \left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^x$ then $\lim\limits_{x\to\infty} f(x)$ is

(a) e^{-4} (b) e^3
(c) e^2 (d) e^4

Ans. (d)

◎ **Solution:** $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^x$

$$= \lim_{x\to\infty}\left(1+\frac{4x+1}{x^2+x+2}\right)^x$$

$$= \lim_{x\to\infty}\left[\left(1+\frac{4x+1}{x^2+x+2}\right)^{\frac{x^2+x+2}{4x+1}}\right]^{\alpha} \qquad (1)$$

where $\alpha = \dfrac{x(4x+1)}{x^2+x+2} = \dfrac{4+1/x}{1+1/x+2/x^2} \to 4$ as $x \to \infty$

$\therefore$ $\quad (1) \Rightarrow \lim\limits_{x\to\infty} f(x) = e^4.$

◉ **Example 36:** $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$ is given by

(a) 0 (b) 1/2
(c) log 2 (d) none of these

Ans. (b)

◎ **Solution:** $\lim\limits_{x\to\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)$

$$= \lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}+1} = \frac{1}{2}.$$

◉ **Example 37:** If $\lim\limits_{x\to -a}\dfrac{x^9+a^9}{x+a} = 9$, then the value of a is

(a) – 7 (b) – 1
(c) 7 (d) none of these

Ans. (b)

◎ **Solution:** $\lim\limits_{x\to -a}\dfrac{x^9+a^9}{x+a} = \lim\limits_{x\to -a}\dfrac{x^9-(-a)^9}{x-(-a)} = 9(-a)^8$

So $\quad 9(-a)^8 = 9 \Rightarrow a^8 = 1$

$\Rightarrow (a-1)(a+1)(a^2+1)(a^4+1) = 0$

$\Rightarrow \quad a = 1$ or -1.

◉ **Example 38:** Let $f(x) = [x] + [-x]$, where $[x]$ denotes the greatest integer not greater than x. Then for any integer m

(a) f is continuous at $x = m$
(b) $\lim\limits_{x\to m+} f(x)$ exists and is equal to $f(m)$
(c) $\lim\limits_{x\to m} f(x)$ exists but is not equal to $f(m)$
(d) f is differentiable at $x = m$

Ans. (c)

◎ **Solution:** For, $m-1 < x < m$, $[x] = m-1$ and $[-x] = -m$ and for $m < x < m+1$, $[x] = m$ and $[-x] = -m-1$. Thus

$\lim\limits_{x\to m} f(x) = m-1-m = -1$ and $\lim\limits_{x\to m+} f(x) = m-m-1 = -1$

and so $\lim\limits_{x\to m} f(x) = -1$. But $f(m) = [m] + [-m] = m - m = 0$.

Hence $\lim\limits_{x\to m} f(x)$ exists but is not equal to $f(m)$.

◉ **Example 39:** Let f be a continuous function on [1, 3]. If f takes only rational values for all x and $f(2) = 10$ then $f(1.5)$ is equal to

(a) 8 (b) $\dfrac{f(1)+f(3)}{3}$
(c) 20 (d) none of these

Ans. (d)

◎ **Solution:** Since f is continuous, so it must take all real values between $f(1)$ and $f(3)$. But since f takes only rational values so f must be a constant function. Hence $f(1.5) = f(2) = 10$.

◉ **Example 40:** Let $f(x) = \begin{cases} |x|\cos(1/x) + 9x^2 & x \neq 0 \\ k & x = 0 \end{cases}$

then f is continuous if the value k is

(a) 9 (b) 6
(c) 0 (d) none of these

Ans. (c)

◎ **Solution:** Since $0 \le ||x|\cos(1/x)| \le |x|$, so $\lim\limits_{x\to 0} |x|\cos 1/x = 0$. Hence

$$k = \lim_{x\to 0} f(x) = 0.$$

◉ **Example 41:** If $f(x) = \tan(\pi/4 - x)/\cot 2x$ for $x \neq \pi/4$. The value of $f(\pi/4)$ so that f is continuous at $x = \pi/4$ is

(a) 1/3 (b) 1/2
(c) 1/4 (d) 1/6

Ans. (b)

◎ **Solution:** For f to be continuous at $x = \pi/4$, we must have $f(\pi/4) = \lim\limits_{x\to\pi/4} f(x) = \lim\limits_{x\to\pi/4}[\tan(\pi/4 - x)/\cot 2x]$

$$= \lim_{x\to\pi/4}\left[\frac{1-\tan x}{1+\tan x}\cdot\frac{2\tan x}{1-\tan^2 x}\right] = \lim_{x\to\pi/4}\left[\frac{2\tan x}{(1+\tan x)^2}\right] = \frac{1}{2}.$$

◉ **Example 42:** Let a function f be defined by $f(x) = \dfrac{x-|x|}{x}$ for $x \neq 0$ and $f(0) = 2$. Then f is

(a) continuous nowhere
(b) continuous everywhere
(c) continuous for all x except $x = 1$
(d) continuous for all x except $x = 0$

Ans. (d)

◎ **Solution:** $f(x) = \begin{cases} 2 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \\ 2 & x = 0 \end{cases}$

Thus $\lim\limits_{x\to 0-} f(x) = 2 \neq 0 = \lim\limits_{x\to 0+} f(x)$. Hence, f is continuous everywhere except at $x = 0$.

◉ **Example 43:** If $f(x)$ is a continuous function satisfying $f(x)f(1/x) = f(x) + f(1/x)$ and $f(1) > 0$ then $\lim\limits_{x\to 1} f(x)$ is equal to

(a) 2 (b) 1

(c) 3 (d) none of these

Ans. (a)

◎ **Solution:** Since f is continuous function so $\lim\limits_{x\to 1} f(x) = f(1)$.

Put $x = 1$ in the given equation we have $(f(1))^2 = 2f(1)$ so $f(1) = 0$ or 2. Since $f(1) > 0$ so $f(1) = 2$.

◉ **Example 44:** The function $f(x) = (x-1)^{1/(2-x)}$ is not defined at $x = 2$. The value of $f(2)$ so that f is continuous at $x = 2$ is

(a) 1 (b) e

(c) $1/e$ (d) $1/e^2$

Ans. (c)

◎ **Solution:** For f to be continuous at $x = 2$, $f(2) = \lim\limits_{x\to 2} (x-1)^{1/(2-x)} = \lim\limits_{x\to 2} (1+(x-2)^{-1/(x-2)} = e^{-1}$.

◉ **Example 45:** Let $f(x) = \dfrac{x + x^2 + \cdots + x^n - n}{x-1}$, $x \neq 1$, the value of $f(1)$ of that f is continuous is

(a) n (b) $\dfrac{n+1}{2}$

(c) $\dfrac{n(n+1)}{2}$ (d) $\dfrac{n(n-1)}{2}$

Ans. (c)

◎ **Solution:** $f(1) = \lim\limits_{x\to 1} \dfrac{x + x^2 + \cdots + x^n - n}{x-1} \left(\dfrac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 1} \frac{1 + 2x + \cdots + n\,x^{n-1}}{1} = 1 + 2 + \ldots + n$$

$$= \frac{n(n+1)}{2}.$$

◉ **Example 46:** Let $y = \dfrac{1}{u^2 + u - 2}$ where $u = \dfrac{1}{x-1}$ then y is discontinuous only at $x =$

(a) 1, 2 (b) 1, – 2

(c) 1, 1/2, 2 (d) none of these

Ans. (c)

◎ **Solution:** $u = \phi(x) = \dfrac{1}{x-1}$ is discontinuous at $x = 1$, $y = f(u) = \dfrac{1}{u^2+u-2} = \dfrac{1}{(u+2)(u-1)}$ is discontinuous at $u = -2$, $u = 1$. If $u = -2$ then $-2 = \dfrac{1}{x-1}$ $\Rightarrow x = 1/2$. If $u = 1$ then $1 = \dfrac{1}{x-1}$ $\Rightarrow x = 2$. Hence the composite function is discontinuous only at $x = 1, 1/2, 2$.

◉ **Example 47:** Let $f(x) = [x]$ and $g(x) = \begin{cases} 0 & , x \in \mathbf{I} \\ x^2 & , x \in \mathbf{R} \sim \mathbf{I} \end{cases}$

Then

(a) g is continuous at $x = 1$

(b) f is continuous at $x = 2$

(c) $g \circ f$ is a continuous function

(d) $f \circ g$ is a continuous function

Ans. (c)

◎ **Solution:** f is not continuous at every $x \in \mathbf{I}$ and g is also not continuous at every $x \in \mathbf{I} \sim \{0\}$.

$$g \circ f(x) = g(f(x)) = g([x]) = 0, \text{ since } [x] \,\varepsilon\, \mathbf{I}$$

$$f \circ g(x) = f(g(x)) = \begin{cases} 0 & \text{if } x \in \mathbf{I} \\ [x^2] & , x \in \mathbf{R} \sim \mathbf{I} \end{cases}$$

Thus $g \circ f$ is a constant function and hence continuous but $f \circ g$ is not continuous at every $x \,\varepsilon\, \mathbf{I} \sim \{0\}$.

◉ **Example 48:** If a, b are chosen from {1, 2, 3, 4, 5, 6, 7} randomly with replacement. The probability that $\lim\limits_{x\to 0}\left(\dfrac{a^x + b^x}{2}\right)^{2/x} = 7$ is

(a) 1/7 (b) 2/49

(c) 4/7 (d) 4/49

Ans. (b)

◎ **Solution:** $\left(\dfrac{a^x + b^x}{2}\right)^{2/x}$

$$= a^2\left(1 + \frac{(b/a)^x - 1}{2}\right)^{\frac{2}{(b/a)^x - 1} \times \frac{(b/a)^x - 1}{x}}$$

So $\lim\limits_{x\to 0}\left(\dfrac{a^x + b^x}{2}\right)^{2/x}$

$$= a^2 \cdot \lim_{u\to 0} (1+u)^{1/u \cdot \lim\limits_{x\to 0} \frac{(b/a)^x - 1}{x}}$$

$$= a^2\, e^{\log b/a} = ab$$

Thus $ab = 7$.

The favourable outcomes are (1, 7) and (7, 1). Therefore, required probability is 2/49.

◉ **Example 49:** $\lim\limits_{x\to \pi/2} \dfrac{\cot x - \cos x}{(\pi - 2x)^3}$ equals:

(a) $\dfrac{1}{16}$ (b) $\dfrac{1}{8}$

(c) $\dfrac{1}{4}$ (d) $\dfrac{\pi}{2}$

Ans. (a)

◎ **Solution:** Put $x - \pi/2 = \theta$, so that

$$\lim_{x\to\pi/2}\frac{\cot x-\cos x}{(\pi-2x)^3}=\lim_{\theta\to 0}\frac{\cot(\pi/2+\theta)-\cos(\pi/2+\theta)}{(-2\theta)^3}$$

$$=\frac{1}{8}\lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\theta^3}$$

$$=\frac{1}{8}\lim_{\theta\to 0}\frac{\sec^2\theta-\cos\theta}{3\theta^2}$$

$$=\frac{1}{24}\lim_{\theta\to 0}\frac{2\sec^2\theta\tan\theta+\sin\theta}{2\theta}\left(\frac{0}{0}\text{ form}\right)$$

$$=\frac{1}{24}\left(\frac{2+1}{2}\right)=\frac{1}{16}.$$

Example 50: If $f(x)=\dfrac{((a-n)nx-\tan x)\sin nx}{x^2}$, $x\neq 0$ and $\lim\limits_{x\to 0}f(x)=0$, n is a non-zero real number, then a is equal to

(a) 0 (b) $\dfrac{n}{n+1}$

(c) n (d) $n+(1/n)$

Ans. (d)

Solution: $\lim\limits_{x\to 0}f(x)=0$

$\Rightarrow$ $\lim\limits_{x\to 0}\dfrac{\sin nx}{nx}\times(n)\left[(a-n)n-\dfrac{\tan x}{x}\right]=0$

$\Rightarrow$ $n((a-n)\,n-1)=0$

$\Rightarrow$ $(a-n)\,n-1=0\ \Rightarrow\ a=n+\dfrac{1}{n}$.

Example 51: The function $f(x)=[x]^n-[x^n]$, $n\geq 2$ (where $[y]$ is the greatest integer less than or equal to y), is discontinuous at all points of

(a) **I** (b) **I** ~ {0, 1}

(c) **I** ~ {0} (d) **I** ~ {1}.

Ans. (d)

Solution: $f(x)=0$ for $x\in\mathbf{I}$

If $0<x<1$, then $0<x^n<1$, so

$[x]=0$ and $[x^n]=0$

$\Rightarrow$ $f(x)=0$ for $0<x<1$

If $1<x<2^{1/n}$ then $1<x^n<2\ \Rightarrow\ [x]=1$

and $[x^n]=1$

Thus $f(x)=[x]^n-[x^n]=0$ if $1<x<2^{1/n}$

So $f(x)=0$ if $0\leq x<2^{1/n}$

Therefore, f is continuous at $x=1$

To the left of any integral value $m\neq 1$ but close to m, $f(x)\neq 0$ but to the right of m and close to m, $f(x)=0$. Hence f is discontinuous for all $m\in\mathbf{I}$ ~ {1}.

Example 52: The value of

$$\lim_{x\to 0}\frac{e^{nx}-\left(1+nx+\dfrac{n^2}{2}x^2\right)}{x^3}\ (n>0)\text{ is}$$

(a) $\dfrac{n^2}{6}$ (b) $\dfrac{n^3}{3}$

(c) $\dfrac{n^3}{6}$ (d) 1/6

Ans. (c)

Solution: Using series expansion of e^{nx}, the required limit is equal to

$$\lim_{x\to 0}\frac{\dfrac{n^3}{6}x^3+\dfrac{n^4}{24}x^4+\cdots}{x^3}=\frac{n^3}{6}.$$

Example 53: Let $f(x)=\dfrac{\left(e^{nx}-1\right)^{2n}}{\sin^n(x/a)(\log(1+(x/a)))^n}$ for $x\neq 0$.

If $f(0)=16^n$ and f is a continuous function, then the value of a is

(a) 16 (b) 2

(c) 8 (d) 4

Ans. (d)

Solution: $\lim\limits_{x\to 0}f(x)=$

$$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^{2n}\times\frac{(x/a)^n}{\sin^n(x/a)}\times\frac{(x/a)^n}{(\log(1+(x/a)))^n}\cdot a^{2n}$$

$$=a^{2n}$$

Since $f(0)=\lim\limits_{x\to 0}f(x)$ so $a^{2n}=16^n=4^{2n}$

Thus $a=4$.

Example 54: Let $f(x)=\begin{cases}x+a & ;\ x<0\\ |x-1| & ;\ x\geq 0\end{cases}$ and

$$g(x)=\begin{cases}x+1 & ;\ \text{if } x<0\\ (x-1)^2+b & ;\ x\geq 0\end{cases}$$

If gof is continuous ($a>0$) then

(a) $a=2, b=0$ (b) $a=2, b=1$

(c) $a=1, b=0$ (d) $a=1, b=1$

Ans. (c)

Solution:

$$(gof)\,(x)=\begin{cases}x+a+1 & ;\quad x<-a\\ (x+a-1)^2+b & ;\quad -a\leq x<0\\ x^2+b & ;\quad 0\leq x<1\\ (x-2)^2+b & ;\quad x\geq 1\end{cases}$$

gof is continuous except possibly at $x=-a, 0, 1$.

$$\lim_{x\to -a-}f(x)=-a+a+1=1,\ gof(-a)=1+b$$

Therefore, $b + 1 = 1 \Rightarrow b = 0.$

$$\lim_{x \to 0-} f(x) = (a-1)^2 + b$$

$$= (a-1)^2,\ gof(0) = b = 0,$$

So $(a-1)^2 = 0 \Rightarrow a = 1.$

◉ **Example 55:** If $f : \mathbf{R} \to \mathbf{R}$ is function defined by $f(x) = [x]^3 \cos\left(\frac{2x-1}{2}\right)\pi$, where $[x]$ denotes the greatest integer function, then f is:

(a) discontinuous only at $x = 0$
(b) discontinuous only at non-zero integral value of x.
(c) continuous only at $x = 0$
(d) continuous for every real x.

Ans. (d)

◎ **Solution:** For $n \in \mathbf{I}$, we have $[x] = n$, for $n \le x < n + 1$,

Thus $f(x) = n^3 \cos\left(\frac{2x-1}{2}\right)\pi$, for $n \le x < n + 1$, Since $[x]$ is continuous for $x \notin I$ and $\cos\left(\frac{2x-1}{2}\right)\pi$ is a continuous. So f is continuous at all $x \notin I$. For $x = n \in \mathbf{I}$,

$$\lim_{x \to n+} f(x) = \lim_{x \to n+} n^3 \cos\left(\frac{2x-1}{2}\right)\pi = 0$$

$$\lim_{x \to n-} f(x) = \lim_{x \to n-} (n-1)^3 \cos\left(\frac{2x-1}{2}\right)\pi = 0$$

Also $f(n) = 0$. So f is continuous on $\mathbf{R}$.

◉ **Example 56:** If $\lim_{x \to 0} (1 + x \log(1 + b^2))^{1/x} = 2b\sin^2\theta$, $b > 0$ and $\theta \in [-\pi, \pi]$, then the value of θ is

(a) $\pm\frac{\pi}{4}$ (b) $\pm\frac{\pi}{3}$
(c) $\pm\frac{\pi}{6}$ (d) $\pm\frac{\pi}{2}$

Ans. (d)

◎ **Solution:** $\lim_{x \to 0} (1 + x \log(1 + b^2))^{1/x}$

$$= \lim_{x \to 0} (1 + x \log(1 + b^2))^{\frac{1}{x \log(1+b^2)} \times \log(1+b^2)}$$

$$= e^{\log(1 + \mathrm{b}^2)} = 1 + b^2.$$

Thus,

$$1 + b^2 = 2b \sin^2\theta$$

$$\Rightarrow \quad \sin^2\theta = \frac{1+b^2}{2b} \ge 1$$

$$\Rightarrow \quad \sin^2\theta = 1 \Rightarrow \sin\theta = \pm 1 \quad \text{or} \quad \theta = \pm \pi/2.$$

◉ **Example 57:** Let $f : \mathbf{R} \to [0, \infty]$ be such that $\lim_{x \to 3} f(x)$ exists and $\lim_{\mathrm{x} \to 3} \frac{(f(x))^2 - 4}{\sqrt{|x-3|}} = 1$

Then $\lim_{x \to 3} f(x)$ equals

(a) 0 (b) 1
(c) 2 (d) 3

Ans. (c)

◎ **Solution:** Since, $\lim_{x \to 3} \frac{(f(x))^2 - 4}{\sqrt{|x-3|}} = 1$ So for $\in = 1$ there is $\delta > 0$ such that

$$0 < \frac{(f(x)^2 - 9)}{\sqrt{|x-5|}} < 2 \text{ whenever } 0 < |x - 3| < \delta$$

$$\Rightarrow 0 < f(x) - 2 < 2\frac{\sqrt{|x-3|}}{f(x)+2} \text{ whenever } 0 < |x - 3| < \delta$$

Since the range of f is $[0, \infty]$ So $\lim_{x \to 3} (f(x) + 2) \ne 0$ and exists. Thus

$$0 \le \lim_{x \to 3} (f(x) - 2) \le 0 \Rightarrow \lim_{x \to 3} f(x) = 2.$$

◉ **Example 58:** The value of $\lim_{x \to 0}\left(\left[\frac{11x}{\sin x}\right] + \left[\frac{21\sin x}{x}\right]\right)$, where $[x]$ is the greatest integer less than or equal to x is

(a) 32 (b) 31
(c) 11 (d) 21

Ans. (b)

◎ **Solution:** Since $x > \sin x$, for $x > 0$ and $\lim_{x \to 0+} \frac{x}{\sin x} = 1$ so $\frac{11x}{\sin x} \to 11$ as $x \to 0+$ but $\frac{11x}{\sin x} > 11$. Thus

$\left[\frac{11x}{\sin x}\right] = 11$ for value $x \to 0+$. Similarly

$\frac{21\sin x}{x} \to 21$ as $x \to 0+$ but $20 \le \frac{21\sin x}{x} < 21$

$\left[\frac{21\sin x}{x}\right] = 20$. Hence

$$\lim_{x \to 0+}\left(\left[\frac{11x}{\sin x}\right] + \left[\frac{21\sin x}{x}\right]\right) = 31$$

Similarly $x < \sin x$ for $x < 0$, so $\left[\frac{11x}{\sin x}\right] = 11$ as $x \to 0-$ and $\left[21\frac{\sin x}{x}\right] = 20$ as $x \to 0-$. Thus

$$\lim_{x \to 0-}\left(\left[\frac{11x}{\sin x}\right] + \left[\frac{21\sin x}{x}\right]\right) = 31$$

◉ **Example 59:** If $\lim_{x \to 0} \frac{x^n - \sin x^n}{x - \sin^n x}$ is nonzero and finite, then n in equal to

(a) 1 (b) 4
(c) 2 (d) 3

Ans. (a)

◎ **Solution:** $\lim_{x \to 0} \frac{x^n - \sin x^n}{x - \sin^n x}\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{nx^{n-1}\left(1-\cos x^n\right)}{1-n\sin^{n-1} x\cos x} \quad \text{(L Hopital Rule)}$$

$$= \frac{0}{1} = 0, \text{ if } n > 1$$

For $n = 0$, $\lim_{x\to 0} \frac{x^n - \sin x^n}{x - \sin^n x} = \lim_{x\to 0} \frac{1-\sin 1}{x-1} = \sin 1 - 1$

For $n = 1$, $\lim_{x\to 0} \frac{x^n - \sin x^n}{x - \sin^n x} = \lim_{x\to 0} \frac{x - \sin x}{x - \sin x} = 1$

Example 60: $\lim_{n\to\infty} \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{2k^2}\right)$ is

(a) $\pi/8$ (b) $\pi/2$

(c) $\pi/4$ (d) $\pi/3$

Ans. (c)

Solution: $\tan^{-1}\left(\frac{1}{2k^2}\right) = \tan^{-1} \frac{2}{1+(2k-1)(2k+1)}$

$$= \tan^{-1} \frac{(2k+1)-(2k-1)}{1+(2k-1)(2k+1)}$$

$$= \tan^{-1}(2k+1) - \tan^{-1}(2k-1)$$

$$\sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{2k^2}\right) = \tan^{-1}(2n+1) - \tan^{-1}(2n-1) + \tan^{-1}(2n-1) - \tan^{-1}(2n-3) + \ldots + \tan^{-1} 3 - \tan^{-1} 1$$

$$= \tan^{-1}(2n+1) - \tan^{-1} 1$$

$$= \tan^{-1}(2n+1) - \frac{\pi}{4}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{2k^2}\right) = \tan^{-1}\infty - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Example 61: If f is a continuous function and $x^3 - \left(\sqrt{5}+1\right)x^2 + \left(\sqrt{5}-2+f(x)\right)x + 2\sqrt{5} - \sqrt{5}\, f(x) = 0$ satisfies for $x \in \mathbf{R}$ then $f\left(\sqrt{5}\right)$ is equal to

(a) $2-\sqrt{5}$ (b) $5+\sqrt{5}$

(c) $3-\sqrt{5}$ (d) $\sqrt{5}-3$

Ans. (d)

Solution: For $x \neq \sqrt{5}$,

$$f(x) = -\frac{x^3 - \left(\sqrt{5}+1\right)x^2 + \left(\sqrt{5}-2\right)x + 2\sqrt{5}}{x-\sqrt{5}}$$

$$= -\frac{(x+1)(x-2)\left(x-\sqrt{5}\right)}{x-\sqrt{5}}$$

$$= -(x+1)(x-2)$$

$$f\left(\sqrt{5}\right) = \lim_{x\to\sqrt{5}} f(x) = -\left(1+\sqrt{5}\right)\left(\sqrt{5}-2\right)$$

$$= -\left(3-\sqrt{5}\right) = \sqrt{5}-3$$

Example 62: If $x = u$ is *a* point of discontinuity of $f(x) = \lim_{n\to\infty} \cos^{2n} x$, then the value of $\cos u$ is

(a) 0 (b) 1/2

(c) $(-1)^n$ (d) 1

Ans. (c)

Solution: $f(x) = \lim_{n\to\infty} \left(\cos^2 x\right)^n$

$$= \begin{cases} 0, & 0 \le \cos^2 x < 1 \\ 1, & \cos^2 x = 1 \end{cases}$$

$$= \begin{cases} 0, & 0 \neq n\pi \\ 1, & x = n\pi \end{cases}$$

Hence $f(x)$ is discontinuous when $u = n\pi$, $n \in \mathbf{I}$

So $\cos u = \cos n\pi = (-1)^n$.

Assertion-Reason Type Questions

Example 63: Let $f(x) = \frac{\sqrt{x^2+1} - \sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1} - \sqrt[5]{x^4+1}}$

Statement-1: $\lim_{x\to\infty} f(x) = 1$

Statement-2: $\lim_{x\to\infty} \frac{1}{x^n} = 0$ for $n > 0$.

Ans. (a)

Solution: $f(x) = \frac{x\left(1+1/x^2\right)^{1/2} - x^{2/3}\left(1+1/x^2\right)^{1/3}}{x\left(1+1/x^4\right)^{1/4} - x^{4/5}\left(1+1/x^4\right)^{1/5}}$

$$= \frac{\left(1+\frac{1}{2}\frac{1}{x^2}+0\left(\frac{1}{x^2}\right)\right) - \frac{1}{x^{1/3}}\left(1+\frac{1}{3x^2}+0\left(\frac{1}{x^2}\right)\right)}{\left(1+\frac{1}{4}\frac{1}{x^4}+0\left(\frac{1}{x^4}\right)\right) - \frac{1}{x^{1/5}}\left(1+\frac{1}{5x^4}+0\left(\frac{1}{x^4}\right)\right)}$$

Since $\lim_{x\to\infty} \frac{1}{x^n} = 0$ for all $n > 0$, so $\lim_{x\to\infty} f(x) = \frac{1-0.1}{1-0.1} = 1$.

Example 64: Let $f(x) = \frac{1}{1-x} - \frac{3}{1-x^3}$, $x \neq 1$

Statement-1: The value of $f(1)$ so that f is a continuous function is 1

Statement-2: $g(x) = \dfrac{x+2}{1+x+x^2}$ is a continuous function

Ans. (d)

Solution: For $x \neq 1, f(x) = \dfrac{1}{1-x}\left[1 - \dfrac{3}{1+x+x^2}\right]$

$$= \frac{(x-1)(x+2)}{(1-x)\left(1+x+x^2\right)} = -\frac{x+2}{1+x+x^2}$$

So, $\quad f(1) = \lim\limits_{x\to 1} f(x) = -\lim\limits_{x\to 1} \dfrac{x+2}{1+x+x^2} = -1.$

Example 65: Let $f(x) = \dfrac{1 - \cos x\sqrt{\cos 2x}}{x^2}$, $x \neq 0$

Statement-1: The value of $f(0)$ so that f is a continuous function is 3/2.

Statement-2: For an even function defined near a point a, $\lim\limits_{x\to a+} f(x) = \lim\limits_{x\to a} f(x)$

Ans. (b)

Solution: $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sin x\sqrt{\cos 2x} + \dfrac{\cos x}{\sqrt{\cos 2x}}\sin 2x}{2x}$

$$= \frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\sqrt{\cos 2x} + \lim_{x\to 0}\frac{\sin 2x}{2x}\lim_{x\to 0}\frac{\cos x}{\sqrt{\cos^2 x}}$$

$$= \frac{1}{2} + 1 = \frac{3}{2}.$$

Statement-2 is also true since $f(x) = f(-x)$.

Example 66:

Statement-1: $\lim\limits_{x\to a}\left(\sin\dfrac{x-a}{2}\tan\dfrac{\pi x}{2a}\right) = \dfrac{a}{\pi}$, $a \neq 0$

Statement-2: Trigonometric functions are continuous on their domain of definition.

Ans. (d)

Solution: $\lim\limits_{x\to a}\left(\sin\dfrac{x-a}{2}\tan\dfrac{\pi}{2}\dfrac{x}{a}\right)$

$$= \lim_{x\to a}\sin\frac{x-a}{2}\cot\frac{\pi}{2}\left(1-\frac{x}{a}\right)$$

$$= \lim_{x\to a}\frac{\sin\dfrac{x-a}{2}}{\dfrac{x-a}{2}} \times \frac{\dfrac{x-a}{2}}{\dfrac{\pi}{2}\dfrac{a-x}{a}} \times \frac{\dfrac{\pi}{2}\left(1-\dfrac{x}{a}\right)}{\sin\dfrac{\pi}{2}\left(1-\dfrac{x}{a}\right)} \times \cos\frac{\pi}{2}\left(1-\frac{x}{a}\right)$$

$$= 1 \times -\frac{a}{\pi} \times 1 \times 1 = -\frac{a}{\pi}$$

Statement 2 is clearly true.

Example 67: Let $f(x) = \dfrac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\tan x}$, $x \neq 0$ and

$$g(x) = \left(1 + \frac{1}{x}\right)^{\frac{x+1}{x}}$$

Statement-1: $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to\infty} g(x)$

Statement-2: Both the limits are equal to 1.

Ans. (a)

Solution: $\lim\limits_{x\to 0} f(x)$

$$= \lim_{x\to 0}\frac{1+\sin x - 1 + \sin x}{\tan x} \times \frac{1}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$$

$$= \lim_{x\to 0} 2\cos x \times \frac{1}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$$

$$= 2 \times 1 \times \frac{1}{1+1} = 1.$$

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^{x\times\left(\frac{1}{x}+\frac{1}{x^2}\right)} = e^0 = 1$$

Example 68: Define F(x) as the product of two real functions $f_1(x) = x$, $x \in \mathbf{R}$, and $f_2(x) = \begin{cases} \sin 1/x & ; \quad \text{if } x \neq 0 \\ 0 & ; \quad \text{if } x = 0 \end{cases}$ as follows:

$$\mathrm{F}(x) = \begin{cases} f_1(x)\, f_2(x) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

Statement-1: F(x) is continuous on **R**

Statement-2: $f_1(x)$ and $f_2(x)$ are continuous on **R**.

Ans. (c)

Solution: Since $|x \sin 1/x| \leq |x|$ so $\lim\limits_{x\to 0} \mathrm{F}(x) = 0 = \mathrm{F}(0)$. Thus F is a continuous function. Since $\lim\limits_{x\to 0}\sin 1/x$ doesn't exist so f_2 is not continuous at $x = 0$.

Example 69:

Statement-1: $\lim\limits_{x\to 0}\left[\dfrac{\tan^{-1} x}{x}\right] = 0$, where $[x]$ represents greatest integer $\leq x$.

Statement-2: $\dfrac{\tan^{-1} x}{x} < 1$ for all $x \neq 0$.

Ans. (a)

Solution: Let $f(x) = \tan^{-1} x - x$

$f'(x) = \dfrac{1}{1+x^2} - 1 = \dfrac{-x^2}{1+x^2} < 0$. Hence f is decreasing function so for $x > 0$, $\tan^{-1} x < x$ and for $x < 0$, $f(x) > f(0) = 0$

ϕι $\tan^{-1} x > x$ ϕι $\dfrac{\tan^{-1} x}{x} < 1$ $(x < 0)$

Thus $\lim_{x \to 0+} \left[\frac{\tan^{-1} x}{x}\right] = 0$ and

$$\lim_{x \to 0-} \left[\frac{\tan^{-1} x}{x}\right] = 0$$

Example 70:

Statement-1: $f(x) = \text{sgn}(x^2 - 5x + 6)$ is continuous for all x

Statement-2: $ax^2 + bx + c = 0$ has no real roots if $b^2 - 4ac < 0$

Ans. (d)

Solution: $f(x) = \text{sgn}\,(x - 2)\,(x - 3)$. Hence f is not continuous at $x = 2$ and $x = 3$

Thus statement 1 is not true but statement 2 is true.

LEVEL 2

Straight Objective Type Questions

Example 71: The value of $\lim_{x \to \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$ where α and β are the roots of $ax^2 + bx + c = 0$ is

(a) $(a - b)^2$ (b) $\frac{(\alpha - \beta)^2}{2}$

(c) $\frac{1}{2}a^2(\alpha - \beta)^2$ (d) none of these

Ans. (c)

Solution: $\lim_{x \to \alpha} \frac{1 - \cos\left(ax^2 + bx + c\right)}{(x - \alpha)^2} \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to \alpha} \frac{(2ax + b)\sin\left(ax^2 + bx + c\right)}{2(x - \alpha)}$$

[using L' Hôpital Rule]

$$= \lim_{x \to \alpha} \frac{(2ax + b)\sin[a(x - \alpha)(x - \beta)]}{2(x - \alpha)}$$

$$= \lim_{x \to \alpha} \frac{(2ax + b)\sin[a(x - \alpha)(x - \beta)]}{2a(x - \alpha)(x - \beta)}\, a\,(x - \beta)$$

$$= \frac{a^2}{2}\left(2\alpha + \frac{b}{a}\right)(\alpha - \beta)$$

$$= \frac{a^2}{2}(2\alpha - \alpha - \beta)(\alpha - \beta)$$

$$= \frac{a^2(\alpha - \beta)^2}{2}.$$

Example 72: The integer n for which $\lim_{x \to 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number is

(a) 1 (b) 2

(c) 3 (d) 4

Ans. (c)

Solution: $(\cos x - 1)\,(\cos x - e^x)$

$$= \left(\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - 1\right)\left[\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots\right)\right]$$

$$= x^2\left(-\frac{1}{2} + \frac{x^2}{4!} \cdots\right)\left(-x - x^2 - \frac{x^3}{3!} \cdots\right)$$

$$= x^3\left(-\frac{1}{2} + \frac{x^2}{4!} \cdots\right)\left(-1 - x - \frac{x^2}{3!} \cdots\right)$$

$$\Rightarrow \frac{(\cos x - 1)\left(\cos x - e^x\right)}{x^3} = \left(\begin{array}{l}\frac{1}{2} + \text{terms containing } x \\ \text{and higher powers of } x\end{array}\right)$$

$\lim_{x \to 0} \frac{(\cos x - 1)\left(\cos x - e^x\right)}{x^3} = \frac{1}{2}$. So $n = 3$.

In case $n = 1, 2, 4$, the limit is zero or does not exist.

Example 73: Let $f(x) = \lim_{n \to \infty} \frac{x^{2n} - 1}{x^{2n} + 1}$, then

(a) $f(x) = 1$ for $|x| = 1$

(b) $f(x) = \begin{cases} 1 & \text{for } |x| > 1 \\ -1 & \text{for } |x| < 1 \end{cases}$

(c) $f(x) = \begin{cases} 1 & \text{for } |x| > 1 \\ -1 & \text{for } |x| \le 1 \end{cases}$

(d) f is not defined for any value of x

Ans. (b)

Solution: For $|x| < 1$, $x^{2n} \to 0$ as $n \to \infty$ and for $|x| > 1$, $\frac{1}{x^{2n}} \to 0$ as $n \to \infty$. For $|x| = 1$ we have $x^2 = 1$. Thus

$$\frac{x^{2n} - 1}{x^{2n} + 1} = \begin{cases} 0, & |x| = 1 \\ \frac{1 - x^{-2n}}{1 + x^{-2n}}, & |x| > 1 \\ \frac{x^{2n} - 1}{x^{2n} + 1}, & |x| < 1 \end{cases}$$

Hence $$f(x)=\begin{cases} 0, & |x|=1 \\ 1, & |x|>1 \\ -1, & |x|<1 \end{cases}$$

Example 74: If $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x^x-a^a}=-1$, then the value of a is

(a) 1 (b) 0

(c) e (d) none of these

Ans. (a)

Solution: $\lim\limits_{x\to a}\dfrac{a^x-x^a}{x^x-a^a}\quad\left(\dfrac{0}{0}\text{form}\right)$

$$=\lim_{x\to a}\frac{a^x\log a-ax^{a-1}}{x^x(1+\log x)}$$

(using L'Hôpital Rule) (1)

Now the numerator tends to a^a (log a – 1) and denominator tends to a^a(1 + log a). If the limit of denominator is 0 then 1 + log a = 0 so $a = e^{-1}$ in this case $a^a(\log a-1)\neq 0$ so the limit in (1) doesn't exist. Thus, the last limit is equal to

$$\frac{a^a(\log a-1)}{a^a(\log a+1)}=\frac{\log a-1}{\log a+1}=-1 \text{ if } 2\log a=0\Rightarrow a=1.$$

Example 75: $\lim\limits_{x\to 0}\left(\dfrac{e^x+e^{-x}-2}{x^2}\right)^{1/x^2}$ is

(a) $e^{1/2}$ (b) $e^{1/4}$

(c) $e^{1/3}$ (d) $e^{1/12}$

Ans. (d)

Solution: For $x\neq 0$, let $u=\dfrac{e^x+e^{-x}-2}{x^2}$

$$=\frac{1}{x^2}\left[\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)+\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}\cdots\right)-2\right]$$

$$=\frac{2}{x^2}\left[\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right]=\left[1+\frac{x^2}{12}+\frac{2x^4}{6!}+\cdots\right]$$

$$\therefore\qquad u^{1/x^2}=\left[1+\frac{x^2}{12}+\cdots\right]^{\frac{1}{x^2}}$$

$$=\left[1+\left(\frac{x^2}{12}+\frac{x^4}{360}-\cdots\right)\right]^{\frac{1}{(x^2/12+x^4/360+\cdots)}\left(\frac{1}{12}+\frac{x^2}{360}+\cdots\right)}$$

$$\lim_{x\to 0}u^{1/x^2}=\lim_{y\to 0}(1+y)^{\frac{1}{y}.\lim\limits_{x\to 0}\left(\frac{1}{12}+\frac{x^2}{360}+\ldots\right)}\quad\left(y=\frac{x^2}{12}+\frac{x^4}{360}+\cdots\right)$$

$$=e^{1/12}$$

Example 76: If $\lim\limits_{x\to\infty}\left(\dfrac{x^2+1}{x+1}-ax-b\right)=0$

then

(a) $a = 1, b = -2$ (b) $a = 1, b = -1$

(c) $a = -1, b = 1$ (d) $a = 2, b = -2$

Ans. (b)

Solution:

$$=\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)$$

$$=\lim_{x\to\infty}\left(\frac{(x^2+1)-(ax+b)(x+1)}{(x+1)}\right)$$

$$=\lim_{x\to\infty}\left(\frac{(1-a)x^2-(a+b)x-b+1}{x+1}\right)$$

This limit does not exist unless 1 – a = 0. In case a = 1, the last limit is equal to

$$\lim_{x\to\infty}\frac{-(1+b)x-b+1}{x+1}=\lim_{x\to\infty}\frac{-(1+b)-b/x+1/x}{1+1/x}$$

$$=-(1+b)$$

Thus $1 + b = 0 \Rightarrow b = -1$.

Example 77: Let $f(x) = g(x)\,\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$, where g is a continuous function then $\lim\limits_{x\to 0}f(x)$ exist if

(a) $g(x) = x + 2$

(b) $g(x) = x^2 + 4$

(c) $g(x) = xh(x)$, $h(x)$ is a polynomial

(d) $g(x)$ is a constant function

Ans. (c)

Solution: $\lim\limits_{x\to 0+}\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}=\lim\limits_{x\to 0+}\dfrac{1-e^{-2/x}}{1+e^{-2/x}}=1$

$$\lim_{x\to 0-}\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}=\lim_{x\to 0-}\frac{e^{2/x}-1}{e^{2/x}+1}=-1$$

Hence $\lim\limits_{x\to 0}f(x)$ exists if $g(x) = xh(x)$ where $h(x)$ is a continuous function. If $g(x) = a$ $(a\neq 0)$ then $\lim\limits_{x\to 0+}f(x)=a$ and $\lim\limits_{x\to 0-}f(x)=-a$. Thus $\lim\limits_{x\to 0}f(x)$ does not exist if $f(x) = x + 2$ or $x^2 + 4$ or is a constant function.

Example 78: The value of $f(0)$, so that the function,

$$f(x)=\frac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}},\ a>0$$

becomes continuous for all x in $[-a, a]$, is given by

(a) $a\sqrt{a}$ (b) $\sqrt{a}$

(c) $-\sqrt{a}$ (d) $-a\sqrt{a}$

Ans. (c)

Solution: Note that f is continuous for all x except possibly at $x = 0$. In order that f becomes continuous at $x = 0$, we must have

$$f(0) = \lim_{x\to 0} f(x)$$

Rationalizing the numerator and denominator, we have

$$f(0) = \lim_{x\to 0} \frac{-2ax\left[\sqrt{a+x}+\sqrt{a-x}\right]}{2x\left[\sqrt{a^2-ax+x^2}+\sqrt{a^2+ax+x^2}\right]} = -\sqrt{a}.$$

Example 79: If $f(x) = \dfrac{2-(256-7x)^{1/8}}{(5x+32)^{1/5}-2}$ $(x \neq 0)$; then for f to be continuous on $[-1, 1]$, $f(0)$ is equal to

(a) -1 (b) 1

(c) 2^6 (d) none of these

Ans. (d)

Solution: f is continuous everywhere except possibly at $x = 0$. For f to be continuous everywhere,

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{2-(256-7x)^{1/8}}{(5x+32)^{1/5}-2}$$

$$= \lim_{x\to 0} \frac{1-\left(1-\dfrac{7x}{256}\right)^{1/8}}{\left(1+\dfrac{5x}{32}\right)^{1/5}-1} = \lim_{x\to 0} \frac{1-\left[1-\dfrac{1}{8}\cdot\dfrac{7x}{256}+0\left(x^2\right)\right]}{\left[1+\dfrac{1}{5}\cdot\dfrac{5x}{32}+0\left(x^2\right)\right]-1}$$

$$= \lim_{x\to 0} \frac{\left[\dfrac{1}{8}\cdot\dfrac{7}{256}-0(x)\right]}{\dfrac{1}{32}+0(x)} = \frac{1}{8}\cdot\frac{7}{256}\times\frac{32}{1} = \frac{7}{64}.$$

Example 80: Let $f(x) = \dfrac{x(1+a\cos x)-b\sin x}{x^3}$, $x \neq 0$ and $f(0) = 1$. The value of a and b so that f is a continuous function are

(a) 5/2, 3/2 (b) 5/2, –3/2

(c) –5/2, –3/2 (d) none of these

Ans. (c)

Solution: Note that f is continuous everywhere except possibly at $x = 0$. For f to be continuous at $x = 0$,

$$1 = f(0) = \lim_{x\to 0} \frac{x(1+a\cos x)-b\sin x}{x^3}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0} \frac{(1+a\cos x)-xa\sin x-b\cos x}{3x^2}$$

$$= \lim_{x\to 0} \frac{1+(a-b)\cos x-xa\sin x}{3x^2}$$

This limit will exist only if $1 + (a - b) = 0$. In this case limit is equal to

$$= \lim_{x\to 0} \frac{-(a-b)\sin x-ax\cos x-a\sin x}{6x}$$

(using L'Hôpital Rule)

$$= \frac{1}{6}\lim_{x\to 0}\left[(b-2a)\frac{\sin x}{x}-a\cos x\right]$$

$$= \frac{1}{6}[b-3a] \Rightarrow b-3a = 6$$

Solving $a - b = -1$ and $b - 3a = 6$, we get $b = -3/2$ and $a = -5/2$.

Example 81: $f(x) = \begin{cases} (3/x^2)\sin 2x^2 & \text{if } x < 0 \\ \dfrac{x^2+2x+c}{1-3x^2} & \text{if } x \geq 0, x \neq \dfrac{1}{\sqrt{3}} \\ 0 & x = 1/\sqrt{3} \end{cases}$

then in order that f be continuous at $x = 0$, the value of c is

(a) 2 (b) 4

(c) 6 (d) 8

Ans. (c)

Solution:

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{3}{x^2}\sin 2x^2 = 6\lim_{x\to 0}\frac{\sin 2x^2}{2x^2} = 6$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{x^2+2x+c}{1-3x^2} = \frac{c}{1} = c$$

Hence for f to be continuous $c = 6$.

Example 82: The function $f(x) = (\sin 2x)^{\tan^2 2x}$ is not defined at $x = \pi/4$. The value of $f(\pi/4)$ so that f is continuous at $x = \pi/4$ is

(a) $\sqrt{e}$ (b) 1

(c) 2 (d) none of these

Ans. (d)

Solution: For f to be continuous at $x = \pi/4$,

$$f(\pi/4) = \lim_{x\to\pi/4} f(x) = \lim_{x\to\pi/4} (1-\cos^2 2x)^{(1/2)\tan^2 2x}$$

$$= \lim_{x\to\pi/4}\left\{(1-\cos^2 2x)^{-(1/\cos^2 2x)}\right\}^{-(\sin^2 2x)/2} = \frac{1}{\sqrt{e}}.$$

Example83: Let $f(x) = \dfrac{\log\left(1+x+x^2\right)+\log\left(1-x+x^2\right)}{\sec x-\cos x}$, $x \neq 0$. Then the value of $f(0)$ so that f is continuous at $x = 0$ is

(a) 1 (b) 0

(c) 2 (d) none of these

Ans. (a)

◎ **Solution:** For f to be continuous at $x = 0$,

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log\left(1+x+x^2\right)+\log\left(1-x+x^2\right)}{\sec x - \cos x}$$

$$= \lim_{x\to 0} \frac{\log\left(\left(1+x^2\right)^2 - x^2\right)}{1-\cos^2 x}\cdot\cos x$$

$$= \lim_{x\to 0} \frac{\log\left(1+\left(x^2+x^4\right)\right)}{\sin^2 x}\cdot\cos x$$

$$= \lim_{x\to 0} \frac{\log\left(1+\left(x^2+x^4\right)\right)}{x^2+x^4}\times\frac{x^2}{\sin^2 x}\times\left(1+x^2\right)\cos x = 1.$$

● **Example 84:** The value of k ($k > 0$) for which the function

$$f(x) = \frac{(e^x-1)^4}{\sin(x^2/k^2)\log\{1+(x^2/2)\}}, \qquad x \neq 0; f(0) = 8$$

may be continuous function is

(a) 1	(b) 4
(c) 2	(d) 3

Ans. (c)

◎ **Solution:** We have $\lim_{x\to 0} f(x) =$

$$\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^4\times\frac{x^2/k^2}{\sin\left(x^2/k^2\right)}\times\frac{x^2/2}{\log\left(1+x^2/2\right)}\times 2k^2$$

$= 2k^2$

So $\quad 2k^2 = 8 \Rightarrow k = \pm 2.$

● **Example 85:** The function $f(x) = [\log(1+ax) - \log(1-bx)]/x$ is not defined at $x = 0$. The value, which should be assigned to f at $x = 0$ so that it is continuous at $x = 0$ is

(a) $a - b$	(b) $a + b$
(c) $\log a + \log b$	(d) none of these.

Ans. (b)

◎ **Solution** $\quad f(0) = \lim_{x\to 0} f(x) =$

$$\lim_{x\to 0}\left[\frac{\log(1+ax)}{x} - \frac{\log(1-bx)}{x}\right]$$

$$= \lim_{x\to 0}\frac{a\log(1+ax)}{ax} - \frac{(-b)\log(1-bx)}{-bx} = a+b.$$

● **Example 86:** Let $f(x) = \dfrac{\log\left(1+x^2\right)}{x^4-26x^2+25}$. Then

(a) f is continuous on [6, 10]
(b) f is continuous on [– 2, 2]
(c) f is continuous on [– 6, 6]
(d) f is continuous on [1, 7]

Ans. (a)

◎ **Solution:** $f(x) = \dfrac{\log\left(1+x^2\right)}{\left(x^2-25\right)\left(x^2-1\right)}$

$$= \frac{\log\left(1+x^2\right)}{(x-5)(x+5)(x-1)(x+1)}$$

Since $\log(1+x^2)$ is continuous on $(-\infty, \infty)$, so f is continuous on $\mathbf{R} \sim \{-5, -1, 1, 5\}$ which clearly contains the interval [6, 10].

● **Example 87:** Let $f(x) = \dfrac{\sin\left(\pi\cos^2 x\right)}{x^2}$, $x \neq 0$. The value of $f(0)$ so that f is a continuous function is

(a) $-\pi$	(b) π
(c) $\pi/2$	(d) 1

Ans. (b)

◎ **Solution:** $f(0) = \lim_{x\to 0}\dfrac{\sin\left(\pi\cos^2 x\right)}{x^2}$

$$= \lim_{x\to 0}\frac{\sin\left(\pi\left(1-\sin^2 x\right)\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{\sin\left(\pi\sin^2 x\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{\sin\left(\pi\sin^2 x\right)}{\pi\sin^2 x}\cdot\frac{\pi\sin^2 x}{x^2}$$

$$= 1\cdot\pi\cdot 1 = \pi.$$

● **Example 88:** Let $f(x) = \begin{cases}\dfrac{e^{\alpha x}-e^x-x}{x^2} & x \neq 0\\ 3/2 & x = 0\end{cases}$

The value of α so that f is a continuous function is

(a) 1	(b) 0
(c) 4	(d) 2

Ans. (d)

◎ **Solution:** $\dfrac{3}{2} = \lim_{x\to 0} f(x) = \lim_{x\to 0}\dfrac{e^{\alpha x}-e^x-x}{x^2}\left(\dfrac{0}{0}\text{form}\right)$

$$= \lim_{x\to 0}\frac{\alpha e^{\alpha x}-e^x-1}{2x}$$

For the last limit to exist, we must have
$\lim_{x\to 0}(\alpha e^{\alpha x} - e^x - 1) = 0$

$\therefore\ \alpha - 1 - 1 = 0 \Rightarrow \alpha = 2$. In this case, the last limit is equal to

$$\lim_{x\to 0}\frac{\alpha^2 e^{\alpha x}-e^x}{2} = \frac{\alpha^2-1}{2} = \frac{3}{2}.$$

Example 89: Let f be a continuous function on **R** such that $f(1/2^n) = (\sin e^n)\ e^{-n^2} + \dfrac{2n^2}{n^2+1}$. Then the value of $f(0)$ is

(a) 1 (b) 1/2
(c) 2 (d) none of these

Ans. (c)

Solution: As f is continuous so $f(0) = \lim\limits_{x\to 0} f(x) = \lim\limits_{n\to\infty} f(x_n)$ for any sequence x_n such that $\lim\limits_{n\to\infty} x_n = 0$.

Thus

$$f(0) = \lim_{n\to\infty} f\left(1/2^n\right) = \lim_{n\to\infty}\left(\left(\sin e^n\right)e^{-n^2} + \frac{2n^2}{n^2+1}\right)$$

$$= \lim_{n\to\infty}\left(\left(\sin e^n\right)e^{-n^2} + \frac{2}{1+1/n^2}\right) = 0 + 2 = 2.$$

Example 90: Let $f(x) = \lim\limits_{n\to\infty} \dfrac{x}{1+(2\sin x)^{2n}}$ then f is discontinuous at

(a) π (b) $\pi/3$
(c) $\pi/4$ (d) $\pi/6$

Ans. (d)

Solution:

$$f(x) = \begin{cases} x & \text{if } \ |\sin x| < 1/2 \text{ i.e} -\dfrac{\pi}{6} + n\pi < x < \dfrac{\pi}{6} + n\pi \\ \dfrac{x}{2} & \text{if } \ |\sin x| = \dfrac{1}{2} \text{ i.e. } x = \pm\dfrac{\pi}{6} + n\pi \\ 0 & \text{if } \ |\sin x| > \dfrac{1}{2} \text{ i.e } \dfrac{\pi}{6} + n\pi < x < \dfrac{5\pi}{6} + n\pi \end{cases}$$

Thus points of discontinuities of f are of the form $x = \pm\dfrac{\pi}{6} + n\pi$. In particular, f is discontinuous at $x = \pi/6$.

Alternatively since the choices are in $[0, \pi]$ so one may look at the graph of $\sin x$ and decide about the point of discontinuity.

Example 91: Let f be a non-zero continuous function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in$ **R**. If $f(2) = 9$ then $f(3)$ is

(a) 1 (b) 27
(c) 9 (d) none of these

Ans. (b)

Solution: Any non-zero continuous function satisfying the given functional equation is of the form a^x for some $a > 0$. Since $f(2) = 9$ so $a^2 = 9$ i.e., $a = 3$. Thus $f(x) = 3^x$ and so $f(3) = 3^3 = 27$.

Example 92: Let f be a function on [0, 1] defined by $f(x) = (1/2)^n$, $(1/2)^{n+1} \le x < (1/2)^n$, $n = 0, 1, 2, \ldots$ Then

(a) f is a continuous function
(b) f is continuous except $x = 1/2$
(c) f is continuous except for finitely many points
(d) The set of points where f is not continuous is infinite.

Ans. (d)

Solution: $\lim\limits_{x\to(1/2)^n-} f(x) = \left(\dfrac{1}{2}\right)^n$ whereas

$\lim\limits_{x\to(1/2)^n+} f(x) = (1/2)^{n-1}$ as $f(x) = (1/2)^{n-1}$ for $(1/2)^n \le x < (1/2)^{n-1}$. So f is discontinuous at 1/2, $(1/2)^2$…. Thus f has infinitely many discontinuities.

Example 93: If $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & , x < 0 \\ c & , x = 0 \\ \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}} & , x > 0 \end{cases}$

is continuous at $x = 0$, then

(a) $a = -3/2$, $b = 0$, $c = 1/2$
(b) $a = -3/2$, $b = 1$, $c = 1/2$
(c) $a = -3/2$, b ε **R**, $c = -1/2$
(d) none of these

Ans. (c)

Solution: $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0} \dfrac{\sin(a+1)x + \sin x}{x}$

$$= \lim_{x\to 0}\left[(a+1)\frac{\sin(a+1)x}{(a+1)x} + \frac{\sin x}{x}x^2\right] = a + 2$$

Also $\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0+} \dfrac{\sqrt{x+bx^2} - \sqrt{x}}{bx^{3/2}}$

$$= \lim_{x\to 0} \frac{\sqrt{1+bx} - 1}{bx} = \left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{b}{2\sqrt{1+bx}} \cdot \frac{1}{b}\ \frac{1}{2}$$

Thus we must have $a + 2 = c = 1/2$.
So $a = -3/2$, $c = 1/2$ and b is any real number.

Example 94: If $f(x) = \begin{cases} \dfrac{72^x - 9^x - 8^x + 1}{\sqrt{2} - \sqrt{1+\cos x}}, & x \ne 0 \\ K \log 2 \log 3, & x = 0 \end{cases}$

is a continuous function then K is equal to

(a) $\sqrt{2}$ (b) 24
(c) $18\sqrt{3}$ (d) $24\sqrt{2}$

Ans. (d)

◎ **Solution:** $K \log 2 \log 3 = f(0) = \lim_{x\to 0} f(x)$

$$= \lim_{x\to 0} \frac{72^x - 9^x - 8^x + 1}{\sqrt{2} - \sqrt{1+\cos x}}$$

$$= \lim_{x\to 0} \frac{(9^x-1)(8^x-1)}{x^2} \frac{x^2}{\sqrt{2}\,(1-\cos x/2)}$$

$$= \lim_{x\to 0} \frac{9^x-1}{x}\cdot\frac{8^x-1}{x}\,\frac{16(x/4)^2}{\sqrt{2}\cdot 2\sin^2 x/4}$$

$$= \frac{16}{2\sqrt{2}} \log 9 \log 8 = \frac{8}{\sqrt{2}}\, 6 \log 3 \log 2$$

$$= 24\sqrt{2}\, \log 3 \log 2$$

Thus $K = 24\sqrt{2}$.

◉ **Example 95:** Let f be a function defined on **R** by $f(x) = [x] + \sqrt{x-[x]}$ then

(a) f is not continuous at every $x \in$ **I**
(b) f is not continuous at every $x \in$ **R** ~ **I**
(c) f is a continuous function
(d) none of these

Ans. (b)

◎ **Solution:** $f(x) = n + \sqrt{x-n}$, $n \le x < n+1$

If $x_0 = K \in$ **I** then $\lim_{x\to K+} f(x)$

$$= \lim_{x\to K+} K + \sqrt{x-K} = K$$

and $\lim_{x\to K-} f(x) = \lim_{x\to K-} (K-1) + \sqrt{x-(K-1)}$

$$= -1 + 1 = 0$$

Thus, f is not continuous at every $x_0 = K \in$ **I**. If $x_0 \in$ **R** ~ **I** then $[x]$ is continuous at x_0 which in turn gives that f is continuous at every $x_0 \in$ **R** ~ **I**.

◉ **Example 96:** The function f : **R** ~ $\{0\} \to$ **R** given by $f(x) = \dfrac{\tan x - x}{x - \sin x}$ can be made continuous at $x = 0$ by defining $f(0)$ as

(a) 1 (b) 0
(c) 2 (d) 4

Ans. (c)

◎ **Solution:** We must define $f(0) = \lim_{x\to 0} f(x)$

Let $u(x) = \tan x - x$, $v(x) = x - \sin x$ so $u(0) = v(0) = 0$.
Now $u'(x) = \sec x^2 - 1$, $v'(x) = 1 - \cos x$ so $u'(0) = v'(0) = 0$
Again $u''(x) = 2\sec^2 x \tan x$, $v''(x) = \sin x$ and $u''(0) = v''(0) = 0$
Next $u''(x) = 2\sec^4 x + 4\sec^3 x \tan^2 x$, $v''(x) = \cos x$, so $u'''(0) = 2$, $v'''(0) = 1$

Hence $\lim_{x\to 0} f(x) = \lim_{x\to 0} \dfrac{u(x)}{v(x)} = \lim_{x\to 0} \dfrac{u'''(x)}{v'''(x)} = \dfrac{2}{1} = 2.$

◉ **Example 97:** For $x \ne 1$, f is defined by $f(x) = \dfrac{1}{\log x} - \dfrac{1}{x-1}$

The value of $f(1)$, so that f is a continuous function is

(a) 1 (b) $\frac{1}{2}$
(c) 0 (d) 2

Ans. (b)

◎ **Solution:** $f(x) = \dfrac{x-1-\log x}{(\log x)(x-1)} = \dfrac{u(x)}{v(x)}$

$u(1) = 0$, $v(1) = 0$

$u'(x) = 1 - \dfrac{1}{x}$, $v'(x) = \log x + \dfrac{x-1}{x}$ but $u'(1) = v'(1) = 0$

$$\frac{u'(x)}{v'(x)} = \frac{\dfrac{x-1}{x}}{\dfrac{x\log x + (x-1)}{x}} = \frac{x-1}{x\log x + (x-1)} = \frac{r(x)}{s(x)}$$

$r(1) = s(1) = 0$ and

$$\frac{r'(x)}{s'(x)} = \frac{1}{1 + \log x + 1}$$

so $\lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{r'(x)}{s'(x)} = \lim_{x\to 1} \dfrac{1}{2 + \log x} = \dfrac{1}{2}$

◉ **Example 98:** The number of points where

$$f(x) = \begin{cases} [\cos \pi x] & , \quad 0 \le x \le 1 \\ |5x-2|[x-2] & , \quad 1 < x \le 2 \end{cases}$$

($[x]$ is the greatest integer less than or equal to x) is discontinuous is

(a) 2 (b) 4
(c) 3 (d) 1

Ans. (b)

◎ **Solution:** f is discontinuous possibly at $x = 1$ and points of discontinuities of $[\cos \pi x]$, $0 \le x \le 1$ and $|5x-2|[x-2]$, $1 < x \le 2$.

$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} [\cos \pi x] = -1$ (As for $x \to 1-$, $\pi x \to \pi -$ so $\cos \pi x > -1$)

$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} |5x-2|\,[x-2] = -\lim_{x\to 1+} |5x-2| = -3$

Hence f is discontinuous at $x = 1$.

For $0 \le x \le 1$, $[\cos \pi x] = \begin{cases} 1 & , \quad x = 0 \\ 0 & , \quad 0 < x \le 1/2 \\ -1 & , \quad 1/2 < x \le 1 \end{cases}$

so $[\cos \pi x]$ is discontinuous at $x = 0, 1/2$

For $1 < x \le 2$, $|5x-2|\,[x-2] = (5x-2)\,[x-2]$

$$= \begin{cases} -(5x-2) & , \quad 1 < x < 2 \\ 0 & , \quad x = 2 \end{cases}$$

This function is discontinuous at 2.
Thus f is discontinuous at 0, 1/2, 1, 2.

Example 99: If $f(x) = \begin{cases} x^2 + Ax + 5 & x \in \mathbf{Q} \\ 1 + x & x \in \mathbf{R} \sim \mathbf{Q} \end{cases}$ is continuous at exactly two points, then the possible values of A are in

(a) $(1, \infty)$ (b) $(-3, \infty)$
(c) $(5, \infty) \cup (-\infty, -3)$ (d) $(-\infty, \infty)$

Ans. (c)

Solution: For f to be continuous, $x^2 + Ax + 5 = 1 + x$ should have real roots

ϕι $x^2 + (A - 1)x + 4 = 0$
i.e. $(A - 1)^2 - 16 \le 0$
ϕι $A - 1 \ge 4$ or $A - 1 \le -4$
ϕι $A \ge 5$ or $A \le -3$
ϕι $A \in [5, \infty) \cup (-\infty, -3]$

Example 100: $\lim\limits_{x\to 1} \dfrac{n\,x^{n+1} - n\,x^n + 1}{(e^x - e^2)\sin \pi x}$ is equal to

(a) $\dfrac{n(n+1)}{e^2 - e}$ (b) $\dfrac{n}{(e^2 - e)\pi}$

(c) $\dfrac{n}{(e - e^2)\pi}$ (d) $\dfrac{n^2(n+1)}{(e^2 - e)\pi}$

Solution: $\lim\limits_{x\to 1} \dfrac{n\,x^{n+1} - n\,x^n + 1}{(e^x - e^2)\sin \pi x}\left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 1} \frac{n(n+1)\,x^n - n^2\,x^n + 1}{e^x \sin \pi\,x + (e^x - e^2)\pi \cos \pi x}$$

$$= \frac{n}{(e^2 - e)\pi}.$$

EXERCISE

Concept-based Straight Objective Type Questions

1. $\lim\limits_{x\to\infty} \dfrac{1}{n^2}\,(1 + 2 + \cdots + n)$ is equal to

   (a) 0 (b) 1
   (c) $\dfrac{1}{2}$ (d) $\dfrac{1}{4}$

2. $\lim\limits_{x\to\infty}\left(\dfrac{1}{1.2} + \dfrac{1}{2.3} + \cdots \dfrac{1}{(n-1)n}\right)$ is equal to

   (a) 0 (b) $\dfrac{1}{2}$
   (c) $\dfrac{1}{3}$ (d) 1

3. $\lim\limits_{x\to 1} \dfrac{x^m - 1}{x^n - 1}$ is equal to (m and n are positive integers)

   (a) $\dfrac{m}{n}$ (b) 1
   (c) 0 (d) $\dfrac{m-1}{n-1}$

4. $\lim\limits_{\alpha\to 0} \dfrac{\sin(\alpha^n)}{(\sin \alpha)^m}$ (m, $n \in \mathbf{I}^+$) is equal to

   (a) 1 if $m > n$ (b) 0 if $n > m$
   (c) 1 if $> m$ (d) 1 if $m = n$

5. $\lim\limits_{x\to\infty}\left(1 + \dfrac{k}{x}\right)^{mx}$ is equal to

   (a) m^k (b) e^m
   (c) e^{mk} (d) $e^{m/k}$

6. Let $f(x) = \begin{cases} a|x^2 - x - 2|, & x < 2 \\ b & , x = 2 \\ \dfrac{x - [x]}{x - 2} & , x > 2 \end{cases}$

   If f is continuous at $x = 2$ then
   (a) $a = 1, b = 1$ (b) $a = 1, b = 2$
   (c) $a = 2, b = 1$ (d) $a = 2, b = 2$

7. Let $f(x) = \dfrac{2x^5 - 8x^2 + 11}{x^4 + 4x^3 + 8x^2 + 8x + 4}$. Then

   $A = \{x : f$ is continuous at $x\}$ is equal to
   (a) $\mathbf{R}$ (b) $\mathbf{R} \sim \{0\}$
   (c) $\mathbf{R} \sim \{-1\}$ (d) $\mathbf{R} \sim \{-1, -2\}$

8. Let $f(x) = \begin{cases} ax - b & , \quad x \le 1 \\ 3x & , \quad 1 < x < 2 \\ bx^2 - a & , \quad x \ge 2 \end{cases}$

   If f is continuous function then (a, b) is equal to
   (a) (1, 2) (b) (1, 3)
   (c) (3, 6) (d) (6, 3)

9. If $f(x) = \begin{cases} \cos^{-1}(\cot x) & , \quad x < \dfrac{\pi}{2} \\ a(\pi[x] - 1) & , \quad x \ge \dfrac{\pi}{2} \end{cases}$

The value of a for f to be continuous at $x = \frac{\pi}{2}$ is

(a) $\frac{\pi}{2(\pi-1)}$ (b) $\pi - 1$

(c) $\frac{\pi}{\pi+1}$ (d) $\frac{\pi}{\pi-1}$

10. Let $f(x) = \begin{cases} \frac{\sin 4x}{\log(1+3x)} &, x \neq 0 \\ A+1 &, x = 0 \end{cases}$

The value of A for f to be continuous at $x = 0$ is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$

(c) 1 (d) $\frac{3}{4}$

LEVEL 1

Straight Objective Type Questions

11. The value of $\lim_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+2} - \tan^{-1}\frac{x}{x+2}\right)$ is

(a) 1 (b) 0

(c) 1/2 (d) 1/4

12. The value of $\lim_{x\to 0}\left(1+\tan^2\sqrt{x}\right)^{1/2x}$

(a) e (b) $e^{1/2}$

(c) $e^{1/4}$ (d) $e^{-1/2}$

13. If $f(x) = \frac{\sin(a+x)-\sin(a-x)}{\tan(a+x)-\tan(a-x)}$, $x \neq 0$ then $\lim_{x\to 0} f(x)$ is

(a) $\cos a$ (b) $\frac{1}{2}\cos a$

(c) $\cos^2 a$ (d) none of these

14. $\lim_{x\to\pi/6}\frac{\sin(x-\pi/6)}{\sqrt{3}/2-\cos x}$ is equal to

(a) 2 (b) 1

(c) – 1 (d) $\sqrt{3}/2$

15. $\lim_{x\to\pi/2}\frac{\cos x}{\sqrt[3]{(1-\sin x)^2}}$ is equal to

(a) 1 (b) 2

(c) 21/3 (d) none of these

16. If $x^2 + 2x^3 \leq x + f(x) \leq x^5 - 2x^3$ for value of x near 0, then $\lim_{x\to 0}\frac{f(x)}{x}$ is

(a) – 1 (b) 0

(c) 1 (d) none of these

17. If $f(x) = \frac{\sin^{-1}([x]+x)}{[x]}$, $[x] \neq 0$

$= 0, [x] = 0$

where $[x]$ denotes the greatest integer less than or equal to x, then $\lim_{x\to 0} f(x)$ is

(a) 1 (b) – 1

(c) 0 (d) none of these

18. The function $f(x) = [x]\cos\frac{2x-1}{2}\pi$, where [ ] denotes the greatest integer function is discontinuous at

(a) all x (b) all integer points

(c) no x (d) x which is not integer

19. Let $f(y) = \sin\frac{y-a}{2}\tan\frac{\pi y}{2a}$, $y \neq a$. The value of $f(a)$ so that f is a continuous function is

(a) π/a (b) $-a/\pi$

(c) $\pi/2a$ (d) none of these

20. Let $f(x) = (1 + \sin x)^{\operatorname{cosec} x}$, the value of $f(0)$ so that f is a continuous function is

(a) e (b) $e^{1/2}$

(c) e^2 (d) 1

21. Let $f(x) = \begin{cases} x+1 &, x \leq 1 \\ 3-ax^2 &, x > 1 \end{cases}$

The value of a so that f is continuous is

(a) 1/2 (b) 1

(c) 2 (d) 3

22. Let $f(x) = [x^2 + 1]$, ($[x]$ is the greatest integer less than or equal to x). Then f is continuous

(a) on [1, 3]

(b) for all x in [1, 3] except four points

(c) for all x in [1, 3] except seven points

(d) for all x in [1, 3] except eight points

23. The function $y = \sqrt{\frac{1}{2}-\cos^2 x}$ is not continuous at

(a) $x = \pi/4$ (b) $x = 3\pi/4$

(c) $x = 5\pi/4$ (d) none of these

24. The number of points at which the function $f(x) = 1/(x - [x])$ is not continuous is

(a) 1 (b) 2

(c) 3 (d) none of these

25. Let $f(x) = x$, $x \in \mathbf{Q}$; $f(x) = 1 - x$, $x \in \mathbf{R} \sim \mathbf{Q}$ then f is continuous only at $x =$
(a) 1/2 (b) 1
(c) 0 (d) 2

26. $\lim\limits_{x\to 0} \dfrac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$ is
(a) 2 (b) – 2
(c) 1/2 (d) – 1/2

27. Let $f(x) = \dfrac{\sqrt{1-\cos 2(x-2)}}{x-2}$, $x \neq 2$. The $\lim\limits_{x\to 2} f(x)$
(a) exists and is equal to $\sqrt{2}$
(b) does not exist because $\lim\limits_{x\to 2+} f(x)$ doesn't exist
(c) equal to 1
(d) doesn't exist because $\lim\limits_{x\to 2+} f(x) \neq \lim\limits_{x\to 2-} f(x)$

28. The value of $f(0)$ so that the function $f(x) = \dfrac{\sqrt{1+x} - \sqrt[3]{1+x}}{x}$ becomes continuous is equal to
(a) 1/6 (b) 1/4
(c) 1/3 (d) 3

29. The set of all points of continuity of *fo fo f*, where $f(x) = \operatorname{sgn} x$ is
(a) $\mathbf{R} \sim \{0\}$ (b) $\mathbf{R} \sim \{1, 0, 1\}$
(c) $\mathbf{R} \sim \{-1, 1\}$ (d) none of these

30. The set of all points of discontinuity of $f(x) = \dfrac{x-1}{x^3+6x^2+11x+6}$
(a) ϕ (b) $\{-1\}$
(c) $\{-1, -2, -3\}$ (d) $\{1, 2, 3\}$

31. The number of continuous functions on $\mathbf{R}$ which satisfy $(f(x))^2 = x^2$ for $x \in \mathbf{R}$ is
(a) 2 (b) 4
(c) 1 (d) infinitely many

32. The value of $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$ is
(a) 3 (b) 4
(c) 1 (d) 2

33. The value of $\lim\limits_{x\to a} \dfrac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$ $(a > b)$ is
(a) $\dfrac{1}{4a}$ (b) $\dfrac{1}{a\sqrt{a-b}}$
(c) $\dfrac{1}{2a\sqrt{a-b}}$ (d) $\dfrac{1}{4a\sqrt{a-b}}$

34. The value of $\lim\limits_{x\to\infty} \left(\sqrt{x^2+1}-\sqrt{x^2-1}\right)$ is
(a) – 1 (b) 1
(c) 0 (d) none of these

35. The value of $\lim\limits_{x\to\infty} x^{3/2}\left(\sqrt{x^3+1}-\sqrt{x^3-1}\right)$ is
(a) 1 (b) – 1
(c) 0 (d) none of these

36. The value of $f(1)$ so that $f(x) = (e^x - e)/(x - 1)$ is continuous at $x = 1$ is
(a) e^{-1} (b) e^2
(c) $e^{1/2}$ (d) e

37. The function $f(x) = a[x + 1] + b[x - 1]$, $(a \neq 0, b \neq 0)$ where $[x]$ is the greatest integer function is continuous at $x = 1$ if
(a) $a = 2b$ (b) $a = b$
(c) $a + b = 0$ (d) $a + 2b = 0$

38. Let $f(x) = [2x^3 - 6]$, where $[x]$ is the greatest integer less than or equal to x. Then the number of points in (1,2) where f is discontinuous is
(a) 5 (b) 7
(c) 13 (d) 12

39. Let f be defined on $\mathbf{R}$ by $f(x) = \lim\limits_{n\to\infty}\left[\sin\left(\dfrac{\pi x}{2}\right)\right]^{2n}$ then
(a) f is continuous on [0, 2]
(b) f is continuous on $\mathbf{R} \sim \{2n, n \in I\}$
(c) f is continuous on $\mathbf{R} \sim \{(2n + 1), n \in \mathbf{I}\}$
(d) $\lim\limits_{x\to 0+} f(x) = 1$

40. If $f(x) = (1/2)x - 1$, then on the interval $[0, \pi]$,
(a) $\tan f(x)$ and $\dfrac{1}{f(x)}$ are both continuous
(b) $\tan f(x)$ and $\dfrac{1}{f(x)}$ are both discontinuous
(c) $\tan f(x)$ and $f^{-1}(x)$ are both continuous
(d) $\tan(f(x))$ is continuous but $\dfrac{1}{f(x)}$ is not

41. If $\lim\limits_{x\to 0}[\cot(\pi/4+x)]^{1/x} = Ae^2$ then the value of A is
(a) e^2 (b) e^{-4}
(c) e (d) e^3

42. Let $f(x) = \dfrac{[1/2+x]-[1/2]}{x}$, $-1 \le x \le 2$ and $f(0) = 0$ ($[x]$ is greatest integer less than or equal to x) then
(a) f is not continuous at $x = 0$
(b) f is a continuous function on [–1/2, 1/2]
(c) f is continuous on [– 1, 0]
(d) f is continuous on [0, 2]

43. Let f and g functions such that $f + g$ is a continuous function then
(a) f and g are continuous function
(b) f or g is a continuous function
(c) f may be discontinuous but g is continuous
(d) both f and g may be discontinuous

44. $\lim\limits_{x\to 0} \dfrac{(1+x^2)^{1/3}-(1-2x)^{1/4}}{x+x^2}$ is

(a) 2 (b) 1/2

(c) 3/2 (d) cannot be determined

45. The set of all points for which $f(x) = |x|\ |x-1| + 1/[x+1]$ ($[x]$ is the greatest integer function) is continuous is

(a) **R** (b) **R** ~ **I**

(c) **R** ~ ($I \cup [-1, 0)$) (d) none of these

46. Let $f(x) = \begin{cases} \left(\left(e^{[x]} - e^{\{x\}}\right)e^{-x} + A\right) & , \quad x < 0 \\ \dfrac{2\sin\{x\}}{\tan\{x\}} & , \quad x > 0 \\ 2 & , \quad x = 0 \end{cases}$

The value of A so that f is continuous at $x = 0$ is ($[x]$ is greatest integer function and $\{x\}$ is the fractional part of x) is

(a) e^{-1} (b) $3 - e^{-1}$

(c) $2 - e^{-1}$ (d) 2

47. If $\lim\limits_{x\to 0}\left(\dfrac{\sin 2x}{x^3} + a + \dfrac{b}{x^2}\right) = 0$ then the value of $3a + b$ is

(a) 2 (b) – 2

(c) – 1 (d) 0

48. For continuous functions f and g on R, let $f(a) = 4$, $f'(a) = 6$, $g(a) = 2$, $g'(a) = 1$. Then the value of

$$\lim_{x\to a} \frac{\sqrt{f(x)\,g(a)} - \sqrt{g(x)\,f(a)}}{(x-a)\left(\sqrt{f(x)\,g(a)} + \sqrt{g(x)\,f(a)}\right)}$$

(a) 0 (b) 3

(c) 1/4 (d) 2/3

49. Let $f(x) = \begin{cases} \dfrac{\sqrt{1+ax} - \sqrt{1-ax}}{x} & , -1 \le x < 0 \\ \dfrac{2x+1}{x-2} & , 0 \le x \le 1 \end{cases}$

The value of a so f is continuous on [– 1, 1] is

(a) 1/2 (b) –1/2

(c) 1 (d) – 1

Assertion-Reason Type Questions

50. Let $f(x) = \dfrac{\sqrt[n]{x} - 1}{\sqrt[m]{x} - 1}$, $x \ne 1$, and $g(x) = \dfrac{\sin mx}{\sin nx}$, $x \ne 0$ and $n \ne m$.

Statement-1: $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 0} g(x)$

Statement-2: Both the limits are equal to $\dfrac{n}{m}$

51. Let $f(x) = \dfrac{\cos x - \sin x}{\cos 2x}$, $x \ne \dfrac{\pi}{4}$.

Statement-1: The value of $f\left(\dfrac{\pi}{4}\right)$ so that f is a continuous is $\dfrac{1}{\sqrt{2}}$

Statement-2: $f(x) = \dfrac{1}{\sqrt{2}} \dfrac{1}{\sin\left(x + \dfrac{\pi}{4}\right)}$

52. Let $f(x) = \dfrac{(x+1)^{10} + (x+2)^{10} \cdots + (x+50)^{10}}{x^{10} + 10^{10}}$

Statement-1: $\lim\limits_{x\to\infty} f(x) = 10$

Statement-2: $f(x) =$

$$\sum_{r=1}^{10} \sum_{\alpha=1}^{50} {}^{10}C_r x^{r-10} \alpha^{10-r} \left(1 + \left(\frac{10}{x}\right)^{10}\right)^{-1}$$

53. Let $f(x) = x$ if x is rational and $f(x) = 1 - x$ if x is irrational

Statement-1: f is discontinuous on [0, 1]

Statement-2: f satisfies the intermediate value property.

54. Let f be a continuous function satisfying $f\left(\dfrac{x+y}{2}\right) = \dfrac{f(x) + f(y)}{2}$, $f(0) = 0$, $f(1) = 1$

Statement-1: There is only one function satisfying the above conditions.

Statement-2: A continuous function satisfying $f(x + y) = f(x) + f(y)$ is of the form $f(x) = \alpha x$.

55. **Statement-1:** $\lim\limits_{x\to 0} \cos^{-1}\{x\}$ does not exist, $\{x\}$being the fractional part of x

Statement-2: $\{x\}$ is discontinuous

56. **Statement-1:** The functions $|x^2 + e^x - \log x|$, $x^2 + e^x - \log x$ are continuous functions

Statement-2: Continuity of $|f|$ implies the continuity of f

57. **Statement-1:** If $|f(x)| \le |x|$ for all $x \in$ **R** then $|f|$ is continuous at $x = 0$

Statement-2: If f is continuous then $|f|$ is also continuous

58. **Statement-1:** $f(x) = \cos x + 2[x]$ is discontinuous at $x = 0$, where $[x]$ denotes the greatest integer $\le x$.

Statement-2: If $u(x)$ is continuous and $v(x)$ is discontinuous at $x = a$ then $u(x) + v(x)$ is discontinuous at $x = a$

59. **Statement-1:** $f(x) = \{\sin x\} - \{\cos x\}$ is discontinuous at $x = \pi/2$, $\{x\}$ is the fractional part of x.

Statement-2: Sum of two discontinuous function is a discontinuous function

60. **Statement-1:** $\lim\limits_{x\to 0} [x]^2 \left\{\dfrac{e^{1/x}-1}{e^{1/x}+1}\right\}$ does not exist

Statement-2: $\lim\limits_{x\to 0+} \left\{\dfrac{e^{1/x}-1}{e^{1/x}+1}\right\} = \lim\limits_{x\to 0+} [x]^2 \left\{\dfrac{e^{1/x}-1}{e^{1/x}+1}\right\}$

LEVEL 2

Straight Objective Type Questions

61. If α and β are the roots of $ax^2 + bx + c = 0$, $c \neq 0$ then

$$\lim_{x\to 1/\alpha} \sqrt{\frac{1-\cos\left(cx^2+bx+a\right)}{2(1-\alpha x)^2}} \text{ is}$$

(a) $c(a - b)$ (b) $\dfrac{c}{2\alpha}\left(\dfrac{1}{\alpha}-\dfrac{1}{\beta}\right)$

(c) $\dfrac{1}{\beta}\left(\dfrac{1}{\alpha}-\dfrac{1}{\beta}\right)$ (d) $\left|\left(\dfrac{1}{\alpha}-\dfrac{1}{\beta}\right)\dfrac{c}{2\alpha}\right|$

62. Let f be a continuous function satisfying $f(x)\, f(y) = f(x) + f(y) + f(xy) - 2$ for all $x, y \in \mathbf{R}$ and $f(2) = 5$ then $\lim\limits_{x\to 4} f(x)$ is

(a) 5 (b) 17

(c) – 5 (d) 21

63. The value of $\lim\limits_{x\to\infty} \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}} \tan(1/x)$ is

(a) 1 (b) 0

(c) – 1 (d) none of these

64. Let f be a continuous function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$ and $f(1) = 5$ then $\lim\limits_{x\to 4} f(x)$ is equal to

(a) 4 (b) 80

(c) 0 (d) none of these

65. Let $f(x) = e^x \operatorname{sgn}(x + [x])$, where sgn is the signum function and $[x]$ is the greatest integer function. Then

(a) $\lim\limits_{x\to 0+} f(x) = 0$ (b) $\lim\limits_{x\to 0+} f(x) = -1$

(c) $\lim\limits_{x\to 0+} f(x) = 1$ (d) $\lim\limits_{x\to 0-} f(x) = 1$

66. The value of $\lim\limits_{x\to -4} \dfrac{\tan \pi x}{x+4} + \lim\limits_{x\to\infty} (1 + 1/x^2)^x$ is

(a) greater than 3 (b) less than 2

(c) equal to 0 (d) equal to 1

67. The value of $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$ is

(a) e (b) 1

(c) e^2 (d) $1/e$

68. $\lim\limits_{x\to 0} \dfrac{(\sin x-\tan x)^2+(1-\cos 2x)^4+x^5}{7\tan^7 x+\sin^6 x+2\sin^5 x}$ is equal to

(a) 0 (b) 1

(c) 1/2 (d) 2

69. Let $f(x) = x - 1$ and $g(x) = 1/x$. Then the set of points where go*f og* is continuous is

(a) $\mathbf{R} \sim \{0\}$ (b) $\mathbf{R} \sim \{1\}$

(c) $(-\infty, \infty)$ (d) $(-\infty, \infty) \sim \{0, 1\}$

70. Let $f(x) = \lim\limits_{n\to\infty} \dfrac{\left[x^2\right]+\left[(2x)^2\right]+\cdots+\left[(nx)^2\right]}{n^3}$ then the set of all points of continuity of f ([x] denotes the greatest integer function)

(a) $(-\infty, \infty) \sim \{0\}$ (b) $(-\infty, \infty) \sim \mathbf{I}$

(c) $(-\infty, \infty)$ (d) $(-\infty, \infty) \sim \{0, 1\}$

71. Let $f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$, then

(a) f is discontinuous for every real x

(b) f is continuous on $\mathbf{R}$

(c) f is continuous at the points where x is rational

(d) f is continuous at the points where x is irrational.

72. Let f: $\mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x, & x \text{ irrational} \\ 1-x, & x \text{ is rational} \end{cases}$$

then f is

(a) discontinuous at $x = 1/2$
(b) continuous at $x = 1/2$
(c) continuous everywhere
(d) discontinuous everywhere

73. Let f: $\mathbf{R} \to \mathbf{R}$ be any function. Define g: $\mathbf{R} \to \mathbf{R}$ by $g(x) = |f(x)|$ for all x. Then g is
(a) one-one if f is one-one
(b) discontinuous if f is discontinuous
(c) continuous if f is continuous
(d) onto if f is onto

74. Let f be a function defined on $\mathbf{R}$ by

$$f(x) = \lim_{n\to\infty} \frac{\log(3+x) - x^{2n}\sin x}{1+x^{2n}} \text{ then}$$

(a) f is continuous on $\mathbf{R}$
(b) f is continuous on $\mathbf{R} \sim \{-1, 1\}$
(c) f is continuous on $\mathbf{R} \sim \{0\}$
(d) none of these

75. If $f(x) = \left(\dfrac{x^2+5x+3}{x^2+x+2}\right)^x$, then $\lim\limits_{x\to\infty} f(x)$ is

(a) e^4	(b) e^3
(c) e^2	(d) 2^4

76. Let $f(x)$ be defined for all $x > 0$ and be continuous. Let $f(x)$ satisfy $f\left(\dfrac{x}{y}\right) = f(x) - f(y)$ for all x, y and $f(e) = 1$, then
(a) f is bounded
(b) $f(1/x) \to 0$ as $x \to 0$
(c) $x f(x) \to 1$ as $x \to 0$
(d) $f(x) = \log x$

77. Given the function $f(x) = 1/(1-x)$, the numbers of discontinuities of $f^{3n} = f \circ f \ldots of$ ($3n$ times) is

(a) 1	(b) 2
(c) 3	(d) infinite

78. If $f(x) = \dfrac{1+\sin x - \cos x}{1-\sin x - \cos x}$, $x \neq 0$. The value of $f(0)$ so that f is a continuous function is

(a) 1	(b) -2
(c) -1	(d) 2

79. If $f(x) = \begin{cases} ax+1 & , x \le \pi/2 \\ \sin x + b & , x > \pi/2 \end{cases}$ is continuous then

(a) $a = 1, b = 0$	(b) $a = b\dfrac{\pi}{2} + 1$
(c) $b = a\dfrac{\pi}{2}$	(d) $a = b = \pi/2$

80. Let $f(x) = \begin{cases} \dfrac{x^3+x^2-16x+20}{(x-2)^2} & , \text{if } x \neq 2 \\ K & , \text{if } x = 2 \end{cases}$

then the value of K so that f is a continuous function is

(a) 5	(b) 7
(c) 6	(d) 4

81. If $f(x) = \begin{cases} \dfrac{2^{x+2}-16}{4^x-16} & , \text{if } x \neq 2 \\ A & , \text{if } x = 2 \end{cases}$

then the value of A so that f is continuous is

(a) 1/2	(b) 1
(c) 2	(d) e

82. If the function $f(x) = \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1}$ is continuous at $x = 0$ then $f(0)$ is equal to

(a) -2	(b) -1
(c) 0	(d) -4

83. If a and b are positive integers then

(a) $\lim\limits_{x\to 0+} \dfrac{x}{a}\left[\dfrac{b}{x}\right] = \dfrac{a}{b}$	(b) $\lim\limits_{x\to 0+} \dfrac{x}{a}\left[\dfrac{b}{x}\right] = ab$
(c) $\lim\limits_{x\to 0+} \dfrac{a}{x}\left[\dfrac{x}{b}\right] = \dfrac{b}{a}$	(d) $\lim\limits_{x\to 0+} \dfrac{x}{a}\left[\dfrac{b}{x}\right] = \dfrac{b}{a}$

84. Let $f(x) = \dfrac{\tan\left[e^2\right]x^2 - \tan\left[-e^2\right]x^2}{\sin^2 x}$ $x \neq 0$. Then the value of $f(0)$ so that f is a continuous function is

(a) 15	(b) 10
(c) 7	(d) 8

85. The value of $f(0)$ so that the function $f(x) = \dfrac{\cos ax - \cos bx}{x^2}$, $x \neq 0$ is continuous is given by

(a) $a - b$	(b) $a^2 - b^2$
(c) $\dfrac{b^2-a^2}{2}$	(d) $\dfrac{a^2+b^2}{2}$

Previous Years' AIEEE/JEE Main Questions

1. f is defined on $[-5, 5]$ as $f(x) = \begin{cases} x \text{ if } x \text{ is rational} \\ -x \text{ if } x \text{ is irrational} \end{cases}$
   - (a) $f(x)$ is continuous at every x, except $x = 0$
   - (b) $f(x)$ is discontinuous at every x, except $x = 0$
   - (c) $f(x)$ is continuous everywhere
   - (d) $f(x)$ is discontinuous everywhere **[2002]**

2. $\lim\limits_{x \to 0} \dfrac{\sqrt{1-\cos 2x}}{\sqrt{2}x}$ is
   - (a) 1 (b) -1
   - (c) 0 (d) does not exist **[2002]**

3. $\lim\limits_{x \to \infty} \left(\dfrac{x^2+5x+3}{x^2+x+3}\right)^x$
   - (a) e^4 (b) e^2
   - (c) e^3 (d) e **[2003]**

4. If $\lim\limits_{x \to 0} \dfrac{\log(3+x)-\log(3-x)}{x} = k$, the value of k is
   - (a) $-\dfrac{1}{3}$ (b) $\dfrac{2}{3}$
   - (c) $-\dfrac{2}{3}$ (d) 0 **[2003]**

5. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\left(1-\tan\dfrac{x}{2}\right)(1-\sin x)}{\left(1+\tan\dfrac{x}{2}\right)(\pi-2x)^3}$ is
   - (a) 0 (b) $\dfrac{1}{32}$
   - (c) ∞ (d) $\dfrac{1}{8}$ **[2003]**

6. If $\lim\limits_{x \to \infty} \left(1+\dfrac{a}{x}+\dfrac{b}{x^2}\right)^{2x} = e^2$, then the values of a and b are
   - (a) $a \in \mathbf{R}, b = 2$ (b) $a = 1, b \in \mathbf{R}$
   - (c) $a \in \mathbf{R}, b \in \mathbf{R}$ (d) $a = 1$ and $b = 2$ **[2004]**

7. Let $f(x) = \dfrac{1-\tan x}{4x-\pi}, x \neq \dfrac{\pi}{4}, x \in \left[0, \dfrac{\pi}{4}\right]$. If $f(x)$ is continuous in $\left[0, \dfrac{\pi}{2}\right]$ then $f\left(\dfrac{\pi}{4}\right)$ is
   - (a) $-\dfrac{1}{2}$ (b) $\dfrac{1}{2}$
   - (c) 1 (d) -1 **[2004]**

8. Let α and β be the distinct roots of $ax^2 + bx + c = 0$, then $\lim\limits_{x \to \alpha} \dfrac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to
   - (a) $-\dfrac{a^2}{2}(\alpha-\beta)^2$ (b) $\dfrac{1}{2}(\alpha-\beta)^2$
   - (c) $\dfrac{a^2}{2}(\alpha-\beta)^2$ (d) 0 **[2005]**

9. The function $f : \mathbf{R} \sim \{0\} \to \mathbf{R}$ given by $f(x) = \dfrac{1}{x} - \dfrac{2}{e^{2x}-1}$ can be made continuous at $x = 0$ by defining $f(0)$ as
   - (a) 2 (b) -1
   - (c) 0 (d) 1 **[2007]**

10. Let f: $R \to R$ be a positive increasing function with $\lim\limits_{x\to\infty} \dfrac{f(3x)}{f(x)} = 1$. Then $\lim\limits_{x\to\infty} \dfrac{f(2x)}{f(x)}$
    - (a) 3/2 (b) 3
    - (c) 1 (d) 2/3 **[2010]**

11. The values of p and q for which function

$$f(x) = \begin{cases} \dfrac{\sin(p+1)x+\sin x}{x} & , x < 0 \\ q & , x = 0 \\ \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}} & , x > 0 \end{cases}$$

is continuous for all x in $\mathbf{R}$, are:
   - (a) $p = \dfrac{1}{2}, \ q = \dfrac{3}{2}$ (b) $p = \dfrac{1}{2}, \ q = -\dfrac{3}{2}$
   - (c) $p = \dfrac{5}{2}, \ q = \dfrac{1}{2}$ (d) $p = -\dfrac{3}{2}, \ q = \dfrac{1}{2}$

[2011]

12. Let $f : \mathbf{R} \to [0, \infty)$ be such that $\lim\limits_{x\to5} f(x)$ exists and $\lim\limits_{x\to5} \dfrac{(f(x))^2-9}{\sqrt{|x-5|}} = 0$. Then $\lim\limits_{x\to5} f(x)$ equals.
    - (a) 0 (b) 1
    - (c) 2 (d) 3 **[2011]**

13. Define F(x) as the product of two real functions $f_1(x)$ $= x$, $x \in \mathbf{R}$, and $f_2(x) = \begin{cases} \sin 1/x & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ as follows:

$$F(x) = \begin{cases} f_1(x).f_2(x) & \text{, if } x \neq 0 \\ 0 & \text{, if } x = 0 \end{cases}$$

Statement-1: F(x) is continuous on **R**.
Statement-2: $f_1(x)$ and $f_2(x)$ are continuous on **R**. **[2011]**

14. $\lim\limits_{x \to 2} \dfrac{\sqrt{1 - \cos 2(x-2)}}{x - 2}$

(a) equals $\dfrac{1}{\sqrt{2}}$ (b) does not exist
(c) equals $\sqrt{2}$ (d) equals $-\sqrt{2}$ **[2011]**

15. If f: **R** → **R** is function defined by $f(x) = [x] \cos \dfrac{2x-1}{2}\pi$ whose $[x]$ denotes the greatest integer function, then f is:
(a) discontinuous only at $x = 0$.
(b) discontinuous only at non-zero integral value of x.
(c) continuous only at $x = 0$.
(d) continuous for every real x. **[2012]**

16. $\lim\limits_{x \to 0} \dfrac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to
(a) 1/2 (b) 1
(c) 2 (d) – 1/4 **[2013]**

17. If the function

$$f(x) = \begin{cases} \dfrac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2} & , \quad x \neq \pi \\ k & , \quad x = \pi \end{cases}$$

is continuous at $x = \pi$ then k equals
(a) 2 (b) 1/4
(c) 1/2 (d) 0 **[2014, online]**

18. $\lim\limits_{x \to 0} \dfrac{\sin(\pi \cos^2 x)}{x^2}$ is equal to
(a) $\pi/2$ (b) 1
(c) $-\pi$ (d) π **[2014]**

19. If $f(x)$ is continuous and $f\left(\dfrac{9}{2}\right) = \dfrac{2}{9}$, then $\lim\limits_{x \to 0} f\left(\dfrac{1 - \cos 3x}{x^2}\right)$ is equal to

(a) $\dfrac{9}{2}$ (b) $\dfrac{2}{9}$
(c) 0 (d) $\dfrac{8}{9}$ **[2014, online]**

20. If $\lim\limits_{x \to 2} \dfrac{\tan(x-2)\{x^2 + (k-2)x - 2x\}}{x^2 - 4x + 4} = 5$, then k is equal to
(a) 0 (b) 1
(c) 2 (d) 3 **[2014, online]**

21. If $\lim\limits_{x \to \infty} \dfrac{e^{x^2} - \cos x}{\sin^2 x}$ is equal to

(a) 3 (b) $\dfrac{3}{2}$
(c) $\dfrac{5}{4}$ (d) 2 **[2015]**

22. If $\lim\limits_{x \to 0} \dfrac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to
(a) 4 (b) 3
(c) 2 (d) $\dfrac{1}{2}$ **[2015]**

23. Let k be a non-zero real number. If

$$f(x) = \begin{cases} \dfrac{(e^x - 1)^2}{\sin\left(\dfrac{x}{k}\right)\log\left(1 + \dfrac{x}{4}\right)} & , x \neq 0 \\ 12 & , x = 0 \end{cases}$$

is a continuous function, then the value of k is
(a) 1 (b) 2
(c) 3 (d) 4 **[2015]**

24. Let $p = \lim\limits_{x \to 0+} \left(1 + \tan^2 \sqrt{x}\right)^{\frac{1}{2x}}$, then $\log p$ is equal to
(a) 2 (b) 1
(c) $\dfrac{1}{2}$ (d) $\dfrac{1}{4}$. **[2016]**

25. If $\lim\limits_{x \to \infty} \left(1 + \dfrac{a}{x} - \dfrac{4}{x^2}\right)^{2x} = e^3$. Then a is equal to

(a) 2 (b) $\dfrac{3}{2}$
(c) $\dfrac{1}{2}$ (d) $\dfrac{2}{3}$ **[2016, online]**

26. Let $a, b \in \mathbf{R}(a \neq 0)$. If the function f defined as

$$f(x) = \begin{cases} \dfrac{2x^2}{a}, & 0 \le x < 1 \\ a, & 1 \le x < \sqrt{2} \\ \dfrac{2b^2 - 4b}{x^3} & \sqrt{2} \le x < \infty \end{cases}$$

is continuous in the interval $[a, \infty)$, then an ordered pair (a, b) is
(a) $\left(-\sqrt{2}, 1 - \sqrt{3}\right)$ (b) $\left(\sqrt{2}, -1 + \sqrt{3}\right)$

(c) $\left(\sqrt{2}, 1-\sqrt{3}\right)$ (d) $\left(-\sqrt{2}, 1+\sqrt{3}\right)$

[2016, online]

27. $L = \lim\limits_{x\to 0} \dfrac{(1-\cos 2x)^2}{2x\tan x - x\tan 2x}$ is

(a) 2 (b) $-\dfrac{1}{2}$

(c) –2 (d) $\dfrac{1}{2}$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. If $\lim\limits_{x\to c}\left(\dfrac{f(x)}{g(x)}\right)$ exists, then

(a) $\lim\limits_{x\to c} f(x)$ exist

(b) neither $\lim\limits_{x\to c} f(x)$ nor $\lim\limits_{x\to c} g(x)$ may exist

(c) $\lim\limits_{x\to c} g(x)$ exist

(d) both $\lim\limits_{x\to c} f(x)$ and $\lim\limits_{x\to c} g(x)$ must exist **[2006]**

2. Let $f : (-\infty, \infty) \to (-\infty, \infty)$ be such that $f(.)$ is continuous at 0 and let

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \text{ for } x, y \in (-\infty, \infty)$$

Statement-1: $f(.)$ is continuous at every point on $(-\infty, \infty)$

Statement-2: $2f(x+h) + f(0) = 2f(x) + f(2h)$ for $x, h \in (-\infty, \infty)$ **[2008]**

3. **Statement-1:** $\lim\limits_{x\to 1} \sin\dfrac{\pi}{4}\left(\dfrac{x|x|-1}{|x|-1}\right)$ exists

Statement-2: $\lim\limits_{x\to 1} \tan\dfrac{\pi}{4}\left(\dfrac{x|x|-1}{|x|-1}\right)$ exists **[2009]**

4. Let $f : \mathbf{R} \to \left(\dfrac{-1}{2}, \dfrac{1}{2}\right)$ be an odd function such that $\lim\limits_{x\to 0} f(x)$ exists. Then, $\lim\limits_{x\to 0} \dfrac{1}{2f(x)-1}$ equals

(a) 0 (b) $\dfrac{1}{2}$

(c) 2 (d) – 1 **[2010]**

5. The values of α and β such that

$$\lim_{x\to\infty}\left[\frac{x^2+1}{x-1} - \alpha x - 2\beta\right] = \frac{3}{2} \text{ are}$$

(a) $\alpha = -1, \beta = \dfrac{3}{4}$ (b) $\alpha = 1, \beta = -\dfrac{1}{4}$

(c) $\alpha = -1, \beta = \dfrac{5}{4}$ (d) $\alpha = 1, \beta = \dfrac{-3}{4}$ **[2011]**

6. Let $f(x) = |x| + [x-1]$, where [.] is greatest integer function, then $f(x)$ is

(a) continuous at $x = 0$ as well as at $x = 1$

(b) continuous at $x = 0$ but not at $x = 1$

(c) continuous at $x = 1$ but not at $x = 0$

(d) neither continuous at $x = 0$ or nor at $x = 1$ **[2012]**

7. If for some real number a, $\lim\limits_{x\to 0} \dfrac{\sin 2x + a\sin x}{x^3}$ exists, then the limit is equal to

(a) – 2 (b) – 1

(c) 1 (d) 2 **[2013]**

8. If $f(x) = \begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & 1 \end{vmatrix}$, then $\lim\limits_{x\to 0} \dfrac{f(x)}{x^2}$ is

(a) 1 (b) – 1

(c) 0 (d) 2 **[2015]**

9. Let $f(x) = [x]$ where $[x]$ be the greatest integer less than or equal to x.

$$g(x) = \begin{cases} 0, & x \in \mathbf{Z} \\ x^2, & x \in \mathbf{R}-\mathbf{Z} \end{cases}$$

z is the set of integers, $\phi(x) = fog(x)$ and $\psi(x) = gof(x)$. Then on the set $\mathbf{R}-\mathbf{Z}$.

(a) both ϕ and ψ are continuous

(b) neither ϕ nor ψ is continuous.

(c) ϕ is continuous and ψ is not continuous.

(d) ψ is continuous and ϕ is not continuous **[2015]**

10. $\lim_{x\to 1}((1-x)+[x-1]+|1-x|)$, where $[x]$ denotes the greatest integer less than or equal to x
    (a) is equal to 0 (b) is equal to 1
    (c) does not exist (d) is equal to –1

[2016]

Answers

Concept-based

1. (c) **2.** (d) **3.** (a) **4.** (b)
5. (c) **6.** (a) **7.** (a) **8.** (d)
9. (a) **10.** (b)

Level 1

11. (c) **12.** (b) **13.** (d) **14.** (a)
15. (d) **16.** (a) **17.** (d) **18.** (c)
19. (b) **20.** (a) **21.** (b) **22.** (d)
23. (d) **24.** (d) **25.** (a) **26.** (c)
27. (d) **28.** (a) **29.** (a) **30.** (c)
31. (b) **32.** (b) **33.** (d) **34.** (c)
35. (a) **36.** (d) **37.** (c) **38.** (c)
39. (c) **40.** (d) **41.** (b) **42.** (b)
43. (d) **44.** (b) **45.** (c) **46.** (b)
47. (a) **48.** (c) **49.** (b) **50.** (c)
51. (b) **52.** (d) **53.** (c) **54.** (a)
55. (c) **56.** (c) **57.** (b) **58.** (a)
59. (c) **60.** (c)

Level 2

61. (d) **62.** (b) **63.** (b)
64. (d) **65.** (c) **66.** (a)
67. (d) **68.** (c) **69.** (d)
70. (c) **71.** (a) **72.** (b)
73. (c) **74.** (b) **75.** (a)
76. (d) **77.** (b) **78.** (c)
79. (c) **80.** (b) **81.** (a)
82. (d) **83.** (d) **84.** (a)
85. (c)

Previous Years' AIEEE/JEE Main Questions

1. (b) **2.** (d) **3.** (a) **4.** (b)
5. (b) **6.** (b) **7.** (a) **8.** (c)
9. (d) **10.** (c) **11.** (d) **12.** (d)
13. (c) **14.** (b) **15.** (d) **16.** (c)
17. (b) **18.** (d) **19.** (b) **20.** (d)
21. (b) **22.** (c) **23.** (c) **24.** (c)
25. (a) **26.** (c) **27.** (a)

Previous Years' B-Architecture Entrance Examination Questions

1. (b) **2.** (a) **3.** (c) **4.** (d)
5. (b) **6.** (d) **7.** (b) **8.** (a)
9. (a) **10.** (c)

Hints and Solutions

Concept-based

1. $\frac{1}{n^2}\ (1+2\cdots+n)=\frac{n(n+1)}{2n^2}=\frac{1}{2}\left(1+\frac{1}{n}\right)$

so $\lim_{n\to\infty}\frac{1}{n^2}\ (1+2+\cdots+n)=\frac{1}{2}$

2. $\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{(n-1)n}=\sum_{k=2}^{n}\frac{1}{(k-1)k}$

$$=\sum_{k=2}^{n}\left[\frac{1}{k-1}-\frac{1}{k}\right]=1-\frac{1}{n}$$

so $\lim_{n\to\infty}\left[\frac{1}{1.2}+\frac{1}{2.3}+\cdots+\frac{1}{(n-1)n}\right]=1$

3. $\lim_{x\to 1}\frac{x^m-1}{x^n-1}=\lim_{x\to 1}\frac{mx^{m-1}}{nx^{n-1}}=\frac{m}{n}$

4. If $n>m$, then $\frac{\sin\alpha^n}{(\sin\alpha)^m}=\frac{\sin\alpha^n}{\alpha^n}\cdot\frac{\alpha^n}{\alpha^m}\left(\frac{\alpha}{\sin\alpha}\right)^m$

$$=\frac{\sin\alpha^n}{\alpha^n}\cdot\alpha^{n-m}\left(\frac{\alpha}{\sin\alpha}\right)^m$$

$\Rightarrow \lim_{\alpha\to 0}\frac{\sin\alpha^n}{(\sin\alpha)^m}=0$

5. $\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^{mx}=\lim_{k\to\infty}\left(1+\frac{k}{x}\right)^{\frac{x}{k}\cdot mk}=e^{mk}$

6. $\frac{x-[x]}{x-2}=1$ for $2<x<3$

So $b=\lim_{x\to 2+}f(x)=1$. For $1<x<2$, $\frac{a|x^2-x-2|}{-(x^2-x-2)}$

$=\frac{a|x-2||x+1|}{-(x-2)(x+1)}=\frac{-a(x-2)(x+1)}{-(x-2)(x+1)}=a$

For f to be continuous $a = b = 1$

7. $x^4 + 4x^3 + 8x^2 + 8x + 4 = (x^2 + 2x + 2)^2$ and $x^2 + x + 2 = (x + 1)^2 + 1$ so $x^4 + 4x^3 + 8x^2 + 8x + 4 \neq 0$ for any x. Hence f is a continuous function

8. $f(1) = a - b$ and $\lim\limits_{x\to 1} 3x = 3$ so for f to be continuous $a - b = 3$. Also $f(2) = 4b - a$, so $\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2-} 3x = 6$. Thus $4b - a = 6$ solving, we have $a = 6$, $b = 3$.

9. $f(\pi/2) = a\left(\pi\left[\dfrac{\pi}{2}\right] - 1\right) = a\,(\pi - 1)$

$$\lim_{x\to\pi/2-} f(x) = \lim_{x\to\pi/2-} \cos^{-1}(\cot x) = \frac{\pi}{2}$$

$\phi\iota\ a\,(\pi - 1) = \dfrac{\pi}{2}$.

10. For $x \neq 0$, $\dfrac{\sin 4x}{\log(1+3x)} = \dfrac{\sin 4x}{4x} \times \dfrac{4}{3} \times \dfrac{3x}{\log(1+3x)}$

so $A + 1 = \lim\limits_{x\to 0} \dfrac{\sin 4x}{\log(1+3x)} = \dfrac{4}{3} \Rightarrow A = \dfrac{1}{3}$

Level 1

11. $x\left(\tan^{-1}\dfrac{x+1}{x+2} - \tan^{-1}\dfrac{x}{x+2}\right)$

$$= x\tan^{1}\frac{\frac{x+1}{x+2}\quad\frac{x}{x+2}}{1+\frac{x(x+1)}{(x+2)^2}}$$

$$= x\tan^{-1}\frac{x+2}{2x^2+5x+4}$$

$$= x\left(\tan^{-1}\frac{\frac{1}{x}+\frac{2}{x^2}}{2+\frac{5}{x}+\frac{4}{x^2}}\right)$$

$$= \frac{\tan^{-1}u}{u}\cdot\frac{1+\frac{2}{x}}{2+\frac{5}{x}+\frac{4}{x^2}},\quad u = \frac{\frac{1}{x}+\frac{2}{x^2}}{2+\frac{5}{x}+\frac{4}{x^2}}$$

$$\to \frac{1}{2} \text{ as } x \to \infty.$$

12. $\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}} = \left(1+\tan^2\sqrt{x}\right)^{\left(\frac{1}{\tan^2\sqrt{x}}\right)\cdot\left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)^2\frac{1}{2}}$

$$\to e^{1/2} \text{ as } x \to 0$$

13. $f(x) = \dfrac{-2\cos a\cos x\cos(a+x)\cos(a-x)}{\sin 2x}$ so $\lim\limits_{x\to 0} f(x)$ does not exist.

14. Required limit $= \lim\limits_{x\to\frac{\pi}{6}} \dfrac{\cos\left(x-\frac{\pi}{6}\right)}{\sin x} = 2$.

15. The limit does not exist.

16. $x + 2x^2 - 1 \le \dfrac{f(x)}{x} \le x^4 - 2x^2 - 1$ so $\lim\limits_{x\to 0}\dfrac{f(x)}{x} = -1$

(Sandwich Theorem)

17. $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to 0-}\dfrac{\sin^{-1}(x-1)}{-1} = \dfrac{\pi}{2}$.

$\lim\limits_{x\to 0+} f(x) = 0$.

18. If $x \in n \in \mathbf{I}$, $f(n) = n\cos\dfrac{2n-1}{2}\pi = 0$.

$$\lim_{x\to n+} f(x) = \lim_{x\to n+} n\cos\frac{2x-1}{2}\pi = 0 = \lim_{x\to n-} f(x)$$

If $x \notin \mathbf{I}$, then $[u]$ is a continuous at x and $\cos\dfrac{2u-1}{2}\pi$ is also continuous. Hence f is a continuous function.

19. $f(a) = \lim\limits_{y\to a}\sin\dfrac{(y-a)}{2}\tan\dfrac{\pi}{2}\dfrac{y}{a}$

$$= \lim_{y\to a}\frac{\sin\frac{y-a}{2}}{\cot\frac{\pi}{2}\frac{y}{a}}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{y\to a}\frac{\left(\frac{1}{2}\right)\cos\frac{y-a}{2}}{\left(-\frac{\pi}{2a}\right)\operatorname{cosec}^2\frac{\pi}{2}\frac{y}{a}} = -\frac{a}{\pi}.$$

20. $f(0) = \lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{\sin x}} = e$

21. $2 = f(1) = \lim\limits_{x\to 1+}(3 - ax^2) = 3 - a \Rightarrow a = 1$.

22. f is discontinuous at x for which $x^2 + 1 \in \mathbf{I}$. For $x \in (1, 3)$ we have $x^2 + 1 \in (2, 10)$ so there are seven values. f is clearly continuous from right at 1 but discontinuous from left at 3.

23. $u = \sqrt{x}$, $v = \left(\dfrac{1}{2}\right) - \cos^2 x$ are continuous for $x > 0$.

24. f is not defined for $x \in \mathbf{I}$.

25. $f\left(\frac{1}{2}\right) = \lim\limits_{x \to \frac{1}{2}+} f(x) = \lim\limits_{x \to \frac{1}{2}-} f(x)$ so f is continuous only at $x = \frac{1}{2}$.

26. $\dfrac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$

$$= \frac{1}{4} x \left[\frac{2 \tan x}{1 - \tan^2 x} - 2 \tan x\right] \frac{1}{\sin^4 x}$$

$$= \frac{1}{4} \frac{2x \tan^3 x}{1 - \tan^2 x} \frac{1}{\sin^4 x} = \frac{1}{2} \frac{\left(\frac{\tan x}{x}\right)^3}{1 - \tan^2 x} \cdot \frac{1}{\left(\frac{(\sin x)}{x}\right)^4}$$

So the required limit = $\frac{1}{2} \cdot \frac{1}{1 - 0} \cdot \frac{1}{1} = \frac{1}{2}$

27. $f(x) = \dfrac{\sqrt{2}\,|\sin(x-2)|}{(x-2)}$ so $\lim\limits_{x \to 2+} f(x) = 1$

and $\lim\limits_{x \to 2-} f(x) = -1$.

28. $f(x) = \frac{1}{x}\left[(1+x)^{1/2} - (1+x)^{1/3}\right]$

$$= \frac{1}{x}\left[\left(1 + \left(\frac{1}{2}\right)x + 0(x^2)\right) - \left(1 + \left(\frac{1}{3}\right)x + 0(x^2)\right)^2\right]$$

$$= \frac{1}{x}\left[\left(\frac{1}{6}\right)x + 0(x^2)\right] = \frac{1}{6} + 0(x) \cdot$$

$\lim\limits_{x \to 0} f(x) = \frac{1}{6}$.

29. $x = 0$ is the only discontinuity of f. $f \circ f(x) = f(x)$, $f \circ f \circ f(x) = f(x)$ so $f \circ f \circ f$ is continuous on **R** ~ {0}.

30. $x^3 + 6x^2 + 11x + 6 = (x + 1)(x + 2)(x + 3)$. Thus, f is discontinuous at $x = -1, -2, -3$.

31. $f(x) = x$, $f(x) = -x$, $f(x) = \begin{cases} x\ , & x \ge 0 \\ -x\ , & x < 0 \end{cases}$

$f(x) = \begin{cases} -x\ , & x \ge 0 \\ x\ , & x < 0 \end{cases}$.

32. $\dfrac{\sqrt{x^2+1} - 1}{\sqrt{x^2+16} - 4} = \dfrac{x^2\left(\sqrt{x^2+16} + 14\right)}{x^2\left(\sqrt{x^2+1} + 1\right)}$, for $x \ne 0$

So the required limit = $\frac{4+4}{1+1}$ = 4.

33. $\dfrac{\sqrt{x-b} - \sqrt{a-b}}{x^2 - a^2}$

$$= \frac{(x-b) - (a-b)}{(x-a)(x+a)} \frac{1}{\sqrt{x-b} + \sqrt{a-b}}$$

$$= \frac{1}{(x+a)\left(\sqrt{x-b} + \sqrt{x-a}\right)}$$

Required limit = $\dfrac{1}{2a.2\sqrt{a-b}} = \dfrac{1}{4a\sqrt{a-b}}$.

34. $\sqrt{x^2+1} - \sqrt{x^2-1} = \dfrac{2}{\sqrt{x^2+1} + \sqrt{x^2-1}}$

$$= \frac{2/x}{\sqrt{1 + \frac{1}{x^3}} + \sqrt{1 - \frac{1}{x^3}}} \to 0 \text{ as } x \to \infty$$

35. $x^{\frac{3}{2}}\left(\sqrt{x^3+1} - \sqrt{x^3-1}\right) = \dfrac{2x^{\frac{3}{2}}}{\sqrt{x^3+1} + \sqrt{x^3-1}}$

$$= \frac{2}{\sqrt{1 + \frac{1}{x^3}} + \sqrt{1 - \frac{1}{x^3}}} \to \frac{2}{1+1} = 1 \text{ as } x \to \infty$$

36. $f(x) = \dfrac{e\left(e^{x-1} - 1\right)}{x - 1}$ so $f(1) = e$.

37. $f(1) = 2a$, $\lim\limits_{x \to 1-} f(x) = a - b$ so $a + b = 0$ for f to be continuous at $x = 1$.

38. For $x \in (1, 2)$, $2 < 2x^3 < 16 \Rightarrow -4 < 2x^3 - 6 < 10$, $2x^3 - 6 = -3, -2, -1, 0, 1, 2, \ldots 9$. So there are 13 points at which the function f is discontinuous.

39. $f(x) = \begin{cases} 0 \text{ if } \left|\sin \frac{\pi}{2} x\right| < 1 \\ 1 \text{ if } \left|\sin \frac{\pi}{2} x\right| = 1 \end{cases}$.

$\left|\sin \frac{\pi}{2} x\right| = 1 \Rightarrow x = (2n + 1)$. So f is continuous on **R** ~ $\{(2n + 1) : n \in \mathbf{I}\}$.

40. $f(x) = \tan\left(\left(\frac{1}{2}\right)x - 1\right)$. For $x \in [0, \pi]$, $\left(\frac{1}{2}\right)x - 1 \in \left[-1, \frac{\pi}{2} - 1\right]$ so f is continuous $[0, \pi]$ but $\dfrac{1}{f(x)} = \dfrac{2}{x-2}$ which is not continuous at $x = 2 \in [0, \pi]$

41. $Ae^2 = \lim\limits_{x\to 0}\left(\dfrac{\cot x-1}{\cot x+1}\right)^{\frac{1}{x}}$

$$= \lim_{x\to 0}\left(1-\frac{2}{\cot x+1}\right)^{-\frac{\cot x+1}{2}\times\frac{-2\tan x}{1+\tan x}} = e^{-2}$$

42. $$f(x) = \begin{cases} -\dfrac{1}{x} & -1 \le x < -\dfrac{1}{2} \\ 0 & -\dfrac{1}{2} \le x < \dfrac{1}{2} \\ \dfrac{1}{x} & \dfrac{1}{2} \le x < \dfrac{3}{2} \\ \dfrac{2}{x} & \dfrac{3}{2} \le x \le 2 \end{cases}$$

43. Take $g = -f$, f any discontinuous function.

44. $$\frac{(1+x^2)^{\frac{1}{3}}-(1-2x)^{\frac{1}{4}}}{x+x^2}$$

$$= \frac{1+\frac{1}{3}x^2+0(x^4)-1+\frac{1}{2}x+0(x^2)}{x(x+1)}$$

$$= \frac{\frac{1}{2}+\text{terms containing } x}{x+1}$$

so the required limit is $\dfrac{1}{2}$.

45. Modulus function is continuous, $[x]$ is continuous on $\mathbf{R} \sim \mathbf{I}$ and $[x+1] = 0$ for $-1 \le x < 0$. So f is continuous on $\mathbf{R} \sim (\mathbf{I} \cup (-1, 0))$.

46. $f(0) = 2$ and $\lim\limits_{x\to 0-} f(x) = \lim\limits_{x\to\infty}\left(e^{[x]}-e^{\{x\}}\right)e^{-x}+\text{A})$
$= (e^{-1}-1) + \text{A}$

47. Required limit $= \lim\limits_{x\to 0}\dfrac{\sin 2x+ax^3+bx}{x^3}\left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{2\cos 2x+3ax^2+b}{3x^2}$$

If this limit exists then, we must have b + 2 = 0
In this case the last limit is equal to
$\lim\limits_{x\to 0}\dfrac{-4\sin 2x+6ax}{6x^2} = -\dfrac{4}{3}+a$, so a $= \dfrac{4}{3}$.

48. Rationalize the numerator and apply $\dfrac{0}{0}$ form

49. $f(0) = -\dfrac{1}{2}$ and $\lim\limits_{x\to 0-} f(x) = a$

50. Use L' Hopital rule to calculate $\lim\limits_{x\to 1} f(x)$, the value of this limit is $\dfrac{m}{n}$. Also $\lim\limits_{x\to 0}\dfrac{\sin mx}{\sin nx} =$
$\lim\limits_{x\to 0}\dfrac{\sin mx}{mx}\times\dfrac{mx}{nx}\times\dfrac{nx}{\sin nx} = \dfrac{m}{n}$.

51. $f(x) = \dfrac{\cos x-\sin x}{\cos^2 x-\sin^2 x}$

$$= \frac{1}{\cos x+\sin x} = \frac{1}{\sqrt{2}}\times\frac{1}{\sin\left(x+\frac{\pi}{4}\right)}$$

52. $$f(x) = \frac{\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+\cdots+\left(1+\frac{50}{x}\right)^{10}}{1+\left(\frac{10}{x}\right)^{10}}$$

so $\lim\limits_{x\to\infty} f(x) = \dfrac{1+\cdots+1}{1}$ (50 times) = 50.

53. f is discontinuous at every point of [0, 1] except at $x = 1/2$. Also f is a one-to-one mapping of [0, 1] onto [0, 1] and hence satisfies the intermediate value property.

54. $f\left(\dfrac{x}{2}\right) = f\left(\dfrac{x+0}{2}\right) = \dfrac{f(x)+f(0)}{2} = \dfrac{f(x)}{2}$

$\dfrac{f(x)+f(y)}{2} = f\left(\dfrac{x+y}{2}\right) = \dfrac{f(x+y)}{2}$

$\Rightarrow\ f(x)+f(y) = f(x+y)$
Using statement-2 we get $f(x) = \alpha x$ but $f(1) = 1$, so $f(x) = x$ is the only function satisfying the given conditions.

55. $\lim\limits_{x\to 0+}\cos^{-1}\{x\} = \cos^{-1} 0 = \dfrac{\pi}{2}$

$\lim\limits_{x\to 0-}\cos^{-1}\{x\} = \cos^{-1} = 0$

(Look at the graph of $\{x\}$)
$\{x\}$ is discontinuous at $x = n$, $n \in \mathbf{I}$
For $x \notin \mathbf{I}$ $\{x\}$ is continuous

56. $g(x) = x^2 + e^x - \log x$ is a continuous function and $u(x) = |x|$ is also continuous so $uog(x) = |x^2 + e^x - \log x|$ is a continuous function. However continuity of $|f|$ does not imply the continuity of f, e.g.
$f(x) = \begin{cases} 2 & x \ge 0 \\ -2 & x < 0 \end{cases}$ is not continuous at $x = 0$ but
$|f|(x) = 2$ for all x is continuous.

57. $|f(x)| \le |x|$ for all $x \in \mathbf{R} \Rightarrow 0 \le |f(0)| \le 0$
so $|f(0)| = 0$. Also

$0 \le \lim\limits_{x\to 0} |f|\,(x) \le \lim\limits_{x\to 0} |x| = 0$ i.e $\lim\limits_{x\to 0} |f|(x) = 0$

so $|f|$ is continuous. The statement-2 is also true but is not a correct explanation.

58. cos x is a continuous function and $[x]$ is discontinuous at $x = n \in \mathbf{I}$. By statement-2 $\cos x + 2[x]$ is discontinuous at $x = 0$. Statement-2 is true for if $u(x) + v(x)$ is continuous at $x = a$ then $v(x) = u(x) + v(x) - u(x)$ is continuous at $x = a$ which is a contradiction.

59. $\lim\limits_{x\to\pi/2+} f(x) = 0 - 0 = 0$

$\lim\limits_{x\to\pi/2-} f(x) = 1 - 0 = 1$

f is discontinuous at $x = \pi/2$. But sum of two discontinuous need not be discontinuous e.g.

$$f(x) = \begin{cases} 1 & , \ x \ge 0 \\ -1 & , \ x < 0 \end{cases} \quad g(x) = \begin{cases} -1 & , \ x \ge 0 \\ 1 & , \ x < 0 \end{cases}$$

$f(x) + g(x) = 0$ for all x.

60. $\lim\limits_{x\to 0+} [x]^2 \left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right) = \lim\limits_{x\to 0+} [x]^2 \left(\dfrac{1-e^{-1/x}}{1+e^{-1/x}}\right)$

$= 0 \times 1 = 0$

$\lim\limits_{x\to 0-} [x]^2 \left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right) = (-1)^2 \times (-1) = -1.$

$\lim\limits_{x\to 0+} \left(\dfrac{e^{1/x}-1}{e^{1/x}+1}\right) = 1.$

Level 2

61. If α, β are the roots of $ax^2 + bx + c = 0$ then $1/\alpha$, $1/\beta$ will be the roots of $cx^2 + bx + a = 0$.

Thus $\lim\limits_{x\to 1/a} \sqrt{\dfrac{1-\cos\left(cx^2+bx+a\right)}{2(1-\alpha x)^2}}$

$$= \lim_{x\to 1/a} \frac{\left|\sin\dfrac{cx^2+bx+a}{2}\right|}{|\alpha||1/\alpha - x|}$$

$$= \lim_{x\to 1/a} \frac{\left|\sin\dfrac{c}{2}(x-1/\alpha)(x-1/\beta)\right|}{|\alpha|\dfrac{|c|}{2}|x-1/\alpha||x-1/\beta} \times \frac{|c|}{2}|x-1/\beta|$$

$$= \left|\frac{c}{2\alpha}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\right|.$$

62. $\lim\limits_{x\to 4} f(x) = f(4)$ as f is continuous. Putting $x = y = 2$ in the given functional equation, we have $(f(2))^2 = 2f(2) + f(4) - 2 \Rightarrow 25 - 10 + 2 = f(4)$. Therefore, $f(4) = 17$.

63. $\lim\limits_{x\to\infty} \dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}} \tan(1/x) = \lim\limits_{x\to\infty} \dfrac{1-e^{-2/x}}{1+e^{-2/x}} \tan 1/x$

$= 0.$

64. $f(2) = f(1 + 1) = f(1) + f(1) = 10$

$f(4) = f(2 + 2) = f(2) + f(2) = 20$

Therefore, $\lim\limits_{x\to 4} f(x) = f(4) = 20$.

65. $\lim\limits_{x\to 0+} f(x) = \lim\limits_{x\to 0+} e^x (x + [x]) = 1.0 = 0.$

66. $\lim\limits_{x\to -4} \dfrac{\tan \pi x}{x+4} = \lim\limits_{x\to -4} \dfrac{\pi \sec^2 \pi x}{1}\left(\dfrac{0}{0} \text{ form}\right).$

$= \lim\limits_{x\to\infty} \left(1+1/x^2\right)^x = \lim\limits_{x\to\infty} \left(1+1/x^2\right)^{x^2.1/x} = e^0 = 1$

So the sum of the two given limits is equal to $\pi + 1 > 3$.

67. $\lim\limits_{x\to 0} \left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$

$= \lim\limits_{x\to 0} \left(1+\left(-1+\dfrac{\sin x}{x}\right)\right)^{\frac{x}{\sin x - x}\times -\frac{\sin x}{x}}$

$= \lim\limits_{u\to 0} (1+u)^{\frac{1}{u}\times \lim\limits_{u\to 0}\left(-\frac{\sin x}{x}\right)}$

$= e^{-1}.$

68. $\lim\limits_{x\to 0} \dfrac{(\sin x - \tan x)^2 + (1-\cos 2x)^4 + x^5}{7\tan^7 x + \sin^6 x + 2\sin^5 x}$

$$= \lim_{x\to 0} \frac{\sin^2 x\left(\dfrac{1-\cos^2 x}{\cos^2 x}\right)^2 + 16\sin^8 x + x^5}{7\dfrac{\sin^7 x}{\cos^7 x} + \sin^6 x + 2\sin^5 x}$$

$$= \lim_{x\to 0} \frac{\left(\dfrac{\sin x}{x}\right)^5 \sin x.\sec^4 x + 16\left(\dfrac{\sin x}{x}\right)^5 \sin^3 x + 1}{7\left(\dfrac{\sin x}{x}\right)^5 \sin^2 x \sec^7 x + \left(\dfrac{\sin x}{x}\right)^5 \sin x + 2\left(\dfrac{\sin x}{x}\right)^5}$$

$$= \frac{1\times 0\times 1 + 16\times 1\times 0 + 1}{7\times 1\times 0\times 1 + 1\times 0 + 2\times 1} = \frac{1}{2}.$$

69. g is not continuous at $x = 0$. $fog\ (x) = f(1/x)$ $= \dfrac{1-x}{x}$. $gofog\ (x) = g\left(\dfrac{1-x}{x}\right) = \dfrac{x}{1-x}$ which is not continuous at $x = 1$. Hence $gofog$ is.

continuous except for $x = 0, 1$ i.e $(-\infty, \infty) \sim \{0, 1\}$

70. $k^2 x^2 - 1 \le [(kx)^2] \le k^2 x^2$ for all k, so

$x^2 (1^2 + 2^2 + \cdots + n^2) - 1 \le [x^2] + [(2x)^2] + \cdots + [(nx)^2]$

$\le x^2 (1^2 + 2^2 \cdots + n^2)$

$$\Rightarrow \frac{x^2 n(n+1)(2n+1)}{6n^3} - \frac{1}{n^3}$$

$$\le \frac{\left[x^2\right] + \left[(2x)^2\right] \cdots + \left[(nx)^2\right]}{n^3}$$

$$\le \frac{x^2 n(n+1)(2n+1)}{6n^3}$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{x^2(1+1/n)(2+1/n)}{6} - \frac{1}{3}\right)$$

$$\le \lim_{n\to\infty}\frac{[x^2]+[(2x)^2]+\cdots+[(nx)^2]}{n^3}$$

$$\le \lim_{n\to\infty}\frac{x^2(1+1/n)(2+1/n)}{6}$$

$\Rightarrow \dfrac{x^2}{6} \le f(x) \le \dfrac{x^2}{6}$ so $f(x) = \dfrac{x^2}{3}$ which is continuous on $(-\infty, \infty)$

71. If $x \in Q$ (the set of rational numbers) then we can find a sequence of irrational numbers x_n converging to x. But $f(x_n) = 0$ and $f(x) = 1$. So f is discontinuous at every rational number. Similarly, if x is irrational number.

72. $\lim\limits_{x\to 1/2} f(x) = \dfrac{1}{2} = f\left(\dfrac{1}{2}\right)$. So f is continuous at $x = \dfrac{1}{2}$. f is discontinuous at every other point

73. Let $f(x) = x$ in (a), $g(x) = |x|$ is not one-one.

Take in (b), $f(x) = \begin{cases} 1 \text{ if } x \ge 0 \\ -1 \text{ if } x < 0 \end{cases}$ then $g(x) = 1$ for all x so g is continuous but not f. If f is continuous then g is composition of two continuous functions so is continuous. Take $f(x) = x$ in (d), f is onto but $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = |x|$ is not onto.

74. If $|x| < 1$ then $x^{2n} \to 0$ as $n \to \infty$ and if $|x| > 1$ then $1/x^{2n} \to 0$ as $n \to \infty$. So,

$$f(x) = \begin{cases} \log(3+x) & \text{for } -1 < x < 1 \\ -\sin x & \text{for } x > 1 \; or \; x < -1 \\ \frac{1}{2}[\log{(3 + \mathrm{x})} - \sin \mathrm{x}] & \text{for x} = 1, -1 \end{cases}$$

$\lim\limits_{x\to 1+} f(x) = -\sin 1$, $\lim\limits_{x\to 1-} f(x) = \log 4$.

$\lim\limits_{x\to -1+} f(x) = \log 2$, $\lim\limits_{x\to 1-} f(x) = \sin 1$

so f is continuous on $\mathbf{R} \sim \{-1, 1\}$.

75. $f(x) = \left(1 + \dfrac{4x+1}{x^2+x+2}\right)^{\frac{x^2+x+2}{4x+1} \times \frac{(4x+1)}{x^2+x+1}}$

$$= (1+u)^{\frac{1}{u} \times \frac{4+1/x}{1+1/x+2/x^2}}, \; u = \frac{4x+1}{x^2+x+2}$$

Since $u \to 0$ as $x \to \infty$ so $\lim\limits_{x\to\infty} f(x) = e^4$.

76. $f(x) = \log x$ satisfy this functional equation.

f is clearly not bounded $f(1/x) = -\log x \to \infty$ as $x \to 0+$, $x \log x \to -\infty$ as $x \to 0+$.

77. The point $x = 1$, is a discontinuity of the function f. If $x \ne 1$, then $u(x) = f(f(x)) = \dfrac{x-1}{x}$ Hence $x = 0$ is a point of discontinuity of u. If $x \ne 0$, $x \ne 1$ then $fofof(x) = x$. Hence $y(x) = f^{3n}(x) = x$ is continuous everywhere except $x = 0, 1$.

78. $f(0) = \lim\limits_{x\to 0}\dfrac{1-\cos x+\sin x}{1-\cos x-\sin x}$

$$= \lim_{x\to 0}\frac{2\sin x/2(\sin x/2+\cos x/2)}{2\sin x/2(\sin x/2-\cos x/2)}$$

$$= \lim_{x\to 0}\frac{\sin x/2+\cos x/2}{\sin x/2-\cos x/2} = -1.$$

79. If f is continuous then $\lim\limits_{x\to \pi/2+} f(x) = f(\pi/2) = \lim\limits_{x\to \pi/2-} f(x)$

so $a\dfrac{\pi}{2} + 1 = 1 + b \Rightarrow a\dfrac{\pi}{2} = b$.

80. $x^3 + x^2 - 16x + 20 = (x-2)^2 (x+5)$. Hence $\lim\limits_{x\to 2} f(x) = 7 = K$.

81. For $x \ne 2$, $f(x) = \dfrac{16\left(2^{x-2}-1\right)}{16\left(4^{x-2}-1\right)}$

$$= \frac{2^{x-2}-1}{x-2} \times \frac{x-2}{4^{x-2}-1}$$

So $\lim\limits_{x\to 2} f(x) = \dfrac{\log 2}{\log 4} = \dfrac{1}{2}$.

82. For $x \neq 0$, $f(x) = \dfrac{-2\sin^2 x}{x^2+1-1} \times \left(\sqrt{x^2+1}+1\right)$

$= 2\dfrac{\sin^2 x}{x^2} \times \left(\sqrt{x^2+1}+1\right)$

Thus $\lim\limits_{x\to 0} f(x)$ = – 2 × 1 × 2 = – 4.

83. $\lim\limits_{x\to 0}\dfrac{x}{a}\left[\dfrac{b}{x}\right] = \lim\limits_{x\to 0}\dfrac{x}{a}\left[\dfrac{b}{x}-\left\{\dfrac{b}{x}\right\}\right]$

($\{x\}$ is the fractional part of x)

$= \lim\limits_{x\to 0}\left(\dfrac{b}{a}-\dfrac{b}{a}\dfrac{\{b/x\}}{b/x}\right)$

$= \dfrac{b}{a} - \dfrac{b}{a}\lim\limits_{y\to\infty}\dfrac{\{y\}}{y}$

$= \dfrac{b}{a}$.

Since $0 \le \{y\} < 1$ so $\dfrac{\{y\}}{y} \le \dfrac{1}{y}$ for $y > 0$

Hence $\lim\limits_{y\to\infty}\dfrac{\{y\}}{y} = 0$.

84. Since $7 < e^2 < 8$ so $[e^2] = 7$ and $[-e^2] = -8$, so

$f(0) = \lim\limits_{x\to 0}\dfrac{\tan 7x^2 - \tan(-8)x^2}{\sin^2 x}$

$= \lim\limits_{x\to 0}\left(7\dfrac{\tan 7x^2}{7x^2}\times\dfrac{x^2}{\sin^2 x}+8.\dfrac{\tan 8x^2}{8x^2}\times\dfrac{x^2}{\sin^2 x}\right)$

= 7 + 8 = 15.

85. $f(0) = \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\dfrac{\cos ax - \cos bx}{x^2}$

$= \lim\limits_{x\to 0}\dfrac{-a\sin ax + b\sin bx}{2x}\left(\dfrac{0}{0}\right)$

$= \lim\limits_{x\to 0}\dfrac{-a^2\cos ax + b^2\cos bx}{2}$

$= \dfrac{b^2-a^2}{2}$.

Previous Years' AIEEE/JEE Main Questions

1. $f(0) = 0 = \lim\limits_{x\to 0} f(x)$, so f is continuous at $x = 0$. If $x_0 \in Q$ then $\lim\limits_{x\to x_0} f(x)$ does not exist similarly if x_0 is irrational.

2. $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos 2x}}{\sqrt{2}x} = \lim\limits_{x\to 0}\dfrac{|\sin x|}{\sqrt{2}x}$ which doesn't exist as $\lim\limits_{x\to 0+}\dfrac{|\sin x|}{\sqrt{2}x} = \dfrac{1}{\sqrt{2}}$ and $\lim\limits_{x\to 0-}\dfrac{|\sin x|}{\sqrt{2}x} = -\dfrac{1}{\sqrt{2}}$.

3. $\lim\limits_{x\to\infty}\left(\dfrac{x^2+5x+3}{x^2+x+3}\right)^x = \lim\limits_{x\to\infty}\left(1+\dfrac{4x}{x^2+x+3}\right)^x$

$= \lim\limits_{x\to\infty}\left(1+\dfrac{4x}{x^2+x+3}\right)^{\frac{x^2+x+3}{4x}\times\frac{4x^2}{x^2+x+3}}$

$= \lim\limits_{x\to\infty}\left(1+\dfrac{4x}{x^2+x+3}\right)^{\frac{x^2+x+3}{4x}\times\frac{4}{1+1/x+3/x^2}}$

$= e^4$.

4. $\lim\limits_{x\to 0}\dfrac{\log(3+x)-\log(3-x)}{x} = \lim\limits_{x\to 0}\dfrac{1}{x}\left[\log\dfrac{3+x}{3-x}\right]$

$= \lim\limits_{x\to 0}\dfrac{1}{x}\left[\log\left(1+\dfrac{2x}{3-x}\right)\right]$

$= \lim\limits_{x\to 0}\dfrac{\log\left(1+\dfrac{2x}{3-x}\right)}{\dfrac{2x}{3-x}}\times\dfrac{2x}{x(3-x)}$

$= 1\times\dfrac{2}{3} = \dfrac{2}{3}$.

5. $\lim\limits_{x\to\pi/2}\dfrac{1-\tan\dfrac{x}{2}}{1+\tan\dfrac{x}{2}}\,\dfrac{1-\sin x}{(\pi-2x)^3}$

$= \dfrac{1}{8}\lim\limits_{x\to\pi/2}\dfrac{\tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)(1-\sin x)}{(\pi/2-x)^3}$

$= \dfrac{1}{8}\lim\limits_{u\to 0}\dfrac{\left(\tan\dfrac{1}{2}u\right)(1-\sin(\pi/2-u)}{u^3}$

$= \dfrac{1}{8}\lim\limits_{u\to 0}\dfrac{1}{2}\dfrac{\tan u/2}{u/2}\times\dfrac{1-\cos u}{u^2}$

$= \dfrac{1}{16}\lim\limits_{u\to 0}\dfrac{\tan u/2}{u/2}\,\dfrac{2\sin^2 u/2}{4(u/2)^2}$

$= \dfrac{1}{32}$.

6. We have $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}+\dfrac{b}{x^2}\right)^{2x} = e^{2a}$

$\therefore\ e^{2a} = e^2$

Thus, $a = 1$ and b may be any real number.

7. We have $\lim\limits_{x\to\pi/4} f(x) = \lim\limits_{x\to\pi/4} \dfrac{1-\tan x}{4x-\pi/4}$

$$= \lim_{x\to\pi/4} \frac{-\sec^2 x}{4}$$

$$= \frac{-(\sqrt{2})^2}{4} = -\frac{1}{2}.$$

8. $\lim\limits_{x\to\alpha} \dfrac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$

$$= \lim_{x\to\alpha} \frac{\sin(ax^2+bx+c)(2ax+b)}{2(x-\alpha)} \quad \text{[L$'$ Hospital Rule]}$$

$$= \lim_{x\to\alpha} \frac{\cos(ax^2+bx+c)(2ax+b)^2+\sin(ax^2+bx+c)(2a)}{2}$$

$$= \frac{\cos(a\alpha^2+b\alpha+c)(2a\alpha+b)^2+\sin(a\alpha^2+b\alpha+c)(2a)}{2}$$

$$= \frac{\cos(0)\left(2\alpha+\dfrac{b}{a}\right)^2 a^2+\sin(0)(2a)}{2}$$

$$= \frac{a^2}{2}(2\alpha-\alpha-\beta)^2 \qquad \left[\because -\frac{b}{a} = \alpha+\beta\right]$$

$$= \frac{1}{2}a^2(\alpha-\beta)^2$$

9. $f(0) = \lim\limits_{x\to 0} f(x)$

$$= \lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{e^{2x}-1}\right)$$

$$= \lim_{x\to 0} \frac{e^{2x}-1-2x}{x(e^{2x}-1)}\left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{2e^{2x}-2}{e^{2x}-1+2xe^{2x}}\left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0} \frac{4e^{2x}}{4e^{2x}(1+x)} = 1.$$

10. As f is an increasing function, and

$x < 2x < 3x\ \forall\ x > 0$,

we get

$f(x) < f(2x) < f(3x)\ \forall\ x > 0$

$$\Rightarrow 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)} \qquad [\because f(x) > 0]$$

Since $\lim\limits_{x\to\infty} 1 = 1$ and

$\lim\limits_{x\to\infty} \dfrac{f(3x)}{f(x)} = 1$ (given), by the Sandwich theorem, we get

$$1 \le \lim_{x\to\infty} \frac{f(2x)}{f(x)} \le 1 \Rightarrow \lim_{x\to\infty} \frac{f(2x)}{f(x)} = 1.$$

11. f is continuous at all points except possibly at $x = 0$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{\sin(p+1)x+\sin x}{x}$$

$$= (p+1)\lim_{x\to 0-} \frac{\sin(p+1)x}{(p+1)x} + \lim_{x\to 0-} \frac{\sin x}{x}$$

$$= (p+1) + 1 = p + 2$$

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}} = \lim_{x\to 0+} \frac{\sqrt{1+x}-1}{x}$$

$$= \lim_{x\to 0+} \frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}+1\right)}{x\left(\sqrt{x+1}+1\right)}$$

$$= \lim_{x\to 0+} \frac{x+1-1}{x}\cdot\frac{1}{\sqrt{x+1}+1} = \frac{1}{2}$$

For continuity at $x = 0$, we must have $p + 2 = 1/2$ $= q \Rightarrow p = -3/2$, $q = 1/2$.

12. As $\lim\limits_{x\to 5} \dfrac{(f(x))^2-9}{\sqrt{|x-5|}} = 0$,

we have

$(f(x))^2 - 9 = k|x-5|^{\alpha}P(x)$

where $\alpha > 1/2$ and $P(x)$ is some function of x.

$$= \lim_{x\to 5} f(x)^2 = \lim_{x\to 5}\,[9 + k\,|x-5|^{\alpha}P(x)]$$

$$= 9 + 0 = 9$$

Thus, only possible answer is $\lim\limits_{x\to 5} f(x) = 3$.

13. f_2 is not continuous at $x = 0$, as $\lim\limits_{x\to 0}\sin\left(\dfrac{1}{x}\right)$ does not exist. $\therefore$ Statement –2 is false.

$$F(x) = \begin{cases} x\sin 1/x, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

$\lim\limits_{x\to 0} F(x) = 0$ so F is continuous on **R**.

14. $1 - \cos(2(x-2)) = 2\sin^2(x-2)$

$$\Rightarrow \sqrt{1-\cos(2(x-2))} = \sqrt{2}\,|\sin(x-2)|$$

Now, $\lim\limits_{x\to 2-} \dfrac{\sqrt{1-\cos(2(x-2))}}{x-2}$

$$= \lim_{x\to 2-} \frac{-\sqrt{2}\sin(x-2)}{x-2}$$

$$= -\sqrt{2}$$

and $\lim\limits_{x\to 2+} \dfrac{\sqrt{1-\cos(2(x-2))}}{x-2}$

$$= \lim_{x\to 2+} \frac{\sqrt{2}\sin(x-2)}{x-2} = \sqrt{2}$$

As $\lim\limits_{x\to 2-} \dfrac{\sqrt{1-\cos(2(x-2))}}{x-2}$

$$\ne \lim_{x\to 2+} \frac{\sqrt{1-\cos(2(x-2))}}{x-2}$$

we get $\lim\limits_{x\to 2} \dfrac{\sqrt{1-\cos(2(x-2))}}{x-2}$ does not exist.

15. $f(x) = [x]\cos\left(\dfrac{2x-1}{2}\right)\pi$

Let n be any integer, then $[x] = n$. For $n \le x < n+1$.

Thus,

$f(x) = n\cos\left(\dfrac{2x-1}{2}\right)\pi$ for $n \le x < n+1$.

$$= \begin{cases} -\cos\left(\dfrac{2x-1}{2}\right)\pi & \text{for} -1 \le x < 0 \\ 0 & \text{for}\, 0 \le x < 1 \\ \cos\left(\dfrac{2x-1}{2}\right)\pi & \text{for}\, 1 \le x < 2 \\ 2\cos\left(\dfrac{2x-1}{2}\right)\pi & \text{for}\, 2 \le x < 3 \end{cases}$$

Note that f is continuous at all points except possibly at …, –3, –2, 0, 1, …

We check the continuity at $x = n \in I$.

$$\lim_{x\to n-} f(x) = \lim_{x\to n-} (n-1)\cos\left(\frac{2x-1}{2}\right)\pi = 0$$

and $\lim\limits_{x\to n+} f(x) = \lim\limits_{x\to n+} n\cos\left(\dfrac{2x-1}{2}\right)\pi = 0$

Thus, f is continuous at each $x \in$ **R**.

16. $\lim\limits_{x\to 0} \dfrac{(1-\cos 2x)(3+\cos x)}{x\tan 4x}$

$$= \lim_{x\to 0} \frac{(2\sin^2 x)(3+\cos x)}{x\tan 4x}$$

$$= \frac{2}{4}\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \frac{4x}{\tan 4x}(3+\cos x)$$

$$= \frac{1}{2}\,(1^2)\,(1)\,(4) = 2$$

17. As f is continuous at $x = \pi$,

$$k = \lim_{x\to\pi} \frac{\sqrt{2+\cos x}-1}{(\pi - x)^2}$$

Put $\pi - x = \theta$, so that

$$k = \lim_{\theta\to 0} \frac{\sqrt{2+\cos(\pi-\theta)}-1}{\theta^2}$$

$$= \lim_{\theta\to 0} \frac{\sqrt{2-\cos\theta}-1}{\theta^2}$$

$$= \lim_{\theta\to 0} \frac{(\sqrt{2-\cos\theta}-1)(\sqrt{2-\cos\theta}+1)}{\theta^2(\sqrt{2-\cos\theta}+1)}$$

$$= \lim_{\theta\to 0} \frac{2-\cos\theta-1}{\theta^2}\cdot\frac{1}{\sqrt{2-\cos\theta+1}}$$

$$= \lim_{\theta\to 0} \frac{(1-\cos\theta)(1+\cos\theta)}{\theta^2}\,\frac{1}{1+\cos\theta}\cdot\frac{1}{\sqrt{2-\cos\theta+1}}$$

$$= 1\left(\frac{1}{2}\right)\left(\frac{1}{1+1}\right) = \frac{1}{4}$$

18. $\lim\limits_{x\to 0} \dfrac{\sin(\pi\cos^2 x)}{x^2} = \lim\limits_{x\to 0} \dfrac{\sin(\pi - \pi\sin^2 x)}{x^2}$

$$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\frac{\pi\sin^2 x}{x^2}$$

$$= \pi.$$

19. As f is continuous

$$\lim_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right) = f\left(\lim_{x\to 0}\frac{1-\cos 3x}{x^2}\right)$$

$$= f\left(\lim_{x\to 0} 9\left(\frac{\sin 3x}{3x}\right)^2 \frac{1}{1+\cos 3x}\right)$$

$$= f\left(\frac{9}{2}\right) = \frac{2}{9}$$

20. $\lim\limits_{x\to 2} \dfrac{\tan(x-2)(x-2)(x+k)}{(x-2)^2} = 5$

$$\Rightarrow \lim_{x\to 2} \frac{\tan(x-2)}{(x-2)}(x + k) = (1)(2 + k) = 5$$

$$\Rightarrow k = 3$$

21. $\lim\limits_{x\to 0} \dfrac{e^{x^2} - \cos x}{\sin^2 x}$

$$= \lim_{x\to 0} \frac{(e^{x^2} - 1)(1 - \cos x)}{\sin^2 x}$$

$$= \lim_{x\to 0} \left[\frac{(e^{x^2} - 1)/x^2}{(\sin x/x)^2} + \frac{1}{1+\cos x}\right]$$

$$= \frac{1}{1} + \frac{1}{2} = \frac{3}{2}.$$

22. Same as Q. 16.

23. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} 4k\left(\dfrac{e^x - 1}{x}\right)^2 \left(\dfrac{x/k}{\sin(x/k)}\right)\left(\dfrac{x/4}{\log(1+x/4)}\right)$

$= (4k)\ (1)\ (1) = 4k$

As f is continuous at $x = 0$,

$12 = f(0) = \lim\limits_{x\to 0} f(x) = 4k \Rightarrow k = 3.$

24. $\log p = \lim\limits_{x\to 0+} \log\left(1 + \tan^2\sqrt{x}\right)^{\frac{1}{2x}}$

$$= \lim_{x\to 0+} \frac{\log\left(1+\tan^2\sqrt{x}\right)}{2x}$$

$$= \frac{1}{2}\lim_{x\to 0+} \frac{\log\left(1+\tan^2\sqrt{x}\right)}{\tan^2\sqrt{x}} \times \left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)^2$$

$$= \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}.$$

25. $e^3 = \lim\limits_{x\to\infty} 4k\left(1 + \dfrac{a}{x} - \dfrac{4}{x^2}\right)^{2x}$

$$= \lim_{x\to\infty}\left(1 + \frac{ax-4}{x^2}\right)^{2x}$$

$$= \lim_{x\to\infty}\left[\left(1 + \frac{ax-4}{x^2}\right)^{\frac{x^2}{ax-4}}\right]^{\left(a-\frac{4}{x}\right)(2)}$$

$$= e^{2a}$$

$$\Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}.$$

26. f is continuous on all points except possibly at $x = 1$ and $x = \sqrt{2}$.

$$\therefore \lim_{x\to 1-} f(x) = \lim_{x\to 1-} \frac{2x^2}{a} = \frac{2}{a}$$

$\lim\limits_{x\to 1+} f(x) = \lim\limits_{x\to 1+} a = a$. By continuity of f at $x = 1$, we have $a = \dfrac{2}{a} \Rightarrow a = \pm\sqrt{2}$.

Also $\lim\limits_{x\to\sqrt{2}-} f(x) = a$ and $\lim\limits_{x\to\sqrt{2}+} f(x)$

$$= \lim_{x\to\sqrt{2}+} \frac{2b^2 - 4b}{x^3} = \frac{2b^2 - 4b}{2\sqrt{2}} = \frac{b^2 - 2b}{\sqrt{2}}$$

Thus $a = \dfrac{b^2 - 2b}{\sqrt{2}}$.

If $a = \sqrt{2}$, then $b^2 - 2b = 2 \Rightarrow b = 1 \pm \sqrt{3}$,

If $a = -\sqrt{2}$ then $b^2 - 2b = -\sqrt{2} \Rightarrow (b - 1)^2 = 1 - \sqrt{2} < 0$, not possible

Thus $(a, b) = (\sqrt{2}, 1-\sqrt{3})$.

27. $L = \lim\limits_{x\to 0} \dfrac{4\sin^4 x}{x\left(2\tan x - \dfrac{2\tan x}{1+\tan^2 x}\right)}$

$$= \lim_{x\to 0} \frac{2\sin^4 x(1+\tan^2 x)}{x.\tan x\tan^2 x}$$

$$= 2\lim_{x\to 0} \frac{\sin x}{x}(1+\tan^2 x)\cos^2 x$$

$= 2.\ 1.\ (1 + 0).1 = 2.$

Previous Years' B-Architecture Entrance Examination Questions

1. Take $f(x) = g(x) = \dfrac{1}{x}$ and $c = 0$.

2. $2f\,(x + h) + f(0) = 2f\left(\dfrac{2x+2h}{2}\right) + f(0)$

$$= 2\,\frac{f(2x) + f(2h)}{2} + f(0)$$

$= f(2x) + f(0) + f(2h)$

$$= 2f\left(\frac{2x+0}{2}\right) + f(2h)$$

$= 2f(x) + f(2h)$

Taking limit on both the sides as $h \to 0$, we have

$$2\lim_{h\to 0} f(x+h) + f(0) = 2f(x) + \lim_{h\to 0} f(2h)$$

$= 2f(x) + f(0)$ (f is continuous at 0)

$\Rightarrow \lim\limits_{h\to 0} f(x+h) = f(x)$

$\Rightarrow f$ is continuous at every point.

3. $$\lim_{x\to 1} \sin\frac{\pi}{4}\left(\frac{x\,|x|-1}{|x|-1}\right) = \lim_{x\to 1} \sin\frac{\pi}{4}\left(\frac{x^2-1}{x-1}\right)$$

$$= \sin\frac{\pi}{4}(1+1) = 1.$$

$$\lim_{x\to 1} \tan\frac{\pi}{4}\left(\frac{x\,|x|-1}{|x|-1}\right) = \lim_{x\to 1} \tan\frac{\pi}{4}\left(\frac{x^2-1}{x-1}\right)$$

$= \lim\limits_{x\to 1} \tan\frac{\pi}{4}(x+1)$, which does not exist.

4. $$\lim_{x\to 0} f(x) = \lim_{x\to 0+} f(x) = -\lim_{x\to 0+} f(-x) = -\lim_{x\to 0-} f(x)$$

$$= -\lim_{x\to 0} f(x)$$

Hence $\lim\limits_{x\to 0} f(x) = 0$.

Thus $$\lim_{x\to 0} \frac{1}{2f(x)-1} = \frac{1}{\lim\limits_{x\to 1}(2f(x)-1)} = -1.$$

5. $$\lim_{x\to\infty}\left[\frac{x^2+1}{x-1} - \alpha x - 2\beta\right]$$

$$= \lim_{x\to\infty} \frac{x^2(1-\alpha) - (2\beta-\alpha)x + 1 + 2\beta}{x-1}$$

Since the last exists so $1 - \alpha = 0 \Rightarrow \alpha = 1$. In this case the last limit is equal to

$$\lim_{x\to\infty} \frac{-(2\beta-\alpha) + \dfrac{1+2\beta}{x}}{1-\dfrac{1}{x}} = -(2\beta - \alpha) = \frac{3}{2}$$

$$\Rightarrow -2\beta = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow \beta = -\frac{1}{4}.$$

6. $[x]$ is not continuous at $x \in I$ and $|x|$ is a continuous function so $|x| + [x - 1]$ is neither continuous at $x = 0$ nor at $x = 1$.

7. $$\lim_{x\to 0} \frac{\sin 2x + a\sin x}{x^3} = \lim_{x\to 0} \frac{2\cos 2x + a\cos x}{3x^2} \left(\frac{0}{0} \text{ form}\right)$$

The last limits will exist if $a = -2$. In this case the last limit is equal to

$$\lim_{x\to 0} \frac{-4\sin 2x + 2\sin x}{6x} = \lim_{x\to 0}\left[-\frac{4}{3}\frac{\sin 2x}{2x} + \frac{1}{3}\frac{\sin x}{x}\right]$$

$$= -\frac{4}{3} + \frac{1}{3} = -1.$$

8. For $x \neq 0$,

$$\frac{f(x)}{x^2} = \begin{vmatrix} \dfrac{\sin x}{x} & \cos x & \tan x \\ x & x & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \lim_{x\to 0} \frac{f(x)}{x^2} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 1$$

9. f is continuous on $\mathbf{R} \sim \mathbf{Z}$ and g is continuous on $\mathbf{R} \sim \mathbf{Z}$ so $\phi(x) = fog(x)$ is continuous on $\mathbf{R} \sim \mathbf{Z}$.

Next, $\psi(x) = (gof)(x) = g(f(x)) = g([x]) = 0 \ \forall\ x \in \mathbf{R}$.
$\Rightarrow \psi$ is continuous on $\mathbf{R}$ in particular on $\mathbf{R} \sim \mathbf{Z}$.

10. For $0 < x < 1$, $-1 < x - 1 < 0$, so

$F(x) = (1 - x) + (-1) + (1 - x) = 1 - 2x$

$\Rightarrow \lim\limits_{x\to 1-} f(x) = 1 - 2.(1) = -1.$

For $1 < x < 2$, $0 < x - 1 < 1$, so

$f(x) = (1 - x) + 0 + (x - 1) = 0$

$\Rightarrow \lim\limits_{x\to 1+} f(x) = 0$. Thus $\lim\limits_{x\to 1} f(x)$ does not exist.

CHAPTER TEN

Differentiability and Differentiation

DIFFERENTIABILITY

Let f be a real-valued function defined on an open interval containing a point x_0.
then f is said to be *differentiable* at x_0 if

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

exists. If this limit exists we call it the derivative of $f(x)$ at x_0 and denote it by $f'(x_0)$ or by

$$\left.\frac{dy}{dx}\right|_{x=x_0}$$

If this limit does not exist, we say that f is not *differentiable* at $x = x_0$.

Equivalently $\lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$ exists. Graphically the graph of f has a non vertical tangent line at $x = x_0$.

Using one-sided limits, we define the *right-hand derivative*, denoted by $f'(x_0+)$, at $x = x_0$ as

$$\lim_{\Delta x \to 0+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

and the *left-hand derivative*, denoted by $f'(x_0-)$, at $x = x_0$ as

$$\lim_{\Delta x \to 0-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

Thus, a function $f(x)$ is derivable at $x = x_0$ if $f'(x_0+) = f'(x_0-)$.

Every differentiable function is continuous but the converse may not be true. That is, a continuous function need not be differentiable. e.g.

Illustration 1

$f(x) = |x|$ at $x = 0$,

$$f'(0+) = \lim_{\Delta x\to 0+} \frac{f(0+\Delta x)-f(0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0+} \frac{|\Delta x|}{\Delta x} = 1$$

$$f'(0-) = \lim_{\Delta x\to 0-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x\to 0-} \frac{-\Delta x}{\Delta x}$$

(as $\Delta x \to 0-$, we have $\Delta x < 0$)

$$= -1$$

Thus f is not differentiable at $x = 0$.

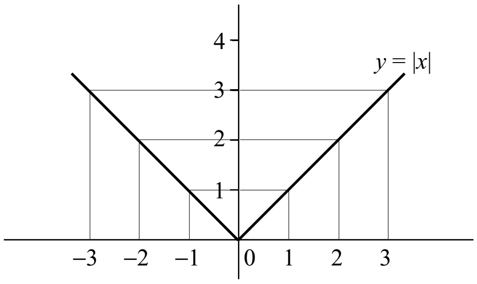


Fig. 10.1

The *absolute value* function is not differentiable at $x = 0$. Clearly f is a continuous function. Similarly if $f(x) = |x - a|$, then f is not differentiable at $x = a$, $f'(a+) = 1$ and $f'(a-) = -1$.

Geometrically, we interpret

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

as the slope of the graph at the point $(x, f(x))$. The line through $(x, f(x))$ which has this slope is called the tangent line at $(x, f(x))$. Thus, if there is no tangent line at a certain point or has a vertical tangent line, the function is not differentiable at that point. In other words, a function is not differentiable at a corner point of a curve, i.e., a point where the curve suddenly changes direction (Fig. 10.2).

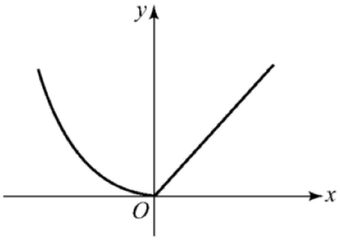


Fig. 10.2

DIFFERENTIABILITY ON AN INTERVAL

A function $f(x)$, defined on $[a, b]$, is said to be differentiable on $[a, b]$ if it is differentiable at every c $\in$ (a, b) and both

$$\lim_{h \to 0+} \frac{f(a+h)-f(a)}{h} \text{ and } \lim_{h \to 0-} \frac{f(b+h)-f(b)}{h}$$

exist.

Continuously Differentiable

A function f is said to be continuously differentiable if the derivative $f'(x)$ exists and is itself a continuous function. e.g.

Illustration 2

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h\to 0} h \sin \frac{1}{h} = 0 .$$

For $x \neq 0$ $f'(x) = 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}$

Since $\lim\limits_{x\to 0} f'(x)$ does not exist. So f is differentiable but not continuously differentiable. All polynomials are continuously differentiable.

HIGHER ORDER DIFFERENTIATION

Let $y = f(x)$ be a differentiable function such that $z = f'(x)$ is also differentiable. Then the second derivative of $y = f(x)$ is denoted by $y_2(x)$, $f''(x)$ or $\mathrm{d}^2y/\mathrm{d}x^2$, and is defined by

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(= \frac{dz}{dx}\right)$$

Similarly, we can define

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = f'''(x) \text{ and } f^{(\mathrm{iv})}(x) = \frac{\mathrm{d}^4 y}{\mathrm{d}x^4}$$

or, in general, $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}\right)$ for any $n \in \mathbf{N}$, $n > 1$.

Many functions, like polynomial, trigonometric, exponential and logarithmic functions, are differentiable at each point of their domain.

SOME FORMULAE OF DIFFERENTIATION

Let $f(x)$ and $g(x)$ be differentiable functions and $\alpha \in \mathbf{R}$.

1. *Sum and Difference Rule*

   $$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) \pm g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) \pm \frac{\mathrm{d}}{\mathrm{d}x}(g(x))$$

2. *Scalar Multiple Rule*

   $$\frac{\mathrm{d}}{\mathrm{d}x}(\alpha f(x)) = \alpha \frac{\mathrm{d}}{\mathrm{d}x} f(x)$$

3. *Product rule.*

   $$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x)) = g(x)\frac{\mathrm{d}}{\mathrm{d}x}(f(x)) + f(x)\frac{\mathrm{d}}{\mathrm{d}x}(g(x))$$

4. *Quotient rule*

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\dfrac{\mathrm{d}}{\mathrm{d}x}(f(x)) - f(x)\dfrac{\mathrm{d}}{\mathrm{d}x}(g(x))}{(g(x))^2}$$

   provided g $(x) \neq 0$

5. *Chain rule* If $y = h(u)$ and $u = f(x)$, then

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

6. *Test of constancy* If at all points of a certain interval, $f'(x) = 0$, then the function $f(x)$ has a constant value within this interval.

DIFFERENTIATION OF SOME ELEMENTARY FUNCTIONS

1. The trigonometric functions have the following derivatives:

   $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$ $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$

   $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$ $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\operatorname{cosec}^2 x$

   $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$

   $\dfrac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

2. If $f(x)$ is a differentiable function

   $\dfrac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$ and in general

   $$\frac{\mathrm{d}}{\mathrm{d}x}(f(x)^n) = n(f(x))^{n-1} f'(x)$$

3. If $f(x)$ is a differentiable function

   $\dfrac{\mathrm{d}}{\mathrm{d}x}(\log x) = \dfrac{1}{x}$ and in general

   $$\frac{\mathrm{d}}{\mathrm{d}x}(\log f(x))) = \frac{1}{f(x)} f'(x)$$

4. $\dfrac{d}{dx}(a^x) = a^x \log a$. In particular,

$$\frac{d}{dx}(e^x) = e^x \quad \text{and} \quad \frac{d}{dx}(e^{f(x)}) = e^{f(x)} f'(x)$$

5. $\dfrac{d}{dx}(\sin^{-1}x) = -\dfrac{d}{dx}(\cos^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$.

At the points $x = \pm 1$, $\sin^{-1} x$ and $\cos^{-1} x$ are not differentiable.

6. $\dfrac{d}{dx}(\tan^{-1} x) = -\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{1}{1+x^2}$, for $x \in \mathbf{R}$

7. $\dfrac{d}{dx}(\sec^{-1}x) = -\dfrac{d}{dx}(\operatorname{cosec}^{-1}x) = \dfrac{1}{|x|\sqrt{x^2-1}}$ for $|x| > 1$.

DIFFERENTIATION OF IMPLICIT FUNCTIONS

To find dy/dx when a differentiable function $y = y(x)$ satisfies the equation $F(x, y) = 0$, we differentiate F with respect to x, considering y as a function of x, and solve the resulting equation to obtain for dy/dx.

Illustration 3

To find $\dfrac{dy}{dx}$ if y satisfies $x^3 + y^3 - 3\,axy = 0$.

Differentiating w.r.t. x, we have

$$3x^2 + 3y^2\frac{dy}{dx} - 3a\left[x\frac{dy}{dx} + y\right] = 0$$

$$\Rightarrow \quad [3y^2 - 3ax]\frac{dy}{dx} = 3ay - 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

DIFFERENTIATION OF FUNCTIONS REPRESENTED PARAMETRICALLY

Let there be given two functions $x = \varphi(t)$ and $y = \psi(t)$, where $t \in (\alpha, \beta)$. If the function $x = \varphi(t)$ has an inverse, $t = \varphi^{-1}(x)$ on the interval (α, β), then the new function defined by $y(x) = \psi(\varphi^{-1}(x))$ is said to represent y parametrically. We find its derivative by using the relation

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

Illustration 4

$$x = e^t \sin t,\ y = e^t \cos t$$

To find $\dfrac{dy}{dx}$, we compute first $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

$$\frac{dx}{dt} = e^t \cos t + e^t \sin t = e^t(\sin t + \cos t)$$

$$\frac{dy}{dt} = -e^t \sin t + e^t \cos t = e^t(\cos t - \sin t)$$

So, $$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{\cos t - \sin t}{\cos t + \sin t}.$$

LOGARITHMIC DIFFERENTIATION

The process of taking logarithms of both sides and then differentiating them is called logarithmic differentiation. It is useful in the following cases:

1. If the given function consists of three or more factors which are functions of x.
2. If the given function is of the form $(f(x))^{h(x)}$
3. If the given function is of the form $h_1(x)h_2(x)/h_3(x)h_4(x)$.

If $h(x) = (f_2(x))^{f_1(x)} f_3(x)$, then take logarithum of both the sides. The right hand side will be take the form

$$f_1(x)\log f_2(x) + \log f_3(x)$$

The last expression can be differentiated easily.

Illustration 5

$$y = (\sin x)^{\cos x}$$

Taking logarithum of both the sides, we have

$$\log y = \cos x \log \sin x$$

Differentiating, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos x\frac{d}{dx}(\log\sin x) + (\log\sin x)\frac{d}{dx}\cos x$$

$$= \cos x\frac{\cos x}{\sin x} - (\log\sin x)\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - (\log\sin x)\sin x\right]$$

Illustration 6

$$y = \sqrt[3]{\frac{x(x^2+1)}{(x^2-1)^2}}$$

Taking logarithum, we obtain

$$\log y = \frac{1}{3}\left[\log x + \log(x^2+1) - 2\log(x^2-1)\right]$$

Differentiating, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x} + \frac{2x}{x^2+1} - \frac{4x}{x^2-1}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt[3]{\frac{x(x^2+1)}{(x^2-1)^2}} \times \frac{1}{3}\left[\frac{1}{x} + \frac{2x}{x^2+1} - \frac{4x}{x^2-1}\right]$$

LEIBNITZ THEOREM AND *n*th DERIVATIVES

Let $f(x)$ and $g(x)$ be functions both possessing derivatives up to nth order. Then,

$$\frac{d^n}{dx^n}(f(x)\,g(x)) = f^n(x)\,g(x) + {}^nC_1 f^{n-1}(x)\,g^1(x) + {}^nC_2 f^{n-2}(x)\,g^2(x) + \dots + {}^nC_r f^{n-r}(x)\,g^r(x) + \dots + {}^nC_n f(x)\,g^n(x).$$

where for any $k > 1$, $f^k(x) = \frac{d^k}{dx^k}(f(x))$ i.e. kth derivative of $f(x)$

$$\frac{d^n}{dx^n}(x^n) = n!;\quad \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{(-1)^n\, n!}{x^{n+1}};$$

$$\frac{d^n}{dx^n}(\sin x) = \sin\left(x + n\frac{\pi}{2}\right),$$

$$\frac{d^n}{dx^n}(\cos x) = \cos\left(x + n\frac{\pi}{2}\right);$$

$$\frac{d^n}{dx^n}(e^{mn}) = m^n e^{mx}.$$

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: If $x = k(t - \sin t)$, $y = k(1 - \cos t)$ $(k \neq 0)$ then

$$\frac{d^2y}{dx^2} \text{ at } t = \frac{\pi}{2}$$

(a) $-\frac{1}{k}$ (b) $\frac{1}{2k}$

(c) $\frac{1}{k^2}$ (d) $2k$

Ans. (a)

Solution:

$$\frac{dx}{dt} = k(1 - \cos t),\quad \frac{dy}{dt} = k\sin t$$

so
$$\frac{dy}{dx} = \frac{k\sin t}{k(1-\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \cot\frac{t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{1}{2}\left(-\text{cosec}^2\left(\frac{t}{2}\right)\right)\frac{1}{k\left(2\sin^2\left(\frac{t}{2}\right)\right)}$$

$$= -\frac{1}{4k}\text{cosec}^4\frac{t}{2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\pi/2} = -\frac{1}{4k}(\sqrt{2})^4 = -\frac{1}{k}$$

Example 2: If $x = 2t + 3t^2$, $y = t^2 + 2t^3$ then $y'^2 + 2y'^3$ $\left(y' = \frac{dy}{dx}\right)$ is equal to

(a) $2y$ (b) y^2

(c) y (d) $3y + 2$

Ans. (c)

Solution: $\frac{dx}{dt} = 2 + 6t$, $\frac{dy}{dt} = 2t + 6t^2$

So that $$y' = \frac{dy}{dx} = \frac{2t+6t^2}{2+6t} = t$$

$$y'^2 + 2\,y'^3 = t^2 + 2t^3 = y$$

Example 3: If function $y = \frac{x-3}{x+4}$ satisfies the relationship $Ay'^2 = (y - 1)y''$ then A is equal to

(a) 1 (b) 3

(c) 4 (d) 2

Ans. (d)

Solution: $y = 1 - \frac{7}{x+4}$

$$\Rightarrow\quad y' = \frac{7}{(x+4)^2} \text{ and } y'' = -\frac{14}{(x+4)^3}$$

Therefore, the L.H.S. $= A\,y'^2 = \frac{49A}{(x+4)^4}$

and R.H.S. $= (y-1)y'' = \left(\frac{x-3-x-4}{x+4}\right)\left(-\frac{14}{(x+4)^3}\right)$

$$= \frac{98}{(x+4)^4}$$

Hence $A = 2$

Example 4: Let $f(x) = \left|x - \left(\frac{\pi}{2}\right)\right|^3 + \sin^2 x$, then $f''\left(\frac{\pi}{2}\right)$

(a) does not exist (b) is equal to 2

(c) is equal to –2 (d) is equal to 0

Ans. (c)

Solution: $f(x) = \begin{cases} \left(x - \frac{\pi}{2}\right)^3 + \sin^2 x & \text{if } x \geq \frac{\pi}{2} \\ -\left(x - \frac{\pi}{2}\right)^3 + \sin^2 x & \text{if } x < \frac{\pi}{2} \end{cases}$

Let $u(x) = \begin{cases} \left(x - \frac{\pi}{2}\right)^3 & x \geq \frac{\pi}{2} \\ -\left(x - \frac{\pi}{2}\right)^3 & x < \frac{\pi}{2} \end{cases}$

$$u'\left(\frac{\pi}{2}+\right) = \lim_{h\to 0+} \frac{\left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)^3 - 0}{h}$$

$$= \lim_{h\to 0+} \frac{h^3}{h} = \lim_{h\to 0+} h^2 = 0$$

$$u'\left(\frac{\pi}{2}-\right) = \lim_{h\to 0-} \frac{-\left(\frac{\pi}{2} + h - \frac{\pi}{2}\right)^3}{h} = 0$$

$$u'(x) = \begin{cases} 3\left(x - \frac{\pi}{2}\right)^2, & x \geq \frac{\pi}{2} \\ -3\left(x - \frac{\pi}{2}\right)^2, & x < \frac{\pi}{2} \end{cases}$$

$$u''(x) = \begin{cases} 6\left(x - \frac{\pi}{2}\right), & x > \frac{\pi}{2} \\ -6\left(x - \frac{\pi}{2}\right), & x < \frac{\pi}{2} \end{cases}$$

$$u''\left(\frac{\pi}{2}+\right) = \lim \frac{3\left(h + \frac{\pi}{2} - \frac{\pi}{2}\right)^2 - 0}{h} = 0$$

If $v(x) = \sin^2 x$, then $v'(x) = 2 \sin x \cos x = \sin 2x$

so $v''(x) = 2\cos 2x$ and $v''\left(\frac{\pi}{2}\right) = -2$

Hence $f''\left(\frac{\pi}{2}\right) = 0 - 2 = -2$

Example 5: If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, then the value of $\frac{\left(1 + y'^2\right)^{3/2}}{y''}$ is equal to

(a) $\frac{a-b}{a+b}$ (b) $\frac{c+a}{c-a}$

(c) 1 (d) $-c$

Ans. (d)

Solution: $2(x - a) + 2(y - b)\,\frac{dy}{dx} = 0$ (i)

$$\Rightarrow \quad y' = \frac{dy}{dx} = -\frac{x-a}{y-b}$$

Differentiating (i) again, we get

$$1 + (y - b)\, y'' + y'^2 = 0$$

$$\Rightarrow \quad y'' = -\frac{1 + y'^2}{(y-b)}$$

So, $$\frac{(1 + y'^2)^{3/2}}{y''} = \frac{\left(1 + \left(\frac{x-a}{y-b}\right)^2\right)^{1/2} \left(1 + y'^2\right)}{-\frac{\left(1 + y'^2\right)}{y-b}}$$

$$= -(y-b)\frac{\left((y-b)^2 + (x-a)^2\right)^{1/2}}{(y-b)}$$

$$= -c$$

Example 6: If $y = \sin(n \sin^{-1} x)$ satisfies $(1 - x^2)y'' - xy' + Ay = 0$ then A is equal to

(a) 1 (b) 2

(c) n^2 (d) n

Ans. (c)

Solution: $y' = \cos\left(n \sin^{-1} x\right) \frac{n}{\sqrt{1 - x^2}}$

Squaring both the sides, we have

$$(1 - x^2)\, y'^2 = n^2 \cos^2 (n \sin^{-1} x)$$

$$= n^2 [1 - \sin^2 (n \sin^{-1} x)]$$

$$= n^2 [1 - y^2]$$

Differentiating again

$$-2xy'^2 + (1 - x^2)\, 2y'y'' = -n^2\, 2yy'$$

$$\Rightarrow \quad (1 - x^2)\, y'' - xy' = -n^2 y$$

Hence $A = n^2$

Example 7: If $y = (x^2 + 1)^{\sin x}$, then $y'(0)$ is equal to

(a) 0 (b) 1

(c) 2 (d) – 1

Ans. (a)

Solution: Using logarithmic differentiation, we have

$$\log y = \sin x\, (\log(x^2 + 1))$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \cos x\, (\log(x^2 + 1)) + \frac{(\sin x)(2x)}{x^2 + 1}$$

$$\Rightarrow \quad \frac{dy}{dx} = (x^2 + 1)^{\sin x} \left[\frac{2x \sin x}{x^2 + 1} + \cos x \left(\log(x^2 + 1)\right)\right]$$

So, $\left.\dfrac{dy}{dx}\right|_{x=0} = 1\,[0 + |\log|]$

$= 0$

Example 8: If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$ then $\dfrac{dy}{dx}$ at $x = 0$ is

(a) 1 (b) –1

(c) 0 (d) 2

Ans. (b)

Solution: Differentiating the given relation implies

$$x\frac{1}{2\sqrt{1+y}}\frac{dy}{dx} + \sqrt{1+y} + \frac{dy}{dx}\sqrt{1+x} + \frac{y}{2\sqrt{1+x}} = 0$$

$$\Rightarrow \left(\frac{x}{2\sqrt{1+y}} + \sqrt{1+x}\right)\frac{dy}{dx} = -\left(\sqrt{1+y} + \frac{y}{2\sqrt{1+x}}\right)$$

Since $y(0) = 0$ So putting these values, we have

$$(0+\sqrt{1})\left.\frac{dy}{dx}\right|_{x=0} = -(\sqrt{1}+0)$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = -1$$

Example 9: If $y = \sin\left(e^{x^2+3x-2}\right)$ then $\left.\dfrac{dy}{dx}\right|_{x=1}$ is equal to

(a) $5e^2 \cos e^2$ (b) $3e^2 \cos e^2$

(c) $e^2 \cos e^2$ (d) $\cos e^2$

Ans. (a)

Solution:

$$\frac{dy}{dx} = \cos\left(e^{x^2+3x-2}\right)\frac{d}{dx}\left(e^{x^2+3x-2}\right)$$

$$= \cos\left(e^{x^2+3x-2}\right)e^{x^2+3x-2}\frac{d}{dx}(x^2+3x-2)$$

$$= \cos\left(e^{x^2+3x-2}\right)e^{x^2+3x-2}(2x+3)$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 5e^2 \cos(e^2)$$

Example 10: If $x = e^{\sin^{-1} y}$ then $\left.\dfrac{dy}{dx}\right|_{x=1}$

(a) –1 (b) 2

(c) 1/2 (d) 1

Ans. (d)

Solution: $x = e^{\sin^{-1} y} \Rightarrow \log x = \sin^{-1} y$

$\Rightarrow \quad y = \sin(\log x)$

$$\frac{dy}{dx} = \cos(\log x)\frac{1}{x},$$

so $\left.\dfrac{dy}{dx}\right|_{x=1} = \cos(\log 1) = \cos 0 = 1$

LEVEL 1

Straight Objective Type Questions

Example 11: If $f'(3) = 2$, then $\displaystyle\lim_{h\to 0}\frac{f(3+h^2)-f(3-h^2)}{2h^2}$ is

(a) 1 (b) 2

(c) 3 (d) 1/2

Ans. (b)

Solution: $\displaystyle\lim_{l-0}\frac{f(3+h^2)-f(3-h^2)}{2h^2}$

$$= \frac{1}{2}\lim_{h\to 0}\frac{f(3+h^2)-f(3)}{h^2} - \frac{1}{2}\lim_{h\to 0}\frac{f(3-h^2)-f(3)}{h^2}$$

$$= \frac{1}{2}[f'(3)+f'(3)] = 2.$$

Example 12: The set of all points where the function $f(x) = \sqrt{1-e^{-x^2}}$ is differentiable is

(a) $(0, \infty)$ (b) $(-\infty, \infty)$

(c) $(-\infty, \infty) \sim \{0\}$ (d) none of these.

Ans. (c)

Solution: For $x \neq 0, f'(x) = \dfrac{1}{2}\dfrac{1}{\sqrt{1-e^{-x^2}}}[-(-2x)e^{-x^2}]$

$$= \frac{x\,e^{-x^2}}{\sqrt{1-e^{-x^2}}}$$

Also $\displaystyle f'(0+) = \lim_{h\to 0+}\frac{f(h)-f(0)}{h}$

$$= \lim_{h \to 0+} \left(\frac{e^{-h^2} - 1}{-h^2} \right)^{1/2} = 1$$

and $$f'(0-) = -\lim_{h \to 0-} \left(\frac{e^{-h^2} - 1}{-h^2} \right)^{1/2} = -1.$$

Hence the set of all points of differentiability is $(-\infty, \infty) \sim \{0\}$.

◉ **Example 13:** If $f(x) = |\log_{10} x|$ then at $x = 1$

(a) f is not continuous
(b) f is continuous but not differentiable
(c) f is differentiable
(d) the derivative is 1.

Ans. (b)

◎ **Solution:** Since $g(x) = |x|$ is continuous but not differentiable at $x = 0$, so f is continuous but not differentiable for $\log_{10} x = 0$ i.e. for $x = 1$.

◉ **Example 14:** $f(x) = \begin{cases} |x-4| & \text{for } x \ge 1 \\ x^3/2 - x^2 + 3x + 1/2 & \text{for } x < 1, \end{cases}$ then

(a) $f(x)$ is continuous at $x = 1$ and at $x = 4$
(b) $f(x)$ is differentiable at $x = 4$
(c) $f(x)$ is continuous and differentiable at $x = 1$
(d) $f(x)$ is only continuous at $x = 1$.

Ans. (a)

◎ **Solution:** Since $g(x) = |x|$ is a continuous function and $\lim\limits_{x \to 1+} f(x) = 3 = \lim\limits_{x \to 1-} f(x)$ so f is a continuous function. In particular f is continuous at $x = 1$ and $x = 4$. f is clearly not differentiable at $x = 4$. Since $g(x) = |x|$ is not differentiable at $x = 0$. Now

$$f'(1+) = \lim_{h \to 0+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0+} \frac{|-3+h| - 3}{h} = -1,$$

$$f'(1-) = \lim_{h \to 0-} \frac{(1/2)(1+h)^3 - (1+h)^2 + 3(1+h) + 1/2 - 3}{h}$$

$$= \lim_{h \to 0-} \frac{(1/2)\left(h^3 + 3h^2 + 3h\right) - \left(h^2 + 2h\right) + 3h}{h}$$

$$= \frac{5}{2}.$$

Hence f is not differentiable at $x = 1$.

◉ **Example 15:** If $f(x) = |x - a| + |x + b|$, $x \in \mathbf{R}$, $b > a > 0$. Then

(a) $f'(a+) = 1$ (b) $f'(a+) = 0$
(c) $f'(-b+) = 0$ (d) $f'(-b+) = 1$

Ans. (c)

◎ **Solution:** $f'(a+) = \lim\limits_{h \to 0+} \dfrac{f(a+h) - f(a)}{h}$

$$= \lim_{h \to 0+} \frac{|h| + |a+b+h| - |a+b|}{h}$$

$$= \lim_{h \to 0+} \frac{h + a + b + h - (a+b)}{h} \text{ as } h > 0, a+b > 0$$

$$= 2$$

$$f'(-b+) = \lim_{h \to 0+} \frac{f(-b+h) - f(-b)}{h}$$

$$= \lim_{h \to 0+} \frac{|-b+h-a| + |h| - |-b-a|}{h}$$

$$= \lim_{h \to 0+} \frac{|a+b-h| + |h| - |a+b|}{h}$$

$$= \lim_{h \to 0+} \frac{a+b-h+h-(a+b)}{h} = 0.$$

◉ **Example 16:** The set of all points where the function $f(x) = 2x|x|$ is differentiable is

(a) $(-\infty, \infty)$ (b) $(-\infty, 0) \cup (0, \infty)$
(c) $(0, \infty)$ (d) $[0, \infty)$

Ans. (a)

◎ **Solution:** $f(x) = \begin{cases} 2x^2, & x \ge 0 \\ -2x^2, & x < 0 \end{cases}$

Since x^2 and $-x^2$ are differentiable functions, $f(x)$ is differentiable except possible at $x = 0$

Now $$f'(0+) = \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{f(h)}{h}$$

$$= \lim_{h \to 0+} \frac{2h^2}{h} = 0$$

and $$f'(0-) = \lim_{h \to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0-} \frac{f(h)}{h}$$

$$= \lim_{h \to 0-} \frac{-2h^2}{h} = 0.$$

Hence f is differentiable everywhere.

◉ **Example 17:** The function $f(x) = \sin^{-1}(\tan x)$ is not differentiable at

(a) $x = 0$ (b) $x = -\pi/6$
(c) $x = \pi/6$ (d) $x = \pi/4$

Ans. (d)

◎ **Solution:** The function $f(x) = \sin^{-1}(\tan x)$ $(-\pi/4 \le x < \pi/4)$ is not differentiable for those x for which $\tan x = 1$ or -1. This happens if $x = \pi/4$ or $-\pi/4$.

◉ **Example 18:** Let $[\cdot]$ denote the greatest integer function and $f(x) = [\tan^2 x]$. Then

(a) $\lim_{x\to 0} f(x)$ does not exist
(b) $f(x)$ is continuous at $x = 0$
(c) $f(x)$ is not differentiable at $x = 0$
(d) $f'(0) = 1$.

Ans. (b)

◎ **Solution:** For $0 < x < \pi/4$, $0 < \tan^2 x < 1 \Rightarrow [\tan^2 x] = 0$ for $0 < x < \pi/4$. As $\tan^2 x$ is an even function, so $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} f(x) = 0$. So f is continuous at $x = 0$. Now

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-(0)}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

Hence f is differentiable at $x = 0$ and $f'(0) = 0$.

◉ **Example 19:** If $f(x)$ differentiable everywhere then

(a) $|f(x)|$ is differentiable everywhere
(b) $|f|^2$ is differentiable everywhere
(c) $f\,|f|$ is not differentiable at some point
(d) none of these

Ans. (b)

◎ **Solution:** If f is differentiable then $|f|^2(x) = f(x)^2$ which is differentiable. For $f(x) = x$, (a) and (c) do not hold.

◉ **Example 20:** Suppose f is differentiable at $x = 1$ and $\lim_{h\to 0}\frac{1}{h}f(1+h) = 5$ then

(a) $f'(1) = 4$ (b) $f'(1) = 3$
(c) $f'(1) = 6$ (d) none of these

Ans. (d)

◎ **Solution:** Since f is differentiable so it is continuous also. Thus

$$f(1) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} h\frac{f(1+h)}{h} = (0)(5) = 0$$

Hence $$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{f(1+h)}{h} = 5.$$

◉ **Example 21:** If $f(a) = a^2$, $\phi(a) = b^2$ $(a > 0, b > 0)$ and $f'(a) = 3\,\phi'(a)$ then $\lim_{x\to a}\frac{\sqrt{f(x)}-a}{\sqrt{\phi(x)}-b}$ is

(a) b^2/a^2 (b) b/a
(c) $2b/a$ (d) none of these

Ans. (d)

◎ **Solution:** $$\lim_{x\to a}\frac{\sqrt{f(x)}-a}{\sqrt{\phi(x)}-b}$$

$$= \lim_{x\to a}\frac{f(x)-a^2}{\phi(x)-b^2}\times\frac{\sqrt{\phi(x)}+b}{\sqrt{f(x)}+a}$$

$$= \lim_{x\to a}\frac{f(x)-f(a)}{\phi(x)-\phi(a)}\times\frac{\sqrt{\phi(x)}+b}{\sqrt{f(x)}+a}$$

$$= \frac{f'(a)}{\phi'(a)}\times\frac{\sqrt{\phi(a)}+b}{\sqrt{f(a)}+a} = \frac{3b}{a}.$$

◉ **Example 22:** Let f and g be differentiable function satisfying $g'(a) = 2$, $g(a) = b$ and $g = f^{-1}$ then $f'(b)$ is equal to

(a) 1/2 (b) 2
(c) 2/3 (d) none of these

Ans. (a)

◎ **Solution:** Since $g = f^{-1}$ so $f \circ g\,(x) = x$. Therefore, $f'(g(x))\,g'(x) = 1$. Thus, $f'(g(a))\,g'(a) = 1$

$\Rightarrow \quad f'(b)2 = 1 \quad \Rightarrow \quad f'(b) = 1/2.$

◉ **Example 23:** Let $f(x) = |x - a|\,\varphi(x)$, where φ is a continuous function and $\varphi(a) \neq 0$. Then

(a) $f'(a+) = \varphi'(a)$
(b) f is differentiable at $x = a$
(c) $f'(a+) = \varphi'(a)$
(d) $f'(a-) = -\varphi(a)$

Ans. (d)

◎ **Solution:** $$f'(a+) = \lim_{h\to 0+}\frac{|a+h-a|\,\varphi(a+h)-0}{h}$$

$$= \lim_{h\to 0+}\frac{|h|\,\varphi(a+h)}{h}$$

$$= \lim_{h\to 0+}\varphi(a+h) = \varphi(a)$$

Similarly $f'(a-) = -\varphi(a)$.

◉ **Example 24:** If $f'(c)$ exists and non-zero then $\lim_{h\to 0}\frac{f(c+h)+f(c-h)-2f(c)}{h}$ is equal to

(a) $f'(c)$ (b) 0
(c) $2f'(c)$ (d) none of these

Ans. (b)

◎ **Solution:** $$\lim_{h\to 0}\frac{f(c+h)+f(c-h)-2f(c)}{h}$$

$$= \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}+\frac{f(c-h)-f(c)}{h}$$

$$= f'(c) - \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}$$

$$= f'(c) - f'(c) = 0$$

◉ **Example 25:** Let $f: \mathbf{R}\to\mathbf{R}$ be any function. Define $g: \mathbf{R}\to\mathbf{R}$ by $g(x) = |f(x)|$ for all x. Then g is

(a) g may be bounded even if f is unbounded
(b) one-one if f is one
(c) continuous if f is continuous
(d) differentiable if f is differentiable

Ans. (c)

◎ **Solution:** Take $f(x) = x$. Since g: $\mathbf{R}\to\mathbf{R}$ given by $g(x) = |x|$ is not one-one so (b) is violated. Also g is not differentiable at $x = 0$. Let $u(x) = |x|$ then u is continuous function and $g(x) = u(f(x)) = u \circ f(x)$. Hence g is continuous if f is continuous. It is easy to see g is bounded if and only if f is bounded.

◉ **Example 26:** The function $f(x) = (x^2 - 1)\,|x^2 - 3x + 2| + \cos|x|$ is not differentiable at

(a) -1 (b) 0
(c) 1 (d) 2

Ans. (d)

◎ **Solution:** $f(x) = (x + 1)(x - 1)\,|x - 1|\,|x - 2| + \cos x$, because

$$\cos|x| = \begin{cases} \cos x, & \text{if } x \ge 0 \\ \cos(-x), & \text{if } x < 0 \end{cases} = \cos x.$$

Since $h(x) = x|x|$ is differentiable at $x = 0$, so $(x-1)\,|x-1|$ is differentiable at $x = 1$. Also $f(x)$ is not differentiable at $x = 2$.

◉ **Example 27:** Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = xf(x)$ then

(a) g is a differentiable function
(b) g is differentiable at 0 if f is continuous at 0
(c) g is one-one if f is one-one
(d) none of these

Ans. (b)

◎ **Solution:** Take $f(x) = \operatorname{sgn} x$ then $g(x) = |x|$, hence g need not be differentiable. Take $f(x) = x$, then $g(x) = x^2$ which is not one-one.

Let f be continuous at 0, then for $h \neq 0$.

$$\frac{g(0+h) - g(0)}{h} = \frac{hf(h)}{h} = f(h). \text{ Therefore, } g'(0) = f(0).$$

◉ **Example 28:** If $y = \tan^{-1}\left(\dfrac{\sqrt{1+x^2} - 1}{x}\right)$, then $y'(0)$ is

(a) 1/2 (b) 0
(c) 1 (d) doesn't exist.

Ans. (a)

◎ **Solution:** Putting $x = \tan\theta$, we get

$$\frac{\sqrt{1+x^2} - 1}{x} = \frac{\sec\theta - 1}{\tan\theta} = \frac{1 - \cos\theta}{\sin\theta} = \tan\left(\frac{\theta}{2}\right)$$

Hence $y = \dfrac{1}{2}\tan^{-1} x$ so $y'(x) = \dfrac{1}{2(1+x^2)}$ and thus $y'(0) = 1/2$.

◉ **Example 29:** If $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x and $f(2) = 4 = f'(2)$ then $(f(24))^2 + (g(24))^2$ is

(a) 32 (b) 24
(c) 64 (d) 48

Ans. (a)

◎ **Solution:** $\dfrac{d}{dx}\left((f(x))^2 + (g(x))^2\right) = 2[f(x)f'(x) + g(x)g'(x)] = 2[f(x)g(x) - g(x)f(x)] = 0.$

Hence $(f(x))^2 + (g(x))^2$ is constant. Thus $(f(24))^2 + (g(24))^2 = (f(2))^2 + (g(2))^2 = 16 + 16 = 32.$

◉ **Example 30:** If $f(x) = \log_{x^2}(\log x)$ then $f'(x)$ at $x = e$ is

(a) 0 (b) 1
(c) e^{-1} (d) $(2e)^{-1}$.

Ans. (d)

◎ **Solution:** $f(x) = \dfrac{\log(\log x)}{\log x^2} = \dfrac{1}{2}\dfrac{\log(\log x)}{\log x}$

$$\text{Therefore } f'(x) = \frac{1}{2}\,\frac{\log x \cdot \dfrac{1}{x\log x} - \dfrac{1}{x}\log(\log x)}{(\log x)^2}$$

and thus $f'(e) = \dfrac{1}{2}\left[\dfrac{1}{e} - 0\right] = (2e)^{-1}$.

◉ **Example 31:** If $x = 2\sin t - \sin 2t$, $y = 2\cos t - \cos 2t$, then the value of $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{2}$ is

(a) 2 (b) $-1/2$
(c) $-3/4$ (d) $-3/2$.

Ans. (b)

◎ **Solution:** $\dfrac{dx}{dt} = 2\cos t - 2\cos 2t = 2\,[2\sin(3t/2)\sin t/2]$

and $\dfrac{dy}{dt} = -2\sin t + 2\sin 2t = 2\left[\sin\dfrac{3t}{2}\cos\dfrac{t}{2}\right].$

Hence $\dfrac{dy}{dx} = \cot(t/2)$ and $\left(\dfrac{d^2y}{dx^2}\right) = -\dfrac{1}{2}\operatorname{cosec}^2(t/2) \times \dfrac{dt}{dx}$

Therefore, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{2}\operatorname{cosec}^2(t/2)\,x \times \dfrac{1}{4\sin(3t/2)\sin(t/2)}$

and $\left.\dfrac{d^2y}{dx^2}\right|_{t=\pi/2} = \left(-\dfrac{1}{2}\right) \times \left(\sqrt{2}\right)^2 \times \dfrac{1}{4 \times (1/2)} = -\dfrac{1}{2}$.

◉ **Example 32:** Let $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log x$. If $F(x) = h(f(g(x)))$, then $F''(x)$ is

(a) $2\operatorname{cosec}^3 x$
(b) $2\cot x^2 - 4x^2\operatorname{cosec}^2 x^2$
(c) $2x\cot x^2$
(d) $-2\operatorname{cosec}^2 x$.

Ans (b)

◎ **Solution:** $F(x) = h(f(g(x))) = h(f(x^2)) = h(\sin x^2) = \log\sin x^2$.

Hence $F'(x) = 2x\cot x^2$. and $F''(x) = 2\cot x^2 - 4x^2\operatorname{cosec}^2 x^2$.

Example 33: Let $y = e^{2x}$. Then $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)$ is

(a) 1　　(b) e^{-2x}

(c) $2\,e^{-2x}$　　(d) $-2e^{-2x}$.

Ans. (d)

Solution: $\frac{dy}{dx} = 2e^{2x}$ and $\frac{d^2y}{dx^2} = 4e^{2x}$. Also $x = \frac{1}{2}\log y$,

$$\text{so}\qquad \frac{dx}{dy} = \frac{1}{2y} \text{ and } \frac{d^2x}{dy^2} = -\frac{1}{2y^2} = -\frac{1}{2}e^{-4x}.$$

$$\text{Hence}\quad \left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = -2e^{-2x}.$$

Example 34: Let $f(x + y) = f(x)f(y)$ for all x and y. If $f(5) = 2$ and $f'(0) = 3$ then $f'(5)$ is equal to

(a) 5　　(b) 8

(c) 0　　(d) none of these

Ans. (d)

Solution: Putting $x = 0$, $y = 5$ in the given equation, we get

$$f(0 + 5) = f(0)f(5) \Rightarrow f(5)\,[f(1) - 0] = 0 \Rightarrow f(0) = 1$$

$$f'(5) = \lim_{h\to 0}\frac{f(5+h) - f(5)}{h} = \lim_{h\to 0}\frac{f(5)(f(h)-1)}{h}$$

$$= f'(5)\lim_{h\to 0}\frac{f(h) - f(0)}{h} = (2)(3) = 6.$$

Example 35: If $f''(x)$ is continuous at $x = 0$ and $f''(0) = 4$, then the value of

$$\lim_{x\to 0}\frac{2f(x) - 3f(2x) + f(4x)}{x^2} \text{ is}$$

(a) 4　　(b) 8

(c) 12　　(d) none of these

Ans. (c)

Solution: Required limit

$$= \lim_{x\to 0}\frac{2f'(x) - 6f'(2x) + 4f'(4x)}{2x}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0}\frac{2f''(x) - 12f''(2x) + 16f''(4x)}{2}$$

$$= 3f''(0) = 12.$$

Example 36: The solution set of $f'(x) > g'(x)$ where $f(x) = (1/2)\,5^{2x+1}$ and $g(x) = 5^x + 4x\log 5$ is

(a) $(1, \infty)$　　(b) $(0, 1)$

(c) $[0, \infty)$　　(d) $(0, \infty)$

Ans. (d)

Solution: $f'(x) = (1/2)\,5^{2x+1}(\log 5)(2) = \log 5\cdot(5^{2x+1})$ also $g'(x) = 5^x\log 5 + 4\log 5$.

So
$$\begin{aligned}\{x : f'(x) > g'(x)\} &= \{x : \log 5\cdot 5^{2x+1} > \log 5\,5^x + 4\log 5\}\\ &= \{x : 5^{2x+1} > 5^x + 4\}\\ &= \{t = 5^x : 5t^2 - t - 4 > 0\}\\ &= \{t = 5^x : (5t + 4)(t - 1) > 0\}\\ &= \{t = 5^x : t > 1 \text{ or } t < -4/5\}\\ &= \{t = 5^x : t > 1\} = (0, \infty).\end{aligned}$$

Example 37: The values of a and b such that the function f defined as

$$f(x) = \begin{cases} ax^2 - b, & |x| < 1\\ -1/|x|, & |x| \ge 1\end{cases} \text{ is differentiable are}$$

(a) $a = 1, b = -1$

(b) $a = 1/2, b = 1/2$

(c) $a = 1/2, b = 3/2$

(d) none of these

Ans. (c)

Solution: Since every differentiable function is continuous, so we must have

$$\lim_{x\to 1-} f(x) = f(1) \quad\Rightarrow\quad a - b = -1.$$

For f to be differentiable, $f'(1-) = f'(1+)$

$$\Rightarrow\quad \lim_{h\to 0-}\left[\frac{a(1+h)^2 - b + 1}{h}\right] = \lim_{h\to 0+}\left[\frac{-1/|1+h| + 1}{h}\right]$$

$$\Rightarrow\quad \lim_{h\to 0-}\left[\frac{a(2h+h^2)}{h}\right] = \lim_{h\to 0+}\frac{h}{h(1+h)} \quad (\text{as } a - b = -1)$$

$\Rightarrow\quad 2a = 1$. Hence $a = 1/2$ and $b = 3/2$.

Example 38: For a whole number n, if $f(x) = x^{n-1}\sin(1/x)$, $x \ne 0$ and $f(0) = 0$ then in order that f is differentiable at all x, the value of n can be

(a) 1　　(b) 2

(c) 3　　(d) 0

Ans. (c)

Solution: f is clearly differentiable except possibly $x = 0$.

$$f'(0) = \lim_{h\to 0}\frac{h^{n-1}\sin(1/h)}{h}$$

$$= \lim_{h\to 0} h^{n-2}\sin(1/h)$$

The last limit exists and is equal to 0 only if $n - 2 \ge 1$, i.e. $n \ge 3$. [Using Sandwich Theorem]

Example 39: $\frac{d}{dx}(\log\,(ax)^x)$, where a is a constant is equal to

(a) 1　　(b) $\log ax$

(c) $1/a$　　(d) $\log(ax) + 1$

Ans. (d)

◎ **Solution:** $\frac{d}{dx}(\log (ax)^x) = \frac{d}{dx}(x \log ax) = \log ax + 1.$

◉ **Example 40:** If $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \log a$ then $\frac{dy}{dx}$ is equal to

(a) y/x (b) x/y

(c) x^2/y^2 (d) y^2/x^2

Ans. (a)

◎ **Solution:** $\cos^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = \log\ a \Rightarrow \frac{x^2 - y^2}{x^2 + y^2} = \cos$ $\log a = A$ (say)

Putting $u = y/x$ and applying componendo and dividendo, we have

$$(y/x)^2 = u^2 = (1 - A)/(1 + A)$$

$$\Rightarrow y/x = \sqrt{(1-A)/(1+A)}$$

$$\Rightarrow \quad x\, dy/dx - y = 0$$

$$\Rightarrow dy/dx = y/x$$

◉ **Example 41:** If $(\sin x)(\cos y) = 1/2$ then d^2y/dx^2 at $(\pi/4, \pi/4)$ is equal to

(a) -4 (b) -2

(c) -6 (d) 0

Ans. (a)

◎ **Solution:** Differentiating the given equation, we have $\cos x \cos y - \sin x \sin y\, dy/dx = 0$. Putting $x = y = \pi/4$, we have $\left.\frac{dy}{dx}\right|_{(\pi/4,\, \pi/4)} = 1$. Differentiating again, we get $-\sin x \cos y - \cos x \sin y\, dy/dx - \cos x \sin y\, dy/dx - \sin x \sin y\, (dy/dx)^2 - \sin x \sin y\, d^2y/dx^2 = 0$. Putting $x = y = \pi/4$, we have

$$\left.\frac{d^2 y}{dx^2}\right|_{(\pi/4, \pi/4)} = -4.$$

◉ **Example 42:** The set onto which the derivative of the function $f(x) = x(\log x - 1)$ maps the ray $[1, \infty)$ is

(a) $[1, \infty)$ (b) $(0, \infty)$

(c) $[0, \infty)$ (d) none of these

Ans. (c)

◎ **Solution:** $f'(x) = \log x - 1 + x\left(\frac{1}{x}\right) = \log x$

Since $\log x$ is an increasing function so f' maps $[1, \infty)$ onto $[0, \infty)$.

◉ **Example 43:** Let $f(x)$ be a function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in \mathbf{R}$ and $f(x) = 1 + x\, g(x)$ where $\lim_{x \to 0} g(x) = 1$ then $f'(x)$ is equal to

(a) $g'(x)$ (b) $g(x)$

(c) $f(x)$ (d) none of these

Ans. (c)

◎ **Solution:** $f'(x)$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h}$$

$$= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} = f(x) \lim_{h \to 0} \frac{1 + hg(h) - 1}{h}$$

$$= f(x) \lim_{h \to 0} g(h) = f(x)$$

◉ **Example 44:** Let $g = f^{-1}$ and $f'(x) = \frac{1}{1 + x^4}$ then $g'(x)$ is equal to

(a) $(1 + (g(x)^4)^{-1}$ (b) $(1 + (f(x))^4)^{-1}$

(c) $1 + (f(x))^4$ (d) $1 + (g(x))^4$

Ans. (d)

◎ **Solution:** Since $g(f(x)) = x$ so $g'(f(x)) f'(x) = 1$ $\Rightarrow g'(f(x)) = 1 + x^4$. Putting $u = f(x)$ so that $g(u) = x$, we have $g'(u)$

$$= 1 + (g(u))^4.$$

◉ **Example 45:** Suppose for a differentiable function f, $f(0) = 0, f(1) = 1$ and $f'(0) = 4 = f'(1)$. If $g(x) = f(e^x)\, e^{f(x)}$ then $g'(0)$ is equal to

(a) 4 (b) 8

(c) 2 (d) none of these

Ans. (b)

◎ **Solution:** $g'(x) = f'(e^x)\, e^x\, e^{f(x)} + f(e^x)\, e^{f(x)} f'(x)$, so $g'(0) = f'(1)\, e^{f(0)} + f(1)\, e^{f(0)} f'(0) = f'(1) + f'(0) = 8$

◉ **Example 46:** If the function f is differentiable and strictly increasing in a neighbourhood of 0, then $\lim_{x \to 0} \frac{f(x^5) - f(x)}{f(x) - f(0)}$ is equal to

(a) -1 (b) 0

(c) 1 (d) $5/2$

Ans. (a)

◎ **Solution:** $\frac{f(x^5) - f(x)}{f(x) - f(0)}$

$$= \frac{f(x^5) - f(0) - (f(x) - f(0))}{f(x) - f(0)}$$

$$= \frac{f(x^5) - f(0)}{f(x) - f(0)} - 1$$

$$= \frac{\frac{f(x^5) - f(0)}{x^5 - 0} \times x^5}{\frac{f(x) - f(0)}{x - 0} \times x} - 1$$

So $$\lim_{x \to 0} \frac{f(x^5) - f(x)}{f(x) - f(0)} = \lim_{x \to 0} x^4 \times \frac{f'(0)}{f'(0)} - 1$$

$$= 0 - 1 = -1\ .\ (f'(0) > 0)$$

Example 47: Let $f(x) = ||x-1|-1|$, then all the points where $f(x)$ is not differentiable is (are)

(a) 1 (b) 1, 2
(c) ± 1 (d) 0, 1, 2

Ans. (d)

Solution: $f(x) = \begin{cases} |x-1|-1 & \text{if } |x-1| \ge 1 \\ 1-|x-1| & \text{if } |x-1| < 1 \end{cases}$

$$= \begin{cases} -x & x \le 0 \\ x & 0 < x \le 1 \\ 2-x & 1 < x < 2 \\ x-2 & x \ge 2 \end{cases}$$

f is not differentiable at 0, 1, 2 as these are corner points on the graph.

Example 48: $\dfrac{d^2x}{dy^2}$ equals

(a) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$
(c) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$ (d) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$

Ans. (d)

Solution: $\dfrac{dx}{dy} = \dfrac{1}{dy/dx}$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{1}{dy/dx}\right) = \frac{d}{dx}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy}$$

$$-\frac{d}{dx}\left(\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)^{-2}\left(\frac{dy}{dx}\right)^{-1}$$

$$= -\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3}$$

Example 49: If $x^2 + y^2 = t + \dfrac{1}{t}$ and $x^4 + y^4 = t^2 + \dfrac{1}{t^2}$ $(xy > 0)$ then $\dfrac{d^2y}{dx^2}$ is equal

(a) $1/x^3$ (b) $2/x^3$
(c) $2\sqrt{2}/x^3$ (d) $2\sqrt{2}/x^4$

Ans. (b)

Solution: $(x^2 + y^2)^2 = \left(t+\dfrac{1}{t}\right)^2 = t^2 + \dfrac{1}{t^2} + 2 = x^4 + y^4 + 2$

$$\Rightarrow \quad 2x^2y^2 = 2 \quad \Rightarrow \quad x^2y^2 = 1 \quad \Rightarrow \quad xy = 1$$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{y}{x} = -\frac{1}{x^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{2}{x^3}.$$

Example 50: If $2f(\sin x) + f(\cos x) = x$ for all $x \in$ **R** then $y = f(x)$ satisfies

(a) $(1 - x^2)\, y'' - xy' = 0$
(b) $x^2 y'' - (1 - x^2)y' = 0$
(c) $(1 + x^2)\, y'' + xy' = 0$
(d) $(1 - x^2)\, y'' + xy' = 0$

Ans. (a)

Solution: Replacing x by $\pi/2 - x$ in the given equation, we get

$$2f(\cos x) + f(\sin x) = \pi/2 - x$$

Solving this equation with the given equation, we obtain

$$f(\cos x) = \pi/3 - x \Rightarrow y = f(x) = \pi/3 - \cos^{-1}x$$

$$\Rightarrow \quad y' = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad y'^2(1 - x^2) = 1$$

Differentiating again we have

$$2\,y'\,y''(1 - x^2) - 2x\,y'^2 = 0$$

$$\Rightarrow \quad (1 - x^2)\,y'' - xy' = 0.$$

Example 51: Let $f(x) = (1 + \sin x)\,(2 + \sin^2 x) \ldots + (n + \sin^n x)$ then $f'(0)$ is equal to

(a) 1 (b) $n!$
(c) $n!/2$ (d) n

Ans. (b)

Solution: $\log f(x) = \displaystyle\sum_{k=1}^{n} \log\left(k + \sin^k x\right)$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = \sum_{k=1}^{n} \frac{k \sin^{k-1} x \cos x}{k + \sin^k x}$$

$f'(0) = f(0) \,.\, 1$ (If $k = 1$, then at $x = 0$ the corresponding term in summation is 1 otherwise it is 0)

$$f'(0) = 1.\,2.\,\ldots n = n! \text{ as } f(0) = n!$$

Example 52: Let f_α be a function defined by $f_\alpha(x) = \begin{cases} x^\alpha \sin 1/x & , x \ne 0 \\ 0 & , x = 0 \end{cases}$

A value of α so that f_α is differentiable on **R** and f'_α is differentiable on **R** is

(a) 2 (b) 5/2
(c) 7/2 (d) 1

Ans. (c)

Solution: For $x \ne 0$

$$f'_\alpha(x) = x^{\alpha-2} \sin 1/x + \alpha x^{\alpha-1} \sin 1/x$$

$$\frac{f_\alpha(x) - f_\alpha(0)}{x - 0} = x^{\alpha-1} \sin 1/x$$

$$\lim_{x \to 0} \frac{f_\alpha(x) - f_\alpha(0)}{x - 0} \text{ exists and is equal 0 if } \alpha > 1$$

For $x \ne 0$, $f''_\alpha(x) = (\alpha - 2)\, x^{\alpha-3} \sin 1/x - x^{\alpha-4} \cos 1/x + \alpha(\alpha - 1)\, x^{\alpha-2} \sin 1/x - \alpha\, x^{\alpha-3} \cos 1/x$

$$\lim_{x\to 0}\frac{f'_\alpha(x)-f'_\alpha(0)}{x-0} = \lim_{x\to 0}\left(x^{\alpha-3}\sin 1/x+\alpha x^{\alpha-2}\sin 1/x\right)$$

The last limit exists if $\alpha > 3$ So $\alpha = 7/2$.

◉ **Example 53:** If function $f(x)$ is differentiable at $x = a$ then $\lim_{x\to a}\frac{x^2 f(a)-a^2 f(x)}{x-a}$ is.

(a) $-a^2 f'(a)$

(b) $af(a) - a^2 f'(a)$

(c) $2af(a) - a^2 f'(a)$

(d) $2af(a) + a^2 f'(a)$

Ans. (c)

◎ **Solution:** For $x \neq a$, $\frac{x^2 f(a)-a^2 f(x)}{x-a}$

$$= \frac{(x^2-a^2)f(a)+a^2(f(a)-f(x))}{x-a}$$

$$= (x+a)f(a) - a^2\frac{f(x)-f(a)}{x-a}$$

Therefore, $\lim_{x\to a}\frac{x^2 f(a)-a^2 f(x)}{x-a} = 2af(a) - a^2 f'(a)$.

◉ **Example 54:** Let f: $(0, 1) \to \mathbf{R}$ be defined by $f(x) = \frac{b-x}{1-bx}$ where b is a constant such that $0 < b < 1$. Then

(a) f is not invertible on (0,1)

(b) $f \neq f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$

(c) $f = f^{-1}$ on (0, 1) and $f'(b) = \frac{1}{f'(0)}$

(d) f^{-1} is differentiable on (0, 1).

Ans. (a)

◎ **Solution:** As $f(0)$ and hence $f'(0)$ are not defined so (*b*) and (*c*) are not possible. Note that $0 < bx < 1$, for $0 < x < 1$. Thus $f(x) > 0$, for $0 < x < b$, $f(b) = 0$ and $f(x) < 0$, for $b < x < 1$. For $y_1, y_2 \in (0, 1)$

$$f(y_1) = f(y_2) \Rightarrow \frac{b-y_1}{1-by_1} = \frac{b-y_2}{1-by_2}$$

$$\Rightarrow \quad b - y_1 - b^2y_2 + by_1y_2 = b - b^2y^1 - y_2 + by_1y_2$$

$$\Rightarrow \quad (y_2 - y_1)(1 - b^2) = 0$$

Since $0 < b < 1$ so $y_2 = y_1$

Thus f is one – one, but f is not onto. The pre-image of any $y \in (1, \infty)$ is $\frac{b-y}{1-by}$ but $\frac{b-y}{1-by} \in (0,1)$ if $0 < b < 1/y$.

◉ **Example 55:** If $y = \operatorname{cosec}(\cot^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

(a) $\frac{1}{\sqrt{2}}$ (b) 1

(c) $-\sqrt{2}$ (d) $-\frac{1}{\sqrt{2}}$

Ans. (a)

◎ **Solution:**

$$\frac{dy}{dx} = -\operatorname{cosec}(\cot^{-1}x)\cot(\cot^{-1}x)\frac{d}{dx}(\cot^{-1}x)$$

$$= \operatorname{cosec}(\cot^{-1}x)\times\frac{1}{1+x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \operatorname{cosec}\frac{\pi}{4}\times\frac{1}{2} = \sqrt{2}\times\frac{1}{2} = \frac{1}{\sqrt{2}}.$$

◉ **Example 56:** Let $f(x) = \begin{cases}\frac{1}{|x|} & \text{if } |x|>2\\ A+Bx^2 & \text{if } |x|\le 2\end{cases}$

then $f(x)$ is differentiable at $x = -2$ for:

(a) $A = \frac{3}{4},\ B = -\frac{1}{16}$

(b) $A = -\frac{1}{4},\ B = \frac{1}{16}$

(c) $A = \frac{1}{4},\ B = -\frac{1}{16}$

(d) $A = \frac{3}{4},\ B = \frac{1}{16}$

Ans. (a)

◎ **Solution:** $f(x) = \begin{cases}-\frac{1}{x}, & x<-2\\ A+Bx^2, & -2\le x\le 2\\ \frac{1}{x}, & x>2\end{cases}$

$$f'(-2+) = \lim_{h\to 0+}\frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h\to 0+}\frac{A+B(-2+h)^2-(A+4B)}{h}$$

$$= B\lim_{h\to 0+}\frac{(4+h^2-4h)-4}{h}$$

$$= B\lim_{h\to 0+}(h-4) = -4B$$

$$f'(2-) = \lim_{h\to 0-}\frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h\to 0-}\frac{-\frac{1}{-2+h}-(A+4B)}{h}$$

$$= -\lim_{h\to 0-}\frac{1+(A+4B)(-2)+(A+4B)h}{(-2+h)h}$$

The last limit exists if $1 + (-2)(A + 4B) = 0$ (i)

In case (i) is satisfied then the last limit is equal to

$$-\lim_{h\to 0-}\frac{A+4B}{(-2+h)} = \frac{1}{2}(A+4B) = \frac{1}{4}$$

Hence $B = -\frac{1}{16}$. Putting this value in (i)

$$1+(-2)\left(A-\frac{1}{4}\right) = 0 \Rightarrow A = \frac{3}{4}$$

Example 57: If $f''(x) = -f(x)$, $h(x) = f'(x)$ and $F(x) = (f(x/2))^2 + (h(x/2))^2$. Given that $F(2) = 4$, the value $F(1)$ is

(a) 2 (b) 4

(c) 1 (d) 0

Ans. (b)

Solution: $h'(x) = f''(x) = -f(x)$.

$$F'(x) = 2f\left(\frac{x}{2}\right)f'\left(\frac{x}{2}\right)\frac{1}{2} + 2h\left(\frac{x}{2}\right)h'\left(\frac{x}{2}\right)\frac{1}{2}$$

$$= f\left(\frac{x}{2}\right)f'\left(\frac{x}{2}\right) + f'\left(\frac{x}{2}\right)\left(-f\left(\frac{x}{2}\right)\right) = 0$$

Hence F is constant function. Therefore, $F(1) = F(2) = 4$.

Example 58: Let f be a differentiable function such that $8f(x) + 6f(1/x) - x = 5$ $(x \neq 0)$ and $y = x^2 f(x)$, then $\frac{dy}{dx}$ at $x = 1$ is

(a) $\frac{15}{14}$ (b) $\frac{17}{14}$

(c) $\frac{19}{14}$ (d) $-\frac{17}{14}$

Ans. (c)

Solution: Differentiating the given expression, we get

$$8f'(x) + 6f'\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) - 1 = 0$$

$\Rightarrow$ $8f'(1) + 6f'(1)(-1) = 1$

$\Rightarrow$ $f'(1) = 1/2$

Also $\frac{dy}{dx} = 2xf(x) + x^2 f'(x)$

so, $\left.\frac{dy}{dx}\right|_{x=1} = 2f(1) + f'(1)$

Putting $x = 1$ in the given equation, we obtain

$8f(1) + 6f(1) - 1 = 5$

$\Rightarrow$ $14f(1) = 6$

$\Rightarrow$ $f(1) = \frac{3}{7}$

Hence $\left.\frac{dy}{dx}\right|_{x=1} = 2,\frac{3}{7} + \frac{1}{2} = \frac{19}{14}$

Example 59: Let $f(x) = \begin{cases} Ax & , x < 1 \\ Ax^2 + Bx + 4 & , x \geq 1 \end{cases}$

If f is differentiable for all x, then

(a) $A = 2, B = -2$ (b) $A = -4, B = 4$

(c) $A = 1, B = -1$ (d) $A = 4, B = -4$

Ans. (d)

Solution:

$$f'(1+) = \lim_{h\to 0+}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0+}\frac{A(1+h)^2 + B(1+h) + 4 - (A+B+4)}{h}$$

$$= \lim_{h\to 0+}\frac{A(h^2+2h)+Bh}{h} = 2A + B$$

$$f'(1-) = \lim_{h\to 0-}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0-}\frac{A(1+h)-(A+B+4)}{h}$$

$$= \lim_{h\to 0-}\frac{Ah-(B+4)}{h}$$

The last limit exist if $B + 4 = 0 \Rightarrow B = -4$. In this case, the last limit is equal to A.

Hence $A = 2A + B \Rightarrow A = 4$

Example 60: A twice differentiable function $f : \mathbf{R} \to \mathbf{R}$ satisfies $\sin x \cos y\,(f(2x + 2y) - f(2x - 2y)) = \cos x \sin y\,(f(2x + 2y) + f(2x - 2y))$ for all $x, y \in \mathbf{R}$ then

(a) $f''(x) = f(x) = 0$ (b) $4f''(x) + f(x) = 0$

(c) $f''(x) + f(x) = 0$ (d) $4f''(x) - f(x) = 0$

Ans. (b)

Solution:

$f(2x + 2y)(\sin x \cos y - \cos x \sin y)$

$= f(2x - 2y)[\sin x \cos y + \cos x \sin y]$

$\Rightarrow$ $f(2x + 2y)\sin(x - y) = f(2x - 2y)\sin(x + y)$

$$\Rightarrow \frac{f(2(x+y))}{\sin(x+y)} = \frac{f(2x-2y)}{\sin(x-y)}$$

Putting $2(x + y) = u$, $2(x - y) = v$

$$\Rightarrow \frac{f(u)}{\sin\frac{u}{2}} = \frac{f(v)}{\sin\frac{v}{2}} = k \text{ for all } u, v \in \mathbf{R}$$

$$\Rightarrow f(u) = k\sin\frac{u}{2}$$

$$\Rightarrow f'(u) = \frac{k}{2}\cos\frac{u}{2} \text{ and } f''(u) = \frac{-k}{4}\sin\frac{u}{2}$$

$$\text{so } 4f''(u) = -k\sin\frac{u}{2} = -f(u)$$

$$\Rightarrow 4f''(u) + f(u) = 0$$

Assertion-Reason Type Questions

Example 61: Let $y = \sqrt{2x - x^2}$, $0 \le x \le 2$

Statement-1: y satisfies $y^3\, y'' + 1 = 0$
Statement-1: y is a differentiable function of x on [0, 2]

Ans. (c)

Solution: $y' = \frac{1}{2}\,\frac{2 - 2x}{\sqrt{2x - x^2}} = \frac{1 - x}{y}$ for $x \ne 0, 2$

$\Rightarrow$ $yy' = 1 - x$. Differentiating again, we have

$$\Rightarrow \quad y'^2 + yy'' = -1 \quad \Rightarrow \quad y^3 y'' = -y^2 - (yy')^2$$
$$= -y^2 - (1 - x)^2$$
$$= -(2x - x^2) - (1 + x^2 - 2x) = -1$$

y is not differentiable at $x = 0, 2$.

Example 62: Let $x = \log t$, $y = t^2 - 1$

Statement-1: $\frac{d^2y}{dx^2}$ at $x = 0$ is 4

Statement-1: $\frac{d^2y}{dx^2} = 4t^2$

Ans. (a)

Solution: $\frac{dx}{dt} = \frac{1}{t}, \frac{dy}{dt} = 2t$ so $\frac{dy}{dx} = 2t^2$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = 4t\ .\ t = 4t^2$$

If $x = 0$ then $t = 1$ so $\frac{d^2y}{dx^2}$ at $x = 0$ is 4.

Example 63: If $e^y + xy = e$ then

Statement-1: $y''(0) = 1/e^2$
Statement-2: $y'(0) = -1$

Ans. (c)

Solution: $e^y\, y' + xy' + y = 0$ (1)

$\Rightarrow$ $e^y\, y'' + e^y\, y'^2 + y' + xy'' + y' = 0$ (2)

For $x = 0$, $e^{y(0)} + 0.y(0) = e \Rightarrow y(0) = 1$

Putting $x = 0$ in (1) $e^y(0)\, y'^{(0)} + y(0) = 0 \Rightarrow y'(0) = -1/e$

Putting $x = 0$ in (2) $e^{y(0)}\, y''(0) + e^{y(0)}\, (y'(0))^2 + 2y'(0) = 0$

$$\Rightarrow \quad ey''(0) + e\ .\ \frac{1}{e^2} - \frac{2}{e} = 0$$

$\Rightarrow$ $y''(0) = 1/e^2$.

Example 64: Let $x = t^3 + 1$, $y = t^2 + t + 1$

Statement-1: $\frac{dy}{dx}$ at (1, 1) is 1

Statement-2: $\frac{dt}{dx}$ is not defined at $x = 1$.

Ans. (d)

Solution: $\frac{dx}{dt} = 3t^2, \frac{dy}{dt} = 2t + 1$

$$\frac{dy}{dx} = \frac{2t + 1}{3t^2}.\ \text{If } x = 1, \text{ then } t = 0$$

$\frac{dt}{dx}$ is not defined at $t = 0$ and also $\frac{dy}{dx}$ is not defined at $t = 0$.

Example 65: If $y = \log\cos\left(\tan^{-1}\frac{e^x - e^{-x}}{2}\right)$

Statement-1: $y'(0) = 0$

Statement-2: $y'(x) = -\frac{e^x - e^{-x}}{1 + x^2}$

Ans. (c)

Solution: $y'(x)$

$$= -\frac{\sin\left(\tan^{-1}\frac{e^x - e^{-x}}{2}\right)}{\cos\left(\tan^{-1}\frac{e^x - e^{-x}}{2}\right)} \times \frac{1}{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2} \times \frac{e^x + e^{-x}}{2}$$

$$= -\tan\left(\tan^{-1}\frac{e^x - e^{-x}}{2}\right)\frac{2}{\left(e^x + e^{-x}\right)^2} \times \left(e^x + e^{-x}\right)$$

$$= -\frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$y'(0) = 0$.

Example 66: Consider the function

$f(x) = |x - 2| + |x - 5|$, x ε **R**

Statement 1. $f'(4) = 0$

Statement 2. f is continuous on [2, 5], differentiable on (2, 5) and $f(2) = f(5)$.

Ans. (b)

Solution: $f(x) = \begin{cases} 7 - 2x & if\ x \le 2 \\ 3 & if\ 2 \le x \le 5 \\ 2x - 7 & if\ x > 5 \end{cases}$

$f'(4) = 0$ as f is a constant function on (2, 5). Also f is continuous on [2, 5] and differentiable on (2, 5) and $f(2) = f(5)$ = 0 but this is not correct explanation of statement-1.

Example 67: Statement 1 : $f(x) = (\sin \pi x)(x-2)^{1/3}$ is differentiable at $x = 2$.

Statement 2: Pointwise product of two differentiable functions in differentiable.

Ans. (b)

Solution:

$$f'(2+) = \lim_{h\to 0+} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0+} \frac{(\sin \pi(2+h))(2+h-2)^{1/3}}{h}$$

$$= \pi \lim_{h\to 0+} \frac{(\sin \pi h)(h)^{1/3}}{\pi h}$$

(h is small but positive)

$$= \pi.1.0 = 0$$

$$f'(2-) = \lim_{h\to 0-} \frac{\sin \pi(2+h)(2+h-2)^{1/3}}{h}$$

$$= \pi \lim_{h\to 0-} \frac{(-\sin \pi h)(h)^{1/3}}{\pi h}$$

(h is small and negative)

$$= -\pi \times 1 \times 0 = 0$$

Hence f is differentiable at $x = 2$. Statement 2 is also correct but since $(x-2)^{1/3}$ is not differentiable at $x = 2$, so statement 2 does not imply Statement 1.

Example 68: Suppose that f is a differentiable function and satisfies $f(x) + f(x-4) = 0$ for all $x \in \mathbf{R}$. If $f'(x) = u$ then $f'(x+800) = u$.

Statement 2: f satisfies $f(x) = f(x+8)$

Ans. (a)

Solution: Replacing x by $x + 4$ in the given equation of statement 1, we have

$$f(x+4) + f(x) = 0 \qquad \ldots\text{(i)}$$

Again replacing x by $x + 4$

$$f(x+8) + f(x+4) = 0$$

$$\Rightarrow \quad f(x+8) - f(x) = 0 \text{ using (i)}$$

so $\quad f(x) = f(x+8)$, for all $x \in \mathbf{R}$

$$f'(x+800) = \lim_{h\to 0} \frac{f(x+800+h)-f(x+800)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$$

Example 69: Suppose that a differentiable function f satisfies $f(x+y^5) = f(x) + f(y^5)$, $x, y \in \mathbf{R}$.

Statement 1: If $f'(9) = 8$ then $f'(-9) = 8$

Statement 2: f is an odd function.

Ans. (a)

Solution: Putting $x = y = 0$ in the given equation, we have $f(0) = f(0) + f(0) \Rightarrow f(0) = 0$

Putting $y = -x^{1/5}$, we have $0 = f(0) = f(x + y^5)$

$$= f(x) + f(y^5)$$

$$= f(x) + f(-x)$$

Hence f is an odd function.

$$f'(-9) = \lim_{h\to 0} \frac{f(-9+h)-f(-9)}{h}$$

$$= \lim_{h\to 0} \frac{-f(9-h)+f(9)}{h}$$

$$= \lim_{h\to 0} \frac{f(9-h)-f(9)}{-h} = f'(9).$$

Example 70: Let $f(x) = \log\left(x+\sqrt{x^2+1}\right) + x^3 + \sin^5 x$

Statement 1: f' is an even function

Statement 2: f is an odd function

Ans. (a)

Solution: $f(-x) = \log(-x+\sqrt{x^2+1}) + (-x^3) + (-\sin^5 x)$

$$= \log\left(\frac{-x^2+x^2+1}{x+\sqrt{x^2+1}}\right) + (-x^3) + (-\sin^5 x)$$

$$= \log \frac{1}{x+\sqrt{x^2+1}} - x^3 - \sin^5 x$$

$$= -f(x)$$

Hence f is an odd function. Morever f is a differentiable function so f' is even.

LEVEL 2

Straight Objective Type Questions

Example 71: If $f(x) = p\,|\sin x| + qe^{|x|} + r|x|^3$ and if $f(x)$ is differentiable at $x = 0$, then

(a) $q + r = 0, p \in \mathbf{R}$ (b) $p + q = 0; r \in \mathbf{R}$

(c) $q = 0; r = 0; p \in \mathbf{R}$ (d) $r = 0; p = 0; q \in \mathbf{R}$

Ans. (b)

Solution: For $-\pi/2 < x \le 0$, $f(x) = -p \sin x + qe^{-x} - rx^3$, so

$$f'(0-) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0-}\left[-\frac{p\sin x}{x} - q\left(\frac{e^{-x}-1}{-x}\right) - rx^2\right]$$

$$= -p - q.$$

For $0 < x < \pi/2, f(x) = p\sin x + q\,e^x + rx^3$,

$$f'(0+) = \lim_{x\to 0+}\frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0+}\left[\frac{p\sin x}{x} + q\left(\frac{e^{x}-1}{x}\right) - rx^2\right]$$

$$= p + q.$$

For f to be differentiable at $x = 0$, we must have

$$p + q = -p - q \Rightarrow p + q = 0.$$

Example 72: Let $h(x) = \min\{x, x^2\}$ for $x \in \mathbf{R}$. Then which of the following is not correct

(a) h is continuous for all x

(b) h is differentiable for all x

(c) $h'(x) = 1$ for all $x > 1$

(d) h is not *a* differentiable function at atleast two points

Ans. (b)

Solution: $h(x) = \begin{cases} x, & x \ge 1 \\ x^2, & 0 \le x < 1 \\ x, & x < 0 \end{cases}$

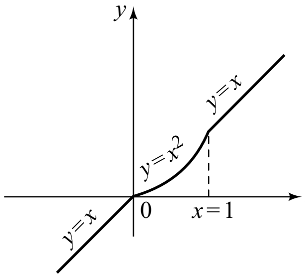


Fig. 10.3

From the graph it is clear that h is continuous. Also h is differentiable except possibly at $x = 0$ and 1.

$$h'(x) = \begin{cases} 1, & x > 1 \\ 2x, & 0 < x < 1 \\ 1 & x < 0 \end{cases}$$

For $x = 1$, $h'(1+) = \lim_{t\to 0+}\frac{h(1+t)-h(1)}{t}$

$$= \lim_{t\to 0+}\frac{1+t-1}{t} = 1$$

but $\quad h'(1-) = \lim_{t\to 0+}\frac{h(1-t)-h(1)}{t} = \lim_{t\to 0+}\frac{(1-t)^2-1}{t}$

$$= -2$$

So h is not differentiable at 1. Similarly $h'(0+) = 0$ but $h'(0-) = 1$

Example 73: Let $f(x) = \begin{cases} \dfrac{\sin|x^2-5x+6|}{x^2-5x+6} & x \ne 2, 3 \\ 1 & x = 2 \text{ or } 3 \end{cases}$

the set of all points where f is differentiable is

(a) $(-\infty, \infty)$

(b) $(-\infty, \infty) \sim \{2\}$

(c) $(-\infty, \infty) \sim \{3\}$

(d) $(-\infty, \infty) \sim \{2, 3\}$.

Ans. (d)

Solution: The function is clearly differentiable except possibly at $x = 2, 3$.

$$f'(2+) = \lim_{h\to 0+}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0+}\frac{\sin h(1-h) + h(1-h)}{h^2(-1+h)}$$

$$= -\lim_{h\to 0+}\left(\frac{\sin h(1-h)}{h(1-h)} + 1\right)\frac{1}{h}$$

The last limit doesn'*t* exist. If this limit then $\lim_{h\to 0+}\frac{1}{h}$ exist, which is not true.

Thus f is not differentiable at $x = 2$. Similarly f is not differentiable at $x = 3$. Hence the set of all points where f is differentiable is $(-\infty, \infty) \sim \{2, 3\}$.

Example 74: If $f(x) = x\cdot\dfrac{(a^{1/x} - a^{-1/x})}{a^{1/x} + a^{-1/x}}$, $x \ne 0$ $(a > 0)$, $f(0) = 0$ then

(a) f is differentiable at $x = 0$

(b) f is not differentiable at $x = 0$

(c) f is not continuous at $x = 0$

(d) none of these.

Ans. (b)

Solution: $\lim_{x\to 0+} f(x) = \lim_{x\to 0+}\dfrac{x\left(a^{1/x} - a^{-1/x}\right)}{a^{1/x} + a^{-1/x}}$

$$= \lim_{x\to 0+}\frac{x\left(1 - a^{-2/x}\right)}{1 + a^{-2/x}}$$

also $\quad \lim_{x\to 0-} f(x) = \lim_{x\to 0-}\dfrac{x\left(a^{2/x} - 1\right)}{a^{2/x} - 1} = 0.$

So f is continuous at $x = 0$.

$$f'(0+) = \lim_{x\to 0+}\frac{h\left(a^{1/h} - a^{-1/h}\right)}{h\left(a^{1/h} + a^{-1/h}\right)}$$

$$= \lim_{x\to 0+}\frac{1 - a^{-2/h}}{1 + a^{-2/h}} = 1$$

Similarly $f'(0-) = -1$. Hence, f is not differentiable at $x = 0$.

Example 75: Suppose that f is a differentiable function with the property that $f(x + y) = f(x) + f(y) + xy$ and $\lim\limits_{h\to 0}\frac{1}{h}f(h) = 3$ then

(a) f is a linear function
(b) $f(x) = 3x + x^2$
(c) $f(x) = 3x + x^2/2$
(d) none of these.

Ans. (c)

Solution: $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x)+f(h)+xh-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}f(h)+x = 3 + x$$

Hence $f(x) = 3x + x^2/2 + C$. Putting $x = y = 0$ in the given equation, we have $f(0) = f(0 + 0) = f(0) + f(0) + 0 \Rightarrow f(0) = 0$. Thus $C = 0$ and $f(x) = 3x + x^2/2$.

Example 76: Let $f(x) = \begin{cases} x^2, & if\ x \le x_0 \\ ax+b, & x > x_0 \end{cases}$

If f is differentiable at x_0 then

(a) $a = x_0, b = -x_0$ (b) $a = 2x_0, b = -x_0^2$
(c) $a = 2x_0, b = x_0^2$ (d) none of these

Ans. (b)

Solution: Since f is differentiable so it is continuous also, therefore
$$x_0^2 = f(x_0) = \lim_{x\to x_0+} f(x) = ax_0 + b$$

Also $$\lim_{h\to 0+}\frac{f(x_0+h)-f(x_0)}{h}$$
$$= \lim_{h\to 0+}\frac{a(x_0+h)+b-x_0^2}{h}$$
$$= \lim_{h\to 0+}\frac{x_0^2+ah-x_0^2}{h} = a$$
$$(\text{since } x_0^2 = ax_0 + b)$$

Thus $$a = f'(x_0-) = \lim_{h\to 0+}\frac{(x_0+h)^2-x_0^2}{h} = 2x_0$$

Hence $$x_0^2 = 2x_0^2 + b \quad\Rightarrow\quad b = -x_0^2$$

Example 77: Let $f(x)$ be *a* polynomial of degree two which is positive for all $x \in \mathbf{R}$. If $g(x) = f(x) + f'(x) + f''(x) + xf'''(x) + x^2 f^{\text{IV}}(x)$, then for any real x

(a) $g(x) < 0$ (b) $g(x) > 0$
(c) $g(x) = 0$ (d) $g(x) \ge 0$

Ans. (b)

Solution: Let $f(x) = ax^2 + bx + c$. As $f(x) > 0$ for all $x \in \mathbf{R}$, we must have, $a > 0$ and $b^2 - 4ac < 0$.

Now $$g(x) = ax^2 + bx + c + (2ax + b) + 2a + x.0 + x^2.0$$
$$= ax^2 + (b + 2a)x + b + c + 2a$$

Discriminant of $g(x) = (b + 2a)^2 - 4a(b + c + 2a)$
$$= -4a^2 + (b^2 - 4ac) < 0$$

Thus $g(x) > 0$ for all $x \in \mathbf{R}$.

Example 78: The left-hand derivative of $f(x) = [x]\sin \pi x$ at $x = k$, k is an integer is

(a) $(-1)^k (k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$
(c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k\pi$

Ans. (a)

Solution: Clearly $f(k) = 0$, so the left-hand derivative is equal to $\lim\limits_{h\to 0-}\frac{f(k+h)}{h}$
$$= \lim_{h\to 0-}\frac{[k+h]\sin(k+h)\pi}{h}$$
$$= \lim_{h\to 0-}\frac{(k-1)\sin(k\pi+h\pi)}{h}$$
$$= \lim_{h\to 0-}\frac{(k-1)(-1)^k\sin h\pi}{h} \qquad (\text{since } h < 0)$$
$$= (k-1)(-1)^k\pi.$$

Example 79: If $y = \tan^{-1}\left(\frac{\log ex^{-2}}{\log ex^2}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$ then $\frac{d^2y}{dx^2}$ is

(a) 2 (b) 1
(c) 0 (d) x

Ans. (c)

Solution: $$y = \tan^{-1}\left(\frac{\log e - 2\log x}{\log e + 2\log x}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$$
$$= \tan^{-1}\left(\frac{1-2\log x}{1+2\log x}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$$
$$= \frac{\pi}{4} - \tan^{-1}(2\log x) + \tan^{-1}3 + \tan^{-1}(2\log x)$$
$$= \frac{\pi}{4} + \tan^{-1}3$$

Thus $$\frac{dy}{dx} = 0 \quad\Rightarrow\quad \frac{d^2y}{dx^2} = 0.$$

Example 80: If $f(x) = x^{1/x}$ then $f''(e)$ is equal to

(a) $e^{1/(e-3)}$ (b) $e^{1/e}$
(c) $e^{1/(e-2)}$ (d) none of these

Ans. (d)

◎ **Solution:** $\log f(x) = \frac{1}{x}\log x$. Differentiating both sides we have

$$\frac{f'(x)}{f(x)} = \frac{1}{x^2} - \frac{\log x}{x^2}$$

$$\Rightarrow \qquad f'(x) = f(x)\,\frac{1-\log x}{x^2}$$

Differentiating again both the sides, we have

$$f''(x) = f'(x)\,\frac{1-\log x}{x^2} + f(x)\left[\frac{x^2(-1/x) - (1-\log x)\,2x}{x^4}\right]$$

thus $f''(e) = -\frac{f(e)}{e^3} = -e^{(1/e)-3}$.

◉ **Example 81:** If $y^2 = P(x)$ is a polynomial of degree 3, then $\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$ is equal to

(a) $P(x) + P''(x)$ (b) $P(x)\,P''(x)$

(c) $P(x)\,P'''(x)$ (d) *a* constant.

Ans. (c)

◎ **Solution:** From $y^2 = P(x)$, we have $2yy_1 = P'(x)$ i.e. $2y_1 = P'(x)/y$

$$\Rightarrow \quad 2y_2 = \frac{yP''(x) - P'(x)y_1}{y^2} = \frac{yP''(x) - P'(x)P'(x)/2y}{y^2}$$

$$= \frac{2y^2P''(x) - (P'(x))^2}{2y^3}$$

$$\Rightarrow \qquad 2y_2\,y^3 = \frac{1}{2}\,[2\,P(x)\,P''(x) - (P'(x))^2]$$

$$\Rightarrow 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = \frac{1}{2}\,[2\{P'(x)\,P''(x) + P(x)\,P'''(x)\} - 2P'(x)\,P''(x)]$$

$$= \frac{1}{2}\,2P(x)\,P'''(x) = P(x)\,P'''(x).$$

◉ **Example 82:** If $f(x) = (1+x)^n$, then the value of $f(0) + f'(0) + \frac{1}{2!}f''(0) + \cdots + \frac{1}{n!}f^n(0)$ is

(a) n (b) $2n$

(c) 2^{n-1} (d) none of these

Ans. (b)

◎ **Solution:** $f(0) = 1, f'(x) = n(1+x)^{n-1}$,

$f''(x) = n\,(n-1)\,(1+x)^{n-2} \cdots f^n(x) = n(n-1)\cdots 1\,(1+x)^{n-n}$.

So $\qquad f'(0) = n, f''(0) = n\,(n-1), \ldots f^n(0) = n!$.

∴ the given expression is equals to

$$1 + n + \frac{n(n-1)}{2!} + \cdots \frac{n!}{n!} = 2^n.$$

◉ **Example 83:** If $f(x) = x^2 + \frac{x^2}{(1+x^2)} + \frac{x^2}{(1+x^2)^2} + \ldots + \frac{x^2}{(1+x^2)^n} + \ldots$ then at $x = 0$

(a) $f(x)$ has no limit

(b) $f(x)$ is discontinuous

(c) $f(x)$ is continuous but not differentiable

(d) $f(x)$ is differentiable

Ans. (b)

◎ **Solution:** For $x = 0, f(0) = 0$ and for $x \neq 0$,

$$f(x) = x^2\left[1 + \frac{1}{1+x^2} + \frac{1}{(1+x^2)^2} + \ldots\right]$$

$$= x^2\,\frac{1}{1 - 1/(1+x^2)} = \frac{x^2(1+x^2)}{x^2} = 1 + x^2,$$

Thus, $\lim_{x\to 0} f(x) = 1$, Hence f is not continuous at $x = 0$, so f is also not differentiable at $x = 0$.

◉ **Example 84:** Let $f(x) = \begin{cases}\int_0^x \{1 + |1-t|\}\,dt & \text{if } x > 2\\ 5x - 7 & \text{if } x \le 2,\end{cases}$ then

(a) f is not continuous at $x = 2$

(b) f is continuous but not differentiable at $x = 2$

(c) f is differentiable everywhere

(d) $f'(2+)$ doesn'*t* exist

Ans. (b)

◎ **Solution:** For $x > 2$,

$$\int_0^x \{1 + |1-t|\}dt = \int_0^1 \{1 + |1-t|\}dt + \int_1^x \{1 + |1-t|\}dt$$

$$= \int_0^1 (2-t)dt + \int_1^x t\,dt = 1 + \frac{x^2}{2}.$$

Thus,. $$f(x) = \begin{cases} 1 + x^2/2, & x > 2\\ 5x - 7, & x \le 2\end{cases}$$

$$\lim_{x\to 2+} f(x) = 1 + 4/2 = 3 = f(2) = \lim_{x\to 2-} f(x)$$

$$f'(2+) = \lim_{h\to 0+}\frac{1 + (1/2)(2+h)^2 - 3}{h}$$

$$= \lim_{h\to 0+}\frac{(1/2)h^2 + 2h}{h} = 2,$$

$$f'(2-) = \lim_{h\to 0-}\frac{5(2+h) - 7 - 3}{h} = 5$$

Hence f is continuous but not differentiable at $x = 2$.

Example 85: Let $f(x) = \underset{k=1}{\pi} (\cos (2k-1)x + \mathrm{i} \sin (2k-1) x)$ then $(\operatorname{Re} f(x))'' + \mathrm{i}\, (\operatorname{Im} f(x))''$ is equal to

(a) $n^2 f(x)$ (b) $-n^4 f(x)$
(c) $-n^2 f(x)$ (d) $n^4 f(x)$

Ans. (b)

Solution: $f(x) = \prod\limits_{k=1}^{n} (\cos (2k-1)x + i \sin (2k-1)x)$

$$= \cos \sum_{k=1}^{n} (2k-1)\, x + i \sin \sum_{k=1}^{n} (2k-1)\, x$$

$= \cos n^2x + \mathrm{i} \sin n^2x$ (Using De Movire's Theorem)

$\therefore$ $(\operatorname{Re} f(x))'' = -n^4 \cos n^2x$ and

$\operatorname{Im} f(x)'' = -n^4 \sin n^2x$. Thus

$(\operatorname{Re} f(x))'' + i\, (\operatorname{Im} f(x))'' = -n^4 [\cos n^2x + i \sin n^2x] = -n^4 f(x)$.

Example 86: Let $f(x)$ be defined for all $x > 0$ and be continuous. Let $f(x)$ satisfy $f(x/y) = f(x) - f(y)$ for all x, y and $f(e) = 1$. Then which of the following may be true

(a) $f(x)$ is bounded
(b) $f(1/x) \to 0$ as $x \to \infty +$
(c) $x f(x) \to 1$ as $x \to 0 +$
(d) $f'(x) = 1/x$

Ans. (d)

Solution: A function satisfying the given functional equation is $f(x) = \log x$. Clearly $f(x)$ is not bounded and $f(1/x) = -\log x \not\to 0$ as $x \to \infty$. Also $x f(x) = x \log x \not\to 1$ as $x \to 0$. But $f'(x) = 1/x$.

In fact, it can be shown that any continuous function satisfying the given functional equation is of the form $k \log x$, for some $k > 0$. Since $f(e) = 1$ so $k = 1$.

Example 87: If $x = \cos \theta$, $y = \sin^3 \theta$, then $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at $\theta = \pi/2$ is

(a) 1 (b) 2
(c) -2 (d) -3

Ans. (d)

Solution: $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin \theta$, $\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 3 \sin^2 \theta \cos \theta \Rightarrow$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -3 \sin \theta \cos \theta$. Differentiating again, we have $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -3 \cos {}^2\theta \dfrac{\mathrm{d}\theta}{\mathrm{d}x} = \dfrac{3 \cos 2\theta}{\sin\theta}$.

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 9 \sin^2 \theta \cos^2 \theta + \sin^3 \theta \frac{3 \cos 2\theta}{\sin\theta}$$

$$= 9 \sin^2 \theta \cos^2 \theta + 3 \sin^2 \theta \cos 2\theta$$

So the value of the expression on L.H.S. at $\theta = \pi/2$ is -3.

Example 88: If $y = \tan^{-1} \dfrac{1}{1 + x + x^2} + \tan^{-1} \dfrac{1}{x^2 + 3x + 3} + \tan^{-1} \dfrac{1}{x^2 + 5x + 7} + \ldots$ + upto n terms, then $y'(0)$ is equal to

(a) $-1/(n^2 + 1)$ (b) $-n^2/(n^2 + 1)$
(c) $n^2/(n^2 + 1)$ (d) none of these

Ans. (b)

Solution: $y = \tan^{-1} \dfrac{1}{1 + x + x^2} + \tan^{-1} \dfrac{1}{x^2 + 3x + 3} + \ldots$ + upto n terms

$$= \tan^{-1} \frac{(x+1) - x}{1 + x(x+1)} + \tan^{-1} \frac{(x+2) - (x+1)}{1 + (x+1)(x+2)} + \ldots n \text{ terms}$$

$= \tan^{-1} (x + 1) - \tan^{-1} x + \tan^{-1} (x + 2) - \tan^{-1} (x + 1) + \ldots$
$+ \tan^{-1} (x + n) - \tan^{-1} (x + (n - 1))$
$= \tan^{-1} (x + n) - \tan^{-1} x$.

$$y'(x) = \frac{1}{1 + (x+n)^2} - \frac{1}{1 + x^2}$$

$$\Rightarrow \quad y'(0) = \frac{1}{1 + n^2} - 1 = -\frac{n^2}{1 + n^2}.$$

Example 89: If $f(x) = \sqrt{x + 2\sqrt{2x - 4}} + \sqrt{x - 2\sqrt{2x - 4}}$ then

(a) f is differentiable at all points of its domain except $x = 4$
(b) f is differentiable on $(2, \infty)$
(c) f is differentiable on $(-\infty, \infty)$
(d) $f'(x) = 0$ for all $x \in [2, 6)$

Ans. (a)

Solution: The domain of f is $[2, \infty)$. Put $t = \sqrt{2x - 4}$

$$f(x) = \sqrt{t^2/2 + 2 + 2t} + \sqrt{t^2/2 + 2 - 2t}$$

$$= \frac{1}{\sqrt{2}} (t + 2) + \frac{1}{\sqrt{2}} |t - 2| = \begin{cases} \dfrac{1}{\sqrt{2}} \times 4 & \text{if } t < 2 \\ \sqrt{2} t & \text{if } t \geq 2 \end{cases}$$

$$= \begin{cases} 2\sqrt{2} & \text{if } x \in (2, 4) \\ 2\sqrt{x - 2} & \text{if } x \in (4, \infty) \end{cases}$$

$$\text{Hence } f'(x) = \begin{cases} 0 & \text{if } x \in (2, 4) \\ \dfrac{1}{\sqrt{x - 2}} & \text{if } x \in (4, \infty) \end{cases}$$

and $f'(4)$ does not exist.

◉ **Example 90:** If $y = \sec^{-1}\left(\dfrac{x+1}{x-1}\right) + \sin^{-1}\left(\dfrac{x-1}{x+1}\right)$, $x \in [0, \infty) \sim \{1\}$ then $\dfrac{dy}{dx}$ is equal to

(a) 1 (b) $\dfrac{x-1}{x+1}$

(c) 0 (d) $\dfrac{x+1}{x-1}$

Ans. (c)

◎ **Solution:** $y = \cos^{-1}\dfrac{x-1}{x+1} + \sin^{-1}\dfrac{x-1}{x+1}$, since $-1 \le \dfrac{x-1}{x+1} \le 1$

$$\therefore \qquad y = \frac{\pi}{2}. \text{ So } \frac{dy}{dx} = 0$$

◉ **Example 91:** The least value of n so that $y_n = y_{n+1}$ where $y = x^2 + e^x$ is

(a) 4 (b) 3

(c) 5 (d) 2

Ans. (b)

◎ **Solution:** $y' = 2x + e^x$, $y'' = 2 + e^x$, $y''' = e^x$, $y^{iv} = e^x$. Thus $n = 3$.

◉ **Example 92:** If $y = \cos^{-1}\dfrac{9-x^2}{9+x^2}$ then $y'(-1)$ is equal to

(a) – 3/5 (b) 3/5

(c) 2/7 (d) 3/8

Ans. (a)

◎ **Solution:** $\cos^{-1}\dfrac{9-x^2}{9+x^2} = \cos^{-1}\dfrac{1-(x/3)^2}{1+(x/3)^2} = 2\tan^{-1}\dfrac{x}{3}$, if $0 \le x < \infty$ and is equal to $-2\tan^{-1} x/3$ if $-\infty < x \le 0$. Hence

$$y'(x) = \begin{cases} \dfrac{2}{1+(x/3)^2}.\dfrac{1}{3} & \text{if } 0 < x < \infty \\ -\dfrac{2}{1+(x/3)^2}.\dfrac{1}{3} & \text{if } -\infty < x < 0 \end{cases}$$

Hence $y'(-1) = -3/5$.

◉ **Example 93:** If $y = (x^2 + 1)^{\sin x}$, then $y'(0)$ is equal to

(a) 1/2 (b) e^2

(c) 0 (d) 3/2

Ans. (c)

◎ **Solution:** $\log y = \sin x \log(x^2 + 1)$.

Differentiating we get $\dfrac{y'}{y} = (\sin x)\dfrac{2x}{x^2+1} + \cos x \log(x^2 + 1)$ so $y'(0) = y(0).0 = 0$.

◉ **Example 94:** If $f(x) = x^n + 4$ then the value of $f(1) + \dfrac{f'(1)}{1!} + \dfrac{f''(1)}{2!} + \ldots + \dfrac{f^n(1)}{n!}$ is

(a) 2^{n-1}

(b) $2^n + 4$

(c) $1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \ldots + \dfrac{1}{n!}$

(d) none of these

Ans. (b)

◎ **Solution:** $f(1) = 5, f'(x) = nx^{n-1}$ so $f'(1) = n$ $f''(1) = n(n-1), \ldots, f^n(1) = 1.2 \ldots n$. Thus

$$f(1) + \frac{f'(1)}{1!} + \ldots + \frac{f^n(1)}{n!}$$

$$= 5 + \frac{n}{1} + \frac{n(n-1)}{2!} + \ldots + \frac{n!}{n!} = (1+1)^n + 4 = 2^n + 4.$$

◉ **Example 95:** Let a function $y = y(x)$ be defined parametrically by $x = 2t - |t|$, $y = t^2 + t|t|$. Then $y'(x)$, $x > 0$

(a) 0 (b) $4x$

(c) $2x$ (d) does not exist

Ans. (b)

◎ **Solution:** $x = 2t - |t| = \begin{cases} t, & t \ge 0 \\ 3t, & t < 0 \end{cases}$, so $t = \begin{cases} x, & x \ge 0 \\ x/3, & x < 0 \end{cases}$

Therefore, we can express $y = t^2 + t|t| = \begin{cases} 2t^2, & t \ge 0 \\ 0 & t < 0 \end{cases}$

$$= \begin{cases} 2x^2, & x \ge 0 \\ 0 & x < 0 \end{cases}. \text{ Hence } y'(x) = \begin{cases} 4x, & x > 0 \\ 0, & x < 0 \end{cases}$$

Note that we cannot find $\dfrac{dx}{dt}$ as the derivative does not exist at $t = 0$.

◉ **Example 96:** $\dfrac{d^2x}{dy^2}$ equals

(a) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$ (b) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$

(c) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$ (d) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$

Ans (a)

◎ **Solution:** $\dfrac{d^2x}{d^2y} = \dfrac{d}{dy}\left(\dfrac{dx}{dy}\right) = \dfrac{d}{dx}\left(\dfrac{dx}{dy}\right)\dfrac{dx}{dy}$

$$= \frac{d}{dx}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy}$$

$$= (-1)\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-1}$$

$$= -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}.$$

◉ **Example 97:** Let $y = f(x) = \log(1+\sin x)^2, x \neq (2n+1)\dfrac{\pi}{2}$, $n \in \mathbf{I}$ then $f''(x) + 2e^{-y/2}$ is equal to

(a) 0 (b) $\dfrac{1}{1+\sin x}$

(c) $\dfrac{1}{(1+\sin x)^2}$ (d) $-\dfrac{1}{1+\sin x}$

Ans. (a)

◎ **Solution:** $y = 2\log(1+\sin x)$

$$y' = \frac{2\cos x}{1+\sin x}$$

$$\Rightarrow \quad y'' = 2\frac{-(1+\sin x)\sin x - \cos^2 x}{(1+\sin x)^2}$$

$$= -2\frac{1+\sin x}{(1+\sin x)^2} = -\frac{2}{1+\sin x}$$

$$f''(x) + 2e^{-y/2} = f''(x) + 2e^{-\log(1+\sin x)}$$

$$= f''(x) + \frac{2}{1+\sin x} = 0.$$

◉ **Example 98:** Let $f(x) = x + \cot x$ and f is the inverse of g, then $g'(x)$ is equal to

(a) $\dfrac{x}{x-g(x)}$ (b) $\dfrac{1}{(x-g(x))^2}$

(c) $-\dfrac{1}{(x-g(x))^2}$ (d) $\dfrac{x}{(x-g(x))^2}$

Ans. (c)

◎ **Solution:** $f(g(x)) = x$

$\Rightarrow \quad g(x) + \cot g(x) = x$

Differentiating both sides, we get

$g'(x) - (\operatorname{cosec}^2 g(x))\, g'(x) = 1$

$$\Rightarrow \quad g'(x) = \frac{1}{1-\operatorname{cosec}^2 g(x)}$$

$$= \frac{1}{1-(1+\cot^2 g(x))} = -\frac{1}{(x-g(x))^2}$$

◉ **Example 99:** If $f(x) = \begin{cases} xe^{-\left\{\frac{1}{|x|}+\frac{1}{x}\right\}} &, x \neq 0 \\ 0 &, x = 0 \end{cases}$ then f is

(a) discontinuous everywhere

(b) continuously differentiable at $x = 0$

(c) differentiable everywhere

(d) continuous but not differentiable

Ans. (d)

◎ **Solution:** $f(x) = \begin{cases} x & x < 0 \\ xe^{-\frac{2}{x}} & x > 0 \\ 0 & x = 0 \end{cases}$

f is clearly a continuous function

$$f'(x) = \begin{cases} 1 &, x < 0 \\ e^{-\frac{2}{x}} + \frac{2}{x}e^{-\frac{2}{x}} &, x > 0 \end{cases}$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h}{h} = 1$$

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{he^{-\frac{2}{h}}}{h} = \lim_{h\to 0^+} e^{-\frac{2}{h}} = 0$$

f is not differentiable at $x = 0$.

◉ **Example 100:** Let f and g be twice differentiable functions such that $\text{fog}(x) = x^3$ and $g(1) = 2$, $g'(1) = 1$, $g''(1) = 4$ then $f''(2)$ is equal to

(a) 0 (b) 2

(c) 4 (d) -6

Ans. (d)

◎ **Solution:** Since $f(g(x)) = x^3$, differentiating both the sides, we get

$$f'(g(x))\, g'(x) = 3x^2$$

$$\Rightarrow \quad f''(g(x))\,(g'(x))^2 + f'(g(x))\, g''(x) = 6x$$

$$\Rightarrow \quad f''(g(x))\,(g'(x))^2 + \frac{3x^2}{g'(x)}\, g''(x) = 6x$$

Putting $x = 1$, we have

$$f''(2)\, 1^2 + \frac{3}{1}\, 4 = 6$$

$$\Rightarrow \quad f''(2) = -6.$$

EXERCISE

Concept-based Straight Objective Type Questions

1. If $f(t) = \dfrac{t^2-5t-1}{t^3}$ then $f'\left(\dfrac{1}{a}\right)$ is given by

(a) $3a^4 + 10a^3 + a^2$ (b) $3a^4 + 10a^3 - a^2$

(c) $a^4 + 5a^3 - a^2$ (d) $a^4 + 5a^2 - a$

2. If $y = \dfrac{1}{\sqrt[3]{2x-1}} + \dfrac{1}{\sqrt[4]{(x^2+2)^3}}$ then $y'(0)$ is given by

(a) $\dfrac{1}{3}$ (b) $\dfrac{2}{3}$

(c) $-\frac{2}{3}$ (d) $-\frac{1}{3}$

3. Let $f(x) = |\log x|$ then
   (a) $f'(1+) = -1$ (b) $f'(1+) = f'(1-)$
   (c) $f'(1-) = 1$ (d) $f'(1+) = 1$

4. For $f(x) = \sin^{-1}\frac{2x}{1+x^2}$, if $A = \{x \in \text{dom } f : f$ is not differentiable at $x\}$ then number of points in A is
   (a) 0 (b) 1
   (c) 2 (d) 3

5. Let $y = \sqrt{1-\sqrt{1-x^2}}$. The set of all points of non differentiability of y is
   (a) $\{0, 1\}$ (b) $\{1, -1\}$
   (c) $\{0, -1\}$ (d) $\{0, -1, 1\}$

6. If $y = \log_2 (\log_3(\log_5 x))$ then $y'(125)$ is equal to
   (a) $\frac{1}{375 \log 2}$ (b) $\frac{1}{125 \log 2 \log 3 \log 5}$
   (c) $\frac{1}{375 \log 3 \log 3}$ (d) $\frac{1}{375 \log 2 \log 3 \log 5}$

7. If $x = \frac{t+1}{t}$, $y = \frac{t-1}{t}$ then $\frac{d^2y}{dx^2}$ at $t = 1$
   (a) 0 (b) 1
   (c) -1 (d) $\frac{1}{\sqrt{2}}$

8. If $x = 3 \cos t$, $y = 4 \sin t$ then $\frac{dy}{dx}$ at $\left(3/\sqrt{2}, 2\sqrt{2}\right)$ is equal to
   (a) $\frac{1}{3}$ (b) $\frac{2}{3}$
   (c) $-\frac{2}{3}$ (d) $-\frac{4}{3}$

9. If $x^4 + y^4 = x^2y^2$ then $\frac{dy}{dx}$ is given by
   (a) $\frac{x}{y}\cdot\frac{y^2-x^2}{2y^2-x^2}$ (b) $\frac{y}{x}\cdot\frac{y^2-2x^2}{y^2-x^2}$
   (c) $\frac{x}{y}\cdot\frac{y^2-2x^2}{2y^2-x^2}$ (d) $\frac{y}{x}\cdot\frac{y^2-2x^2}{2y^2-x^2}$

10. Let $f(x) = |x^2 - 4|x| + 3|$, then $A = \{x : f$ is not differentiable at $x\}$ is equal to
    (a) $\{-1, 1, -3, 3\}$ (b) $\{1, 3\}$
    (c) $\{1, -1\}$ (d) $\{3, -3\}$

LEVEL 1

Straight Objective Type Questions

11. The function
$$f(x) = \begin{cases} x^4 + x^2 - x + 2, & \text{for } x \le 1 \\ 3x^3 - x^2 + x, & \text{for } x > 1 \end{cases}$$
is
    (a) continuous everywhere
    (b) differentiable everywhere
    (c) differentiable at $x = 1$
    (d) such that f' exists everywhere but f'' is not continuous at $x = 1$.

12. If
$$f(x) = \begin{cases} x\sin(1/x), & \text{for } x \ne 0 \\ 0, & \text{for } x = 0 \end{cases}$$
then
    (a) f is a continuous function
    (b) $f'(0+)$ exists but $f'(0-)$ does not exist
    (c) $f'(0+) = f'(0-)$
    (d) $f'(0-)$ exists but $f'(0+)$ does not exist.

13. If $y = \log_{e^x}(x-2)^2$ for $x \ne 0, 2$, then $y'(3)$ is equal to
    (a) 1/3 (b) 2/3
    (c) 4/3 (d) none of these

14. The function f defined by
$$f(x) = \begin{cases} \frac{\sin x^2}{x}, & \text{for } x \ne 0 \\ 0, & \text{for } x = 0 \end{cases}$$
is
    (a) continuous and derivable at $x = 0$
    (b) neither continuous nor derivable at $x = 0$
    (c) continuous but not derivable at $x = 0$
    (d) none of these.

15. The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = 0$ is
    (a) 1/4 (b) 1/8
    (c) 1/2 (d) 1

16. Let $f(x) = x^n$, n being a non-negative integer. The value of n for which the equality $f'(a + b) = f'(a) + f'(b)$ is valid for all $a, b > 0$ is
    (a) 3 (b) 1
    (c) 2 (d) none of these

17. The function $f(x) = |x^3|$ is
    (a) differentiable everywhere
    (b) continuous but not differentiable at $x = 0$
    (c) not a continuous function
    (d) a function with range $(0, \infty)$
18. The set of all points of differentiability of the function $f(x) = e^{-|x|}$ is
    (a) $(0, \infty)$ (b) $(-\infty, \infty)$
    (c) $[0, \infty)$ (d) $(-\infty, \infty) \sim \{0\}$
19. If $x = \log t$ and $y = t^2 - 1$, then $y''(1)$ at $t = 1$ is
    (a) 2 (b) 4
    (c) 3 (d) none of these
20. For a differentiable function f on $\mathbf{R}$ there is an x_0 with
    $$f'(x_0) = \frac{1}{2}[f'(0) + f'(1)]$$
    (a) only if f is constant
    (b) only if f is increasing
    (c) if f is decreasing
    (d) if f is continuously differentiable
21. Let f be defined by $f(x) = |x - 2| + |x| + |x + 2|$ for $x \in \mathbf{R}$ then
    (a) $f'(-2+)$ doesn't exist
    (b) $f'(2-)$ doesn't exist
    (c) $f'(2+) = 3$
    (d) $f'(0+) = 2$
22. Let f be defined on $\mathbf{R}$ by $f(x) = x^4 \sin(1/x)$, if $x \neq 0$ and $f(0) = 0$ then
    (a) $f'(0)$ doesn't exist
    (b) $f'(2-)$ doesn't exist
    (c) f'' is not continous at $x = 0$
    (d) $f''(0)$ exist but f'' is not continuous at $x = 0$
23. If $x^y = e^{x-y}$ then
    (a) $\dfrac{dy}{dx}$ doesn't exist at $x = 1$
    (b) $\dfrac{dy}{dx} = 0$ when $x = 1$
    (c) $\dfrac{dy}{dx} = \dfrac{1}{2}$ when $x = e$
    (d) none of these
24. If f is one-one and satisfies $f'(x) = \sqrt{1-(f(x))^2}$ then $(f^{-1})'(x)$
    (a) is equal to $\dfrac{1}{\sqrt{1-x^2}}$
    (b) may not exist for every $x \in \mathbf{R}$
    (c) may not be known explicitly
    (d) is equal to $\sin^{-1}(f(x))$
25. The function $y = \sin^{-1} x$ satisfies
    (a) $(1 - x^2)\, y'' = xy''$ (b) $(1 - x^2)\, y'' = xy'$
    (c) $(1 - x^2)\, y'' = x^2y'$ (d) $(1 - x^2)\, y' = 2xy'$
26. Let f be a one-one function satisfying $f'(x) = f(x)$ then $(f^{-1})''(x)$ is equal to
    (a) $-1/x^3$ (b) $-1/x^2$
    (c) $f(x)$ (d) $f^{-1}(x)$
27. If $y = \cos^{-1}(x^{-1})$ then $y'(-2)$ is equal to
    (a) $\dfrac{1}{2\sqrt{3}}$ (b) $-\dfrac{1}{2\sqrt{3}}$
    (c) $\dfrac{1}{2\sqrt{5}}$ (d) $-\dfrac{1}{2\sqrt{5}}$
28. If $x = \sin^{-1}(t^2 - 1)$, $y = \cos^{-1}(2t)$ then $\dfrac{dy}{dx}$ at $t = 0$ is
    (a) $-\sqrt{2}$ (b) $-\dfrac{1}{\sqrt{2}}$
    (c) $\sqrt{2}$ (d) $\dfrac{1}{\sqrt{2}}$
29. If $\tan^{-1} y - y + x = 0$ then $\dfrac{d^2y}{dx^2}$ is equal to
    (a) $\dfrac{-2(1+y^2)}{y^5}$ (b) $\dfrac{1+y^2}{y^5}$
    (c) $\dfrac{2(1+y^2)}{y^4}$ (d) $\dfrac{2(1+y^2)}{y^5}$
30. If $F(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$ then $F'(x)$ is equal to
    (a) $6x^3$ (b) $x^3 + 6x^2$
    (c) $3x$ (d) $6x^2$
31. If $y = \dfrac{\sin^2 x}{1+\cot x} + \dfrac{\cos^2 x}{1+\tan x}$ then $y'(x)$ is equal to
    (a) $\cos 2x$ (b) $-\cos 2x$
    (c) $2\cos^2 x$ (d) $\cos^3 x$
32. If $z(t) = \sqrt{t^3+1}$, then $z'(2)$ is equal to
    (a) 1/2 (b) – 1
    (c) 2 (d) – 1
33. If $y = \cos^{-1}\left(\dfrac{4\cos x - 5\sin x}{\sqrt{41}}\right)$, then $\dfrac{dy}{dx}$ is equal to
    (a) 0 (b) 1
    (c) – 1 (d) none of these
34. If $y = \log\cos\left(\tan^{-1}\dfrac{e^x - e^{-x}}{2}\right)$ then $y'(0)$ is equal to
    (a) $e + e^{-1}$ (b) $e - e^{-1}$
    (c) $\dfrac{e+e^{-1}}{2}$ (d) none of these

35. If the function $y = \log \frac{1}{1+x}$ satisfies the relation $xy' + 1 = f(y)$ then $f(y)$ is equal to

(a) y (b) $y^2 + 1$
(c) e^y (d) e^{-y}

36. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ satisfies the relation $(1 - x^2)y' - xy = k$ then the value of k is

(a) 1 (b) 0
(c) –1 (d) 2

37. If $x \sin y = \sin(y + a)$ and $\frac{dy}{dx} = \frac{A}{1+x^2-2x\cos a}$ then the value of A is

(a) 2 (b) $\cos a$
(c) $\sin a$ (d) none of these

38. If $f(x) = \cot^{-1} \frac{3x-x^3}{1-3x^2}$ and $g(x) = \sin^{-1} \frac{1-x^2}{1+x^2}$ then $\lim\limits_{x\to t} \frac{f(x)-f(t)}{g(x)-g(t)}$ is

(a) $\frac{3}{2(1+t^2)}$ (b) $\frac{-5}{2(1+t^2)}$
(c) $\frac{5}{2}$ (d) $\frac{3}{2}$

39. Let f be a function defined for every x, such that $f'' = -f$, $f(0) = 0$, $f'(0) = 1$ then $f(x)$ is equal to

(a) $\tan x$ (b) $e^x - 1$
(c) $\sin x$ (d) $2 \sin x$

40. If $y = \sin^{-1}(\cos x)$ and $y'(x)$ is identically equal to $g(x)$ on $\mathbf{R} \sim \{kp, k \in \mathbf{I}\}$ then $g(x)$ is equal to

(a) $|\sin x|$ (b) $-\operatorname{sgn}(\sin x)$
(c) $\operatorname{sgn}(\sin x)$ (d) none of these

41. If $f(x) = \cos(x^2 - 4[x])$ for $0 < x < 1$, where $[x]$ denotes the greatest integer $\le$ x, then $f'\left(\sqrt{\pi}/2\right)$ is equal

(a) $-\sqrt{\frac{\pi}{2}}$ (b) $\sqrt{\frac{\pi}{2}}$
(c) 0 (d) $\frac{\sqrt{\pi}}{4}$

42. If $x = a \cos 2t$, $y = b \sin^2 t$ then $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{a}{b} \cos 2t$ (b) 0
(c) 1 (d) $\frac{b}{a} \sin 2t$

43. If $f(x) = \tan^{-1} \sqrt{(1+\sin x)/(1-\sin x)}$, $0 \le x < \pi/2$, then $f'(\pi/6)$ is

(a) – 1/4 (b) – 1/2
(c) 1/4 (d) 1/2

44. If $2^x + 2^y = 2^{x+y}$, then the value of $\frac{dy}{dx}$ at $x = y = 1$ is

(a) 0 (b) – 1
(c) 1 (d) 2

45. If $t = \sin^{-1} 2^s$ then $\frac{ds}{dt}$ is equal to

(a) $\frac{\log 2}{\sqrt{1-t^2}}$ (b) $\frac{\sin t}{\log 2}$
(c) $\frac{\cot t}{\log 2}$ (d) none of these

46. Let (x, y) be any point on the unit circle with centre at the origin, then

(a) $yy'' - 2(y')^2 + 1 = 0$
(b) $yy'' + y'^2 + 1 = 0$
(c) $yy'' + y'^2 - 1 = 0$
(d) $yy'' + 2y'^2 + 1 = 0$

47. Let $f(x) = \log \sin x^2$, $0 < x < \pi/2$ then $f''\left(\sqrt{\pi}/2\right)$ is equal to

(a) 2π (b) $2(1 + \pi)$
(c) $2(1 - \pi)$ (d) $4(1 - \pi)$

48. Let $g(x)$ be the inverse of $f(x)$ and $f'(x) = \frac{1}{1+x^5}$ then $\frac{d^2}{dx^2}(g(x))$ is equal to

(a) $\frac{1}{1+(g(x))^5}$ (b) $5(g(x))^4 (1 + (g(x))^5$
(c) $\frac{g'(x)}{1+(g(x))^5}$ (d) $\frac{5(g'(x))^4}{1+(g(x))^5}$

49. The derivative of $f(\tan x)$ with respect to $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'\left(\sqrt{2}\right) = 4$, is:

(a) $1/\sqrt{2}$ (b) $\sqrt{2}$
(c) 1 (d) 0

50. If $y = \left(\log_{\cos x} \sin x\right) \left(\log_{nx} \cos x\right) + \sin^{-1} \frac{2x}{1+x^2}$, then $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$ is equal to:

(a) $\frac{8}{\left(4+\pi^2\right)}$ (b) 0
(c) $-\frac{8}{\left(4+\pi^2\right)}$ (d) 1

51. If $y = \sin^{-1}\left(\dfrac{5x + 12\sqrt{1 - x^2}}{13}\right)$, then $\dfrac{dy}{dx}$ is equal to:

(a) $-\dfrac{1}{\sqrt{1 - x^2}}$ (b) $\dfrac{1}{\sqrt{1 - x^2}}$

(c) $\dfrac{3}{\sqrt{1 - x^2}}$ (d) $\dfrac{x}{\sqrt{1 - x^2}}$

52. Let f: $(-1, 1) \to \mathbf{R}$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0) =$

(a) 0 (b) -2

(c) 4 (d) -4

53. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals

(a) log 2 (b) $-$ log 2

(c) -1 (d) 1

54. Let $f(x) = \begin{cases} (x - 1)\sin\dfrac{1}{x - 1} & if \;\; x \neq 1 \\ 0 & if \;\; x = 1 \end{cases}$

Then which one of the following is true

(a) f is neither differentiable at $x = 0$ nor at $x = 1$

(b) f is differentiable at $x = 0$ and at $x = 1$

(c) f is differentiable at $x = 0$ but not at $x = 1$

(d) f is differentiable at $x = 1$ but not at $x = 0$.

Assertion-Reason Type Questions

55. Let $y = \dfrac{1}{3}\log\dfrac{x + 1}{\sqrt{x^2 - x + 1}} + \dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2x - 1}{\sqrt{3}}$

Statement-1: $\dfrac{dy}{dx}$ at $x = 0$ is 1

Statement-2: $\dfrac{dy}{dx} = \dfrac{2}{x^3 + 2}$

56. Let $y = \log\dfrac{1}{1 + x}$

Statement-1: $y'(1) = -1/2$

Statement-2: $xy' + 1 = e^y$.

57. Suppose that $\cos(xy) = x$

Statement-1: $\dfrac{dy}{dx} = \dfrac{1 + y\sin(xy)}{x\sin(xy)}$

Statement-2: $\dfrac{dy}{dx} < 0$ for $x, y > 0$ such that $0 < xy < \pi$.

58. Suppose that $x = t - t^4$, $y = t^2 - t^3$

Statement-1: $\dfrac{dy}{dx}$ at (0, 0) is equal to $\dfrac{1}{3}$ or 0

Statement-2: $\dfrac{dy}{dx} = \dfrac{2t - 3t^2}{1 - 4t^3}$.

59. If $y = \sin(3\sin^{-1} x)$

Statement-1: $y''(0) = 0$

Statement-2: $y'(0) = 3$.

60. **Statement-1:** Let $f : \mathbf{R} \to \mathbf{R}$ be a real valued function satisfying $|f(x) - f(y)| \le A|x - y|^{\alpha}$, for some constant A and $\alpha > 1$, for every $x, y \in \mathbf{R}$ then f is constant function.

Statement-2: $f'(x) = 0$ for all $x \in \mathbf{R}$ then f is a constant function.

61. Suppose that $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, $\pi/2 < x < \pi$

Statement-1: $\dfrac{d^2y}{dx^2} = 0$

Statement-2: y is a linear function of x.

62. Let f is a twice differentiable function satisfying $f'(x) = f(1 - x)$ for $x \in \mathbf{R}$. $f(\pi/2) = 1$

Statement-1: $f(x) = \left(-\dfrac{\cos 1}{1 + \sin 1}\right)\cos x + \sin x$

Statement-2: f satisfies $f''(x) - f(x) = 0$.

LEVEL 2

Straight Objective Type Questions

63. Let $f(x)$ be defined by

$$f(x) = \begin{cases} \sin 2x & \text{if } 0 < x \le \pi/6 \\ ax + b & \text{if } \pi/6 < x \le 1 \end{cases}$$

The values of a and b such that f and f' are continuous, are

(a) $a = 1, b = 1/\sqrt{2} + \pi/6$

(b) $a = 1/\sqrt{2}, b = 1/\sqrt{2}$

(c) $a = 1, b = \sqrt{3}/2 - \pi/6$

(d) none of these

64. Let
$$f(x) = \begin{cases} 1/|x| & \text{for } |x| \ge 1 \\ ax^2 + b & \text{for } |x| < 1 \end{cases}$$
The coefficients a and b so that f is continuous and differentiable at any point, are equal to
(a) $a = -1/2, b = 3/2$ (b) $a = 1/2, b = -3/2$
(c) $a = 1, b = -1$ (d) none of these.

65. For a real number y, let $[y]$ denote the greatest integer less than or equal to y. Then
$$f(x) = \frac{\tan(\pi[x - \pi])}{1 + [x]^2}$$
is
(a) discontinuous at some x
(b) continuous at all x, but the derivative $f'(x)$ does not exist for some x
(c) $f'(x)$ exists for all x but the second derivative $f''(x)$ does not exist
(d) $f'(x)$ exists for all x.

66. If
$$f(x) = \begin{cases} (\cos x)^{1/\sin x} & \text{for } x \ne 0 \\ k & \text{for } x = 0 \end{cases}$$
the value of k, so that f is differentiable at $x = 0$ is
(a) 0 (b) 1
(c) 1/2 (d) none of these.

67. If $y = \sin^{-1}\left(\dfrac{\sin\alpha \sin x}{1 - \cos\alpha \sin x}\right)$, then $y'(0)$ is
(a) 1 (b) $2\tan\alpha$
(c) $(1/2)\tan\alpha$ (d) $\sin\alpha$.

68. The set of all points of differentiability of the function
$$f(x) = \frac{\sqrt{x+1} - 1}{\sqrt{x}}$$
for $x \ne 0$ and $f(0) = 0$ is
(a) $(-\infty, \infty)$ (b) $[0, \infty)$
(c) $(0, \infty)$ (d) $(-\infty, \infty) \sim \{0\}$.

69. If $x = \sin^{-1} t$ and $y = \log(1 - t^2)$; then $\left.\dfrac{d^2y}{dx^2}\right|_{t=1/2}$ is
(a) $-8/3$ (b) $8/3$
(c) $3/4$ (d) $-3/4$

70. If $y^2 + x^2 = R^2$ and $k = 1/R$, then k is equal to
(a) $\dfrac{|y''|}{\sqrt{1 + y'^2}}$ (b) $\dfrac{|y''|}{\sqrt{(1 + y'^2)^3}}$
(c) $\dfrac{2|y''|}{\sqrt{1 + y'^2}}$ (d) $\dfrac{|y''|}{2\sqrt{(1 + y'^2)^3}}$.

71. Let $f : \mathbf{R} \to \mathbf{R}$. Then f is differentiable on $\mathbf{R}$ if
(a) $|f(x) - f(y)| \le k(x - y)$ for all $x, y \in \mathbf{R}$ and some $k > 0$
(b) $|f(x) - f(y)| \le k|x - y|^{1/2}$ for all $x, y \in \mathbf{R}$ and some $k > 0$
(c) $|f(x) - f(y)| \le k(x - y)^2$ for all $x, y \in \mathbf{R}$ and some $k > 0$
(d) f^2 is differentiable on $\mathbf{R}$

72. $f : \mathbf{R} \to \mathbf{R}$ be such that $|f(x) - f(y)| \le |x - y|^3$ for all $x, y \in \mathbf{R}$ then the value of $f'(x)$ is
(a) $f(x)$
(b) constant possibly different from zero
(c) $(f(x))^2$
(d) 0

73. Let f be the function on [0, 1] given by $f(x) = x\sin\dfrac{\pi}{x}$ for $x \ne 0$ and $f(0) = 0$. Then f is
(a) discontinuous but bounded
(b) differentiable
(c) continuous but not bounded
(d) continuous and bounded

74. Let $f(x) = \dfrac{|x|\left(3e^{1/|x|} + 4\right)}{2 - e^{1/|x|}}$, $x \ne 0$ and $f(0) = 0$ then
(a) f is not continuous
(b) f is continuous but not differentiable at $x = 0$
(c) $f'(0)$ exist
(d) $f'(0+) = 2$

75. Let $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ k(x - 1), & x > 1 \end{cases}$ then
(a) f is continuous for only finitely many values of k
(b) f is discontinuous at $x = 1$
(c) f is differentiable only when $k = 2$
(d) there are infinitely many values of k for which f is differentiable

76. Let f be a function defined on $[0, \infty)$ by
$$f(x) = \begin{cases} 0 & x = 0 \\ (x + a/\sqrt{2})(1/\sqrt{2}) & 0 < x \le a \\ (x/2 + a)(1/\sqrt{2}) & x \ge a \end{cases}$$
then
(a) $f'(x)$ is discontinuous at $x = 0, a$
(b) f is a continuous function
(c) f is discontinuous at $x = a$
(d) none of these

77. The number of times the function $y = (x^2 + 1)^{80}$ to be differentiated to result in a polynomial of degree 50 is
(a) 70 (b) 80
(c) 110 (d) 30

78. The derivative of the function $y = \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right)$ doesn't exist for
(a) all values of x for which $|x| < 1$
(b) $x = -1, 1$

(c) all values of x for which $|x| > 1$
(d) none of these

79. If $0 < \alpha < 1$ and f is defined as

$$f(x) = \begin{cases} x^{\alpha} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{then}$$

(a) f is differentiable at $x = 0$
(b) f is not continuous
(c) f is not differentiable at $x = 0$
(d) f is bounded

80. Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$

then
(a) f is continuous on **R**
(b) f is differentiable on **R** ~ **Q**
(c) f is differentiable only at one point
(d) f is continuous at infinitely many points of **R**

81. A value of k such that $f(x) = x^k$ is $(n - 1)$ times differentiable at 0 but not n- times differentiable at 0 is

(a) $\dfrac{2n-3}{3}$ (b) $\dfrac{3n-2}{3}$
(c) $\dfrac{4n-3}{3}$ (d) none of these

82. The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \dfrac{1}{2}(|x| - 1) & \text{if } |x| > 1 \end{cases} \quad \text{is}$$

(a) **R** ~ {0} (b) **R** ~ {1}
(c) **R** ~ {– 1} (d) **R** ~ {– 1, 1}

83. The derivative of $\sin^{-1}\left(\dfrac{1-x^{2n}}{1+x^{2n}}\right)$ with respect to x^{2n} is

(a) $-\dfrac{1}{x^n + x^{3n}}$ (b) $\dfrac{1}{x^n + x^{3n}}$
(c) $\dfrac{1}{x^n + x^{2n}}$ (d) $-\dfrac{1}{x^n + x^{2n}}$

84. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$ and $\dfrac{dy}{dx} = f(x, y)\sqrt{\dfrac{1-y^6}{1-x^6}}$ then the value of $f(x, y)$ is

(a) y/x (b) y^2/x^2
(c) $2y^2/x^2$ (d) x^2/y^2

85. If $f(x) = x^2 + \tan x$ and f is inverse of g, then $g'(x)$ is equal to

(a) $\dfrac{1}{2+[g(x)-x]^2}$ (b) $\dfrac{1}{2g(x)+(g(x)-x)^2}$
(c) $\dfrac{1}{2g(x)-(g(x)-x)^2}$
(d) $\dfrac{1}{2g(x)+1+(x-(g(x))^2)^2}$

86. Number of points of non-differentiability of $f(x) = \sin \pi(x - [x])$ in $(-\pi/2, \pi/2)$, where [.] denotes greatest integer function is

(a) 4 (b) 5
(c) 3 (d) 2

87. Suppose that f is differentiable function with the property $f(x + y) = f(x) + f(y) + x^2y^2$ and $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = 100$ then $f'(x)$ is equal to

(a) 100 (b) 20
(c) 30 (d) none of these

Previous Years' AIEEE/JEE Main Questions

1. $f(x)$ and $g(x)$ are two differential function on [0, 2] such that $f''(x) - g''(x) = 0$, $f'(1) = 2g'(1) = 4$, $f(2) = 3g(2) = 9$ then $f(x) - g(x)$ at $x = \dfrac{3}{2}$ is

(a) 0 (b) 2
(c) 10 (d) 5 **[2002]**

2. If $f(1) = 1, f'(1) = 2$ then $\lim\limits_{x \to 1} \dfrac{\sqrt{f(x)} - 1}{\sqrt{x} - 1}$ is

(a) 2 (b) 4
(c) 1 (d) $\dfrac{1}{2}$ **[2002]**

3. If $y = \left(x + \sqrt{1+x^2}\right)^n$, then $(1+x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx}$ is

(a) n^2y (b) $-n^2y$
(c) $-y$ (d) $2x^2y$ **[2002]**

4. If $f(x) = x^n$, then the value of

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \cdots + \frac{(-1)^n f^n(1)}{n!} \text{ is}$$

(a) 2^{n-1} (b) 0
(c) 1 (d) 2^n **[2003]**

5. If $f(x) = \begin{cases} x\, e^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ then $f(x)$ is

(a) continuous for all x, but not differentially at $x = 0$
(b) neither differential nor continuous at $x = 0$
(c) discontinuous everywhere
(d) continuous as well as differentiable for all x. **[2003]**

6. Let $f(a) = g(a) = k$ and their nth derivatives $f^n(a)$, $g^n(a)$ exist and are not equal for some n. Further if

$$\lim_{x \to a} \frac{f(a)g(x) - f(a) - g(a)f(x) + g(a)}{g(x) - f(x)} = 4 \text{ then}$$

the value of k is

(a) 2 (b) 1
(c) 0 (d) 4 **[2003]**

7. If $x = e^{y + e^{y + \ldots\infty}},$ $x > 0$, then $\frac{dy}{dx}$ is

(a) $\frac{1-x}{x}$ (b) $\frac{1}{x}$
(c) $\frac{x}{1+x}$ (d) $\frac{1+x}{x}$ **[2004]**

8. If f is a real valued differentiable function satisfying $|f(x) - f(y)| \leq (x - y)^2$, $x, y \in \mathbf{R}$ and $f(0) = 0$, then $f(1)$ equals

(a) 2 (b) 1
(c) – 1 (d) 0 **[2005]**

9. Suppose $f(x)$ is differentiable at $x = 1$ and $\lim_{h \to 0} \frac{1}{h} f(1 + h) = 5$, then $f'(1)$ equals

(a) 5 (b) 6
(c) 3 (d) 4 **[2005]**

10. The set of points where $f(x) = \frac{x}{1 + |x|}$ is differentiable is

(a) $(0, \infty)$
(b) $(-\infty, 0) \cup (0, \infty)$
(c) $(-\infty, -1) \cup (-1, \infty)$
(d) $(-\infty, \infty)$ **[2006]**

11. If $x^m y^n = (x + y)^{m+n}$, then $\frac{dy}{dx}$ is

(a) $\frac{x}{y}$ (b) $\frac{y}{x}$
(c) $\frac{x+y}{xy}$ (d) xy **[2006]**

12. Let $f : \mathbf{R} \to \mathbf{R}$ be a function defined by $f(x) = \min\{x + 1, |x| + 1\}$. Then which of the following is true.

(a) $f(x) \geq 1$ for $x \in \mathbf{R}$
(b) $f(x)$ is not differentiable at $x = 1$
(c) $f(x)$ is differentiable everywhere
(d) $f(x)$ is not differentiable at $x = 0$ **[2007]**

13. Let $f(x) = \begin{cases} (x-1)\sin\left(\frac{1}{x-1}\right), & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$

Then which one of the following is true

(a) f is neither differentiable at $x = 0$ nor at $x = 1$
(b) f is differentiable at $x = 0$ and at $x = 1$
(c) f is differentiable at $x = 0$ but not at $x = 1$
(d) f is differentiable at $x = 1$ but not at $x = 0$ **[2008]**

14. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then $y'(1)$ equals

(a) log 2 (b) – log 2
(c) – 1 (d) 1 **[2009]**

15. Let $f : (-1, 1) \to \mathbf{R}$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$.

Then $g'(0) =$

(a) 0 (b) – 2
(c) 4 (d) – 4 **[2010]**

16. $\frac{d^2x}{dy^2}$ equals

(a) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2y}{dx^2}\right)^{-1}$
(c) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

[2011]

17. If function $f(x)$ is differentiable at $x = a$ than $\lim_{x \to a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$ is

(a) $-a^2 f'(a)$ (b) $a^2 f'(a)$
(c) $2a f(a) - a^2 f'(a)$ (d) $2a f(a) + a^2 f'(a)$

[2011]

18. Consider the function , $f(x) = |x - 2| + |x - 5|$, $x \in \mathbf{R}$

Statement 1: $f'(4) = 0$

Statement 2: f is continuous is [2,5], differentiable in (2,5)and $f(2) = f(5)$ **[2012]**

19. Let f be a differentiable function such that $8f(x) + 6f\left(\frac{1}{x}\right) - x = 5$ $(x \neq 0)$ and $y = x^2 f(x)$, then $\frac{dy}{dx}$ at $x = -1$ is

    (a) $\frac{15}{14}$ (b) $-\frac{15}{14}$

    (c) $-\frac{1}{14}$ (d) $\frac{1}{14}$ **[2013, online]**

20. For $a > 0$, $t \in \left(0, \frac{\pi}{2}\right)$, let $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$. Then $1 + \left(\frac{dy}{dx}\right)^2$ equals

    (a) $\frac{x^2}{y^2}$ (b) $\frac{y^2}{x^2}$

    (c) $\frac{x^2 + y^2}{y^2}$ (d) $\frac{x^2 + y^2}{x^2}$ **[2013, online]**

21. If $y = \sec(\tan^{-1} x)$, then $\left.\frac{dy}{dx}\right|_{x=1}$ is equal to

    (a) $\frac{1}{2}$ (b) 1

    (c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$ **[2013]**

22. Let $f(x) = \frac{x^2 - x}{x^2 + 2x}$, $x \neq 0, -2$. Then $\frac{d}{dx}(f^{-1}(x))$ (wherever defined) is equal to

    (a) $\frac{-1}{(1-x)^2}$ (b) $\frac{3}{(1-x)^2}$

    (c) $\frac{1}{(1-x)^2}$ (d) $\frac{-3}{(1-x)^2}$ **[2013, online]**

23. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to

    (a) $1 + x^5$ (b) $5x^4$

    (c) $\frac{1}{1+(g(x))^5}$ (d) $1 + (g(x))^5$ **[2014]**

24. Let $f(x) = x|x|$, $g(x) = \sin x$ and $h(x) = go\,f(x)$. Then

    (a) $h(x)$ is not differentiable at $x = 0$

    (b) $h(x)$ is differentiable at $x = 0$, but $h'(x)$ is not continuous at $x = 0$

    (c) $h'(x)$ continuous at $x = 0$ but it is not differentiable at $x = 0$

    (d) $h'(x)$ is differentiable at $x = 0$

    [2014, online]

25. If $f(x) = x^2 - x + 5$, $x > \frac{1}{2}$, and $g(x)$ is its inverse function, then $g'(7)$ equals

    (a) $-\frac{1}{3}$ (b) $\frac{1}{13}$

    (c) $\frac{1}{3}$ (d) $-\frac{1}{13}$ **[2014, online]**

26. Let $f : \mathbf{R} \to \mathbf{R}$ be function such that $|f(x)| \leq x^2$ for all $x \in \mathbf{R}$. Than at $x = 0$, f is

    (a) continuous but not differentiable

    (b) continuous as well as differentiable

    (c) neither continuous nor differentiable

    (d) differentiable but not continuous **[2014, online]**

27. If $y = e^{nx}$, then $\left(\frac{d^2y}{dx^2}\right)\frac{d^2x}{dy^2}$ is equal to

    (a) ne^{nx} (b) ne^{-nx}

    (c) 1 (d) $-ne^{-nx}$ **[2014]**

28. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx + 2, & 3 < x \leq 5 \end{cases}$ is differentiable, then the value of $k + m$ is:

    (a) 2 (b) $\frac{16}{5}$

    (c) $\frac{10}{3}$ (d) 4 **[2015]**

29. For $x \in \mathbf{R}$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then

    (a) g is not differentiable at $x = 0$

    (b) $g'(0) = \cos(\log 2)$

    (c) $g'(0) = -\cos(\log 2)$

    (d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$ **[2016]**

30. If $f(x)$ is differentiability function in the internal $(0, \infty)$ such that $f(1) = 1$ and $\lim_{t \to x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$, then $f\left(\frac{3}{2}\right)$ is equal to

    (a) $\frac{23}{18}$ (b) $\frac{13}{6}$

    (c) $\frac{25}{9}$ (d) $\frac{31}{8}$ **[2016, online]**

31. If the function $f(x) = \begin{cases} -x, & x < 1 \\ a + \cos^{-1}(x + b) & 1 \leq x \leq 2 \end{cases}$ is differentiable at $x = 1$, then $\frac{a}{b}$ is equal to

    (a) $\frac{\pi + 2}{2}$ (b) $\frac{\pi - 2}{2}$

    (c) $\frac{-\pi - 2}{2}$ (d) $-1 - \cos^{-1}(2)$

Previous Years' B-Architecture Entrance Examination Questions

1. Let $f(x) = \begin{cases} ax & , \quad x < 2 \\ ax^2 + bx + 3, & x \geq 2 \end{cases}$

If f is differentiable for all x, then the value of (a, b) is equal to

(a) $(1, 2)$ (b) $\left(\frac{3}{2}, \frac{9}{2}\right)$

(c) $\left(\frac{3}{4}, -\frac{9}{2}\right)$ (d) $\left(\frac{3}{4}, -\frac{9}{4}\right)$ **[2007]**

2. Let $f : [0, \infty) \to [0, \infty)$ be given by

$$f(x) = \sum_{i=1}^{10} \frac{x^i}{i}.$$

If $f'(\cdot)$ denotes the derivative of $f(\cdot)$, then the value of $\lim_{x \to 1} \frac{f'(x) - 10}{x - 1}$ is

(a) 35 (b) 40

(c) 45 (d) 55 **[2008]**

3. Let $f : (-\infty, \infty) \to (-\infty, \infty)$ be given by

$$f(x) = x|x| + |x - 1| - |x - 2|^2 + |x - 3|^3$$

If A denotes the set of points where $f(\cdot)$ is not differentiable, then $A =$

(a) $\{3\}$ (b) $\{1\}$

(c) $\{0\}$ (d) $\{2\}$ **[2008]**

4. Let $f(\cdot)$ and $g(\cdot)$ be differentiable functions on $(-\infty, \infty)$ and let $f'(\cdot)$ and $g'(\cdot)$ denote derivatives of $f(\cdot)$ and $g(\cdot)$ respectively. If $f(0) = \frac{1}{2}$, $g(0) = \frac{1}{3}$, $f'(0) = 1$ and $g'(0) = 2$, then the value of $\lim_{x \to 0} \frac{2f(2x^2 + 3x) - 1}{3g(x) - 1}$ is

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{3}$ (d) $\frac{1}{4}$ **[2008]**

5. Let $f : (-\infty, \infty) \to (-\infty, \infty)$ be defined by

$$f(x) = \begin{cases} x^3, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

and let $f'(\cdot)$ denote the derivative of $f(\cdot)$

Statement-1: $\lim_{x \to 0} f'(x) = f'(0)$

Statement-2: $f'(x)$ exists, for each $x \in (-\infty, \infty)$ and $f'(x)$ is continuous at $x = 0$ **[2008]**

6. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be functions defined by $f(x) = \text{sgn}(\sin x)$, $g(x) = \sin(\text{sgn}\, x)$ where

$$\text{sgn}\, \alpha = \begin{cases} 1 & \text{if } \alpha > 0 \\ -1 & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0 \end{cases}$$

If $A = f'(\pi)$ and $B = g'(\pi)$, then

(a) A does not exist and $B = 0$

(c) $A = 0$ and $B = 0$

(c) both A and B do not exist

(d) $A = 0$ and B does not exist **[2010]**

7. Let $f(x) = 2\tan^{-1}\left(\frac{1+x}{1-x}\right) + \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \in \mathbf{R}$, $x \neq 1$

Statement-1: $f''(x) = 0$

Statement-2: The range of $f(x)$ is $\{\pi\}$ **[2010]**

8. If $f(x) = \begin{cases} 1 - x^2, & x \leq -1 \\ 2x + 2, & x > -1 \end{cases}$ then the derivative of $f(x)$ at $x = -1$ is

(a) 2 (b) 0

(c) $\frac{1}{2}$ (d) 3 **[2011]**

9. Amongst the following functions, a function that is differentiable at $x = 0$ is

(a) $\cos(|x|) - |x|$ (c) $\cos(|x|) + |x|$

(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$ **[2012]**

10. Let f be a differentiable function such that $8f(x) + 6f\left(\frac{1}{x}\right) - x = 5$ $(x \neq 0)$ and $y = x^2 f(x)$ then $\frac{dy}{dx}$ at $x = -1$ is

(a) $\frac{15}{14}$ (c) $-\frac{15}{14}$

(c) $-\frac{1}{14}$ (d) $\frac{1}{14}$ **[2013]**

11. If $f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > 2 \\ a + bx^2 & \text{if } |x| \leq 2 \end{cases}$

then $f(x)$ is differentiable at $x = -2$ for

(a) $a = \frac{3}{4}$ and $b = -\frac{1}{16}$

(b) $a = -\frac{1}{4}$ and $b = \frac{1}{16}$

(c) $a = \frac{1}{4}$ and $b = -\frac{1}{16}$

(d) $a = \frac{3}{4}$ and $b = \frac{1}{16}$ **[2014]**

12. Let $y(x) = (1 + x)(1 + x^2)(1 + x^4) \ldots(1 + x^{32})$. Then $\frac{dy}{dx}$ at $x = \frac{1}{2}$ is

(a) $1-65\left(\frac{1}{2}\right)^{62}$ (b) $1-63\left(\frac{1}{2}\right)^{64}$

(c) $4-65\left(\frac{1}{2}\right)^{62}$ (d) $4-63\left(\frac{1}{2}\right)^{64}$ **[2015]**

13. If $y(x) = \begin{vmatrix} \sin x & \cos x & \sin x + \cos x + 1 \\ 23 & 17 & 13 \\ 1 & 1 & 1 \end{vmatrix}$, $x \in \mathbf{R}$, then $\frac{d^2y}{dx^2} + y$ is equal to

(a) 4 (b) –10
(c) 0 (d) 6 **[2016]**

Answers

Concept-based

1. (b) **2.** (c) **3.** (d) **4.** (c)
5. (d) **6.** (d) **7.** (a) **8.** (d)
9. (c) **10.** (a)

Level 1

11. (a) **12.** (a) **13.** (b) **14.** (a)
15. (a) **16.** (c) **17.** (a) **18.** (d)
19. (b) **20.** (d) **21.** (c) **22.** (d)
23. (b) **24.** (a) **25.** (b) **26.** (b)
27. (a) **28.** (a) **29.** (a) **30.** (d)
31. (b) **32.** (c) **33.** (b) **34.** (d)
35. (c) **36.** (a) **37.** (c) **38.** (d)
39. (c) **40.** (b) **41.** (a) **42.** (b)
43. (d) **44.** (b) **45.** (c) **46.** (b)
47. (c) **48.** (b) **49.** (a) **50.** (a)
51. (b) **52.** (d) **53.** (d) **54.** (c)
55. (c) **56.** (a) **57.** (d) **58.** (a)
59. (b) **60.** (a) **61.** (a) **62.** (c)

Level 2

63. (c) **64.** (a) **65.** (d) **66.** (b)
67. (d) **68.** (c) **69.** (a) **70.** (b)
71. (c) **72.** (d) **73.** (d) **74.** (b)
75. (c) **76.** (a) **77.** (c) **78.** (b)
79. (c) **80.** (c) **81.** (b) **82.** (d)
83. (a) **84.** (d) **85.** (d) **86.** (c)
87. (a)

Previous Years' AIEEE/JEE Main Questions

1. (d) **2.** (a) **3.** (a) **4.** (b)
5. (a) **6.** (d) **7.** (a) **8.** (d)
9. (a) **10.** (d) **11.** (b) **12.** (c)
13. (a) **14.** (d) **15.** (d) **16.** (a)
17. (c) **18.** (b) **19.** (c) **20.** (d)
21. (d) **22.** (b) **23.** (d) **24.** (c)
25. (c) **26.** (b) **27.** (d) **28.** (a)
29. (b) **30.** (c) **31.** (a)

Previous Years' B-Architecture Entrance Examination Questions

1. (d) **2.** (c) **3.** (b) **4.** (a)
5. (a) **6.** (a) **7.** (a) **8.** (a)
9. (d) **10.** (d) **11.** (a) **12.** (c)
13. (d)

Hints and Solutions

Concept-based

1. $f(t) = \frac{1}{t} - \frac{5}{t^2} - \frac{1}{t^3} \Rightarrow f'(t) = -\frac{1}{t^2} + \frac{10}{t^3} + \frac{3}{t^4}$

So $f'\left(\frac{1}{a}\right) = -a^2 + 10a^3 + 3a^4$

2. $y = (2x - 1)^{-1/3} + 5\,(x^2 + 2)^{-3/4}$

$$y'(x) = -\frac{2}{3}\frac{1}{(2x-1)^{4/3}} - \frac{15x}{2(x^2+2)^{7/4}}$$

$$\Rightarrow \quad y'(0) = -\frac{2}{3}$$

3. $f'(1+) = \lim_{h\to 0+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0+} \frac{\log(1+h)-\log 1}{h}$

$$= \lim_{h\to 0+} \frac{1}{h}\log(1+h) = \lim_{h\to 0+} \log(1+h)^{1/h}$$

$$f'(1-) = \lim_{h\to 0-} \frac{f(1+h)}{h} = \lim_{h\to 0-} \frac{\log(1+h)}{h}$$

$$= \lim_{h\to 0-} \log(1+h)^{-1/h(-1)} \qquad \begin{pmatrix} h \to 0-, \\ 1/h \to -\infty \end{pmatrix}$$

$$= \log e^{-1} = -1$$

4. Since $\sin^{-1} x$ is not differentiable at $x = \pm 1$, so f is not differentiable at all x for which $\left|\frac{2x}{1+x^2}\right| = 1$

$\Rightarrow \quad 2|x| = 1 + x^2 \Rightarrow (|x| - 1)^2 = 0$

$\Rightarrow \quad |x| = 1$

Hence f is not differentiable at $x = 1, -1$.

5. The domain of definition of the function is $-1 \le x \le 1$

$$y' = \frac{1}{2\sqrt{1-\sqrt{1-x^2}}} \cdot \frac{-1}{2\sqrt{1-x^2}}(-2x) \text{ at } x \ne 0, \pm 1$$

As $x \to 1-$ or $x \to -1+$, we have $y' \to \infty$. At $x = 0$,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\sqrt{1-\sqrt{1-x^2}}}{x}$$

Since $(1 - x^2)^{1/2} - 1 = -\frac{1}{2}x^2 + 0(x^4)$, so

$$\lim_{x\to 0} \frac{\sqrt{1-\sqrt{1-x^2}}}{x} = \lim_{x\to 0} \frac{\sqrt{\frac{1}{2}x^2}}{x} = \begin{cases} \frac{1}{\sqrt{2}} & \text{as } x \to 0+ \\ -\frac{1}{\sqrt{2}} & \text{as } x \to 0- \end{cases}$$

6. $$y'(x) = \frac{1}{\log^2 \log_3(\log_5 x)} \frac{d}{dx}(\log_3(\log_5 x))$$

$$= \frac{1}{\log_2 \log_3(\log_5 x)} \frac{1}{\log^3 \log_5 x} \frac{d}{dx}(\log_5 x)$$

$$= \frac{1}{x(\log 2 \log 3 \log 5)\log_3(\log_5 x)\log_5 x}$$

$$y'(125) = y'(5^3) = \frac{1}{125(\log 2 \log 3 \log 5)3}$$

$$= \frac{1}{375 \log 2 \log 3 \log 5}$$

7. $x = 1 + \frac{1}{t} \Rightarrow \frac{dx}{dt} = -\frac{1}{t^2}$ and $\frac{dy}{dt} = \frac{1}{t^2}$

So $\frac{dy}{dx} = -1$ for all t. Hence $\frac{d^2y}{dx^2} = 0$

8. $\frac{dx}{dt} = -3 \sin t$, $\frac{dy}{dt} = 4 \cos t$ so that

$\frac{dy}{dx} = -\frac{4}{3} \cot t$. For $x = \frac{3}{\sqrt{2}}$,

we have $\cos t = \frac{1}{\sqrt{2}}$ and $y = 2\sqrt{2}$, $\sin t = \frac{1}{\sqrt{2}}$.

Hence $\cot t = 1$. So, $\frac{dy}{dx} = -\frac{4}{3}$.

9. $4x^3 + 4y^3 \frac{dy}{dx} = 2xy^2 + 2yx^2 \frac{dy}{dx}$

$\Rightarrow \quad (4y^3 - 2yx^2)\frac{dy}{dx} = 2xy^2 - 4x^3$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x(y^2 - 2x^2)}{2y(2y^2 - x^2)} = \frac{x}{y} \cdot \frac{y^2 - 2x^2}{2y^2 - x^2}$$

10. $f(x) = |(|x| - 1)(|x| - 3)|$. Hence f is not differentiable at $|x| = 1$, $|x| = 3$ i.e. at $x = 1, -1, 3, -3$

Level 1

11. f being represented by polynomials is differentiable everywhere except possibly $x = 1$

$$f'(1+) = \lim_{h\to 0+} \frac{3(1+h)^3 - (1+h)^2 + (1+h) - 3}{h}$$

$$= \lim_{h\to 0} \frac{3[h^3 + 3h^2 + 3h] - (h^2 + 2h) + h}{h} = 8$$

$$f'(1-) = \lim_{h\to 0} \frac{(1+h)^4 + (1+h)^2 - (1+h) + 2 - 3}{h}$$

$$= \lim_{h\to 0} \frac{[h^4 + 4h^3 + 6h^2 + 4h] + (h^2 + 2h) - h}{h} = 5$$

$\lim_{x\to 1+} f(x) = 3 = \lim_{x\to 1} f(x) = f(1)$.

So f is continuous everywhere.

12. f is continuous, $f'(0+) = \lim_{x\to 0+} \sin\frac{1}{x}$ which does not exist. Similarly $f'(0-)$ does not exist.

13. $y = \frac{2}{x} \log(x - 2)$, $y'(x)$

$= 2\left[\frac{1}{x(x-2)} - \frac{1}{x^2}\log(x-2)\right]$, $y'(3) = \frac{2}{3}$.

14. f is continuous, since $\lim_{x\to 0} f(x) = 0 = f(0)$,

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\sin x^2}{x^2} = 1.$$

15. Putting $x = \tan \theta$, $\frac{\sqrt{1 + x^2 - 1}}{x} = \frac{\sec\theta - 1}{\tan\theta}$

$$= \frac{1 - \cos\theta}{\sin\theta} = \tan\frac{\theta}{2}.$$

Thus $x = \tan^{-1}\frac{\sqrt{1+x^2} - 1}{x} = \frac{\theta}{2} = \frac{1}{2}\tan^{-1} x$.

Putting $x = \sin \theta$, we obtain $v = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$
$= 2 \sin^{-1} x$

$$\frac{du}{dv} = \frac{\left(\frac{1}{2}\right)\frac{1}{1+x^2}}{\frac{2}{\sqrt{1-x^2}}} \quad \Rightarrow \quad \left.\frac{du}{dv}\right|_{x=0} = \frac{1}{4}.$$

16. $n(a + b)^{n-1} = n(a^n - 1 + b^n - 1)$ is valid if $n = 2$.

17. $f(x) = \begin{cases} x^3, & x \geq 0 \\ x^3, & x < 0 \end{cases}$, $f'(0+) = 0 = f'(0-)$

18. $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ e^{x}, & x < 0 \end{cases}$, $f'(0+) = -1, f'(0-) = 1$.

19. $\frac{dy}{dx} = \frac{2t}{1/t} = 2t^2$, $y''(x) = \frac{d}{dt}(2t^2)\frac{dt}{dx} = 4t.\ t$

So $y''(1) = 4$.

20. For continuously differentiable function, apply intermediate value theorem to f' for the inequality

$$f'(0) \leq \frac{1}{2}\,(f'(0) + f'(1)) \leq f'(1) \text{ if } f'(0) \leq f'(1),$$

$$f'(1) \leq \frac{1}{2}\,(f'(0) + f'(1)) \leq f'(0) \text{ if } f'(1) \leq f'(0).$$

21. $f(x) = \begin{cases} -3x & x < -2 \\ -x + 4 & -2 \leq x < 0 \\ x + 4 & 0 \leq x \leq 2 \\ 3x & x > 2 \end{cases}$

$f'(-2+) = -1, f'(2-) = 1, f'(0+) = 1, f'(2+) = 3$.

22. $f'(x) = \begin{cases} 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$f''(x) = \begin{cases} 12x^2 \sin\left(\frac{1}{x}\right) - 4x\sin\frac{1}{x} + \sin\frac{1}{x} - 2x\cos\frac{1}{x}, & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Since $f''(0) = \lim\limits_{x \to 0} \frac{4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x}}{x} = 0$.

Clearly $\lim\limits_{x \to 0} f''(x)$ doesn't exist.

23. $y \log x = x - y \quad \Rightarrow \quad \frac{y}{x} + (\log x)\frac{dy}{dx} = 1 - \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x(1 + \log x)} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=1} = 1 - y(1) = 0$.

24. Let $u = f^{-1}(x) \Rightarrow f(u) = x \Rightarrow f'(u)\frac{du}{dx} = 1$

$$\Rightarrow \quad \frac{du}{dx} = \frac{1}{\sqrt{1 - (f(u))^2}} = \frac{1}{\sqrt{1 - x^2}}.$$

25. $y = \sin^{-1} x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$

$\Rightarrow \quad y'^2(1 - x^2) = 1$

Differentiating w.r.t. x, we get

$2y'y''(1 - x^2) - 2xy'^2 = 0$

$\Rightarrow \quad y''(1 - x^2) = xy'$.

26. Let $u = f^{-1}(u) \quad \Rightarrow \quad f(u) = x \quad \Rightarrow \quad f'(u)\frac{du}{dx} = 1$

$$\Rightarrow \quad f''(u)\left(\frac{du}{dx}\right)^2 + f'(u)\frac{d^2u}{dx^2} = 0$$

$$\Rightarrow \quad f'(u)\left(\frac{1}{f(u)}\right)^2 + f'(u)\frac{d^2u}{dx^2} = 0$$

$$\Rightarrow \quad \frac{d^2u}{dx^2} = -\frac{1}{(f(u))^2} = -\frac{1}{x^2}.$$

27. $y'(x) = \frac{1}{\sqrt{1 - x^{-2}}}\, x^{-2}$ so $y'(2) = \frac{1}{2\sqrt{3}}$.

28. $$\frac{dy}{dx} = -\frac{2}{\sqrt{1 - 4t^2}} \Bigg/ \frac{2t}{\sqrt{1 - (t^2 - 1)^2}}$$

$$= -\frac{\sqrt{2t^2 - t^4}}{\sqrt{1 - 4t^2}\,.t} = -\frac{\sqrt{2 - t^2}}{\sqrt{1 - 4t^2}}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{t=0} = -\sqrt{2}$$

29. $$\frac{1}{1 + y^2}\frac{dy}{dx} - \frac{dy}{dx} + 1 = 0 \quad \Rightarrow \quad \frac{dy}{dx} = 1 + \frac{1}{y^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{y^3}\frac{dy}{dx} = -\frac{2(1 + y^2)}{y^5}.$$

30. $$f'(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

$= 0 + 0 + 6x^2 = 6x^2$.

31. $$y = (\tan x \sin^2 x + \cos^2 x)\frac{1}{1 + \tan x}$$

$$= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$$

$$= 1 - \frac{1}{2}\sin 2x$$

$$\frac{dy}{dx} = -\cos 2x.$$

32. $z'(t) = \frac{1}{2}\frac{3t^2}{(t^3 + 1)^{1/2}}$, $z'(2) = \frac{1}{2}.\frac{3.4}{3} = 2$.

33. $y = \cos^{-1}(\cos(\theta + x))$, $4 = r\cos\theta$, $5 = r\sin\theta$

$= \theta + x$ so $\frac{dy}{dx} = 1$.

34. $\frac{dy}{dx} = \frac{1}{\cos\left(\tan^{-1}\frac{e^x - e^{-x}}{2}\right)}\left(-\sin\left(\tan^{-1}\left(\frac{e^x - e^{-x}}{2}\right)\right)\right)\times$

$\frac{d}{dx}\left(\tan^{-1}\left(\frac{e^x - e^{-x}}{2}\right)\right)$

$\left.\frac{dy}{dx}\right|_{x=0} = 0$.

35. $y = -\log(1 + x)$ so $x\frac{dy}{dx} + 1 = -\frac{x}{1+x} + 1 = \frac{1}{1+x}$

$= e^y$.

36. $(1 - x^2)y^2 = (\sin^{-1} x)^2 \Rightarrow (1 - x^2)2yy' + (-2x)$

$y^2 = \frac{2\sin^{-1} x}{\sqrt{1-x^2}} = 2y$

$\Rightarrow (1 - x^2)y' - xy = 1$

37. $\frac{dy}{dx} = \frac{\sin^2 y}{\sin a} = \frac{\sin a \sin^2 y}{-\sin^2 y + \sin^2 a + \sin^2 y}$

$= \frac{\sin a \sin^2 y}{\sin(a+y)\sin(a-y) + \sin^2 y}$

$= \frac{\sin a \sin^2 y}{\sin(a+y)[\sin(a+y) - 2\sin y\cos a + \sin^2 y]}$

$= \frac{\sin a}{1 + \frac{\sin^2(a+y)}{\sin^2 y} - 2\cos a\frac{\sin(a+y)}{\sin y}}$

$= \frac{\sin a}{1 + x^2 - 2x\cos a}$.

38. $f(x) = \cot^{-1}\frac{3x - x^3}{1 - 3x^2} = \frac{\pi}{2} + 3\tan^{-1} x$

$f'(x) = \frac{3}{1+x^2}$ and $g(x) = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$= 2\tan^{-1} x$

so $g'(x) = \frac{2}{1+x^2}$. The required limit is equal to $\frac{f'(t)}{g'(t)}$

$= \frac{3}{2}$.

39. Verify the answers or consider $g(x) = \frac{f(x)}{\sin x}$ show that $g''(x) = -2\cot x\, g'(x) \Rightarrow g'(x) = c\,\text{cosec}^2 x$. Thus $\sin x f'(x) - \cos x f(x) = C$ but $f(0) = 0$, $f'(0) = 1$ so const $= 0$. $g(x) =$ const $\Rightarrow f(x) = C\sin x$ but $f'(0) = 1$ so $C = 1$.

40. $y'(x) = -\frac{\sin x}{\sqrt{1-\cos^2 x}} = -\frac{\sin x}{|\sin x|} = -\text{sgn}(\sin x)$.

41. $f(x) = \cos x^2$ as $[x] = 0$ for $0 < x < 1$ so $f'(x) = -2x\sin x^2$ hence $f'\left(\frac{\sqrt{\pi}}{2}\right) = -\frac{\sqrt{\pi}}{\sqrt{2}}$

42. $\frac{dy}{dx} = \frac{2b\sin t\cos t}{-2a\sin 2t} = -\frac{b}{a} \Rightarrow \frac{d^2y}{dx^2} = 0$.

43. $f(x) = \frac{\pi}{2} - \cot^{-1}\left(\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)\right) = \frac{\pi}{4} + \frac{x}{2}$

Thus $f'(x) = \frac{1}{2}$.

44. $2^x\log 2 + 2^y\log 2\frac{dy}{dx} = 2^{x+y}\log 2\left(1 + \frac{dy}{dx}\right)$

Putting $x = y = 1$, we obtain

$(2 - 2^2)\frac{dy}{dx} = 2^2 - 2$

$\Rightarrow \frac{dy}{dx} = -1$

45. $2^s = \sin t \Rightarrow 2^s\log 2\frac{ds}{dt} = \cos t$

$\Rightarrow \frac{ds}{dt} = \frac{\cos t}{\sin t}\frac{1}{\log 2} = \frac{\cot t}{\log 2}$.

46. $x^2 + y^2 = 1 \Rightarrow 2x + 2yy' = 0 \Rightarrow x + yy' = 0$

Differentiating again $1 + y'^2 + yy'' = 0$

47. $f'(x) = 2x\cot x^2$, $f''(x) = 2[\cot x^2 - 2x^2\,\text{cosec}^2\,x^2]$

$f''\left(\sqrt{\pi}/2\right) = 2\left[\cot\pi/4 - \frac{2\pi}{4}\text{cosec}^2\,\pi/4\right]$

$= 2(1 - \pi)$.

48. $f(g(x) = x \Rightarrow f'(g(x))\,g'(x) = 1$

$\Rightarrow g'(x) = \frac{1}{f'(g(x))} = 1 + (g(x))^5$

$\Rightarrow g''(x) = 5(g(x))^4 g'(x) = 5(g(x))^4(1 + (g(x))^5)$

49. $u = f(\tan x)$, $v = g(\sec x)$

$\frac{du}{dx} = f'(\tan x)\sec^2 x$, $\frac{du}{dx} = g'(\sec x)\sec x\tan x$

$\therefore \frac{du}{dv} = \frac{du/dx}{dv/dx}$

$\Rightarrow \left.\frac{du}{dv}\right]_{x=\pi/4} = \frac{f'(1)(2)}{g'\left(\sqrt{2}\right)\left(\sqrt{2}\right)} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$

50. $y = (\log_{\cos x} \sin x)\,(\log_{nx} \cos x) + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$= \left(\frac{\log \sin x}{\log \cos x}\right)\left(\frac{\log \cos x}{\log nx}\right) + 2\tan^{-1} x$$

$$= \frac{\log \sin x}{\log(nx)} + 2\tan^{-1} x$$

$$\frac{dy}{dx} = \frac{\log(nx)(\cot x) - \frac{1}{x}\log \sin x}{(\log nx)^2} + \frac{2}{1+x^2}$$

$$\left.\frac{dy}{dx}\right]_{x=\pi/2} = 0 + \frac{2}{1+\pi^2/4} = \frac{8}{4+\pi^2}$$

51. Put $x = \sin\theta$ and $5/13 = \cos\alpha$, so that

$$y = \sin^{-1}[\sin(\theta+\alpha)] = \theta + \alpha$$

$$y = \sin^{-1} x + \cos^{-1}(5/13)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

52. $g'(x) = 2[f(2\,f(x) + 2)]\;[f'(2\,f(x) + 2)\;(2\,f'(x))$

$$g'(0) = 4[f(2\,f(0) + 2)]\;[f'(2\,f(0) + 2)\;(2\,f'(0))$$

$$= 4f(0)\,f'(0)\,f'(0) = -4.$$

53. Put $x = 1$ in $x^{2x} - 2x^x \cot y - 1 = 0$, we have $1 - 2\cot y - 1 = 0 \Rightarrow \cot y = 0 \Rightarrow y = \pi/2$.

Differenting the given expression,

$2x \,.\, x^{2x-1} + 2\log x \,.\, x^{2x} - 2[x.\, x^{x-1} + (\log x)\, x^x]\cot y + 2x^x(\operatorname{cosec}^2 y)\, y'(x) = 0$

Putting $x = 1$, $y = \pi/2$, we have $y'(1) = -1$

54. For $x \neq 1$, $f'(x) = \sin\frac{1}{(x-1)} - \frac{(x-1)}{(x-1)^2}\cos\frac{1}{x-1}$

∴ f is differentiable at $x = 0$

$f'(1)$ doesn't exist.

55. $$\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x+1} \times \frac{1}{2}\frac{3-3x}{x^2-x+1}\right] + \frac{1}{\sqrt{3}}\,\frac{2/\sqrt{3}}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2} = \frac{1}{x^3+1}$$

56. $y = -\log(1+x) \;\Rightarrow\; y' = -\frac{1}{1+x}$

$\Rightarrow\; (1+x)y' = -1$

$xy' = -1 - y' = -1 + \frac{1}{1+x} = -1 + e^y$. Since $y(1) = -\log 2$, so putting $x = 1$ in $xy' + 1 = e^y$, we get $y'(1) + 1 = \frac{1}{2} \;\Rightarrow\; y'(1) = -\frac{1}{2}$.

57. $$\frac{dy}{dx} = -\frac{1 + y\sin(xy)}{x\sin(xy)}$$

58. $\frac{dy}{dx} = \frac{2t - 3t^2}{1 - 4t^3}$ but $(x, y) = (0, 0)$ if and only if $t = 0$ or $t = 1$

59. $y'(x) = \cos(3\sin^{-1} x)\,\frac{3}{\sqrt{1-x^2}} \;\Rightarrow\; y'(0) = 3$

$(1 - x^2)\,y'^2 = 9\cos^2(3\sin^{-1} x) = 9(1 - y^2)$

$2(1 - x^2)\,y''y' - 2xy'^2 = -18yy'$

$\Rightarrow\; (1 - x^2)y'' - xy' + 9y = 0$

Putting $x = 0$, $y''(0) = -9y(0) = 0$.

60. $$\left|\frac{f(x) - f(y)}{x - y}\right| \le A|x - y|^{\alpha - 1}$$

Since $\alpha - 1 > 0$ so $\lim_{y\to x}|x - y|^{\alpha-1} = 0$.

Therefore $\lim_{y\to x}\frac{f(x) - f(y)}{x - y} = 0$

$\Rightarrow f'(x) = 0$ for $x \in \mathbf{R}$.

$\Rightarrow f$ is a constant function

61. For $\pi/2 < x < \pi$, $\tan x < 0$, so

$$y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$$

$$= \tan^{-1}(\cot x) + \tan^{-1}(\cot x) + \pi$$

$$= 2\tan^{-1}(\cot x) + \pi$$

$$= -\cos^{-1}\left(\frac{1 - \cot^2 x}{1 + \cot^2 x}\right) + \pi$$

$$= -\cos^{-1}(\sin^2 x - \cos^2 x) + \pi$$

$$= -(\pi - \cos^{-1}(\cos 2x)) + \pi$$

$$= 2x$$

Hence $\frac{d^2y}{dx^2} = 0$ and y is a linear function of x.

62. Differentiating $f'(x) = f(1 - x)$, we get

$$f''(x) = -f'(1 - x) = -f(1 - (1 - x)) = -f(x)$$

$\Rightarrow\quad f(x) = A\cos x + B\sin x$

$f(\pi/2) = 1 \;\Rightarrow\; B = 1$

Also $f'(\pi/2) = f\left(1 - \frac{\pi}{2}\right) = A\cos\left(\frac{\pi}{2} - 1\right) + \sin\left(\frac{\pi}{2} - 1\right)$

$= A\sin 1 + \cos 1$

But $f'(x) = -A\sin x + B\cos x$, so $f'\left(\frac{\pi}{2}\right) = -A$

$\Rightarrow\; A(1 + \sin 1) + \cos 1 = 0$ i.e. $A = -\frac{\cos 1}{1 + \sin 1}$

Hence $f(x) = -\frac{\cos 1}{1 + \sin 1}\cos x + \sin x$.

Level 2

63. For f to be continuous $f(\pi/6) = \lim\limits_{x \to \pi/6+} f(x)$

$$\sin \frac{\pi}{3} = a\,\frac{\pi}{6} + b \text{ i.e. } \frac{\sqrt{3}}{2} = a\,\frac{\pi}{6} + b$$

$$f'(\pi/6+) = \lim_{h \to 0+} \frac{f(\pi/6+h) - f(\pi/6)}{h}$$

$$= \lim_{h \to 0+} \frac{a(\pi/6+h) + b - \sin \pi/3}{h}$$

$$= \lim_{h \to 0+} \frac{a(\pi/6+h) + b - (a\,\pi/6 + b)}{h}$$

$= a$.

$$f'\,(\pi/6-) = \lim_{h \to 0-} \frac{f(\pi/6+h) - f\,(\pi/6)}{h}$$

$$= \lim_{h \to 0-} \frac{\sin 2(\pi/6+h) - \sin \pi/3}{h}$$

$$= 2 \cos 2\,\frac{\pi}{6} = 2\frac{1}{2} = 1,$$

so $a = 1$, $b = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$. For these values of a and b,

$$f'(x) = \begin{cases} 2\cos 2x & 0 < x \le \pi/6 \\ 1 & \pi/6 < x \le 1 \end{cases} \text{ which is continuous.}$$

64. $f(1) = 1 = \lim\limits_{x \to 1-} f(x) = \lim\limits_{x \to 1-} ax^2 + b = a + b$

$$f'(1\,+) = \lim_{h \to 0+} \frac{f\,(1+h) - f(1)}{h} = \lim_{h \to 0+} \frac{(1/(1+h)) - 1}{h}$$

$$= \lim_{h \to 0+} \frac{1-(1+h)}{h\,(1+h)} = -1$$

$$f'(1-) = \lim_{h \to 0-} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0-} \frac{a(1+h)^2 + b - 1}{h}$$

$$= \lim_{h \to 0-} \frac{a + b - 1 + ah^2 + 2ah}{h}$$

$= 2a$.
so $a = -1/2$, $b = 3/2$

65. Since $[x - \pi]$ is an integer, so $\tan(\pi\,[x - \pi]) = 0$ for all x. Hence $f\,(x) = 0$ for all x, so $f'(x)$ exists for all x.

66. Since differentiability implies continuity so

$$f\,(0) = \lim_{x \to 0} f(x)$$

$$= \lim_{x \to 0} \left(1 + (-1 + \cos x)\right)^{\frac{1}{-1+\cos x} \times \frac{-2\sin^2 x/2}{2\sin x/2 \cos x/2}}$$

$$= \lim_{x \to 0} \left(1 + (-1 + \cos x)\right)^{\frac{1}{-1+\cos x} \times -\tan x/2}$$

$= e^0 = 1.$

67. $y(0) = \sin^{-1} 0 = 0$. Also

$$\sin y = \sin \alpha\, \frac{\sin x}{1 - \cos\alpha \sin x}$$

$$\cos y \frac{dy}{dx}$$

$$= \sin \alpha\, \frac{(1 - \cos\alpha \sin x)\cos x + \sin^2 x \cos\alpha}{(1 - \cos\alpha \sin x)^2}$$

At $x = 0$, $\left.\dfrac{dy}{dx}\right|_{x=0} = \sin \alpha$.

68. f is defined on $[0, \infty)$.

$$\lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{\sqrt{h+1} - 1}{\sqrt{h}\;h}$$

$$= \lim_{h \to 0+} \frac{1}{\sqrt{h}\,(\sqrt{h+1} + 1)}$$

which does not exist. f is clearly differentiable on $(0, \infty)$.

69. $\dfrac{dx}{dt} = \dfrac{1}{\sqrt{1-t^2}}$, $\dfrac{dy}{dt} = -\dfrac{2t}{1-t^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{-2t}{\sqrt{1-t^2}}$

$$\frac{d^2y}{dx^2} = -2\frac{d}{dt}\left(\frac{t}{\sqrt{1-t^2}}\right)\frac{dt}{dx}$$

$$= -2\,\frac{\left(\sqrt{1-t^2} + \dfrac{t^2}{\sqrt{1-t^2}}\right) \times \sqrt{1-t^2}}{(1-t^2)}$$

$$= -2\,\frac{1}{1-t^2}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=1/2} = -2 \cdot \frac{4}{3} = -\frac{8}{3}.$$

70. $2yy' + 2x = 0 \Rightarrow y' = -x/y$

$y'^2 + yy'' + 1 = 0 \Rightarrow yy'' = -(1 + x^2/y^2)$

so $|y''| = \dfrac{x^2 + y^2}{y^3}$

$$\frac{|\,y''\,|}{\sqrt{(1+y'^2)^3}} = \frac{x^2+y^2}{y^3\sqrt{(1+x^2/y^2)^3}} = \frac{x^2+y^2}{(x^2+y^2)^{3/2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{R} = k.$$

71. For (*a*), let $f(x) = |x|$ then $||x| - |y|| \le |x - y|$ for all $x, y \in \mathbf{R}$ so $|f(x) - f(y)| \le |x - y|$ but f is not differentiable at $x = 0$

For (*b*), let $h(x) = \begin{cases} x^{1/2}, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$

Since for $x \in [0, 1]$ $|x^{1/2} - y^{1/2}| \le |x^{1/2} + y^{1/2}|$ so

$|x^{1/2} - y^{1/2}|^2 \le |x - y|$. Thus $|h(x) - h(y)| \le |x - y|^{1/2}$ but h is not differentiable.

If f satisfies $|f(x) - f(y)| \le k\,|x - y|^2$ then

$$\left|\frac{f(x) - f(y)}{x - y}\right| \le |x - y|$$

$$\Rightarrow \left|\lim_{y \to x} \frac{f(x) - f(y)}{x - y}\right| \le \lim_{y \to x} |x - y| = 0$$

$\Rightarrow f$ *is* differentiable and $f'(x) = 0$

For (*d*) Let $f(x) = |x|$, $f^2(x) = x^2$ is differentiable

on $\mathbf{R}$ but f is not differentiable.

72. $\left|\dfrac{f(x) - f(y)}{x - y}\right| \le |x - y|^2 \Rightarrow f'(x) = 0$

73. Since $\left|x \sin \dfrac{\pi}{x}\right| \le |x|$ so f is continuous and

bounded on [0, 1]

$\lim\limits_{x \to 0} \dfrac{x \sin \pi/x - 0}{x - 0} = \lim\limits_{x \to 0} \sin \pi/x$ which does not exist so f is not differentiable.

74. $$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{|x|(3e^{1/|x|} + 4)}{2 - e^{1/|x|}}$$

$$= \lim_{x \to 0+} \frac{x\,(3e^{1/|x|} + 4)}{2 - e^{1/x}} = \lim_{x \to 0+} \frac{x\,(3 + 4.e^{-1/x})}{2\,.\,e^{-1/x} - 1}$$

$$= 0.\ \frac{3}{-1} = 0$$

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{-x\,(3e^{-1/x} + 4)}{2 - e^{-1/x}}$$

$$= -\lim_{x \to 0-} \frac{x\,(3 + 4e^{-1/x})}{2\,e^{-1/x} - 1} = 0$$

Thus f is continuous at $x = 0$

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{x\,(3e^{1/x} + 4)}{x\left(2 - e^{1/x}\right)}$$

$$= \lim_{x \to 0+} \frac{(3 + 4e^{-1/x})}{2\,e^{-1/x} - 1} = -3$$

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{-(3 + 4e^{1/x})}{\left(2\,e^{1/x} - 1\right)} = 3.$$

75. $\lim\limits_{x \to 1+} f(x) = k \lim\limits_{x \to 1+} (x - 1) = 0$

$\lim\limits_{x \to 1-} f(x) = \lim\limits_{x \to 1-} (x^2 - 1) = 0$

f is continuous for any value of k

$$\lim_{h \to 0+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0+} \frac{k(1 + h - 1)}{h} = k.$$

$$\lim_{h \to 0-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0-} \frac{(1 + h)^2 - 1)}{h} = 2$$

f is differentiable only when $k = 2$.

76. $\lim\limits_{x \to a-} f(x) = \lim\limits_{x \to a-} \left(1/\sqrt{2}\right)\left(x + a/\sqrt{2}\right)$

$$= \frac{1}{\sqrt{2}}\left(a + a/\sqrt{2}\right)$$

$$= \frac{\sqrt{2} + 1}{2}\,a$$

$$\lim_{x \to a+} f(x) = \lim_{x \to a+} \left(1/\sqrt{2}\right)\left(\frac{x}{2} + a\right) = \frac{3a}{2\sqrt{2}}$$

f is clearly not conttnuous at $x = 0$,

so f' is not defined at 0 and a, hence not continuous.

77. Given function is a polynomial of degree 160. Differentiation one-time reduces a polynomial by one degree. Thus to obtain polynomial of degree 50, we has to differentiate 110 times.

78. The derivative of $y = \sin^{-1}\left(\dfrac{2x}{1 + x^2}\right)$ doesn't exist

for $\left|\dfrac{2x}{1 + x^2}\right| = 1$. So $2\,|x| = x^2 + 1$. For $x \ge 0$,

$(x - 1)^2 = 0 \Rightarrow x = 1$, if $x < 0$, then $(x + 1)^2 = 0$

so $x = -1$.

79. $\lim\limits_{x \to 0+} f(x) = \lim\limits_{x \to 0+} x^\alpha = 0$, f is continuous at $x = 0$

$$\lim_{h \to 0+} \frac{f(h) - f(0)}{h - 0} = \lim_{h \to 0+} \frac{h^\alpha}{h} = \lim_{h \to 0+} \frac{1}{h^{1-\alpha}}$$

which does not exist.

80. $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = 0$ when x approach through

rational or irrational so f is differentiable at

$x = 0$. If $a \neq 0$ then

$\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ does not exist. Morever f is continuous at $x = 0$ only.

81. $f(x) = x^{k}$ is $(n - 1)$ times differentiable at $x = 0$ if $k - (n - 1) \geq 0$ and not n-times differentiable if $k - n < 0$ so $n - 1 \leq k < n$, $k = \dfrac{3n-2}{3}$ satisfy this inequality.

82. We have

$$f(x) = \begin{cases} (-\tfrac{1}{2})(x+1) & \text{if} \quad x<-1 \\ \tan^{-1}x & \text{if} \quad -\leq x \leq 1 \\ (\tfrac{1}{2})(x-1) & \text{if} \quad x>1 \end{cases}$$

Since $f(-1) = \dfrac{\pi}{4}$, $f(1) = \dfrac{\pi}{4}$ and $\lim\limits_{x\to 1-} f(x) = -1$, $\lim\limits_{x\to 1+} f(x) = 0$ so f is not continuous at $x = -1, 1$ and hence not differentiable at $-1, 1$. Also

$$f'(x) = \begin{cases} -\tfrac{1}{2} & \text{if} \quad x<-1 \\ \dfrac{1}{1+x^2} & \text{if} \quad -1<x<1 \\ \tfrac{1}{2} & \text{if} \quad x>1 \end{cases}$$

Thus the domain of f' is $\mathbf{R} \sim \{-1, 1\}$

83. Let $y = \sin^{-1}\left(\dfrac{1-x^{2n}}{1+x^{2n}}\right)$ and $u = x^{2n}$, then

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$$

$$= \frac{1}{\sqrt{1-\left(\dfrac{1-x^{2n}}{1+x^{2n}}\right)^2}}\,\frac{d}{dx}\left(\frac{1-x^{2n}}{1+x^{2n}}\right)\frac{1}{2\,n\,x^{2n-1}}$$

$$= \frac{1+x^{2n}}{\sqrt{4x^{2n}}}\,\frac{-(1+x^{2n})2nx^{2n-1}-(1-x^{2n})2n\,x^{2n-1}}{(1+x^{2n})^2}$$

$$\times\frac{1}{2n\,x^{2n-1}}$$

$$= \frac{1}{2\,x^{n}}\times\frac{-4\,n\,x^{2n-1}}{(1+x^{2n})}\,\frac{1}{2n\,x^{2n-1}}$$

$$= -\,\frac{1}{x^{n}\,(1+x^{2n})} = -\frac{1}{x^{n}+x^{3n}}.$$

84. Put $x^3 = \cos\theta$, $y^3 = \cos\phi$ so that given equation reduces to $\sin\theta + \sin\phi = a(\cos\theta - \cos\phi)$

$$\Rightarrow 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = -\,a\;2\sin\frac{\theta-\phi}{2}\sin\frac{\theta+\phi}{2}$$

$$\Rightarrow \cos\frac{\theta-\phi}{2} = -\text{ a }\sin\frac{\theta-\phi}{2}$$

$$\Rightarrow \theta - \phi = 2\tan^{-1}(-1/a)$$

$$\Rightarrow \frac{d\theta}{dx}-\frac{d\phi}{dx} = 0$$

$$\frac{d}{dx}(\cos^{-1}x^3) - \frac{d}{dx}(\cos^{-1}y^3) = 0$$

$$\Rightarrow -\frac{3x^2}{\sqrt{1-x^6}}+\frac{3y^2}{\sqrt{1-y^6}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$$

85. $f(g(x)) = x \Rightarrow f'(g(x))\,g'(x) = 1$

Also $f'(x) = 2x + \sec^2 x = 2x + 1 + \tan^2 x$

$= 2x + 1 + (f(x) - x^2)^2$

$f'(g(x)) = 2g(x) + 1 + (f(g(x)) - (g(x)^2)^2$

$= 2\,g(x) + 1 + (x - (g(x))^2)^2$

Thus $g'(x) = \dfrac{1}{f'(g(x))}$

$$= \frac{1}{2\,g(x)+1+(x-(g(x))^2)^2}$$

86. $[x]$ is not continuous at integral points.

The number of integers lying in $(-\pi/2, \pi/2)$ is 3.

87. $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)+f(h)+x^2h^2-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(h)}{h}+x^2\lim_{h\to 0}h = 10.$$

Previous Years' AIEEE/JEE Main Questions

1. $f''(x) - g''(x) = 0 \Rightarrow f'(x) - g'(x) =$ constant. Putting $x = 1$, we get $f'(1) - g'(1) = C \Rightarrow C = 2g'(1) - g'(1) = g'(1) = 2$.

So $f'(x) - g'(x) = 2 \Rightarrow f(x) - g(x) = 2x + C'$. Putting $x = 2$ we have $f(2) - g(2) = 4 + C' \Rightarrow 4 + C' = 3g(2) - g(2) = 2g(2) = 6 \Rightarrow C' = 6 - 4 = 2$. Hence $f(x) - g(x) = 2x + 2$. At $x = \frac{3}{2}$

$$f\left(\frac{3}{2}\right) - g\left(\frac{3}{2}\right) = 2 \cdot \frac{3}{2} + 2 = 5.$$

2. $$\lim_{x\to 1} \frac{\sqrt{f(x)} - 1}{\sqrt{x} - 1} = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} \times \frac{\sqrt{x} + 1}{\sqrt{f(x)} + 1}$$

$$= f'(1) \times \frac{2}{\sqrt{f(1)} + 1} = 2 \times \frac{2}{1+1} = 2$$

3. $$\frac{dy}{dx} = n\left(x + \sqrt{1+x^2}\right)^{n-1}\left(1 + \frac{x}{\sqrt{1+x^2}}\right) = \frac{ny}{\sqrt{1+x^2}}$$

$$(1 + x^2)\left(\frac{dy}{dx}\right)^2 = n^2y^2$$

$$\Rightarrow (1 + x^2)\ 2\frac{dy}{dx}\frac{d^2y}{dx} + 2x\left(\frac{dy}{dx}\right)^2 = 2n^2y\frac{dy}{dx}$$

$$\Rightarrow (1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = n^2y.$$

4. $f'(x) = nx^{n-1} \Rightarrow f'(1) = n$

Also $f''(x) = n(n - 1)x^{n-2} \Rightarrow f''(1) = n(n - 1)$

$\Rightarrow f^k(1) = n(n - 1) \ldots (n - k + 1)$

So $f(1) - \frac{f'(1)}{1!} + \frac{f''}{2!} \cdots + \frac{(-1)^n f^n(1)}{n!}$

$$= 1 - \frac{n}{1!} + \frac{n(n-1)}{2!} + \cdots + \frac{(-1)^n n!}{n!}$$

$= (1 - 1)^n = 0.$

5. $$f(x) = \begin{cases} x & x < 0 \\ xe^{-2/x} & x > 0 \\ 0 & x = 0 \end{cases}$$

$$\lim_{x\to 0-} f(x) = 0 \text{ and } \lim_{x\to 0+} f(x) = \lim_{x\to 0+} xe^{-2/x} = 0$$

$$\lim_{x\to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0+} \frac{xe^{-2/x}}{x} = \lim_{x\to 0+} e^{-2/x} = 0$$

$$\lim_{x\to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0-} \frac{x - 0}{x} = 1.$$

So f is not differentiable at $x = 0$ but is continuous for all x.

6. $$\lim_{x\to a} \frac{f(a)g(x) - f(a) - g(a)f(x) + g(a)}{g(x) - f(x)}$$

$$= k \lim_{x\to a} \frac{g(x) - 1 - f(x) + 1}{g(x) - f(x)}$$

$= k$, so $k = 4$

7. We have $x = e^{y + x}$

$$\Rightarrow \log x = y + x \Rightarrow \frac{1}{x} = \frac{dy}{dx} + 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x} - 1 = \frac{1-x}{x}$$

8. For $x \neq y$,

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq |x - y|$$

$$\Rightarrow \left|\frac{f(x+h) - f(x)}{h}\right| \leq |h|$$

Taking limit as $h \to 0$, we get

$|f'(x)| \leq 0$

But $|f'(x)| \geq 0$

$\therefore |f'(x)| = 0 \Rightarrow f'(x) = 0 \ \forall\ x$

Thus, $f(x)$ is a constant function.

Now, $f(0) = 0 \Rightarrow f(1) = 0$

9. As f is differentiable at $x = 1$, f is continuous at $x = 1$. Therefore,

$$f(1) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} h\left\{\frac{f(1+h)}{h}\right\}$$

$$= \left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0} \frac{f(1+h)}{h}\right) = (0)\ (5) = 0$$

Now,

$$f^1(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{f(1+h)}{h} = 5$$

10. We have

$$f(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } x < 0 \\ \dfrac{x}{1+x} & \text{if } x \geq 0 \end{cases}$$

The function f is differentiable at all points except possibly at $x = 0$.

We have

$$Lf'(0) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{1}{1-x} = 1$$

and $$Rf'(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0+} \frac{1}{1+x} = 1$$

As $Lf'(0) = Rf'(0) = 1$, we get f is differentiable at $x = 0$. Thus f is differentiable for all $x \in (-\infty, \infty)$.

11. $m \log x + n \log y = (m + n) \log (x + y)$

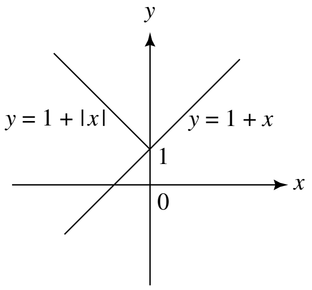
$$\Rightarrow \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{n}{y} - \frac{m+n}{x+y}\right)\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{y}$$

$$\Rightarrow \frac{nx-my}{y(x+y)}\frac{dy}{dx} = \frac{nx-my}{(x+y)x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}.$$

12. $f(x) = \min \{x + 1, |x| + 1\} = x + 1$, so is differentiable everywhere.



13. For $x \neq 1$,

$$f'(x) = \sin\left(\frac{1}{x-1}\right) - \frac{1}{x-1}\cos\left(\frac{1}{x-1}\right)$$

$\therefore$ f is differentiable at $x = 0$

$\lim\limits_{x\to 1} \dfrac{f(x)-f(0)}{x-1} = \lim\limits_{x\to 1} \sin\dfrac{1}{x-1}$ which does not exist.

$\therefore$ f is differentiable at $x = 0$ but not at $x = 1$.

14. Putting $x = 1$ in $x^{2x} - 2x^x \cot y - 1 = 0$, we get $1 - 2 \cot y - 1 = 0$

$\Rightarrow \cot y = 0 \Rightarrow y = \pi/2$.

Using the formula,

$\dfrac{d}{dx}\left[f(x)^{g(x)}\right] = g(x)\, f(x)^{g(x)-1} f'(x) + (g'(x) \log f(x))\, f(x)^{g(x)}$, we have

$2x.x^{2x-1}(1) + (2 \log x)x^{2x} - 2[x.x^{x-1}(1) + (\log x)\, x^x] \cot y + 2x^x (\operatorname{cosec}^2 y)\, y'(x) = 0$

Putting $x = 1$, $y = \pi/2$, we get

$2 + 0 - 2 (1) (0) + 2 \operatorname{cosec}^2 (\pi/2)\, y'(1) = 0$

$\Rightarrow y'(1) = -1.$

15. $g'(x) = 2[f(2f(x) + 2)]f'(2f(x) + 2)\,(2f'(x))$

$\Rightarrow g'(0) = 2[f(2f(0) + 2)]f'\,(2f(0) + 2)\,(2f'(0))$

$= 4(f(0))\, f'(0)^2 = 4(-1)(1)^2 = -4$

16. $$\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{1}{dy/dx}\right) = \frac{d}{dy}\left(\frac{1}{dy/dx}\right)\frac{dx}{dy}$$

$$= \frac{-1}{(dy/dx)^2}\left(\frac{d^2y}{dx^2}\right)\frac{dx}{dy}$$

$$= \frac{d^2y/dx^2}{(dy/dx)^3}$$

17. $$\lim_{x\to a} \frac{x^2 f(a) - a^2 f(x)}{x-a}$$

$$= \lim_{x\to a} \frac{(x^2-a^2)f(a) + a^2(f(a)-f(x))}{x-a}$$

$$= \lim_{x\to a} \left[(x+a)f(a) - a^2\frac{f(x)-f(a)}{x-a}\right]$$

$= 2a\, f\, (a) - a^2 f'(a)$

18. We have

$$f(x) = \begin{cases} (2-x)+(5-x) & \text{if } x \le 2 \\ (x-2)+(5-x) & \text{if } 2 < x \le 5 \\ (x-2)+(x-5) & \text{if } x > 5 \end{cases}$$

$$= \begin{cases} 7-2x & \text{if } x \le 2 \\ 3 & \text{if } 2 < x \le 5 \\ 2x-7 & \text{if } x > 5 \end{cases}$$

Note that $f'(4) = 0$ as f is a constant function in the interval $(2, 5)$.

Thus, the statement-1 is true.

Statement-2 is also true. But it is not the correct explanation for the statement-1.

19. $8f(x) + 6f\left(\frac{1}{x}\right) - x = 5.$ (i)

Replace x by $\frac{1}{x}$, we have $8f\left(\frac{1}{x}\right) + 6f(x) - \frac{1}{x} = 5$ (ii)

Solving (i) and (ii) for $f(x)$ and $f(1/x)$, we have

$$14f(x) = 5 - \frac{3}{x} + 4x$$

$$\Rightarrow 14f'(x) = \frac{3}{x^2} + 4$$

So $f(-1) = \frac{1}{14}\,[5 + 3 - 4] = \frac{2}{7}$

and $f'(-1) = \frac{1}{14}\,[3 + 4] = \frac{1}{2}$

Differentiating $y = x^2 f(x)$, we have $\frac{dy}{dx} = 2xf(x) + x^2 f'(x)$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x=-1} = -2f(-1) + f'(-1) = -\frac{4}{7} + \frac{1}{2} = -\frac{1}{14}.$$

20. $\frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1} t}\right)^{-\frac{1}{2}} \cdot a^{\sin^{-1} t}\,\frac{1}{\sqrt{1-t^2}}\ \log a$

$$= \frac{1}{2}\frac{\left(a^{\sin^{-1} t}\right)^{\frac{1}{2}}}{\sqrt{1-t^2}}\ \log a = \frac{1}{2}\frac{x}{\sqrt{1-t^2}} \log a$$

$$\frac{dy}{dt} = -\frac{1}{2}\left(a^{\cos^{-1} t}\right)^{-\frac{1}{2}} \cdot a^{\cos^{-1} t} \cdot \frac{1}{\sqrt{1-t^2}}\ \log a$$

$$= -\frac{1}{2}\frac{y}{\sqrt{1-t^2}}\ \log a$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x} \text{ so } 1 + \left(\frac{dy}{dx}\right)^2 = \frac{x^2 + y^2}{x^2}.$$

21. $y = \sec\,(\tan^{-1}x) = \sqrt{1 + (\tan(\tan^{-1} x))^2}$

$$\Rightarrow y = \sqrt{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=1} = \frac{1}{\sqrt{2}}$$

22. Let $y = f(x) = \frac{x^2 - x}{x^2 + 2x} = 1 - \frac{3x}{x(x+2)} = 1 - \frac{3}{x+2}$

$$\Rightarrow x = -\frac{3}{y-1} - 2 \text{ so } f^{-1}(x) = -\frac{3}{x-1} - 2$$

$$\frac{d}{dx}(f^{-1}(x)) = \frac{3}{(x-1)^2}.$$

23. $g(f(x)) = x \Rightarrow g'(f(x))f'(x) = 1$

$$\Rightarrow g'(f(x)) = 1 + x^5$$

$$\Rightarrow g'(x) = 1 + (f^{-1}(x))^5 = 1 + (g(x))^5.$$

24. $h(x) = (gof)\,(x) = g(f(x)) = \sin\,(x|x|)$

$$= \begin{cases} -\sin(x^2); & \text{if } \ x < 0 \\ \sin(x^2); & \text{if } \ x \ge 0 \end{cases}$$

We have

$$h'(x) = \begin{cases} -2\,x\cos(x^2); & \text{if } \ x < 0 \\ 2x\cos(x^2); & \text{if } \ x > 0 \end{cases}$$

Also, $Lh'(0) = \lim_{x\to 0-} \frac{h(x) - h(0)}{x - 0}$

$$= \lim_{x\to 0-} \frac{-\sin(x^2)}{x} = 0$$

and $Rh'(0) = 0$

Thus, $h'(x) = \begin{cases} -2\cos\,(x^2) & \text{if } \ x < 0 \\ 2x\cos\,(x^2) & \text{if } \ x \ge 0 \end{cases}$

Note that $h'(x)$ is continuous at all points except possibly at $x = 0$.

Now, $\lim_{x\to 0-} h'(x) = \lim_{x\to 0-} [-2x\cos\,(x^2)] = 0 = h'(0)$

and $\lim_{x\to 0+} h'(x) = \lim_{x\to 0+} [2x\cos\,(x^2)] = 0 = h'(0)$

This shows that h' is continuous at $x = 0$.

Also, h' is differentiable at all points except possibly at $x = 0$.

Now, $Lh''(0) = \lim_{x\to 0-} \frac{h'(x) - h'(0)}{x - 0} = -2$

and $Rh''(0) = 2$

This shows that h' is differentiable at all points except at $x = 0$.

25. We have $g(f(x)) = x$

$\Rightarrow g'(f(x))\, f'(x) = 1$

Now, $f(x) = 7 \Rightarrow x^2 - x + 5 = 7$

$\Rightarrow (x - 2)(x + 1) = 0 \Rightarrow x = 2$ as $x > 1/2$.

Thus, $g'(7)\, f'(2) = 1$

$$\Rightarrow g'(7) = \frac{1}{2(2)-1} = \frac{1}{3}.$$

26. For $x \in \mathbf{R}$ $0 \le |f(x)| \le x^2$

$$\Rightarrow \lim_{x\to 0} 0 \le \lim_{x\to 0} |f(x)| \le \lim_{x\to 0}(x^2)$$

$$\Rightarrow \lim_{x\to 0} |f(x)| = 0 \Rightarrow \left|\lim_{x\to 0} f(x)\right| = 0$$

$$\Rightarrow \lim_{x\to 0} f(x) = 0.$$

Also, $0 \le |f(0)| \le 0^2 \Rightarrow f(0) = 0$

$$\therefore \lim_{x\to 0} f(x) = 0 = f(0)$$

$\Rightarrow f$ is continuous at $x = 0$.

Also, for $x \ne 0$,

$$0 \le \left|\frac{f(x)-f(0)}{x}\right| \le \left|\frac{x^2-0}{x}\right| = |x|$$

$$\therefore \lim_{x\to 0}\left|\frac{f(x)-f(0)}{x-0}\right| = 0$$

$\Rightarrow f'(0) = 0$

27. $y = e^{nx} \Rightarrow$ In $y = nx$

$$\text{Now, } \frac{d^2y}{dx^2} = n^2e^{nx} = n^2y$$

$$\text{and } \frac{d^2x}{dy^2} = \frac{1}{ny^2}$$

$$\therefore \left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right) = \frac{n}{y} = -ne^{-nx}$$

28. As g is differentiable at $x = 3$, g is continuous at $x = 3$.

$$\therefore \lim_{x\to 3-} g(x) = \lim_{x\to 3+} g(x) = g(3)$$

$$= k\sqrt{3+1} = 3m + 2 = k\sqrt{3+1}$$

$\Rightarrow 2k = 3m + 2$

Also, as g is differentiable at $x = 3$,

$$\lim_{x\to 3-} \frac{g(x)-g(3)}{x-3} = \lim_{x\to 3+} \frac{g(x)-g(3)}{x-3}$$

$$\Rightarrow \frac{k}{2(2)} = m \Rightarrow k = 4m.$$

$\therefore m = 2/5$ and $k = 8/5$

$\Rightarrow k + m = 2$

29. $f(x) = |\log 2 - \sin x|$, $x \in \mathbf{R}$

Let $a = \sin^{-1}(\log 2)$

For $-a < x < a$, $f(x) = \log 2 - \sin x$

$\Rightarrow g(x) = f(f(x)) = \log 2 - \sin(f(x))$

$= \log 2 - \sin(\log 2 - \sin x)$

$\Rightarrow g'(x) = (-\cos(\log 2 - \sin x))\, (-\cos x)$

$\therefore g'(0) = (-\cos(\log 2\ ))\, (-1) = \cos(\log 2)$

30. $$\lim_{t\to x} \frac{t^2 f(x) - x^2 f(t)}{t-x} = 1$$

$$\Rightarrow \lim_{t\to x} \frac{(t^2-x^2)f(x) - x^2(f(t)-f(x))}{t-x} = 1$$

$$\Rightarrow \lim_{t\to x}\left[(t+x)f(x) - x^2\frac{f(t)-f(x)}{t-x}\right] = 1$$

$\Rightarrow 2xf(x) - x^2f'(x) = 1$

$\Rightarrow x^2f'(x) - 2xf(x) = -1$

$$\Rightarrow f'(x) - \frac{2}{x}f(x) = \frac{1}{x^2}$$

$$\Rightarrow \frac{1}{x^2}f'(x) - \frac{2}{x^3}f(x) = \frac{1}{x^4}$$

$$\Rightarrow \frac{d}{dx}\left[\frac{1}{x^2}f(x)\right] = \frac{1}{x^4}$$

$$\Rightarrow \frac{1}{x^2}f(x) = \int x^{-4}dx = -\frac{1}{3}x^{-3} + C$$

$$\Rightarrow f(x) = -\frac{1}{3x} + Cx^2$$

$$\text{Now, } 1 = f(1) = -\frac{1}{3} + C \Rightarrow C = \frac{4}{3}$$

$$\text{Thus, } f(x) = -\frac{1}{3x} + \frac{4}{3}x^2$$

$$\therefore\ f\left(\frac{3}{2}\right) = -\frac{2}{9}+\frac{4}{3}\cdot\frac{9}{4} = \frac{25}{9}$$

31. As f is continuous at $x = 1$,

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1+} f(x)$$

$$\Rightarrow -1 = a + \cos^{-1}(1 + b)$$

Also, as f is differentiable at $x = 1$, (1)

$$\lim_{x\to 1-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+}\frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow -1 = \frac{-1}{\sqrt{1-(1+b)^2}}$$

$$\Rightarrow 1 - (1 + b)^2 = 1 \Rightarrow 1 + b = 0 \qquad (2)$$

From (1) and (2)

$$-1 = a + \cos^{-1}(0) = a+\frac{\pi}{2}$$

$$\Rightarrow a = -1 -\pi/2$$

Thus, $\dfrac{a}{b} = \dfrac{2+\pi}{2}$

Previous Years' B-Architecture Entrance Examination Questions

1. $\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2-} ax = 2a$ and $\lim\limits_{x\to 2+} f(x) = \lim\limits_{x\to 2+} (ax^2 + bx + 3) = 4a + 2b + 3$

Since f is differentiable so continuous, hence

$$2a + 2b + 3 = 0 \qquad \text{(i)}$$

Also $\lim\limits_{h\to 0+}\dfrac{f(2+h)-f(2)}{h}$

$$= \lim_{h\to 0+}\frac{a(2+h)^2+b(2+h)+3-(4a+2b+3)}{h}$$

$$= \lim_{h\to 0+}\frac{ah^2+4ah+bh}{h}$$

$$= 4a + b$$

$$\lim_{h\to 0-}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0-}\frac{a(2+h)-(4a+2b+3)}{h}$$

$= a$ (using (i))

So $a = 4a + b \Rightarrow -3a = b$. Putting this in (i), we get $a = \dfrac{3}{4}$, $b = -\dfrac{9}{4}$.

2. $f'(x) = \sum\limits_{i=1}^{10} x^{i-1} \Rightarrow f'(1) = 10$ and $f''(x) = \sum\limits_{i=2}^{10}(i-1)x^{i-2}$.

Therefore,

$$\lim_{x\to 1}\frac{f'(x)-10}{x-1} = \lim_{x\to 1}\frac{f'(x)-f'(1)}{x-1} = f''(1) = \sum_{i=2}^{10}(i-1)$$

$$= 1 + \ldots + 9 = \frac{9\times 10}{2} = 45.$$

3. $h(x) = x|x| = \begin{cases} x^2, & x\ge 0\\ -x^2, & x<0\end{cases}$ is a differentiable function

$u(x) = |x - 2|^2 = (x - 2)^2$ is a differentiable function

$v(x) = |x - 3|^3 = \begin{cases} (x-3)^3, & x\ge 3\\ -(x-3)^3, & x<3\end{cases}$ is a differentiable function

$w(x) = |x - 1|$ is not differentiable at $x = 1$.

f is differentiable everywhere except at $x = 1$.

Hence $A = \{1\}$

4. $$\lim_{x\to 0}\frac{2f(2x^2+3x)-1}{3g(x)-1}\left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0}\frac{2(4x+3)f'(2x^2+3x)}{3g'(x)}$$

$$= \frac{2(0+3)f'(0)}{3g'(0)} = \frac{2.3.1}{3.2} = 1$$

5. $f'(x) = \begin{cases} 3x^2, & x>0\\ 0, & x<0\end{cases}$ and $\lim\limits_{x\to 0+}\dfrac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to 0+}\frac{x^3}{x} = 0.$$

$$\lim_{x\to 0-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-}\frac{0-0}{x} = 0.$$

$f'(x)$ exists for all x and is continuous so

$$\lim_{x\to 0} f'(x) = f'(0)$$

6. $f(x) = \begin{cases} 1 & \text{if} \quad \sin x > 0 \\ -1 & \text{if} \quad \sin x < 0 \\ 0 & \text{if} \quad \sin x = 0 \end{cases}$ so

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in (0, \pi) \\ -1 & \text{if} \quad x \in (\pi, 2\pi) \\ 0 & \text{if} \quad x = \pi \end{cases}$$

$\lim\limits_{h \to 0+} \dfrac{f(\pi + h) - f(\pi)}{h} = \lim\limits_{h \to 0+} \dfrac{-1}{h}$ which doesn't exist

So A doesnot exist.

$g(x) = \begin{cases} \sin 1, & x > 0 \\ \sin(-1), & x < 0 \\ 0 & x = 0 \end{cases}$ so $\lim\limits_{h \to 0+} \dfrac{g(\pi + h) - g(\pi)}{h}$

$$\lim_{h \to 0+} \frac{\sin 1 - \sin 1}{h} = 0.$$

and $\lim\limits_{h \to 0-} \dfrac{g(\pi + h) - g(\pi)}{h} = \lim\limits_{x \to 0-} \dfrac{\sin 1 - \sin 1}{h} = 0$.

Thus $B = 0$.

7. $u = 2\tan^{-1}\dfrac{1+x}{1-x} = 2\left(\dfrac{\pi}{4} + \tan^{-1} x\right)$ and

$= \dfrac{\pi}{2} - \cos^{-1}\dfrac{1-x^2}{1+x^2}$

Hence $f(x) = u + v = \pi$ for all x

$f'(x) = 0 \Rightarrow f''(x) = 0$.

8. $\lim\limits_{h \to 0-} \dfrac{f(-1+h) - f(-1)}{h} = \lim\limits_{h \to 0-} \dfrac{1 - (-1+h)^2 - 0}{h}$

$= \lim\limits_{h \to 0-} \dfrac{1 - (1 + h^2 - 2h)}{h} = 2$

$\lim\limits_{h \to 0+} \dfrac{f(-1+h) - f(-1)}{h} = \lim\limits_{h \to 0+} \dfrac{2(-1+h) + 2 - 0}{h}$

$= 2$.

So $f'(-1) = 2$.

9. $\cos |x| - |x| = \cos x - |x|$ which is not differentiable at $x = 0$.

$\cos |x| + |x| = \cos x + |x|$ which is not differentiable at $x = 0$

$u(x) = \sin(|x|) + |x| = \begin{cases} \sin x + x & \text{if } x \geq 0 \\ -(\sin x + x) & \text{if } x < 0 \end{cases}$

$\lim\limits_{h \to 0+} \dfrac{u(h) - u(0)}{h} = \lim\limits_{h \to 0+} \dfrac{\sin h + h}{h} = 2$

$\lim\limits_{h \to 0-} \dfrac{u(h) - u(0)}{h} = -\lim\limits_{h \to 0-} \dfrac{\sin h + h}{h} = -2$

So u is not differentiable at $x = 0$

Let $v(x) = \sin(|x|) - |x| = \begin{cases} \sin x - x, & \text{if } x \geq 0 \\ -\sin x + x, & \text{if } x < 0 \end{cases}$

$\lim\limits_{h \to 0+} \dfrac{v(h) - v(0)}{h} = \lim\limits_{h \to 0+} \dfrac{\sin h - h}{h} = 0$

$\lim\limits_{h \to 0-} \dfrac{v(h) - v(0)}{h} = \lim\limits_{h \to 0-} \dfrac{-\sin h + h}{h} = 0$

v is a differentiable function.

10. Same as Q.19 in AIEEE/JEE Question.

11. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad x > 2 \\ -\dfrac{1}{x} & \text{if} \quad x < -2 \\ a + bx^2 & \text{if} \quad -2 \leq x \leq 2 \end{cases}$

$\lim\limits_{x \to -2-} f(x) = \lim\limits_{x \to -2-} -\dfrac{1}{x} = \dfrac{1}{2}$

$\lim\limits_{x \to -2+} f(x) = \lim\limits_{x \to -2+} (a + bx^2) = a + 4b$

Since f is differentiable so continuous, hence $a + 4b = \dfrac{1}{2}$

$\lim\limits_{h \to 0+} \dfrac{f(-2+h) - f(-2)}{h}$

$= \lim\limits_{h \to 0+} \dfrac{a + b(-2+h)^2 - (a + 4b)}{h}$

$= \lim\limits_{h \to 0+} \dfrac{bh^2 - 4bh}{h} = -4b$

$\lim\limits_{h \to 0-} \dfrac{f(-2+h) - f(-2)}{h} = \lim\limits_{h \to 0-} \dfrac{-\dfrac{1}{-2+h} - (a + 4b)}{h}$

$= \lim\limits_{h \to 0-} \dfrac{-\dfrac{1}{-2+h} - \dfrac{1}{2}}{h} = -\lim\limits_{h \to 0-} \dfrac{2 - 2 + h}{2h(-2+h)}$

$= -\lim\limits_{h \to 0-} \dfrac{1}{2(-2+h)}$

$= \frac{1}{4}.$

So $b = \frac{-1}{16}$ and $a = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$

12. For $0 < x < 1$

$(1 - x)\, y(x) = (1 - x)(1 + x)(1 + x^2) \ldots (1 + x^{32})$

$= (1 - x^2)(1 + x^2) \ldots (1 + x^{32})$

$= (1 - x^4)(1 + x^4) \ldots (1 + x^{32})$

$= \ldots = 1 - x^{64}$

$\Rightarrow y(x) = \frac{1 - x^{64}}{1 - x}$

$$y'(x) = \frac{(1-x)(-64x^{64}) - (-1)(1 - x^{64})}{(1-x)^2}$$

$$\Rightarrow y'\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)(-64)\left(\frac{1}{2}\right)^{63} + 1 - \left(\frac{1}{2}\right)^{64}}{\left(1 - \frac{1}{2}\right)^2}$$

$$= 4\left[1 - 65\left(\frac{1}{2}\right)^{64}\right] = 4 - 65\left(\frac{1}{2}\right)^{62}$$

13. Using $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$y(x) = \begin{vmatrix} \sin x & \cos x - \sin x & \cos x + 1 \\ 23 & -6 & -10 \\ 1 & 0 & 0 \end{vmatrix}$$

$= -10(\cos x - \sin x) + 6 \cos x + 6$

$= 10 \sin x - 4 \cos x + 6$

$\frac{dy}{dx} = 10 \cos x + 4 \sin x$

$\frac{d^2y}{dx^2} = -10 \sin x + 4 \cos x = -y + 6$

$\Rightarrow \frac{d^2y}{dx^2} + y = 6.$

CHAPTER ELEVEN

Applications of Derivatives

THE DERIVATIVE AS A RATE OF CHANGE

In case of a linear function $y = mx + b$ the graph is a straight line and the slope m measures the steepness of the line by giving the rate of climb of the line, the rate of change of y with respect to x.

As x changes from x_0 to x_1, y changes m times as much:

$$y_1 - y_0 = m(x_1 - x_0)$$

Thus the slope $m = (y_1 - y_0)/(x_1 - x_0)$ gives the change in y per unit change in x.

In more general case of a differentiable function $y = f(x)$, the difference quotient

$$\frac{f(x+h) - f(x)}{x + h - x} = \frac{f(x+h) - f(x)}{h},\ h \neq 0$$

gives the average rate of change of y (or f) with respect to x. The limit as h approaches zero is the derivative $dy/dx = f'(x)$, which can be interpreted as the *instantaneous rate of change of* y with respect to x. Since the graph is a curve, the rate of change of y can vary from point to point.

Illustration 1

Find the rate of change of volume of a sphere with respect to its radius when $r = 4$ cm

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dr} = 4\pi r^2$$

when $\quad r = 4, \qquad \dfrac{dV}{dr} = 64\pi$

Illustration 2

A point is in motion along a curve $12y = x^3$. Which of its coordinate change faster?

Differentiating both the coordinates with respect to t we have

$$12\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$

$$\Rightarrow \qquad \frac{y'_t}{x'_t} = \frac{x^2}{4}$$

Hence, if

(i) $-2 < x < 2$, then $\dfrac{y'_t}{x'_t} < 1$, i.e. the rate of change of the ordinate is less than that of the abscissa.

(ii) For $x > 2$ or $x < -2$, $\dfrac{y'_t}{x'_t} > 1$ i.e. the rate of change of the ordinate is greater than that of the abscissa.

(iii) For $x = \pm 2$, the rate of change of ordinate is equal to that of the abscissa.

Velocity and Acceleration

Suppose that an object is moving along a straight line and that, for each time t during a certain time interval, the object has (coordinate) $x(t)$. Then at time $t + h$, the position of the object is $x(t + h)$, and $x(t + h) - x(t)$ is the change in position that the object experienced during the time period t to $t + h$. The ratio

$$\frac{x(t+h) - x(t)}{t + h - t} = \frac{x(t+h) - x(t)}{h}$$

gives the average velocity of the object during this time period. If

$$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h} = x'(t)$$

exists, then $x'(t)$ gives the (instantaneous) rate of change of position with respect to time. This rate of change of position is called the velocity of the object. If the velocity function is itself differentiable, then its rate of change with respect t time is called the acceleration, in symbols

$$a(t) = v'(t) = x''(t)$$

the speed is by definition the absolute value of the velocity:

$$\text{speed at time } t = |v(t)|$$

If the velocity and acceleration have the same sign then the object is speeding up, but if the velocity and acceleration have opposite signs then the object is slowing down.

A sudden change in acceleration is called a *jerk*. Thus, *Jerk* is the derivative of acceleration. If a body's position at the time t is $x(t)$, the body's jerk at time t is

$$j = \frac{da}{dt} = \frac{d^3x}{dt^3}$$

Illustration 3

A point moves in a straight line so that its distance from the start in t sec is equal to $s = \frac{1}{4}t^4 - 4t^3 + 16t^2$. What will be acceleration and at what times is its velocity equal to zero?

$$v = \frac{ds}{dt} = t^3 - 12t^2 + 32t$$

$$a = \frac{d^2s}{dt^2} = 3t^2 - 24t + 32$$

$$\begin{aligned} v = 0 \quad &\Rightarrow \quad t^3 - 12t^2 + 32t = 0 \\ &\Rightarrow \quad t(t^2 - 12t + 32) = 0 \\ &\Rightarrow \quad t(t-4)(t-8) = 0 \end{aligned}$$

at $t = 0$, $t = 4$, $t = 8$ the velocity will be zero and the corresponding acceleration will be 32, – 16, 32 respectively.

Illustration 4

A body whose mass is 3 kg performs rectilinear motion according to the formula $s = 1 + t + t^2$, where s is measured in cm and t in secs. Determine the kinetic energy $\left(\frac{mv^2}{2}\right)$ of the body in 5*s* after the start.

$$\frac{ds}{dt} = 1 + 2t, \quad v = \left.\frac{ds}{dt}\right|_{t=5} = 11 \text{ cm/s}$$

$$\begin{aligned} \text{K.E.} = \frac{mv^2}{2} = \frac{3000 \times 121}{2} &= 181500 \text{ gm cm}^2/\text{s}^2 \\ &= 181.5 \times 10^3 \text{ erg.} \end{aligned}$$

Differentials Let $y = f(x)$ be a differentiable function. Let $h \neq 0$. The difference $f(x + h) - f(x)$ is called the increment of f from x to $x + h$, and is denoted by Δf.

$$\Delta f = f(x + h) - f(x)$$

The product $f'(x)\,h$ is called the differential of f at x with increment h, and is denoted by df

$$df = f'(x)\,h$$

The change in f from x to $x + h$ can be approximated by $f'(x)\,h$;

$$f(x + h) - f(x) \simeq f'(x)\,h.$$

Illustration 5

Find the increment and differential of the function

$$y = 3x^3 + x - 1 \quad \text{at } x = 1, \Delta x = 0.1.$$

$$dy = (9x^2 + 1)\,\Delta x$$

$$\begin{aligned} \text{At} \quad x = 1, \Delta x = 0.1, dy &= (9 + 1)(0.1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \Delta y &= [3(x + \Delta x)^3 + (x + \Delta x) - 1] - (3x^3 + x - 1) \\ &= 9x^2\,\Delta x + 9x\,\Delta x^2 + 3\Delta x^3 + \Delta x \end{aligned}$$

$$\Delta y - dy = 9x\,\Delta x^2 + 3\Delta x^3$$

$$\text{At} \quad x = 1, \Delta x = 0.1$$

$$\Delta y - dy = 0.09 + 0.003 = 0.093.$$

TANGENT AND NORMAL

Let $y = f(x)$ be the equation of a curve, and let $P(x_0, y_0)$ be a point on it. Let PT be the tangent, PN the normal and PM the perpendicular to the x-axis (Fig. 11.1).

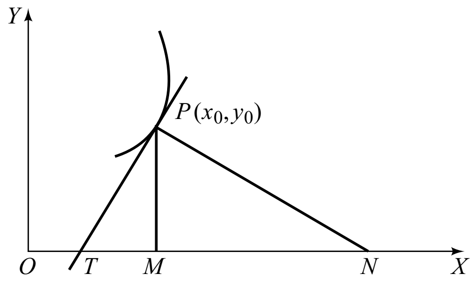


Fig. 11.1

The slope of the tangent to the curve $y = f(x)$ at P is given by $f'(x_0)$ Thus the equation of the tangent to the curve $y = f(x)$ at (x_0, y_0) is

$$y - y_0 = f'(x_0)(x - x_0)$$

Since PN is perpendicular to PT, it follows that, if $f'(x_0) \neq 0$, the slope of PN is $-1/f'(x_0)$. Hence the equation of the normal to the curve $y = f(x)$ at (x_0, y_0) is

$$f'(x_0)(y - y_0) + (x - x_0) = 0$$

An equation of the normal parallel to the x-axis is $y = y_0$. The *length of the tangent* at (x_0, y_0) is PT, and it is equal to

$$|y_0|\sqrt{1 + \frac{1}{(f'(x_0))^2}}\,.$$

The *length of the normal* is PN and it is equal to

$$|y_0|\sqrt{1 + (f'(x_0))^2}\,.$$

TM is called subtangent and the *length of the subtangent* is equal to

$$\left|\frac{1}{f'(x_0)}y_0\right|$$

MN is called subnormal and the *length of the subnormal* is equal to $|y_0 f'(x_0)|$.

Illustration 6

Derive the equation of tangent and normal at (x_0, y_0) of the curve $y = \log x$.

$$\frac{dy}{dx} = \frac{1}{x} \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=x_0} = \frac{1}{x_0}$$

Hence equation of tangent at (x_0, y_0) is

$$y - \log x_0 = \frac{1}{x_0}(X - x_0)$$

Equation of normal is

$$Y - \log x_0 = -x_0(X - x_0)$$

Length of tangent = $|\log x_0|\sqrt{1 + x_0^2}$

Length of normal = $|\log x_0|\sqrt{1 + \dfrac{1}{x_0^2}}$

Length of subtangent = $|x_0 \log x_0|$

Length of subnormal = $|y_0|x_0|$

ANGLE BETWEEN TWO CURVES

An angle of intersection of two curves is defined as the angle between the tangents to the two curves at their point of intersection. Let $y = f(x)$ and $y = g(x)$ be two curves, and let $P(x_0, y_0)$ be their point of intersection. Also, let ψ and φ be the angles of inclination of the two tangents at P with the x-axis, and let θ be the angle between the two tangents. Then

$$\pm \tan\theta = \frac{\tan\varphi - \tan\psi}{1 + \tan\varphi \tan\psi} = \frac{g'(x_0) - f'(x_0)}{1 + f'(x_0)\, g'(x_0)}$$

Orthogonal Curves

If the angle of intersection between two curves is a right angle then the two curves are said to be intersecting orthogonally. Two curves $y = f(x)$ and $y = g(x)$ cut orthogonally if $f'(x)\, g'(x) = -1$.

THE ROLLE'S AND LAGRANGE'S THEOREMS

Rolle's theorem Let $f(x)$ be a function defined on a closed interval $[a, b]$, such that (i) $f(x)$ is continuous on $[a, b]$, (ii) $f(x)$ is derivable on $]a, b[$, and (iii) $f(a) = f(b)$. Then there exists a $c \in\ ]a, b[$ such that $f'(c) = 0$.

Rolle's theorem for polynomials If $\phi(x)$ is any polynomial, then between any pair of roots of $\phi(x) = 0$ lies a root of $\phi'(x) = 0$.

If a polynomial equation $P(x) = 0$ has at least n real roots, then $P'(x) = 0$ has at least $(n-1)$ real roots, $P''(x) = 0$ has at least $(n-2)$ real roots and so on.

Lagrange's Mean Value theorem Let $f(x)$ be a function defined on $[a, b]$, such that (i) $f(x)$ is continuous on $[a, b]$, and (ii) $f(x)$ is derivable on $]a, b[$. Then there exists a $c \in\ ]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Illustration 7

Check the validity of Rolle's theorem for $y = 4^{\sin x}$ in the interval $[0, \pi]$.

Being exponential function, y is a continuous function x and $y' = (4^{\sin x})(\log 4)\cos x$. Also $y(0) = 4^{\circ} = 1$ and $y(\pi) = 4^{\sin\pi} = 4^{\circ} = 1$. Hence all the hypothesis of Rolle's theorem are satisfied. For conclusion,

$$y'(\pi/2) = 0 \quad \text{and} \quad \pi/2 \in [0, \pi]$$

Illustration 8

Show that the equation $x^3 - 3x + C = 0$ cannot have two different roots in (0, 1).

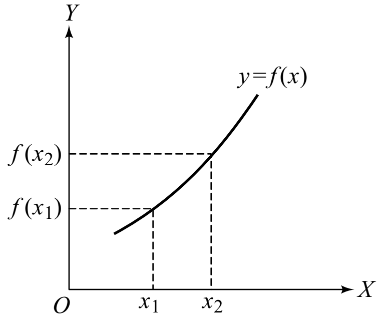
Consider $f(x) = x^3 - 3x + C$. Suppose f has two distinct roots say x_1 and x_2 in (0,1). f is a polynomial so continuous on $[x_1, x_2]$ and differentiable in (x_1, x_2) and $f(x_1) = f(x_2) = 0$ so by Rolle's theorem there is $\alpha \in (x_1, x_2) \subseteq (0, 1)$ such that

$$f'(\alpha) = 0 \quad \Rightarrow \quad 3\alpha^2 - 3 \neq 0 \quad \Rightarrow \quad \alpha = \pm 1 \notin (0, 1).$$

So $x^3 - 3x + c = 0$ cannot have two different roots in (0, 1).

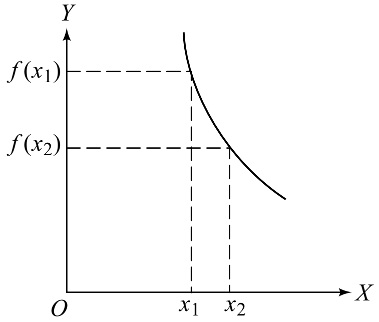
MONOTONICITY

A function $f(x)$ defined on a set D is said to be non-decreasing, increasing, non-increasing and decreasing respectively, if for any $x_1, x_2 \in D$ and $x_1 < x_2$, we have $f(x_1) \leq f(x_2)$, $f(x_1) < f(x_2)$, $f(x_1) \geq f(x_2)$ and $f(x_1) > f(x_2)$, respectively (Figs 11.2 and 11.3). The function $f(x)$ is said to be monotonic if it possesses any of these properties.



An increasing function

Fig. 11.2



A decreasing function

Fig. 11.3

As an immediate consequence of Lagrange's Mean theorem, we have

Testing monotonicity Let $f(x)$ be continuous on $[a, b]$ and differentiable on $]a, b[$. Then

(i) for $f(x)$ to be non-decreasing (non-increasing) on $[a, b]$ it is necessary and sufficient that $f'(x) \geq 0$ ($f'(x) \leq 0$) for all $x \in\ ]a, b[$.

(ii) for $f(x)$ to be increasing (decreasing) on $[a, b]$ it is necessary and sufficient that $f'(x) > 0$ ($f'(x) < 0$) for all $x \in]a, b[$.

Illustration 9

Show that the function $y = \sqrt{2x - x^2}$ increases in the interval (0, 1) and decreases in (1, 2)

$$y'(x) = \frac{1}{2}\frac{2-2x}{\sqrt{2x-x^2}} = \frac{1-x}{\sqrt{(2-x)x}} \quad x \neq 0, 2.$$

y' is not defined at $x = 0$, $x = 2$.

$y'(x) > 0$ if $1 - x > 0$ so y increases on (0, 1) and $y'(x) < 0$ for $x \in (1, 2)$, so y decreases on (1, 2).

Illustration 10

Find the interval of monotonicity of $y = \dfrac{1-x+x^2}{1+x+x^2}$

$$y'(x) = \frac{(1+x+x^2)(-1+2x)-(1-x+x^2)(1+2x)}{(1+x+x^2)^2}$$

$$= \frac{-2+2x^2}{(1+x+x^2)^2} = \frac{2(x^2-1)}{(1+x+x^2)^2}$$

Since the denominator is always positive so the sign of $y'(x)$ according to the sign of $x^2 - 1$.

Thus $y'(x) > 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow x > 1$ or $x < -1$

Hence y increases on $(1, \infty) \cup (-\infty, -1)$ and decreases on $(-1, 1)$.

MAXIMA AND MINIMA

A function has a local *maximum* at the point x_0 if the value of the function $f(x)$ at that point is greater than its values at all points other than x_0 of a certain interval containing the point x_0. In other words, a function $f(x)$ has a maximum at x_0 if it is possible to find an interval (α, β) containing x_0, i.e., with $\alpha < x_0 < \beta$, such that for all points different from x_0 in (α, β), we have $f(x) < f(x_0)$.

A function $f(x)$ has a local minimum at x_0 if there exists an interval (α, β) containing x_0 such that $f(x) > f(x_0)$ for $x \in (\alpha, \beta)$ and $x \neq x_0$.

One should not confuse the local maximum and local minimum of a function with its largest and smallest values over a given interval. The local maximum of a function is the largest value only in comparison to the values it has at all points sufficiently close to the point of local maximum. Similarly, the local minimum is the smallest value only in comparison to the values of the function at all points sufficiently close to the local minimum point.

The general term for the maximum and minimum of a function is *extremum*, or the extreme values of the function. A necessary condition for the existence of an extremum at the point x_0 of the function $f(x)$ is that where $f'(x_0) = 0$, or $f'(x_0)$ does not exist. The points at which $f'(x) = 0$, or where $f'(x)$ does not exist, are called *critical points* of the function f.

First derivative test

(i) If $f'(x)$ changes sign from positive to negative at x_0, i.e., $f'(x) > 0$ for $x < x_0$, and $f'(x) < 0$ for $x > x_0$, then the function attains a local maximum at x_0.

(ii) If $f'(x)$ changes sign from negative to positive at x_0, i.e., $f'(x) < 0$ for $x < x_0$, and $f'(x) > 0$ for $x > x_0$, then the function attains a local minimum at x_0.

(iii) If the derivative does not change sign in moving through the point x_0, there is no extremum at that point.

Second derivative test Let f be twice differentiable, and let c be a root of the equation $f'(x) = 0$. Then

(i) c is a local maximum point if $f''(c) < 0$

(ii) c is a local minimum point if $f''(c) > 0$.

However, if $f''(c) = 0$, then the following result is applicable. Let $f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0$ (where f^r denotes the rth derivative), but $f^{(n)}(c) \neq 0$.

(i) If *n is even* and $f^{(n)}(c) < 0$, there is a local maximum at c, while if $f^{(n)}(c) > 0$, there is a local minimum at c.

(ii) If n is *odd*, there is no extremum at the point c.

The *greatest* (*least*) value of continuous function $f(x)$ on the interval $[a, b]$ is attained either at the critical points or at the end-points of the interval. To find the greatest (least) value of the function, we have to compute its values at all the critical points on the interval $]a, b[$, and the values $f(a)$,

$f(b)$ of the function at the end-points of the interval, and choose the greatest (least) out of the numbers so obtained.

Illustration 11

Find the extrema of $y = 2x^3 - 3x^2$. y is differentiable function and $y' = 6x^2 - 6x = 6x(x - 1)$.

So, points extremum are 0 and 1 $y'' = 6(2x - 1)$ $y''(1) = 6 > 0$, $y''(0) = -6 < 0$.

Hence $y_{\max} = y(0) = 0$

and $y_{\min} = y(1) = 2 - 3 = -1$

Illustration 12

Find the greatest and least value of $y = x^3 - 3x^2 + 6x - 2$ on $[-1, 1]$. y is a differentiable function of x and $y'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 2) = 3((x - 1)^2 + 1) > 0$.

Hence y increases on $[-1, 1]$. Thus the greatest value $= y(1) = 1 - 3 + 6 - 2 = 2$, least value $= y(-1) = -1 - 3 - 6 - 2 = -12$.

Point of Inflection

A point $x = c$ is said to be point of inflection for a curve $y = f(x)$ if $f''(c) = 0$ or is not defined and $f'''(c) \neq 0$

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: The approximate value of cos 31° is

(a) 0.52 (b) 0.851

(c) 0.641 (d) 0.681

Ans. (b)

Solution: Known value near to cos 31° is cos 30°

$$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Let $y = \cos x \quad y'(x) = -\sin x$

$$\cos 31 = \cos\left(\frac{\pi}{6} + \frac{\pi}{180}\right). \text{ So putting } x = \frac{\pi}{6} \text{ and } \Delta x = \frac{\pi}{180},$$

we have $y'\left(\dfrac{\pi}{6}\right) = -\sin\dfrac{\pi}{6} = -\dfrac{1}{2}$

$\cos(x + \Delta x) \simeq \cos x + f'(x)\,\Delta x$. Putting $x = \dfrac{\pi}{6}$ and $\Delta x = \dfrac{\pi}{180}$, we have

$$\cos 31 \simeq \frac{\sqrt{3}}{2} - \frac{1}{2}\frac{\pi}{180} = \frac{1.732}{2} - \frac{1}{2} \times \frac{3.14}{180}$$

$$= 0.851$$

Example 2: The tangent line at (2, 4) to the curve $y = x^3 - 3x + 2$ meets the x-axis at

(a) (2, 0) (b) $\left(\dfrac{7}{2}, 0\right)$

(c) $\left(\dfrac{11}{9}, 0\right)$ (d) $\left(\dfrac{14}{9}, 0\right)$

Ans. (d)

Solution: $\dfrac{dy}{dx} = 3x^2 - 3 = 3(x^2 - 1)$

$$\left.\frac{dy}{dx}\right|_{x=2} = 3(4 - 1) = 9$$

Equation of tangent at (2, 4) is

$$Y - 4 = 9(X - 2)$$

This meets x-axis if $Y = 0$

$$\Rightarrow \quad \frac{-4}{9} = X - 2 \quad \Rightarrow \quad X = 2 - \frac{4}{9} = \frac{14}{9}.$$

Example 3: The slope of the tangent to the curve

$$x = t^2 + 3t - 8$$
$$y = 2t^2 - 2t - 5$$

at the point (2, –1) is

(a) 2/3 (b) 6/7

(c) 4/5 (d) 3/2

Ans. (b)

Solution: We need to determine the value of t corresponding to the given point, so

$$\left.\begin{array}{l} 2 = t^2 + 3t - 8 \\ -1 = 2t^2 - 2t - 5 \end{array}\right] \begin{array}{l} \Rightarrow \\ \Rightarrow \end{array} \left.\begin{array}{l} t^2 + 3t - 10 = 0 \\ t^2 - t - 2 = 0 \end{array}\right] \begin{array}{l} \Rightarrow \quad t = 2, -5 \\ \Rightarrow \quad t = 2, -1 \end{array}$$

Hence common value is 2

$$y'|_{x=2} = \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)_{t=2} = \left(\frac{4t - 2}{2t + 3}\right)_{t=2} = \frac{6}{7}.$$

Example 4: The interval in which $y = \dfrac{1}{4x^3 - 9x^2 + 6x}$ is increasing is

(a) $(-\infty, \infty)$ (b) (0, 1/2)

(c) (1/2, 1) (d) $(1, \infty)$

Ans. (c)

Solution: $$y'(x) = -\frac{12x^2 - 18x + 6}{(4x^3 - 9x^2 + 6x)^2}$$

$$= -\frac{6(2x - 1)(x - 1)}{(4x^3 - 9x^2 + 6x)^2}$$

$$y'(x) > 0 \Leftrightarrow (2x-1)(x-1) < 0$$
$$\Leftrightarrow \text{(i)}\ \ 2x-1>0,\quad x-1<0$$
$$\text{(ii)}\ \ 2x-1<0,\quad x-1>0$$

(i) $\Rightarrow$ $x \in (1/2, 1)$ case (ii) is not possible. Hence y increases on $(1/2, 1)$

Example 5: Let $y = x - \log(1 + x)$, the minimum value of y is

(a) 1 (b) 0

(c) –1 (d) $\frac{1}{2}$

Ans. (b)

Solution: $y'(x) = 1-\frac{1}{1+x}=\frac{x}{1+x}$. The domain y is $x > -1$. so $y'(x) = 0 \Rightarrow x = 0$

$y''(x) = \frac{1}{(1+x)^2} > 0$. Hence y is min at $x = 0$ and $y_{\min} = 0 - \log 1 = 0$

Example 6: A covered box of volume 72 cm^3 and the base sides in a ratio of 1:2 is to be made. The length all sides so that the total surface area is the least possible is

(a) 2, 4, 9 (b) 8, 3, 3

(c) 6, 6, 2 (d) 6, 3, 4

Ans. (d)

Solution: Let l, b, h be the dimensions $l = 2b$, so $V = lbh = 2b^2h$

$$\Rightarrow \quad 72 = 2b^2h \quad \Rightarrow \quad h = \frac{36}{b^2}$$

The surface area $S = 2(lb + bh + lh)$

$$= 2\left(2b^2 + \frac{3b}{b} + 2b \times \frac{36}{b^2}\right)$$
$$= 2\left(2b^2 + \frac{108}{b}\right)$$
$$= 4\left(b^2 + \frac{54}{b}\right)$$

$\frac{dS}{db} = 4\left(2b - \frac{54}{b^2}\right)$, $\frac{dS}{db}$ is zero if $b = 3$ and

$$\frac{d^2S}{db^2} = 4\left(2 + \frac{108}{b^3}\right) > 0$$

Hence S is minimum when $b = 3$. So the dimensions are $6, 3, \frac{36}{9} = 6, 3, 4$

Example 7: A point on the curve $y = x^3 - 3x + 5$ at which the tangent line is parallel to $y = -2x$ is

(a) (1, 3) (b) (0, 5)

(c) $\left(\frac{1}{\sqrt{3}}, 5-\frac{8\sqrt{3}}{9}\right)$ (d) $\left(\frac{1}{\sqrt{2}}, 0\right)$

Ans. (c)

Solution: Since $y'(x) = 3x^2 - 3$, so by the condition of parallelism

$$3x^2 - 3 = -2$$
$$x = \pm\frac{1}{\sqrt{3}}$$

If $x = \frac{1}{\sqrt{3}}$, then $y = \frac{1}{3\sqrt{3}} - \frac{3}{\sqrt{3}} + 5$

$$= 5 - \frac{8}{3\sqrt{3}} = 5 - \frac{8\sqrt{3}}{9}.$$

Example 8: The greatest value of $y = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is

(a) $\frac{\pi}{2}$ (b) 1

(c) 2 (d) $-\frac{\pi}{2}$

Ans. (a)

Solution $y' = 2\cos 2x - 1 \Rightarrow y'(x) = 0 \Leftrightarrow \cos^2 x = \frac{1}{2}$

So, $y'(x) = 0$ if $x = \frac{\pi}{6}$. Thus the critical points of y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ are $-\frac{\pi}{2}, \frac{\pi}{6}, \frac{\pi}{2}$. Now $y\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$, $y\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$ and $y\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{2}$. The greatest value is $\frac{\pi}{2}$.

Example 9: The point of inflection of $y = x^3 - 5x^2 + 3x - 5$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{4}$

(c) $\frac{7}{4}$ (d) $\frac{5}{4}$

Ans. (d)

Solution: Since y is twice differentiable function so for inflection points $y''(x) = 0$, $y'''(x) \neq 0$

$$y'(x) = 3x^2 - 10x + 3$$
$$y'' = 6x - 10. \quad \text{So, } y''(x) = 0 \text{ if } x = \frac{5}{4}$$
$$y'''(x) = 6 \neq 0. \quad \text{So, } x = \frac{5}{4} \text{ is a point of inflection.}$$

Example 10: The minimal rate of change of the function $f(x) = 3x^5 - 5x^3 + 5x - 7$ is

(a) $\frac{3}{4}$ (b) $\frac{5}{4}$

(c) $\frac{2}{3}$ (d) $\frac{3}{2}$

Ans. (b)

Solution Rate of change of $f(x)$ is $f'(x)$

$$= 15x^4 - 15x^2 + 5 = 15\left(x^4 - x^2 + \frac{1}{3}\right)$$

$$= 15\left(\left(x^2 - \frac{1}{2}\right)^2 + \frac{1}{12}\right) \ge \frac{15}{12} = \frac{5}{4}$$

The minimum vale of f' is attained at $x = +\frac{1}{\sqrt{2}}$ and equals $\frac{5}{4}$.

LEVEL 1

Straight Objective Type Questions

Example 11: A spherical balloon is expanding. If the radius is increasing at the rate of 5 inch per minute, the rate at which the volume increases (in cubic inches per minute) when the radius is 10 inch is

(a) $100\,\pi$ (b) $1000\,\pi$
(c) $2000\,\pi$ (d) $500\,\pi$

Ans. (c)

Solution: The volume $V = \frac{4}{3}\pi r^3$, r being the radius. It is given that $\frac{dr}{dt}$ = 5 inch/min and we have to find $\frac{dV}{dt}$ when r = 5 inches.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \qquad \left.\frac{dV}{dt}\right|_{r=10} = 4\pi\,(100)5 = 2000\,\pi.$$

Example 12: An object is moving in the clockwise direction around the unit circle $x^2 + y^2 = 1$. As it passes through the point $(1/2, \sqrt{3}/2)$, its y-coordinate is decreasing at the rate of 3 units per second. The rate at which the x-coordinate changes at this point is (in units per second)

(a) 2 (b) $3\sqrt{3}$
(c) $\sqrt{3}$ (d) $2\sqrt{3}$

Ans. (b)

Solution: We find $\frac{dx}{dt}$ when $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$ given that $\frac{dy}{dt}$ = – 3 units/s and $x^2 + y^2 = 1$.

Differentiating $x^2 + y^2 = 1$, we have

$$2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0.$$

Putting $x = 1/2$, $y = \sqrt{3}/2$ and $dy/dt = -3$, we have

$$\frac{1}{2}\frac{dx}{dt} + \frac{\sqrt{3}}{2}(-3) = 0 \quad \Rightarrow \quad \frac{dx}{dt} = 3\sqrt{3}.$$

Example 13: An approximate value of cos 40° is

(a) 0.7688 (b) 0.7071
(c) 0.7117 (d) 0.7

Ans. (a)

Solution:

Let $f(x) = \cos x$. 40° = 45° – 5°

$$= \frac{\pi}{4} - \frac{\pi}{180} \times 5 = \frac{\pi}{4} - \frac{\pi}{36} \text{ radians}$$

We use a differential to estimate the change in cos x when x decreases from $\pi/4$ to $\pi/4 - (\pi/36)$

$$f'(x) = -\sin x \text{ and } df = f'(x)\,h = -h\sin x$$

With $x = \pi/4$ and $h = -\pi/36$, df is given by

$$df = -f'(x)h = -\left(-\frac{\pi}{36}\right)\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{36}\cdot\frac{\sqrt{2}}{2} = \frac{\pi\sqrt{2}}{72} = 0.0617$$

$$\cos 40 \cong \cos 45 + 0.0617 \cong 0.7071 + 0.0617$$
$$= 0.7688.$$

Example 14: The values of x for which the tangents to the curves $y = x\cos x$, $y = (\sin x)/x$ are parallel to the axis of x are roots of (respectively)

(a) $\sin x = x$, $\tan x = x$ (b) $\cot x = x$, $\sec x = x$
(c) $\cot x = x$, $\tan x = x$ (d) $\tan x = x$, $\cot x = x$

Ans. (c)

Solution: Let $y = f(x) = x\cos x$ and $y = g(x) = (\sin x)/x$. Now $f'(x) = -x\sin x + \cos x$ and $g'(x) = (x\cos x - \sin x)/x^2$. Since the tangents are parallel to x-axis so $f'(x) = 0$ and $g'(x) = 0$, which is turn give $\cot x = x$ and $\tan x = x$ respectively.

Example 15: The length of the subtangent to the ellipse $x = a\cos t$, $y = b\sin t$ at $t = \pi/4$ is

(a) a (b) b
(c) $b/\sqrt{2}$ (d) $a/\sqrt{2}$

Ans. (d)

Solution: $\frac{dx}{dt} = -a \sin t$ and $\frac{dy}{dt} = b \cos t$, therefore $\left.\frac{dy}{dx}\right|_{t=\pi/4} = -\frac{b}{a} \cot(\pi/4) = -\frac{b}{a}$. The length of the subtangent = $\left|y_0 \frac{dx}{dy}\right|_{t=\pi/4} = \left|b \sin\frac{\pi}{4} \times -\frac{a}{b}\right| = \frac{a}{\sqrt{2}}$.

Example 16: Tangent of the angle at which the curve $y = a^x$ and $y = b^x$ $(a \neq b > 0)$ intersect is given by

(a) $\frac{\log ab}{1+\log ab}$
(b) $\frac{\log a/b}{1+(\log a)(\log b)}$
(c) $\frac{\log ab}{1+(\log a)(\log b)}$
(d) none of these

Ans. (b)

Solution: Intersection of the two curves is given by $a^x = b^x$ which implies $x = 0$. If α is the angle at which the two curves intersect then

$$\tan\alpha = \frac{m_1 - m_2}{1+m_1m_2} = \left.\frac{a^x \log a - b^x \log b}{1+a^x b^x (\log a)(\log b)}\right|_{x=0}$$

$$= \frac{(\log a/b)}{1+(\log a)(\log b)}. \qquad \text{(Putting } x = 0\text{)}$$

Example 17: For the parabola $y^2 = 16x$, the ratio of the length of the subtangent to the abscissa is

(a) 2 : 1
(b) 1 : 1
(c) $x : y$
(d) $x^2 : y$

Ans. (a)

Solution: Differentiating, $2y\frac{dy}{dx} = 16$ so $\frac{dy}{dx} = \frac{8}{y}$. Thus the length of the subtangent is $y.\frac{dx}{dy} = \frac{y^2}{8} = \frac{16x}{8} = 2x$. Hence length of the subtangent: abscissa = $2x : x = 2 : 1$.

Example 18: If the tangent to the curve $x^3 - y^2 = 0$ at $(m^2, -m^3)$ is parallel to $y = -\frac{1}{m}x - 2m^3$, then the value of m^2 is

(a) 1/3
(b) 1/6
(c) 2/3
(d) – 2/3

Ans. (c)

Solution: Differentiating $x^3 - y^2 = 0$, we have

$$\frac{dy}{dx} = \frac{3x^2}{2y} \Rightarrow \left.\frac{dy}{dx}\right|_{(m^2-m^3)} = -\frac{3m^4}{2m^3} = -\frac{3m}{2}.$$

According to the given condition, $-\frac{3}{2}m = -\frac{1}{m}$

$$\Rightarrow m^2 = \frac{2}{3}$$

Example 19 If the function $f(x) = \frac{cx+d}{(x-1)(x-4)}$ has a turning point at the point (2, – 1) then

(a) $c = 2, d = 0$
(b) $c = 1, d = 0$
(c) $c = 1, d = -1$
(d) $c = 1, d = 1$

Ans. (b)

Solution: $f'(x) = \frac{c(x-1)(x-4)-(cx+d)(2x-5)}{(x-1)^2(x-4)^2}$

So $\quad 0 = f'(2) = \frac{-2c+(2c+d)}{4} = \frac{d}{4} \quad \Rightarrow \quad d = 0$

Also $\quad -1 = f(2) = \frac{2c+d}{-2} = -c \quad \Rightarrow \quad c = 1.$

Example 20: The distance between the origin and the normal to the curve $y = e^{2x} + x^2$ at the point whose abscissa is 0 is

(a) $1/\sqrt{5}$
(b) $2/\sqrt{5}$
(c) $3/\sqrt{5}$
(d) $2/\sqrt{3}$

Ans. (b)

Solution: The point on the curve corresponding to $x = 0$ is (0,1)

$$\frac{dy}{dx} = 2e^{2x} + 2x \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=0} = 2$$

Hence the equation of the normal at the point (0, 1) is

$$y - 1 = (-1/2)(x-0) \Rightarrow 2y + x - 2 = 0$$

Therefore, the distance of the point (0, 0) from this line is $2/\sqrt{5}$.

Example 21: The function $f(x) = 1 + x(\sin x)[\cos x]$, $0 < x \leq \pi/2$

(a) is continuous on $(0, \pi/2)$
(b) is strictly decreasing in $(0, \pi/2)$
(c) is strictly increasing in $(0, \pi/2)$
(d) has global maximum value 2.

Ans. (a)

Solution: For $0 < x \leq \pi/2$, $[\cos x] = 0$. Hence $f(x) = 1$ for all $x \in (0, \pi/2]$. Trivially $f(x)$ is continuous on $(0, \pi/2)$. This function is neither strictly increasing nor strictly decreasing and its global maximum is 1.

Example 22: Let f be an increasing function on $[a, b]$ and g a decreasing function from $[a, b]$ to $[a, b]$, then

(a) $f \circ g$ is a decreasing function
(b) $g \circ f$ is an increasing function
(c) $f \circ g$ is an increasing function
(d) none of these

Ans. (a)

Solution: For $x_1 < x_2$, $x_1, x_2 \in [a, b]$, since g is decreasing so $g(x_1) > g(x_2) \Rightarrow f(g(x_1)) > f(g(x_2))$ i.e. $f \circ g$ is decreasing function on $[a, b]$. Similarly $g \circ f$ is also decreasing function.

Example 23: If $a < 0$ and $f(x) = e^{ax} + e^{-ax}$. Suppose that $S = \{x : f(x) \text{ is monotonically decreasing}\}$ then

(a) $S = \{x : x > 0\}$ (b) $S = \{x : x < 0\}$
(c) $S = \{x : x > 1\}$ (d) $S = \{x : x < 1\}$

Ans. (b)

◎ **Solution:** $f'(x) = a(e^{ax} - e^{-ax})$. So f is monotonically decreasing if and only if $e^{ax} - e^{-ax} > 0$, i.e. $e^{2ax} > 1$ which is true if and only if $2ax > 0$. Since $a < 0$, we must have $x < 0$.

◉ **Example 24:** The equation of the horizontal tangent to the graph of the function $y = e^x + e^{-x}$ is

(a) $y = -2$ (b) $y = -1$
(c) $y = 2$ (d) none

Ans. (c)

◎ **Solution:** The slope of the horizontal tangent is 0, so $0 = dy/dx = e^x - e^{-x} \Rightarrow e^{2x} = 1 \Rightarrow x = 0$. If $x = 0$ then $y = 2$. Hence the equation of horizontal tangent at (0, 2) is $y = 2$.

◉ **Example 25:** Let $f(x)$ and $g(x)$ be defined and differentiable for $x \geq x_0$ and $f(x_0) = g(x_0)$, $f'(x) > g'(x)$ for $x > x_0$ then

(a) $f(x) < g(x)$ for some $x > x_0$
(b) $f(x) = g(x)$ for some $x > x_0$
(c) $f(x) > g(x)$ for all $x > x_0$
(d) none of these

Ans. (c)

◎ **Solution:** Consider the function $\phi(x) = f(x) - g(x)$ on the interval $[x_0, x]$. By Lagrange's theorem we have $\phi(x) - \phi(x_0) = \phi'(z)(x - x_0)$ for some $z \in (x_0, x)$. Since $\phi(x_0) = 0$, $\phi'(z) = f'(z) - g'(z) > 0$. So $\phi(x) - \phi(x_0) = \phi(x) = (f'(z) - g'(z))(x - x_0) > 0 \Rightarrow \phi(x) > 0$

Thus $f(x) > g(x)$ for all $x > x_0$.

◉ **Example 26:** If $y = 2x + \cot^{-1} x + \log\left(\sqrt{1+x^2} - x\right)$, then y

(a) decreases on $(-\infty, \infty)$
(b) decreases on $[0, \infty)$
(c) neither decreases nor increases on $[0, \infty)$
(d) increases on $(-\infty, \infty)$

Ans. (d)

Solution:

$$y'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2} - x}\frac{d}{dx}\left(\sqrt{1+x^2} - x\right)$$

$$= 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2} - x} \times \left(\frac{x}{\sqrt{1+x^2}} - 1\right)$$

$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} \geq 0$$

since $1/(1+x^2)$ and $1/\sqrt{1+x^2}$ are less than or equal to 1 for all x. So $f(x)$ increases on $(-\infty, \infty)$.

◉ **Example 27** The equation $x^4 - 7x + 2 = 0$ has

(a) exactly two real and distinct solutions
(b) has four real roots
(c) no real root
(d) all the four roots lie between 0 and 2

Ans. (a)

◎ **Solution:** Let $f(x) = x^4 - 7x + 2$. Since $f(0) = 2$, $f(1) = -4$ and $f(2) = 4$

so by Intermediate value theorem there is $x_1 \in (0, 1)$ and $x_2 \in (1, 2)$ such that $f(x_1) = f(x_2) = 0$. Also $f'(x) = 4x^3 - 7$, so $f(x)$ decreases for $x < (7/4)^{1/3}$ and increases for $x > (7/4)^{1/3}$ and $1 < (7/4)^{1/3} < 2$. Thus $f(x)$ cannot be zero at any other point.

◉ **Example 28:** The maximum value of $x^{1/x}$ is

(a) $(1/e)^e$ (b) $e^{1/e}$
(c) e (d) $1/e$

Ans. (b)

◎ **Solution:** Let $f(x) = x^{1/x} \Rightarrow \log f(x) = (1/x)\log x$. Differentiating both the sides, we have $f'(x) = f(x)\left[\frac{1 - \log x}{x^2}\right]$.

So $f'(x) = 0 \Leftrightarrow x = e$. Also $f'(x) > 0$ for $0 < x < e$ and $f'(x) < 0$ for $e < x < \infty$. Thus, $f(x)$ has a maximum at $x = e$ and $\max f(x) = e^{1/e}$

◉ **Example 29:** Let $P(x) = a_0 + a_1 x^2 + a_2 x^4 + \dots + a_n x^{2n}$ be a polynomial in a real variable x with $0 < a_0 < a_1 \dots < a_n$. The function $P(x)$ has

(a) neither a maximum nor a minimum
(b) only one maximum
(c) only one minimum
(d) none of these

Ans. (c)

◎ **Solution:** Since $P(x)$ is a polynomial, it is differentiable, and its extremum values are given by $P'(x) = 0$

$$\Rightarrow \quad 2a_1x + 4a_2x^3 + \dots + 2na_n x^{2n-1} = 0$$

$$\Rightarrow \quad 2x(a_1 + 2a_2x^2 + \dots + na_n x^{2n-2}) = 0$$

But, $a_1 > 0$ and other terms of $2a_2 x^2 + \dots + n\, a_n x^{2n-2}$ are non negative so we have $x = 0$. Also $P''(x) = 2a_1 + 12a_2 x^2 + \dots + 2n(2n-1)\, a_n x^{2n-2}$ so that $P''(0) = 2a_1 > 0$. Therefore, $P(x)$ has only one minimum at $x = 0$.

◉ **Example 30:** If $f(x) = \dfrac{x}{\sin x}$ and $g(x) = \dfrac{x}{\tan x}$, where $0 < x \leq 1$, then in this interval

(a) $f(x)$ and $g(x)$ are increasing functions
(b) both $f(x)$ and $g(x)$ are decreasing functions
(c) $f(x)$ is an increasing function
(d) $g(x)$ is an increasing function

Ans. (c)

◎ **Solution:** Let $f(x) = \dfrac{x}{\sin x} \Rightarrow f'(x) = \dfrac{\sin x - x\cos x}{\sin^2 x}$

Let $u(x) = \sin x - x\cos x$ so $u'(x) = \cos x - \cos x + x\sin x = x\sin x > 0$ for $0 < x \le 1$. Hence u is an increasing so $u(x) > u(0) = 0$. Thus $f'(x) > 0$ for $0 < x \le 1$. i.e. f is an increasing function. Now

$$g'(x) = \frac{\tan x - x\sec^2 x}{\tan^2 x}.$$

Let $v(x) = \tan x - x\sec^2 x$. Since $v'(x) = \sec^2 x - \sec^2 x - 2x\sec^2 x\tan x = -2x\sec^2 x\tan x < 0$ for $0 < x \le 1$. Hence $v(x) < v(0) = 0$ so $g'(x) < 0$ for $0 < x \le 1$. Thus g is a decreasing function.

Example 31: If $f(x) = x^{2/3}$ then

(a) (0, 0) is a point of maximum
(b) (0, 0) is not a point of minimum
(c) (0, 0) is a critical point
(d) There is no critical point

Ans. (c)

Solution: $\frac{dy}{dx} = \frac{2}{3}x^{-1/3}$. This derivative is never zero, but there is no derivative for $x = 0$. So (0, 0) is a critical point. If $x < 0$ then $\frac{dy}{dx} < 0$ and if $x > 0$ then $\frac{dy}{dx} > 0$. Thus (0, 0) is a point of minimum.

Example 32: Let $f(x) = (ax + b)/(cx + d)$ $(da - cb \ne 0, c \ne 0)$ then $f(x)$ has

(a) a critical point (b) no point of inflection
(c) a maximum (d) a minimum

Ans. (b)

Solution: $f'(x) = \frac{da - cb}{(cx + d)^2} \ne 0$ for any $x \in \mathbf{R}$ so f has no critical point in particular no minimum and no maximum. Now

$f''(x) = \frac{-2(da - cb)c}{(cx + d)^3} \ne 0$ for any $(x, f(x))$. Hence f has no point of inflection.

Example 33: Let $f(x) = (x - a)^m (x - b)^n$, where $m, n \in \mathbf{I}$ and $m, n > 1$. Then

(a) $(a, 0)$, $(b, 0)$ are the only critical points of f
(b) there are $m + n$ critical points of f
(c) there are exactly three critical points of f
(d) none of these

Ans. (c)

Solution: $f'(x) = m(x - a)^{m-1}(x - b)^n + n(x - a)^m(x - b)^{n-1}$

$= (x - a)^{m-1}(x - b)^{n-1}[(m + n)x - (mb + na)]$. Thus $f'(x) = 0 \Rightarrow x = a, x = b, x = \frac{mb + na}{m + n}$. So f has exactly three critical points.

Example 34: A ball is dropped from a platform 19.6 m high. Its position function is

(a) $x = -4.9t^2 + 19.6 (0 \le t \le 1)$
(b) $x = -4.9t^2 + 19.6 (0 \le t \le 2)$
(c) $x = -9.8t^2 + 19.6 (0 \le t \le 2)$
(d) $x = -4.9t + 19.6 (0 \le t \le 2)$

Ans. (b)

Solution: We have $a = \frac{d^2x}{dt^2} = -9.8$. The initial conditions are $x(0) = 19.6$ and $v(0) = 0$.

So $\quad v = \frac{dx}{dt} = -9.8t + v(0) = -9.8t$

$\therefore \quad x = -4.9t^2 + x(0) = -4.9t^2 + 19.6$

Now, the domain of the function is restricted since the ball hits the ground after a certain time. To find this time we set $x = 0$ and solve for t.

$$0 = -4.9t^2 + 19.6 \quad \Rightarrow \quad t = 2$$

Thus $\quad x = -4.9t^2 + 19.6 \quad (0 \le t \le 2)$

Example 35: Let $f(n) = 20n - n^2$ $(n = 1, 2, 3\ldots)$, then

(a) $f(n) \to \infty$ as $n \to \infty$
(b) $f(n)$ has no maximum
(c) The maximum value of $f(n)$ is greater than 200.
(d) None of these

Ans. (d)

Solution: Consider $f(x) = 20x - x^2$, defined for all real number x. $f'(x) = 20 - 2x$ and $f''(x) = -2$. Hence $x = 10$ is a point of maximum. Thus the maximum value of $f(n)$ is $200 - 100 = 100$.

Example 36: The point of the curve $y = x^2$ that is closest to $(4, -1/2)$ is

(a) (1, 1) (b) (2, 4)
(c) (2/3, 4/9) (d) (4/3, 16/9)

Ans. (a)

Solution: Let the required point be (x, y) on the curve so $d = \sqrt{(x-4)^2 + (y + 1/2)^2}$ should be minimum. It is enough to consider

$$D = (x - 4)^2 + (y + 1/2)^2 = (x - 4)^2 + (x^2 + 1/2)^2.$$

$D' = 4x^3 + 4x - 8$. Now for critical points

$D' = 0$ so $x^3 + x - 2 = 0 \quad \Rightarrow \quad x = 1$. Clearly D'' at $x = 1$ is $16 > 0$. Thus D is minimum when $x = 1$. So the required point is (1, 1)

Example 37: The smallest value of M such that $|x^2 - 3x + 2| \le M$ for all x in the interval $[1, 5/2]$ is

(a) 1/4 (b) 3/4
(c) 5/4 (d) 5/16

Ans. (b)

Solution: Consider $f(x) = x^2 - 3x + 2$ on $[1, 5/2]$. Now $f'(x) = 2x - 3$ so the only critical point is 3/2. Since $f(1) = 0$, $f(3/2) = -5/4$ and $f(5/2) = 3/4$. min $\{f(x): x \in [1, 5/2]\} = 0$, hence the

$\max\{|f(x)|: x \in [1, 5/2]\} = \max\{f(x): x \in [1, 5/2]\} = 3/4$. Thus $M = 3/4$.

◉ **Example 38:** The number of solutions of the equation $a^{f(x)} + g(x) = 0$, where $a > 0$, $g(x) \neq 0$ and $g(x)$ has minimum value 1/2, is

(a) one (b) two
(c) infinitely many (d) zero

Ans. (d)

◎ **Solution:** $a^{f(x)} = -g(x) \leq -1/2$, since $g(x) \geq 1/2$ for all x. But this is not possible as $a^{f(x)} > 0$ for all x. Thus the number of solution is zero.

◉ **Example 39:** The minimum value of $f(x) = |3 - x| + |2 + x| + |5 - x|$ is

(a) 0 (b) 7
(c) 8 (d) 10

◎ **Solution:**

$$f(x) = \begin{cases} 6 - 3x, & x < -2 \\ 10 - x, & -2 \leq x < 3 \\ x + 4, & 3 \leq x < 5 \\ 3x - 6, & x \geq 5 \end{cases}$$

The function f is decreasing for $x \in (-\infty, 3)$ and increases on $(3, \infty)$. Hence $x = 3$ is a point of minimum and $\min f(x) = 7$.

◉ **Example 40:** If $f(x) = x(x - 2)(x - 4)$, $1 \leq x \leq 4$, then a number satisfying the conclusion of the mean value theorem is

(a) 1 (b) 2
(c) 5/2 (d) 7/2

Ans. (a)

◎ **Solution:** $f'(x) = (x - 2)(x - 4) + x(x - 4) + x(x - 2) = 3x^2 - 12x + 8$. Also $f(4) = 0$ and $f(1) = 3$. Thus $\dfrac{f(4) - f(1)}{4 - 1} = -1$. We must have $-1 = f'(x)$

$$\Rightarrow \quad 3x^2 - 12x + 9 = 0$$
$$\Rightarrow \quad x^2 - 4x + 3 = 0$$
$$\Rightarrow \quad x = 1 \quad \text{or} \quad x = 3.$$

◉ **Example 41:** The sum of the intercepts of a tangent to $\sqrt{x} + \sqrt{y} = \sqrt{a}$, $a > 0$ upon the coordinate axes is

(a) $2a$ (b) a
(c) $a/2$ (d) $\sqrt{a}$

Ans. (b)

◎ **Solution:** $\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}$.

Equation of tangent at any point (x, y) of the curve is $Y - y = -\dfrac{\sqrt{y}}{\sqrt{x}}(X - x)$. So intercepts of X-axis and Y-axis are $x + \sqrt{xy}$ and $y + \sqrt{xy}$.

Therefore, the sum of intercepts $= x + y + 2\sqrt{xy} = \left(\sqrt{x} + \sqrt{y}\right)^2 = a$.

◉ **Example 42:** Let x and y be two real numbers such that $x > 0$ and $xy = 1$. The minimum value of $x + y$ is

(a) 1 (b) 1/2
(c) 2 (d) 1/4

Ans. (c)

◎ **Solution:** If $xy = 1$ then $S = x + y = x + 1/x$ and $S'(x) = 1 - 1/x^2 \Rightarrow S'(x) = 0 \Leftrightarrow x = 1$ (as $x > 0$). Also $S''(x) = 2/x^3 > 0$ for $x = 1$. Hence minimum value of $x + y$ is 2.

Alternatively $S = x + \dfrac{1}{x} = \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2 + 2 \geq 2$ and $S = 2$ for $x = 1$. Hence the minimum value of S is 2.

◉ **Example 43:** The function $\dfrac{\sin(x + \alpha)}{\sin(x + \beta)}$ has no maximum or minimum if (k an integer)

(a) $\beta - \alpha = k\pi$ (b) $\beta - \alpha \neq k\pi$
(c) $\beta - \alpha = 2k\pi$ (d) none of the above

Ans. (b)

◎ **Solution:** Let $f(x) = \dfrac{\sin(x + \alpha)}{\sin(x + \beta)}$, so that

$$f'(x) = \frac{\sin(x + \beta)\cos(x + \alpha) - \cos(x + \beta)\sin(x + \alpha)}{\sin^2(x + \beta)}$$
$$= \frac{\sin(\alpha - \beta)}{\sin^2(x + \beta)}$$

Thus, $f(x)$ has no maximum or minimum if $f'(x) \neq 0$. i.e. if $\sin(\beta - \alpha) \neq 0$ equivalently $\beta - \alpha \neq k\pi$, $k \in \mathbf{I}$.

◉ **Example 44:** The tangent to the curve $y = x^3 - 6x^2 + 9x + 4$, $0 \leq x \leq 5$ has maximum slope at x which is equal to

(a) 2 (b) 3
(c) 4 (d) none of these

Ans. (d)

◎ **Solution:** $u = \dfrac{dy}{dx} = 3x^2 - 12x + 9$. For u to be maximum or minimum $\dfrac{du}{dx} = 0$ which is true if and only if $6x - 12 = 0 \Leftrightarrow x = 2$. Now $u(0) = 9$, $u(2) = -3$ and $u(5) = 24$. Thus the maximum of $u(x)$, $0 \leq x \leq 5$ is $u(5)$ so $x = 5$.

◉ **Example 45:** The set of all values of the parameters a for which the points of minimum of the function $y = 1 + a^2x - x^3$ satisfy the inequality $\dfrac{x^2 + x + 2}{x^2 + 5x + 6} \leq 0$ is

(a) an empty set

(b) $\left(-3\sqrt{3}, -2\sqrt{3}\right)$

(c) $\left(2\sqrt{3}, 3\sqrt{3}\right)$

(d) $\left(-3\sqrt{3},-2\sqrt{3}\right)\cup\left(2\sqrt{3},3\sqrt{3}\right)$

Ans. (c)

◎ **Solution:** $\frac{dy}{dx}=a^2-3x^2=0 \Leftrightarrow x=\pm a/\sqrt{3}$. Since $\frac{d^2y}{dx^2} = -6x$ so y is minimum for $x=-a/\sqrt{3}$.

Since $x^2+x+2>0$ for all x so for $\frac{x^2+x+2}{x^2+5x+6}\le 0$, we must have $x^2+5x+6<0$. If $x=-a/\sqrt{3}$, we have $a^2/3-5a/\sqrt{3}+6<0$ i.e. $a^2-5\sqrt{3}\,a+18<0 \Leftrightarrow (a-2\sqrt{3})(a-3\sqrt{3})<0$ i.e. $a\in(2\sqrt{3},3\sqrt{3})$.

◉ **Example 46:** The point of intersection of the tangents drawn to the curve $x^2y=1-y$ at the points where it is met by the curve $xy=1-y$ is given by

(a) $(0,-1)$
(b) $(1,1)$
(c) $(0,1)$
(d) none of these

Ans. (c)

◎ **Solution:** The intersection of $x^2y=1-y$ and $xy=1-y$ are given by $x^2y=xy$. Since $y\neq 0$ on any of two curves so we must have $x=0,1$. Thus the curves intersects at $(0,1)$; $(1,1/2)$.

Now, differentiating $x^2y=1-y$, we have $\frac{dy}{dx}=-\frac{2xy}{1+x^2}$. Thus $\left.\frac{dy}{dx}\right|_{(0,1)}=0$ and $\left.\frac{dy}{dx}\right|_{(1,1/2)}=-\frac{1}{2}$. Equations of tangents are $Y=1$ and $Y-1/2=(-1/2)(X-1)$. Their intersection is given by $(0,1)$.

◉ **Example 47:** The equation of the tangent to the curve $y=(2x-1)\,e^{2(1-x)}$ at the point of its maximum is

(a) $y=1$
(b) $x=1$
(c) $x+y=1$
(d) $x-y=-1$

Ans. (a)

◎ **Solution:** $y'(x)=-2\,(2x-1)\,e^{2(1-x)}+2e^{2(1-x)}=2e^{2(1-x)}(-2x+2)$

Thus $y'(x)=0 \Rightarrow x=1$. Since, $y''(1)<0$ so $(1,1)$ is the point of maximum and an equation of tangent to the curve is $y-1=0\,(x-1)$, i.e. $y=1$.

◉ **Example 48:** If the function $f(x)=x^2+\alpha/x$ has a local minimum at $x=2$, then the value of α is

(a) 8
(b) 18
(c) 16
(d) none of these

Ans. (c)

◎ **Solution:** $f'(x)=2x-\alpha/x^2$, so $f'(x)=0 \Leftrightarrow 2x=\alpha/x^2 \Leftrightarrow x=(\alpha/2)^{1/3}$. Clearly for $x=(\alpha/2)^{1/3}$, we have $f''(x)>0$. So $2=(\alpha/2)^{1/3}\Rightarrow \alpha=16$.

◉ **Example 49:** Three normals are drawn to the parabola $y^2=4x$ from the point $(c,0)$. These normals are real and distinct when

(a) $c=0$
(b) $c=1$
(c) $c=2$
(d) $c=3$

Ans. (d)

◎ **Solution:** Any point on $y^2=4x$ is $(t^2,2t)$. Since $\frac{dy}{dx}=\frac{2}{y}$ so $\left.\frac{dy}{dx}\right|_{(t^2,2t)}=\frac{2}{2t}=\frac{1}{t}$. Hence equation of normal at $(t^2,2t)$ is

$$Y-2t=-t\,(X-t^2)$$

This passes through $(c,0)$ if $-2t=-t\,(c-t^2)$

$$\Rightarrow \quad t=0,\ t^2=c-2$$

Thus the roots are real and distinct if $c>2$ so $c=3$ is the correct choice.

◉ **Example 50:** The function $f(x)=(\log(x-1))^2\,(x-1)^2$ has

(a) local extremum at $x=1$
(b) point of inflection at $x=1$
(c) local extremum at $x=2$
(d) point of inflection at $x=2$

Ans. (c)

◎ **Solution:** $f'(x)=(x-1)^2\,2\log(x-1)\cdot 1/(x-1)+2(\log(x-1))^2(x-1)$

$$=2\,(x-1)\,[\log(x-1)]\,[1+\log(x-1)]$$

$f'(x)=0 \Rightarrow x=2$ or $1+e^{-1}$, since the domain of f is $\{x : x>1\}$.

$$f''(x)=2\log(x-1)\,(1+\log(x-1))+2\,(x-1)\times\left[\frac{1}{x-1}+\frac{2}{x-1}\log(x-1)\right]$$

$f''(2)=2>0$. Hence $x=2$ is a point of extremum.

◉ **Example 51:** If $f(x)=\log x$ satisfies Lagrange's theorem on $[1,e]$ then value of $c\in(1,e)$ such that the tangent at c is parallel to line joining $(1,f(1))$ and $(e,f(e))$ is

(a) $e-\frac{3}{2}$
(b) $\frac{1+e}{2}$
(c) $e-1$
(d) $e-\frac{1}{2}$

Ans. (c)

◎ **Solution:** $f'(c)=\frac{1}{c}$

So $\frac{f(e)-f(1)}{e-1}=f'(c)=\frac{1}{c}$

$$\Rightarrow \quad \frac{\log e-\log 1}{e-1}=\frac{1}{c}$$

$$\Rightarrow \quad \frac{1}{e-1}=\frac{1}{c} \quad \Rightarrow \quad c=e-1$$

Example 52: The value of c for which the conclusion of Lagrange's theorem holds for the function $f(x) = \sqrt{a^2 - x^2}, a > 1$ on the interval [1, a] is

(a) $\dfrac{a(a+1)}{2}$ (b) $\dfrac{1+a}{2}$

(c) $\dfrac{\sqrt{a(a+1)}}{2}$ (d) $\dfrac{a(a-1)}{2}$

Ans. (c)

Solution: $\dfrac{f(a) - f(1)}{a-1} = f'(c) = \dfrac{-c}{\sqrt{a^2 - c^2}}$

$$\frac{\sqrt{a^2 - a^2} - \sqrt{a^2 - 1}}{a-1} = \frac{-c}{\sqrt{a^2 - c^2}}$$

$$\Rightarrow \quad -\sqrt{\frac{a+1}{a-1}} = -\frac{c}{\sqrt{a^2 - c^2}}$$

$$\Rightarrow \quad \frac{a+1}{a-1} = \frac{c^2}{a^2 - c^2}$$

$$\Rightarrow \quad (a + 1)(a^2 - c^2) = c^2(a - 1)$$

$$\Rightarrow \quad c^2(-a - 1 + 1 - a) = -(a + 1)a^2$$

$$\Rightarrow \quad c^2(2a) = (a + 1)a^2$$

$$\Rightarrow \quad c^2 = \frac{(a+1)a}{2}$$

Since $c \in (1, a)$ so $c = \sqrt{\dfrac{(a+1)a}{2}}$

Example 53: Let $f(x) = \begin{cases} |x-2| + a, & \text{if } x \le 2 \\ 4x^2 + 3x + 1, & \text{if } x > 2 \end{cases}$

If $f(x)$ has a local minimum at $x = 2$, then

(a) $a > 21$ (b) $a \le 21$

(c) $a > 30$ (d) $a > 24$

Ans. (b)

Solution: Since f has a local minimum at $x = 2$

So $f(2) \le f(x)$ for x in an open interval around 2.

$$\Rightarrow \quad f(2) \le \lim_{x \to 2+} f(x)$$

Thus, $a = f(2) \le \lim\limits_{x \to 2} 4x^2 + 3x + 1 = 21$

Example 54 If $y = mx + 2$ is parallel to a tangent to curve $e^{4y} = 1 + 16x^2$ then

(a) $|m| \le 1$ (b) $|m| < 1$

(c) $|m| > 1$ (d) $|m| \ge 1$

Ans. (a)

Solution Differentiating, we have

$$4e^{4y}\frac{dy}{dx} = 32x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{8x}{1+16x^2}$$

According to the given condition

$$m = \frac{8x}{1+16x^2}$$

$$|m| = \left|\frac{8x}{1+16x^2}\right| \le 1 \ \ (\text{since } 16x^2 - 8\,|x| + 1 = (4\,|x| - 1)^2 \ge 0)$$

Example 55: Given the function $f(x) = x^2 e^{-2x}$, $x > 0$. Then $f(x)$ has the maximum value equal to

(a) e^{-2} (b) $(2e)^{-1}$

(c) e^{-1} (d) none of these.

Ans. (a)

Solution: $f'(x) = 2xe^{-2x} - 2x^2e^{-2x} = 2(1 - x)x\,e^{-2x}$. Now, $f'(x) = 0$ $\Leftrightarrow$ $x = 1, 0$. Also $f''(x) = 2(1 - x)e^{-2x} - 2xe^{-2x} - 4(1 - x)xe^{-2x}$, so $f''(1) = -2e^{-2} < 0$ and $f''(0) > 0$. Thus max $f(x) = f(1) = e^{-2}$.

Example 56: Let $f(x) = (x - 4)(x - 5)(x - 6)(x - 7)$ then

(a) $f'(x) = 0$ has four real roots

(b) three roots of $f'(x) = 0$ lie in $(4, 5) \cup (5, 6) \cup (6, 7)$

(c) the equation $f'(x) =$ has only two roots

(d) three roots of $f'(x) = 0$ lie in $(3, 4) \cup (4, 5) \cup (5, 6)$

Ans. (b)

Solution: Since $f(4) = f(5) = f(6) = f(7) = 0$, so by Rolle's theorem applied to the intervals [4, 5], [5, 6], [6, 7] there exist $x_1 \in (4, 5)$, $x_2 \in (5, 6)$, $x_3 \in (6,7)$ such that $f'(x_1) = f'(x_2) = f'(x_3) = 0$. Since f' is a polynomial of degree 3 so cannot have four roots.

Example 57: If $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ for every real number x, then the minimum value of f

(a) does not exist because f is unbounded

(b) is not attained even though f is bounded

(c) is equal to 1

(d) is equal to – 1

Ans. (d)

Solution: We have $f(x) = \dfrac{x^2 - 1}{x^2 + 1} = 1 - \dfrac{2}{x^2 + 1}$

$f(x)$ will attain its minimum value when $2/(x^2 + 1)$ is maximum i.e. when $x^2 + 1$ is minimum i.e. at $x = 0$. Thus min $f(x) = f(0) = -1$. Since $-1 \le f(x) < 1$ so f is bounded.

Example 58: For all $x \in (0, 1)$

(a) $e^x < 1 + x$ (b) $\log_e(1 + x) < x$

(c) $\sin x > x$ (d) $\log_e x > x$

Ans. (b)

Solution: Since $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} > 1 + x$ for $x > 0$ so (*a*) is not true. Also $\log x$ is an increasing function so $x > \log(1 + x)$. Again $e^x > x$ for $x > 0$ so $x > \log_e x$. We know that

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots = x - \left(\frac{x^3}{3!} - \frac{x^5}{5!} + \ldots\right)$ The terms inside the brackets are positive since $x \in (0, 1)$ so $\sin x < x$.

◉ **Example 59:** Let f be a differentiable function on **R** and $h(x) = f(x) - (f(x))^2 + (f(x))^3$ for all $x \in$ **R**. Then

(a) h increases whenever f decreases
(b) h decreases whenever f increases
(c) h increases or decreases accordingly as f increases or decreases
(d) nothing can be claimed in general

Ans. (c)

◎ **Solution:** $h'(x) = f'(x) - 2f(x)f'(x) + 3(f(x))^2 f'(x)$

$$= f'(x)\,[1 - 2f(x) + 3(f(x))^2]$$

Since $1 - 2f(x) + 3(f(x))^2 = 3\left[\left(f(x) - \frac{1}{3}\right)^2 + \frac{2}{9}\right] > 0$ for all x

so $h'(x) > 0$ or < 0 accordingly as $f'(x) > 0$ or < 0. Hence h increases or decreases according as f increases or decreases.

◉ **Example 60:** Let $f(x) = ax^3 + bx^2 + cx + d$, $b^2 - 3ac > 0$, $a > 0$, $c < 0$. Then $f(x)$ has

(a) local maximum at some $x \in \mathbf{R}^+$
(b) a local maximum at some $x \in \mathbf{R}^-$
(c) a local minima at $x = 0$
(d) local minima at some $x \in \mathbf{R}^-$, $\mathbf{R}^+ = (0, \infty)$. $\mathbf{R}^- = (-\infty, 0)$

Ans. (b)

◎ **Solution:** $f'(x) = 3ax^2 + 2bx + C$. The discriminant $D = 4b^2 - 12ac = 4(b^2 - 3ac) > 0$

So $f'(x) = 0$ has exactly two real roots say α, β so $f'(x) = 3a(x - \alpha)(x - \beta)$. But $\frac{c}{a} < 0 \Rightarrow \alpha\beta < 0$. So we have $\alpha < 0$, $\beta > 0$ or $\alpha > 0$, $\beta > 0$. If $\alpha < 0$, $\beta > 0$, for $x < \alpha$, $f'(x) > 0$ and for $x > \alpha$ but near to α, $f'(x) > 0$. Also for $x < \beta$, but close to β, $f'(x) < 0$ and for $x > \beta$, $f'(x) > 0$. Thus f has local maxima at $x = \alpha$ which is $\mathbf{R}^-$ and f has local minima at $x = \beta \in \mathbf{R}^+$
In case $\alpha > 0$, $\beta < 0$, our conclusion is same *i.e.*, local minima at $x = \alpha$ and local maxima at $x = \beta$.

◉ **Example 61:** If $f(x) = \begin{cases} 3 - x^2, & x \le 2 \\ \sqrt{a + 14} - |x - 48|, & x > 2 \end{cases}$ and $f(x)$ has a local maxima at $x = 2$, then

(a) a cannot be determined
(b) least value of a is 2011
(c) greatest value of a is 2011
(d) $a \ge 3010$

Ans. (c)

◎ **Solution:** Since f has local maxima at $x = 2$, so $\lim_{h \to 0} f(2 + h) \le f(2)$

Since $\lim_{h \to 0-} f(2 + h) = \lim_{h \to 0-} 3 - (2 + h)^2 = f(2)$

So we must have $\lim_{h \to 0+} f(2 + h) \le f(2)$

$$\Rightarrow \quad \lim_{h \to 0+} \sqrt{a + 14} - |2 + h - 48| \le -1$$

$$\Rightarrow \quad \sqrt{a + 14} \le 46 - 1 = 45$$

$$\Rightarrow \quad a \le 2011.$$

The greatest value of a is 2011.

◉ **Example 62:** The total number of local maxima and local minima of the function

$$f(x) = \begin{cases} (2 + x)^3, & -3 < x \le -1 \\ x^{2/3}, & - < x < 2 \end{cases}$$

(a) 0 (b) 1
(c) 2 (d) 3

Ans. (c)

◎ **Solution:** Since $\lim_{x \to -1-} f(x) = \lim_{x \to -1+} f(x) = 1$

So, f is continuous on $(-3, 2)$. Also

$$f'(x) = \begin{cases} 3(x + 2)^2, & -3 < x < -1 \\ (2/3)\,x^{-1/3}, & -1 < x < 0, 0 < x < 2 \end{cases}$$

$f'(0)$ does not exist

$$\mathrm{L}f'(-1) = \lim_{x \to -1-} \frac{f(x) - f(1)}{x + 1} = \lim_{x \to -1-} \frac{(x + 2)^3 - 1}{x + 1}$$

$$= \lim_{x \to -1-} \frac{(x + 1)\left[(x + 2)^2 + (x + 2) + 1\right]}{x + 1} = 3.$$

$$\mathrm{R}f'(-1) = \lim_{x \to -1+} \frac{x^{2/3} - 1}{x + 1}$$

$$= \lim_{x \to 1+} \frac{\left(x^{1/3} - 1\right)\left(x^{1/3} + 1\right)}{\left(x^{1/3} + 1\right)\left(x^{2/3} + x^{1/3} + 1\right)} = -2$$

So f is not differentiable at $x = -1$
For $x \ne 0, -1$, $f'(x) \ne 0$. Critical points of f are -1, 0. At $x = -1$, f has a local maximum and at $x = 0$, f has a local minimum.

◉ **Example 63:** Let $g : \mathbf{R} \to (-\pi/2, \pi/2)$ be given by $g(u) = 2\tan^{-1}(e^u) - \pi/2$, then g is

(a) even and is strictly increasing in $(0, \infty)$
(b) odd and is strictly decreasing in $(-\infty, \infty)$
(c) odd and is strictly increasing in $(-\infty, \infty)$
(d) neither even nor odd, but is strictly increasing in $(-\infty, \infty)$.

Ans. (c)

◎ **Solution:** $g(-u) = 2\tan^{-1}\left(\frac{1}{e^u}\right) - \pi/2$

$$= 2\cot^{-1} e^u - \pi/2$$

$$= 2\left[\pi/2 - \tan^{-1} e^u\right] - \pi/2$$

$$= \pi/2 - \tan^{-1} e^u = -g(u)$$

Also, $g'(u) = \dfrac{2\,e^u}{1+e^{2u}} > 0 \ \forall\ u \in \mathbf{R}$

Thus g is an odd function which is strictly increasing on **R**.

◉ **Example 64:** Let $f(x) = \begin{cases} x^{\alpha} \log x & , \text{ if } x > 0 \\ 0 & , \text{ if } x = 0 \end{cases}$

If Rolle's theorem can be applied to f on [0, 1] then value of α can be

(a) – 1 (b) – 1/2
(c) 0 (d) 1/2

Ans. (d)

◎ **Solution:** For the Rolle's theorem to be applicable on [0, 1], we must have that f is continuous on [0, 1] and in particular at $x = 0$ from the right.

So $\lim\limits_{x \to 0+} x^{\alpha} \log x = f(0) = 0$

$$\Rightarrow \quad \lim_{x \to 0+} \frac{\log x}{x^{-\alpha}} = 0$$

$$\Rightarrow \quad \lim_{x \to 0+} \frac{(1/x)}{(-\alpha) x^{-\alpha-1}} = 0 \quad \Rightarrow \quad \lim_{x \to 0+} \frac{x^{\alpha}}{-\alpha} = 0$$

This is possible only if $\alpha > 0$.

◉ **Example 65:** Suppose that $f(x) = x^3 - px + q$ has three distinct real roots where $p, q > 0$. Then which one of the following hold?

(a) $f(x)$ has minima at $\sqrt{\dfrac{p}{3}}$ and maxima at $-\sqrt{\dfrac{p}{3}}$.

(b) $f(x)$ has minima at $-\sqrt{\dfrac{p}{3}}$ and maxima at $\sqrt{\dfrac{p}{3}}$.

(c) $f(x)$ has minima at both $\sqrt{\dfrac{p}{3}}$ and $-\sqrt{\dfrac{p}{3}}$.

(d) $f(x)$ has maxima at both $\sqrt{\dfrac{p}{3}}$ and $-\sqrt{\dfrac{p}{3}}$.

Ans. (a)

◎ **Solution:** As $f(x) = 0$ has three real and distinct zero so $f'(x) = 3x^2 - p = 0$ has two real and distinct zeros by Rolle's Theorem.

So $x = \pm\sqrt{\dfrac{p}{3}}$.

$f'(x) > 0$ if $x < -\sqrt{\dfrac{p}{3}}$; $f'(x) < 0$ if $-\sqrt{\dfrac{p}{3}} < x < \sqrt{\dfrac{p}{3}}$;

$f'(x) > 0$ if $x > \sqrt{\dfrac{p}{3}}$.

Thus $f(x)$ has a local maximum at $x = -\sqrt{\dfrac{p}{3}}$ and a local minimum at $x = \sqrt{\dfrac{p}{3}}$.

◉ **Example 66:** Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$.
If $P(-1) < P(1)$, then in the interval [– 1, 1]

(a) $P(-1)$ is the minimum but $P(1)$ is not the maximum of P
(b) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P
(c) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
(d) $P(-1)$ is not minimum but $P(1)$ is the maximum of P

Ans. (d)

◎ **Solution:** $P'(x) = 4x^3 + 3ax^2 + 2\,bx + c$

As $P'(0) = 0$ so $c = 0$

$\therefore$ $P'(x) = x(4x^2 + 3ax + 2b)$

As $x = 0$ is the only real root of $P'(x) = 0$, roots of $4x^2 + 3ax + 2b$ must have imaginary roots,
therefore $4x^2 + 3ax + 2\,b > 0 \ \forall\ x \in \mathbb{R}$

Thus $P'(x) < 0$ for $x < 0$
> 0 for $x > 0$

Therefore, $x = 0$ is a point of local minimum at $x = 0$ As $P(-1) < P(1)$, we get $P(1)$ is maximum but $P(-1)$ is not minimum of P on [– 1, 1].

◉ **Example 67:** Let $f: \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} k - 2x & , \text{ if } x \le -1 \\ 2x + 3 & , \text{ if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k is

(a) –1/2 (b) – 1
(c) 1 (d) 0

Ans. (b)

◎ **Solution:** If f has a local minimum at $x = -1$ then $\lim\limits_{h \to 0} f(-1+h) \ge f(-1) = k + 2$

$$\Rightarrow \quad \lim_{h \to 0+} f(-1+h) \ge k + 2$$

$$\Rightarrow \quad \lim_{h \to 0+} 2(-1+h) + 3 \ge k + 2$$

$\Rightarrow \quad 1 \ge k + 2 \Rightarrow k \le -1.$

◉ **Example 68:** The value of a in order that $f(x) = \sin x - \cos x - ax + b$ decreases for all real values of x is given by

(a) $a \ge \sqrt{2}$ (b) $a < \sqrt{2}$
(c) $a \ge 1$ (d) $a < 1$

Ans. (a)

Solution: $f'(x) = \cos x + \sin x - a$

$$= \sqrt{2}\sin(x + \pi/4) - a$$

$f'(x) \le 0$ for all x if

$$a \ge \max_{x \in \mathbf{R}} \sqrt{2}\sin(x + \pi/4) = \sqrt{2}$$

Example 69: The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact, is given by :

(a) $2y - 3x = 0$ (b) $y = \dfrac{6}{x}$

(c) $x^2 + y^2 = 13$ (d) $\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{3}\right)^2 = 2$

Ans. (b)

Solution: The equation of tangent at (x, y) to the curve $y = f(x)$ is given by

$$Y - y = \frac{dy}{dx}(X - x)$$

This tangent meets the axes at $A\left(x - y\dfrac{dx}{dy}, 0\right)$ and $B\left(0, y - x\dfrac{dy}{dx}\right)$. We are given mid point of AB as (x, y) so,

$$\frac{1}{2}\left(x - y\frac{dx}{dy}\right) = x \text{ and } \frac{1}{2}\left(y - \frac{xdy}{dx}\right) = y$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} = 0 \quad \Rightarrow \quad \log|xy| = C$$

$$\Rightarrow \quad xy = \pm e^c = A$$

As it passes through (2, 3) so $A = 6$

$\therefore$ $xy = 6$ or $y = 6/x$

Example 70: A spherical balloon is filled with 4500 π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72 π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is :

(a) 7/9 (b) 2/9

(c) 9/2 (d) 9/7

Ans. (b)

Solution: Let r be the radius of the balloon and V be its volume. It is given that

$$\frac{dV}{dt} = -72\pi$$

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -72\pi \Rightarrow \frac{4}{3}\pi r^3 = -72\pi t + C$$

When $t = 0$, $\dfrac{4}{3}\pi r^3 = V = 4500\pi$

$$\therefore \quad \frac{4}{3}\pi r^3 = 4500\pi - 72\pi t$$

When $t = 49$, $\dfrac{4}{3}\pi r^3 = 4500\pi - (72\pi)(49) = 972\pi$

$\Rightarrow$ $r^3 = 729$ so $r = 9$

Also, $\dfrac{d}{dt}\left(\dfrac{4}{3}\pi r^3\right) = -72\pi \Rightarrow 4\pi r^2 \dfrac{dr}{dt} = -72\pi$

$$\Rightarrow \quad \frac{dr}{dt} = -\frac{72\pi}{4\pi r^2} = -\frac{18}{r^2}. \text{ So } \left.\frac{dr}{dt}\right|_{r=9} = -\frac{2}{9}.$$

Assertion-Reason Type Questions

Example 71: Let $f(x) = 2\sqrt{x}$ and $g(x) = 3 - 1/x$, $x > 1$

Statement-1: $f(x) > g(x)$ $(x > 1)$

Statement-2: $f(x) - g(x)$ increases on $(1, \infty)$

Ans. (a)

Solution: $f'(x) - g'(x) = \dfrac{x^{3/2} - 1}{x^2} > 0$ for $x > 1$

Hence $f(x) - g(x) > f(1) - g(1) = 0$, $x > 1$

Therefore $f(x) > g(x)$, $x > 1$.

Example 72: Let $f(x) = x^4 - 2x^2 + 5$ be defined on $[-2, 2]$

Statement-1: The range of $f(x)$ is [2, 13]

Statement-2: The greatest value of f is attained at $x = 2$

Ans. (d)

Solution: $f'(x) = 4x^3 - 4x = 4x(x - 1)(x + 1)$. The critical points of f are 0, – 1, 1 But $f(0) = 5$, $f(1) = 4$, $f(-1) = 4$, $f(2) = 13$. So the range of f is [4, 13] and grea, test value of f is at $x = 2$.

Example 73: Let $f(x) = 2x^3 + 3x^2 - 12x + 1$

Statement-1: f decreases on $(-2, 1)$

Statement-2: The solution set of $x^2 + x - 2 < 0$ is $(-2, 1)$.

Ans. (a)

Solution: $f'(x) = 6x^2 + 6x - 12 = 6(x - 1)(x + 2)$

So $f'(x) < 0$ if and only if $x \in (-2, 1)$.

Example 74: Let $y = x + \dfrac{a^2}{x}$ $(a > 0)$

Statement-1: $y_{\max} = -2a$ (local mix)

Statement-2: $y_{\min} = 2a$ (local min)

Ans. (b)

◎ **Solution:** $y'(x) = 1 - \frac{a^2}{x^2} = \frac{(x-a)(x+a)}{x^2}$

⇒ $y'(x) > 0$ for $x \in (a, \infty) \cup (-\infty, -a)$ and $y'(x) < 0$ for $x \in (-a, a)$. Therefore y is local maximum at $x = -a$ and loval minimum at $x = a$.

◉ **Example 75:** Let $f(x) = (x-1)(x-2)(x-3)(x-4)$

Statement-1: $f'(x) = 0$ has three roots lying in $(1, 2) \cup (2, 3) \cup (3, 4)$

Statement-2: Rolle's theorem is valid for f on $[1, 4]$

Ans. (b)

◎ **Solution:** Applying Rolle's theorem on [1, 2], [2, 3], [3, 4], we get three roots of $f'(x) = 0$ on (1, 2) (2, 3), (3, 4).

◉ **Example 76:** Let f be a function defined by

$$f(x) = \begin{cases} \frac{\tan x}{x} & , \text{ if } x \neq 0 \\ 1 & , \text{ if } x = 0 \end{cases}$$

Statement-1: $x = 0$ is point of maxima of f

Statement-2: $f'(0) = 0$

Ans (b)

◎ **Solution:** Let $g(x) = \tan x - x$, $-\pi/2 < x < x/2$

⇒ $g'(x) = \text{see}^2 x - 1 = \tan^2 x > 0$ for $-\pi/2 < x < \pi/2$

⇒ g increases on $(-\pi/2, \pi/2)$

⇒ $\tan x < x$ for $-\pi/2 < x < \pi/2$

Thus $f(x) = \frac{\tan x}{x} < 1$ for $-\pi/2 < x < \pi/2$, $x \neq 0$

and $f(1) = 1$, ∴ $x = 0$ in point of maxima.

Also $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{\tan x - x}{x^2}$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{2x} = \frac{1}{2} \lim \frac{\tan^2 x}{x} = 0$$

Hence $f'(0) = 0$

Thus, both the statements are true. However statement: 2 in not a correct explanation of statement: 1.

◉ **Example 77:** Let $a, b \in \mathbf{R}$ be such that the function f given by $f(x) = \log |x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$

Statement-1: f has local maximum at $x = -1$ and at $x = 2$

Statement-2: $a = 1/2$ and $b = -1/4$.

Ans. (a)

◎ **Solution:** $f'(x) = \frac{1}{x} + 2bx + a$, $x \neq 0$

As $x = -1$ and $x = 2$ are extreme value of f, $f'(-1) = 0$ and $f'(2) = 0$

⇒ $-1 - 2b + a = 0$ and $\frac{1}{2} + 4b + a = 0$

Solving these equations, we get $a = 1/2$, $b = -1/4$

Thus $f'(x) = \frac{1}{x} - \frac{1}{2}x + \frac{1}{2} = -\frac{1}{2x}(x+1)(x-2)$.

We have $f'(x) > 0$, if $x < -1$; $f'(x) < 0$, if $-1 < x < 0$; $f'(x) > 0$, if $0 < x < 2$ and $f'(x) < 0$, if $x > 2$.

Therefore, $f(x)$ has a local maximum at $x = -1$ and $x = 2$. Thus statement-1 and statement-2 are True and statement-2 is a correct explanation for statement :1.

◉ **Example 78:** **Statement 1:** $x^y > y^x$, $e < x < y$

Statement 2: $f(x) = \frac{\log x}{x}$ is a decreasing function for $x > e$.

Ans. (a)

◎ **Solution:** Let $f(x) = \frac{\log x}{x}$

$$f'(x) = \frac{1 - \log x}{x^2}$$

Thus $f'(x) > 0$ if $1 - \log x > 0$ i.e. $\log x < 1$ i.e. $x < e$ and $f'(x) < 0$ if $x > e$.

So f increases on $(0, e)$ and decreases on (e, ∞)

$$e < x < y \Rightarrow f(x) > f(y) \quad \Rightarrow \frac{\log x}{x} > \frac{\log y}{y}$$

⇒ $y \log x > x \log y$ ⇒ $x^y > y^x$.

◉ **Example 79:** Let $f(x) = |x - 7| + |x - 10| + |x - 12|$

Statement 1: $f(x)$ has a minimum at $x = 12$

Statement 2: f is not differentiable at $x = 12$

Ans. (d)

◎ **Solution:** $f(x) = \begin{cases} 29 - 3x & , \quad x < 7 \\ 15 - x & , \quad 7 \leq x < 10 \\ x - 5 & , \quad 10 \leq x < 12 \\ 3x - 29 & , \quad x \geq 12 \end{cases}$

Since $f'(x)$ changes sign around $x = 10$ from negative to positive f has minimum at $x = 10$.

f is clearly not differentiable at $x = 12$.

◉ **Example 80:** Let b, c be two non-zero real numbers such that $b^2 \leq 3c$

Let $f(x) = x^3 + bx^2 + cx + d$, $x \in \mathbf{R}$.

Statement 1: f is 1–1 function

Statement 2: f is strictly function

Ans. (c)

◎ **Solution:** $f'(x) = 3x^2 + 2bx + c$

The discriminant of f' is $4b^2 - 12c = 4(b^2 - 3c) < 0$.

So, $f'(x) > 0$. Hence f is an increasing function.

If there exists x_1, x_2 s.t. $x_1 \neq x_2$ (say $x_1 < x_2$) and $f(x_1) = f(x_2)$ then by Rolle's theorem there is $c \in (x_1, x_2)$ such that $f'(c) = 0$ but $f'(x) > 0$ for all x. So f is 1 – 1.

LEVEL 2

Straight Objective Type Questions

Example 81: The point M (x, y) of the graph of the function $y = e^{-|x|}$ so that area bounded by the tangent at M and the coordinate axes is greatest is

(a) $(1, e^{-1})$ (b) $(2, e^{-2})$
(c) $(-2, e^{2})$ (d) $(0, 1)$

Ans. (a)

Solution: For $x \geq 0$, $y = e^{-x}$. The equation of tangent is $Y - y = -e^{-x}(X - x)$. This will intersect coordinate axes at $(x + ye^{x}, 0)$ and $(0, y + xe^{-x})$. Hence the area of the required triangle A is $\frac{1}{2}(y + xe^{-x})(x + ye^{x})$

$$= \frac{1}{2}(1 + x)^2 e^{-x} \qquad [\because y = e^{-x}]$$

Now $$\frac{dA}{dx} = \frac{1}{2}[-(1 + x)^2 e^{-x} + 2(1 + x)e^{-x}]$$

$$= \frac{1}{2}(1 + x)e^{-x}(1 - x)$$

Note that $\frac{dA}{dx} = 0 \Rightarrow \quad x = 1, -1$

Also, $\frac{dA}{dx} > 0$, if $0 \leq x < 1$ and $\frac{dA}{dx} < 0$ if $x < 1$. Hence A is maximum when $x = 1$ so $y = e^{-1}$. Since y is even function other possibility of M is $(-1, e^{-1})$.

Example 82: The abscissa of the point on the curve $9y^2 = x^3$, the normal at which cuts off equal intercepts on the coordinate axes is

(a) 2 (b) 4
(c) – 4 (d) – 2

Ans. (b)

Solution: Differentiating $9y^2 = x^3$ we have $\frac{dy}{dx} = \frac{x^2}{6y}$.

Any point on the curve is of the form $(t^2, t^3/3)$ and so $\frac{dy}{dx}$ at this point is $t/2$. Thus an equation of normal is

$$y - t^3/3 = (-2/t)(X - t^2)$$

This will intersect coordinate axes at $(0, 2t + t^3/3)$ and $\left(\frac{t^4}{6} + t^2, 0\right)$. Hence we must have

$$2t + \frac{t^3}{3} = t^2 + \frac{t^4}{6} \Rightarrow 2 + \frac{t^2}{3} = t + \frac{t^3}{6}$$

Clearly $t = 2$ satisfies, the last equation. Hence the abscissa of the required point is 4. (For $t = 0$, the normal meets both the axes only at origin.)

Example 83: A curve passes through the point (2, 0) and the slope of the tangent at any point (x, y) is $x^2 - 2x$ for all value of x. The point of maximum ordinate on the curve is

(a) (0, 4/3) (b) (0, 2/3)
(c) (1, 2/3) (d) (2, 4/3)

Ans. (a)

Solution: $\frac{dy}{dx} = x^2 - 2x = \frac{d}{dx}\left(\frac{x^3}{3} - x^2 + c\right)$

$$y = \frac{x^3}{3} - x^2 + c$$

Since the curve passes through (2, 0), we get $0 = 8/3 - 4 + c$, i.e., $c = 4/3$

Hence the equation of the curve is

$$y = \frac{x^3}{3} - x^2 + \frac{4}{3}$$

Now, from $dy/dx = 0$, we get $x = 0$ or 2. Also

$$\frac{d^2y}{dx^2} = 2x - 2 \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=0} = -2 \textit{ and } \left.\frac{d^2y}{dx^2}\right|_{x=2} = 2$$

Hence at $x = 0$, y has a maximum. Thus the required point is (0, 4/3).

Example 84: If the tangent to the curve $2y^3 = ax^2 + x^3$ at the point (a, a) cuts off intercepts α and β on the coordinate axes, where $\alpha^2 + \beta^2 = 61$ then the value of $|a|$ is

(a) 16 (b) 28
(c) 30 (d) 31

Ans. (c)

Solution: The slope of the tangent is

$$\frac{dy}{dx} = \frac{2ax + 3x^2}{6y^2}$$

and the value of this slope at (a, a) is 5/6. Therefore, the equation

$$y - a = \frac{5}{6}(x - a) \Rightarrow \frac{x}{-a/5} + \frac{y}{a/6} = 1,$$

represents the tangent. Thus the x-intercept α is $-a/5$, and the y-intercept β is $a/6$. From $\alpha^2 + \beta^2 = 61$, we now get

$$\frac{a^2}{25} + \frac{a^2}{36} = 61 \Rightarrow a^2 = 25 \times 36 \Rightarrow |a| = 30.$$

Example 85: The coordinates of the point on the parabola $y^2 = 8x$, which is at minimum distance from the circle $x^2 + (y + 6)^2 = 1$ are

(a) (2, – 4) (b) (18, – 12)
(c) (2, 4) (d) none of these

Ans. (a)

◎ **Solution:** A point on the parabola is at a minimum distance from the circle if and only if it is at a minimum distance from the centre of the circle. Any point on the parabola $y^2 = 8x$ is of the form $P(2t^2, 4t)$. The centre of the circle $x^2 + (y + 6)^2 = 1$ is $O\ (0, -6)$

$$OP^2 = 4t^4 + (-6 - 4t)^2 = 4\,(t^4 + 4t^2 + 12t + 9)$$

Let $$A = t^4 + 4t^2 + 12t + 9$$

$$\frac{dA}{dt} = 4t^3 + 8t + 12 = 4\,(t^3 + 2t + 3) = 4(t + 1)\,(t^2 - t + 3)$$

So $$\frac{dA}{dt} = 0 \text{ if } t = -1. \text{ Moreover,}$$

$$\left.\frac{d^2A}{dt^2}\right|_{t=-1} = 4\,(3\,(-1)^2 + 2) > 0.$$

Hence required point is $P(2, -4)$.

◉ **Example 86:** The equation $e^{x-8} + 2x - 17 = 0$ has

(a) two real roots (b) one real root
(c) eight real roots (d) four real roots

Ans. (b)

◎ **Solution:** Clearly $x = 8$ satisfies the given equation. Assume that $f(x) = e^{x-8} + 2x - 17 = 0$ has a real root α other than $x = 8$. We may suppose that $\alpha > 8$ (the case for $\alpha < 8$ is exactly similar). Applying Rolle's theorem on $[8, \alpha]$, we get $\beta \in (8, \alpha)$, such that $f'(\beta) = 0$. But $f'(\beta) = e^{\beta-8} + 2$, so that $e^{\beta-8} = -2$ which is not possible. Hence there is no real root other than 8.

◉ **Example 87:** The maximum and minimum value of $f(x) = ab \sin x + b\sqrt{1-a^2}\, \cos x + c$ lie in the interval (assuming $|a| < 1, b > 0$)

(a) $[b - c, b + c]$ (b) $(b - c, b + c)$
(c) $[c - b, b + c]$ (d) none of these

Ans. (c)

◎ **Solution:** $f'(x) = ab \cos x - b\sqrt{1-a^2}\, \sin x = b \cos (x + \phi)$ where $\phi = \cos^{-1} a$. So $f'(x) = 0 \Rightarrow x = (2m + 1)(\pi/2 - \phi)$

Also $f''(x) = -b \sin (x + \phi)$. Thus $f''(\pi/2 - \phi) = -b < 0$ and $f''\,(3\pi/2 - \phi) = b > 0$. Hence f has maximum at $\pi/2 - \phi$ and minimum at $3\pi/2 - \phi$. Moreover, $\max f = ab \sin (\pi/2 - \phi) + b\sqrt{1-a^2}\, \cos (\pi/2 - \phi) + c = a^2b + b(1 - a^2) + c = b + c$ and $\min f = c - b$.

> *Alternatively* $f(x) = b \sin (x + \phi) + c$, $\phi = \cos^{-1} a$ so
> $\max f(x) = b.1 + c$ and $\min f(x) = -b + c = c - b$

◉ **Example 88:** The maximum area of the rectangle whose sides pass through the angular points of a given the rectangle is of sides a and b is

(a) $(1/2)\,(ab)^2$ (b) $(1/2)\,(a + b)$
(c) $(1/2)\,(a + b)^2$ (d) none of these

Ans. (c)

◎ **Solution:** Let $ABCD$ be the given rectangle of sides a and b and $EFGH$ be any rectangle, whose sides pass through A, B, C, D.

$$A = \text{Area } EFGH = (b \sin \theta + a \cos \theta)\,(a \sin \theta + b \cos \theta) = ab + (a^2 + b^2) \sin \theta \cos \theta.$$

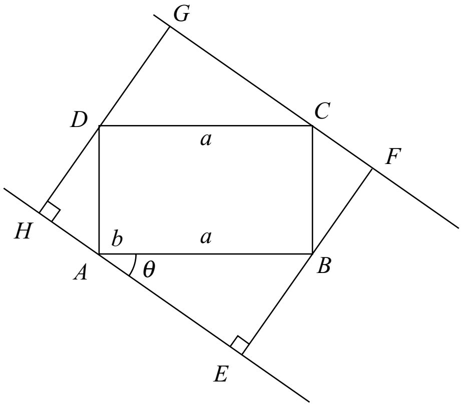


Fig. 11.4

$$dA/d\theta = (a^2 + b^2) \cos 2\theta \text{ so } dA/d\theta = 0 \quad \Rightarrow \quad \theta = \pi/4$$

$$\Rightarrow \quad \frac{d^2A}{d\theta^2} = -2\,(a^2 + b^2) \sin 2\theta, \text{ so } \left.\frac{d^2A}{d\theta^2}\right|_{\theta=\pi/4} < 0$$

Hence $A_{\max} = (1/2)\,(a + b)^2$.

◉ **Example 89:** The image of the interval $[-1, 3]$ under $f(x) = 4x^3 - 12x$ is

(a) $[-2, 0]$ (b) $[-8, 72]$
(c) $[-8, 0]$ (d) $[8, 72]$

Ans. (b)

◎ **Solution:** To find the image of the given interval, we must find the set of values of $f(x)$ for $x \in [-1, 3]$. By virtue of the continuity of $f(x)$, the image is the interval

$$\left[\min_{x\in[-1,3]} f\,(x), \max_{x\in[-1,3]} f\,(x)\right]$$

The critical points of $f(x)$ are given by $f'(x) = 12x^2 - 12 = 12\,(x^2 - 1) = 0$. That is, $x = \pm 1$, so that $f(1) = -8$, $f(-1) = 8$ and $f(3) = 72$.

$$\therefore \quad \max_{x\in[-1,3]} f(x) = f(3) = 72 \text{ and } \min_{x\in[-1,3]} f(x) = f(1) = -8$$

Hence the image of $[-1, 3]$ under the mapping $f(x)$ is $[-8, 72]$.

◉ **Example 90:** The difference between the greatest and least values of the function $f(x) = \cos x + (1/2) \cos 2x - (1/3) \cos 3x$ is

(a) 3/8 (b) 2/3
(c) 8/7 (d) 9/4

Ans. (d)

◎ **Solution:** The given function is periodic, with period 2π. So the difference between the greatest and least values of the function is the difference between these values on the interval $[0, 2\pi]$. We have

$$f'(x) = -(\sin x + \sin 2x - \sin 3x)$$
$$= -4 \sin x \sin (3x/2) \sin (x/2).$$

Hence $x = 0, 2\pi/3, \pi$ and 2π are the critical points. Also, $f(0) = 1 + 1/2 - 1/3 = 7/6$, $f(2\pi/3) = -13/12$, $f(\pi) = -1/6$ and $f(2\pi) = 7/6$. Hence the greatest value is 7/6 and the least value is $-13/12$. Thus the difference is

$$7/6 - (-13/12) = 27/12 = 9/4.$$

◉ **Example 91:** The longest distance of the point $(a, 0)$ from the curve $2x^2 + y^2 - 2x = 0$ is

(a) $\sqrt{1-2a+2a^2}$ (b) $\sqrt{1-2a+a^2}$

(c) $\sqrt{1+2a+2a^2}$ (d) $\sqrt{1+a+a^2}$

Ans. (a)

◎ **Solution** Let (x, y) be any point on the curve $2x^2 + y^2 - 2x = 0$. Its distance S from the point $(a, 0)$ is given by

$$T = S^2 = (x-a)^2 + y^2 = (x-a)^2 + (2x - 2x^2)$$
$$= -x^2 + 2x(1-a) + a^2 \qquad (1)$$

For S to be maximum, we have $dT/dx = 0$
$\Rightarrow \; -2x + 2(1-a) = 0 \;\Rightarrow\; x = 1-a$. Since $d^2T/dx^2 < 0$, so S^2 is maximum when $x = 1-a$ and this maximum value is given by $-(1-a)^2 + 2(1-a)^2 + a^2 = 1 - 2a + 2a^2$. Hence $S_{\max} = (1 - 2a + 2a^2)^{1/2}$.

◉ **Example 92:** The sides of the rectangle of the greatest area, that can be inscribed in the ellipse $x^2 + 2y^2 = 8$, are given by

(a) $4\sqrt{2}, 4$ (b) $4, 2\sqrt{2}$

(c) $2, \sqrt{2}$ (d) $2\sqrt{2}, 2$

Ans. (b)

◎ **Solution:** Any point on the ellipse $\dfrac{x^2}{8} + \dfrac{y^2}{4} = 1$ is $(2\sqrt{2}\cos\theta, 2\sin\theta)$. [see Fig. 11.5]

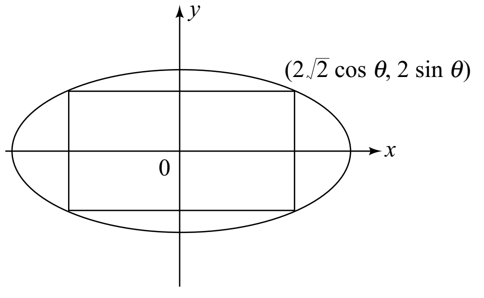


Fig. 11.5

$$A = \text{Area of the inscribed rectangle}$$
$$= 4(2\sqrt{2}\cos\theta)(2\sin\theta)$$
$$= 8\sqrt{2}\sin 2\theta$$
$$\frac{dA}{d\theta} = 16\sqrt{2}\cos 2\theta = 0 \;\Rightarrow\; \theta = \pi/4$$

Also $\dfrac{d^2A}{d\theta^2} = -32\sqrt{2}\sin 2\theta < 0$ for $\theta = \pi/4$.

Hence, the inscribed rectangle is of largest area if the sides are $4\sqrt{2}\cos \pi/4$ and $4 \sin(\pi/4)$ i.e. 4 and $2\sqrt{2}$.

◉ **Example 93:** An equation of the circle that is tangent to $y = x^3$ at (1, 1) and has the same second derivative there is

(a) $x^2 + y^2 + 24x - 28y + 2 = 0$

(b) $2(x^2 + y^2) + 12x - 8y - 8 = 0$

(c) $3(x^2 + y^2) - 24x + 10y + 8 = 0$

(d) none of these

Ans. (d)

◎ **Solution:** $\dfrac{dy}{dx} = 3x^2$ so $\left.\dfrac{dy}{dx}\right|_{(1,1)} = 3$ and $\left.\dfrac{d^2y}{dx^2}\right|_{(1,1)} = 6$

Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$.
Since this should pass through (1, 1) so we have

$$2 + 2g + 2f + c = 0 \qquad (1)$$

Also $\quad 2x + 2y\dfrac{dy}{dx} + 2g + 2f\dfrac{dy}{dx} = 0$

Putting $x = 1$, $y = 1$ and $\dfrac{dy}{dx} = 3$, we have

$$1 + 3 + g + 3f = 0$$

Again differentiating (1), we have

$$1 + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} + f\frac{d^2y}{dx^2} = 0$$

Putting $y = 1$, $\dfrac{dy}{dx} = 3$ and $\dfrac{d^2y}{dx^2} = 6$, we have

$$16 + 6f = 0 \qquad \Rightarrow \qquad f = -8/3$$

so $g = 4$ and $c = -14/3$. Thus required circle is

$$3(x^2 + y^2) + 24x - 16y - 14 = 0.$$

◉ **Example 94:** Let $f(x) = 6x^{4/3} - 3x^{1/3}$, $x \in [-1, 1]$. Then

(a) The maximum value of $f(x)$ on $[-1, 1]$ is 3

(b) The maximum value of $f(x)$ on $[-1, 1]$ is 9

(c) The minimum value of $f(x)$ on $[-1, 1]$ is 0

(d) none of these

Ans. (b)

◎ **Solution:** $f'(x) = \dfrac{8x-1}{x^{2/3}}$. Thus $f'(x) = 0$ when $x = 1/8$ and $f'(x)$ does not exist when $x = 0$. Now $f(-1) = 9$, $f(0) = 0$, $f(1/8) = -9/8$ and $f(1) = 3$.
The maximum value of $f(x)$ is 9 and the minimum value is $-9/8$ on $[-1, 1]$.

Example 95: Let $g(x) = (\log(1 + x))^{-1} - x^{-1}, x > 0$ then

(a) $1 < g(x) < 2$ (b) $-1 < g(x) < 0$
(c) $0 < g(x) < 1$ (d) none of these

Ans. (c)

Solution: For $x > 0$, consider the function $f(x) = \log(1 + x)$ on the interval $[0, x]$. Since the function is differentiable on $(0, x)$, by Lagrange's theorem, there is *a* $\xi \in (0, x)$ such that

$$\frac{\log(1+x) - \log(1+0)}{x - 0} = f'(\xi) = \frac{1}{1+\xi}$$

$$\Rightarrow \quad \frac{\log(1+x)}{x} = \frac{1}{1+\xi} < 1$$

$$\Rightarrow \quad \log(1 + x) < x \quad \Rightarrow \quad g(x) > 0$$

Also $\quad \xi \in (0, x) \quad \Rightarrow \quad \xi < x \quad \Rightarrow \quad \dfrac{1}{1+\xi} > \dfrac{1}{1+x}$

$$\Rightarrow \quad \frac{\log(1+x)}{x} > \frac{1}{1+x}$$

$$\Rightarrow \quad \log(1 + x) > \frac{x}{1+x} \Rightarrow g(x) < 1.$$

Example 96: The maximum value of $\dfrac{x^2 - x + 1}{x^2 + x + 1}$ for all real values of x is

(a) 1/2 (b) 1
(c) 2 (d) 3

Ans. (d)

Solution: Let $S = \dfrac{x^2 - x + 1}{x^2 + x + 1} = 1 - \dfrac{2x}{x^2 + x + 1}$

$$S'(x) = -2\,\frac{\left(x^2 + x + 1\right) - x\,(2x + 1)}{\left(x^2 + x + 1\right)^2}$$

$$= -2\,\frac{-x^2 + 1}{\left(x^2 + x + 1\right)^2}$$

$S'(x) = 0 \Leftrightarrow x = \pm 1$. Since $S'(x) > 0$ for $x < -1$ and $S'(x) < 0$ for $-1 < x < 1$, also $S'(x) > 0$ for $x > 1$. So S is maximum when $x = -1$. Hence $S_{\max} = 1 + 2/1 = 3$.

Example 97: If the tangent at (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P, then P is

(a) (4, 4) (b) (– 1, 2)
(c) (9/4, 3/8) (d) none of these

Ans. (c)

Solution: $2y\dfrac{dy}{dx} = (2 - x)^2 - 2x(2 - x)$

$$= 3x^2 - 8x + 4. \text{ So } \left.\frac{dy}{dx}\right|_{(1,\,1)} = -1/2.$$

An equation of tangent at (1, 1) is $Y - 1 = (-1/2)(X - 1)$ i.e. $Y = (-1/2)x + 3/2$. The intersection of this line with the given curve is given by $(-x/2 + 3/2)^2 = x(2 - x)^2$

$$\Rightarrow \quad x^2 - 6x + 9 = 16x + 4x^3 - 16x^2. \text{ So,}$$

$$4x^3 - 17x^2 + 22x - 9 = 0$$

$$\Rightarrow \quad (x - 1)(4x - 9)(x + 1) = 0.$$

Thus $x = 1, 9/4, -1$. But $x = -1$ cannot lie on the given curve so required point is (9/4, 3/8).

Example 98: If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$, cut each other at right angles then the value of b is

(a) 2 (b) 4
(c) 9/2 (d) none of these

Ans. (c)

Solution: The intersection of the two curves is given by

$$9x^2 + 6bx - x = x \qquad \text{(i)}$$

Differentiating $y^2 = 6x$, we have $\dfrac{dy}{dx} = \dfrac{3}{y}$.

Differentiating $9x^2 + 6y^2 = 16$, we have $\dfrac{dy}{dx} = -\dfrac{9x}{by}$.

For curves to intersect at right angles, we must have at the points of intersection $\dfrac{3}{y}\left(\dfrac{-9x}{by}\right) = -1 \quad \Rightarrow \quad 27x = by^2$. Thus we must have

$$9x^2 + 9y^2 = 16 \quad \Rightarrow \quad 9x^2 + 27x - 16 = 0 \qquad \text{(ii)}$$

(i) and (ii) must be identical so $27 = 6b \Rightarrow b = 9/2$.

Example 99: The distance of that point on $y = x^4 + 3x^2 + 2x$ which is nearest to the line $y = 2x - 1$ is

(a) $4/\sqrt{5}$ (b) $3/\sqrt{5}$
(c) $2/\sqrt{5}$ (d) $1/\sqrt{5}$

Ans. (d)

Solution: Distance of any point (x, y) from $y = 2x - 1$ is: $\left|\dfrac{y - 2x + 1}{\sqrt{5}}\right|$. If (x, y) is on $y = x^4 + 3x^2 + 2x$ then this distance is $S = \dfrac{x^4 + 3x^2 + 1}{\sqrt{5}}$

$$\frac{dS}{dx} = \frac{4x3 + 6x}{\sqrt{5}} \quad \Rightarrow \quad \frac{dS}{dx} = 0 \quad \Rightarrow \quad x = 0.$$

Also, $S'(x) < 0$ for $x < 0$ and $S'(x) > 0$ for $x > 0$.
Thus S is minimum when $x = 0$, and min. S is $1/\sqrt{5}$.

Example 100: A given right circular cone has a volume p, and the largest right circular cylinder that can be inscribed in the cone has a volume q. Then $p : q$ is

(a) 9 : 4 (b) 8 : 3
(c) 7 : 2 (d) none of these

Ans. (a)

◎ **Solution:** Let H be the height of the cone and α be its semi vertical angle. Suppose that x is the radius of the inscribed cylinder and h be its height $h = QL = OL - OQ = H - x \cot \alpha$

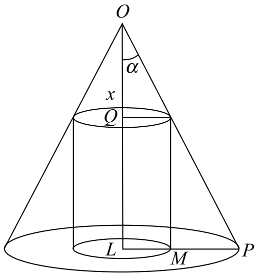


Fig. 11.6

$$V = \text{volume of the cylinder}$$

$$= \pi x^2 (H - x \cot \alpha)$$

Also $p = \dfrac{1}{3} \pi (H \tan \alpha)^2 H$ (i)

$$\frac{dV}{dx} = \pi (2Hx - 3x^2 \cot \alpha)$$

so $\dfrac{dV}{dx} = 0 \Leftrightarrow x = 0,$

$$x = \frac{2}{3} H \tan \alpha, \left.\frac{d^2V}{dx^2}\right|_{x=\frac{2}{3}H\tan\alpha} = -2\pi H < 0. \text{ so}$$

V is maximum when $x = \dfrac{2}{3} H \tan \alpha$ and $q = V_{\max} = \pi \dfrac{4}{9} H^2 \tan^2 \alpha \dfrac{1}{3} H = \dfrac{4}{9} p$. [using (i)]

Hence $p : q = 9 : 4$.

◉ **Example 101:** The set of all values of a for which $f(x) = (a^2 - 3a + 2)(\cos^2(x/4) - \sin^2(x/4)) + (a - 1)x + \sin 1$ doesn't possess critical points is

(a) $[1, \infty)$ (b) $(-2, 4)$

(c) $(1, 3) \cup (3, 5)$ (d) $(0, 1) \cup (1, 4)$

Ans. (d)

◎ **Solution:** The given function can be written as

$$f(x) = (a - 1)(a - 2) \cos (x/2) + (a - 1) x + \sin 1.$$

It is clearly differentiable, so its critical points are given by

$$f'(x) = (-1/2)(a - 1)(a - 2) \sin (x/2) + (a - 1) = 0$$

If $a = 1$, $f'(x) = 0$ for all x, while for values of a other than 1, $f'(x)$ will be zero if

$$(a - 2) \sin (x/2) = 2$$

In order not to have critical point, a must therefore satisfy $a \neq 1$ and $\left|\dfrac{2}{a-2}\right| > 1$ which is the same as saying $a \in (0, 1) \cup (1, 4)$.

◉ **Example 102:** Let $x, p \in \mathbf{R}$, $x + 1 > 0$, $p \neq 0, 1$. Then

(a) $(1 + x)^p > 1 + px$ for $p > 0$

(b) $(1 + x)^p > 1 + px$ for $p \in (-\infty, 0) \cup (1, \infty)$

(c) $(1 + x)^p > 1 + px$ for $0 < p < 1$

(d) $(1 + x)^p < 1 + px$ for $p < 1$

Ans. (b)

◎ **Solution:** Let $f(x) = (1 + x)^p - (1 + px)$ so $f(0) = 0$. Now $f'(x) = p(1 + x)^{p-1} - p$ and $f''(x) = p(p - 1)(1 + x)^{p-2}$. Let $p \in (-\infty, 0) \cup (1, \infty)$ then $f''(x) > 0$ so f' is increasing. This implies $f' < 0$ on $(-1, 0)$ and $f' > 0$ on $(0, \infty)$ as $f'(0) = 0$. Therefore f decreases on $(-1, 0)$ and increases on $(0, \infty)$ but $f(0) = 0$ so $f(x) > 0$ if $x > -1$ $(x \neq 0)$. Thus $(1 + x)^p > 1 + px$ for $p \in (-\infty, 0) \cup (1, \infty)$. Similarly it follows that $(1 + x)^p < 1 + px$, if $0 < p < 1$.

◉ **Example 103:** If $f(x) = \dfrac{a \sin x + b \cos x}{c \sin x + d \cos x}$ decreases for all x if

(a) $ad - bc < 0$ (b) $ad - bc > 0$

(c) $ab - cd > 0$ (d) $ab - cd < 0$

Ans. (a)

Solution: $$f'(x) = \frac{(a \cos x - b \sin x)(c \sin x + d \cos x) - (a \sin x + b \cos x)(c \cos x - d \sin x)}{(c \sin x + d \cos x)^2}$$

$$= \frac{ad(\cos^2 x + \sin^2 x) - bc(\cos^2 x + \sin^2 x)}{(c \sin x + d \cos x)^2}$$

$$= \frac{ad - bc}{(c \sin x + d \cos x)^2}$$

f decreases for all x if and only if $f'(x) < 0$ for all x i.e. $ad - bc < 0$.

◉ **Example 104:** On the interval $[0, 1]$ the function $x^{25}(1 - x)^{75}$ takes its maximum value at the point

(a) 0 (b) 1/3

(c) 1/2 (d) 1/4

Ans. (d)

◎ **Solution:** Let $f(x) = x^{25}(1 - x)^{75}$. The critical points of f are given by $f'(x) = 0$. But

$$f'(x) = x^{24}(1 - x)^{74}[25 - 25x - 75x]$$

$$= 25x^{24}(1 - x)^{74}(1 - 4x)$$

Thus the critical points are 0, 1, 1/4

Since $f(0) = 0$, $f(1) = 0$ and $f(1/4) = (1/4)^{25}(3/4)^{75}$

so f takes its maximum value at $x = 1/4$.

◉ **Example 105:** For $a > 0$, the value of a for which the equation $ax^2 = \log x$ possess a single root is

(a) 1/2 (b) $1/2e$

(c) $1/e$ (d) $2e^{-1}$

Ans. (b)

◎ **Solution:** For $a > 0$, the curves $y_1 = ax^2$ and $y_2 = \log x$ can have only one point in common if they touch each other. At the point of tangency $y'_1(x) = y'_2(x) \Rightarrow 2ax = 1/x \Rightarrow x = 1/\sqrt{2a}$ (clearly x cannot be negative). Putting this value in $ax^2 = \log x$, we have $(1/2) = \log (2a)^{-1/2} \Rightarrow \log 2a = -1 \Rightarrow a = 1/2e$.

EXERCISE

Concept-based
Straight Objective Type Questions

1. The length of the tangent to the curve $x = a \sin^3 t$, $y = a \cos^3 t$ $(a > 0)$ at an arbitrary is
   (a) $a \cos^2 t$ (b) $a \sin^2 t$
   (c) $\dfrac{a \sin^2 t}{\cos t}$ (d) $\dfrac{a \cos^2 t}{\sin t}$

2. Equation of normal to $x = 2e^t$, $y = e^{-t}$ at $t = 0$ is
   (a) $x + y - 4 = 0$
   (b) $x + 2y - 4 = 0$
   (c) $2x - y - 3 = 0$
   (d) $x - 2y - 3 = 0$

3. A point moves according $s = \dfrac{2}{9} \sin \dfrac{\pi}{2} t + s_0$. The acceleration at the end of first second is
   (a) $-\dfrac{\pi}{18}$ (b) $-\dfrac{\pi^2}{18}$
   (c) $\dfrac{\pi}{18}$ (d) $\dfrac{\pi^2}{18}$

4. Let $f(x) = x \log x - x + 1$ then the set $\{x : f(x) > 0\}$ is equal to
   (a) $(1, \infty)$ (b) $(1/e, \infty)$
   (c) $[e, \infty)$ (d) $(0, 1) \cup (1, \infty)$

5. On the curve $y = x^3$, the point at which the tangent line in parallel to the chord through the point $(-1, -1)$ and $(2, 8)$ is
   (a) $(1, 1)$ (b) $\left(\dfrac{1}{2}, \dfrac{1}{8}\right)$
   (c) $\left(\dfrac{1}{3}, \dfrac{1}{27}\right)$ (d) $\left(-\dfrac{1}{2}, -\dfrac{1}{8}\right)$

6. Let $f(x) = 2x^2 - \log x$, then
   (a) f increases on $(0, \infty)$
   (b) f decrease on $\left(\dfrac{1}{2}, \infty\right)$
   (c) f increases on $\left(\dfrac{1}{2}, \infty\right)$
   (d) f decreases on $(0, 1)$

7. Let $f(x) = \dfrac{3}{4}x^4 - x^3 - 9x^2 + 7$, then the number of critical points in $[-1, 4]$ is
   (a) 4 (b) 3
   (c) 2 (d) 1

8. On the curve $x^3 = 12y$, the values of x for which the abscissa changes at a faster rate than the ordinate is
   (a) $(-2, 2) \sim \{0\}$ (b) $(-3, 3) \sim \{0\}$
   (c) $(1, 4)$ (d) $(2, 4)$

9. The value of $k > 0$ for which the curves $\dfrac{x^2}{k^2} + \dfrac{y^2}{4} = 1$ and $y^2 = 16x$ cut each other orthogonally is
   (a) 1 (b) $\dfrac{2\sqrt{3}}{3}$
   (c) $3\sqrt{3}$ (d) $5\sqrt{5}$

10. The least value of $g(t) = 8t - t^4$ on $[-2, 1]$ is
    (a) -16 (b) -20
    (c) -32 (d) 7

LEVEL 1

Straight Objective Type Questions

11. Let $f(x) = \tan^{-1} x$ and $g(x) = \dfrac{x}{1 + x^2}, x > 0$ then
    (a) $f(x) < g(x)$, on $(0, \infty)$
    (b) $f(x) \le g(x)$ on $[1, \infty)$
    (c) $g(x) < f(x)$ on $(0, \infty)$
    (d) none of these

12. Let $f(x) = (x - 2)(x - 3)(x - 4)(x - 5)(x - 6)$ then
    (a) $f'(x) = 0$ has five real roots
    (b) four roots of $f'(x) = 0$ lie in $(2, 3) \cup (3, 4) \cup (4, 5) \cup (5, 6)$
    (c) the equation $f'(x)$ has only three roots
    (d) four roots of $f'(x) = 0$ lie in $(1, 2) \cup (2, 3) \cup (3, 4) \cup (4, 5)$

13. Let $f(x) = (x - 3)^5 (x + 1)^4$ then
    (a) $x = -1$ is point of minima
    (b) $x = -1$ is point of maxima
    (c) $x = 7/9$ is a point of maxima
    (d) $x = -1$ is neither a point of maxima and minima.

14. The normal to the curve represented parametrically by $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$ at any point θ, is such that it
    (a) makes a constant angle with the x-axis
    (b) is at a constant distance from the origin
    (c) does not touch a fixed circle
    (d) passes through the origin.

15. The value of a for which the equation $x^3 - 3x + a = 0$ has two distinct roots in [0, 1], is given by
    (a) – 1
    (b) 1
    (c) 3
    (d) none of these

16. If the sum of the squares of the intercepts on the axes cut off by the tangent to the curve $x^{1/3} + y^{1/3} = a^{1/3}$ $(a > 0)$ at $(a/8, a/8)$ is 2, then a has the value
    (a) 1
    (b) 2
    (c) 4
    (d) 8

17. The value of m for which the area of the triangle included between the axes and any tangent to the curve $x^m y = b^m$ is constant, is
    (a) 1/2
    (b) 1
    (c) 3/2
    (d) 2

18. If the tangent at any point on the curve $x^4 + y^4 = a^4$ cuts off intercepts p and q on the coordinate axes, the value of $p^{-4/3} + q^{-4/3}$ is
    (a) $a^{-4/3}$
    (b) $a^{-1/2}$
    (c) $a^{1/2}$
    (d) none of these

19. The interval of increase of the function $y = x - 2\sin x$ if $0 \le x \le 2\pi$, is
    (a) (0, π)
    (b) (0, π)
    (c) (π/2, π)
    (d) (π/3, 5π/3)

20. The greatest value of $y = (x + 1)^{1/3} - (x - 1)^{1/3}$ on [0, 1] is
    (a) 1
    (b) 2
    (c) 3
    (d) $2^{1/3}$

21. Let f be a function defined by $f(x) = 2x^2 - \log|x|$, $x \neq 0$ then
    (a) f increase on $[-1/2, 0] \cup [1/2, \infty)$
    (b) f decreases on $(-\infty, 0)$
    (c) f increases on $(-\infty, -1/2)$
    (d) f decreases on $[1/2, \infty]$

22. The shortest distance of (0, 0) from the curve $y = \dfrac{e^x + e^{-x}}{2}$ is
    (a) 1/2
    (b) 1/3
    (c) 2
    (d) none of these

23. The normal to the circle $x^2 + y^2 - 2x - 2y = 0$ passing through (2, 2) is
    (a) $x = y$
    (b) $2x + y - 6 = 0$
    (c) $x + 2y - 6 = 0$
    (d) $x + y - 4 = 0$

24. If $f(0) = 0$ and $f''(x)$ exists and > 0, for all $x > 0$ then $f(x)/x$
    (a) decreases on (0, ∞)
    (b) increases on (0, ∞)
    (c) decreases on (1, ∞)
    (d) neither increases nor decreases on (0, ∞)

25. The value of k so that the equation $x^3 - 12x + k = 0$ has distinct roots in [0, 2] is
    (a) 4
    (b) 2
    (c) –2
    (d) none of these

26. Let $f(x) = 6x^{4/3} - 3x^{1/3}$ defined on [– 1, 1] then
    (a) maximum value of f is 7
    (b) maximum value of f is 5
    (c) maximum value of f is 9
    (d) minimum value of f is – 3/2

27. An equation of tangent line at an inflection point of $f(x) = x^4 - 6x^3 + 12x^2 - 8x + 3$ is
    (a) $y = 3x + 4$
    (b) $y = 4$
    (c) $y = 3x + 2$
    (d) none of these

28. The number of real roots of the equation $2x^3 - 3x^2 + 6x + 6 = 0$ is
    (a) 1
    (b) 2
    (c) 3
    (d) none of these

29. Let $f(x) = (x - 2)(x^4 - 4x^3 + 6x^2 - 4x + 1)$ then value of local minimum of f is
    (a) – 2/3
    (b) $-(4/5)^4$
    (c) $-4^4/5^5$
    (d) $-(4/5)^5$

30. Let $f(x) = x^2 - 2|x| + 2$, $x \in [-1/2, 3/2]$ then
    (a) $\min\limits_{x \in [-1/2, 3/2]} f(x) = 1/2$
    (b) $\min\limits_{x \in [-1/2, 3/2]} f(x) = 1$
    (c) $\max\limits_{x \in [-1/2, 3/2]} f(x) = 3/2$
    (d) none of these

31. A critical point of the function $f(x) = \dfrac{|x-1|}{x^2}$ is
    (a) – 1
    (b) 3
    (c) 2
    (d) 1/2

32. The function $f(x) = x^x$ decreases on the interval
    (a) $(0, e)$
    (b) (0, 1)
    (c) $(0, 1/e)$
    (d) none of these

33. The interval of increase of the function $y = x - e^x + \tan(\pi/7)$ is
    (a) (– ∞, 1)
    (b) (0, ∞)
    (c) (– ∞, 0)
    (d) (1, ∞)

34. Let $f(x) = x^2 + px + q$. The value of (p, q) so that $f(1) = 3$ is an extreme value of f on [0, 2] is
    (a) (– 2, 2)
    (b) (1, 4)
    (c) (– 2, 4)
    (d) (– 2, 3)

35. The number of inflection points of a function given by a third degree polynomial is exactly

(a) 2 (b) 1
(c) 3 (d) 0

36. Let $f(x) = 2\tan^{-1} x + \sin^{-1} \dfrac{2x}{1+x^2}$ then
(a) $\max f(x) = \pi/2$ (b) $\min f(x) = \pi/4$
(c) $\max f(x) = \pi$ (d) none of these

37. If the normal to the curve $x^3 = y^2$ at the point $(m^2, -m^3)$ is $y = mx - 2m^3$, then the value of m^2 is
(a) 1 (b) 1/2
(c) 1/3 (d) 2/3

38. Let $f(x) = 2\sin x + \cos 2x$ $(0 \le x \le 2\pi)$ and $g(x) = x + \cos x$ then
(a) g is a decreasing function
(b) f increases on $(0, \pi/2)$
(c) f increases on $(0, \pi/6) \cup (\pi/2, 5\pi/6)$
(d) f decreases on $(0, \pi/2)$

39. The minimum value of the function $f(x) = \tan x + \cot x$ in the interval $(0, \pi/2)$ is
(a) 1 (b) 0
(c) 2 (d) 1/2

40. The number of points of extremum of the function $f(x) = 3x^4 - 4x^3 + 6x^2 + b$ for any value of b is
(a) 4 (b) 3
(c) 1 (d) 2

41. The shortest distance of the line $y - x - 1 = 0$ from $x = y^2$ is
(a) 3/8 (b) $3\sqrt{2}/4$
(c) 3/4 (d) $3\sqrt{2}/8$

42. The value of a for which the extremum of the function $f(x) = x^3 - 3ax^2 + 3(a^2 - 1)x + 1$ lie in the interval $(-2, 4)$ lie in
(a) $(-1, 0)$ (b) $(-2, 4)$
(c) $(-1, 5)$ (d) $(-1, 3)$

43. If $A > 0$, $B > 0$ and $A + B = \pi/3$ then the maximum value of $\tan A \tan B$ is
(a) 1/3 (b) 1/2
(c) $1/\sqrt{2}$ (d) $\sqrt{3}/2$

44. The maximum value of $|x \log x|$ for $0 < x \le 1$ is
(a) 0 (b) $1/e$
(c) $2e^{-1}$ (d) none of these

45. The greatest value of the function $\log_x 1/9 - \log_3 x^2$ $(x > 1)$ is
(a) 2 (b) 0
(c) -4 (d) -2

46. Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \ge 2$ for $x \in [1, 6]$ then
(a) $f(6) < 8$ (b) $f(6) \ge 8$
(c) $f(6) \ge 10$ (d) $f(6) \le 5$

47. An extremum value of the function
$f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$ $(-1 < x < 1)$ is
(a) $7\pi^3/8$ (b) $\pi^3/8$
(c) $\pi^3/32$ (d) $\pi^3/16$

48. Let $f(x) = x \log x + 3x$. Then
(a) f increases on (e^{-4}, ∞)
(b) f increases on $(0, \infty)$
(c) f decreases on $(0, \infty)$
(d) f decreases on $(0, e^{-2})$

49. Let $f(x) = x^2 e^{-x}$ then
(a) $\max f(x) = e^{-1}$ (b) $\max f(x) = 4e^{-2}$
(c) $\min f(x) = e^{-1}$ (d) $\min f(x) > 0$

50. The minimum value of $f(x) = |3 - x| + |2 + x| + |5 - x|$ is
(a) 0 (b) 7
(c) 8 (d) 10

51. Let $f(x) = 2 + 2x - 3x^{2/3}$ on $[-1, 10/3]$. Then f has
(a) Absolute maximum at an end point
(b) Absolute minimum at an interior point
(c) Absolute minimum is $f(10/3)$
(d) Absolute minimum is $f(-1)$

52. If f and g are defined on $[0, \infty)$ by
$$f(x) = \lim_{n\to\infty} \frac{x^n - 1}{x^n + 1} \text{ and } g(x) = \int_0^x f(t)\, dt.$$
Then
(a) g has local maximum at $x = 1$
(b) g has local minimum at $x = 1$
(c) g is an increasing function on $(0, \infty)$
(d) g is a decreasing function on $(0, \infty)$.

53. Let $f(x) = \sin x + \cos x$ then
(a) $x = 17\pi/4$ is a point of minima
(b) $x = 13\pi/4$ is a point of maxima
(c) $x = 21\pi/4$ is a point of minima
(d) $x = 29\pi/4$ is a point of maxima

54. If $f(x) = x\, e^{x(1-x)}$, then $f(x)$ is
(a) increasing on $[-1/2, 1]$
(b) decreasing on **R**
(c) increasing on **R**
(d) decreasing on $[-1/2, 1]$

55. The tangent to the curve $y = e^x$ drawn at the point (c, e^c) intersects the line joining the points $(c - 1, e^{c-1})$ and $(c + 1, e^{c+1})$
(a) on the left of $x = c$ (b) on the right of $x = c$
(c) at no point (d) at all points

Assertion-Reason Type Questions

56. **Statement-1:** $e^{\pi} > \pi^{e}$
Statement-2: The function $x^{1/x}$ $(x > 0)$ has local maximum at $x = e$.

57. **Statement-1:** $|\cot x - \cot y| \le |x - y|$ for all $x, y \in (-\pi/2, \pi/2)$
Statement-2: If f is differentiable on an open interval and $|f'(x)| \le M$ then $|f(x) - f(y)| \le M |x - y|$.

58. **Statement-1:** The function $f(x) = 2 \sin x + \cos 2x$ $(0 \le x \le 2\pi)$ has minimum at $x = \pi/3$ and maximum at $5\pi/3$.
Statement-2: The function $f(x)$ above decreases on $(0, \pi/3)$, increases on $(\pi/3, 5\pi/3)$ and decreases on $(5\pi/2, 2\pi)$

59. Let $f(x) = \tan^{-1} \dfrac{1-x}{1+x}$
Statement-1: The difference between the greatest and smallest value of $f(x)$ on $[0, 1]$ is $\pi/4$
Statement-2: If a function g decreases on $[a, b]$ then the greatest value of $g = g(a)$ and least value of g is $g(b)$.

60. Let $f(x) = \dfrac{x}{\log x}$
Statement-1: The minimum value of $f(x)$ is e
Statement-2: $\log x > 1$ for $x > e$ and < 1 for $x < e$.

LEVEL 2

Straight Objective Type Questions

61. The critical points of the function $f(x) = (x - 2)^{2/3} (2x + 1)$ are
(a) -1 and 2 (b) 1
(c) 1 and $-1/2$ (d) 1 and $1/2$

62. The function $f(x) = (1/4)x^3 - \sin \pi x + 3$ on $[-2, 2]$ takes the value
(a) 1 (b) $16/3$
(c) 6 (d) 8

63. The greatest value of $f(x) = \tan^{-1} x - \dfrac{1}{2} \log x$ on $[1/\sqrt{3}, \sqrt{3}]$ is
(a) $\pi/2 + (1/2) \log 3$ (b) $\pi/6 + (1/4) \log 3$
(c) $\pi/6 + (1/2) \log 3$ (d) $\pi/4 - (1/4) \log 3$

64. Equations of those tangents to $4x^2-9y^2=36$ which are perpendicular to the straight line $2y + 5x = 10$; are
(a) $5(y - 3) = \left(x - \sqrt{117/4}\right)$
(b) $5(y - 2) = 2(x - \sqrt{18})$
(c) $5(y + 2) = 2(x - \sqrt{18})$
(d) none of these

65. If $a, b, c, \in \mathbf{R}$, then
$$f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} \text{ decrease on}$$
(a) $(-(2/3)(a^2 + b^2 + c^2), 0)$
(b) $(0, (2/3)(a^2 + b^2 + c^2))$
(c) $((1/3)(a^2 + b^2 + c^2), 0)$
(d) none of these

66. A channel 27 m wide falls at a right angle into another channel 64 m wide. The greatest length of the log that can be floated along this system of channels is
(a) 120 (b) 125
(c) 100 (d) 110

67. For $a \in [\pi, 2\pi]$, the function
$$f(x) = \frac{1}{3} \sin a \tan^3 x + (\sin a - 1) \tan x + \frac{\sqrt{a-2}}{\sqrt{8-a}}$$
has
(a) $x = n\pi$ $(n \in \mathbf{I})$ as critical points
(b) no critical points
(c) $x = 2n\pi$ $(n \in \mathbf{I})$ as critical points
(d) $x = (2n + 1)\pi$ $(n \in \mathbf{I})$ as critical points.

68. The value of a for which the function
$f(x) = (4a - 3)(x + \log 5) + 2(a - 7) \cot (x/2) \sin^2 (x/2)$ does not possess critical point is
(a) $(-\infty, -4/3]$ (b) $(-\infty, -1)$
(c) $[1, \infty)$ (d) $(0, \infty)$

69. The interval to which b may belong so that the function
$$f(x) = \left(1 - \frac{\sqrt{21-4b-b^2}}{b+1}\right)x^3 + 5x + \sqrt{6}$$
is increasing at every point of its domain is
(a) $[-7, 0]$ (b) $[-6, 0]$
(c) $[1, 4]$ (d) $[2, 3]$

70. A tangent is drawn at a variable point on an ellipse $x^2/a^2 + y^2/b^2 = 1$, then minimum area of the triangle formed by the tangent and the coordinate axes is

(a) ab (b) $(a^2 + b^2)/2$
(c) $(a + b)^2/4$ (d) $2\,ab$

71. The set of all x for which $\log(1 + x) \leq x$ is

(a) $(1, \infty)$ (b) $(0, \infty)$
(c) $(-1, \infty)$ (d) none of these

72. The minimum value of $2^{(x^2-3)^3} + 27$ is

(a) 2^{27} (b) 2
(c) 1 (d) none of these

73. Let $f(x) = \begin{cases} |x| & \text{for } 0 < |x| \leq 2 \\ 1 & \text{for } x = 0 \end{cases}$

Then at $x = 0$, f has

(a) a local maximum (b) no local maximum
(c) a local minimum (d) no extremum

74. If $f(x) = xe^{x(1-x)}$, then $f(x)$ is

(a) increasing on $[-1/2, 1]$
(b) decreasing on **R**
(c) increasing on **R**
(d) decreasing on $[-1/2, 1]$

75. Let $f(x) = \begin{cases} x^{\alpha} \log x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$

Rolle's theorem can be applied to f on $[0, 1]$ then value of α can be

(a) -1 (b) $-1/2$
(c) 0 (d) 1/2

76. A cone is made from a circular sheet of radius $\sqrt{3}$ by cutting out a sector and gluing the cut edges of the remaining piece together. The maximum volume attainable for the cone is

(a) $\pi/3$ (b) $\pi/6$
(c) $2\pi/3$ (d) $3\sqrt{3}\pi$

77. The dimensions of the rectangle of maximum area that can be inscribed in the ellipse $(x/4)^2 + (y/3)^2 = 1$ are

(a) $\sqrt{8}, \sqrt{2}$ (b) 4, 3
(c) $2\sqrt{8}, 3\sqrt{2}$ (d) none of these

78. The condition for $y = ax^4 + bx^3 + cx^2 + dx + e$ to have points of inflection is

(a) $b^2 - 4ac > 0$ (b) $3b^2 - 8ac = 0$
(c) $3b^2 - 8ac > 0$ (d) $3b^2 - 8ac < 0$

79. The largest value of m such that $|x^2 - 3x + 2| \geq m$ for all x in the interval $[3/2, 7/4]$ is

(a) 3/4 (b) 3/8
(c) 3/16 (d) 7/4

80. The point in the interval $[0, \pi]$ for which the curve $y = (1/2)x$ and $y = \sin x$ are farthest apart is

(a) $\pi/2$ (b) $\pi/4$
(c) $\pi/6$ (d) π

81. The points at which the tangents to the curve $ax^2 + 2hxy + by^2 = 1$ is parallel to y-axis is

(a) $(0, 0)$
(b) where $hx + by = 0$ meets it
(c) where $ax + hy$ meets it
(d) none of these

82. If the point on $y = x \tan \alpha - \dfrac{ax^2}{2u^2 \cos^2 \alpha}$ $(0 < \alpha < \pi/2)$ where the tangent is parallel to $y = x$ has an ordinate $u^2/4a$ then the value of α is

(a) $\pi/2$ (b) $\pi/6$
(c) $\pi/3$ (d) none of these

83. Let $f(x) = \begin{cases} |x-1| + a, & x \leq 1 \\ 2x + 3, & x > 1 \end{cases}$

If $f(x)$ has local minimum at $x = 1$ and $a \geq 5$ then the value of a is

(a) 5 (b) 6
(c) 11/2 (d) 15/2

84. Let $g(x) = \int_0^x f(t)\,\mathrm{d}t$ and $f(x)$ satisfies the equation $f(x+y) = f(x) + f(y) + 2xy - 1$ for all $x, y \in \mathbf{R}$ and $f'(0) = 2$ then

(a) g increases on $(0, \infty)$ and decreases on $(-\infty, -0)$
(b) g increases on $(0, \infty)$
(c) g decreases on $(0, \infty)$ and increases $(-\infty, 0)$
(d) g decreases on $(-\infty, \infty)$

85. The area of the triangle formed by the positive x-axis and the normal and the tangent to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ is

(a) $2\sqrt{3}$ (b) $\sqrt{3}$
(c) $4\sqrt{3}$ (d) 3

86. The interval into which the function $y = \dfrac{x-1}{x^2 - 2x + 3}$ transforms the entire real line is

(a) $[1/3, 2]$ (b) $[-1/3, 2]$
(c) $[-1/3, 1]$ (d) none of these

87. The angle at which $x^2 + y^2 = 16$ can be seen from the point $(8, 0)$ is

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/2$ (d) $\pi/3$

88. The critical points of the function $f(x) = (x + 2)^{2/3}(2x - 1)$ are

(a) -1 and 2 (b) 1
(c) 1 and $-1/2$ (d) -1 and -2

89. The function $f(x) = \dfrac{\log(\pi + x)}{\log(e + x)}$ is

(a) increasing on $[0, \infty)$
(b) decreasing on $[0, \infty)$
(c) increasing on $[0, \pi/e)$ and decreasing on $[\pi/e, \infty)$
(d) decreasing on $[0, \pi/e)$ and increasing on $[\pi/e, \infty)$

90. A rectangle with perimeter 32 cm has greatest area if its length is

(a) 12 (b) 10
(c) 8 (d) 14

91. The greatest value of the function

$f(x) = \cot^{-1} x + (1/2) \log x$ on $[1, \sqrt{3}]$ is

(a) $(\pi/6) + 0.25 \log 3$ (b) $(\pi/3) - 0.25 \log 3$
(c) $\pi/4$ (d) $\tan^{-1} e - 1/2$

92. A particle is moving along the parabola $y^2 = 4(x + 2)$. As it passes through the point (7, 6) its y-coordinate is increasing at the rate of 3 units per second. The rate at which x-coordinate change at this instant is (in units/sec)

(a) 4 (b) 6
(c) 8 (d) 9

93. The perimeter of a rectangle is fixed at 24 cm. If the length l of the rectangle is increasing at the rate of 1 cm per second, the value of l for which the area of rectangle start to decrease is

(a) 2 cm (b) 6 cm
(c) 4 cm (d) 8 cm

94. The rate at which fluid level inside vertical cylindrical tank of radius r drop if we pump fluid out at the rate of 3cm^3 l/min is

(a) $-\dfrac{1}{\pi r^2}$ (b) $\dfrac{3}{\pi r^2}$
(c) $\dfrac{2}{\pi r^2}$ (d) $\dfrac{4}{\pi r}$

95. The length l of a rectangle is decreasing at the rate of 2 cm/sec while the widths w is increasing at the rate of 2 cm/sec. When $l = 12$ and $w = 5$, the rate of change of area is (in cm^2/sec)

(a) 14 (b) 12
(c) 8 (d) 4

96. Let f be twice differentiable function such that $f(x) = x^2$, $x = 1, 2, 3$. Then

(a) $f''(x) = 2\ \forall\ x \in (1, 3)$
(b) $f''(x) = 2$ for some $x \in (1, 3)$
(c) $f''(x) = 3\ \forall\ x \in (2, 3)$
(d) $f''(x) = f'(x)$ for some $x \in (2, 3)$

97. A tangent drawn to the curve $y = f(x)$ at $P(x, y)$ cuts the x-axis and y-axis at A and B respectively such that $BP : AP = 3 : 1$, given that $f(1) = 1$ then

(a) equation of the curve is $x\dfrac{dy}{dx} - 3y = 0$.
(b) normal at (1, 1) is $x + 3y = 4$.
(c) curve passes through (2, 1/8)
(d) equation of the curve is $x\dfrac{dy}{dx} + 4y = 0$.

98. If $0 < b^2 < c$ then $f(x) = x^3 + bx^2 + cx + d$

(a) has no local minima
(b) has no local maxima
(c) is strictly increasing on **R**
(d) is strictly decreasing on **R**

Previous Years' AIEEE/JEE Main Questions

1. If $2a + 3b + 6c = 0 (a, b, c \in \mathbf{R})$ then the quadratic equation $ax^2 + bx + c = 0$ has

(a) at least one root in [0, 1]
(b) at least one root in [2, 3]
(c) at least one root.
(d) none of these **[2002, 2004]**

2. The maximum distance from origin of a point on the curve $x = a \sin t - b \sin\left(\dfrac{at}{b}\right)$, $y = a \cos t - b \cos\left(\dfrac{at}{b}\right)$, both $a, b > 0$ is

(a) $a - b$ (b) $a + b$
(c) $\sqrt{a^2 + b^2}$ (d) $\sqrt{a^2 - b^2}$ **[2002]**

3. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals

(a) 1 (b) 2
(c) 1/2 (d) 3 **[2003]**

4. If $u = \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta} + \sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}$

then the difference between the maximum and minimum values of u^2 is given by

(a) $(a+b)^2$ (b) $2\sqrt{a^2+b^2}$

(c) $2(a^2+b^2)$ (d) $(a-b)^2$ **[2004]**

5. A function $y = f(a)$ has a second order derivative $f''(x) = 6(x-1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is

(a) $(x+1)^3$ (b) $(x-1)^3$

(c) $(x-1)^2$ (d) $(x+1)^2$ **[2004]**

6. The normal to the curve $x = a(1 + \cos\theta)$, $y = a\sin\theta$ at θ always passes through the fixed point

(a) (0, 0) (b) $(0, a)$

(c) $(a, 0)$ (d) (a, a) **[2004]**

7. A function is matched below against an interval where it is supported to be increasing which is of the following pairs in incorrectly matched?

	Interval	Function
(a)	$(-\infty, 1/3)$	$3x^2 - 2x + 1$
(b)	$(-\infty, -4)$	$x^3 + 6x^2 + 6$
(c)	$(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(d)	$(2, \infty)$	$2x^3 - 3x^2 - 12x + 6$

[2005]

8. The normal to the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at any point Q is such that

(a) it passes through $\left(\dfrac{a\pi}{2}, -a\right)$

(b) it is at constant distance from origin

(c) it passes through origin

(d) it makes angle $\dfrac{\pi}{2} + \theta$ with the x-axis **[2005]**

9. Let f be differentiable for all x. If $f(1) = -2$ and $f'(x) \geq 2$ for $x \in [1, 6]$ then

(a) $f(6) < 5$ (b) $f(6) = 5$

(c) $f(6) \geq 8$ (d) $f(6) < 8$ **[2005]**

10. A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of 50 cm^3/min. When the thickness of ice is 5 cm, then the rate at which the thickness of ice decreases, is

(a) $1/54\pi$ cm/min (b) $5/6\pi$ cm/min

(c) $1/36\pi$ cm/min (d) $1/8\pi$ cm/min **[2005]**

11. The function $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$ has a local minimum at

(a) $x = 1$ (b) $x = 2$

(c) $x = -2$ (d) $x = 0$ **[2006]**

12. A value of c for which the conclusion of mean value theorem holds for the function $f(x) = \log x$ on the interval [1, 3] is

(a) $2\log_3 e$ (b) $(1/2)\log 3$

(c) $\log_3 c$ (d) $\log 3$. **[2007]**

13. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function is

(a) $(\pi/4, \pi/2)$ (b) $(-\pi/2, \pi/4)$

(c) $(0, \pi/2)$ (d) $(-\pi/2, \pi/2)$ **[2007]**

14. Suppose the cube $x^3 - px + q$ has three distinct real roots where $p > 0$ and $q > 0$. Then which one of the following holds?

(a) The cubic has minima at $\sqrt{\dfrac{p}{3}}$ and maxima at $-\sqrt{\dfrac{p}{3}}$

(b) The cubic has minima at $-\sqrt{\dfrac{p}{3}}$ and maxima at $\sqrt{\dfrac{p}{3}}$

(c) The cubic has minima at both $\sqrt{\dfrac{p}{3}}$ and $-\sqrt{\dfrac{p}{3}}$

(d) The cubic has maxima at both $\sqrt{\dfrac{p}{3}}$ and $-\sqrt{\dfrac{p}{3}}$

[2008]

15. Given $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$.

If $P(-1) < P(1)$, then in the interval $[-1, 1]$

(a) $P(-1)$ is the minimum but $P(1)$ is not the maximum of P

(b) neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

(c) $P(-1)$ is the minimum and $P(1)$ is the maximum of P

(d) $P(-1)$ is not minimum but $P(1)$ is the maximum of P. **[2009]**

16. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} k - 2x, & \text{if } x \leq -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$

If f has a local minimum at $x = -1$, then a possible value of k is

(a) $-1/2$ (b) -1

(c) 1 (d) 0 **[2010]**

17. The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact, is given by:

(a) $2y - 3x = 0$ (b) $y = 6/x$

(c) $x^2 + y^2 = 13$ (d) $\left(\dfrac{x}{2}\right)^2 + \left(\dfrac{y}{3}\right)^2 = 2$

[2011]

18. Let f be a function defined by

$$f(x) = \begin{cases} \dfrac{\tan x}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$$

Statment-1: $x = 0$ is point of maxima of f **[2011]**

Statement-2: $f'(0) = 0$.

19. Let $a, b \in \mathbf{R}$ be such that the function f given by $f(x) = \log|x| + bx^2 + ax$, $x \neq 0$ has extreme values at $x = -1$ and $x = 2$

Statement-1: f has local maximum at $x = -1$ and at $x = 2$

Statement-2: $a = 1/2$ and $b = -1/4$. **[2012]**

20. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is

(a) 7/9 (b) 2/9

(c) 9/2 (d) 9/7 **[2012]**

21. The cost of running a bus from A to B is Rs. $\left(av + \dfrac{b}{v}\right)$, where v km/h is the average speed of bus. When the bus travels at 30 km/h, the cost comes out to be Rs. 75 while at 40 km/h, it is Rs. 65. Then the most economical speed in km/h of the bus is

(a) 45 (b) 50

(c) 60 (d) 40 **[2013, online]**

22. If the surface area of a sphere of radius r is increasing uniformly at the rate of 8 cm^2/s then the rate of change of its volume is

(a) constant (b) proportional to $\sqrt{r}$

(c) proportional to r^2 (d) proportional to r **[2013, online]**

23. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in $[0, 1]$

(a) lies between 2 and 3

(b) lies between –1 and 0

(c) does not exist

(d) lies between 1 and 2 **[2013]**

24. The maximum area of a right angled triangle with hypotenuse h is

(d) $\dfrac{h^2}{2\sqrt{2}}$ (b) $\dfrac{h^2}{2}$

(c) $\dfrac{h^2}{\sqrt{2}}$ (d) $\dfrac{h^2}{4}$ **[2013, online]**

25. **Statement-1:** The equation $x \log x = 2 - x$ is satisfied by at least one value of x lying between 1 and 2

Statement-2: The function $f(x) = x \log x$ is an increasing function in $[1, 2]$ and $g(x) = 2 - x$ is a decreasing function in $[1, 2]$ and the graphs represented by these functions intersect at a point in $[1, 2]$.**[2013, online]**

26. **Statement-1:** The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$

Statement 2: The function x^2e^x and x^2e^{-x} are increasing for all $x > 0$ and sum of two increasing functions in any interval (a, b) is an increasing function in (a, b) **[2013, online]**

27. If f and g are differentiable function in $[0, 1]$ satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$ then for some $c \in (0, 1)$

(a) $2f'(c) = g'(c)$ (b) $2f'(c) = 3g'(c)$

(c) $f'(c) = g'(c)$ (d) $f'(c) = 2g'(c)$ **[2014]**

28. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$ then:

(a) $\alpha = -6, \beta = \dfrac{1}{2}$ (b) $\alpha = -6, \beta = -\dfrac{1}{2}$

(c) $\alpha = 2, \beta = -\dfrac{1}{2}$ (d) $\alpha = 2, \beta = \dfrac{1}{2}$ **[2014]**

29. If the volume of a spherical ball is increasing at the rate of 4π cc/sec, then the rate of increase of its radius (in cm/sec), when the volume is 288π cc is

(a) $\dfrac{1}{9}$ (b) $\dfrac{1}{6}$

(c) $\dfrac{1}{36}$ (d) $\dfrac{1}{24}$ **[2014, online]**

30. If non-zero real number b and c are such that $\min f(x) > \max g(x)$ where $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ $(x \in \mathbf{R})$ then $\left|\dfrac{c}{b}\right|$ lies in the interval:

(a) $\left[\dfrac{1}{\sqrt{2}}, \sqrt{2}\right]$ (b) $\left[0, \dfrac{1}{2}\right]$

(c) $\left[\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}\right]$ (d) $\left[\sqrt{2}, \infty\right]$ **[2014, online]**

31. Let f and g be two differentiable functions on $\mathbf{R}$ such that $f'(x) > 0$ and $g'(x) < 0$ for all $x \in \mathbf{R}$. Then for all x

(a) $g(f(x)) > g(f(x-1))$

(b) $f(g(x)) > f(g(x+1)$

(c) $f(g(x)) > f(g(x-1))$

(d) $g(f(x)) < g(f(x+1))$ **[2014 online]**

32. If Rolle's theorem holds for the function $f(x) = 2x^3 + ax^2 + bx$ in the interval $[-1, 1]$ for the point $c = \frac{1}{2}$, then the value of $2a + b$ is

(a) 1 (b) – 1
(c) 2 (d) – 2
[2014, 2015 online]

33. Let $f(x)$ be a polynomial of degree four having extreme value at $x = 1$ and $x = 2$. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right] = 3$, then $f(2)$ is equal to

(a) –8 (b) – 4
(c) 0 (d) 4 **[2015]**

34. The equation of a normal to the curve $\sin y = x\sin\left(\frac{\pi}{3}+y\right)$ at $x = 0$ is

(a) $2x + \sqrt{3}y = 0$ (b) $2y - \sqrt{3}x = 0$
(c) $2y + \sqrt{3}x = 0$ (d) $2x - \sqrt{3}y = 0$
[2015, online]

35. Let k and K be the minimum and maximum values of the function $f(x) = \frac{(1+x)^{0.6}}{1+x^{0.6}}$ in $[0, 1]$ respectively, then the order pair (k, K) is equal to

(a) $(1, 2^{0.6})$ (b) $(2^{-0.4}, 2^{0.6})$
(c) $(2^{-0.6}, 1)$ (d) $(2^{-0.4}, 1)$ **[2015, online]**

36. From the top of a 64 metres high tower, a stone is thrown upward vertically with the velocity of 48m/s. The greatest height (in metres) attained by stone, assuming the value of the gravitational acceleration $g = 32$ m/s^2, is:

(a) 100 (b) 88
(c) 128 (d) 112 **[2015, online]**

37. The distance, from the origin, of the normal to the curve, $x = 2\cos t + 2t\sin t$, $y = 2\sin t - 2t\cos t$ at $t = \frac{\pi}{4}$, is

(a) 4 (b) $2\sqrt{2}$
(c) 2 (d) $\sqrt{2}$ **[2015, online]**

38. Let the tangents drawn to the circle, $x^2 + y^2 = 16$ from the point $P(0, h)$ meet the x-axis at points A and B. If the area of ΔAPB is minimum, then h is equal to:

(a) $4\sqrt{3}$ (b) $3\sqrt{3}$
(c) $3\sqrt{2}$ (d) $4\sqrt{2}$ **[2015]**

39. Consider

$$f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right),\ x\in\left(0,\frac{\pi}{2}\right).$$

A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point:

(a) (0, 0) (b) $\left(0,\frac{2\pi}{3}\right)$
(c) $\left(\frac{\pi}{6},0\right)$ (d) $\left(\frac{\pi}{4},0\right)$ **[2016]**

40. If m and M are are the minimum and the maximum values of $4+\frac{1}{2}\sin^2 2x - 2\cos^4 x$, $x \in \mathbf{R}$, then $M - m$ is equal to

(a) $\frac{9}{4}$ (b) $\frac{15}{4}$
(c) $\frac{7}{4}$ (d) $\frac{1}{4}$ **[2016, online]**

41. If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 - 1$, $t \in \mathbf{R}$ meets the curve again at a point Q, then the coordinates of Q are:

(a) $(16t^2 + 3, -64t^3 - 1)$ (b) $(4t^2 + 3, -8t^3 - 1)$
(c) $(t^2 + 3, t^3 - 1)$ (d) $(t^2 + 3, -t^3 - 1)$
[2016, online]

42. Let $f(x) = \sin^4 x + \cos^4 x$. Then f is an increasing function in the interval

(a) $\left]\frac{5\pi}{8},\frac{3\pi}{4}\right[$ (b) $\left]\frac{\pi}{2},\frac{5\pi}{8}\right[$
(c) $\left]\frac{\pi}{4},\frac{\pi}{2}\right[$ (d) $\left]0,\frac{\pi}{4}\right[$ **[2016, online]**

43. Let C be a curve given by $y(x) = 1+\sqrt{4x-3}$, $x>\frac{3}{4}$. If P is a point on C, such that the tangent at P has slope $\frac{2}{3}$, then a point through which the normal at P passes, is

(a) (1, 7) (b) (3, –4)
(c) (4, –3) (d) (2, 3) **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. The slope of the normal to curve $y = x^3 - 4x^2$ at $(2, -1)$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$

(c) 4 (d) –4 **[2006]**

2. For the curve $x = t^2 - 1$, $y = t^2 - t$, the tangent line is perpendicular to the x-axis when

(a) $t = 0$ (b) $t = 1$

(c) $t = \frac{1}{\sqrt{3}}$ (d) $t = \frac{1}{2}$ **[2006]**

3. If $f(x) = 4^{\sin x}$ satisfies the Rolle's theorem on $[0, \pi]$, then the value of $c \in (0, \pi)$ for which $f'(c) = 0$ is

(a) $c = \frac{\pi}{6}$ (b) $c = \frac{\pi}{4}$

(c) $c = \frac{\pi}{2}$ (d) $c = \frac{\pi}{3}$ **[2006]**

4. The value of c for which the conclusion of Lagrange's mean value theorem holds for the function $f(x) = \sqrt{25 - x^2}$ on the interval $[1, 5]$ is

(a) $\sqrt{3}$ (b) $\sqrt{5}$

(c) $\sqrt{15}$ (d) 2 **[2007]**

5. Let $f(x) = \begin{cases} |x-1| + a & \text{if } x \le 1 \\ 2x + 3 & \text{if } x > 1 \end{cases}$

If $f(x)$ has a local minimum at $x = 1$, then

(a) $a > 5$ (b) $0 < a \le 5$

(c) $a \le 5$ (d) $a = 5$ **[2007]**

6. If m is the slope of a tangent to the curve $e^{2y} = 1 + 4x^2$ then

(a) $|m| \le 1$ (b) $|m| > 1$

(c) $|m| \ge 1$ (d) $|m| < 1$ **[2007]**

7. Let $f : (-\infty, \infty) \to (-\infty, \infty)$ be a continuous and differentiable function and let $f'(\cdot)$ denote the derivative of $f(\cdot)$. If $f(0) = -2$ and $f'(x) \le 3$ for each $x \in [0, 2]$, then the largest possible value of $f(2)$ is

(a) 1 (b) 2

(c) 3 (d) 4 **[2008]**

8. Let f: $[-1, 2] \to (-\infty, \infty)$ be given by

$$f(x) = \frac{x^4 + 3x^2 + 1}{x^2 + 1}$$

Then the maximum possible value of $f(\cdot)$ on $[-1, 2]$ is

(a) 1 (b) $\frac{29}{5}$

(c) $\frac{21}{5}$ (d) $\frac{28}{5}$ **[2008]**

9. Let $y = f(x)$ be a curve which passes through $(3, 1)$ and is such that normal at any point on it passes through $(1, 1)$. Then $y = f(x)$ describes

(a) a circle of area π

(b) an ellipse of area 2π

(c) an ellipse of area 3π

(d) a circle of area 4π **[2008]**

10. Let $f(x) = \begin{cases} x \sin\frac{\pi}{x} & , \quad 0 < x \le 1 \\ 0 & , \quad x = 0 \end{cases}$

then $f'(x) = 0$ for

(a) exactly two values of x

(b) no value of x

(c) infinitely many values of x

(d) exactly one value of x **[2009]**

11. Let $f(x) = [1 - x^2]$, $x \in \mathbf{R}$, where [ ] is the greatest integer function. Then

(a) f is increasing

(b) $x = 0$ is the point of maxima of f

(c) f is continuous at $x = 0$

(d) f is decreasing **[2009]**

12. A particle is constrained to move along the curve $y = \sqrt{x}$ starting at the origin at time t = 0. The point on the curve where the abscissa and the ordinate are changing at the same rate is:

(a) $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(\frac{1}{8}, \frac{1}{2\sqrt{2}}\right)$

(c) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (d) $(1, 1)$ **[2009]**

13. If the tangent and normal to the hyperbola $x^2 - y^2 = 4$ at a point cut off intercepts a_1 and a_2 respectively on x-axis and b_1 and b_2 respectively on y-axis then the value of $a_1 a_2 + b_1 b_2$ is

(a) –1 (b) 0

(c) 4 (d) 1 **[2010]**

14. Let f be a differentiable function defined on **R** such that $f(0) = -3$. If $f'(x) \le 5$ for all x then

(a) $f(2) > 7$ (b) $f(2) \leq 7$
(c) $f(2) > 8$ (d) $f(2) = 8$ **[2010]**

15. Let f be a function defined on $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ by $f(x) = 3\cos^4 x - 6\cos^3 x - 6\cos^2 x - 3$. Then the range of $f(x)$ is
(a) $[-12, -3]$ (b) $[-6, -3]$
(c) $[-6, 3]$ (d) $(-12, 3]$ **[2010]**

16. The function $f(x) = xe^{-x}$ has
(a) neither maximum nor minimum at $x = 1$
(b) a minimum at $x = 1$
(c) a maximum at $x = 1$
(d) a maximum at $x = -1$ **[2011]**

17. Each side of a square is increasing at the uniform rate of 1m/sec. If after sometime the area of the square is increasing at the rate of 8 m^2/sec, then the area of square at that time in sq. meters is
(a) 4 (b) 9
(c) 16 (d) 25 **[2012]**

18. The rate of change of the volume of a sphere with respect to its surface area when the radius is 2 units is:
(a) 4 (b) 3
(c) 2 (d) 1 **[2013]**

19. If m is the slope of a tangent to curve $e^y = 1 + x^2$ at the point (x, y) on the curve then all possible values of m lie in the interval:
(a) $[0, 1]$ (b) $(1, \infty)$
(c) $(-\infty, -1)$ (d) $[-1, 1]$ **[2013]**

20. $f(x) = |x \log x|$, $x > 0$ is monotonically decreasing in:
(a) $\left(0, \dfrac{1}{e}\right)$ (b) $\left[\dfrac{1}{e}, 1\right]$
(c) $(1, e)$ (d) (e, ∞) **[2014]**

21. Let $f(x) = |x - x_1| + |x - x_2|$, where x_1 and x_2 are distinct real numbers. Then the number of points at which $f(x)$ is minimum
(a) 1 (b) 2
(c) 3 (d) more than 3 **[2014]**

22. The maximum value of $f(x) = 2\sin x + \sin 2x$, in the interval $\left[0, \dfrac{3}{2}\pi\right]$ is
(a) $\sqrt{2} + 1$ (b) $2\sqrt{3}$
(c) $\dfrac{3\sqrt{3}}{2}$ (d) $\sqrt{3}$ **[2015]**

23. The abscissa of a point, tangent at which to the curve $y = e^x \sin x$, $x \in [0, \pi]$, has maximum slope, is:
(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$
(c) π (d) 0 **[2016]**

24. Let $p(x)$ be a real polynomial of degree 4 having extreme values at $x = 1$ and $x = 2$. If $\lim\limits_{x \to 0} \dfrac{p(x)}{x^2} = 1$, then $p(4)$ is equal to:
(a) 16 (b) 32
(c) 64 (d) 8 **[2016]**

Reasoning Type

25. Let b, c be two non-zero real numbers such that $b^2 \leq 3c$. Let $f(x) = x^3 + bx^2 + cx + d$, $x \in \mathbf{R}$.

Statement-1: f is a 1 – 1 function.
Statement-2: f is a strictly decreasing function on **R**.
[2009]

26. **Statement-1:** The function f defined on **R** as $f(x) = \min\{x, x^2)$ is not differentiable at $x = 1$.

Statement-2: The smaller angle between the tangents to the curve $y = x$ and $y = x^2$ at $x = 1$ is $\tan^{-1}\dfrac{1}{3}$.
[2010]

27. **Statement-1:** If $f(x) = e^{(x-1)(x-3)}$ then Rolle's theorem is applicable to $f(x)$ in the interval $[1, 3]$.

Statement-2: Mean value theorem is applicable to $f(x) = e^{(x-1)(x-3)}$ in the interval $[1, 4]$. **[2011]**

28. Let $f(x) = x^2 - 8x + 12$, $x \in [2, 6]$.

Statement 1: $f'(c) = 0$ for some $c \in (2, 6)$.

Statement 2: f is continuous on $[2, 6]$ and differentiable on $(2, 6)$ with $f(2) = f(6)$. **[2012]**

29. Let $a, b, c \in \mathbf{R}$, $a > 0$ and the function $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = ax^2 + bx + c$

Statement-1: $b^2 < 4ac \Rightarrow f(x) > 0$ for every value of x.

Statement-2: f is strictly decreasing in the interval $\left(-\infty, -\dfrac{b}{2a}\right)$ and strictly increasing in the interval $\left(-\dfrac{b}{2a}, \infty\right)$ **[2012]**

Answers

Concept-based

1. (a) **2.** (c) **3.** (b) **4.** (d)
5. (a) **6.** (c) **7.** (c) **8.** (a)
9. (b) **10.** (c)

Level 1

11. (c) **12.** (b) **13.** (b) **14.** (b)
15. (d) **16.** (c) **17.** (b) **18.** (a)
19. (d) **20.** (c) **21.** (a) **22.** (d)
23. (a) **24.** (b) **25.** (d) **26.** (c)
27. (d) **28.** (a) **29.** (c) **30.** (b)
31. (c) **32.** (c) **33.** (c) **34.** (c)
35. (b) **36.** (c) **37.** (d) **38.** (c)
39. (c) **40.** (c) **41.** (d) **42.** (d)
43. (a) **44.** (b) **45.** (c) **46.** (b)
47. (c) **48.** (a) **49.** (b) **50.** (b)
51. (d) **52.** (b) **53.** (c) **54.** (a)
55. (a) **56.** (d) **57.** (d) **58.** (a)
59. (a) **60.** (a)

Level 2

61. (b) **62.** (a) **63.** (b) **64.** (d)
65. (a) **66.** (b) **67.** (b) **68.** (a)
69. (d) **70.** (a) **71.** (c) **72.** (d)
73. (a) **74.** (a) **75.** (d) **76.** (c)
77. (c) **78.** (c) **79.** (c) **80.** (d)
81. (b) **82.** (d) **83.** (a) **84.** (b)
85. (a) **86.** (d) **87.** (d) **88.** (d)
89. (b) **90.** (c) **91.** (a) **92.** (d)
93. (b) **94.** (b) **95.** (a) **96.** (b)
97. (c) **98.** (c)

Previous Years' AIEEE/JEE Main Questions

1. (a) **2.** (b) **3.** (b) **4.** (a)
5. (b) **6.** (c) **7.** (a) **8.** (b)
9. (c) **10.** (d) **11.** (b) **12.** (a)
13. (b) **14.** (a) **15.** (d) **16.** (b)
17. (b) **18.** (b) **19.** (a) **20.** (b)
21. (c) **22.** (d) **23.** (c) **24.** (c)
25. (a) **26.** (c) **27.** (d) **28.** (c)
29. (c) **30.** (d) **31.** (b) **32.** (b)
33. (c) **34.** (a) **35.** (d) **36.** (a)
37. (c) **38.** (d) **39.** (b) **40.** (a)
41. (d) **42.** (c) **43.** (a)

Previous Years' B-Architecture Entrance Examination Questions

1. (a) **2.** (b) **3.** (c) **4.** (c)
5. (c) **6.** (a) **7.** (d) **8.** (b)
9. (d) **10.** (c) **11.** (d) **12.** (c)
13. (b) **14.** (b) **15.** (a) **16.** (c)
17. (c) **18.** (d) **19.** (d) **20.** (b)
21. (b) **22.** (c) **23.** (b) **24.** (a)
25. (c) **26.** (b) **27.** (b) **28.** (a)
29. (b)

Hints and Solutions

Concept-based

1. $$\frac{dy}{dx} = \frac{-a\cos^2 t+\sin t}{a\sin^2 t\cos t} = -\frac{\cos t}{\sin t}$$

Length of the tangent $= |y|\sqrt{1+\frac{\sin^2 t}{\cos^2 t}}$

$$= \frac{a\cos^3 t}{\cos t} = a\cos^2 t$$

2. $\frac{dy}{dx} = -\frac{e^{-t}}{2e^{t}} = -\frac{1}{2}e^{-2t}$. If $t = 0$ then $X = 2$, $y = 1$

Equation of normal is

$$Y - 1 = 2e^{2t}\Big|_{t=0}(X-2)$$

$\Rightarrow$ $Y - 1 = 2(X - 2) \Rightarrow 2X - Y - 3 = 0.$

3. $\frac{ds}{dt} = \frac{\pi}{9}\cos\frac{\pi}{2}t$ and $a = \frac{d^2s}{dt^2} = \frac{-\pi^2}{18}\sin\frac{\pi}{2}t$

$a(\mathrm{t} = 1) = -\frac{\pi^2}{18}$

4. $f'(x) = 1 + \log x - 1 = \log x > 0$ for $x > 1$. So f is increasing for $x > 1 \Rightarrow f(x) > f(1) = 0$ for $x > 1$. $f'(x) < 0$ for $0 < x < 1 \Rightarrow f$ is decreasing for $0 < x < 1 \Rightarrow f(x) > f(1) = 0$ for $0 < x < 1$.

5. $\frac{dy}{dx} = 3x^2$, we need x such that $3x^2 = \frac{8+1}{2+1} = \frac{9}{3} = 3$ $\Rightarrow x^2 = 1$ so $x = 1$ or -1. Thus two points are $(-1, -1)$ and $(1, 1)$.

6. $$f'(x) = 4x - \frac{1}{x}$$

The function f increases if $4x - \frac{1}{x} > 0$ *i.e.* $x > 1/2$

The function f decreases if $4x - \frac{1}{x} < 0$ *i.e.* $0 < x < \frac{1}{2}$

7. f is differentiable function, so the critical points are given by $f'(x) = 0$

   $f'(x) = 3x^3 - 3x^2 - 18x = 3x\,(x + 2)\,(x - 3)$

   The critical points of f are 0, –2, 3 but only 0, 3 $\in$ [–1 4].

8. $3x^2 \dfrac{dx}{dy} = 12 \Rightarrow \dfrac{dx}{dy} = \dfrac{4}{x^2}$. We need $\left|\dfrac{dx}{dy}\right| > 1$

   $\Rightarrow \dfrac{4}{x^2} > 1 \Rightarrow x^2 < 4, x \neq 0 \Rightarrow x \in (-2, 2) \sim \{0\}$.

9. For $\dfrac{x^2}{k^2} + \dfrac{y^2}{4} = 1$, $\dfrac{dy}{dx} = \dfrac{-4x}{k^2 y}$ and for $y^3 = 16x$,

   $\dfrac{dy}{dx} = \dfrac{16}{3y^2}$. For orthogonality $\left(\dfrac{-4x}{k^2 y}\right)\left(\dfrac{16}{3y^2}\right) = -1$

   $\Rightarrow \quad 64x = 3k^2 y^3 = 48k^2 x$

   $\Rightarrow \quad k^2 = \dfrac{4}{3} \Rightarrow k = \dfrac{2\sqrt{3}}{3}$

10. $g'(t) = 0 \Rightarrow 8 - 4t^3 = 0 \Rightarrow t^3 = 2$ so $t = 2^{1/3}$ but $2^{1/3} \notin [-2, 1]$. Now $g(-2) = -32$, $g(1) = 7$ So –32 in the least value.

Level 1

11. Let $h(x) = \tan^{-1} x - \dfrac{x}{1+x^2}$, $h'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{x^2+1} + \dfrac{2x^2}{(x^2+1)^2} > 0$ for all x. For $x > 0$ $h(x) > h(0) = 0$

    $\Rightarrow f(x) > g(x)$ on $(0, \infty)$

12. $f(2) = f(3) = f(4) = f(5) = f(6) = 0$,

    so, by Rolle's theorem applied on [2, 3], [3, 4], [4, 5], [5, 6] there are $x_1 \in (2, 3)$, $x_2 \in (3,4)$, $x_3 \in (4, 5)$, $x_4 \in (5, 6)$ such that $f'(x_i) = 0$, $i = 1, 2, 3, 4$. Since f' is polynomial of degree 4 so cannot have five roots.

13. $f'(x) = (x - 3)^4\,(x + 1)^3\,(9x - 7)$

    For $x < -1$, $f'(x) > 0$ and for $-1 < x < 0$, $f'(x) < 0$. $x = -1$ is a point of maxima.

14. $\dfrac{dy}{dx} = \tan\theta$, so equation the equation of normal is $y - a(\sin\theta - \theta\cos\theta) = -\cot\theta(x - a(\cos\theta + \theta\sin\theta)$ which simplifies to $x\cos\theta + y\sin\theta = a$. The distance of the normal from origin is $|\,a\,|$.

15. Apply Rolle's theorem to show that the given equation cannot have two real distinct roots.

16. $\left(\dfrac{dy}{dx}\right)$ $(a/8, a/8) = -1$. The equation of tangent at $(a/8, a/8)$ is $y + x = a/4$. The x and y intercepts of this line both equal $a/4$, so

    $2 = \dfrac{a^2}{16} + \dfrac{a^2}{16} \Rightarrow a = 4$.

17. The slope of tangent to $x^m\, y = b^m$ is $\dfrac{dy}{dx} = -\dfrac{my}{x}$. So the equation of tangent is $Y - y = -\dfrac{my}{x}(X - x)$ which simplifies to $\dfrac{X}{x + \frac{x}{m}} + \dfrac{Y}{y + my} = 1$. Thus the base of the triangle is the X-intercept, $x + x/m$ and the height is the y-intercept, $y + my$. So the area of the triangle is

    $$(1/2)\left(x + \frac{x}{m}\right)(y + my)$$

    $$= \frac{xy\,(1+m)^2}{2m} = \frac{b^m\,(1+m)^2}{2m\,n^{m-1}}$$

    which is constant if $m = 1$.

18. The slope of the tangent is $-x^3/y^3$ and its equation is $\dfrac{X}{a^4/x^3} + \dfrac{Y}{a^4/y^3} = 1$.

    Thus the required intercepts are $p = a^4/x^3$ and $q = a^4/y^3$ so that

    $p^{-4/3} + q^{-4/3} = a^{-16/3}\,(x^4 + y^4) = a^{-16/3}\,a^4 = a^{-4/3}$

19. $1 - 2\cos x > 0$ if $\cos x < 1/2 \Rightarrow x \in \left(\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right)$.

20. For $x \neq -1, 1$, $\dfrac{dy}{dx} = \dfrac{(x-1)^{2/3} - (x+1)^{2/3}}{3(x^2-1)^{2/3}} = 0$ only at $x = 0$. $y(0) = 2$, $y(-1) = y(1) = 2^{1/3}$, so the greatest value of y is 2.

21. $f'(x) = 4x - \dfrac{1}{x} = \dfrac{4x^2 - 1}{x} = \dfrac{(2x-1)\,(2x+1)x}{x^2}$

	$2x - 1$	$2x + 1$	x	$f'(x)$
$x < -1/2$	–	–	–	–
$-1/2 < x < 0$	–	+	–	+
$0 < x < 1/2$	–	+	+	–
$x > 1/2$	+	+	+	+

    Thus f increases on $[-1/2, 0] \cup [1/2, \infty]$.

22. Let (x, y) be any point on $y = \dfrac{e^x + e^{-x}}{2}$

$U = S^2$ = square of distance of (x, y) from $(0, 0)$

$$= x^2 + \frac{e^{2x}+e^{-2x}}{4} + \frac{1}{2}$$

$$\frac{dU}{dx} = 2x + \frac{e^{2x}-e^{-2x}}{2} = 0 \text{ for } x = 0$$

and $\frac{d^2U}{dx^2} > 0$ at $x = 0$. Hence S^2 is min for $x = 0$ and min $S^2 = \frac{1}{2}+\frac{1}{2} = 1$.

23. $2x + 2y\frac{dy}{dx} - 2 - 2\frac{dy}{dx} = 0 \Rightarrow \left.\frac{dy}{dx}\right|_{(2,2)} = -1$
Required equation of normal is $y - 2 = (x - 2)$ $\Rightarrow$ $y = x$.

24. Let $g(x) = \frac{f(x)}{x} \cdot g'(x) = \frac{xf'(x)-f(x)}{x^2}$.
Let $h(x) = x f'(x) - f(x)$, $h'(x) = xf''(x) > 0$
So $h(x) > h(0) = 0$ for $x > 0'$. Hence $g'(x) > 0$, $x > 0$ g increases on $(0, \infty)$.

25. If α and β are real and distinct roots of $x^3 - 12x + k = 0$. Then $f(x) = x^3 - 12x + k$ satisfies $f(\alpha) = f(\beta) = 0$ then there $\gamma \in [\alpha, \beta] \subseteq [0, 2]$ such that $f'(\gamma) = 0$ i.e., $3v^2 - 12 = 0 \Rightarrow \gamma = \pm 2$. Not possible.

26. $f'(x) = 8x^{1/3} - x^{-2/3} = \frac{8x-1}{x^{2/3}} = 0$ if $x = 1/8$
$f(1) = 3$, $f(-1) = 6 + 3 = 9$, $f\left(\frac{1}{8}\right) = -9/8$ thus max $f = 9$.

27. $f'(x) = 4x^3 - 18x^2 + 24x - 8$, $f''(x) = 12x^2 - 36x + 24 = 12(x^2 - 3x + 2) = 12(x - 1)(x - 2)$, $f''(x) = 0 \Rightarrow x = 1, 2$
If $x = 1$ then $y = 2$. If $x = 2$, $y = 3$. So equation of tangent at $(1, 2)$ is $y - 2 = f'(1)(x - 1) = 2(x - 1)$. Equation of the tangent at $(2, 3)$ is $y - 3 = f'(2)(x - 2) = 0$.

28. The odd degree has at least one real root if the equation has at two real roots then by Rolle's theorem there is $\gamma \in [\alpha, \beta]$ such that $6\gamma^2 - 6\gamma + 6 = 0 \Rightarrow \gamma^2 - \gamma + 1 = 0$ which is not possible for $\gamma \in \mathbf{R}$.

29. $f(x) = (x - 2)(x - 1)^4 \Rightarrow f'(x) = (x - 1)^4 + 4(x - 1)^3(x - 2) = (x - 1)^3(5x - 9)$ so for $1 < x < 9/5$, $f'(x) < 0$ and for $x > 9/5$, $f'(x) > 0$. Thus $x = 9/5$
f has local minimum $f(9/5) = -\frac{1}{5}\left(\frac{4}{5}\right)^4$.

30. $f(x) = \begin{cases} x^2+2x+2, & -1/2 \le x \le 0 \\ x^2-2x+2, & 0 < x \le 3/2 \end{cases}$
and $f'(1) = 0$
$f'(0)$ doesn't exist. Since $f(0) = 2$, $f(-1/2) = 5/4$, $f(3/2) = 5/4$, $f(1) = 1$ so Max. $f(x) = 2$ and min $f(x) = 1$.

31. The function is not differentiable at $x = 0$ and $x = 1$. Now

$$f(x) = \begin{cases} \frac{x-1}{x^2} & x \ge 1 \\ \frac{1-x}{x^2}, & x < 1 \\ & x \ne 0 \end{cases} \Rightarrow f'(x) = \begin{cases} \frac{-x+2}{x^3} \text{ if } x > 1 \\ \frac{x-2}{x^3}, \text{ if } & x < 1 \\ & x \ne 0 \end{cases}$$

so we get a third critical point $x = 2$.

32. The derivative of the function, $x^x(\log x + 1)$, is negative when $\log < -1$ i.e., when $x < e^{-1}$.

33. $y'(x) = 1 - e^x$ is positive when $e^x < 1$ i.e., when $x < 0$

34. $3 = f(1) = 1 + p + q$ and $f'(1) = 0 \Rightarrow 2 + p = 0$ $\Rightarrow p = -2$, $q = 4$.

35. Let $f(x) = ax^3 + bx^2 + cx + d$, $a \ne 0$, $f''(x) = 6ax + 2b$, $f''(x) = 0$
$\Rightarrow x = \frac{-b}{3a}$ $f'''(x) = 6a \ne 0$. So only one point of inflection.

36. $f(x) = \begin{cases} 4\tan^{-1} x & \text{if } -1 \le x \le 1 \\ 4\tan^{-1} x - \pi & \text{if } x > 1 \\ 4\tan^{-1} x + \pi & \text{if } x < -1 \end{cases}$
So max $f(x) = 4\frac{\pi}{2} - \pi = \pi$

37. $\left.\frac{dy}{dx}\right|_{(m^2, -m^3)} = -\frac{3m}{2}$ The equation of normal at $(m^2, -m^3)$ is $Y + m^3 = (2/3m)(X - m^2)$. Simplifying we have $Y = (2/3m)X - (2m/3 + m^3)$ which is identical with $y = 2mx - 2m^3$ if $m^2 = 2/3$.

38. $f'(x) = 2\cos x - 2\sin 2x = 2\cos x(1 - 2\sin x) > 0$
if (i) $\cos x > 0$, $\sin x < 1/2$ (ii) $\cos x < 0$, $\sin x > 1/2$
i.e., if $x \in (0, \pi/6) \cup (3\pi/2, 2\pi)$ or $x \in (\pi/2, 5\pi/6)$.
So f increases on $(0, \pi/6) \cup (\pi/2, 5\pi/6)$.

39. $f'(x) = \sec^2 x - \operatorname{cosec}^2 x = \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x} = \frac{-\cos 2x}{\sin^2 x \cos^2 x}$ $f'(\pi/4) = 0$, $f'(3\pi/4) = 0$. For $0 < x < \pi/4$, $f'(x) < 0$ and for $\pi/4 < x < \pi/2$, $f'(x) > 0$. Min $f(x) = f(\pi/4) = 2$.

40. $f'(x) = 12x(x^2 - x + 1)$, $f'(x) = 0$ only for $x = 0$.

41. Distance of (t^2, t) on $y^2 = x$ from $y - x - 1 = 0$ is $\frac{|t-t^2-1|}{\sqrt{2}} = \frac{t^2-t+1}{\sqrt{2}} + 1$. This is minimum if $t = 1/2$. So the shortest distance $= 3\sqrt{2}/8$.

42. $f'(x) = 3(x - (a + 1))\,(x - (a - 1)) = 0 \Rightarrow x = a \pm 1$.
$a - 1 \in (-2, 4) \Rightarrow a \in (-1, 5)$, $a + 1 \in (-2, 4) \Rightarrow a \in (-3, 3)$

43. $y = \tan A \tan (\pi/3 - A) = \dfrac{x(\sqrt{3} - x)}{1 + \sqrt{3}x}$, where $x = \tan A$.

$\dfrac{dy}{dx} = \dfrac{-(\sqrt{3}x - 1)\,(x + \sqrt{3})}{(1 + \sqrt{3}x)^2}$. As $0 < A < \pi/3$, $x = \tan A > 0$

$\dfrac{dy}{dx} = 0 \Rightarrow x = \dfrac{1}{\sqrt{3}} \cdot \dfrac{dy}{dx} > 0$ for $0 < x < \dfrac{1}{\sqrt{3}}$ and is < 0 for $x > \dfrac{1}{\sqrt{3}}$. So $y_{\max} = y = \left(x = 1/\sqrt{3}\right) = 1/3$.

44. $f(x) = -x \log x$, for $0 < x < 1$, and $f'(x) = -(1 + \log x)$. So $f'(x) = 0 \Rightarrow x = e^{-1}$. It is easy to see max $f(x) = f(e^{-1}) = e^{-1}$.

45. $f(x) = \log_x (1/9) - \log_3 x^2 = -\dfrac{\log 9}{\log x} - \dfrac{2\log x}{\log 3}$

$f'(x) = \dfrac{2((\log 3)^2 - (\log x)^2)}{x(\log x)^2 \log 3}$ so $f'(x) = 0 \Rightarrow \log x = \pm \log 3$
$\Rightarrow x = 3$ or $1/3$. One can see by first derivative test that max $f(x) = f(3) = -4$.

46. Applying Lagrange's mean value theorem there is $c \in (1, 6)$ such that $\dfrac{f(6) - +(1)}{5} = f'(c) \ge 2$
$\Rightarrow f(6) \ge f(1) + 10 = 8$.

47. $f'(x) = 3[(\sin^{-1} x)^2 - (\cos^{-1} x)^2]\sqrt{1 - x^2}$
$f'(x) = 0 \Rightarrow \sin^{-1} x = \cos^{-1} x$ or $\sin^{-1} x = -\cos^{-1} x$ but
$\sin^{-1} x + \cos^{-1} x = \pi/2$ so $\sin^{-1} x = \cos^{-1} x \Rightarrow \sin^{-1} x = \pi/4$
$\Rightarrow x = 1/\sqrt{2}$. Thus $f(1/\sqrt{2}) = \pi^3/32$.

48. $f'(x) = \log x + 4$ so $f'(x) > 0$ if $\log x > -4$ i.e., $x > e^{-4}$.

49. $f'(x) = (2 - x)xe^{-x}$. $f'(x) = 0 \Rightarrow x = 0, 2$
For $0 < x < 2$, $f'(x) > 0$ and for $x > 2$, $f'(x) < 0$
So max $f(x) = f(2) = 4e^{-2}$.

50. $f(x) =$

$$\begin{cases} 6 - 3x & , x \le -2 \\ 10 - x & , \ 2 < x \le 3 \\ 4 + x & , 3 < x \le 5 \\ 3x - 6 & , x > 5 \end{cases} \qquad \min f(x) = f(3) = 7.$$

51. f is not differentiable at $x = 0$. For $x \ne 0$, $f'(x) = 2 - 3x^{-1/3}$. So $f'(x) = 0$
$\Rightarrow x = \dfrac{27}{8} > \dfrac{10}{3}$, hence $f'(x)$ is never zero, for $x \ne 0$.

Now $f(-1) = -3$, $f(0) = 2$

$f(10/3) = \dfrac{26}{3} - 3\left(\dfrac{10}{3}\right)^{2/3} > 0$ but is less then 2.

Hence absolute minimum is $f(-1)$ and absolute maximum is $f(0)$.

52. $f(x) = \displaystyle\lim_{n \to \infty} \frac{x^n - 1}{x^n + 1} = \begin{cases} -1, & \text{if} \quad 0 \le x < 1 \\ 1, & \text{if} \quad x > 1 \\ 0, & \text{if} \quad x = 1 \end{cases}$

$g'(x) = f(x)$. Hence g has local minimum at $x = 1$.

53. $f'(x) = \cos x - \sin x = 0 \Rightarrow x = n\pi + \pi/4$
$f''(x) = -\sin x - \cos x = -(\sin x + \cos x)$

$$f''\left(n\pi + \frac{\pi}{4}\right) = (-1)^{n+1}\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)$$
$$= (-1)^{n+1}\sqrt{2}$$

If n is even then $f(x)$ has maxima and if n is odd then $f(x)$ has minima.

54. Let $f(x) = xe^{x(1-x)}$, we can write
$f'(x) = -(x - 1)(2x + 1)\,e^{x(1-x)}$
since $e^{x(1-x)} > 0\ \forall\, x \in \mathbf{R}$ so
$f'(x) < 0$ for $x < -1/2$; $f'(x) > 0$ for $-1/2 < x < 1$ and is < 0 for $x > 1$.
Thus $f(x)$ increases on $[-1/2, 1]$

55. Equation tangent at $P(c, e^c)$ is $y - e^c = e^c\,(x - c)$ (1)
The equation of line joining $(c + 1, e^{c+1})$ and (c^{-1}, e^{c-1}) is

$$y - e^{c+1} = \frac{1}{2}\,(e^{c+1} - e^{c-1})\,(x - c - 1) \qquad (2)$$

Subtracting (2) from (1)
$e^c\,(e - 1) =$

$$e^c\left[(x - c)\left\{1 - \frac{1}{2}\left(e - e^{-1}\right)\right\} + \frac{1}{2}\left(e - e^{-1}\right)\right]$$

$$\Rightarrow e - 1 = (x - c)\left[1 - \frac{1}{2}\left(e - e^{-1}\right)\right] + \frac{1}{2}\left(e - e^{-1}\right)$$

$$\Rightarrow c - x = \frac{e + e^{-1} - 2}{e - e^{-1} - 2} > 0 \quad \Rightarrow \quad x < c$$

Thus two lines meet on the left of $x = c$.

56. Consider $f(x) = x^{1/x}$

$$f'(x) = x^{1/x}\left(\frac{1 - \log x}{x^2}\right)$$

f has local maximum at $x = e$
Since $\pi > e$ and f decreases on (e, ∞) so $f(\pi) < f(e) \Rightarrow \pi^e < e^\pi$.

57. Applying Lagrange's mean value on $[x, y]$, we have $c \in (x, y)$ such that

$$f'(c) = \frac{f(y) - f(x)}{y - x}$$

$| f(y) - f(x) | = | f'(c) | \, | y - x | \le M \, | y - x |$
(if $| f'(x) | \le M$)
and $| f'(y) - f(x) | \ge M \, | x - y |$ (if $|f'(x) | \ge M$)
Putting $f(x) = \cot x$, $f'(x) = -\operatorname{cosec}^2 x$, $| f'(x) | \ge 1$
Hence $| \cot x - \cot y | \ge | x - y |$

58. $f'(x) = 2 \cos x - 2 \sin 2x = 2 \cos x (1 - 2 \sin x)$
Now determine the sign of $f'(x)$ on $[0, 2\pi]$.

59. $f(x) = \pi/4 - \tan^{-1} x \;\Rightarrow\; f'(x) = -\dfrac{1}{1+x^2} < 0$ for all x. So the greatest value is $f(0) = \pi/4$ and least value is $f(1) = 0$.

60. $f'(x) = \dfrac{1 - \log x}{(\log x)^2}$.

Level 2

61. $f'(x) = (2/3)(x - 2)^{-1/3}(2x + 1) + 2(x - 2)^{2/3}$

$= \left(\dfrac{10}{3}\right)(x - 1)(x - 2)^{-2/3}$

$f'(x) = 0 \Rightarrow x = 1$ also $f'(x)$ does not exist at $x = 2$, hence the critical points are $x = 1$ and

$x = 2$.

62. $f(-2) = 1$

63. $f'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{2x} = \dfrac{-(1-x)^2}{\left(1+x^2\right)2x} < 0$

for $x > 0$ i.e. f decreases on $\left[1/\sqrt{3}, \sqrt{3}\right]$
Therefore, the greatest value is $f\left(1/\sqrt{3}\right)$

$= \tan^{-1} \dfrac{1}{\sqrt{3}} - \dfrac{1}{2} \log \dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6} + \dfrac{1}{4} \log 3.$

64. $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1 \Rightarrow \dfrac{2x}{9} - \dfrac{y}{2}\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{4x}{9y}$

The tangent will be perpendicular to $2y + 5x = 10$ if $\dfrac{4x}{9y} \times \left(\dfrac{-5}{2}\right) = -1 \Rightarrow 10x = 9y$

Substituting the value of x in $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$, we have $\left(\dfrac{9}{100} - \dfrac{1}{4}\right) y^2 = 1$, which is not possible. Thus there is no point on the curve at which the tangent is perpendicular to $2y + 5x = 10$.

65. $$f'(x) = \begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$= (x + b^2)(x + c^2) - b^2c^2 + (x + a^2)(x + c^2)$
$- a^2 c^2 + (x + a^2)(x + b^2) - a^2 b^2$
$= 3x^2 + 2(a^2 + b^2 + c^2) x$
$= x (3x + 2(a^2 + b^2 + c^2))$

f decreases if $x < 0$ and $x > -\dfrac{2}{3}(a^2 + b^2 + c^2)$
i.e. on $((-2/3)(a^2 + b^2 + c^2), 0)$.

66. Let 'a' be the part of log in the channel of width 27 m and 'b' be the part of log in the channel of width 64 m.

So $l = a + b$

$= 27 \operatorname{cosec} \theta + 64 \sec \theta$

$\dfrac{dl}{d\theta} = -27 \operatorname{cosec} \theta \cot \theta + 64 \sec \theta \tan \theta$

$\dfrac{dl}{d\theta} = 0 \Rightarrow \tan \theta = \dfrac{3}{4}$

One can verify that $\dfrac{d^2l}{d\theta^2} < 0$ for $\theta = \tan^{-1} \dfrac{3}{4}$

$\max l = 27 \times \dfrac{5}{3} + 64 \times \dfrac{5}{4} = 125.$

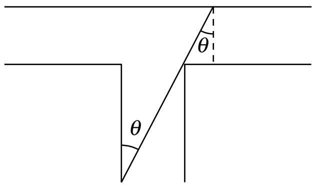


Fig. 11.7

67. $f'(x) = \sin a \tan^2 x \sec^2 x + (\sin a - 1) \sec^2 x$

$= \sec^2 x (\sin a \tan^2 x + (\sin a - 1))$

Since $\sec^2 x \neq 0$ so $\tan^2 x = \dfrac{1 - \sin a}{\sin a}$

Now $a \in [\pi, 2\pi]$ so $\sin a < 0$. Hence L.H.S. is ≥ 0 but R.H.S. < 0. Thus there is no critical point.

68. The derivative of $f(x)$ is $(4a - 3) + (a - 7) \cos x$. It is

zero when $\cos x = \dfrac{4a-3}{7-a} \Rightarrow \left|\dfrac{4a-3}{7-a}\right| \le 1$

$$\Rightarrow -1 \le \frac{4a-3}{7-a} \le 1 \Rightarrow a - 7 \le 4a - 3 \le 7 - a$$

i.e. when $a \ge -4/3$ or $a \le 2$. Hence the interval when f has no critical points are $(-\infty - 4/3]$ and $[2, \infty)$.

69. $f'(x) = 3\left(1 - \frac{\sqrt{21-4b-b^2}}{b+1}\right) x^2 + 5 > 0$ if

$$x^2 > \frac{-5}{3\left(1 - \frac{\sqrt{21-4b-b^2}}{b+1}\right)} \text{ if}$$

$$1 - \frac{\sqrt{21-4b-b^2}}{b+1} > 0$$

which is trivially true if $b + 1 < 0$ i.e. $b < -1$

If $b + 1 > 0$ then

$(b + 1)^2 > 21 - 4b - b^2 \Rightarrow (b - 2)(b + 5) > 0$

i.e. $b > 2$ or $b < -5$

so $b \in [2, 3]$.

70. Any point on the ellipse is given by $(a \cos \theta, b \sin \theta)$

Also $\frac{dy}{dx} = -\frac{b^2x}{a^2y} = \frac{-b}{a} \cot \theta$

The equation of tangent at $(a \cos \theta. b \sin \theta)$ is $Y - b \sin \theta = \frac{-b}{a} \cot\theta (X - a \cos \theta)$

This tangent cuts the coordinate axes at the points $\left(0, \frac{b}{\sin\theta}\right)$ and $\left(\frac{a}{\cos\theta}, 0\right)$

The area of the triangle formed by the tangent and the coordinate axes is

$$A = \frac{1}{2}\frac{ab}{\sin\theta\cos\theta} = ab \operatorname{cosec} 2\theta$$

But $\operatorname{cosec} 2\theta$ is smallest when $\theta = \pi/4$

$A_{\min} = ab.$

71. Let $f(x) = x - \log(1 + x)$. The domain of f is $(-1, \infty)$. $f'(x) = 1 - \frac{1}{1+x} = \frac{1+x-1}{1+x} = \frac{x}{1+x}$. Since $1 + x > 0$ for $x \in$ dom f so $f'(x) > 0$ if $x > 0$ and $f'(x) < 0$ for $(-1, 0)$. Thus f increases $(0, \infty)$. Morever, $f(0) = 0$. Hence $f(x) \ge f(0) = 0 \Rightarrow x \ge \log(1 + x)$ for $x \in (0, \infty)$.

For $x \in (-1, 0)$, f decreases so

$f(x) \ge f(0)$. Hence $x \ge \log(1 + x)$ on $(-1, \infty)$.

72. $2^{(x^2-3)^2} + 27 \ge 28$ and the value 28 is attained at $x = \sqrt{3}$. Hence the minimum value of $2^{(x^2-3)^2} + 27$ is 28.

73. $f(x) = \begin{cases} -x & -2 < x < 0 \\ 1 & x = 0 \\ x & 0 < x < 2 \end{cases}$

f is not continuous at $x = 0$. Morever

$f(0) > f(a)$ for $a \in (-1/2, 1/2) \sim \{0\}$. f has local maximum at $x = 0$.

74. $f'(x) = e^{x(1-x)} + x(1 - 2x)e^{x(1-x)}$

$= (1 + x - 2x^2)e^{x(1-x)} = -(x - 1)(2x + 1)e^{x(1-x)}$

Since $e^{x(1-x)} > 0$ for all x, so $f'(x) > 0$ if and only if $(x - 1)(1 + 2x) < 0$ i.e $-1/2 = \min(1, -1/2) < x < \max(1, -1/2) = 1$.

Thus f increases on $[-1/2, 1]$.

75. Clearly $f(0) = f(1) = 0$. For f to be continuous on $[0, 1]$, we must have $\lim\limits_{x\to 0+} x^{\alpha} \log x = 0$. This is possible only if $\alpha > 0$. Thus $\alpha = 1/2$ is only such value.

76. Let h be the height of cone, then the radius r of the cone satisfies $r^2 = 3 - h^2$

$V = \frac{1}{3}\pi(3 - h^2)h$ so $\frac{dV}{dh} = \frac{1}{3}\pi(-3h + 3)$

$\frac{dV}{dh} = 0 \Rightarrow h = 1$. Also $\frac{d^2V}{dh^2} = -\pi < 0$

so $V_{\max} = \frac{1}{3}\pi(3 - 1) = \frac{2\pi}{3}$.

77. Any point on the ellipse is of the form $(4 \cos \theta, 3 \sin \theta)$, $0 \le \theta \le 2\pi$. Area A of a rectangle inscribed in the ellipse $= 4 \times 4 \cos\theta \times 3 \sin \theta = 24 \sin 2\theta$. This will be maximum when $2\theta = \pi/2$ or $\theta = \pi/4$. Hence the dimensions of the ellipse are

$2 \cdot 4 \cdot \frac{1}{\sqrt{2}}$ and $2 \cdot 3 \cdot \frac{1}{\sqrt{2}}$ i.e. $2\sqrt{8}$ and $3\sqrt{2}$.

78. $y' = 4ax^3 + 3bx^2 + 2cx + d$

$y'' = 12ax^2 + 6bx + 2c$, $y'' = 0$ should have real distinct roots. This is true if $36b^2 - 96ac > 0$

i.e. $3b^2 - 8ac > 0$.

79. $f(x) = |x^2 - 3x + 2|$

$= -(x^2 - 3x + 2)$, $3/2 \le x \le 7/4$

$f'(x) = -(2x - 3)$. So the critical points are

3/2, 7/4 . $f(3/2) = \dfrac{1}{4}$, $f(7/4) = \dfrac{3}{16}$. Thus

$f(x) \geq \dfrac{3}{16}$. So $m = \dfrac{3}{16}$

80. We are looking for $x \in [0, \pi]$ for which $|(1/2)x - \sin x|$ is maximum.

$$S = \begin{cases} (1/2)x - \sin x & \text{if } (1/2)x \geq \sin x \\ \sin x - (1/2)x & \text{if } \sin x < (1/2)x \end{cases}$$

$\dfrac{dS}{dx} = 1/2 - \cos x = 0 \Rightarrow \cos x = 1/2 \Rightarrow x = \pi/3$

Thus critical points are $x = 0, \pi/3, \pi$. $S(0) = 0$,

$S(\pi/3) = \dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}$, $S(\pi) = \dfrac{\pi}{2}$. Thus

$S_{\max} = \pi/2$ at $x = \pi$.

81. $2ax + 2h\left(x\dfrac{dy}{dx} + y\right) + 2\,by\,\dfrac{dy}{dx} = 0$

$\Rightarrow \dfrac{dy}{dx} = -\dfrac{ax + hy}{hx + by}$

The tangent will be || to Y-axis if $hx + by = 0$

82. $\dfrac{dy}{dx} = \tan\alpha - \dfrac{ax}{u^2\cos^2\alpha}$. The tangent will be parallel to $y = x$ if $\tan\alpha - \dfrac{ax}{u^2\cos^2\alpha} = 1$

$\Rightarrow x = \dfrac{(\tan\alpha - 1)\,u^2\cos^2\alpha}{a}$

$= \dfrac{u^2}{a}(\sin\alpha - \cos\alpha)\cos\alpha$

The corresponding ordinate is given by

$y = \dfrac{u^2}{a}(\sin\alpha - \cos\alpha)\cos\alpha\tan\alpha$

$- \dfrac{a}{2u^2\cos^2\alpha}\dfrac{u^4}{a^2}(\sin\alpha - \cos\alpha)^2\cos^2\alpha$

$= \dfrac{u^2}{a}\left[\sin^2\alpha - \sin\alpha\cos\alpha - \dfrac{1}{2}(1 - 2\sin\alpha\cos\alpha)\right]$

$= \dfrac{u^2}{a}(\sin^2\alpha - 1/2)$

According to given condition

$\dfrac{u^2}{a}(\sin^2\alpha - 1/2) = \dfrac{u^2}{4a}$.

$\Rightarrow \sin^2\alpha = 3/4$

83. $f(x) = \begin{cases} 1 - x + a, & x \leq 1 \\ 2x + 3, & x > 1 \end{cases}$

If $f(x)$ has local minimum at $x = 1$, then $f(1) \leq f(1 + h)$, $h > 0$ i.e. $a \leq 2\,(1 + h) + 3$

Hence $a \leq 5$. So $a = 5$.

84. $f'(x) = \lim\limits_{h\to0}\dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to0}\dfrac{f(x) + f(h) + 2xh - 1 - f(x)}{h}$

$= \lim\limits_{h\to0}\dfrac{f(h) - f(0)}{h} + 2x$

(Putting $x = y = 0$ we obtain $f(0) = 1$)

$= f'(0) + 2x = 2\,(x + 1)$

$f(x) = (x + 1)^2 + C$. Putting $x = 0$, we obtain $C = 0$. Thus $f(x) = (x + 1)^2$. But $g'(x) = f(x) = (x + 1)^2 > 0$ so g increases on $(-\infty, \infty)$ in particular on $(0, \infty)$.

85. $\dfrac{dy}{dx} = -\dfrac{x}{y}$, $\left.\dfrac{dy}{dx}\right|_{(1,\sqrt{3})} = -\dfrac{1}{\sqrt{3}}$

Equation of tangent at $(1, \sqrt{3})$ is
$Y - \sqrt{3} = \left(1/\sqrt{3}\right)(X - 1)$

This will intersect X – axis at (4, 0). Equation of the normal at $(1, \sqrt{3})$ is

$Y - \sqrt{3} = \sqrt{3}\,(X - 1)$

This will intersect X-axis at (0, 0) so the required area of the triangle

$= \dfrac{1}{2}4\sqrt{3} = 2\sqrt{3}$.

86. $y' = \dfrac{x^2 - 2x + 3 - 2(x-1)^2}{(x^2 - 2x + 3)^2} = \dfrac{-(x-1)^2 + 2}{(x^2 - 2x + 3)^2}$

$y' = y\ 0$ for $x = 1 \pm \sqrt{2}$

$y' > 0$ for $1 < x < 1 + \sqrt{2}$ and $y' < 0$ for $x > 1 + \sqrt{2}$

$y_{\max} = y\left(1 + \sqrt{2}\right)$. Similarly $y_{\min} = y\left(1 - \sqrt{2}\right)$

$y\left(1 + \sqrt{2}\right) = \dfrac{\sqrt{2}}{4}$ and $y\left(1 - \sqrt{2}\right) = -\dfrac{\sqrt{2}}{4}$.

87. Equation of tangent at any point (x_1, y_1) is
$xx_1 + yy_1 = 16$

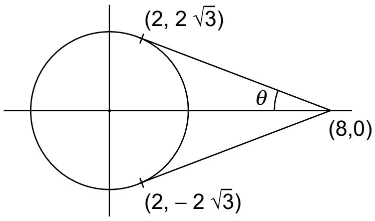


Fig. 11.8

This passes through (8, 0) if $8x_1 = 16$ i.e. $x_1 = 2$ so $y_1 = 2\sqrt{3}$. The slope of tangent is $-\sqrt{3}$.

Therefore $\theta = \pi/6$ and $2\theta = \pi/3$.

88. For $x \neq -2$,

$$f'(x) = \frac{2}{3}(x+2)^{-1/3}(2x-1) + (x+2)^{2/32}\,2$$

$$= \frac{(10/3)(x+1)}{(x+2)^{1/3}}$$

$f'(x) = 0 \Rightarrow x = -1$ clearly f is not differentiable at $x = -2$. Thus the critical points are – 1, – 2.

89. $$f'(x) = \frac{\log(e+x) \times \frac{1}{\pi + x} - \log(\pi + x) \times \frac{1}{e+x}}{(\log(e+x))^2}$$

$$= \frac{\log(e+x)\times(e+x) - (\pi+x)\log(\pi+x)}{(\pi+x)(e+x)(\log(e+x))^2}$$

Since log function is an increasing function and $e < \pi$, $\log(e+x) < \log(\pi + x)$

$\Rightarrow (e + x)\log(e + x) < (e + x)\log(\pi + x) < (\pi + x)\log(\pi + x)$ for all $x > 0$. Thus $f'(x) < 0$ for all $x > 0$.

$\Rightarrow f$ decreases on $[0, \infty)$.

90. $2(1 + b) = 32 \Rightarrow l + b = 16$.

Area of rectangle $A = lb = l(16 - l)$

$\frac{dA}{dl} = 16 - 2l$ so $\frac{dA}{dl} = 0 \Rightarrow l = 8$

$\frac{dA}{dl} = -2 < 0$. Thus A is maximum when $l = 8$.

91. $$f'(x) = -\frac{1}{1+x^2} + \frac{1}{2x} = \frac{(1-x)^2}{(1+x^2)2x} > 0 \text{ for } x \in (1, \sqrt{3})$$

f increases on $[1, \sqrt{3}]$. The greatest value of f is equal to $f(\sqrt{3}) = \cot^{-1}\sqrt{3} + \frac{1}{2}\log\sqrt{3}$

$$= \frac{\pi}{6} + \frac{1}{4}\log 3.$$

92. $2y\frac{dy}{dt} = 4\frac{dx}{dt}$. According to the given condition

$\left.\frac{dy}{dt}\right|_{(7,6)} = 3$. So

$2 \times 6 \times 3 = 4\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 9$ unit/s.

93. $2(l + b) = 24 \Rightarrow l + b = 12$

$\frac{dl}{dt} + \frac{db}{dt} = 0$. If $\frac{dl}{dt}$ is 1 cm/s

then $\frac{db}{dt} = -1$. $\frac{dA}{dt} = l\frac{db}{dt} + b\frac{dl}{dt} = b - l$

$\frac{dA}{dt}$ will start decreasing if $b - l \le 0$ but $b + l = 12$ so $l = 6$.

94. $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = -\frac{3}{\pi r^2}$.

95. $\frac{dA}{dt} = l\frac{db}{dt} + b\frac{dl}{dt}$

$\left.\frac{dA}{dt}\right|_{l = 12, b = 5} = 12 \times 2 + 5 \times (-2) = 14.$

96. Let $g(x) = f(x) - x^2$. As f is twice differentiable so g is also twice differentiable.

Also

$g(1) = f(1) - 1^2 = 1 - 1 = 0$

$g(2) = f(2) - 2^2 = 2^2 - 2^2 = 0$

$g(3) = f(3) - 3^2 = 3^2 - 3^2 = 0$

By Rolle's theorem $\exists\ \alpha \in (1, 2)$ and $\beta \in (2, 3)$ such that $g'(\alpha) = 0$, $g'(\beta) = 0$.

Again applying Rolle's theorem $\exists\ \gamma \in (\alpha, \beta)$ such that $g''(\gamma) = 0$

As $\alpha \in (1, 2)$, $\beta \in (2, 3)$, $(\alpha, \beta) \subset (1, 3)$

$g''(\gamma) = f''(\gamma) - 2 = 0 \Rightarrow f''(\gamma) = 2$

So there is some $x \in (1, 3)$ such that $f''(x) = 2$.

97. Equation of tangent at $P(x, y)$

$Y - y = f'(x)(X - x)$

so $A = \left(x - y\dfrac{1}{f'(x)}, 0\right)$ and $B = (0, y - x f'(x))$.

$3{:}1 = BP{:}AP$

$= \sqrt{x^2 + x^2(f'(x))^2} : \sqrt{y^2/(f'(x))^2 + y^2}$

$$\frac{9}{1} = \frac{x^2(1+(f'(x))^2)}{y^2(1+(f'(x))^2)} \quad (f'(x))^2 = \frac{x^2(f'(x))^2}{y^2}$$

$$\Rightarrow 9\frac{y^2}{x^2} = (f'(x^2)) \Rightarrow 3\frac{y}{x} = \pm\frac{dy}{dx}$$

$$\Rightarrow \frac{x^3}{y} = \text{Const or } x^3y = C$$

Since $f(1) = 1$, so Const $= 1$

Thus the curve is either $x^3 = y$ or $x^3y = 1$. The last curve passes through $(2, 1/8)$

98. f is a differentiable function and

$$f'(x) = 3x^2 + 2bx + c$$

$$= 3\left[\left(x + \frac{b}{3}\right)^2 + \frac{c}{3} - \frac{b^2}{9}\right]$$

$$= 3\left[(x + b/3)^2 + \frac{3c - b^2}{9}\right]$$

Since $b^2 < c < 3c$ so $3c - b^2 > 0$ and hence

$f'(x) > 0$ i.e. f is strictly increasing on **R**.

Previous Years' AIEEE/JEE Questions

1. Let $f(x) = \dfrac{1}{3}ax^3 + \dfrac{1}{2}bx^2 + cx \cdot$

Note that f is continuous and differentiable on **R**. We have

$f(0) = 0$ and $f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c = 0$.

$\therefore$ By the Rolle's theorem $\exists\ \alpha \in (0, 1)$ such that $f'(\alpha) = 0$.

i.e. $a\alpha^2 + b\alpha + c = 0$.

2. $$\frac{dx}{dt} = a\cos t - b\cdot\frac{a}{b}\cos\left(\frac{at}{b}\right) = a\left[\cos t - \cos\frac{at}{b}\right]$$

$$\frac{dy}{dt} = -a\sin t + b\cdot\frac{a}{b}\sin\left(\frac{at}{b}\right) = a\left[\sin\frac{at}{b} - \sin t\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin\frac{at}{b} - \sin t}{\cos t - \cos\frac{at}{b}} = \frac{2\cos\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\sin\left(\left(\frac{a}{b}-1\right)\frac{t}{2}\right)}{-2\sin\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\sin\left(\left(1-\frac{a}{b}\right)\frac{t}{2}\right)}$$

$$= \cot\left(\frac{a}{b}+1\right)\frac{t}{2}$$

Equation tangent at any point is

$$Y - \left(a\cos t - b\cos\frac{at}{b}\right)$$

$$= \cot\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\left(X - \left(a\sin t - b\sin\frac{a}{b}t\right)\right) \qquad \text{(i)}$$

Distance of origin from (i) is

$$S = \left|\frac{\left(a\cos t - b\cos\frac{at}{b}\right) - \cot\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\left(a\sin t - b\sin\frac{a}{b}t\right)}{\sqrt{1+\cot^2\left(\frac{a}{b}+1\right)\frac{t}{2}}}\right|$$

$$= \left|\sin\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\left(a\cos t - b\cos\frac{at}{b}\right) - \cos\left(\left(\frac{a}{b}+1\right)\frac{t}{2}\right)\left(a\sin t - b\sin\frac{a}{b}t\right)\right|$$

$$= \left|a\sin\left(\left(\frac{a}{b}+1\right)\frac{t}{2} - t\right) - b\sin\left(-\frac{a}{b}t + \left(\frac{a}{b}+1\right)\frac{t}{2}\right)\right|$$

$$= \left|a\sin\left(-\frac{t}{2} + \frac{a}{b}\frac{t}{2}\right) + b\sin\left(\frac{t}{2}\frac{a}{b} - \frac{t}{2}\right)\right|$$

$$= \left|(a+b)\sin\left(\frac{a}{b}\frac{t}{2} - \frac{t}{2}\right)\right|$$

So the maximum value of S is $a + b$.

3. $f'(x) = 6x^2 - 18ax + 12a^2 = 6(x^2 - 3ax + 2a^2)$

$f''(x) = 6(2x - 3a)$. According to the given condition $f'(p) = 0\ f'(q)$ and $f''(p) < 0$, $f''(q) > 0$.

$p + q = 3a$, $pq = 2a^2$. Since $p^2 = q$, so $p^3 = 2a^2$ and $p + p^2 = 3a$ and $p < \dfrac{3a}{2}$ and $q > \dfrac{3a}{2}$. $p = \dfrac{-1 \pm \sqrt{1+12a}}{2}$

thus $\left(\dfrac{-1 \pm \sqrt{1+12a}}{2}\right)^3 = 2a^2$. Clearly $a = 2$ satisfy this equation as for $a = 1$, $1/2$, 3 the L.H.S. is irrational and R.H.S. is a rational number.

4. We have

$$u = \sqrt{\frac{a^2+b^2}{2} + \frac{a^2-b^2}{2}\cos 2\theta} + \sqrt{\frac{a^2+b^2}{2} - \frac{a^2-b^2}{2}\cos 2\theta}$$

Squaring, we get

$$u^2 = a^2 + b^2 + 2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2 - \left(\frac{a^2-b^2}{2}\right)^2 \cos^2 2\theta}$$

Thus, $\max (u^2) = a^2 + b^2 + 2\left(\frac{a^2+b^2}{2}\right)$

$= 2(a^2 + b^2)$

and $\min (u^2) = a^2 + b^2 + 2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2 - \left(\frac{a^2-b^2}{2}\right)^2}$

$= a^2 + b^2 + 2ab = (a + b)^2$

$\therefore \max (u^2) - \min (u^2) = (a - b)^2.$

5. $f''(x) = 6(x - 1) \Rightarrow f'(x) = 3(x - 1)^2 + C_1$

But $f'(x)]_{(2,\,1)} = \left.\frac{dy}{dx}\right]_{(2,1)} = 3$

$\therefore 3 = 3(2 - 1)^2 + C_1 \Rightarrow C_1 = 0$

Thus, $f'(x) = 3(x - 1)^2$

$\Rightarrow f(x) = (x - 1)^3 + C_2$

As $y = f(x)$ passes through $(2, 1)$

$1 = (2 - 1)^3 + C_2 \Rightarrow C_2 = 0$

$\therefore f(x) = (x - 1)^3$

6. $\frac{dx}{d\theta} = -a \sin \theta,\ \frac{dy}{d\theta} = a \cos \theta$

$\therefore \frac{dy}{dx} = \frac{-a\cos\theta}{a\sin\theta} \Rightarrow$ slope of normal at θ is $\frac{\sin\theta}{\cos\theta}$.

Thus, equation of normal at θ is

$y - a \sin \theta = \frac{\sin\theta}{\cos\theta}[x - a(1 + \cos \theta)]$

This clearly passes through $(a, 0)$

7. If $f(x) = 3x^2 - 2x + 1$, then

$f'(x) = 6x - 2 = 6\left(x - \frac{1}{3}\right) < 0$ for $x < \frac{1}{3}$

Thus, $f(x)$ decreases on the interval $(-\infty, 1/3]$.

If $f(x) = x^3 + 6x^2 + 6$,

$f'(x) = 3x^2 + 12x = 3x (x + 4) > 0$ if $x < -4$

$\Rightarrow f(x)$ increases on $(-\infty, -4]$

If $f(x) = x^3 - 3x^2 + 3x + 3$, then

$f'(x) = 3x^2 - 6x + 3 = 3(x - 1)^2 \geq 0\ \forall\ x$

$f(x)$ increases on $(-\infty, \infty)$

If $f(x) = 2x^3 - 3x^2 - 12x + 6$, then

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

$= 6(x - 2)(x + 1) > 0$ for $x > 2$

$\Rightarrow f(x)$ increases on $[2, \infty)$

8. $x = a(\cos \theta + \theta \sin \theta),\ y = a(\sin \theta - \theta \cos \theta)$

$\Rightarrow \frac{dx}{d\theta} = a(-\sin \theta + \sin \theta + \theta \cos \theta) = a\theta \cos \theta$

and $\frac{dy}{d\theta} = a(\cos\theta - \cos \theta + \theta \sin \theta) = a\theta \sin \theta$

$\therefore \frac{dy}{dx} = \frac{\sin\theta}{\cos\theta}$

Equation of normal at θ

$y - a(\sin \theta - \theta \cos \theta) = -\frac{\cos\theta}{\sin\theta}\{x - a(\cos \theta + \theta \sin \theta)\}$

$\Rightarrow x \cos \theta + y \sin \theta = a\{\cos^2 \theta + \theta \cos \theta \sin \theta + \sin^2\theta - \theta \cos \theta \sin \theta\}$

$\Rightarrow x \cos \theta + y \sin \theta = a$

This line is at a distance a from the origin.

9. By the Lagrange's mean value theorem,

$\frac{f(6) - f(1)}{6 - 1} = f'(\alpha)$ for some $\alpha \in (1, 6)$

$\Rightarrow f(6) + 2 = 5f'(\alpha) \geq 10$

$\Rightarrow f(6) \geq 8$

10. Let thickness of ice at time t be r cm. Then volume of ice:

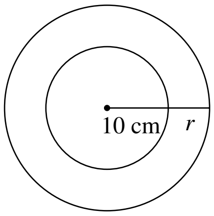


Fig. 11.9

$V = \frac{4}{3}\pi(10 + r)^3 - \frac{4}{3}\pi(10)^3$

$\frac{dV}{dt} = 4\pi(10 + r)^2 \frac{dr}{dt}$

$\Rightarrow -50 = 4\pi(10 + r)^2 \frac{dr}{dt}$

$$\Rightarrow \frac{dr}{dt} = \frac{-50}{4\pi(10+r)^2}$$

$$\Rightarrow \left.\frac{dr}{dt}\right]_{r=5} = \frac{-50}{4\pi(15)^2}$$

$$= -\frac{1}{18\pi}$$

$\therefore$ thickness is decreasing at the rate $\frac{1}{18\pi}$ cm/min.

11. $f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{(x-2)(x+2)}{2x^2}$

For local maximum/minimum, set $f'(x) = 0$

$\Rightarrow x = \pm 2$.

As $f'(x) < 0$ for $0 < x < 2$

and $f'(x) > 0$ for $2 < x < \infty$,

$f(x)$ has a local minimum at $x = 2$.

12. $\frac{f(3)-f(1)}{3-1} = f'(c)$

$\Rightarrow \frac{\log_e 3}{2} = \frac{1}{c}$

$\Rightarrow c = 2\log_3 e$

13. $f'(x) = \frac{\cos x - \sin x}{1+(\sin x + \cos x)^2}$

$f'(x) > 0$ if and only if $\cos x > \sin x$

This is true if $x \in \left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$

14. As $f(x) = x^3 - px + q$ has three real and distinct, roots, $f'(x) = 3x^2 - p$ has two distinct zeros by Rolle's theorem.

Now $f'(x) = 0 \Rightarrow x = \pm\sqrt{\frac{p}{3}}$

We have $f'(x) > 0$ if $x < -\sqrt{\frac{p}{3}}$

< 0 if $-\sqrt{\frac{p}{3}} < x < \sqrt{\frac{p}{3}}$

> 0 if $x > \sqrt{\frac{p}{3}}$

Thus $f(x)$ has a local maximum at $x = -\sqrt{\frac{p}{3}}$ and local minimum at $x = \sqrt{\frac{p}{3}}$

15. $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$\Rightarrow P'(x) = 4x^3 + 3ax^2 + 2bx + c$

As $P'(0) = 0$, we get $c = 0$

$\therefore P'(x) = x(4x^2 + 3ax + 2b)$

As $x = 0$ is the only real root of $P'(x) = 0$, roots of $4x^2 + 3ax + 2b = 0$ must be imaginary, therefore

$4x^2 + 3ax + 2b > 0 \ \forall\ x \in \mathbf{R}$

Thus, $P'(x) < 0$ for $x < 0$

> 0 for $x > 0$.

$\therefore x = 0$ is a point of local minimum at $x = 0$.

Graph of $y = P(x)$ is given in Fig. 11.10.

As $P(-1) < P(1)$, we get $P(1)$ is maximum but $P(-1)$ is not minimum of P on $[-1, 1]$.

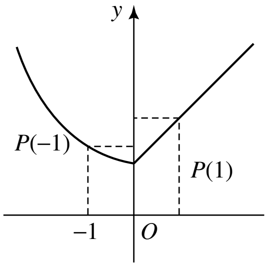


Fig. 11.10

16. f will be continuous at $x = -1$ if

$\Rightarrow \lim_{x\to -1-} f(x) = \lim_{x\to -1+} f(x) = f(-1)$

$\Rightarrow k + 2 = 2(-1) + 3 = k + 2$

$\Rightarrow k = -1$

For this value of k, f is continuous at $x = -1$, $f'(-1)$ does not exist and $f'(x) < 0$ for $x < -1$

and $f'(x) > 0$ for $x > -1$,

therefore, f has a local minimum at $x = -1$.

17. Equation of tangent at (x, y) to the curve $y = f(x)$ is given by

$Y - y = \frac{dy}{dx}(X - x)$

This tangent meets the axes at $A\left(x - y\frac{dx}{dy}, 0\right)$ and $B\left(0, y - x\frac{dy}{dx}\right)$.

We are given mid-point of $AB = (x, y)$

$\Rightarrow \frac{1}{2}\left(x - y\frac{dx}{dy}\right) = x$ and $\frac{1}{2}\left(y - x\frac{dy}{dx}\right) = y$

$\Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$

$\Rightarrow \log |xy| = c \Rightarrow xy = \pm e^c = A$.

As it passes through (2, 3), we get $6 = A$

$\therefore xy = 6$ or $y = 6/x$

18. Let $g(x) = \tan x - x, -\pi/2 < x < \pi/2$

$\Rightarrow g'(x) = \sec^2 x - 1 = \tan^2 x > 0$ for $-\pi/2 < x < \pi/2$.

Thus, $g(x)$ is a strictly increasing function on $(-\pi/2, \pi/2)$.

Therefore, $\tan x < x$ for $-\pi/2 < x < \pi/2$

$\Rightarrow f(x) = \frac{\tan x}{x} < 1$ for $-\pi/2 < x < \pi/2, x \neq 0$.

$\therefore x = 0$ is a point of maxima.

Also,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{\tan x}{x} - 1}{x}$$

$$= \lim_{x\to 0} \frac{\tan x - x}{x^2} \quad \left[\frac{0}{0}\text{form}\right]$$

$$= \lim_{x\to 0} \frac{\sec^2 x - 1}{2x}$$

$$= \frac{1}{2}\lim_{x\to 0} \frac{(\tan x)^2}{x} = 0$$

Thus, both the statements are true. However, statement-2 is not a correct explanation of statement-1.

19. $f'(x) = \frac{1}{x} + 2bx + a, x \neq 0$.

As $x = -1$ and $x = 2$ are extreme values of f so

$f'(-1) = 0 = f'(2)$

$\Rightarrow -1 - 2b = a = 0$ and $\frac{1}{2} + 4b + a = 0$

Solving we get, $a = \frac{1}{2}$ and $b = -\frac{1}{4}$

So $f'(x) = \frac{1}{x} - \frac{1}{2}x + \frac{1}{2} = -\frac{1}{2x}(x + 1)(x - 2)$

We have

$$\begin{aligned} f'(x) &> 0 \text{ if } x < -1 \\ &< 0 \text{ if } -1 < x < 0 \\ &> 0 \text{ if } 0 < x < 2 \\ &< 0 \text{ if } x > 2 \end{aligned}$$

Thus $f(x)$ has a local maximum at $x = -1$ and $x = 2$

$\therefore$ statement-1 and statement-2 are both true and statement-2 is a correct explanation for the statement-1.

20. Let r be the radius of balloon and V be its volume.

It is given that $\frac{dV}{dt} = -72\pi$

$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -72\pi$

$\Rightarrow \frac{4}{3}\pi r^3 = -72\pi t + k$

When $t = 0$, $k = \frac{4}{3}\pi r^3 = V = 4500\pi$

$\therefore \frac{4}{3}\pi r^3 = 4500\pi - 72\pi t$

When $t = 49$, $\frac{4}{3}\pi r^3 = 4500\pi - (72\pi)(49) = 972\pi$

$\Rightarrow r^3 = 729 \Rightarrow r = 9$

Also, $\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -72\pi \Rightarrow 4\pi r^2 \frac{dr}{dt} = -72\pi$

$\frac{dr}{dt} = \frac{-72\pi}{4\pi r^2} = \frac{-18}{r^2} \Rightarrow \left.\frac{dr}{dt}\right|_{r=9} = -\frac{2}{9}$.

21. $C = av + \frac{b}{v}$. According to the given conditions,

$75 = a.30 + \frac{b}{30}$ and $65 = a.40 + \frac{b}{40}$

Solving $a = \frac{1}{2}$, $b = 1800$. Thus

$$C = \frac{1}{2}v + \frac{1800}{v}$$

$\frac{dC}{dv} = \frac{1}{2} - \frac{1800}{v^2}$ so $\frac{dc}{dv} = 0 \Rightarrow v = 60$

$\frac{d^2C}{dv^2} = \frac{3600}{v^3} > 0$. Thus C is minimum when $v = 60$.

22. $S = 4\pi r^2$, $8 = \frac{dS}{dt} = 8\pi r\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{\pi r}$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} = 4r \text{ so } \frac{dV}{dt}\alpha r.$$

23. Let $f(x) = 2x^3 + 3x + k$

Suppose $\alpha, \beta \in [0, 1]$, $\alpha < \beta$ be such that $f(\alpha) = 0 = f(\beta)$.

As f is differentiable on $[\alpha, \beta]$,

By the Rolle's theorem there exists $r \in (a, b)$ such that $f'(r) = 0 \Rightarrow 6r^2 + 3 = 0$.

But $6r^2 + 3 = 0$ is not true for any real r.

Thus, there is no real value of k, for which $2x^3 + 3x + k = 0$ has two real and distinct roots in $[0, 1]$.

24. $A = \frac{1}{2}pb$, p is the perpendicular and b is the base

$p^2 + b^2 = h^2$ so $A = \frac{1}{2}p\sqrt{h^2 - p^2}$

$$\frac{dA}{dp} = \frac{1}{2}\sqrt{h^2 - p^2} + \frac{1}{4}p\frac{(-2p)}{\sqrt{h^2 - p^2}} = \frac{1}{2}\frac{h^2 - p^2 - p^2}{\sqrt{h^2 - p^2}}$$

$\frac{dA}{dp} = 0 \Rightarrow p = \frac{h}{\sqrt{2}}$. It is easy to see that A is maximum for $p = \frac{h}{\sqrt{2}}$, so $A_{\max} = \frac{1}{2}\frac{h}{\sqrt{2}}\sqrt{h^2 - \frac{h^2}{2}}$

$= \frac{h^2}{4}$.

25. If $f(x) = x \log x$ then $f'(x) = 1 + \log x > 0$ for $x \in [1, 2]$ so f increases on $[1, 2]$. For $g(x) = 2 - x$, $g'(x) = -1$ so g decreases on $[1, 2]$. Therefore $g(x) = f(x)$ for some $x \in [1, 2]$. Hence the statement-1 follows.

26. Let $f(x) = x^2e^x$, $f'(x) = 2xe^x + x^2e^x > 0$ for $x > 0$.

$g(x) = x^2e^{-x}$, $g'(x) = 2xe^{-x} - x^2e^{-x} = (2x - x^2)\, e^{-x}$

$g'(x) > 0$ for $x < 2$ and $g'(x) < 0$ for $x > 2$. Hence g decreases for $x > 2$ and increases on $(0, 2)$. Statement-2 is not true. If $h(x) = x^2(e^x + e^{-x})$ then $h'(x) = 2x(e^x + e^{-x}) + x^2\,(e^x - e^{-x}) > 0$ for $x > 0$. Hence h increases for $x > 0$.

27. Let $F(x) = f(x) - 2g(x)$

$F(0) = f(0) - 2g(0) = 2 - 0 = 2.$

$F(1) = f(1) - 2g(1) = 6 - 4 = 2$

F is continuous on $[0, 1]$ and differentiable on $(0, 1)$ with $F(0) = F(1)$, so by Rolle's theorem there is $c \in (0, 1)$ s. t. $f(c) = 0 \Rightarrow f'(c) = 2g'(c)$.

28. $f'(x) = \frac{\alpha}{x} + 2\beta x + 1$

As $x = -1$ and $x = 2$ are extreme points so $f'(-1) = f'(2) = 0$ i.e.

$-\alpha - 2\beta + 1 = 0$

$\frac{\alpha}{2} + 4\beta + 1 = 0$

Solving $\alpha = 2$, $\beta = -\frac{1}{2}$.

29. $V = \frac{4}{3}\pi r^3$. When $V = 288\pi$ C.C then $288\pi = \frac{4}{3}\pi r^3$

$\Rightarrow r = 6$. Also $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow 4\pi = 4\pi 36\frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{36}$.

30. $f(x) = x^2 + 2bx + 2c^2 = (x + b)^2 + 2c^2 - b^2$ so $\min f(x) = 2c^2 - b^2$. Also $g(x) = -x^2 - 2cx + b^2 = b^2 + c^2 - (x + c)^2 \Rightarrow \max g(x) = b^2 + c^2$

Thus $2c^2 - b^2 > b^2 + c^2$

$\Rightarrow \left|\frac{c}{b}\right| > \sqrt{2} \Rightarrow \left|\frac{c}{b}\right| \in \left(\sqrt{2}, \infty\right)$.

31. Note that f is a strictly increasing function and g is a strictly decreasing function.

Now, $x < x + 1 \Rightarrow g(x) > g(x + 1)$

$\Rightarrow f(g(x)) > f(g(x + 1))$

32. $f(-1) = f(1)$

$\Rightarrow -2 + a - b = 2 + a + b$

$\Rightarrow b = -2$

Also, $f'(c) = 0 \Rightarrow 6c^2 + 2ac + b = 0$

$\Rightarrow \frac{3}{2} + a - 2 = 0 \Rightarrow a = \frac{1}{2}$

Thus, $2a + b = 1 - 2 = -1$

33. $\lim\limits_{x\to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$

$\Rightarrow \lim\limits_{x\to 0}\frac{f(x)}{x^2} = 2$

$\therefore f(x)$ must be of the form

$f(x) = ax^4 + \text{b}x^3 + 2x^2$

$\Rightarrow f'(x) = 4ax^3 + 3bx^2 + 4x$

As $f(x)$ has extreme values at $x = 1$ and $x = 2$,

$4a + 3b + 4 = 0$

$32a + 12b + 8 = 0$

Solving these we get $a = \frac{1}{2}$, $b = -2$

Thus, $f(2) = \frac{1}{2}(2^4) - 2(2^3) + 2(2^2) = 0.$

34. When $x = 0$, $\sin y = 0 \Rightarrow y = 0$ (since all lines given pass through origin)

$$\cos y \frac{dy}{dx} = \sin\left(\frac{\pi}{3}+y\right) + x\cos\left(\frac{\pi}{3}+y\right)\frac{dy}{dx}$$

When $x = 0$, $y = 0$, therefore

$$\left.\frac{dy}{dx}\right]_{(0,0)} = \sin\left(\frac{\pi}{3}\right) + 0 = \frac{\sqrt{3}}{2}$$

Thus, slope of normal at (0, 0) is $-2/\sqrt{3}$ and its equation is

$$y = \left(-2/\sqrt{3}\right)x \text{ or } 2x + \sqrt{3}y = 0$$

35. Let $m = 0.6$,

$$f'(x) = \frac{(1+x^m)(m)(1+x)^{m-1} - mx^{m-1}(1+x)^m}{(1+x^m)^2}$$

$$= \frac{m(1+x)^{m-1}(1+x^m - x^{m-1} - x^m)}{(1+x^m)^2}$$

$$= \frac{m(1+x)^{m-1}(1-x^{m-1})}{(1+x^m)^2}$$

For $0 < x < 1$, $f'(x) < 0$

as $x^{m-1} = x^{-0.4} = 1/x^{0.4} > 1$ for $0 < x < 1$.

Thus, $f'(x) < 0$ for $0 < x < 1$

$\Rightarrow f(x)$ is strictly decreasing on [0, 1]

$\therefore$ $k = \max f(x) = f(0) = 1$

and $k = \min f(x) = f(1) = 2^{-0.4}$

Hence, $(k, K) = (2^{-0.4}, 1)$

36. If s is the height at any instant, then

$-2g(s - 64) = v^2 - u^2$

where v is the initial speed and v is the speed at that instant.

At the highest point,

$-2(32)(s - 64) = 0 - 48^2$

$$\Rightarrow s = 64 + \frac{48^2}{64} = 100.$$

37. Let $x_1 = x(\pi/4)$ and $y_1 = y(\pi/4)$.

$$\frac{dx}{dt} = -2\sin t + 2\sin t + 2t\cos t = 2t\cos t$$

$$\frac{dy}{dt} = 2\cos t - 2\cos t + 2t\sin t = 2t\sin t$$

$$\therefore \frac{dy}{dx} = \tan t \Rightarrow \left.\frac{dy}{dx}\right]_{t=\pi/4} = 1$$

$\Rightarrow$ slope of normal at $t = \pi/4$ is -1.

Thus, equation of normal at (x_1, y_1) is

$y - y_1 = -(x - x_1)$ or $x + y - (x_1 + y_1) = 0$

Its distance from the origin is

$$d = \frac{|x_1 + y_1|}{\sqrt{2}}$$

$$\text{But } x_1 + y_1 = 2\left(\frac{1}{\sqrt{2}}\right)\left(1+\frac{\pi}{4}\right) + 2\left(\frac{1}{\sqrt{2}}\right)\left(1-\frac{\pi}{4}\right)$$

$$= 2\sqrt{2}$$

$\therefore$ $d = 2$

38. Let coordinates of A be $(k, 0)$ then equation of tangent PA is

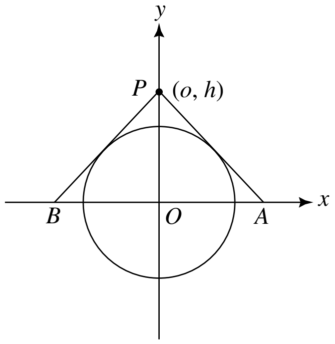


Fig. 11.11

$$\frac{x}{k} + \frac{y}{h} = 1, \text{ where } |h|, |k| > 4.$$

$$\frac{1}{\sqrt{\frac{1}{k^2}+\frac{1}{h^2}}} = 4 \quad (1)$$

$$\Rightarrow \frac{|h||k|}{\sqrt{h^2+k^2}} = 4$$

Let Δ = Area of ΔPAB = 2(area of ΔOAP)

$$= 2\left(\frac{1}{2}\right)(|h|\,|k|) = 4\sqrt{h^2+k^2}$$

$$\text{But from (1), } k^2 = \frac{16h^2}{h^2-16}$$

$$\therefore\ \Delta^2 = 16(h^2 + k^2) = 16\left(h^2 + \frac{16h^2}{h^2 - 16}\right)$$

$$= \frac{16h^4}{h^2 - 16}$$

But Δ is minimum if and only if Δ^2 is, we therefore minimize, Δ^2.

We have

$$\frac{d}{dh}(\Delta^2) = 16\left[\frac{4h^3(h^2 - 16) - h^4(2h)}{(h^2 - 16)^2}\right]$$

$$= \frac{32h^3(h - 4\sqrt{2})(h + 4\sqrt{2})}{(h^2 - 16)^2}$$

As $\frac{d}{dh}(\Delta^2) = 0$ for $h = 4\sqrt{2}$ and $\frac{d}{dh}(\Delta^2) < 0$ if $4 < h < 4\sqrt{2}$ and is > 0 if $h > 4\sqrt{2}$

Thus Δ^2 is minimum when $h = 4\sqrt{2}$.

39. For $x \in (0, \pi/2)$

$1 + \sin x = 1 - \cos(\pi/2 + x)$

$= 2\sin^2(\pi/4 + x/2)$

and $1 - \sin x = 1 + \cos(\pi/2 + x)$

$= 2\cos^2(\pi/4 + x/2)$

$$\therefore\ f(x) = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right] = \frac{\pi}{4} + \frac{x}{2}$$

$\Rightarrow f'(x) = 1/2$

An equation of normal to $y = f(x)$ at $x = \pi/6$ is

$$y - \left(\frac{\pi}{4} + \frac{\pi}{12}\right) = -2\left(x - \frac{\pi}{6}\right)$$

$$\Rightarrow y - \frac{\pi}{3} = -2x + \frac{\pi}{3}$$

$\Rightarrow 2x + y = 2\pi/3$

It passes through $(0, 2\pi/3)$

40. Let $E = 4 + \frac{1}{2}\sin^2 2x - 2\cos^4 x$

$$= 4 + \frac{1}{2}\sin^2 2x - \frac{1}{2}(2\cos^2 x)^2$$

$$= 4 + \frac{1}{2}(1 - \cos^2 2x) - \frac{1}{2}(1 + \cos x)^2$$

$$= 4 + \frac{1}{2}(1 + \cos 2x)\,[1 - \cos 2x - 1 - \cos 2x]$$

$$= 4 - \frac{1}{2}(1 + \cos 2x)(2\cos 2x)$$

$$= 4 - \cos 2x - \cos^2 2x$$

$$= \frac{9}{4} - \left(\frac{1}{2} + \cos 2x\right)^2$$

Note that

$m = 0$ when $\cos 2x = 1$

and $M = \frac{9}{4}$ when $\cos 2x = -1/2$.

So $M - m = \frac{9}{4}$.

41. An equation of tangent at $P(4t^2 + 3, 8t^3 - 1)$ is

$$y - (8t^3 - 1) = \frac{24t^2}{8t}(x - (4t^3 + 3)) \text{ as } \frac{dy}{dx} = \frac{24t^2}{8t}$$

It will meet the curve again at

$Q(4t_1^{\,2} + 3, 8t_1^{\,3} - 1)$ if

$8t_1^{\,3} - 8t^3 = 3t[4t_1^{\,2} - 4t^2]$

$\Rightarrow 2(t_1 - t)(t_1^{\,2} + t^2 + tt_1) = 3(t_1 - t)(tt_1 + tt^2)$

$\Rightarrow (t_1 - t)(2t_1^{\,2} - tt_1 - t^2) = 0$

$$\Rightarrow (t_1 - t)(t_1 - t)(2t_1 + t) = 0 \Rightarrow t_1 = t,\ -\frac{1}{2}t.$$

Thus, coordinates of Q are $(t^2 + 3, -t^3 - 1)$

42. $f(x) = \sin^4 x + \cos^4 x$, since f is differentiable and

$f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x$

$= -2\sin x\cos x(\cos^2 x - \sin^2 x)$

$= -\sin(4x) > 0$

$$\forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$\Rightarrow f(x) \text{ increases on } \left]\frac{\pi}{4}, \frac{\pi}{2}\right[$$

43. $$\frac{dy}{dx} = \frac{4}{2\sqrt{4x - 3}} = \frac{2}{\sqrt{4x - 3}} = \frac{2}{3}$$

$\Rightarrow \sqrt{4x - 3} = 3 \Rightarrow x = 3$

For $x = 3$, $y = 1 + \sqrt{(4)(3) - 3} = 4$

The equation of normal at $(3, 4)$ is

$$y - 4 = -\frac{3}{2}(x - 3)$$

It passes through $(1, 7)$.

Previous Years' B-Architecture Entrance Examination Questions

1. $\frac{dy}{dx} = 3x^2 - 8x$. $\left.\frac{dy}{dx}\right|_{(2,-1)} = 12 - 16 = -4$. So the slope of the normal is $\frac{1}{4}$.

2. $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 2t - 1 \Rightarrow \frac{dy}{dx} = \frac{2t-1}{2t}$ so tangent line is perpendicular to x-axis if $t = 0$.

3. $f(0) = f(\pi) = 1$. $f'(x) = \cos x\, 4^{\sin x}$ so $f'(c) = 0 \Rightarrow \cos c = 0 \Rightarrow c = \frac{\pi}{2}$.

4. $f(5) = 0$, $f(1) = \sqrt{24}$. $\frac{f(5)-f(1)}{5-1} = \frac{1}{4}\sqrt{24} = \sqrt{\frac{3}{2}}$

   $f'(x) = \frac{-x}{\sqrt{25-x^2}}$ so $f'(c) = \sqrt{\frac{3}{2}} \Rightarrow \frac{c^2}{25-c^2} = \frac{3}{2}$

   $\Rightarrow 2c^2 = 75 - 3c^2 \Rightarrow c^2 = 15$. For $c \in (1, 5)$, we have $c = \sqrt{15}$.

5. $f(x) = \begin{cases} 1-x+a & \text{if} \quad x \le 1 \\ 2x+3 & \text{if} \quad x > 1 \end{cases}$, $f(1) = a$

   So we must have $\lim\limits_{x\to 1+} f(x) \ge a$ i.e. $5 \ge a$ or $a \le 5$

6. $2y = \log(1 + 4x^2) \Rightarrow 2\frac{dy}{dx} = \frac{8x}{1+4x^2}$

   $\Rightarrow \frac{dy}{dx} = \frac{4x}{1+4x^2}$

   Since $4x \le 1 + 4x^2$ so $|m| \le 1$

7. We have $\frac{f(2)-f(0)}{2-0} = f'(c)$ for some $c \in (0, 2)$

   $f(2) = 2f'(c) + f(0) \le 6 - 2 = 4$.

8. $f'(x) = \frac{(x^2+1)(4x^3+6x)-(x^4+3x^2+1)2x}{(x^2+1)^2}$

   $= \frac{2x(x^2(x^2+2)+2)}{(x^2+1)^2}$

   $f'(x) = 0$ for $x = 0$, $f'(x) < 0$, for $x < 0$, $f'(x) > 0$ for $x > 0$

   $f(-1) = \frac{5}{2}$, $f(2) = \frac{29}{5}$, $f(0) = 1$ so the maximum value of f is $\frac{29}{5}$.

9. Equation of normal at any point is $Y - y = -\frac{1}{f'(x)}(X - x)$.

   This passes through (1, 1), so $1 - y = -\frac{1}{f'(x)}(1 - x)$

   $\Rightarrow 1 - y = -\frac{dx}{dy}(1 - x)$

   $\Rightarrow \frac{-(1-y)^2}{2} = \frac{(1-x)^2}{2} + c$

   Since it passes through (3, 1) so

   $0 = 2 + C \Rightarrow C = -2$

   $\Rightarrow (x - 1)^2 + (y - 1)^2 = 4$ which is circle of radius 2 so the area is 4π.

10. $f'(x) = \sin\frac{\pi}{x} - \frac{\pi}{x}\cos\frac{\pi}{x}$, $0 < x \le 1$

    $f'(0)$ does not exist. $f'(x) = 0 \Rightarrow \tan\frac{\pi}{x} = \frac{\pi}{x}$

    Since $0 < x \le 1$ so $\pi \le \frac{\pi}{x} < \infty$. By drawing graph of $\tan x$ and x, we can see that there are infinitely many solutions.

11. Since $1 - x^2$ is a decreasing function and $[x]$ is an increasing function so $f(x) = [1 - x^2]$ is a decreasing function.

12. $\frac{dy}{dt} = \frac{1}{2\sqrt{x}}\frac{dx}{dt}$. If $\frac{dy}{dt} = \frac{dx}{dt}$ then $2\sqrt{x} = 1$

    or $x = \frac{1}{4}$ so $y = \frac{1}{2}$.

13. $2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$.

    Equation of tangent at (x, y)

    $Y - y = \frac{x}{y}(X - x)$

    So $a_1 = -\frac{y^2}{x} + x = \frac{4}{x}$, $b_1 = y - \frac{x^2}{y} = -\frac{4}{y}$

    Equation of normal at (x, y)

    $Y - y = \frac{y}{x}(X - x)$

    So $a_2 = 2x$, $b_2 = 2y$

    $a_1a_2 + b_1b_2 = \frac{4}{x} \times 2x - \frac{4}{y} \times 2y = 0$.

14. $\dfrac{f(2)-f(0)}{2-0} = f'(c)$ for some $c \in (0, 2)$

$f(2) = f(0) + 2f'(c) \le -3 + 2.5 = 7.$

15. $f\left(\dfrac{\pi}{2}\right) = f\left(-\dfrac{\pi}{2}\right) = -3$. Also $f'(x) =$

$6 \cos x \sin x\ (2 \cos x + 1)\ (\cos x - 2)$

For $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, $2 \cos x + 1 \ne 0$, $\cos x - 2 \ne 0$ so

$f'(x) = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0.$

$f(0) = 3 - 6 - 6 - 3 = -12$. Hence range of $f(x)$ is $[-12, -3]$.

16. $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1 - x)$, $f'(x) = 0 \Rightarrow x = 1$

$f''(x) = -e^{-x}(1 - x) + e^{-x}(-1)$, $f''(1) = -e^{-1} < 0$

f has maximum at $x = 1$.

17. Area $A(x) = x^2 \Rightarrow \dfrac{dA}{dt} = 2x\dfrac{dx}{dt}$

$\Rightarrow 8 = 2x\dfrac{dx}{dt} = 2x$

$\Rightarrow x = 4 \Rightarrow A = 16$

18. $V = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dr} = 4\pi r^2$ and $S = 4\pi r^2 \Rightarrow \dfrac{dS}{dr} = 8\pi r$

$\dfrac{dV}{dS} = \dfrac{4\pi r^2}{8\pi r} = \dfrac{1}{2}r$, $\left.\dfrac{dV}{dS}\right|_{r=2} = 1.$

19. $e^y = 1 + x^2 \Rightarrow y = \log(1 + x^2)$, so $m = \dfrac{dy}{dx} = \dfrac{2x}{1+x^2}$

$|m| \le 1 \Rightarrow m \in [-1, 1]$

20. $f(x) = |x \log x| = x\, |\log x| = \begin{cases} -x\log x, & 0 < x \le 1 \\ x\log x, & x > 1 \end{cases}$

$f'(x) = \begin{cases} -(1+\log x), & 0 < x < 1 \\ 1+\log x, & x > 1 \end{cases}$

$f'(x) < 0$ if $1 + \log x > 0$ for $0 < x < 1 \Rightarrow \log x > -1 \Rightarrow x > e^{-1}$ for $0 < x < 1$

So $x \in \left[\dfrac{1}{e}, 1\right]$.

21. By drawing the graph, we get two points of minimum.

22. $f(x) = 2 \sin x + \sin 2x = 2 \sin x + 2 \sin x \cos x$

$= 2 \sin x\ (1 + \cos x)$

$f'(x) = 2 \cos x\ (1 + \cos x) - 2 \sin x \sin x$

$= 2\ [\cos x + \cos^2 x - (1 - \cos^2 x)]$

$= 2\ [2\cos^2 x + \cos x - 1]$

$= 2\ (\cos x + 1)\ (2 \cos x - 1)$

Now, $f'(x) = 0 \Rightarrow x = \pi/3$ ($\because\ 0 < x < \pi$)

We have $f(0) = 0$, $f\left(\dfrac{\pi}{3}\right) = 2\left(\dfrac{\sqrt{3}}{2}\right) + \dfrac{\sqrt{3}}{2} = \dfrac{3}{2}\sqrt{3}$,

and $f(\pi) = 0$.

Thus, maximum value is $\dfrac{3}{2}\sqrt{3}$.

23. $m = \dfrac{dy}{dx} = e^x(\sin x + \cos x)$

$\dfrac{dm}{dx} = 2e^x \cos x = 0 \Rightarrow x = \dfrac{\pi}{2}$

$\dfrac{d^2m}{dx^2} = 2e^x\ (\cos x - \sin x)$

$\left.\dfrac{d^2m}{dx^2}\right]_{x=\pi/2} = 2e^{\pi/2}(-1) < 0$

Thus m is maximum when $x = \pi/2$

24. As $\lim\limits_{x\to 0} \dfrac{p(x)}{x^2} = 1$, $p(x)$ must be of the from

$p(x) = x^2 + ax^3 + bx^4\ \ a, b \in \mathbf{R}$

$p'(x) = 2x + 3ax^2 + 4bx^3$

As $p'(1) = 0$, $p'(2) = 0$, we get

$2 + 3a + 4b = 0$ and $4 + 12a + 32b = 0$

$\Rightarrow 3a + 4b = -2$, $3a + 8b = -1$

$\Rightarrow a = -1$, $b = \dfrac{1}{4}$

$\therefore\ p(4) = 4^2(1 + 4a + 16b) = 16(1 - 4 + 4) = 16$

25. $f'(x) = 3x^2 + 2bx + c$. Discriminant of $f'(x) = 4(b^2 - 3ac) \le 0$

So $f'(x) > 0$ and hence f increases. Since f is continuous and increasing so f is $1 - 1$.

26. $\min(x, x^2) = \begin{cases} x^2 & 0 < x \le 1 \\ x & x > 1 \end{cases}$, $\lim\limits_{h\to 0+} \dfrac{f(1+h)-f(1)}{h} =$

$\lim\limits_{h\to 0+} \dfrac{1+h-1}{h} = 1$

$\lim\limits_{h\to 0-} \dfrac{f(1+h)-f(1)}{h} = \lim\limits_{h\to 0-} \dfrac{(1+h)^2-1}{h} = 2$. So f is not differentiable at $x = 1$.

For $y = x$, $\frac{dy}{dx} = 1$ so $\tan\theta_1 = 1$. For $y = x^2$, $\frac{dy}{dx} = 2x$ so $\left.\frac{dy}{dx}\right]_{x=1} = 2$. $\tan\theta = \frac{2-1}{1+2\times 1} = \frac{1}{3}$

27. f in statement-1 is continuous function on [1, 3] and differentiable on (1, 3) $f(3) = 1 = f(1)$ so Rolle's theorem is applicable. Again the mean value is applicable so statement-2 is true but is not a correct explanation for statement-1.

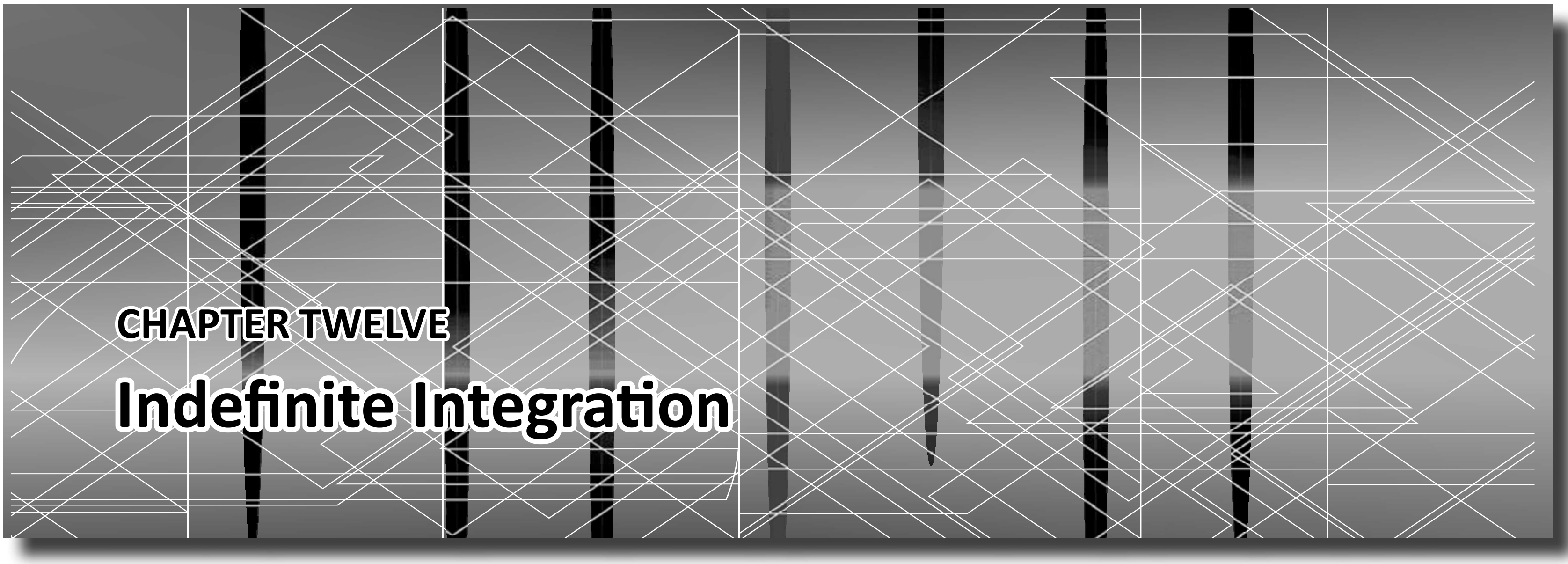
28. f is continuous on [2, 6] and differentiable on (2, 6) being a polynomial also $f(2) = 0 = f(6)$ so by Rolle's theorem there is $c \in (2, 6)$ such that $f'(c) = 0$.

29. $f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right) > 0$$

$f'(x) = 2ax + b$. So $f'(x) > 0$ if $x > -\frac{b}{2a}$ and

$f'(x) < 0$ if $x < -\frac{b}{2a}$.



CHAPTER TWELVE

Indefinite Integration

A function $F(x)$ is an antiderivative of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The set of all antiderivatives of f is the indefinite integral of f with respect to x, denoted by

$$\int f(x)\, dx$$

The function f is the integrand of the integral and x is the variable of integration. By mean value theorem, once we have found one antiderivative, the other antiderivative of f differ from F by a constant. We indicate this by

$$\int f(x)\, dx = F(x) + C$$

The constant C is the constant of integration or arbitrary constant.

TABLE OF BASIC FORMULA

S.No.	*f(x)*	$\int f(x)\,dx$*
1.	0	C
2.	$x^n \ (n \neq -1)$	$\frac{1}{n+1} x^{n+1}$
3.	$\frac{1}{x}$	$\log \lvert x \rvert$
4.	e^x	e^x
5.	$a^x \ (a > 0, a \neq 1)$	$\frac{a^x}{\log a}$
6.	$\cos x$	$\sin x$
7.	$\sin x$	$-\cos x$
8.	$\sec^2 x$	$\tan x$
9.	$\operatorname{cosec}^2 x$	$-\cot x$
10.	$\sec x \tan x$	$\sec x$
11.	$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$
12.	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$ or $-\cos^{-1} x$
13.	$\frac{1}{1+x^2}$	$\tan^{-1} x$ or $-\cot^{-1} x$
14.	$\tan x$	$\log \lvert \sec x \rvert$ or $-\log \lvert \cos x \rvert$
15.	$\cot x$	$\log \lvert \sin x \rvert$ or $-\log \lvert \operatorname{cosec} x \rvert$
16.	$\sec x$	$\log \lvert \sec x + \tan x \rvert$ or $\log \left\lvert \tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \right\rvert$
17.	$\operatorname{cosec} x$	$\log \lvert \operatorname{cosec} x - \cot x \rvert$ or $\log \left\lvert \tan \frac{x}{2} \right\rvert$
18.	$\frac{1}{\sqrt{a^2-x^2}} \ (a \neq 0)$	$\sin^{-1}\left(\frac{x}{a}\right)$
19.	$\frac{1}{\sqrt{x^2+a^2}}$	$\log \left\lvert x + \sqrt{x^2+a^2} \right\rvert$
20.	$\frac{1}{\sqrt{x^2-a^2}}$	$\log \left\lvert x + \sqrt{x^2-a^2} \right\rvert$
21.	$\sqrt{a^2-x^2} \ (a \neq 0)$	$\frac{x}{2}\sqrt{a^2-x^2} +$ $\frac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right)$
22.	$\sqrt{x^2+a^2}$	$\frac{x}{2}\sqrt{x^2+a^2}$ $+ \frac{1}{2} a^2 \log \left\lvert x + \sqrt{x^2+a^2} \right\rvert$
23.	$\sqrt{x^2-a^2}$	$\frac{x}{2}\sqrt{x^2-a^2}$ $- \frac{1}{2} a^2 \log \left\lvert x + \sqrt{x^2-a^2} \right\rvert$
24.	$\frac{1}{x^2-a^2} \ (a \neq 0)$	$\frac{1}{2a} \log \left\lvert \frac{x-a}{x+a} \right\rvert$

*Always add a constant of integration

S.No.	*f(x)*	*∫ f(x) dx**
25.	$\dfrac{1}{a^2 - x^2}$ $(a \neq 0)$	$\dfrac{1}{2a}\log\left\|\dfrac{a+x}{a-x}\right\|$
26.	$\dfrac{1}{a^2 + x^2}$ $(a \neq 0)$	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)$
27.	Integration by parts $u(x)\, v(x)$	$v(x) \int u(x)dx -$ $\int\left[\int \dfrac{dv}{dx} \int u(x)\, dx\right] dx$

TIP

Remembering formulae 21, 22 and 23.

$$\int (\text{integrand})\, dx = \frac{x}{2}\,(\text{integrand}) \pm \frac{a^2}{2}\int \frac{dx}{\text{integrand}}$$

Note: Use a^2 or $-a^2$ as given in the integrand.

*Always add a constant of integration

A LIST OF EVALUATION TECHNIQUES

S.No.	*Form of Integrate*	*Value/Evaluation Technique*
1.	$\int f'(x+a)\, dx$	$f(x+a) + C$
2.	$\int f'(ax+b)dx,\ a \neq 0$	$\dfrac{1}{a} f(ax+b) + C$
3.	$\int \dfrac{f'(x)}{f(x)}\, dx$	$\log \|f(x)\| + C$
4.	$\int (f(x))^n f'(x)\, dx,\ n \neq -1$	$\dfrac{1}{n+1} (f(x))^{n+1} + C$
5.	$\int f'(g(x))\, g'(x)\, dx$	$f(g(x)) + C$
6.	$\int \cos mx \cos nx\, dx$	Use product to sum identities $\cos A \cos B = \dfrac{1}{2}[\cos(A-B) + \cos(A+B)]$
	$\int \sin mx \sin nx\, dx$	$\sin A \sin B = \dfrac{1}{2}[\cos(A-B) - \cos(A+B)]$
	$\int \sin mx \cos nx\, dx$	$\sin A \cos B = \dfrac{1}{2}[\sin(A-B) + \sin(A+B)]$
7.	$\int \sin^m x \cos^n x\, dx$ where $m, n \in N$	If m is odd put $\cos x = t$ If n is odd put $\sin x = t$ If both m and n are odd, put $\sin x = t$ if $m \geq n$ and $\cos x = t$ otherwise. If both m and n are even, use power reducing formulae $\sin^2 x = \dfrac{1}{2}(1 - \cos 2x)$ $\cos^2 x = \dfrac{1}{2}(1 + \cos 2x)$

S.No.	*Form of Integrate*	*Value/Evaluation Technique*
		If $m + n$ is a negative even integer, put $\tan x = t$
8.	$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}dx$ $\int \frac{ae^x + be^{-x}}{pe^x + qe^{-x}}\,dx$ ($p, q \neq 0$ and at least one of $a, b \neq 0$)	Write Num = α (DEN) + $\beta\,\frac{d}{dx}$ (DEN) + γ Find α, β and by comparing the coefficients of $\cos x$, $\sin x$ and constant term (or e^x, e^{-x}) write the integral as $\alpha\int dx + \beta\int\frac{\frac{d}{dx}(DEN)}{DEN}dx + \gamma\int\frac{dx}{\text{DEN}}$
9.	$\int \frac{a\tan x + b}{p\tan x + q}\,dx$ $\int \frac{ae^{2x} + b}{pe^{2x} + q}dx$ ($p, q \neq 0$, at least one of $a, b \neq 0$)	Change to $\int \frac{a\sin x + b\cos x}{p\sin x + q\cos x}\,dx$ or $\int \frac{ae^x + be^{-x}}{be^x + qe^{-x}}\,dx$ and use (8) above
10.	$\int \frac{dx}{a\sin x \pm b\cos x}$ (Both $a, b \neq 0$)	Use $a = r\cos\alpha$, $b = r\sin\alpha$ to put the integral in the form $\frac{1}{r}\int\frac{dx}{\sin(x \pm \alpha)}$. Now, use formula for $\int \operatorname{cosec} x\,dx$
11.	$\int\frac{dx}{\sqrt{Q}}$ or $\int\frac{L}{\sqrt{Q}}\,dx$ or $\int\frac{dx}{Q}$ or $\int\frac{L}{Q}\,dx$ or $\int\sqrt{Q}$ or $\int L\sqrt{Q}$ where $L = px + q$, $p \neq 0$ and $Q = ax^2 + bx + c$, $a \neq 0$	Write $Q = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ and put $t = x + \frac{b}{2a}$
12.	$\int\sqrt{\frac{L_1}{L_2}}\,dx$ where $L_1 = ax + b$, $a \neq 0$ $L_2 = px + q$, $p \neq 0$	Write as $\int\sqrt{\frac{L_1}{L_1 L_2}}\,dx$ to reduce to form $\int\frac{L}{\sqrt{Q}}\,dx$
13.	$\int\frac{dx}{L\sqrt{Q}}$	Put $L = \frac{1}{t}$ to reduce to form $\int\frac{dx}{\sqrt{Q}}$

S.No.	*Form of Integrate*	*Value/Evaluation Technique*
	where $L = px + q, p \neq 0$ and $Q = ax^2 + bx + c, a \neq 0$	
14.	$\int \frac{\sqrt{Q}}{L}\,dx$ where $L = px + q, p \neq 0$ $Q = ax^2 + bx + c,$ $a \neq 0$	Rewrite the integrand as $\frac{Q}{L\sqrt{Q}}$ Divide Q by L to obtain $Q = L_1\,(L) + \alpha$ where L_1 is a linear expression in x and α is a constant. The integral reduces to $\int \frac{L_1}{\sqrt{Q}}\,dx + \alpha\int \frac{dx}{L\sqrt{Q}}$.
15.	$\int \frac{Q_1}{\sqrt{Q_2}}\,dx$ where $Q_1 = ax_2 + bx + c, a \neq 0$ $Q_2 = px^2 + qx + r, p \neq 0$	Write $Q_1 \equiv \alpha\, Q_2 + \beta\, \frac{d}{dx}\,[Q_2] + \gamma$ Find α, β, γ and write the integral as $\alpha \int \sqrt{Q_2}\,dx + \beta \int \frac{\frac{d}{dx}}{\sqrt{Q_2}}\,[Q_2]\,dx + \gamma \int \frac{dx}{\sqrt{Q_2}}$
16.	$\int \frac{dx}{L_1\sqrt{L_2}}$ where $L_1 = ax + b$ and $L_2 = px + q,$ with $a, p \neq 0, \quad a \neq p.$	Put $L_2 = t^2$
17.	$\int \frac{dx}{Q_1\sqrt{Q_2}}$ where $Q_1 = ax^2 + b, Q_2 = px^2 + q$ with $a, p \neq 0, p \neq 0$	First put $x = 1/t$ and simplify then put $p + qt^2 = u^2$
18.	$\int R(x^{1/p,}\, x^{1/q,}\, x^{1/r} \ldots)\,dx$ where R is a rational function.	Let $l = l\text{cm}\,(p, q, r, \ldots)$ and put $x = t^l$
19.	$\int e^x\,(f(x) + f'(x))\,dx$	Write as $\int e^x f(x)\,dx + \int e^x f'(x)\,dx$ Integrate the first integral just once by parts taking e^x as the second function to obtain $e^x f(x) - \int e^x f'(x)\,dx + \int e^x f'(x)\,dx$ Cancel the last two integrals.
20.	$\int e^{ax} \cos\,(bx + c)\,dx$ or $\int e^{ax} \sin\,(bx + c)\,dx$	Integrate by parts taking e^{ax} as the second function to obtain $\frac{1}{a} e^{ax} \cos\,(bx + c) + \frac{b}{a} \int e^{ax} \cos\,(bx + c)\,dx$ Again integrate by parts taking e^{ax} as the second function and then bring a multiple of $\int e^{ax} \cos\,(bx + c)\,dx$ to the left.

S.No.	*Form of Integrate*	*Value/Evaluation Technique*
21.	$\int \frac{P(x)}{Q(x)}\,dx$ where $P(x)$ and $Q(x)$ are polynomials in x	If deg $P(x) \geq$ deg $Q(x)$ first isolate the quotient $A(x)$ when $P(x)$ is divided by $Q(x)$. Now, $\frac{P(x)}{Q(x)} = A(x) + \frac{B(x)}{Q(x)}$ where deg $B(x) <$ deg $Q(x)$ Now use techniques 22 to 29
22.	$\int \frac{P(x)}{Q(x)}\,dx$ where deg $P(x) <$ deg $Q(x)$ and $Q(x)$ is a product of distinct linear factors, that is, $Q(x) = A(x - \alpha_1)(x - \alpha_2)\dots(x - \alpha_r)$ where $a_i'\,s$ are distinct and $A \neq 0$	Write $\frac{P(x)}{Q(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \dots + \frac{A_n}{x-\alpha_n}$ Evaluate $A_i'\,s$ and use $\int \frac{dx}{x-a} = \log\lvert x-a\rvert$
23.	$\int \frac{P(x)}{(x-a)^n}\,dx$ where $n \geq 1$ and $P(x)$ is a polynomial in x.	Put $x - a = t$, express $P(x)$ in terms of t_s and then integrate.
24.	$\int \frac{P(x)\,dx}{(x-a)^m\,(x-b)^n}$ where $m, n \geq 1$, $a \neq b$, and deg $P(x) < m + n$.	Write the integrand as $\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots \frac{A_m}{(x-a)^m}$ $+ \frac{B_1}{x-b} + \frac{B_2}{(x-b)^2} + \dots + \frac{B_n}{(x-b)^n}$ Evaluate A_i's and B_j's and the integral.
25.	$\int \frac{P(x)\,dx}{(x-a)(x^2+bx+c)}$ where $b^2 - 4c < 0$ and deg $P(x) < 3$	Write the integrand as $\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$
26.	$\int \frac{x^2+1}{x^4+kx^2+1}\,dx$	Divide the numerator by x^2 to obtain $\int \frac{(1+1/x^2)\,dx}{x^2+1/x^2+k}$ Now, put $x - 1/x = t$

S.No.	*Form of Integrate*	*Value/Evaluation Technique*
27.	$\int \frac{x^2-1}{x^4+kx^2+1}\,dx$	Divide the numerator by x^2 to obtain
		$\int \frac{(1-1/x^2)\,dx}{x^2+1/x^2+k}$ Now, put $x + 1/x = t$
28.	$\int \frac{xP(x^2)}{Q(x^2)}\,dx$	Put $x^2 = t$
	where P and Q are polynomials in x	
29.	$\int \frac{P(x^2)}{Q(x^2)}\,dx$	Put $x^2 = t$
	where P and Q are polynomials in x	for partial fractions *not for integration*
30.	$\int \frac{1}{(x^2+a^2)^n}\,dx = A_n$	Begin with A_{n-1}
	where $n > 1$	$A_{n-1} = \int 1.\frac{1}{(x^2+a^2)^{n-1}}\,dx$
		Integrate by parts and obtain a reduction formula.
31.	$\int R(\sin x, \cos x)\,dx$	Universal substitution
	where R is a rational function	$\tan(x/2) = t$.
	Special Cases	
	(a) If $R(-\sin x, \cos x)$ $= -R(\sin x, \cos x)$	Put $\cos x = t$
	(b) If $R(\sin x, -\cos x)$ $= -R(\sin x, \cos x)$	Put $\sin x = t$
	(c) If $R(-\sin x, -\cos x)$ $= R(\sin x, \cos x)$	Put $\tan x = t$
32.	$\int x^m (a + bx^n)^p\,dx$	
	where m, n, p are rational numbers.	
	(a) p is a positive integer	Expand $(a + bx^n)^p$ by using binomial theorem
	(b) p is a negative integer,	Put $x = t^k$
	$m = \frac{a}{b}, n = \frac{c}{d},$	where $k = l$ cm (b, d)
	$a, b, c, d \in \mathbf{I}, b > 0, d > 0$	
	(c) $\frac{m+1}{n}$ is an integer	Put $a + bx^n = t^s$ where $p = r/s$, $r, s \in \mathbf{I}$, $s > 0$.
	(d) $\frac{m+1}{n} + p$ is an integer	Put $a + bx^n = x^n t^s$ where $p = r/s$, $r, s \in \mathbf{I}$, $s > 0$.

S.No.	*Form of Integrate*	*Value/Evaluation Technique*
33.	$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx$ where $P_n(x)$ is a polynomial of degree n in x.	Write $\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx = Q_{n-1}(x)\sqrt{ax^2+bx+c} + k \int \frac{dx}{\sqrt{ax^2+bx+c}}$ where $Q_{n-1}(x)$ is a polynomial of degree $(n-1)$ in x and k is a constant. Differentiate both the sides w.r.t. x and multiplying by $\sqrt{ax^2+bx+c}$ to get the identity $P_n(x) = Q'_{n-1}(x)(ax^2+bx+c) + \frac{1}{2} Q_{n-1}(x)(2ax+b) + k$ Compare coefficients to obtain $Q_{n-1}(x)$ and k.

SOME TIPS FOR SIMPLIFYING COMPUTATIONS

1. In case of $\frac{1}{\sqrt{Q}}$, $\sqrt{Q}$, or $\frac{1}{Q}$, where $Q = ax^2 + bx + c$, the substitution $x + \frac{b}{2a} = t$ may be used instead of completing the square in the quadratic Q.

   If $Q = (x - \alpha)(x - \beta)$ with $\alpha \neq \beta$, then $x - \frac{1}{2}(\alpha + \beta) = t$ can be used.

2. In case of $\frac{L}{\sqrt{Q}}$, $L\sqrt{Q}$ and $\frac{L}{Q}$
   where $L = px + q$ and $Q = ax^2 + bx + c$, $p \neq 0$, $q \neq 0$, the *following steps in unnecessary*
   Express

   $$L = \alpha \frac{d}{dx}[Q] + \beta$$

   The problem can be easily solved by putting

   $$x + \frac{b}{2a} = t$$

3. To evaluate

   $$\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$$

   where R is a rational function of x and

   $$\sqrt{ax^2+bx+c},$$

   we may use the following substitutions.

   (i) $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$ if $a > 0$

   (ii) $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$ if $c > 0$

   (iii) $\sqrt{ax^2+bx+c} = t(x - \alpha)$
   where α is a real root of $ax^2 + bx + c = 0$

4. To evaluate

   $$I = \int \frac{dx}{a + b\sin x}$$

   we rewrite

   $$I = \int \frac{dx}{a - b\cos(\pi/2 + x)}$$

   and put $\pi/2 + x = \theta$.
   This will lead to simpler computation then direct evaluation.

SOME TIPS FOR PARTIAL FRACTIONS

1. Partial fractions

   $$f(x) = \frac{P(x)}{(x - a_1)\ldots(x - a_n)}$$

   (where deg $P(x) < n$ and $a_1, a_2 \ldots a_n$ are distinct) is given by

   $$\frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \ldots + \frac{A_n}{x - a_n}$$

   where

$$A_i = \frac{P(a_i)}{(a_i - a_1)\dots(a_i - a_{i-1})(a_i - a_{i+1})\dots(a_i - a_n)}$$

= value of the expression $(x - a_i) f(x)$ at $x = a_i$.
In other words, A_i is the value obtained by substituting $x = a_i$ everywhere in $f(x)$ except in the factor $x - a_i$.
For instance,

$$\frac{2x-1}{(x-1)(x+1)(x-2)} \equiv \frac{2(1)-1}{(1+1)(1-2)}\frac{1}{x-1}$$

$$+ \frac{2(-1)-1}{(-1-1)(-1-2)}\frac{1}{x+1}$$

$$+ \frac{2(2)-1}{(2-1)(2+1)}\frac{1}{x-2}$$

$$\equiv -\frac{1}{2}\frac{1}{x-1} - \frac{1}{2}\frac{1}{x+1} + \frac{1}{x-2}$$

Justification

There exists one and only polynomial of degree two which takes value 2(1) – 1 or 1 at $x = 1$, 2(– 1) – 1 or – 3 at $x = – 1$ and 2(2) – 1 or 3 at $x = 2$.
Indeed, it is the polynomial*

$$\frac{1\,(x+1)(x-2)}{(1+1)(1-2)} + \frac{(-3)(x-1)(x-2)}{(-1-1)(-1-2)}$$

$$+ \frac{3(x-1)(x-2)}{(2-1)(2+1)}$$

2. If $P(x) = b_0 + b_1 x + \dots + b_n x^n$, the partial fraction of

$$\frac{P(x)}{(x-a_1)\dots(x-a_n)}$$

where, $a_1, a_2, \dots a_n$ are distinct, is given by

$$b_n + \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_n}{x-a_n}$$

where A_i's are given in (1) above for instance, if a, b, c are distinct

$$\frac{x^3}{(x-a)(x-b)(x-c)} \equiv 1 + \frac{a^3}{(a-b)(a-c)}\frac{1}{x-a}$$

$$+ \frac{b^3}{(b-a)(b-c)}\frac{1}{x-b}$$

$$+ \frac{c^3}{(c-a)(c-b)}\frac{1}{x-c}$$

3. To resolve

$$\frac{1}{(x-a)^r (x-b)}$$

where $a \neq b$, put $x - a = y$, so that

$$\frac{1}{(x-a)^r (x-b)} = \frac{1}{y^r (y + a - b)}$$

$$= \frac{1}{(a-b)\,y^r}\left[1 + \frac{y}{a-b}\right]^{-1}$$

Now expand $\left(1 + \dfrac{y}{a-b}\right)^{-1}$ upto y^{r-1} and divide by y^r to obtain

$$\frac{C_0}{y^r} + \frac{C_1}{y^{r+1}} + \dots + \frac{C_{r+1}}{y}.$$

The value $\boldsymbol{C_k}$ is the coefficient of $\dfrac{1}{(x-a)^{r-k}}$.

Coefficient of $\dfrac{1}{x-b}$ is $\dfrac{1}{(b-a)^r}$

For instance, if $x - 1 = y$, then

$$\frac{1}{(x-1)^3 (x-2)} = \frac{1}{y^3 (y-1)} = \frac{1}{y^3}(1-y)^{-1}$$

$$= \frac{1}{y^3}(1 + y + y^2 + \dots)$$

Thus,

$$\frac{1}{(x-1)^3 (x-2)} = \frac{1}{x-2} - \frac{1}{x-1}$$

$$- \frac{1}{(x-1)^2} - \frac{1}{(x-1)^3}$$

Illustration Resolve

$$\frac{1}{(x+1)^3 (x+2)^2}$$

into partial fraction. First, put $x + 1 = y$,

$$\frac{1}{(x+1)^3 (x+2)^2} = \frac{1}{y^3 (y+1)^2}$$

$$= \frac{1}{y^3}(1 - 2y + 3y^2 - \dots)$$

Next, put $x + 2 = z$

$$\frac{1}{(x+1)^3 (x+2)^2} = \frac{1}{z^2 (z-1)^3} = -\frac{1}{z^2}(1 - z)^{-3}$$

$$= -\frac{1}{z^2}(1 + 3z + \dots)$$

Thus,

$$\frac{1}{(x+1)^3 (x+2)^3} = \frac{3}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}$$

$$- \frac{3}{x+2} - \frac{1}{(x+2)^2}$$

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* It is known is Lagranges's interpolation polynomial.

4. To obtain partial fractions of

$$\frac{1}{(x-a)(x-b)(x^2+c^2)}$$

where $a \neq b$.

Coefficients corresponding to $\frac{1}{x-a}$ and $\frac{1}{x-b}$ can be obtained as explained in (1).
For instance, we can write

$$\frac{1}{x(x+1)(x^2+2)} \equiv \frac{1}{2x} - \frac{1}{3(x+1)} + \frac{Ax+B}{x^2+2}$$

To obtain A and B, we rewrite above expression as

$$1 \equiv \frac{1}{2}(x+1)(x^2+2) - \frac{1}{3}x(x^2+2)$$
$$+ (Ax+B)\,x(x+1)$$

Now, A and B can be obtained by comparing coefficients of like powers of x.

SOLVED EXAMPLES

Concept-based Straight Objective Type Questions

Example 1: If $f(x) = \frac{\log x}{x^3}$, then its antiderivative $F(x)$ given by

(a) $C - \frac{1}{2x^2}\log\left(x\sqrt{e}\right)$ (b) $C + \frac{1}{2x^2}\log x$

(c) $C - \frac{1}{2x^2}\log(xe)$ (d) $C + \frac{1}{2x^3}\log(xe)$

Ans. (a)

Solution: $F(x) = \int \frac{\log x}{x^3}\,dx = \log x \int \frac{1}{x^3}\,dx - \int \frac{1}{x}\left(\int \frac{1}{x^3}\,dx\right)dx$ (integration by parts)

$$= -\frac{1}{2x^2}\log x + \frac{1}{2}\int \frac{1}{x^3}\,dx + \text{Constant}$$

$$= -\frac{1}{2x^2}\log x - \frac{1}{4x^2} + C$$

$$= -\frac{1}{2x^2}\left[\log x + \frac{1}{2}\log e\right] + C$$

$$= -\frac{1}{2x^2}\log x\sqrt{e} + C$$

Example 2: The value of $\int \frac{1+\sin x}{1-\sin x}\,dx$ is

(a) $2\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) + C$

(b) $2\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) + x + C$

(c) $2\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) - x + C$

(d) $2\tan^2\left(\frac{x}{2}+\frac{\pi}{4}\right) - x + C$

Ans. (c)

Solution: $\int \frac{1+\sin x}{1-\sin x}\,dx = \int \frac{(1+\sin x)^2}{1-\sin^2 x}\,dx$

$$= \int (\sec x + \tan x)^2\,dx$$

$$= \int \sec^2 x + \int (\sec^2 x - 1)\,dx + 2\int \sec x \tan x\,dx$$

$$= 2\tan x - x + 2\sec x + C$$

$$= 2\left(\frac{1+\sin x}{\cos x}\right) - x + C$$

$$= 2\left(\frac{1-\cos\left(x+\frac{\pi}{2}\right)}{\sin\left(x+\frac{\pi}{2}\right)}\right) - x + C$$

$$= 2\,\frac{2\sin^2\left(\frac{x}{2}+\frac{\pi}{4}\right)}{2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)} - x + C$$

$$= 2\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) - x + C$$

Example 3: The value of $\int \frac{dx}{\sqrt{4x-3-x^2}}$ is equal to

(a) $\sin^{-1}(x-1) + C$

(b) $\log\left|(x-2) + \sqrt{4x-3-x^2}\right|$

(c) $\log\left|(x-1) + \sqrt{4x-3-x^2}\right| + C$

(d) $\sin^{-1}(x-2) + C$

Ans. (d)

Solution: $\sqrt{4x-3-x^2} = \sqrt{-\left(x^2-4x+3\right)}$

$$= \sqrt{-\left((x-2)^2-1\right)} = \sqrt{1-(x-2)^2}$$

So, $\displaystyle\int \frac{dx}{\sqrt{4x-3-x^2}} = \int \frac{dx}{\sqrt{1-(x-2)^2}} = \sin^{-1}(x-2) + C$

Example 4: $\displaystyle\int \frac{x^4}{x^2+1}\,dx$ is equal to

(a) $\dfrac{x^3}{3} + x + \tan^{-1} x + C$

(b) $\dfrac{x^3}{3} - x + \tan^{-1} + C$

(c) $\dfrac{x^2}{2} - x + 2\tan^{-1} x + C$

(d) $\dfrac{x^3}{3} - x - \tan^{-1} x + C$

Ans. (b)

Solution: Since degree of numerator is greater than the degree of denominator, we divide them

$$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}$$

so $\displaystyle\int \frac{x^4}{x^2+1}\,dx = \frac{x^3}{3} - x + \tan^{-1} x + C$

Example 5: If $\displaystyle\int \frac{2x - \sqrt{\sin^{-1} x}}{\sqrt{1-x^2}}\,dx = C - 2\sqrt{1-x^2} - \frac{2}{3}\sqrt{f(x)}$ then $f(x)$ is equal to

(a) $\sin^{-1}x$ (b) $2\sin^{-1} x$

(c) $(\sin^{-1}x)^3$ (d) $3(\sin^{-1}x)^3$

Ans. (c)

Solution: The integrand is equal to $\dfrac{2x}{\sqrt{1-x^2}} - \dfrac{\sqrt{\sin^{-1} x}}{\sqrt{1-x^2}}$.

The first part is of form $\dfrac{-g'(x)}{\sqrt{g(x)}}$ where $g(x) = (1-x^2)$

and other part is of the form $\sqrt{h(x)}\; h'(x)$, $h(x) = \sin^{-1} x$ Hence the required integral

$C - 2\sqrt{1-x^2} - \dfrac{2}{3}\left(\sin^{-1} x\right)^{3/2}$. Thus $f(x) = (\sin^{-1} x)^3$

Example 6: If $\displaystyle\int \frac{dx}{1+\sqrt[3]{x+1}} = \frac{3}{2}(x+1)^{2/3} - 3(x+1)^{1/3}$ $+ f(x) + C$ then $f(x)$ is equal to

(a) $\log\left|1+\sqrt[3]{x+1}\right|$ (b) $3\log\left|1+\sqrt[3]{x+1}\right|$

(c) $\dfrac{2}{3}\log\left|1+\sqrt[3]{x+1}\right|$ (d) $\dfrac{1}{3}\log\left|1+\sqrt[3]{x+1}\right|$

Ans. (b)

Solution: Put $x + 1 = t^3$, we have

$$\int \frac{dx}{1+\sqrt[3]{x+1}} = \int \frac{3t^2\,dt}{1+t} = 3\int\left(t - 1 + \frac{1}{1+t}\right)dt$$

$$= 3\left[\frac{t^2}{2} - t + \log|1+t|\right] + C$$

$$= \frac{3}{2}(x+1)^{2/3} - 3(x+1)^{1/3} + 3\log\left|1+\sqrt[3]{x+1}\right| + C$$

Example 7: An antiderivative of the integral $\displaystyle\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ is

(a) $e^x(1+x^2)^2$ (b) $\dfrac{-xe^x}{\left(1+x^2\right)^2}$

(c) $\dfrac{e^x(1-x)}{\left(1+x^2\right)^2}$ (d) $\dfrac{e^x}{1+x^2}$

Ans. (d)

Solution: $e^x\left(\dfrac{1-x}{1+x^2}\right)^2 = e^x\left[\dfrac{1}{1+x^2} - \dfrac{2x}{\left(1+x^2\right)^2}\right]$

so $\displaystyle\int e^x\left(\frac{1-x}{1+x}\right)^2 dx$ (see Page 12.4(19))

$$= \int \frac{e^x}{1+x^2}\,dx - \int e^x \frac{2x}{\left(1+x^2\right)^2}\,dx$$

$$= \frac{e^x}{1+x^2} + \int e^x \frac{2x}{\left(1+x^2\right)^2}\,dx - \int e^x \frac{2x}{\left(1+x^2\right)^2}\,dx + C$$

$$= \frac{e^x}{1+x^2} + C$$

Example 8: $\displaystyle\int \frac{3^{x+1} - 7^{x-1}}{21^x}\,dx = K_1\,3^{-x} + K_2 7^{-x} + C$

(a) $K_1 = \dfrac{1}{7\log 3}$, $K_2 = \dfrac{1}{3\log 7}$

(b) $K_1 = \dfrac{1}{7\log 3}$, $K_2 = -\dfrac{1}{3\log 7}$

(c) $K_1 = \dfrac{1}{7\log 3}$, $K_2 = -\dfrac{3}{\log 7}$

(d) $k_1 = \dfrac{3}{\log 7}$, $k_2 = \dfrac{-7}{\log 3}$

Ans. (c)

◎ **Solution:** $\int \frac{3^{x+1} - 7^{x-1}}{21^x} dx = 3\int 7^{-x}\, dx - \frac{1}{7}\int 3^{-x}\, dx$

$$= -\, 3\frac{7^{-x}}{\log 7} + \frac{1}{7}\frac{3^{-x}}{\log 3} + C$$

$$K_1 = \frac{1}{7\log 3},\ K_2 = -\frac{3}{\log 7}$$

◉ **Example 9:** If $\int \cos x \cos 2x \cos 5x\, dx$
$= A_1 \sin 2x + A_2 \sin 4x + A_3 \sin 6x$
$+ A_4 \sin 8x + C$

then

(a) $A_1 = \frac{1}{2}, A_2 = \frac{1}{4}$ (b) $A_1 = \frac{1}{4}, A_2 = \frac{1}{8}$

(c) $A_2 = \frac{1}{16}, A_3 = \frac{1}{8}$ (d) $A_1 = \frac{1}{8}, A_4 = \frac{1}{32}$

Ans. (d)

◎ **Solution:** $(\cos x \cos 2x) \cos 5x$

$$= \frac{1}{2}(\cos x + \cos 3x) \cos 5x$$

$$= \frac{1}{4}[\cos 4x + \cos 6x] + \frac{1}{4}[\cos 2x + \cos 8x]$$

$$= \frac{1}{4}\cos 2x + \frac{1}{4}\cos 4x + \frac{1}{4}\cos 6x + \frac{1}{4}\cos 8x$$

$$\int \cos x \cos 2x \cos 5x\, dx = \frac{1}{8}\sin 2x + \frac{1}{16}\sin 4x + \frac{1}{24}\sin 6x + \frac{1}{32}\sin x + C$$

◉ **Example 10:** If $\int \frac{dx}{1+e^x} = x + \mathrm{K}\log(1+e^x) + C$ then K is equal to

(a) 1 (b) –1
(c) 2 (d) –2

Ans. (b)

◎ **Solution:** Substituting $1 + e^x = t$, we have

$$\int \frac{dx}{1+e^x} = \int \frac{dt}{t(t-1)} = \int \frac{1}{t-1}dt - \int \frac{dt}{t}$$

$$= \log[t-1] - \log t + C$$

$$= \log e^x - \log(1+e^x) + C$$

$$= x - \log(1+e^x) + C$$

LEVEL 1

Straight Objective Type Questions

◉ **Example 11:** Let $f(x) = 1 + 3^x \log 3$ and $F(x)$ be its antiderivative which assume the value 7 for $x = 2$. The value of x for which the curve $F(x)$ cuts the abscissa axis is

(a) 0 (b) 1
(c) 2 (d) 3

Ans. (b)

◎ **Solution:** $F(x) = \int f(x)\,\mathrm{d}x + C = x + 3^x + C$ but $F(2) = 7$ so $7 = 2 + 9 + C$

$\Rightarrow C = -4$. Hence $F(x) = x + 3^x - 4$. Now $F(x) = 0$
$\Rightarrow x = 1$.

◉ **Example 12:** If $f(x) = \frac{1}{\cos^2 x\sqrt{1+\tan x}}$ then its antiderivative $F(x)$ (given that $F(0) = 4$) is

(a) $\sqrt{1+\tan x} + 4$ (b) $\frac{2}{3}(1+\tan x)^{3/2}$

(c) $2\left(\sqrt{1+\tan x} + 1\right)$ (d) none of these

Ans. (c)

◎ **Solution:** $F(x) = \int \frac{\mathrm{d}x}{\cos^2 x\sqrt{1+\tan x}} + C$

$$= \int \frac{1}{\sqrt{1+t}}\,\mathrm{d}t + C\ (t = \tan x)$$

$$= 2(\sqrt{1+t}) + C$$

$$= 2\sqrt{1+\tan x} + C$$

Since $4 = F(0) = 2 + C \Rightarrow C = 2$. Hence

$F(x) = 2\left(\sqrt{1+\tan x} + 1\right)$.

◉ **Example 13:** Let $f(x) = \frac{1}{4 - 3\cos^2 x + 5\sin^2 x}$ and if its antiderivative $F(x) = (1/3)\tan^{-1}(g(x)) + C$ then $g(x)$ is equal to

(a) $3 \tan x$ (b) $(\sqrt{2})\tan x$
(c) $2 \tan x$ (d) none of these

Ans. (a)

◎ **Solution:** $F(x) = \int \frac{\mathrm{d}x}{4 - 3\cos^2 x + 5\sin^2 x} + C$

$$= \int \frac{\sec^2 x \ \mathrm{d}x}{4\left(1 + \tan^2 x\right) - 3 + 5\tan^2 x} + C$$

$$= \int \frac{\mathrm{d}t}{9t^2 + 1} + C \qquad (t = \tan x)$$

$$= \frac{1}{3}\tan^{-1} 3t + C$$

Therefore $g(x) = 3 \tan x$.

◉ **Example 14:** Let $f(x) = \frac{\sqrt{\tan x}}{\sin x \cos x}$ and $F(x)$ is its anti-derivative, if $F(\pi/4) = 6$ then $F(x)$ is equal to

(a) $2\left(\sqrt{\tan x} + 1\right)$ (b) $2\left(\sqrt{\tan x} + 3\right)$

(c) $2\left(\sqrt{\tan x} + 2\right)$ (d) none of these

Ans. (c)

◎ **Solution:** $F(x) = \int \frac{\sqrt{\tan x}}{\sin x \cos x}\mathrm{d}x = \int \frac{\sqrt{\tan x}}{\tan x}\sec^2 x \ \mathrm{d}x$ $(t = \tan x)$

$$= \int \frac{1}{\sqrt{t}}\,\mathrm{d}t + C = 2\sqrt{t} + C = 2\sqrt{\tan x} + C$$

Since $6 = F(\pi/4) = 2 + C$ so $C = 4$. Hence

$$F(x) = 2\left(\sqrt{\tan x} + 2\right)$$

◉ **Example 15:** The value $\int \frac{\mathrm{d}x}{x^4 + 5x^2 + 4}$ is

(a) $(1/3)\tan^{-1} x - (1/2)\tan^{-1}(x/2) + C$
(b) $3\tan^{-1}(x/3) + 2\tan^{-1}(x/2) + C$
(c) $(1/3)\tan^{-1} x + (1/6)\tan^{-1}(x/2) + C$
(d) $(1/3)\tan^{-1} x - (1/6)\tan^{-1}(x/2) + C$

Ans. (d)

◎ **Solution:** $I = \int \frac{\mathrm{d}x}{x^4 + 5x^2 + 4} = \int \frac{\mathrm{d}x}{\left(x^2 + 1\right)\left(x^2 + 4\right)}$

$$= \int \frac{1}{3}\left[\frac{1}{x^2 + 1} - \frac{1}{x^2 + 4}\right]dx$$

Therefore, $I = \frac{1}{3}\tan^{-1} x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C.$

◉ **Example 16:** If $\int \frac{(x - 1)^2}{\left(x^2 + 1\right)^2}\,\mathrm{d}x = \tan^{-1} x + g(x) + K$ then $g(x)$ is equal to

(a) $\tan^{-1}(x/2)$ (b) $\frac{1}{x^2 + 1}$

(c) $\frac{1}{2\left(x^2 + 1\right)}$ (d) none of these

Ans. (b)

◎ **Solution:** $\frac{(x - 1)^2}{\left(x^2 + 1\right)^2} = \frac{x^2 - 2x + 1}{\left(x^2 + 1\right)^2}$

$$= \frac{x^2 + 1}{\left(x^2 + 1\right)^2} - \frac{2x}{\left(x^2 + 1\right)^2}$$

$$= \frac{1}{x^2 + 1} - \frac{2x}{\left(x^2 + 1\right)^2}$$

so $\int \frac{(x - 1)^2}{\left(x^2 + 1\right)^2}\,\mathrm{d}x = \int \frac{dx}{x^2 + 1} - \int \frac{2x}{\left(x^2 + 1\right)^2}\,\mathrm{d}x$

$$= \tan^{-1} x + \frac{1}{x^2 + 1} + K$$

Therefore, $g(x) = \frac{1}{x^2 + 1}$.

◉ **Example 17:** If the antiderivative of $f(x) = \frac{\sin x}{\sin^2 x + 4\cos^2 x}$ is $\left(1/\sqrt{3}\right)\tan^{-1}\left(\left(1/\sqrt{3}\right)g(x)\right) + C$ then $g(x)$ is equal to

(a) $\sec x$ (b) $\tan x$
(c) $\sin x$ (d) $\cos x$

Ans. (a)

◎ **Solution:** $f(x) = \frac{\sin x}{\sin^2 x + 4\cos^2 x} = \frac{\tan x \sec x}{\tan^2 x + 4}$

$$= \frac{\tan x \sec x}{\sec^2 x + 3}.$$

Putting $\sec x = t$, $\mathrm{d}x \sec x \tan x = \mathrm{d}t$ so

$$\int f(x)\,dx = \int \frac{\mathrm{d}t}{t^2 + 3} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{\sec x}{\sqrt{3}} + C.$$

◉ **Example 18:** If $\int \frac{x}{x^2 - 4x + 8}\,\mathrm{d}x = K \log\left(x^2 - 4x + 8\right) + \tan^{-1}\left(\frac{x - 2}{2}\right) + C$ then the value of K is

(a) 1/2 (b) 1

(c) 2 (d) none of these

Ans. (a)

◎ **Solution:** Completing the square, we get

$$x^2 - 4x + 8 = (x - 2)^2 + 4 \text{ so put } x - 2 = u$$

$$\int \frac{x}{x^2-4x+8}\,dx = \int \frac{u+2}{u^2+4}\,du$$

$$= \frac{1}{2}\int \frac{2u}{u^2+4}\,du + 2\int \frac{du}{u^2+4}$$

$$= \frac{1}{2}\log(u^2+4) + \tan^{-1}\frac{u}{2} + C$$

$$= \frac{1}{2}\log(x^2-4x+8) + \tan^{-1}\frac{x-2}{2} + C.$$

Example 19: The value of integral $\int \frac{dx}{x\sqrt{1-x^3}}$ is given by

(a) $\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}+1}{\sqrt{1-x^3}-1}\right| + C$

(b) $\frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^2}+1}\right| + C$

(c) $\frac{2}{3}\log\left|\frac{1}{\sqrt{1-x^3}}\right| + C$

(d) $\frac{1}{3}\log\left|1-x^3\right| + C$

Ans. (b)

Solution: Put $1 - x^3 = t^2$. Then $-3x^2dx = 2t\,dt$ and the integral becomes

$$-\frac{1}{3}\int\frac{-3x^2\,dx}{x^3\sqrt{1-x^3}} = -\frac{1}{3}\int\frac{2t\,dt}{\left(1-t^2\right)t} = \frac{2}{3}\int\frac{dt}{t^2-1}$$

$$= \frac{2}{3}\left(\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|\right) + C$$

$$= \frac{1}{3}\log\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + C.$$

Example 20: The integral $\int\frac{\cos 4x - 1}{\cot x - \tan x}$ is equal to

(a) $-\frac{1}{2}\cos 4x + C$ (b) $-\frac{1}{4}\cos 4x + C$

(c) $-\frac{1}{2}\sin 2x + C$ (d) none of these

Ans. (d)

Solution: Put $\cos 2x = t$. Then $-2\sin 2x\,dx = dt$ and the integral becomes

$$\int\left(\frac{-2\sin^2 2x}{\cos^2 x - \sin^2 x}\right)\cos x \sin x\,dx$$

$$= -\int\left(\frac{1-\cos^2 2x}{\cos 2x}\right)\sin 2x\,dx$$

$$= \frac{1}{2}\int\frac{1-t^2}{t}\,dt = \frac{1}{2}\left(\log|t| - \frac{t^2}{2}\right) + C$$

$$= \frac{1}{2}\log|\cos 2x| - \frac{1}{4}\cos^2 2x + C.$$

Example 21: The integral $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$ is equal to

(a) $\frac{e^{\tan^{-1}x}}{1+x^2} + C$ (b) $\frac{x\,e^{\tan^{-1}x}}{1+x^2} + C$

(c) $x\,e^{\tan^{-1}x} + C$ (d) none of these

Ans. (c)

Solution: Putting $\tan^{-1}x = u$, we have $\frac{dx}{1+x^2} = du$

$$\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx = \int e^u\,(1+\tan u + \tan^2 u)\,du$$

$$= \int e^u\,(\sec^2 u + \tan u)\,du$$

$$= \tan u\, e^u + C = x\,e^{\tan^{-1}x} + C.$$

$$\left(\text{using } \int e^x(f(x)+f'(x)\,dx = e^x f(x) + C\right)$$

Example 22: $\int\frac{dx}{\sqrt{2-3x-x^2}} = fog\,(x) + C$ then

(a) $f(x) = \sin^{-1}x$, $g(x) = \frac{2x-3}{\sqrt{17}}$

(b) $f(x) = \tan^{-1}x$, $g(x) = \frac{2x+3}{\sqrt{17}}$

(c) $f(x) = \sin^{-1}x$, $g(x) = \frac{2x+3}{\sqrt{17}}$

(d) none of these

Ans. (c)

Solution: $\int\frac{dx}{\sqrt{2-3x-x^2}} = \int\frac{dx}{\sqrt{-\left(x^2+3x-2\right)}}$

$$= \int\frac{dx}{\sqrt{-\left((x+3/2)^2 - 17/4\right)}}$$

$$= \int\frac{dx}{\sqrt{17/4 - (x+3/2)^2}}$$

$$= \sin^{-1}\frac{x+3/2}{\sqrt{17}/2} + C = \sin^{-1}\frac{2x+3}{\sqrt{17}} + C$$

Therefore, $g(x) = \frac{2x+3}{\sqrt{17}}$ and $f(x) = \sin^{-1}x$.

Example 23: If $\int \frac{d\,x}{x\sqrt{5\,x^2-3}} = K\tan^{-1} f(x) + C$ then

(a) $f(x) = \sqrt{\frac{5}{3}x^2-1}, K = \frac{1}{\sqrt{5}}$

(b) $f(x) = \sqrt{\frac{5}{3}x^2-1}, K = \frac{1}{\sqrt{3}}$

(c) $f(x) = \frac{1}{2}\sqrt{5\,x^2-3}, K = \frac{1}{\sqrt{5}}$

(d) none of these

Ans. (b)

Solution: Puting $5x^2 - 3 = t^2$ so $5x\,\mathrm{d}x = t\,\mathrm{d}t$

$$\int \frac{\mathrm{d}x}{x\sqrt{5x^2-3}} = \int \frac{x\,\mathrm{d}x}{x^2\sqrt{5x^2-3}} = \frac{1}{5}\int \frac{5t\,\mathrm{d}t}{\left(t^2+3\right)t}$$

$$= \int \frac{\mathrm{d}t}{t^2+3} = \frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\frac{5x^2-3}{3}} + C.$$

Example 24: The value of $\int \frac{\sin x - \cos x}{\sqrt{\sin 2x - 1/2}}\,\mathrm{d}x$ is

(a) $\log|(\sin x + \cos x) + \sqrt{\sin 2x}| + C$

(b) $-\log|(\sin x + \cos x) + \sqrt{\sin 2x - 1/2}| + C$

(c) $-\frac{1}{\sqrt{2}}\log|(\sin x + \cos x) + \sqrt{\sin 2x - 1/2}| + C$

(d) none of these

Ans. (b)

Solution: The given integral can be written as

$$\int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2 - 3/2}}\,dx = -\int \frac{\mathrm{d}t}{\sqrt{t^2-3/2}}$$

$(t = \sin x + \cos x)$

$$= -\log\left|t + \sqrt{t^2 - 3/2}\right| + C$$

$$= -\log|\sin x + \cos x + \sqrt{\sin 2x - 1/2}| + C$$

Example 25: If $\int \frac{\cos 4\,x + 1}{\cot x - \tan x} = K\cos 4x + C$, then

(a) $K = -1/2$	(b) $K = -1/8$
(c) $K = -1/5$	(d) none of these

Ans. (b)

Solution: $\int \frac{\cos 4x + 1}{\cot x - \tan x}\,\mathrm{d}x$

$$= \int \frac{2\cos^2 2x}{\cos^2 x - \sin^2 x}\cdot \sin x \cos x\,\mathrm{d}x$$

$$= \int \cos 2x \sin 2x\,\mathrm{d}x = \frac{1}{2}\int \sin 4x\,\mathrm{d}x$$

$$= -\frac{1}{8}\cos 4x + C$$

Hence $K = -1/8$.

Example 26: $\int \frac{e^x(1+\sin x)}{1+\cos x}\,\mathrm{d}x$ is equal to

(a) $\log|\tan x| + C$	(b) $e^x \tan x/2 + C$
(c) $e^x \cot x + C$	(d) $\sin \log x + C$

Ans. (b)

Solution: $\int \frac{e^x(1+\sin x)}{1+\cos x}\,\mathrm{d}x$

$$= \int \frac{e^x\left[1 + 2\sin(x/2)\cos(x/2)\right]}{2\cos^2(x/2)}\,\mathrm{d}x$$

$$= \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)\mathrm{d}x$$

$$= \frac{1}{2}\int e^x \sec^2\frac{x}{2}\,\mathrm{d}x + \int e^x \tan\frac{x}{2}\,\mathrm{d}x$$

$$= \frac{1}{2}\int e^x\sec^2\frac{x}{2}\mathrm{d}x + e^x\tan\frac{x}{2} - \frac{1}{2}\int e^x\sec^2\frac{x}{2} + C$$

[integrating by parts]

$$= e^x\tan\frac{x}{2} + C$$

Example 27: If $\int \sqrt{1+\sec x}\,\mathrm{d}x = K\sin^{-1}(f(x)) + C$ then

(a) $f(x) = \sqrt{2}\sin(x/2)$

(b) $f(x) = \sqrt{2}\cos(x/2), K = 2$

(c) $f(x) = \sqrt{2}\tan(x/2), K = 2$

(d) $f(x) = \sqrt{2}\sin(x/2), K = \sqrt{2}$

Ans. (a)

Solution: $\sqrt{1+\sec x} = \sqrt{\frac{1+\cos x}{\cos x}} = \frac{\sqrt{2}\cos x/2}{\sqrt{1-2\sin^2 x/2}}$ so

putting $\sin x/2 = t$, we have

$$\int \sqrt{1+\sec x}\;dx = \int \frac{\sqrt{2}\cos x/2}{\sqrt{1-2\sin^2 x/2}}\,\mathrm{d}x$$

$$= 2\sin^{-1}\sqrt{2}t + C\ (t = \sin x/2)$$

$$= 2\sin^{-1}(\sqrt{2}\sin x/2) + C$$

so $\quad f(x) = \sqrt{2}\sin x/2$ and $K = 2$

Example 28: The antiderivative of $f(x) = \log(\log x) + (\log x)^{-2}$ whose graph passes through (e, e) is

(a) $x(\log(\log x) + (\log x)^{-1})$

(b) $x(-\log(\log x) + (\log x)^{-1}) + e$

(c) $x\,(\log(\log x) - (\log x)^{-1}) + 2e$

(d) none of these

Ans. (c)

◎ **Solution:** An antiderivative of $f(x) = F(x) =$

$$\int (\log(\log x) + (\log x)^{-2})\,\mathrm{d}x + C$$

$$= x \log(\log x) - \int \frac{x}{x \log x}\,\mathrm{d}x + \int (\log x)^{-2}\,\mathrm{d}x + C$$

(integrating by parts the first term)

$$= x \log(\log x) - [x(\log x)^{-1} + \int (\log x)^{-2}\,\mathrm{d}x] + \int (\log x)^{-2}\,\mathrm{d}x + C$$

(again integrating by parts $(\log x)^{-1}$)

$$= x \log(\log x) - x(\log x)^{-1} + C$$

Putting $x = e$, we have $e = 0 - e + C$ so $C = 2e$

Thus $F(x) = x\,(\log(\log x) - (\log x)^{-1}) + 2e$.

● **Example 29:** $\displaystyle\int \frac{\mathrm{d}x}{a^2\cos^2 x + b^2 \sin^2 x}$ $(a, b > 0)$ is equal to

(a) $\sin^{-1}((a/b)\tan x) + C$

(b) $\tan^{-1}((b/a)\tan x) + C$

(c) $\dfrac{1}{ab}\tan^{-1}\left(\left(\dfrac{b}{a}\right)\tan x\right) + C$

(d) none of these

Ans. (c)

◎ **Solution:** Putting $t = \tan x$, we get

$$\int \frac{\mathrm{d}x}{a^2\cos^2 x + b^2\sin^2 x} = \int \frac{\sec^2 x\,\mathrm{d}x}{a^2 + b^2\tan^2 x} = \int \frac{\mathrm{d}t}{a^2 + b^2 t^2}$$

$$= \frac{1}{b^2}\int \frac{\mathrm{d}t}{t^2 + (a^2/b^2)} = \frac{1}{b^2}\cdot\frac{b}{a}\tan^{-1}\left(\frac{b}{a}t\right) + C$$

$$= \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + C.$$

● **Example 30:** If $\int x \log(x + 1/x)\,\mathrm{d}x = f(x)\log(x+1) + g(x)\,x^2 + Lx + C$, then

(a) $f(x) = (1/2)\,(x^2 - 1)$ (b) $g(x) = \log x$

(c) $L = 1$ (d) none of these

Ans. (a)

◎ **Solution:** The given integral equals

$$\int x \log\left(\frac{x+1}{x}\right)\mathrm{d}x = \int x \log(x+1)\,\mathrm{d}x - \int x \log x\,\mathrm{d}x$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\,\mathrm{d}x - \frac{x^2}{2}\log x + \frac{1}{2}\int \frac{x^2}{x}\,\mathrm{d}x$$

$$= \frac{x^2}{2}\log(x+1) - \frac{x^2}{2}\log x - \frac{1}{2}\int\left(x - 1 + \frac{1}{x+1}\right)\mathrm{d}x + \frac{1}{2}\int x\,\mathrm{d}x$$

$$= \frac{x^2}{2}\log(x+1) - \frac{x^2}{2}\log x + \frac{x}{2} - \frac{1}{2}\log(x+1) + C$$

$$\Rightarrow f(x) = \frac{x^2}{2} - \frac{1}{2},\ g(x) = -\frac{1}{2}\log x \text{ and } L = \frac{1}{2}.$$

● **Example 31:** The function f whose graph passes through $(0, 7/3)$ and whose derivative is $x\sqrt{1 - x^2}$ is given by

(a) $f(x) = (1/3)\,((1 - x^2)^{3/2} + 7)$

(b) $f(x) = (3/2)\,[\sin^{-1} x + 6]$

(c) $f(x) = -(1/3)\,[(1 - x^2)^{3/2} - 8]$

(d) $f(x) = -(2/3)\,[(1 - x^2)^{3/2} - 8]$

Ans. (c)

◎ **Solution:** $f(x) = \int x\sqrt{1 - x^2}\,\mathrm{d}x$

$$= -\frac{1}{2}\int(-2x)\sqrt{1 - x^2}\,\mathrm{d}x$$

$$= -\frac{1}{3}(1 - x^2)^{3/2} + C$$

So $7/3 = -1/3 + C \Rightarrow C = 8/3$.

Therefore, $f(x) = -(1/3)\,[(1 - x^2)^{3/2} - 8]$.

● **Example 32:** The value of $\displaystyle\int \frac{\sin\alpha}{\sqrt{1 + \cos\alpha}}\,\mathrm{d}\alpha$ is

(a) $2\sqrt{2}\cos(\alpha/2) + C$

(b) $-2\sqrt{2}\cos(\alpha/2) + C$

(c) $\sqrt{2}\cos(\alpha/2) + C$

(d) $-\sqrt{2}\cos(\alpha/2) + C$

Ans. (b)

◎ **Solution:** $\displaystyle\int \frac{\sin\alpha}{\sqrt{1 + \cos\alpha}}\,\mathrm{d}\alpha = -\int \frac{\mathrm{d}x}{\sqrt{x}}$ $(x = 1 + \cos\alpha)$

$$= -2(1 + \cos\alpha)^{1/2} + C$$

$$= -2\sqrt{2}\cos\alpha/2 + C.$$

● **Example 33:** The value of $\displaystyle\int \frac{\mathrm{d}x}{5 + 4\cos x}$ is

(a) $\dfrac{1}{3}\tan^{-1}\left(\dfrac{1}{3}\tan x\right) + C$

(b) $\dfrac{1}{3}\tan^{-1}\left(\dfrac{1}{3}\tan\left(\dfrac{x}{2}\right)\right) + C$

(c) $\dfrac{2}{3}\tan^{-1}\left(\dfrac{1}{3}\tan x\right) + C$

(d) $\dfrac{2}{3}\tan^{-1}\left(\dfrac{1}{3}\tan\left(\dfrac{x}{2}\right)\right) + C$

Ans. (d)

◎ **Solution:** $\displaystyle\int \frac{\mathrm{d}x}{5 + 4\cos x} = \int \frac{2\,\mathrm{d}t}{5(1 + t^2) + 4(1 - t^2)}$

$(t = \tan(x/2))$

$$= 2\int \frac{dt}{9+t^2} = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{x}{2}\right)\right) + C$$

Example 34: $\int \frac{dx}{x(x^4+1)}$ is equal to

(a) $\frac{1}{4}\log\frac{x^4+1}{x^4}+C$ (b) $\frac{1}{4}\log\left(\frac{x^4}{x^4+1}\right)+C$

(c) $\frac{1}{4}\log(x^4+1)+C$ (d) none of these

Ans. (b)

Solution: $\int \frac{dx}{x\left(x^4+1\right)} = \frac{1}{4}\int \frac{4x^3\,dx}{x^4\left(x^4+1\right)}$

$$= \frac{1}{4}\int \frac{dt}{t(t+1)} \qquad (t = x^4)$$

$$= \frac{1}{4}\int\left(\frac{1}{t}-\frac{1}{t+1}\right)dt = \frac{1}{4}\log\frac{x^4}{x^4+1} + C.$$

Example 35: $\int \frac{dx}{\sin x + \cos x}$ is equal to

(a) $\log\tan(\pi/4 + x/8) + C$

(b) $\sqrt{2}\,\log\tan(x/2 + \pi/8) + C$

(c) $\frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right| + C$

(d) $\sqrt{2}\,\log\tan\left(\frac{x}{4}-\frac{\pi}{8}\right) + C$

Ans. (c)

Solution: $\int \frac{dx}{\sin x + \cos x}$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sin(\pi/4)\sin x + \cos(\pi/4)\cos x}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\cos(x-\pi/4)} = \frac{1}{\sqrt{2}}\int \sec(x-\pi/4)\,dx$$

$$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\pi}{8}\right)\right| + C$$

$$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{\pi}{8}+\frac{x}{2}\right)\right| + C.$$

Example 36: The primitive of $\frac{1}{(x-a)^{3/2}(b-x)^{1/2}}$ is

(a) $\frac{1}{b-a}\left[\frac{b-x}{x-a}\right]^{1/2} + C$

(b) $\frac{3}{4(b-a)}\left[\frac{b-x}{x-a}\right]^{1/2} + C$

(c) $\frac{1}{b-a}\left[\frac{x-a}{b-a}\right]^{1/2} + C$

(d) none of these

Ans. (d)

Solution: Putting $x - a = (1/t)$, we have $dx = (-1/t^2)\,dt$ and the given integral can be written as

$$\int \frac{-\frac{1}{t^2}\,dt}{(1/t)^{3/2}\left(b-a-\frac{1}{t}\right)^{1/2}} = -\int \frac{dt}{\sqrt{(b-a)t-1}}$$

$$= -2\,\frac{\sqrt{(b-a)t-1}}{b-a} + C.$$

$$= \frac{-2}{b-a}\sqrt{\frac{b-a-(x-a)}{x-a}} + C$$

$$= \frac{2}{a-b}\sqrt{\frac{b-x}{x-a}} + C.$$

Example 37: If $f(x)$ is the primitive of

$\frac{\sin\sqrt[3]{x}\,\log(1+3x)}{\left(\tan^{-1}\sqrt{x}\right)^2\left(e^{\sqrt[3]{x}}-1\right)}$ $(x \neq 0)$ then $\lim_{x\to 0} f'(x)$ is

(a) 0 (b) 3/5

(c) 5/3 (d) none of these

Ans. (d)

Solution: $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\frac{\sin\sqrt[3]{x}\,\log(1+3x)}{\left(\tan^{-1}\sqrt{x}\right)^2\left(e^{\sqrt[3]{x}}-1\right)}$

$$= \lim_{x\to 0} 3\,\frac{\sin\sqrt[3]{x}}{\sqrt[3]{x}} \times \frac{\log(1+3x)}{3x} \times \left(\frac{\sqrt{x}}{\tan^{-1}\sqrt{x}}\right)^2 \frac{\sqrt[3]{x}}{e^{\sqrt[3]{x}}-1}$$

$$= 3(1)\,(1)\,(1)^2(1) = 3$$

Example 38: If the primitive of $f(x) = \frac{1}{3\sin x + \sin^3 x}$

is equal to $\frac{1}{6}\log\left|\frac{t-1}{t+1}\right| + \frac{1}{12}\log\left|\frac{2+t}{2-t}\right| + C$ then

(a) $t = \cos x$ (b) $t = \tan x/2$

(c) $t = 2\cos x$ (d) $t = \sin x$

Ans. (a)

Solution: $\int \frac{dx}{3\sin x + \sin^3 x} = \int \frac{\sin x\,dx}{\left(3+\sin^2 x\right)\sin^2 x}$

$$= \int \frac{\sin x\,dx}{\left(4-\cos^2 x\right)\left(1-\cos^2 x\right)}$$

$$= -\int \frac{\mathrm{d}t}{\left(4-t^2\right)\left(1-t^2\right)}$$

$$= -\frac{1}{3}\int\left(\frac{1}{t^2-4}-\frac{1}{t^2-1}\right) \qquad (t = \cos x)$$

$$= \frac{1}{6}\log\left|\frac{t-1}{t+1}\right| + \frac{1}{12}\log\left|\frac{2+t}{2-t}\right| + C.$$

Example 39: The value of $\int \frac{\cos x}{\sin x + \cos x}\,\mathrm{d}x$ is

(a) $\frac{x^2}{2} + \log|\sin x + \cos x| + C$
(b) $\frac{1}{2}(x + \log|\sin x + \cos x|) + C$
(c) $2\log|\sin x + \cos x| + x + C$
(d) none of these

Ans. (b)

Solution: $\int \frac{\cos x}{\sin x + \cos x}\,\mathrm{d}x$

$$= \frac{1}{2}\int \frac{\cos x + \sin x + \cos x - \sin x}{\sin x + \cos x}\,\mathrm{d}x$$

$$= \frac{1}{2}\int\left(1 + \frac{\cos x - \sin x}{\sin x + \cos x}\right)\mathrm{d}x$$

$$= (1/2)\,(x + \log|\sin x + \cos x|) + C.$$

Example 40: If the primitive of $\frac{1}{\left(e^x-1\right)^2}$ is $f(x) - \log|g(x)| + C$ then

(a) dom $f = \mathbf{R}$ (b) $g(x) = 1 - e^x$
(c) dom $f = \mathbf{R} \sim \{0\}$ (d) $f(x) = 1 - e^{-x}$

Ans. (c)

Solution: $\int \frac{1}{\left(e^x-1\right)^2}\,\mathrm{d}x = \int \frac{e^x\,\mathrm{d}x}{e^x\left(e^x-1\right)^2} = \int \frac{\mathrm{d}t}{t(t-1)^2}$

$(t = e^x)$

$$= \int \frac{1}{t-1}\left[\frac{1}{t-1} - \frac{1}{t}\right]\mathrm{d}t$$

$$= \int\left(\frac{1}{(t-1)^2} - \frac{1}{t-1} + \frac{1}{t}\right)\mathrm{d}t$$

$$= \frac{1}{1-t} - \log|t-1| + \log|t| + C$$

$$= (1-e^x)^{-1} - \log|1-e^{-x}| + C$$

Hence $f(x) = (1-e^x)^{-1}$ and $g(x) = 1 - e^{-x}$

The domain of $f = \mathbf{R} \sim \{0\}$

Example 41: If $\int \frac{\mathrm{d}x}{\sin^4 x + \cos^4 x} = \frac{1}{\sqrt{2}}\tan^{-1} f(x) + C$ then

(a) $f(x) = \tan x - \cot x$
(b) $f(\pi/4) = 0$
(c) $f(x)$ is continuous on **R**
(d) $f(x) = \frac{1}{2}(\tan x - \cot x)$

Ans. (b)

Solution: Dividing the numerator and denominator by $\cos^4 x$, the given integral can be written as

$$\int \frac{\sec^2 x\left(1+\tan^2 x\right)}{\tan^4 x + 1}\,\mathrm{d}x = \int \frac{1+t^2}{1+t^4}\,\mathrm{d}t \ (t = \tan x)$$

$$= \int \frac{1+1/t^2}{t^2+1/t^2}\,\mathrm{d}t = \int \frac{1+1/t^2}{\left(t-\frac{1}{t}\right)^2+2}\,\mathrm{d}t$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\left(t-\frac{1}{t}\right) + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{\tan x - \cot x}{\sqrt{2}} + C$$

Thus $f(x) = \frac{\tan x - \cot x}{\sqrt{2}}$ so $f\left(\frac{\pi}{4}\right) = 0.$

Example 42: The $\int \frac{\sin^2 x}{\cos^6 x}\,\mathrm{d}x$ is a

(a) is a polynomial of degree 5 in $\sin x$
(b) is a polynomial of degree 4 in $\tan x$
(c) is a polynomial of degree 5 in $\tan x$
(d) is a polynomial of degree 5 in $\cos x$

Ans. (c)

Solution: Putting $\tan x = t$, the given integral can be written as $\int t^2(1+t^2)\,\mathrm{d}t = \frac{t^3}{3} + \frac{t^5}{5} + C = \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + C$

which is a polynomial of degree 5 in $\tan x$.

Example 43: If $\int \tan^7 x\,\mathrm{d}x = f(x) + C$ then

(a) $f(x)$ is a polynomial of degree 8 in $\tan x$
(b) $f(x)$ is a polynomial of degree 5 in $\tan x$
(c) $f(x) = \frac{1}{6}\tan^6 x - \frac{1}{4}\tan^4 x + \frac{1}{2}\tan^2 x + \log|\cos x| + C$
(d) $f(x)$ is a polynomial of degree 6 in $\tan x$

Ans. (c)

Solution: Putting $\tan x = t$, we have $\mathrm{d}x = \frac{\mathrm{d}t}{1+t^2}$, so

$$\int \tan^7 x\,\mathrm{d}x = \int \frac{t^7}{1+t^2}\,\mathrm{d}t$$

$$= \int \left(t^5 - t^3 + t - \frac{t}{1+t^2} \right) \mathrm{d}t$$

$$= \frac{t^6}{6} - \frac{t^4}{4} + \frac{t^2}{2} - \frac{1}{2} \log (1 + t^2) + C$$

$$= \frac{1}{6} \tan^6 x - \frac{1}{4} \tan^4 x + \frac{1}{2} \tan^2 x - \frac{1}{2} \log \sec^2 x + C$$

$$= \frac{1}{6} \tan^6 x - \frac{1}{4} \tan^4 x + \frac{1}{2} \tan^2 x + \log |\cos x| + C.$$

Example 44: If $\int \frac{\mathrm{d}x}{\sin x \cos x} = \log |f(x)| + C$ then

(a) $f(x) = \sin x + \cos x$ (b) $f(x) = \tan x$

(c) $f(x) = \sec^2 x$ (d) none of these

Ans. (b)

Solution: $\int \frac{\mathrm{d}x}{\sin x \cos x} = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} \mathrm{d}x$

$$= \int (\tan x + \cot x)\, \mathrm{d}x$$

$$= -\log |\cos x| + \log |\sin x| + C$$

$= \log |\tan x| + C$. Thus $f(x) = \tan x$.

Example 45: If polynomials P and Q satisfies

$$\int ((3x - 1) \cos x + (1 - 2x) \sin x)\, \mathrm{d}x$$

$$= P \cos x + Q \sin x$$

(ignoring the constant of integration) then

(a) $P = 3x - 2$ (b) $Q = 2 + x$

(c) $P = 3(x - 1)$ (d) $Q = 3(x - 1)$

Ans. (d)

Solution: Integrating by parts, the given integral is equal to

$$(3x - 1) \sin x - 3 \int \sin x\, \mathrm{d}x - (1 - 2x) \cos x - 2 \int \cos x\, \mathrm{d}x$$

$= (3x - 1) \sin x + 3 \cos x - (1 - 2x) \cos x - 2 \sin x$

$= (3x - 3) \sin x + (2 + 2x) \cos x$

Hence $Q = 3(x - 1)$ and $P = 2(1 + x)$

Example 46: Let $f(x) = \int e^{x^2} (x - 2)(x - 3)(x - 4)\, dx$ then f increases on

(a) $(-\infty, 2)$ (b) $(2, \infty)$

(c) $(2, 3)$ (d) $(3, \infty)$

Ans. (c)

Solution: $f'(x) = e^{x^2} (x - 2)(x - 3)(x - 4)$

As $e^{x^2} > 0$ for all $x \in \mathbf{R}, f'(x) > 0$

if $(x - 2)(x - 3)(x - 4) > 0$ i.e., if $2 < x < 3$ or $x > 4$

Thus $f(x)$ increases on the interval $(2, 3)$

Example 47: $I = \int \frac{x^2 - 1}{x^3 \sqrt{2x^4 - 2x^2 + 1}}\, dx$ equal to

(a) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + C$

(b) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^3} + C$

(c) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{x} + C$

(d) $\frac{\sqrt{2x^4 - 2x^2 + 1}}{2x^2} + C$

Ans. (d)

Solution: $I = \int \frac{\left((1/x^3) - (1/x^5)\right)}{\sqrt{2 - (2/x^2) - (1/x^4)}}\, dx$

Put $2 - \frac{2}{x^2} - \frac{1}{x^4} = t^2 \Rightarrow \left(\frac{4}{x^3} + \frac{4}{x^5}\right) dx = 2t\, dt$

$\therefore$ $I = \frac{1}{2} \int \frac{t\, dt}{t} = \frac{1}{2} t + C$

$$= \frac{1}{2} \sqrt{2 - \frac{2}{x^2} - \frac{1}{x^4}} + C$$

$$= \frac{\sqrt{2x^4 - 2x^2 - 1}}{2x^2} + C$$

Example 48: Let $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1}\, dx$, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1}\, dx$. Then for an arbitrary constant C, the value of $I - J$ equals

(a) $\frac{1}{2} \log \left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1} \right) + C$

(b) $\frac{1}{2} \log \left(\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1} \right) + \mathrm{C}$

(c) $\frac{1}{2} \log \left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right) + C$

(d) $\frac{1}{2} \log \left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1} \right) + C$

Ans. (b)

Solution: $I - J = \int \frac{e^x}{e^{4x} + e^{2x} + 1}\, dx - \int \frac{e^{3x}}{e^{4x} + e^{2x} + 1}\, dx$

$$= \int \frac{e^x\left(1-e^{2x}\right)}{e^{4x}+e^{2x}+1}\,dx$$

Put $e^x = t$, $e^x\,dx = dt$

$$I - J = \int \frac{\left(1-t^2\right)}{t^4+t^2+1}\,dt = \int \frac{\frac{1}{t^2}-1}{t^2+\frac{1}{t^2}+1}\,dt$$

$$= -\int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-1} = \int \frac{du}{1-u^2},$$

$$u = t + \frac{1}{t}$$

$$= \frac{1}{2}\log\left|\frac{1+u}{1-u}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{e^x+e^{-x}+1}{e^x+e^{-x}-1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{e^{2x}+e^x+1}{e^{2x}-e^x+1}\right| + C.$$

Example 49: The value of $I = \sqrt{2}\int \frac{\sin x}{\sin(x-\pi/4)}\,dx$ is

(a) $x + \log|\cos(x-\pi/4)| + C$

(b) $x - \log|\sin(x-\pi/4)| + C$

(c) $x + \log|\sin(x-\pi/4)| + C$

(d) $x - \log|\cos(x-\pi/4)| + C$

Ans. (c)

Solution: Put $x - \pi/4 = t$

$$I = \sqrt{2}\int \frac{\sin(\pi/4+t)}{\sin t}\,dt = \int(\cot t + 1)\,dt$$

$$= t + \log|\sin t| + C'$$

$$= x + \log|\sin(x-\pi/4)| + C,\ C = C' + \pi/4.$$

Example 50: If $\int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx$

$= x[f(x) - g(x)] + C$, then :

(a) $f(x) = \log(\log x)$; $g(x) = \frac{1}{\log x}$

(b) $f(x) = \log x$; $g(x) = \frac{1}{\log x}$

(c) $f(x) = \frac{1}{\log x}$; $g(x) = \log(\log x)$

(d) $f(x) = \frac{1}{x\log x}; g(x) = \frac{1}{\log x}$

Ans. (a)

Solution: $\int \log(\log x)dx$

$$= x\log(\log x) - \int x\,\frac{1}{\log x}\cdot\frac{1}{x}\,dx$$

$$= x\log(\log x) - \int\frac{dx}{\log x}$$

$$= x\log(\log x) - x\,\frac{1}{\log x} + \int x\,\frac{(-1)}{(\log x)^2}\,\frac{1}{x}\,dx$$

$$\Rightarrow \int\left[\log(\log x) + \frac{1}{(\log x)^2}\right]dx = x[f(x) - g(x)] + C$$

where $f(x) = \log(\log x)$

and $g(x) = \frac{1}{\log x}$

Example 51: If $f\left(\frac{3x-4}{3x+4}\right) = x + 2$, then $\int f(x)\,dx$ is:

(a) $e^{x+2}\log\left|\frac{3x-4}{3x+4}\right| + c$

(b) $-\frac{8}{3}\log|1-x| + \frac{2}{3}x + c$

(c) $\frac{8}{3}\log|1-x| + \frac{x}{3} + c$

(d) $e^{[(3x-4)(3x+4)]} - \frac{x^2}{2} - 2x + c$

Ans. (b)

Solution: Put $x = \frac{3t-4}{3t+4} = 1 - \frac{8}{3t+4}$, $dx = \frac{24}{(3t+4)^2}\,dt$

Now
$$\int f(x)\,dx = \int f\left(\frac{3t-4}{3t+4}\right)\frac{24}{(3t+4)^2}\,dt$$

$$= \int \frac{t+2}{(3t+4)^2}(24)\,dt$$

$$= 8\int\frac{3t+4+2}{(3t+4)^2}\,dt$$

$$= 8\int\left[\frac{1}{3t+4} + \frac{2}{(3t+4)^2}\right]dt$$

$$= \frac{8}{3}\log|3t+4| - \frac{2}{3}\,\frac{8}{3t+4} + C_1$$

$$= \frac{8}{3}\log\left|\frac{8}{1-x}\right| - \frac{2}{3}(1-x) + C_1$$

$$= -\frac{8}{3}\log|1-x| + \frac{2}{3}x + C$$

Example 52: $I = \int \cos\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\} dx$ is equal to

(a) $\frac{1}{2}x^2 + C$ (b) $\frac{1}{8}\left(x^2-1\right)+C$

(c) $\frac{1}{2}x + C$ (d) $\frac{1}{4}\left(x^2+1\right)+C$

Ans. (a)

Solution: Put $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$

$$I = \int \cos\left(2\tan^{-1}\tan\theta/2\right)(-\sin\theta)d\theta$$

$$= -\int \cos\theta \sin\theta\, d\theta$$

$$= \frac{\cos^2\theta}{2} + C = \frac{x^2}{2} + C\,.$$

Example 53: Let f be thrice differential function and if $u = -f''(\theta)\sin\theta + f'(\theta)\cos\theta$ and $v = f''(\theta)\cos\theta + f'(\theta)\sin\theta$ then $I = \int\left[\left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2\right]^{1/2} d\theta$ is equal to

(a) $f(\theta) - f''(\theta) + C$

(b) $f(\theta) + f''(\theta) + C$

(c) $f'(\theta) + f''(\theta) + C$

(d) $f'(\theta) - f''(\theta) + C$

Ans. (b)

Solution: $\frac{du}{d\theta} = -f''(\theta)\cos\theta - f'''(\theta)\sin\theta + f''(\theta)\cos\theta - f'(\theta)\sin\theta$

$$= -(f'''(\theta) + f'(\theta))\sin\theta$$

$$\frac{dv}{d\theta} = f'''(\theta)\cos\theta - f''(\theta)\sin\theta + f''(\theta)\sin\theta + f'(\theta)\cos\theta$$

$$= (f'''(\theta) + f'(\theta))\cos\theta$$

$$\left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2 = (f'''(\theta) + f'(\theta))^2$$

$$I = \int (f'''(\theta) + f'(\theta))\, d\theta$$

$$= f''(\theta) + f(\theta)) + C$$

Example 54: If the integral $\int \frac{5\tan x}{\tan x - 2} dx = x + a \log|\sin x - 2\cos x| + K$, then a is equal to

(a) -2 (b) 1

(c) 2 (d) -1

Ans. (c)

Solution: $I = \int \frac{5\tan x}{\tan x - 2} dx = \int \frac{5\sin x}{\sin - 2\cos x} dx$

Let $5\sin x = A(\sin x - 2\cos x) + B(\cos x + 2\sin x)$
Equating coefficients of $\sin x$ and $\cos x$, we get

$5 = A + 2B$ and $0 = -2A + B$

$\Rightarrow A = 1, B = 2$

$$\therefore \quad I = \int \frac{(\sin x - 2\cos x) + 2(\cos x + 2\sin x)}{\sin x - 2\cos x} dx$$

$$= x + 2\log|\sin x - 2\cos x| + K$$

$\therefore \quad a = 2$

Example 55:

$$\mathbf{I} = \int \frac{\sin^{10}x - \cos^8 x\sin^2 x + \sin^8 x\cos^2 x - \cos^{10} x}{1 - 2\sin^2 x\cos^2 x} dx$$ is equal to

(a) $\frac{1}{2}\sin 2x + C$ (b) $-\frac{1}{2}\sin 2x + C$

(c) $-\frac{1}{2}\sin x + C$ (d) $-\sin^2 x + C$

Ans. (b)

Solution: $\sin^{10}x - \cos^8 x\sin^2 x + \sin^8 x\cos^2 x - \cos^{10}x$

$$= \sin^2 x(\sin^8 x - \cos^8 x) + \cos^2 x(\sin^8 x - \cos^8 x)$$

$$= (\sin^2 x + \cos^2 x)(\sin^8 x - \cos^8 x)$$

$$= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x)$$

$$= (\sin^2 x - \cos^2 x)(1 - 2\sin^2 x\cos^2 x)$$

$$= -\cos 2x(1 - 2\sin^2 x\cos^2 x)$$

$$I = -\int \cos 2x\, dx = -\frac{\sin 2x}{2} + C$$

Example 56: The integral

$$\mathbf{I} = \int \frac{\sin^2 x\cos^2 x}{\left(\sin^5 x + \cos^3 x\sin^2 x + \sin^3 x\cos^2 x + \cos^5 x\right)^2}$$ is equal to

(a) $\frac{1}{3\left(1 - \tan^2 x\right)} + C$

(b) $\frac{1}{3\left(1 + \tan^3 x\right)} + C$

(c) $\frac{1}{3\left(2 + \tan^3 x\right)} + C$

(d) $-\frac{1}{3\left(1 + \tan^3 x\right)} + C$

Ans. (d)

◎ **Solution:**

$$\frac{\sin^2 x\cos^2 x}{\left(\sin^5 x+\cos^3 x\sin^2 x+\sin^3 x\cos^2 x+\cos^5 x\right)^2}$$

$$=\frac{\sin^2 x\cos^2 x}{\left(\sin^3 x+\cos^3 x\right)^2\left(\sin^2 x+\cos^2 x\right)^2}$$

$$=\frac{\sin^2 x\cos^2 x}{\left(\sin^3 x+\cos^3 x\right)^2}$$

$$=\frac{\tan^2 x\sec^2 x}{\left(1+\tan^3 x\right)^2}$$

So $$I=\int\frac{\tan^2 x\sec^2 x}{\left(1+\tan^3 x\right)^2}dx$$

$$=\int\frac{t^2}{\left(1+t^3\right)^2}dt \qquad (t=\tan x)$$

$$=\frac{1}{3}\int\frac{du}{u^2}\left(u=1+t^3\right)$$

$$=-\frac{1}{3u}+C=-\frac{1}{3\left(1+\tan^3 x\right)}+C.$$

Assertion-Reason Type Questions

Example 57: Let $f(x)=\dfrac{xe^x}{(1+x)^2}$, $x\neq -1$

Statement-1: The antiderivative F of $f(x)$ satisfying $F(0)=1$ is $\dfrac{1}{1+x}e^x$

Statement-2: $f(x)$ is of the form $(g(x)+g'(x))e^x$, for some $g(x)$

Ans. (a)

◎ **Solution:** $$F(x)=\int\frac{xe^x}{(1+x)^2}dx$$

$$=\int\left(\frac{1}{1+x}-\frac{1}{(1+x)^2}\right)e^x\,dx$$

$$=\frac{1}{1+x}e^x+C$$

$$F(0)=1 \quad\Rightarrow\quad C=0 \text{ so } F(x)=\frac{1}{1+x}e^x.$$

Example 58: Let $f(x)=\dfrac{2-\sin x}{2+\cos x}$

Statement-1: The antiderivative of f is a periodic function of period π

Statement-2: The period of the function $g(x)=\log(2+\cos x)+A\tan^{-1}(B\tan x/2)+C$, A, B, C being constants is 2π.

Ans. (d)

◎ **Solution:** $F(x)=2I_1+I_2$, $I_1=\displaystyle\int\frac{dx}{2+\cos x}$ and

$$I_2=-\int\frac{\sin x}{2+\cos x}dx$$

$= \log(2+\cos x)+$ constant.

$$I_1=\int\frac{2dt}{\left(1+t^2\right)\left(2+\dfrac{1-t^2}{1+t^2}\right)}=2\int\frac{dt}{3+t^2}$$

$(t=\tan(x/2))$

$$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan(x/2)\right)+C$$

Hence $F(x)=\log(2+\cos x)+\dfrac{4}{\sqrt{3}}\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\tan(x/2)\right)+C$ which is periodic with period 2π.

Example 59: Let $f(x)=\dfrac{1}{1-\sin^4 x}$ and F be an antiderivative of f.

Statement-1: $F(x)=\dfrac{1}{2}\tan x+\dfrac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)+C$

Statement-2: F is a one-one function of $\tan x$

Ans. (b)

◎ **Solution:** $$f(x)=\frac{1}{\cos^2 x\left(1+\sin^2 x\right)}=\frac{\sec^2 x\left(1+\tan^2 x\right)}{1+2\tan^2 x}$$

So, $$F(x)=\int\frac{1+t^2}{1+2t^2}dt\ (t=\tan x)$$

$$=\int\left(\frac{1}{2}+\frac{1}{4}\frac{1}{t^2+1/2}\right)dt$$

$$= \frac{1}{2}t + \frac{1}{4}\sqrt{2}\tan^{-1}\left(\sqrt{2}\tan x\right) + C$$

$$= \frac{1}{2}\tan x + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right) + C$$

which is a 1 – 1 function of tan x.

Example 60: Let $f(x) = \dfrac{1}{x^6 + x^4}$ and F be a antiderivative of f such that $F(1) = \dfrac{\pi}{4} + \dfrac{2}{3}$

Statement-1: $F\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

Statement-2: $F(x) = \tan^{-1} x + \dfrac{1}{3x} - \dfrac{1}{x^3}$

Ans. (c)

Solution: $f(x) = \dfrac{1}{x^4\left(x^2+1\right)} = \dfrac{1}{x^4} - \dfrac{1}{x^2} + \dfrac{1}{x^2+1}$

So $\quad F(x) = -\dfrac{1}{3x^3} + \dfrac{1}{x} + \tan^{-1} x + C$

$$F(1) = \frac{2}{3} + \frac{\pi}{4} + C \text{ so } C = 0$$

$$F(x) = \tan^{-1} x + \frac{1}{x} - \frac{1}{3x^3} \quad \Rightarrow \quad F\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Example 61: Let $F(x) = \int e^{-x^2} x^5 \, dx$

Statement-1: If $F(0) = 0$, then $F(1) = 1 - \dfrac{5}{2}e^{-1}$

Statement-2: F increases on $(0, \infty)$

Ans. (b)

Solution: Put $x^2 = t$, $\int e^{-x^2} x^5 \, dx = \dfrac{1}{2}\int e^{-t} t^2 \, dt$

$$= C - \frac{1}{2}e^{-x^2}\ (x^4 + 2x^2 + 2)$$

$$F(0) = 0 \quad \Rightarrow \quad C = 1$$

Hence $F(x) = 1 - \dfrac{1}{2}e^{-x^2}\ (x^4 + 2x^2 + 2)$

$$F'(x) = e^{-x^2}\ x^5 > 0\ x \in (0, \infty).$$

Example 62: Statement 1: $\int e^x \sin x \, dx = \dfrac{e^x}{2}(\sin x - \cos x) + C$

Statement 2: If the integrand is of the form $e^x(f(x) + f'(x))$ then the integral in $e^x f(x)$

Ans. (b)

Solution: Let $I = \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$
$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx + C$

$$\Rightarrow \quad 2I = e^x (\sin x - \cos x) + C$$

$$\Rightarrow \quad I = \frac{e^x}{2}\ (\sin x - \cos x) + C$$

Statement 2: is also true but is not correct explanation of statement 1.

Example 63: Statement 1: For $-1 < a < 4$,
$\displaystyle\int \frac{dx}{x^2 + 2(a-1)x + a + 5} = \lambda \log |g(x)| + C$, where λ and C are constants

Statement 2: For $-1 < a < 4$, $f(x) = \dfrac{1}{x^2 + 2(a-1)x + a + 5}$ is a continuous function

Ans. (d)

Solution: The discriminant of $x^2 + 2(a-1)x + a + 5 = 0$ is $4(a-1)^2 - 4(a+5) = 4(a^2 - 3a - 4) < 0$. Hence the integral
$\displaystyle\int \frac{dx}{x^2 + 2(a-1)x + a + 5} = \lambda \tan^{-1} g(x) + C$

Since the denominator is not zero for any value of x so f is a continuous function.

Example 64: Statement 1: $I =$
$\displaystyle\int \frac{f(x)\,g'(x) - f'(x)\,g(x)}{f(x)\,g(x)}\ \{\log g(x) - \log f(x)\}\ dx$

$$= \frac{1}{2}\left\{\log \frac{g(x)}{f(x)}\right\}^2 + C$$

Statement 2: $\int (\phi(x))^n\, \phi'(x) dx = \dfrac{(\phi(x))^{n+1}}{n+1} + C$

Ans. (a)

Solution: $I = \displaystyle\int \log\left(\frac{g(x)}{f(x)}\right) \frac{d}{dx}\left(\log \frac{g(x)}{f(x)}\right) dx$

$$= \frac{1}{2}\left(\log \frac{g(x)}{f(x)}\right)^2 + C$$

Put $\phi(x) = t$ in integral of Statement 2.

LEVEL 2

Straight Objective Type Questions

Example 65: If $\int \frac{dx}{x^3\,(x-1)^{1/2}} = \frac{\sqrt{x-1}\,(3x+2)}{4x^2} +$ $K\tan^{-1}\sqrt{x-1} + C$ then the value of K is

(a) 1/2 (b) 1

(c) 1/4 (d) 3/4

Ans. (d)

Solution: Put $x - 1 = t^2$ so that $dx = 2t\,dt$, and

$$\int \frac{dx}{x^3\,(x-1)^{1/2}} = \int \frac{2t\,dt}{\left(t^2+1\right)^3 t} = 2\int \frac{dt}{\left(t^2+1\right)^3}$$

Now we use formula of $\int \frac{dy}{(y^2+k^2)^n}$ to write

$$\int \frac{dt}{\left(t^2+1\right)^3} = \frac{t}{4\left(t^2+1\right)^2} + \frac{3}{4}\int \frac{dt}{\left(t^2+1\right)^2} \text{ and}$$

$$\int \frac{dt}{\left(t^2+1\right)^2} = \frac{t}{2\left(t^2+1\right)} + \frac{1}{2}\int \frac{dt}{t^2+1}$$

$$= \frac{t}{2\left(t^2+1\right)} + \frac{1}{2}\tan^{-1} t$$

Hence $\int \frac{dx}{x^3\,(x-1)^{1/2}}$

$$= \frac{t}{2\left(t^2+1\right)^2} + \frac{3t}{4\left(t^2+1\right)} + \frac{3}{4}\tan^{-1} t + C$$

$$= \frac{1}{4\left(t^2+1\right)^2}\,(2t + 3t^3 + 3t) + \frac{3}{4}\tan^{-1} t + C$$

$$= \frac{\sqrt{x-1}\,(3x+2)}{4x^2} + \frac{3}{4}\tan^{-1}\sqrt{x-1} + C.$$

Example 66: The value of $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$ is

(a) $3\sqrt{x} + 3(\sqrt[3]{x}) - 6(\sqrt[6]{x}) + 6\log\left(\sqrt[6]{x} + 1\right) + C$

(b) $2\sqrt{x} + 6\sqrt[6]{x} - 6\log\left(\sqrt[6]{x} + 1\right) + C$

(c) $2\sqrt{x} - 3(\sqrt[3]{x}) + 6(\sqrt[6]{x}) - 6\log\left(\sqrt[6]{x} + 1\right) + C$

(d) none of these

Ans. (c)

Solution: We want a substitution that will allow us to find both the square root and the cube root without getting fractional exponents. Thus we want a substitution of the form $x = u^k$ where k is a multiple of 2 and 3. Let us use the least common multiple 6.

Let $x = u^6,\ dx = 6u^5\,du$

$$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3 + u^2} = 6\int \frac{u^3}{u+1}\,du$$

$$= 6\int\left(u^2 - u + 1 - \frac{1}{u+1}\right)du$$

$$= 2u^3 - 3u^2 + 6u - 6\log(u+1) + C$$

$$= 2\sqrt{x} - 3(\sqrt[3]{x}) + 6\,(\sqrt[6]{x}) - 6\log\left(\sqrt[6]{x} + 1\right) + C$$

Example 67: The value of

$\int \frac{x^5 + x^4 + 4x^3 + 4x^2 + 4x + 4}{\left(x^2+2\right)^5}$ is equal to

(a) $\frac{4x-3}{\left(x^2+1\right)^2} + \frac{3}{8}\frac{x}{x^2+2} + \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$

(b) $\frac{1}{12}\frac{2x-3}{\left(x^2+2\right)^2} + \frac{3}{16}\frac{x}{x^2+2} + \frac{3}{16\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$

(c) $\frac{2x-3}{\left(x^2+2\right)^2} + \frac{3}{8}\frac{x}{x^2+2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$

(d) none of these

Ans. (d)

Solution: $x^5 + x^4 + 4x^3 + 4x^2 + 4x + 4$

$$= x^3(x^2 + 2) + x^2(x^2 + 2) + 2x(x^2 + 2) + 2(x^2 + 2)$$

$$= (x^2 + 2)\,(x^3 + x^2 + 2x + 2)$$

$$= (x^2 + 2)\,(x(x^2 + 2) + (x^2 + 2))$$

$$= (x^2 + 2)^2\,(x + 2). \text{ Therefore,}$$

$$f(x) = \frac{x^5 + x^4 + 4x^3 + 4x^2 + 4x + 4}{\left(x^2+2\right)^5}$$

$$= \frac{x+2}{\left(x^2+2\right)^3} = \frac{1}{2}\frac{2x}{\left(x^2+2\right)^3} + \frac{2}{\left(x^2+2\right)^3}$$

Now $\int \frac{\mathrm{d}x}{\left(x^2+2\right)^3} = \frac{1}{8}\frac{x}{\left(x^2+2\right)^2} + \frac{3}{8}\int \frac{\mathrm{d}x}{\left(x^2+2\right)^2}$

$$\left(\text{using} \int \frac{\mathrm{d}y}{(y^2+k^2)^n}\right)$$

$$\int \frac{\mathrm{d}x}{\left(x^2+2\right)^2} = \frac{1}{4}\frac{x}{x^2+2} + \frac{1}{4}\frac{1}{\sqrt{2}} = \tan^{-1}\frac{x}{\sqrt{2}} + C$$

$$\therefore \quad \int f(x)\,dx = -\frac{1}{4\left(x^2+2\right)^2} + \frac{1}{4}\frac{x}{\left(x^2+2\right)^2}$$

$$+\frac{3}{4}\left[\frac{1}{4}\frac{x}{x^2+2} + \frac{1}{4\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}\right] + C$$

$$= \frac{x-1}{4\left(x^2+2\right)^2} + \frac{3}{16}\frac{x}{x^2+2} + \frac{3}{16\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

Example 68: If $\int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}\,\mathrm{d}x =$

$\sqrt{1+x^2}\; f(x) + K\log\left(x+\sqrt{x^2+1}\right) + C$ then

(a) $f(x) = \tan^{-1}x$, $K = -1$
(b) $f(x) = \tan^{-1}x$, $K = 1$
(c) $f(x) = 2\tan^{-1}x$, $K = -1$
(d) $f(x) = 2\tan^{-1}x$, $K = 1$

Ans. (a)

Solution: $\int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}x$

$$= \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} = \frac{1}{2}\cdot\frac{\sqrt{t}}{1/2} = \sqrt{t} = \sqrt{1+x^2}$$

[where $1 + x^2 = t$]

We integrate by parts, taking $\tan^{-1}x$ as the first function and $x/\sqrt{1+x^2}$ as the second.

$$\Rightarrow \quad \int \frac{x\tan^{-1}x}{\sqrt{1+x^2}}\,\mathrm{d}x$$

$$= (\tan^{-1}x)\,\sqrt{1+x^2} - \int \frac{1}{1+x^2}\cdot\sqrt{1+x^2}\,\mathrm{d}x$$

$$= \sqrt{1+x^2}\,\tan^{-1}x - \int \frac{\mathrm{d}x}{\sqrt{1+x^2}}$$

$$= \sqrt{1+x^2}\,\tan^{-1}x - \log\left(x+\sqrt{x^2+1}\right) + C$$

Thus $f(x) = \tan^{-1}x$, $K = -1$.

Example 69: The value of $\int \frac{\sin x}{\sin 4x}\,\mathrm{d}x$ is

(a) $\frac{1}{4}\log\left|\frac{\sin x - 1}{\sin x + 1}\right| - \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}\sin x - 1}{\sqrt{2}\sin x + 1}\right| + C$

(b) $\frac{1}{8}\log\left|\frac{\cos x - 1}{\cos x + 1}\right| - \frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}\right| + C$

(c) $\frac{1}{8}\log\left|\frac{\sin x - 1}{\sin x + 1}\right| - \frac{1}{4\sqrt{2}}\log\left|\frac{\sqrt{2}\sin x - 1}{\sqrt{2}\sin x + 1}\right| + C$

(d) none of these

Ans. (c)

Solution: The given integrand can be written as

$$\frac{1}{4\cos x\cos 2x} = \frac{\cos x}{4\left(1-\sin^2 x\right)\left(1-2\sin^2 x\right)}, \text{ hence}$$

$$\int \frac{\sin x}{\sin 4x}\,\mathrm{d}x = \frac{1}{4}\int \frac{\mathrm{d}t}{\left(1-t^2\right)\left(1-2t^2\right)} \qquad (t = \sin x)$$

$$= \frac{1}{4}\int\left[\frac{1}{t^2-1} - \frac{1}{t^2-1/2}\right]\mathrm{d}t$$

$$= \frac{1}{4}\left[\frac{1}{2}\log\left|\frac{t-1}{t+1}\right| - \frac{1}{\sqrt{2}}\log\left|\frac{t-1/\sqrt{2}}{t+1/\sqrt{2}}\right|\right] + C$$

$$= \frac{1}{8}\log\left|\frac{\sin x-1}{\sin x+1}\right| - \frac{1}{4\sqrt{2}}\log\left|\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1}\right| + C.$$

Example 70: If $\int \frac{\mathrm{d}x}{(x+1)\sqrt{2x^2+3x+4}}$ $= K\log f(x) + C$ then

(a) $K = -\frac{1}{\sqrt{3}}, f(x)$
$= \frac{5-x}{\sqrt{23}\,(x+1)} + \sqrt{\left(\frac{5-x}{\sqrt{23}\,(x+1)}\right)^2 + \frac{23}{26}}$

(b) $K = \frac{1}{\sqrt{3}}\;\; f(x) = \frac{4-x}{(x+1)} + \sqrt{\left(\frac{4-x}{(x+1)}\right)^2 + \frac{23}{16}}$

(c) $K = 2, f(x) = y + \sqrt{y^2+1}, y = \frac{5-x}{x+1}$

(d) none of these

Ans. (d)

Solution: Putting $x + 1 = 1/y$, the given integral can be written as $-\int \frac{\mathrm{d}y}{\sqrt{3y^2-y+2}}$

$$= -\frac{1}{\sqrt{3}}\int \frac{\mathrm{d}y}{\sqrt{(y-1/6)^2+\left(\sqrt{23}/6\right)^2}}$$

$$= -\frac{1}{\sqrt{3}}\log\left|(y-1/6)+\sqrt{(y-1/6)^2+\left(\sqrt{23/6}\right)^2}\right| + C$$

$$= -\frac{1}{\sqrt{3}}\log\left|\frac{5-x}{6\,(x+1)}+\sqrt{\left(\frac{5-x}{6\,(x+1)}\right)^2+\left(\frac{\sqrt{23}}{6}\right)^2}\right| + C.$$

Example 71: If $\int \frac{\mathrm{d}x}{(1+x^2)\sqrt{1-x^2}} = F(x)$ and $F(1) = 0$, then for $x > 0$, $F(x)$ is equal to

(a) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1+x^2}}\right)+\frac{\pi}{\sqrt{2}}$

(b) $\frac{1}{2}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1+x^2}}\right)-\frac{\pi}{2\sqrt{2}}$

(c) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1-x^2}}\right)+\frac{\pi}{2\sqrt{2}}$

(d) none of these

Ans. (b)

Solution: Putting $x = \frac{1}{t}$ so that $\mathrm{d}x = -\frac{1}{t^2}\,\mathrm{d}t$

We have

$$\int \frac{\mathrm{d}x}{(1+x^2)\sqrt{1-x^2}} = \int \frac{-t\,\mathrm{d}t}{(1+t^2)\sqrt{t^2-1}}$$

$$= -\int \frac{\mathrm{d}x}{2+u^2}\;(t^2-1=u^2)$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\frac{u}{2}+C$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\frac{\sqrt{1-x^2}}{x}\right)+C$$

Since $F(1) = 0$, so $C = 0$. Hence

$$F(x) = -\frac{1}{\sqrt{2}}\left[\frac{\pi}{2}-\cot^{-1}\left(\frac{1}{\sqrt{2}}\frac{\sqrt{1-x^2}}{x}\right)\right]$$

$$[\tan^{-1}\theta+\cot^{-1}\theta=\pi/2]$$

$$= -\frac{\pi}{2\sqrt{2}}+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1-x^2}}\right)$$

$$\left(\begin{array}{l}\cot^{-1}\theta=\tan^{-1}\frac{1}{\theta},\text{for }\theta>0\text{ and}\\ \cot^{-1}\theta=\tan^{-1}\frac{1}{\theta}+\pi,\text{ for }\theta<0\end{array}\right)$$

Example 72: Let $f(x)$ be a polynomial of degree three such that $f(0) = 1$, $f(1) = 2$ and 0 is a critical point of $f(x)$ such that $f(x)$ does not have a local extremum at 0. Then $\int \frac{f(x)}{x^2+1}\mathrm{d}x$ is equal to

(a) $x - \log(x^2+1) + \tan^{-1}x + C$

(b) $x + (1/2)\log(x^2+1) - \tan^{-1}x + C$

(c) $(1/2)x^2 + (1/2)\log(x^2+1) - \tan^{-1}x + C$

(d) $(1/2)(x^2 - \log(x^2+1)) + \tan^{-1}x + C$

Ans. (d)

Solution: Let $f(x) = ax^3 + bx^2 + cx + d$. Since $f(0) = 1$, so $d = 1$. Moreover, $f(1) = 2$ implies that $a + b + c = 1$ since 0 is a critical point, $0 = f'(0) = 3a\,.0 + 2b.0 + C$ i.e., $C = 0$. Also $f''(x) = 6ax + 2b$ and since $f(x)$ does not have extremum at 0 so $0 = f''(0) = b$. Hence $a = 1$ and

$$\int \frac{f(x)}{x^2+1}\mathrm{d}x = \int \frac{x^3+1}{x^2+1}\mathrm{d}x = \int\left(x-\frac{1}{2}\frac{2x}{x^2+1}+\frac{1}{x^2+1}\right)\mathrm{d}x$$

$$= \frac{1}{2}\,[x^2-\log(x^2+1)]+\tan^{-1}x+C.$$

Example 73: If $\int \frac{\log\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\mathrm{d}x = g\ o\ f\ (x) +$ Const. then

(a) $f(x) = \log\left(x+\sqrt{x^2+1}\right)$

(b) $f(x) = \log\left(x+\sqrt{x^2+1}\right)$ and $g(x) = x^2$

(c) $f(x) = \log\left(x+\sqrt{x^2+1}\right)$ and $g(x) = x^2/2$

(d) $f(x) = x^2/2$ and $g(x) = \log\left(x+\sqrt{x^2+1}\right)$

Ans. (c)

Solution: Putting $\log\left(x+\sqrt{1+x^2}\right) = t$, we have

$$\frac{1}{x+\sqrt{1+x^2}}\times\left(1+\frac{x}{\sqrt{1+x^2}}\right)\mathrm{d}x=\mathrm{d}t\ \text{ i.e. }\ \frac{\mathrm{d}x}{\sqrt{1+x^2}}=\mathrm{d}t$$

So $$\int \frac{\log\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\mathrm{d}x = \int t\,\mathrm{d}t=\frac{1}{2}t^2+C$$

$$=\frac{1}{2}\left(\log\,(x+\sqrt{x^2+1})\right)^2+C$$

Thus $f(x) = \left(\log\,(x+\sqrt{x^2+1})\right)$ and $g(x) = x^2/2$.

Example 74: The value of $\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}\mathrm{d}x$ is

(a) $\sin x - 6\tan^{-1}(\sin x) + C$

(b) $\sin x - 2(\sin x)^{-1} + C$

(c) $\sin x - 2(\sin x)^{-1} + 5\tan^{-1}(\sin x) + C$
(d) none of these

Ans. (d)

Solution: Since the integrand satisfies.

$R(\sin x, -\cos x) = -R(\sin x, \cos x)$

Therefore, we substitute $\sin x = t$, so that $\cos x\, dx = dt$.

$$\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$$

$$= \int \frac{\cos x\left[(1-\sin^2 x) + (1-\sin^2 x)^2\right]}{\sin^2 x + \sin^4 x} dx$$

$$= \int \frac{(1-t^2)(2-t^2)}{t^2 + t^4} dt$$

$$= \int \left(1 + \frac{2}{t^2} - \frac{6}{1+t^2}\right) dt$$

$$= t - \frac{2}{t} - 6\tan^{-1} t + C$$

$$= \sin x - 2(\sin x)^{-1} - 6\tan^{-1}(\sin x) + C.$$

Example 75: If $f(x) = \dfrac{x+2}{2x+3}$. Then $\displaystyle\int \left(\frac{f(x)}{x^2}\right)^{1/2} dx$ is equal to $\dfrac{1}{\sqrt{2}}\, g\left(\dfrac{1+\sqrt{2f(x)}}{1-\sqrt{2f(x)}}\right) - \sqrt{\dfrac{2}{3}}\, h\left(\dfrac{\sqrt{3f(x)}+\sqrt{2}}{\sqrt{3f(x)}-\sqrt{2}}\right)$ $+ C$ where

(a) $g(x) = \tan^{-1} x$, $h(x) = \log|x|$
(b) $g(x) = \log|x|$, $h(x) = \tan^{-1} x$
(c) $g(x) = h(x) = \tan^{-1} x$
(d) $g(x) = \log|x|$, $h(x) = \log|x|$

Ans. (d)

Solution: Putting $y^2 = f(x) = \dfrac{x+2}{2x+3}$, we have

$$x = \frac{3y^2-2}{1-2y^2} \text{ and } dx = -\frac{2y}{(1-2y^2)^2} dy.$$

So $$\int (f(x))^{1/2} \frac{dx}{x}$$

$$= -\int y \cdot \frac{2y}{(1-2y^2)^2} \cdot \frac{1-2y^2}{3y^2-2} dy$$

$$= 2\int \frac{y^2}{(2y^2-1)(3y^2-2)} dy$$

$$= -2\int \left[\frac{1}{2y^2-1} - \frac{2}{3y^2-2}\right] dy$$

$$= \frac{1}{\sqrt{2}} \log\left|\frac{1+\sqrt{2}y}{1-\sqrt{2}y}\right| - \sqrt{\frac{2}{3}} \log\left|\frac{\sqrt{3}y+\sqrt{2}}{\sqrt{3}y-\sqrt{2}}\right| + C$$

Thus $g(x) = \log|x|$ and $h(x) = \log|x|$.

Example 76: The value of $\displaystyle\int \frac{\sec x\, dx}{\sqrt{\sin(2x+\theta) + \sin\theta}}$ is

(a) $\sqrt{(\tan x + \tan\theta)\sec\theta} + C$
(b) $\sqrt{2(\tan x + \tan\theta)\sec\theta} + C$
(c) $\sqrt{2(\sin x + \tan\theta)\sec\theta} + C$
(d) none of these

Ans. (b)

Solution: The given integral can be written as

$$\int \frac{\sec x\, dx}{\sqrt{2\sin(x+\theta)\cos x}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\sec^{3/2} x\, dx}{\sqrt{\sin x\cos\theta + \sin\theta\cos x}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\sec^2 x\, dx}{\sqrt{\tan x\cos\theta + \sin\theta}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{t\cos\theta + \sin\theta}} \qquad (t = \tan x)$$

$$= \frac{2}{\sqrt{2}} \frac{\sqrt{t\cos\theta + \sin\theta}}{\cos\theta} + C$$

$$= \sqrt{2(\tan x\sec\theta + \tan\theta\sec\theta)} + C$$

Example 77: If $\displaystyle\int \frac{e^{4x}-1}{e^{2x}} \log\left(\frac{e^{2x}+1}{e^{2x}-1}\right) dx$

$$= \frac{t^2}{2}\log t - \frac{t^2}{4} + \frac{u^2}{2}\log u - \frac{u^2}{4} + C \text{ then}$$

(a) $t = e^{-x} - e^x$ $u = e^x + e^{-x}$
(b) $t = e^x - e^{-x}$, $u = e^x + e^{-x}$
(c) $t = e^x + e^{-x}$, $u = e^x - e^{-x}$
(d) none of these

Ans. (c)

Solution: The integrand can be written as

$$(e^{2x} - e^{-2x})\log(e^x + e^{-x}) - (e^{2x} - e^{-2x})\log(e^x - e^{-x})$$

$$= (e^x + e^{-x})(e^x - e^{-x})\log(e^x + e^{-x})$$

$$- (e^x + e^{-x})(e^x - e^{-x})\log(e^x - e^{-x}).$$

Hence the given integral is equal to

$$\int t\log t\, dt - \int u\log u\, du$$

$(t = e^x + e^{-x}, u = e^x - e^{-x})$

$$= \frac{t^2}{2}\log t - \frac{1}{2}\int t\, dt + \frac{u^2}{2}\log u - \frac{1}{2}\int u\, du$$

(integration by parts)

$$= \frac{t^2}{2}\ \log t - \frac{t^2}{4} + \frac{u^2}{2}\ \log u - \frac{u^2}{4} + C,$$

where $t = e^x + e^{-x}$ and $u = e^x - e^{-x}$.

◉ **Example 78:** If the primitive of $\sin^{-3/2} x \sin^{-1/2} (x + \theta)$ is $-2 \operatorname{cosec} \theta \sqrt{f(x)} + C$ then

(a) $f(x) = \dfrac{\sin x}{\sin (x+\theta)}$

(b) $f(x) = \tan (x + \theta)$

(c) $f(x) = \dfrac{\sin (x+\theta)}{\sin x}$

(d) $f(x) = \dfrac{\tan (x+\theta)}{\tan x}$

Ans. (c)

◎ **Solution:** The primitive of the given function is

$$\int \frac{dx}{\sqrt{\sin^3 x \sin (x+\theta)}}$$

$$= \int \frac{dx}{\sqrt{\sin^3 x (\sin x \cos \theta + \cos x \sin \theta)}}$$

$$= \int \frac{\operatorname{cosec}^2 x\, dx}{\sqrt{\cos \theta + \cot x \sin \theta}}$$

$$= -\frac{1}{\sqrt{\sin \theta}} \int \frac{dt}{\sqrt{\cot \theta + t}} \quad (t = \cot x)$$

$$= -\frac{2}{\sqrt{\sin \theta}}\ (\cot \theta + t)^{1/2} + C$$

$$= -\frac{2}{\sin \theta} (\cos \theta + \sin \theta\, t)^{1/2} + C$$

$$= \frac{-2 \operatorname{cosec} \theta\, (\sin x \cos \theta + \sin \theta \cos x)^{1/2}}{\sqrt{\sin x}} + C$$

$$= -2 \operatorname{cosec} \theta \left(\frac{\sin (\theta + x)}{\sin x}\right)^{1/2} + C.$$

◉ **Example 79:** If the primitive of $\dfrac{\sin x}{\sqrt{1 + \sin x}}$ is $-2\sqrt{f(x)} + \sqrt{2}\ \log |\tan g(x)| + C$ then

(a) $f(x) = 1 + \sin x$

(b) $g(x) = (3\pi/8) - (x/4)$

(c) $f(x) = 2(1 - \sin x)$

(d) none of these

Ans. (b)

◎ **Solution:** $\displaystyle\int \frac{\sin x}{\sqrt{1+\sin x}}\, dx$

$$= -\int \frac{\cos u}{\sqrt{1 + \cos u}}\, du \left(x = \left(\frac{\pi}{2}\right) - u\right)$$

$$= -\int \frac{2\cos^2 (u/2) - 1}{\sqrt{2} \cos (u/2)}\, du$$

$$= -\sqrt{2} \int \cos (u/2)\, du + \frac{1}{\sqrt{2}} \int \sec (u/2)\, du$$

$$= -2\sqrt{2} \sin (u/2)$$
$$+ \sqrt{2} \log |\tan (\pi/4) + (u/4)| + C.$$

$$= -2\sqrt{2} \sin \left(\frac{\pi}{4} - \frac{x}{2}\right)$$
$$+ \sqrt{2} \log \left|\tan \left(\frac{3\pi}{8} - \frac{x}{4}\right)\right| + C.$$

◉ **Example 80:** If $\displaystyle\int \frac{(x^2 - 1)\, dx}{(x^4 + 3x^2 + 1) \tan^{-1} \frac{x^2+1}{x}} = \log |\tan^{-1} f(x)| + C$ then

(a) $f(x) = x^2 + 1$ (b) $f(x) = \dfrac{x^2+1}{2x}$

(c) $f(x) = \dfrac{1}{2}\ (x^2 + 1)$ (d) $f(x) = \dfrac{x^2+1}{x}$

Ans. (d)

◎ **Solution:** Dividing the numerator and denominator by x^2, the given integral transforms into

$$\int \frac{(1 - 1/x^2)\, dx}{((x + 1/x)^2 + 1) \tan^{-1} (x + 1/x)}$$

$$= \int \frac{dt}{(t^2 + 1) \tan^{-1} t} \qquad (t = (x + 1/x))$$

$$= \int \frac{du}{u}, \qquad (u = \tan^{-1} t)$$

$$= \log \left|\tan^{-1} \frac{x^2+1}{x}\right| + C.$$

Thus $f(x) = \dfrac{x^2+1}{x}$.

◉ **Example 81:** If $f(x) = \displaystyle\lim_{n \to \infty} \frac{x^n - x^{-n}}{x^n + x^{-n}}$, $0 < x < 1$, $n \in N$ then

$\int (\sin^{-1} x) f(x)\, dx$ is equal to

(a) $-[x \sin^{-1} x + \sqrt{1 - x^2}\ ] + C$

(b) $x \sin^{-1} x + \sqrt{1 - x^2} + C$

(c) constant

(d) none of these

Ans. (a)

Solution: $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n}-1}{x^{2n}+1} = -1 \ (0 < x < 1)$, so

$$\int \sin^{-1} x\,(f(x))\,\mathrm{d}x = -\int \sin^{-1} x\,\mathrm{d}x$$
$$= -[x\sin^{-1} x + \sqrt{1-x^2}\,] + C$$

Example 82: If $f(x) = \lim\limits_{n\to\infty} [2x + 4x^3 + \ldots + 2nx^{2n-1}]$ $(0 < x < 1)$ then $\int f(x)\,\mathrm{d}x$ is equal to

(a) $-\sqrt{1-x^2}$ (b) $\dfrac{1}{\sqrt{1-x^2}}$

(c) $\dfrac{1}{x^2-1}$ (d) $\dfrac{1}{1-x^2}$

Ans. (d)

Solution: Let $g_n(x) = 1 + x^2 + x^4 + \ldots + x^{2n} = \dfrac{x^{2n+2}-1}{x^2-1}$

so $\quad h_n(x) = g'_n(x) = \dfrac{2x\left(nx^{2n+2}-(n+1)x^{2n}+1\right)}{\left(x^2-1\right)^2}$

Now $\ f(x) = \lim\limits_{n\to\infty} h_n(x) = \dfrac{2x}{\left(x^2-1\right)^2}$ as $0 < x < 1$.

Thus $\ \int f(x)\,\mathrm{d}x = \int \dfrac{2x}{\left(x^2-1\right)^2}\,\mathrm{d}x = -\dfrac{1}{x^2-1} = \dfrac{1}{1-x^2}$.

Example 83: If $f(x) = \lim\limits_{n\to\infty} e^{x\tan(1/n)\log(1/n)}$, and $\displaystyle\int \frac{f(x)}{\sqrt[3]{\sin^{11} x\cos x}}\,\mathrm{d}x = g(x) + C$ (C being the constant of integration), then

(a) $g(\pi/4) = 3/2$

(b) $g(x)$ is continuous for all $x \in \mathbf{R}$

(c) $g(\pi/4) = -15/8$

(d) $g(\pi/4) = 1/2$

Ans. (c)

Solution: $f(x) = \lim\limits_{n\to\infty} e^{x\tan(1/n)\cdot\log(1/n)}$.

But $\lim\limits_{n\to\infty} \tan(1/n)\log(1/n) = \lim\limits_{n\to\infty} -\dfrac{\log n}{n}\,\dfrac{\tan(1/n)}{1/n} = 0$

So $\quad f(x) = e^0 = 1$. Hence

$$\int \frac{f(x)}{\sqrt[3]{\sin^{11} x\cos x}}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{\sqrt[3]{\sin^{11} x\cos x}}$$

$$= \int \frac{1+t^2}{t^{11/3}}\,\mathrm{d}t \ (t = \tan x)$$

$$= \int (t^{-11/3} + t^{-5/3})\,\mathrm{d}t$$

$$= -\frac{3}{8}\,t^{-8/3} - \frac{3}{2}\,t^{-2/3} + C$$

$$= -\frac{3}{8}\,\frac{\left(1+4\tan^2 x\right)}{\tan^2 x\sqrt[3]{\tan^2 x}} + C$$

Thus $\quad g(x) = -\dfrac{3}{8}\,\dfrac{\left(1+4\tan^2 x\right)}{\tan^2 x\sqrt[3]{\tan^2 x}}$, $g(\pi/4) = -\dfrac{15}{8}$.

Clearly g is not defined at $x = 0$.

Example 84: If $f(x) = \lim\limits_{n\to\infty} n^2\left(x^{1/n} - x^{1/(n+1)}\right)$, $x > 0$ then $\int x f(x)\,\mathrm{d}x$ is equal to

(a) $x^2/2$ (b) 0

(c) $x^2\log x - \dfrac{1}{2}x^2 + C$ (d) none of these

Ans. (d)

Solution: $f(x) = \lim\limits_{n\to\infty} n^2 x^{1/(n+1)}\left[x^{\frac{1}{n}-\frac{1}{n+1}} - 1\right]$

$$= \lim_{n\to\infty} \frac{x^{1(n+1)}\left(x^{\frac{1}{n(n+1)}} - 1\right)}{\dfrac{1}{n(n+1)} \times \dfrac{n(n+1)}{n^2}} = \log x.$$

Hence $\ \int x f(x)\,\mathrm{d}x = \int x\log x\,\mathrm{d}x = \dfrac{x^2}{2}\log x - \dfrac{1}{4}x^2 + C.$

Example 85: Let $f(x) = \int e^x(x-1)(x-2)\,\mathrm{d}x$. Then f decreases in the interval

(a) $(-\infty, -2)$ (b) $(-2, -1)$

(c) $(1, 2)$ (d) $(2, \infty)$

Ans. (c)

Solution: We have $f'(x) = e^x(x-1)(x-2)$. Since $e^x > 0$ for $x \in \mathbf{R}$ so $f'(x) < 0$ if and only if $(x-1)(x-2) < 0$ i.e. if $1 < x < 2$.

Example 86: If $\displaystyle\int \frac{\sqrt{1+\sqrt[3]{x}}}{x^{2/3}}\,\mathrm{d}x = 2f(x)^{3/2} + C$ then $f(x)$ is equal to

(a) $1 + x^{2/3}$ (b) $1 + x^{1/3}$

(c) $1 - x^{1/3}$ (d) $1 - x^{2/3}$

Ans. (b)

◎ **Solution:** $I = \int x^{-2/3}(1 + x^{1/3})^{1/2}\,dx$.

Here $m = -2/3$, $n = 1/3$, $p = 1/2$; $\dfrac{m+1}{n} = \dfrac{(-2/3+1)}{1/3} = 1$ (see theorem). Put $1 + x^{1/3} = t^2$, $(1/3)x^{-2/3}dx = 2t\,dt$.

Hence $\quad I = 6\int t^2 dt = 2t^3 + C = 2(1+x^{1/3})^{3/2} + C$.

Example 87: Let *f* and *g* be two polynomials then $\int (f(x)\,g''(x) - f''(x)\,g(x))\,dx$ is equal to (ignoring the constant of integration)

(a) $\dfrac{f(x)}{g'(x)}$

(b) $f'(x)\,g(x) - f(x)\,g'(x)$

(c) $f(x)\,g'(x) - f'(x)\,g(x)$

(d) $f'(x)\,g'(x) + f'(x)\,g(x)$

Ans. (c)

◎ **Solution:** $\int (f(x)g''(x) - f''(x)g(x))\,dx$

$$= f(x)\,g'(x) - \int f'(x)\,g'(x)\,dx - [g(x)\,f(x) - \int f'(x)g'(x)\,dx]$$

$$= f(x)\,g'(x) - g(x)\,f'(x)$$

EXERCISE

Concept-based Straight Objective Type Questions

1. If $\int(1 - \tan 3x)^2\,dx = \dfrac{1}{3}[\tan 3x + \log f(x)] + C$ then $f(x)$ is given by

(a) $\cos 3x$ (b) $\cos^2 3x$

(c) $\sin^2 3x$ (d) $\cos^3 3x$

2. If $\int \dfrac{dx}{x^4 - x^2} = \dfrac{1}{x} + \log|f(x)| + C$ then $f(x)$ is given by

(a) $\dfrac{x+1}{x-1}$ (b) $\dfrac{x-1}{x+1}$

(c) $\left(\dfrac{x-1}{x+1}\right)^{1/2}$ (d) $\left(\dfrac{x-1}{x+1}\right)^2$

3. $\int \dfrac{x^3 - 1}{4x^3 - x}$ is equal to

(a) $\dfrac{1}{4}x + \log|x| - \dfrac{3}{16}\log|2x-1| - \dfrac{9}{16}\log|2x+1| + C$

(b) $\dfrac{1}{4}x - \log|x| - \dfrac{5}{16}\log|2x-1| - \dfrac{7}{16}\log|2x+1| + C$

(c) $\dfrac{1}{4}x - \log|x| - \dfrac{7}{16}\log|2x-1| + \dfrac{9}{16}\log|2x+1| + C$

(d) $\dfrac{1}{4}x + \log|x| - \dfrac{7}{16}\log|2x-1| - \dfrac{9}{16}\log|2x+1| + C$

4. $\int x^5\sqrt[3]{(1+x^3)^2}\,dx$ is equal to

(a) $\dfrac{1}{8}u^{8/3} - \dfrac{1}{5}u^{5/3} + C,\ u = 1 + x^3$

(b) $\dfrac{3}{8}u^{8/3} - \dfrac{3}{5}u^{5/3} + C,\ u = 1 + x^3$

(c) $-\dfrac{3}{8}u^{8/3} + \dfrac{3}{5}u^{5/3} + C,\ u = 1 + x^3$

(d) $-\dfrac{1}{8}u^{8/3} + \dfrac{1}{5}u^{5/3} + C,\ u = 1 + x^3$

5. $\int \dfrac{dx}{\sin^5 x\cos^5 x}$ is equal to

(a) $\dfrac{1}{4}(\tan^4 x - \cot^4 x) + 2(\tan^2 x - \cot^2 x) + C$

(b) $\dfrac{1}{4}(\tan^4 x - \cot^4 x) + 2(\tan^2 x - \cot^2 x) + 6\log|\tan x| + C$

(c) $\dfrac{1}{4}(\tan^4 x - \cot^4 x) + 2(\cot^2 x - \tan^2 x) + 6\log|\cot x| + C$

(d) $\dfrac{1}{4}(\cot^4 x - \tan^4 x) + 2(\cot^2 x - \tan^2 x) + 6\log|\tan x| + C$

6. If $\int \dfrac{dx}{5 - 4\sin x + 3\cos x}$ is equal to $\dfrac{1}{2 + f(x)} + C$ then $f(x)$ is equal to

(a) $\tan x$ (b) $\tan\dfrac{x}{2}$

(c) $-\tan\dfrac{x}{2}$ (d) $\tan x - \tan^2 x$

7. If $\int \dfrac{dx}{1+e^x} = x + f(x) + C$, then $f(x)$ is equal to

(a) $\log(1 + e^x)$ (b) $\log\dfrac{1}{1+e^x}$

(c) $2\log(1 + e^x)$ (d) $\dfrac{x+e^x}{1+e^x}$

8. If $\int e^x\cos 4x\,dx = A\,e^{5x}\left(\sin 4x + \dfrac{5}{4}\cos 4x\right) + C$ then A is equal to

(a) $\frac{4}{41}$ (b) $\frac{3}{41}$

(c) $\frac{5}{41}$ (d) $\frac{9}{41}$

9. The integral of $f(x) = 3^x \cos x$ is given by

(a) $3^x (\sin x + \cos x \log 3) + C$

(b) $3^x (\log 3 \sin x + \cos x) + C$

(c) $\frac{3^x (\sin x + \cos \log 3)}{\left(1 + (\log 3)^2\right)}$

(d) $\frac{3^x (\log 3 \sin x + \cos x)}{\left(1 + (\log 3)^2\right)} + C$

10. If $\int \frac{dx}{\sin 2x - 2\sin x} = C - \frac{1}{4} \log |f(x)| + \frac{1}{8 \sin^2 \frac{x}{2}}$ then $f(x)$ is equal to

(a) $\tan^2 x$ (b) $2 \tan \frac{x}{2}$

(c) $\tan \frac{x}{2}$ (d) $\sin \frac{x}{2}$

LEVEL 1

Straight Objective Type Questions

11. If $f_n(x) = \log \log \ldots \log x$ (log is repeated n times) then $\int [x f_1(x) f_2(x) \ldots f_n(x)]^{-1}$ dx is equal to

(a) $f_{n+1}(x) + C$ (b) $\frac{f_{n+1}(x)}{n+1} + C$

(c) $n f_n(x) + C$ (d) none of these

12. $\int 3^{3^{3^x}} 3^{3^x} 3^x \, dx$ is equal to

(a) $\frac{x^{3^x}}{(\log 3)^3} + C$ (b) $3^{3^{3^x}} (\log 3)^3 + C$

(c) $\frac{3^{3^{3^x}}}{(\log 3)^3} + C$ (d) none of these

13. The value of $\int \log (1 - \sqrt{x}) \, dx$ is

(a) $x \log (1 - \sqrt{x}) - \frac{1}{2} x - \frac{3}{2} \sqrt{x} + C$

(b) $(x - 1) \log (1 - \sqrt{x}) - \frac{1}{2} x - \sqrt{x} + C$

(c) $x^2 \log\left(1 - \sqrt{x}\right) - \frac{1}{2} \sqrt{x} + C$

(d) none of these

14. The value of $\int \frac{(x+1)}{x(1 + xe^x)} \, dx$ is

(a) $\log |1 + x e^x| + x + C$

(d) $\frac{1}{2} \log \left| \frac{x(1 + xe^x)}{x+1} \right| + C$

(c) $\log \left| \frac{xe^x}{1 + xe^x} \right| + C$

(d) $\log \left| \frac{(x+1) e^x}{1 + xe^x} \right| + C$

15. If $\int (\sin 2x - \cos 2x) \, dx = \frac{1}{\sqrt{2}} \sin (2x + k) + C$ then

(a) $k = \pi/4$ (b) $k = 5\pi/4$

(c) $k = 3\pi/4$ (d) none of these

16. If $\int \frac{x + \sin x}{1 + \cos x} dx = f(x) \tan \frac{x}{2} + C$

(a) $f(x) = x^2$ (b) $f(x) = x$

(c) $f(x) = 1 + \cos x$ (d) none of these

17. If $\int \frac{\sin 2x}{\cos^4 x + \sin^4 x} dx = \tan^{-1}(f(x)) + C$ then

(a) $f(x) = x^2$ (b) $f(x) = \cos x$

(c) $f(x) = \sin x$ (d) $f(x) = \tan^2 x$

18. The function f whose graph passes through $(\pi/4, 0)$ and whose derivative is $\frac{\log (\tan x)}{\sin x \cos x}$ is given by

(a) $(\log \tan x)^2 + \frac{\pi}{4}$ (b) $\frac{1}{2} (\log \cos x)^2$

(c) $\frac{1}{2} (\log \sin x)^2$ (d) $\frac{1}{2} (\log \tan x)^2$

19. The function f whose graph passes through $(4, -20)$ and whose derivative is $\cos \left(\sqrt{4 - x}\right)$ is given by

(a) $\sqrt{4-x} \sin \sqrt{4-x} + (4 - 6x) \cos \sqrt{4-x}$

(b) $\sqrt{4-x} \sin \sqrt{4-x} + \sqrt{4-x} \cos \sqrt{4-x} - 20$

(c) $-2\left(\sqrt{4-x} \sin \sqrt{4-x} + \cos \sqrt{4-x}\right) - 18$

(d) $-(2x + 12) \sqrt{4-x} \sin \sqrt{4-x}$

$+(4 - 6x) \cos \sqrt{4-x}$

20. If it is known that at the point $x = 1$ two antiderivative of $f(x) = e^x$ differ by 2. The difference of these antiderivatives at $x = 100$ is

(a) e^{100} (b) 100
(c) 2 (d) none of these

21. If $\int \frac{dx}{\cos^6 x + \sin^6 x} = \tan^{-1}(k \tan 2x) + C$ then

(a) $k = 1/2$ (b) $k = 1/4$
(c) $k = 1/6$ (d) k depends on x

22. If $\int \frac{\sin^2 x}{1 + \sin^2 x} dx = x - K \tan^{-1}(M \tan x) + C$ then

(a) $M = \frac{1}{\sqrt{2}}$ (b) $K = \frac{1}{\sqrt{2}}$
(c) $M = -\frac{1}{\sqrt{2}}$ (d) $K = -\frac{1}{\sqrt{2}}$

23. If $\int \tan^4 x\, dx = K \tan^3 x + L \tan x + f(x)$, then

(a) $K = 2/3$ (b) $L = -2$
(c) $f(x) = x + C$ (d) $K = 4/3$

24. If $\int \log(\sqrt{1-x} + \sqrt{1+x})\ dx = xf(x) + Lx + M \sin^{-1} x + C$, then

(a) $f(x) = \log(\sqrt{1-x} + \sqrt{1+x})$
(b) $L = -1/3$
(c) $M = -2/3$ (d) $M = -1/2$.

25. If $\int x \log\left(1 + \frac{1}{x}\right) dx = f(x) \log(x + 1) + g(x)x^2 + Lx + C$, then

(a) $f(x) = (1/2)\, x^2$ (b) $g(x) = \log x$
(c) $L = 1$ (d) none of these.

26. If $\int \frac{dx}{(x^2 + 1)(x^2 + 4)} = K \tan^{-1} x + L \tan^{-1} \frac{x}{2} + C$, then

(a) $K = 2/3$ (b) $L = 2/3$
(c) $K = -2/3$ (d) $L = -1/6$.

27. If $\int \frac{\cos^4 x}{\sin^2 x} dx = K \cot x + M \sin 2x + L \frac{x}{2} + C$, then

(a) $L = 1$ (b) $K = -2$
(c) $M = -1/4$ (d) none of these.

28. If $\int \frac{xe^x}{\sqrt{1 + e^x}} dx = f(x)\sqrt{1 + e^x} - 2 \log |g(x)| + C$, then

(a) $g(x) = \frac{\sqrt{1+e^x} + 1}{\sqrt{1+e^x} - 1}$ (b) $g(x) = \frac{\sqrt{1+e^x} - 1}{\sqrt{1+e^x} + 1}$
(c) $f(x) = x - 1$ (d) $f(x) = 3(x - 2)$

29. The value of the integral $\int \frac{\log(x+1) - \log x}{x(x+1)} dx$ is

(a) $-\frac{1}{2}[\log |(x + 1)|]^2 - \frac{1}{2}(\log |x|)^2 + \log |x + 1| \log |x| + C$
(b) $-[\{\log(x + 1)\}^2 - (\log x)^2] + \log(x + 1) \log x + C$
(c) $C - (1/2)[\log(1 + 1/x)]$
(d) none of these.

30. If $\int \operatorname{cosec} 2x\, dx = f |g(x)| + C$ then

(a) range of $g(x) = [0, \infty)$
(b) dom $f = (0, \infty)$
(c) $g'(x) = -\operatorname{cosec}^2 x$
(d) $f'(x) = 1/x$ for all $x \in (0, \infty)$

31. If $\int f(x)\, dx = -2 \cos \sqrt{x} + C$ then $f(x)$ is equal to

(a) $\sin \sqrt{x}$ (b) $\frac{\sin \sqrt{x}}{\sqrt{x}}$
(c) $2 \cos \sqrt{x}$ (d) none of these

32. If $\int \frac{dx}{2 \sin x - \cos x + 5} = A \tan^{-1} \frac{1}{\sqrt{5}} f(x) + C$, then

(a) $A = 1/\sqrt{5}$
(b) $A = \sqrt{5}$
(c) $f(x) = 5 \tan(x/2) + 1$
(d) $f(x) = 2 \tan(x/2) + 1$

33. The antiderivative of $\frac{1}{\sin^2 x + \tan^2 x}$ is

(a) $(-1/2)\left[\tan x + (1/\sqrt{2}) \tan^{-1}(1/\sqrt{2}) \tan x\right] + C$
(b) $(-1/2)\left[\cot x + \left(1/\sqrt{2}\right) \tan^{-1}\left((1/\sqrt{2}) \tan x\right)\right] + C$
(c) $-\left[\cot x + (1/\sqrt{2}) \tan^{-1}\left((1/\sqrt{2}) \tan x\right)\right]$
(d) none of these

34. If the antiderivative of $\frac{1}{\sqrt{x + x^{3/2}}}$ is $A\sqrt{1 + \sqrt{x}} + C$ then A is equal to

(a) 2 (b) 1
(c) 3 (d) 4

35. If the antiderivative of $\frac{e^{3x} + e^x}{e^{4x} - e^{2x} + 1}$ is $f(e^x - e^{-x}) + C$ then $f(x)$ is equal to

(a) $\sin x$ (b) $\cos^{-1} x$
(c) $\tan^{-1} x$ (d) $\tan x$

36. The value of $\int e^{\sec x}\sec^3 x\,(\sin^2 x + \cos x + \sin x + \sin x \cos x)\,dx$ is
(a) $e^{\sec x}(\sec x + \tan x) + C$
(b) $e^{\sec x}(\sec^2 x + \sec x \tan x) + C$
(c) $e^{\sec x}(\sec^2 x + \tan x) + C$
(d) none of these

37. If $\int \frac{3\cos x + 2\sin x}{4\sin x + 5\cos x}\,dx = Ax + B\log|4\sin x + 5\cos x| + C$ then
(a) $A = 23/41, B = 1/41$
(b) $A = 23/41, B = 2/41$
(c) $A = 11/23, B = 2/23$
(d) $A = 12/23, B = 2/23$

38. If $\int \frac{dx}{4 - 3\cos^2 x + 5\sin^2 x} = \frac{1}{3} f(3\tan x) + C$, then $f(x)$ is equal to
(a) $3\tan^{-1} x$ (b) x^2
(c) $x^2 + 1$ (d) $\tan^{-1} x$

39. If $f(x) = \frac{x^2}{1+x^2}$ and $g(x) = \sin x$ then $\int fo\, g(x)\cos x\,dx$ is
(a) $\sin x - \tan^{-1}(\sin x) + C$
(b) $\cos x + \tan^{-1}(\cos x) + C$
(c) $\sin x + 2\tan^{-1}(\sin x) + C$
(d) none of these

40. If $P(x) = \int \frac{x^3}{x^3 - x^2}\,dx$, $Q(x) = \int \frac{1}{x^3 - x^2}\,dx$ and $(P + Q)(2) = 5/2$ then $P(3) + Q(3)$ is equal to
(a) $5/2$ (b) $\frac{5}{4} + \log\frac{4}{3}$
(c) $\frac{10}{3} + \log\frac{4}{3}$ (d) none of these

41. $\int \frac{x^4 - 1}{x^2\left(x^4 + x^2 + 1\right)^{1/2}}\,dx$ is equal to
(a) $\sqrt{\frac{x^4 + x^2 + 1}{x}} + C$
(b) $\frac{x^2}{\sqrt{x^4 + x^2 + 1}} + C$
(c) $x\left(x^4 + x^2 + 1\right)^{3/2} + C$
(d) $\frac{\sqrt{x^4 + x^2 + 1}}{x} + C$

42. $\int \frac{1}{x}\log_{ex} e\,dx$ is equal to
(a) $\log_e\left(1 - \log_e x\right) + C$
(b) $\log_e\left(\log_e ex - 1\right) + C$
(c) $\log_e\left(\log_e x - 1\right) + C$
(d) $\log_e\left(1 + \log_e x\right) + C$

43. $\int \frac{x + \left(\cos^{-1} 3x\right)^2}{\sqrt{1 - 9x^2}}\,dx$ is equal to
(a) $C - \left(\frac{1}{9}\sqrt{1 - 9x^2} + \left(\cos^{-1} 3x\right)^3\right)$
(b) $C + \left(\frac{1}{9}\sqrt{1 - 9x^2} + \left(\cos^{-1} 3x\right)^3\right)$
(c) $C - \left(\sqrt{1 - 9x^2} + \left(\cos^{-1} 3x\right)^3\right)$
(d) $\frac{1}{9}\sqrt{1 - 9x^2} - \left(\cos^{-1} 3x\right)^3 + C$

44. If $I = \int \frac{\log\left(x + \sqrt{1 + x^2}\right)}{\sqrt{1 + x^2}}\,dx$, then I equals
(a) $(1/2)\log\left(x + \sqrt{1 + x^2}\right) + C$
(b) $(1/2)\left(\log\left(x + \sqrt{1 + x^2}\right)\right)^2 + C$
(c) $x\log\left(x + \sqrt{1 + x^2}\right) + C$
(d) $\log\left(x + \sqrt{1 + x^2}\right) + C$

45. If $I = \int \sec^{-1}\sqrt{x}\,dx$, then I equals
(a) $x\sec^{-1}\sqrt{x} - \log(1 + x) + C$
(b) $\sec^{-1}\sqrt{x} - \tan^{-1}\sqrt{x} + C$
(c) $x\sec^{-1}\sqrt{x} - \sqrt{x - 1} + C$
(d) $x\sec^{-1}\sqrt{x} + \log(1 + x) + C$

Assertion-Reason Type Questions

46. Let $I = \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ and $J = \int \frac{dx}{e^x\sqrt{1-e^{-2x}}}$ then

    Statement-1: $I + J = 2e^{\sqrt{x}} - \sin^{-1} e^{-x} + C$

    Statement-2: Suitable substitution is $\sqrt{x} = t$ in $I + J$

47. Let $f(x) = \frac{1}{\sqrt{x - x^2}}$ and F be the antiderivative of f satisfying $F(0) = 0$

    Statement-1: $F(1) = \pi$

    Statement-2: $F(x) = 4\tan^{-1}\sqrt{x}$

48. **Statement-1:** $I = \int \frac{(x+1)}{x\left(1+xe^x\right)}\,dx = \log\left|\frac{xe^x}{1+xe^x}\right| + C$

    Statement-2: Suitable substitution for computing I is $xe^x = t$.

49. Let $f(x) = \frac{1}{\sin 2x - 2\sin x}$ and $F(x)$ be an antiderivative of $f(x)$

    Statement-1: $F(x)$ is a periodic function with period 2π

    Statement-2: $F(x) = C - \frac{1}{4}\log|\cos x| + \frac{1}{8\sin^2\frac{x}{2}}$

50. If $\int xe^{x^2}\left(x^2+1\right)dx = f(x)e^{x^2} + C$

    Statement-1: $f(x)$ is of the form $\frac{x^n}{2}$ for some n

    Statement-2: f is an even function

LEVEL 2

Straight Objective Type Questions

51. If $f(x) = \int (x^3 - 2x^2 + 3)e^{3x}\,dx$ then the number of critical points of f are
    (a) 1
    (b) 2
    (c) 3
    (d) 4

52. If $f(x) = \int \frac{3x+2}{x^4 - x^3 + x^2 - 1}\,dx$, then $f(x)$ has
    (a) one critical point
    (b) two critical points
    (c) three critical points
    (d) no critical point

53. The function $f(x) = \int \frac{(x-2)\,dx}{x^2 - 7x + 12}$
    (a) decreases on **R**
    (b) increases on **R** ~ (2, 3)
    (c) increases on [2, 3) ∪ (4, ∞)
    (d) decreases on (2, ∞)

54. If $f(x) = \sin x$ then $\int \frac{e^{f(x)}\left(x\cos^3 x - f(x)\right)}{1-(f(x))^2}\,dx$ is equal to
    (a) $e^{f(x)}(x - \sin x) + C$
    (b) $e^{f(x)}(x - \sec x) + C$
    (c) $e^{f(x)}(x + \sec x) + C$
    (d) $e^{f(x)}(x + \sec^2 x) + C$

55. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\,dx = Ax + B\log(9e^{2x} - 4) + C$ then
    (a) $A = 3/2, B = 35/36$
    (b) $A = -5/4, B = 23/24$
    (c) $A = -3/2, B = 35/36$
    (d) $A = 5/4, B = -35/36$

56. Let f be a continuous function satisfying $f(x + y) = f(x)f(y)$ $(x, y \in \mathbf{R})$ with $f(1) = e$ then the value of $\int \frac{xf(x)}{\sqrt{1+f(x)}}\,dx$ is
    (a) $2x\sqrt{1+e^x} - 4\sqrt{1+e^x} - 2\log\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right| + C$
    (b) $2\sqrt{1+e^x} - 4\sqrt{1+f(x)} - 2\log\left|\sqrt{1+f(x)} - 1\right| + C$
    (c) $2\sqrt{1+f(x)} - 4\sqrt{1+f(x)} - 2\log\left|\frac{\sqrt{1+f(x)}-1}{\sqrt{1+f(x)}+1}\right| + C$
    (d) none of these.

57. If $\int \frac{x^7}{\left(1+x^4\right)^2} \mathrm{d}x = \frac{1}{4}\left[\log(1+x^4) + f(x)\right] + C$ then

(a) $f(x) = 1 + x^4$

(b) $f(x) = \frac{1}{\left(1+x^4\right)^2}$

(c) $f(x) = \frac{1}{1+x^4}$

(d) $f(x) = \tan^{-1}(1 + x^4)$

58. The function f whose graph passes through (0, 0) and whose derivative is $\frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}$ is given by

(a) $\frac{1}{4}\log\left|\frac{1+\tan x}{1-\tan x}\right| + \frac{1}{2}\sin x \cos x$

(b) $\log\left|\frac{\cos x + \sin x}{\cos x - \sin x}\right| + \sin 2x$

(c) $\log\left|\frac{\cos x - \sin x}{\cos x + \sin x}\right| + \frac{1}{2}\sin x \cos x + x$

(d) none of these

59. Let $P(x)$ be a polynomial of degree n with leading coefficient 1. Let $v(x)$ be any function and $v_1(x) = \int v(x)\, \mathrm{d}x$,

$$v_2(x) = \int v_1(x)\, \mathrm{d}x \ \ldots\ v_{n+1}(x) = \int v_n(x)\, \mathrm{d}x.$$

then $\int P(x)\, v(x)\, \mathrm{d}x$ is equal to

(a) $P(x)\, v_1(x) + \frac{P'(x)v_2(x)}{2!} + \frac{P''(x)v_3(x)}{3!} \ldots + \frac{v_{n+1}(x)}{(n+1)!}$

(b) $P(x)\, v_1(x) - P'(x)\, v_2(x) + P''(x)\, v_3(x) \ldots + (-1)^n\, n!\, v_{n+1}(x)$

(c) $P(x)\, v_1(x) + P'(x)\, v_2(x) + P''(x)\, v_3(x) \ldots + n\, v_{n+1}(x)$

(d) $P(x)\, v_1(x) - \frac{P'(x)v_2(x)}{2!} + \frac{P''(x)v_3(x)}{3!} \ldots + (-1)^n \frac{v_{n+1}(x)}{(n+1)!}$

60. If $\int \frac{x+1}{\left(x^2+x+1\right)} \frac{\mathrm{d}x}{\sqrt{x^2+x+1}} = k\frac{x-1}{\sqrt{x^2+x+1}} + C$ then the value of k is

(a) 1/2 (b) 1/3

(c) 2/3 (d) none of these

61. Let f be a continuous function satisfying $f(x + y) = f(x) + f(y)$, for each $x, y \in \mathbf{R}$ and $f(1) = 2$ then $\int \frac{f(x)\tan^{-1} x}{\left(1+x^2\right)^2} \mathrm{d}x$ is equal to

(a) cannot be determined explicitly

(b) $C - \frac{\tan^{-1} x}{2(1+x^2)} + \frac{1}{4}\tan^{-1} x + \frac{f(x)}{1+(f(x))^2}$

(c) $C - \frac{\tan x}{2(1+x^2)} + \frac{\tan^{-1} x}{4} + \frac{x}{4(1+x^2)}$

(d) $C - \frac{1}{(1+x^2)}\tan^{-1} x + \frac{1}{2}\tan^{-1} x + \frac{x}{2(1+x^2)}$

62. The value of $\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$ is

(a) $3(x-1)^{1/4} - (5/3)(x+2)^{3/4} + C$

(b) $4\left(\sqrt[4]{x-1}\right) - 5\left(\sqrt[4]{(x+2)^5}\right) + C$

(c) $\frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}} + C$

(d) none of these

63. The integral $\int \frac{\cos 4x - 1}{\cot x - \tan x}$ is equal to

(a) $-\frac{1}{2}\cos 4x + C$ (b) $-\frac{1}{4}\cos 4x + C$

(c) $-\frac{1}{2}\sin 2x + C$ (d) none of these.

64. If $f(x) = \lim\limits_{x\to\infty} \frac{x^n - x^{-n}}{x^n + x^{-n}}, x > 1$ then $\int \frac{x f(x) \log\left(x + \sqrt{1+x^2}\right)}{\sqrt{1+x^2}} \mathrm{d}x$ is

(a) $\log\left(x + \sqrt{1+x^2}\right) - x + C$

(b) $\left(\frac{1}{2}\right)x^2\left(\log\left(x+\sqrt{1+x^2}\right) - 1\right) + C$

(c) $(x-1)\log\left(x + \sqrt{1+x^2}\right) + C$

(d) none of these

65. The value of $\int \sin\sqrt[3]{x}\, dx$ is

(a) $\{(2 - x^{2/3})\cos x^{1/3} + 2x^{1/3}\sin x^{1/3}\} + C$

(b) $3\{(2 - x^{2/3})\cos x^{1/3} + 2x^{1/3}\sin x^{1/3}\} + C$

(c) $3\,\{(2 - x^{2/3}) \sin x^{1/3} - 2\,x^{1/3} \cos x^{1/3}\} + C$

(d) none of these

66. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ and $f(0) = 2$. Then $\int \frac{f(x)}{3+4f(x)}\,dx$ is equal to

(a) $(1/4) \log (3 + 8e^x) + C$

(b) $(1/2) \log (3 + 5e^x) + C$

(c) $(1/4) \log (3 + 4e^{2x}) + C$

(d) none of these

67. If $\int \frac{f(x)}{x^2 - x + 1}\,dx = \frac{3}{2} \log (x^2 - x + 1) + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}} + C$ then $f(x)$ is equal to

(a) $3x$ (b) $3x - 4$

(c) $3x - 1$ (d) $\frac{1}{1+x^2}$

68. If the antiderivative of $\frac{1}{x^2\sqrt{1+x^2}}$ is $-\frac{\sqrt{f(x)}}{x} + C$ then $f(x)$ is equal to

(a) $1 + 1/x^2$ (b) $\tan^{-1} x$

(c) $1 + x^2$ (d) $2\sqrt{1+x^2}$

69. The value of the integeral $\int \frac{dx}{\cos^3 x\sqrt{\sin 2x}}$ can be expressed as

(a) a rational function of tan x

(b) an irrational function tan x

(c) a rational function of cos x

(d) a rational function of sin x

70. If $f(x) = \sqrt{x}$, $g(x) = e^x - 1$ and $h(x) = \tan^{-1} x$ then the antiderivative of $f \text{ o } g(x)$ is

(a) $2\,[g \text{ o } f(x) - g \text{ o } f \text{ o } h(x)] + C$

(b) $2\,[f \text{ o } g(x) - f \text{ o } g \text{ o } h(x)] + C$

(c) $f \text{ o } g(x) + f \text{ o } g \text{ o } h(x) + C$

(d) $2\,[f \text{ o } g(x) - h \text{ o } f \text{ o } g(x)] + C$

71. If $f(x) = \sqrt{4x^2 + 4x - 3}$ then $\int \frac{x+3}{f(x)}\,dx$ is equal

(a) $(1/4) f(x) + (2/3) \log |2x + 1 + f(x)| + C$

(b) $(1/4) f(x) + (5/4) \log |2x + 1 + f(x)| + C$

(c) $(1/3) f(x) + (2/3) \log |(x + 3) + f(x)| + C$

(d) $(2/3) f(x) + (1/3) \log |(x + 3) + f(x)|$

72. If $f(x) = \frac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}$ and $g(x) = \tan \frac{x}{2}$ and $h(x) = \log |x|$ then $\int \frac{dx}{\sin x(2\cos^2 x - 1)}$ is equal to

(a) $(1/\sqrt{2}) \log |f(x)| + g \text{ o } h(x) + C$

(b) $\sqrt{2}\; h \text{ o } f(x) + h \text{ o } g(x) + C$

(c) $h \text{ o } (f + g)(x) + C$

(d) $(1/\sqrt{2})\; f \text{ o } (g + h)(x) + C$

73. If $f(x) = \sqrt{x^2 + 4}$ then $\int \frac{f(x)}{x^6}\,dx$ is equal to

(a) $\frac{(f(x))^3\left((f(x))^2 - 10\right)}{120x^5} + C$

(b) $\frac{(f(x))^3\left((f(x))^2 - 4\right)}{6x^5} + C$

(c) $\frac{(f(x))^2\left((f(x))^3 - 2\right)}{20x^4} + C$

(d) $\frac{(f(x))^3\left(x^2 - 6\right)}{6x^5} + C$

74. If $f(x) = \cos x$ and $g(x) = \sin x$ then $\int \frac{\log f(x)}{(f(x))^2}\,dx$ is

(a) $f(x)(\log f(x) + 1) + x + C$

(b) $\frac{g(x)}{f(x)} (\log f(x) + 1) + \frac{x^2}{2} + C$

(c) $\frac{f(x)}{g(x)} (\log g(x) + 1) + x + C$

(d) $\frac{g(x)}{f(x)} [\log f(x) + 1] - x + C$

75. If $f(x) = \cos x$ then $\int (2(f(x))^2 - 1)(4(f(x))^3 - 3 f(x))\,dx$ is equal to

(a) $\frac{1}{10} \sin 5x + \frac{1}{2} \sin x + C$

(b) $\frac{1}{5} \sin 5x + \sin x + C$

(c) $\frac{1}{15} \sin 5x - \frac{1}{2} \sin x + C$

(d) none of these

76. If $I_n = \int \cot^n x\,dx$, and $I_0 + I_1 + 2\,(I_2 + \ldots + I_8) + I_9 + I_{10} = A\left(u + \frac{u^2}{2} + \ldots + \frac{u^9}{9}\right)$, where $u = \cot x$ then

(a) $A = 1$ (b) $A = -1$

(c) $A = 1/2$ (d) $A = -1/2$

77. $f(x) = \int \frac{dx}{\sin^4 x}$ is a

(a) polynomial of degree 3 in $\cot x$
(b) polynomial of degree 4 in $\cot x$
(c) polynomial of degree 4 in $\sin x$
(d) polynomial of degree 3 in $\tan x$

78. If $f(x) = \sqrt{x^2 - a^2}$ then $\int \frac{x^2}{f(x)} dx$ is

(a) $x f(x) + \frac{a^2}{2} \log|x + f(x)| + C$

(b) $\frac{x^2}{2} f(x) - \frac{a^2}{2} \log|x + f(x)| + C$

(c) $\frac{x^2}{2} f(x) + a^2 \log|x - f(x)| + C$

(d) $\frac{x}{2} f(x) + \frac{a^2}{2} \log|x + f(x)| + C$

79. Let $f(x) = \frac{x}{(1 + x^n)^{1/n}}$ for $n \geq 2$ and $g(x) = \underbrace{\frac{f \circ f \ldots \circ f}{n\ times}}$. Then $\int x^{n-2} g(x)\, dx$ equals

(a) $\frac{1}{n(n-1)} (1 + n x^n)^{1 - 1/n} + k$

(b) $\frac{1}{n-1} (1 + n x^n)^{1 - 1/n} + k$

(c) $\frac{1}{n(n-1)} (1 + n x^n)^{1 + 1/n} + k$

(d) $\frac{1}{n-1} (1 + n x^n)^{1 + 1/n} + k.$

80. If $\int f(x)\, dx = \phi(x)$ i.e. ϕ is a function such that $\phi'(x) = f(x)$, then $\int x^9 f(x^5)\, dx$ is equal to

(a) $\frac{1}{5} [x^5 \phi(x^5) - \int x^4 \phi(x^5) dx] + C$

(b) $\frac{1}{5} x^5 \phi(x^5) - 5 \int x^5 \phi(x^5)\, dx + C$

(c) $\frac{1}{5} x^5 \phi(x^5) - \int x^4 \phi(x^5)\, dx + C$

(d) $\frac{1}{5} [x^5 \phi(x^5) - \int x^5 \phi(x^5) dx] + C$

Previous Years' AIEEE/JEE Main Questions

1. If $\int \frac{\sin x}{\sin(x - \alpha)} dx = Ax + B \log \sin(x - \alpha) \sin + C$ then the value of (A, B) is

(a) $(-\sin\alpha, \cos\alpha)$
(b) $(\cos\alpha, \sin\alpha)$
(c) $(\sin\alpha, \cos\alpha)$
(d) $(-\cos\alpha, \sin\alpha)$ **[2004]**

2. $\int \frac{dx}{\cos x - \sin x}$ is equal to

(a) $\frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{x}{2} - \frac{3\pi}{8}\right)\right| + C$

(b) $\frac{1}{\sqrt{2}} \log\left|\cot\frac{x}{2}\right| + C$

(c) $\frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)\right| + C$

(d) $\frac{1}{\sqrt{2}} \log\left|\tan\left(\frac{x}{2} + \frac{3\pi}{8}\right)\right| + C$ **[2004]**

3. $\int \left\{\frac{(\log x - 1)}{1 + (\log x)^2}\right\}^2 dx$ is equal to

(a) $\frac{x e^x}{1 + x^2} + C$ (b) $\frac{x}{(\log x)^2 + 1} + C$

(c) $\frac{\log x}{(\log x)^2 + 1} + C$ (d) $\frac{x}{x^2 + 1} + C$ **[2005]**

4. $\int \frac{dx}{\cos x + \sqrt{3} \sin x}$ equals

(a) $\frac{1}{2} \log \tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$

(b) $\frac{1}{2} \log \tan\left(\frac{x}{2} - \frac{\pi}{12}\right) + C$

(c) $\log \tan\left(\frac{x}{2} + \frac{\pi}{12}\right) + C$

(d) $\log \tan\left(\frac{x}{2} - \frac{\pi}{12}\right) + C$ **[2007]**

5. The value of $\sqrt{2}\int \dfrac{\sin x}{\sin\left(x-\dfrac{\pi}{4}\right)}$ is

(a) $x+\log\left|\cos\left(x-\dfrac{\pi}{4}\right)\right|+C$

(b) $x-\log\left|\sin\left(x-\dfrac{\pi}{4}\right)\right|+C$

(c) $x+\log\left|\sin\left(x-\dfrac{\pi}{4}\right)\right|+C$

(d) $x-\log\left|\cos\left(x-\dfrac{\pi}{4}\right)\right|+C$ **[2008]**

6. If the integral $\int \dfrac{5\tan x}{\tan x-2}\,dx$ = $x + a\log|\sin x - 2\cos x| + k$, then a is equal to

(a) –2 (b) 1
(c) 2 (d) –1 **[2012]**

7. If $\int \dfrac{x^2-x+1}{x^2+1}\, e^{\cot^{-1}x}\,dx = A(x)\,e^{\cot^{-1}x} + C$

(a) $-x$ (b) x
(c) $\sqrt{1-x}$ (d) $\sqrt{1+x}$ **[2013, online]**

8. The integral $\int \dfrac{x\,dx}{2-x^2+\sqrt{2-x^2}}$ equals

(a) $\log\left|1+\sqrt{2+x^2}\right|+C$

(b) $-\log\left|1+\sqrt{2-x^2}\right|+C$

(c) $-x\log\left|1-\sqrt{2-x^2}\right|+C$

(d) $x\log\left|1-\sqrt{2+x^2}\right|+C$ **[2013, online]**

9. If $\int \dfrac{dx}{x+x^7} = p(x)$ then, $\int \dfrac{x^6}{x+x^7}\,dx$ is equal to

(a) $\log|x| - p(x) + C$
(b) $\log|x| + p(x) + C$
(c) $x - p(x) + C$
(d) $x + p(x) + C$ **[2013, online]**

10. If $\int f(x)\,dx = \psi(x)$, then $\int x^5 f(x^3)\,dx$ is equal to

(a) $\dfrac{1}{3}\,x^3\,\psi(x)^3 - 3\int x^3\,\psi(x^3)\,dx + C$

(b) $\dfrac{1}{3}\,x^3\,\psi(x^3) - \int x^2\,\psi(x^3)\,dx + C$

(c) $\dfrac{1}{3}\,[x^3\,\psi(x^3) - \int x^3\,\psi(x^3)\,dx] + C$

(d) $\dfrac{1}{3}\,[x^3\,\Psi(x^3) - \int x^2\,\Psi(x^3)\,dx] + C$ **[2013]**

11. The integral $\int \left(1+x-\dfrac{1}{x}\right)e^{x+\frac{1}{x}}\,dx$ is equal to

(a) $(x-1)\,e^{x+\frac{1}{x}}+C$

(b) $x\,e^{x+\frac{1}{x}}+C$

(c) $(x+1)\,e^{x+\frac{1}{x}}+C$

(d) $-x\,e^{x+\frac{1}{x}}+C$ **[2014]**

12. **I** = $\int \dfrac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}\,dx$ is equal to

(a) $\dfrac{1}{2}\sin 2x + C$

(b) $-\dfrac{1}{2}\sin 2x + C$

(c) $-\dfrac{1}{2}\sin x + C$

(d) $-\sin^2 x + C$ **[2014, online]**

13. The integral $\int x\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)dx$ $(x > 0)$ is equal to

(a) $-x + (1+x^2)\tan^{-1}x + C$
(b) $x - (1+x^2)\cot^{-1}x + C$
(c) $-x + (1+x^2)\cot^{-1}x + C$
(d) $x - (1+x^2)\tan^{-1}x + C$ **[2014, online]**

14. The integral $\int \dfrac{\sin^2 x\cos^2 x}{\left(\sin^3 x+\cos^3 x\right)^2}\,dx$ is equal to

(a) $\dfrac{1}{1+\cot^3 x}+C$

(b) $-\dfrac{1}{3\left(1+\tan^3 x\right)}+C$

(c) $\dfrac{\sin^3 x}{1+\cos^3 x}+C$

(d) $\dfrac{\cos^3 x}{3\left(1+\sin^3 x\right)}+C$ **[2014, online]**

15. If m is non zero number and

$\int \dfrac{x^{5m-1}+2x^{4m-1}}{\left(x^{2m}+x^m+1\right)^3}\,dx = f(x) + C$, then $f(x)$ is

(a) $\dfrac{x^{5m}}{2m\left(x^{2m}+x^m+1\right)^2}$ (c) $\dfrac{2m\left(x^{5m}+x^{4m}\right)}{\left(x^{2m}+x^m+1\right)^2}$

(b) $\dfrac{x^{4m}}{2m\left(x^{2m}+x^m+1\right)^2}$ (d) $\dfrac{x^{5m}+x^{4m}}{2m\left(x^{2m}+x^m+1\right)^2}$

[2014, online]

16. The integral $\int \dfrac{dx}{x^2(x^4+1)^{3/4}}$ equals

(a) $\left(\dfrac{x^4+1}{x^4}\right)^{\frac{1}{4}}+C$ (b) $\left(x^4+1\right)^{\frac{1}{4}}+C$

(c) $-\left(x^4+1\right)^{\frac{1}{4}}+C$ (d) $-\left(\dfrac{x^4+1}{x^4}\right)^{\frac{1}{4}}+C$ **[2015]**

17. The integral $I = \int \dfrac{dx}{(x+1)^{3/4}(x-2)^{5/4}}$ is equal to

(a) $4\left(\dfrac{x+1}{x-2}\right)^{1/4}+C$ (b) $4\left(\dfrac{x-2}{x+2}\right)^{1/4}+C$

(c) $-\dfrac{4}{3}\left(\dfrac{x+1}{x-2}\right)^{1/4}+C$ (d) $-\dfrac{4}{3}\left(\dfrac{x-2}{x+1}\right)^{1/4}+C$

[2015, online]

18. If $I = \int \dfrac{\log\left(t+\sqrt{1+t^2}\right)}{\sqrt{1+t^2}}dt = \dfrac{1}{2}(g(t))^2 + C$, where C is a constant, then $g(2)$ is equal to

(a) $2\log(2+\sqrt{5})$ (b) $\log(2+\sqrt{5})$

(c) $\dfrac{1}{\sqrt{5}}\log(2+\sqrt{5})$ (d) $\dfrac{1}{2}\log(2+\sqrt{5})$

[2015, online]

19. The integral $I = \int \dfrac{2x^{12}+5x^9}{(x^5+x^3+1)^3}dx$ is equal to

(a) $\dfrac{-x^5}{(x^5+x^3+1)^2}+C$

(b) $\dfrac{x^{10}}{2(x^5+x^3+1)^5}+C$

(c) $\dfrac{x^5}{2(x^5+x^3+1)^2}+C$

(d) $\dfrac{-x^{10}}{2(x^5+x^3+1)^2}+C$ **[2016]**

20. The integral $I = \int \dfrac{dx}{(1+\sqrt{x})\sqrt{x-x^2}}$ is equal to (where C is a constant of integration)

(a) $-2\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}+C$ (b) $-\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}+C$

(c) $-2\sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}+C$ (d) $2\sqrt{\dfrac{1+\sqrt{x}}{1-\sqrt{x}}}+C$

[2016, online]

21. If $I = \int \dfrac{dx}{\cos^3 x\sqrt{2\sin 2x}} = (\tan x)^A + C\,(\tan x)^B + k$, where k is a constant of integration, then $A + B + C$ equals:

(a) $\dfrac{16}{5}$ (b) $\dfrac{27}{10}$

(c) $\dfrac{7}{10}$ (d) $\dfrac{21}{5}$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. An antiderivative of the integral $\int e^x\left(\dfrac{1-x}{1+x^2}\right)^2 dx$ is

(a) $e^x(1+x^2)^2$ (b) $\dfrac{-xe^x}{\left(1+x^2\right)^2}$

(c) $\dfrac{e^x(1-x)}{\left(1+x^2\right)^2}$ (d) $\dfrac{e^x}{1+x^2}$ **[2007]**

2. Let $f : (11, \infty) \to (0, \infty)$ be given by

$f(x) = \prod_{l=1}^{10} \dfrac{1}{(x-l)}$

where for real number $a_1 \cdots a_n$

$\prod_{l=1}^{n} a_l$ denotes the product $a_1 \times a_2 \cdots \times a_n$

Statement 1: $\int f(x)\,dx = \sum_{l=1}^{10} \frac{(-1)^l \log|x-l|}{(l-1)!\,(10-l)!}$

Statement 2: For $x \in [11, \infty)$ $f(x) = \sum_{l=1}^{10} \frac{A_l}{x-l}$

where $A_1 = \prod_{j=1}^{10} \frac{j}{l-j}$ $l = 1, 2, 10$ **[2008]**

3. $\int \frac{2^{x+1} - 5^{x-1}}{10^x}\,dx$ equals

(a) $\frac{1}{5\log 2} 2^{x+1} - \frac{2}{\log 5} 5^{x-1} + C$

(b) $\frac{2(x+1)}{5^{x+1}} - \frac{x}{2^x} + C$

(c) $\frac{2(x+1)}{5^{x+1}} + \frac{x}{2^x} + C$

(d) $\frac{1}{5\log 2} 2^{-x} - \frac{2}{\log 5} 5^{-x} + C$ **[2009]**

4. The integral $\int \frac{dx}{(a^2 - b^2x^2)^{3/2}}$ equals

(a) $\frac{x}{\sqrt{a^2 - b^2x^2}} + C$

(b) $\frac{x}{a^2\sqrt{a^2 - b^2x^2}} + C$

(c) $\frac{ax}{\sqrt{a^2 - b^2x^2}} + C$

(d) $\frac{1}{a^2\sqrt{a^2 - b^2x^2}} + C$ **[2012]**

5. $\int \frac{7x^{13} + 5x^{15}}{(x^7 + x^2 + 1)^3}$ equals

(a) $\frac{x^{14}}{(x^7 + x^2 + 1)^2} + C$

(b) $\frac{x^{14}}{2(x^7 + x^2 + 1)^2} + C$

(c) $\frac{x^7}{(x^7 + x^2 + 1)^2} + C$

(d) $\frac{x^7}{2(x^7 + x^2 + 1)^2} + C$ **[2013]**

6. If $\int \frac{1 - 5\sin^2 x}{\cos^5 x \sin^2 x}\,dx = \frac{f(x)}{\cos^5 x} + C$, then $f(x)$:

(a) $-\operatorname{cosec} x$ (b) $\operatorname{cosec} x$
(c) $\cot x$ (d) $-\cot x$ **[2014]**

7. If $\int \frac{1 - x^9}{x(1 + x^9)}\,dx = A \log|x| + B \log|1 + x^9| + C$, then the ratio $A : B$ is equal to
(a) $-2 : 9$ (b) $2 : 9$
(c) $9 : -2$ (d) $9 : 2$ **[2015]**

8. If $I = \int \frac{dx}{x^3(1 + x^6)^{2/3}} = f(x)\,(1 + x^{-6})^{1/3} + C$, where C is a constant of integration, then $f(x)$ is equal to:

(a) $-\frac{1}{6}$ (b) $-\frac{6}{x}$

(c) $-\frac{x}{2}$ (d) $-\frac{1}{2}$ **[2016]**

Answers

Concept-based

1. (b) **2.** (c) **3.** (d) **4.** (a)
5. (b) **6.** (c) **7.** (b) **8.** (a)
9. (c) **10.** (c)

Level 1

11. (a) **12.** (c) **13.** (b) **14.** (c)
15. (b) **16.** (b) **17.** (d) **18.** (d)
19. (d) **20.** (c) **21.** (a) **22.** (b)
23. (c) **24.** (a) **25.** (d) **26.** (d)
27. (a) **28.** (b) **29.** (a) **30.** (b)
31. (b) **32.** (a) **33.** (b) **34.** (d)
35. (c) **36.** (a) **37.** (b) **38.** (d)
39. (a) **40.** (d) **41.** (d) **42.** (d)
43. (a) **44.** (b) **45.** (c) **46.** (c)
47. (c) **48.** (a) **49.** (c) **50.** (b)

Level 2

51. (a) **52.** (c) **53.** (c) **54.** (b)
55. (c) **56.** (a) **57.** (c) **58.** (a)
59. (b) **60.** (c) **61.** (d) **62.** (c)
63. (d) **64.** (d) **65.** (b) **66.** (a)
67. (c) **68.** (c) **69.** (b) **70.** (d)
71. (b) **72.** (b) **73.** (a) **74.** (d)
75. (a) **76.** (b) **77.** (a) **78.** (d)
79. (a) **80.** (c)

Previous Years' AIEEE/JEE Main Questions

1. (b) **2.** (d) **3.** (b) **4.** (a)
5. (c) **5.** (c) **6.** (c) **7.** (b)
8. (b) **9.** (a) **10.** (b) **11.** (b)
12. (b) **13.** (a) **14.** (b) **15.** (b)
16. (d) **17.** (c) **18.** (b) **19.** (b)
20. (c) **21.** (a)

Previous Years' B-Architecture Entrance Examination Questions

1. (d) **2.** (c) **3.** (d) **4.** (b)
5. (b) **6.** (d) **7.** (c) **8.** (d)

Hints and Solutions

Concept-based

1. $\int(1 - \tan 3x)^2 = \int(1 + \tan^2 3x - 2\tan 3x)\, dx$

$= \int(\sec^2 3x - 2 \tan 3x)\, dx$

$= \dfrac{\tan 3x}{3} - \dfrac{2}{3} \log|\sec 3x| + C$

$= \dfrac{1}{3} [\tan 3x + \log \cos^2 3x] + C$

2. $\dfrac{1}{x^4 - x^2} = \dfrac{1}{x^2\,(x-1)\,(x+1)} = \dfrac{1}{2}\dfrac{1}{x^2}\left[\dfrac{1}{x-1} - \dfrac{1}{x+1}\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{x-1} - \dfrac{1}{x} - \dfrac{1}{x^2} - \left[\dfrac{1}{x^2} - \left(\dfrac{1}{x} - \dfrac{1}{x+1}\right)\right]\right]$

$= \dfrac{1}{2}\left[\dfrac{1}{x-1} - \dfrac{2}{x^2} - \dfrac{1}{x+1}\right]$

$\int\dfrac{dx}{x^4 - x^2} = \dfrac{1}{2}\left[\log|x-1| - \log|x+1| + \dfrac{1}{x}\right] + C$

$= \dfrac{1}{x} + \log\left|\dfrac{x-1}{x+1}\right|^{1/2} + C.$

3. $\dfrac{x^3 - 1}{4x^3 - x} = \dfrac{1}{4} + \dfrac{\frac{1}{4}x - 1}{x(2x-1)(2x+1)}$

$= \dfrac{1}{4} + \dfrac{1}{x} - \dfrac{7}{8}\dfrac{1}{2x-1} - \dfrac{9}{8}\dfrac{1}{2x+1}$

$\int\dfrac{x^3 - 1}{4x^3 - x}\, dx = \dfrac{1}{4} + \log|x| - \dfrac{7}{16} \log|2x - 1|$

$- \dfrac{9}{16} \log|2x + 1| + C.$

4. Put $1 + x^3 = t^3 \Rightarrow x^2\, dx = t^2\, dt$

$$\int x^5 \sqrt[3]{(1 + x^3)^2}\, dx = \int(t^3 - 1)\, t^4\, dt$$

$= \dfrac{t^8}{8} - \dfrac{t^5}{5} + C$

$= \dfrac{1}{8}(1 + x^3)^{8/3} - \dfrac{1}{5}(1 + x^3)^{5/3} + C.$

5. Substitute $\tan x = t$, $\sec^2 x\, dx = dt \Rightarrow dx = \dfrac{dt}{1+t^2}$

$\int\dfrac{dx}{\sin^5 x \cos^5 x} = \int\dfrac{dt}{(1+t^2)\left(\dfrac{t}{\sqrt{1+t^2}}\right)^5\left(\dfrac{1}{\sqrt{1+t^2}}\right)^5}$

$= \int\dfrac{(1+t^2)^4}{t^5}\, dt$

$= \int\left(\dfrac{1}{t^5} + \dfrac{4}{t^3} + 6\dfrac{1}{t} + 4t + t^3\right) dt$

$= -\dfrac{t^4}{4} - 2t^{-2} + 6 \log|t| + 4\dfrac{t^2}{2} + \dfrac{t^4}{4} + C$

$= \dfrac{1}{4}(\tan^4 x - \cot^4 x) + 2(\tan^2 x - \cot^2 x) + 6 \log|\tan x|$
$+ C$

6. Put $\tan\dfrac{x}{2} = t \Rightarrow dx = \dfrac{2dt}{1+t^2}$

$\int\dfrac{dx}{5 - 4\sin x + 3\cos x}$

$= \int\dfrac{2dt}{(1+t^2)\left(5 - \dfrac{8t}{1+t^2} + \dfrac{3(1-t^2)}{1+t^2}\right)}$

$= 2\int\dfrac{dt}{2t^2 - 8t + 8}$

$= \int\dfrac{dt}{t^2 - 4t + 4} = \int\dfrac{dt}{(t-2)^2}$

$= -\dfrac{1}{t-2} + C = \dfrac{1}{2 - \tan\dfrac{x}{2}} + C$

7. Let $1 + e^x = t \Rightarrow e^x = t - 1 \Rightarrow x = \log(t - 1)$

so $dx = \dfrac{1}{t-1}\, dt$

Thus $I = \int\dfrac{dx}{1+e^x} = \int\dfrac{dt}{t\,(t-1)} = \int\left(\dfrac{1}{t-1} - \dfrac{1}{t}\right) dt$

$= \log(t - 1) - \log t + C$

$= \log e^x - \log(1 + e^x) + C$

$= x - \log(1 + e^x) + C$

$= x + \log\dfrac{1}{1+e^x} + C$

8. Integrating by parts, taking e^{4x} as first function, we have

$$\mathbf{I} = \int e^{5x} \cos 4x \, dx = \frac{1}{4} e^{5x} \sin 4x - \frac{5}{4} \int e^{5x} \sin 4x \, dx$$

$$= \frac{1}{4} e^{5x} \sin 4x - \frac{5}{4} \left[-e^{5x} \frac{\cos 4x}{4} + \frac{5}{4} \int e^{5x} \cos 4x \, dx \right]$$

$$= \frac{1}{4} e^{5x} \sin 4x + \frac{5}{16} e^{5x} \cos 4x - \frac{25}{16} \mathbf{I}$$

$$\Rightarrow \left(1 + \frac{25}{16}\right) \mathbf{I} = \frac{4}{16} e^{5x} \left(\sin 4x + \frac{5}{4} \cos 4x\right)$$

$$\Rightarrow \mathbf{I} = \frac{4}{41} e^{5x} \left(\sin 4x + \frac{5}{4} \cos x\right) + C$$

9. $\mathbf{I} = \int (3^x \cos x \, dx$

$$= 3^x \sin x - \log 3 \int 3^x \sin x \, dx$$

$$= 3^x \sin x - \log 3 \left[- 3^x \cos x + (\log 3) \int 3^x \cos x \, dx\right]$$

$$= 3^x \sin x + (\log 3), 3^x \cos x - (\log 3)^2$$

$$\Rightarrow \quad (1 + (\log 3)^2) \mathbf{I} = 3^x (\sin x + (\log 3) \cos x)$$

$$\mathbf{I} = \frac{3^x (\sin x + (\log 3) \cos x)}{\left(1 + (\log 3)^2\right)}.$$

10. $$\int \frac{dx}{\sin 2x - 2 \sin x} = \frac{1}{2} \int \frac{dx}{\sin x (\cos x - 1)}$$

$$= \frac{1}{2} \int \frac{\sin x \, dx}{\left(1 - \cos^2 x\right) (\cos x - 1)}$$

$$= \frac{1}{2} \int \frac{-\sin x \, dx}{(1 - \cos x)^2 \, (1 + \cos x)}$$

$$= \frac{1}{2} \int \frac{dt}{(1 - t)^2 \, (1 + t)} \qquad (t = \cos x)$$

$$= \frac{1}{4} \int \left(\frac{1}{2} \frac{t}{1 + t} + \frac{1}{2} \frac{1}{1 - t} + \frac{1}{(1 - t)^2} \right) dt$$

$$= \frac{1}{4} \left[\frac{1}{2} \log |1 + t| - \frac{1}{2} \log |1 - t| + \frac{1}{1 - t} \right] + C$$

$$= \frac{1}{4} \left[\frac{1}{1 - \cos x} + \frac{1}{2} \log \left| \frac{1 + \cos x}{1 - \cos x} \right| \right] + C$$

$$= \frac{1}{8} \frac{1}{\sin^2 \frac{x}{2}} - \frac{1}{8} \log \left| \frac{2 \sin^2 x/2}{2 \cos^2 x/2} \right| + C$$

$$= \frac{1}{8} \frac{1}{\sin^2 x/2} - \frac{1}{4} \log |\tan x/2| + C$$

Level 1

11. Put $f_n(x) = t$, $\dfrac{dt}{dx} = \dfrac{1}{x \, f_1(x) \ldots f_{n-1}(x)}$

So $\int [x \, f_1(x) \ldots f_n(x)]^{-1} \, dx = \int \frac{dt}{t} = \log t + C$

$= f_{n+1}(x) + C.$

12. Put $3^{3^{3^x}} = t \Rightarrow (\log 3)^3 \; 3^{3^{3^x}} \,.\, 3^{3^x} \,.\, 3x = \dfrac{dt}{dx}$

So given integral is equal to $\displaystyle\int \frac{dt}{(\log 3)^3} = \frac{t}{(\log 3)^3} + C.$

13. Integrating by parts taking $\log (1 - \sqrt{x})$ as the first function, the given integral can be written as

$$x \log (1 - \sqrt{x}) + \frac{1}{2} \int \frac{\sqrt{x}}{1 - \sqrt{x}} \, dx$$

$$= x \log (1 - \sqrt{x}) + \int \frac{t^2}{1 - t} \, dt$$

$$= x \log (1 - \sqrt{x}) + \int \left(-t - 1 + \frac{1}{1 - t} \right) dt$$

$$= (x - 1) \log (1 - \sqrt{x}) \frac{x}{2} - \sqrt{x} + C.$$

14. Write the integrand as $\dfrac{1}{x} + 1 - \dfrac{(x + 1) \, e^x}{1 + x \, e^x}$.

So the given integral is equal to

$\log x + x - \log (1 + x \, e^x) + C$

$= \log x + \log e^x - \log (1 + x \, e^x) + C$

$= \log \dfrac{xe^x}{1 + xe^x} + C.$

15. Write the given integral as $- \sqrt{2} \int \cos (2x + x/4) \, dx$.

$$= -\frac{1}{\sqrt{2}} \sin \left(2x + \frac{\pi}{4}\right) + C = \frac{1}{\sqrt{2}} \sin \left(2x + \frac{5\pi}{4}\right) + C.$$

16. $$\int \frac{x + \sin x}{1 + \cos x} dx = \frac{1}{2} \int x \sec^2 \frac{x}{2} \, dx + \int \frac{\sin x}{1 + \cos x} dx$$

$$= \frac{1}{2} \left[2x \tan \frac{x}{2} - 2 \int \tan \frac{x}{2} \right] + \int \tan \frac{x}{2} \, dx$$

$$= x \tan \frac{x}{2} + C.$$

17. $$\frac{\sin 2x}{\cos^4 x + \sin^4 x} = \frac{2 \sin x \cos x}{\cos^4 x + \sin^4 x} = \frac{2 \tan x \sec^2 x}{1 + \tan^4 x}$$

Now put $\tan^2 x = t$.

18. $f(x) = \displaystyle\int \frac{\log (\tan x)}{\sin x \cos x} = \frac{1}{2} (\log \tan x)^2 + C$

$0 = f \left(\dfrac{\pi}{4}\right) = \dfrac{1}{2} (\log 1)^2 + C \Rightarrow C = 0$

19. $f(x) = \displaystyle\int \cos \sqrt{4 - x} \; dx + C$

$= - 2 \displaystyle\int t \cos t \, dt, t = \sqrt{4 - x}$

$$= -2\left(\sqrt{4-x}\sin\sqrt{4-x}+\cos\sqrt{4-x}\right)+C$$

$-20 = f(4) = -2 + C \Rightarrow C = -18.$

20. Let the two antiderivates be $e^x + C_1$ and $e^x + C_2$
At $x = 1$, $C_1 - C_2 = 2$
At $x = 100$ it is $e^{100} + C_1 - e^{100} - C_2$
$= C_1 - C_2 = 2.$

21. $$\frac{1}{\cos^6 x+\sin^6 x}=\frac{1}{1-3\sin^2 x\cos^2 x}$$

$$= \frac{\sec^4 x}{\sec^4 x-3\tan^2 x}=\frac{(1+\tan^2 x)\sec^2 x}{(1+\tan^2 x)^2-3\tan^2 x}$$

Putting $\tan x = t$, the given integral reduces to

$$\int\frac{1+t^2}{1+t^4-t^2}\,dt=\int\frac{1+1/t^2}{t^2+1/t^2-1}\,dt$$

$$= \int\frac{du}{u^2+1},\ u = t - 1/t$$

$$= \tan^{-1}(\tan x - \cot x) + C$$

$$= \cot^{-1}\left(-\frac{1}{2}\tan 2x\right) + C$$

$$= \tan^{-1}\left(\frac{1}{2}\tan 2x\right) + \text{Const.}$$

22. $$\frac{\sin^2 x}{1+\sin^2 x}=1-\frac{1}{1+\sin^2 x}=1-\frac{\sec^2 x}{1+2\tan^2 x}$$

So $$\int\frac{\sin^2 x}{1+\sin^2 x} = x - \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C.$$

23. $$\int\tan^4 x\,dx=\int\tan^2 x(\sec^2 x-1)dx$$

$$= \frac{\tan^3 x}{3}-\int(\sec^2 x-1)\,dx$$

$$= \frac{\tan^2 x}{3} - \tan x + x + C.$$

24. Integrating by parts, the given integral equals

$$x\log\left(\sqrt{1-x}+\sqrt{1+x}\right)-I$$

where $$I = \int x\frac{d}{dx}\left(\log\left(\sqrt{1-x}+\sqrt{1+x}\right)\right)dx$$

$$= \frac{1}{2}\int\frac{x}{\sqrt{1-x^2}}\left(\frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\right)dx$$

$$= \frac{1}{2}\int\frac{x}{\sqrt{1-x^2}}\left(\frac{2-2\sqrt{1-x^2}}{-2x}\right)dx \text{ (Rationalizing)}$$

$$= -\frac{1}{2}\int\frac{dx}{\sqrt{1-x^2}}+\frac{1}{2}\int dx=-\frac{1}{2}\sin^{-1}x+\frac{1}{2}x+C$$

25. The given integral is equal to

$$\int x\log(x+1)dx-\int x\log x\,dx$$

$$= \frac{x^2}{2}\log(x+1)-\frac{1}{2}\int\frac{x^2}{x+1}\,dx$$

$$-\frac{x^2}{2}\log x+\frac{1}{2}\int\frac{x^2}{x}\,dx$$

$$= \frac{x^2}{2}\log(x+1)-\frac{x^2}{2}\log x$$

$$-\frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)dx+\frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\log(x+1)-\frac{x^2}{2}\log x+\frac{x}{2}-\frac{1}{2}\log(x+1)+C$$

$$\Rightarrow f(x) = \frac{x^2}{2}-\frac{1}{2},\ g(x) = -\frac{1}{2}\log x,\ L = \frac{1}{2}.$$

26. The integrand is equal $\dfrac{1}{3}\left(\dfrac{1}{x^2+1}-\dfrac{1}{x^2+4}\right)$

The integral can now be easily evaluated to give

$$\frac{1}{3}\tan^{-1}x-\frac{1}{6}\tan^{-1}\frac{x}{2}+C.$$

27. The integrand can be written as

$$\text{cosec}^2 x+\frac{1-\cos 2x}{2}-2.$$

So the integral is $-\cot x - \dfrac{1}{4}\sin^2 x-\dfrac{3}{2}+C.$

28. Putting $1 + e^x = t^2$. The given integral reduces to

$$2\int\log(t^2-1)\,dt$$

$$= 2\left[t\log(t^2-1)-2\int\left(1+\frac{1}{t^2-1}\right)dt\right]$$

$$= 2\left[t\log(t^2-1)-2\left(t+\frac{1}{2}\log\frac{t-1}{t+1}\right)\right]+C$$

$$= 2x\sqrt{1+e^x}-4\sqrt{1+e^x}-2\log\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}+C$$

So $$f(x) = 2(x-2),\ g(x) = \frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}.$$

29. Put $t = 1 + \frac{1}{x}$, the given integral reduces to

$$-\int\frac{\log t}{t} = -\frac{1}{2}(\log t)^2 + C$$

$$= -\frac{1}{2}[\log(x+1)]^2 - \frac{1}{2}(\log x)^2$$

$$+ \log(x+1)\log x + C.$$

30. The given integral is equal to $\frac{1}{2}$ [log | tan x | + C

So $f(x) = \log x$ and $g(x) = \tan x$, dom $f = (0, \infty)$.

31. Differentiating both sides, we get

$$f(x) = \frac{\sin\sqrt{x}}{\sqrt{x}}.$$

32. Putting $t = \tan\frac{x}{2}$, the given integral reduces to

$$\int\frac{dt}{3t^2+2t+2} = \frac{1}{3}\int\frac{dt}{(t+1/3)^2+(\sqrt{5}/3)^2}$$

$$= \frac{1}{\sqrt{5}}\tan^{-1}\frac{3\tan x/2+1}{\sqrt{5}} + C.$$

33. The integrand can be written as $\frac{\sec^2 x}{\tan^2 x(2+\tan^2 x)}$

putting $\tan x = t$, the integral reduces to

$$\int\frac{dt}{t^2(2+t^2)} = \frac{1}{2}\int\left(\frac{1}{t^2}-\frac{1}{2+t^2}\right)dt$$

$$= \frac{1}{2}\left[-\frac{1}{t}-\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}\right] + C$$

$$= \left(-\frac{1}{2}\right)\left[\cot x+\left(\frac{1}{\sqrt{2}}\right)\tan^{-1}\left(\left(\frac{1}{\sqrt{2}}\right)\tan x\right)\right]+C$$

34. Put $x^{1/2} = t$, the given integral reduces to

$$2\int\frac{t\,dt}{\sqrt{t^2+t^3}} = 2\int\frac{dt}{\sqrt{t+1}} = 4\sqrt{1+x^{1/2}} + C$$

35. Putting $t = e^x$, the integral reduces to

$$\int\frac{t^2+1}{t^4-t^2+1} = \int\frac{1+1/t^2}{(t-1/t)^2+1}\,dt$$

$$= \int\frac{du}{u^2+1}\quad(u = t - 1/t)$$

$$= \tan^{-1}(e^x - e^{-x}) + C$$

36. The given integral is equal to

$$\int e^{\sec x}\sec x(\sec x\tan x + \tan^2 x + \sec x + \tan x)dx$$

$$= \int e^{\sec x}\sec x\tan x(\sec x + \tan x) + e^{\sec x}(\sec^2 x + \sec x\tan x)\,dx$$

$$= \int\frac{d}{dx}(e^{\sec x}(\sec x + \tan x))dx$$

$$= e^{\sec x}(\sec x + \tan x) + C.$$

37. Writing $3\cos x + 2\sin x = A(4\sin x + 5\cos x) + B(4\cos x - 5\sin x)$ and comparing the coefficients we obtain $A = \frac{23}{41}$ and $B = \frac{2}{41}$. Thus the given integral reduces to

$$\frac{23}{41}\int dx + \frac{2}{41}\int\frac{4\cos x - 5\sin x}{4\sin x + 5\cos x}dx$$

$$= \frac{23}{41}x + \frac{2}{41}\log|4\sin x + 5\cos x| + C.$$

38. Dividing numerator by $\cos^2 x$, the given integral reduces to

$$\int\frac{\sec^2 x\,dx}{4(1+\tan^2 x) - 3 + 5\tan^2 x} = \int\frac{dt}{9t^2+1}$$

$$= \frac{1}{9}\int\frac{dt}{t^2+(1/3)^2} = \frac{1}{3}\tan^{-1}(3t) + C.$$

39. $f\,o\,g(x) = \frac{\sin^2 x}{1+\sin^2 x}$, put $\sin x = t$, the given integral reduces to $\int\frac{t^2}{1+t^2}\,dt$

$$= \int 1\,dt - \int\frac{dt}{1+t^2} = \sin x - \tan^{-1}(\sin x) + C$$

40. $P(x) + Q(x) = \int\frac{x^3+1}{x^3-x^2}\,dx$

$$= \int\left(1+\frac{2}{x-1}-\frac{1}{x}-\frac{1}{x^2}\right)dx$$

$$= x + 2\log|x-1| - \log|x| + \frac{1}{x} + C$$

Since

$(P + Q)(2) = \frac{5}{2}$ so $C = \log 2$.

Hence

$(P + Q)(3) = \frac{10}{3} + \log\frac{8}{3}$.

41. $$\frac{x^4-1}{x^2\left(1+x^2+x^4\right)^{1/2}} = \frac{\left(x^2-1\right)\left(x^2+1\right)}{x^3\left(x^2+1+1/x^2\right)^{1/2}}$$

$$= \frac{(x+1/x)\left(1-1/x^2\right)}{\left((x+1/x)^2-1\right)^{1/2}}$$

$$\int \frac{x^4 - 1}{x^2\left(1 + x^2 + x^4\right)^{1/2}}\, dx = \int \frac{u}{(u^2 - 1)^{1/2}}\, du,$$

$$u = x + 1/x$$

$$= \sqrt{u^2 - 1} + C$$

$$= \sqrt{x^2 + \frac{1}{x^2} + 1} + C = \sqrt{\frac{x^4 + x^2 + 1}{x}} + C.$$

42. $\int \frac{1}{x} \log_{ex} e\, dx = \int \frac{1}{x} \frac{\log_c e}{\log_e ex}\, dx$

$$= \int \frac{1}{x} \frac{1}{1 + \log_e x}\, dx = \log_e\left(1 + \log_e x\right) + C.$$

43. The given integral is equal to $I_1 + I_2$ where

$$I_1 = \int \frac{x}{\sqrt{1 - 9x^2}}\, dx = -\frac{1}{18} \int \frac{dt}{\sqrt{t}} \left(t = 1 - 9x^2\right)$$

$$= \frac{-1}{9} \sqrt{1 - 9x^2} + C$$

$$I_2 = \int \frac{\left(\cos^{-1} 3x\right)^2}{\sqrt{1 - 9x^2}}\, dx = \frac{-1}{3} \int t^2\, dt, t = \cos^{-1} 3x$$

$$= -\frac{\left(\cos^{-1} 3x\right)^3}{9}.$$

44. Put $\log\left(x + \sqrt{1 + x^2}\right) = t,\ dt = \frac{dx}{\sqrt{1 + x^2}}$

45. Put $\sqrt{x} = \sec \theta$, given integral reduces to

$$\int \theta\, 2\sec^2 \theta \tan \theta\, d\theta = \int \theta \frac{d}{d\theta}\left(\sec^2 \theta\right) d\theta$$

$$= \theta \sec^2 \theta - \int \sec^2 \theta\, d\theta$$

$$= \theta \sec^2 \theta - \tan \theta + C$$

46. Put $\sqrt{x} = t$ in I and $e^{-x} = t$ in J.

47. Put $x = \sin^2 u$.

48. Put $x e^x = t$.

49. $F(x) = C - \frac{1}{4} \log\left|\tan \frac{x}{2}\right| + \frac{1}{8 \sin^2 \frac{x}{2}}$

(Put $\tan x/2 = t$)

50. $\int x e^x \left(x^2 + 1\right) dx = (1/2) \int e^t (t + 1)\, dt \quad (t = x^2)$

$$= (1/2)\ e^t.t + c = e^{x^2} . \frac{x^2}{2} + c.$$

Level 2

51. We have

$$f'(x) = (x^3 - 2x^2 + 3)\ e^{3x}$$

$$= (x + 1)\ (x^2 - 3x + 3)\ e^{3x}$$

As e^{3x} and $x^2 - 3x + 3 > 0\ \forall\ x \in \mathbf{R}$

We get $f'(x) = 0 \Rightarrow x = -1$.

Thus, number of critical points of f is 1.

52. We have

$$f'(x) = \frac{3x + 2}{x^4 - x^3 + x^2 - 1} = \frac{3x + 2}{(x - 1)(x^3 + x + 1)}$$

Note that $x^3 + x + 1 = 0$ has exactly one real root, say α.

$\therefore f'(x)$ does not exist for $x = 1$ and $x = \alpha$.

Also, $f'(x) = 0$ for $x = -2/3$.

Thus, f has 3 critical points.

53. $f'(x) = \frac{x - 2}{x^2 - 7x + 12} = \frac{x - 2}{(x - 3)(x - 4)}$

Note that

$$f'(x) < 0 \text{ for } x < 2$$
$$> 0 \text{ for } 2 < x < 3$$
$$< 0 \text{ for } 3 < x < 4$$
$$> 0 \text{ for } x > 4$$

Thus, $f(x)$ increases on $(2, 3) \cup (4, \infty)$.

54. Let $I = \int e^{\sin x} \frac{(x \cos^3 x - \sin x)}{1 - \sin^2 x}\, dx$

$$= \int e^{\sin x} (x \cos x - \tan x \sec x)\, dx.$$

$$= \int x e^{\sin x} \cos x\, dx - \int e^{\sin x} \tan x \sec x\, dx$$

$$= x e^{\sin x} - \int e^{\sin x}\, dx$$

$$- e^{\sin x} \sec x + \int e^{\sin x} \cos x \sec x\, dx$$

$$= e^{\sin x} (x - \sec x) + C$$

55. Let $I = \int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}}\, dx$

$$= \int \frac{4e^{2x} + 6}{9e^{2x} - 4}\, dx$$

Put $e^{2x} = t$, so that

$$I = \int \frac{4t + 6}{9t - 4} \cdot \frac{1}{2t}\, dt$$

$$= \int\left[\frac{-3}{4}\frac{1}{t} + \frac{35}{4}\frac{1}{9t-4}\right]dt$$

$$= \frac{-3}{4}\ln|t| + \frac{35}{36}\ln|9t-4| + C$$

$$= \frac{-3}{2}x + \frac{35}{36}\ln\left(9e^{2x}-4\right) + C$$

$\therefore\ A = \frac{-3}{2},\ B = \frac{35}{36}.$

Alternatively differentiate both the sides w.r.t. x, we get

$$\frac{4\,e^{x}+6\,e^{-x}}{9\,e^{x}-4\,e^{-x}} = A + \frac{18\,Be^{2x}}{9\,e^{2x}-4}$$

$\Rightarrow 4e^{x} + 6e^{-x} = A\ (9e^{x} - 4^{e-x}) + 18\ Be^{x}$

Equating the coefficients of e^{x} and $e,^{-x}$

We get $4 = 9A + 18\ B$ and $6 = -4A$

$\Rightarrow A = \frac{-3}{2},\ B = \frac{35}{36}$

56. First show that $f(x) = e^{x}$

Let $I = \int\frac{x\,e^{x}}{\sqrt{1+e^{x}}}dx$

$$= 2x\sqrt{1+e^{x}} - 2I_1$$

where $I_1 = \int\sqrt{1+e^{x}}\,dx$

Put $1 + e^{x} = t^{2}$, so that

$$I_1 = \int\frac{t\,2t}{t^{2}-1}dt = 2\int\left[1+\frac{1}{t^{2}-1}\right]dt$$

$$= 2\left[t+\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|\right]$$

$$= 2\sqrt{1+e^{x}} + \log\left|\frac{\sqrt{1+e^{x}}-1}{\sqrt{1+e^{x}}+1}\right|$$

Thus,

$$I = 2x\sqrt{1+e^{x}} - 4\sqrt{1+e^{x}} - 2\log\left|\frac{\sqrt{1+e^{x}-1}}{\sqrt{1+e^{x}+1}}\right| + C$$

57. Put $1 + x^{4} = t$, so that

$$\int\frac{x^{7}}{\left(1+x^{4}\right)^{2}}dx = \frac{1}{4}\int\frac{(t-1)}{t^{2}}dt$$

$$= \frac{1}{4}\ \log|t| + \frac{1}{4}\frac{1}{t} + C$$

$$= \frac{1}{4}\left[\log(1+x^{4}) + \frac{1}{1+x^{4}}\right] + C$$

$$\therefore\ f(x) = \frac{1}{1+x^{4}}$$

58. We have

$$f(x) = \int\frac{\cos^{4}x+\sin^{4}x}{\cos^{2}x-\sin^{2}x}dx$$

Using, $2(\cos^{4}x + \sin^{4}x)$

$= (\cos^{2}x + \sin^{2}x)^{2} + (\cos^{2}x - \sin^{2}x)^{2}$, we can write

$$f(x) = \frac{1}{2}\int\frac{1}{\cos^{2}x-\sin^{2}x}dx + \frac{1}{2}\int\left(\cos^{2}x-\sin^{2}x\right)dx$$

$$= \frac{1}{2}\int\sec 2x\,dx + \frac{1}{2}\int\cos 2x\,dx$$

$$= \frac{1}{4}\log\left|\tan\left(x+\frac{\pi}{4}\right)\right| + \frac{1}{4}\ \sin 2\,x + C$$

$$= \frac{1}{4}\log\left|\frac{1+\tan x}{1-\tan x}\right| + \frac{1}{2}\sin x\ \cos x + C$$

Since $f(0) = 0$, we get $C = 0$.

59. Integrating by parts repeatedly, we get

$\int P(x)\ v\ (x)dx = P(x)\ v_1\ (x) - \int P'(x)\ v_1\ (x)dx$

$= P(x)\ v_1\ (x) - P'(x)\ v_2\ (x) + \int P''(x)v_2\ (x)\ dx$

$= P(x)\ v_1\ (x) - P'(x)\ v_2\ (x) + P''\ (x)\ v_3\ (x) - \ldots.$

$+ (-1)^{n}\int P^{(n)}\ (x)\ v_n\ (x)\ dx$

Now, use $P^{(n)}\ (x) = n!$

60. Differentiating both the sides w · r · t · x, and simplifying, we get

$$\frac{x+1}{(x^{2}+x+1)^{3/2}} = \frac{3k}{2}\frac{x+1}{(x^{2}+x+1)^{3/2}} \Rightarrow k = \frac{2}{3}.$$

61. First show that $f(x) = 2x$. Now,

$$I = \int\frac{f\ (x)\tan^{-1}x}{(1+x^{2})^{2}}dx = \int\frac{2x}{(1+x^{2})^{2}}\ \tan^{-1}x\ dx$$

Put $x = \tan\ \theta$, so that

$$I = \int\frac{2\tan\theta}{\sec^{4}\theta}\theta\sec^{2}\theta\,d\theta$$

$$= \int\theta\sin(2\theta)\,d\theta$$

$$= -\frac{1}{2}\theta\cos 2\theta + \frac{1}{4}\ \sin 2\theta + C$$

$$= C - \frac{1}{2}\theta\frac{1-\tan^{2}\theta}{1+\tan^{2}\theta} + \frac{1}{2}\frac{\tan\theta}{1+\tan^{2}\theta}$$

$$= C - \frac{1}{2}\frac{1-x^{2}}{1+x^{2}}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^{2}}$$

$$= C - \frac{1}{1+x^2}\tan^{-1} x + \frac{1}{2}\tan^{-1} x + \frac{1}{2}\frac{x}{1+x^2}.$$

62. $I = \int \frac{dx}{4\sqrt{(x-1)^3\,(x+2)^5}}$

$$= \int \left(\frac{x-1}{x+2}\right)^{-3/4} \frac{1}{(x+1)^2}\,dx$$

Now, put $\frac{x-1}{x+1} = t$, to obtain

$$I = \frac{1}{3}\int t^{-3/4}\,dt = \frac{4}{3}t^{1/4} + C\cdot$$

63. We can write

$$I = \int \frac{-2\sin^2 2x\cos x\sin x}{\cos^2 x - \sin^2 x}\,dx.$$

$$= -\int \frac{\sin^3 2x}{\cos 2x}\,dx = -\int \frac{\sin^3 2x}{1-\sin^2 2x}\cos 2x\,dx$$

Put $\sin 2x = t$, to obtain

$$I = -\frac{1}{2}\int \frac{t^3}{1-t^2}\,dt = \frac{1}{2}\int \frac{t(1-t^2)-t}{1-t^2}\,dt$$

$$= \frac{1}{4}t^2 + \frac{1}{4}\log|1-t^2| + C$$

$$= \frac{1}{4}\sin^2 2x + \frac{1}{2}\log|\cos 2x| + C.$$

64. $f'(x) = \lim_{n\to\infty} \frac{x^n - x^{-n}}{x^n + x^{-n}} = \lim_{n\to\infty} \frac{1-(1/x)^{2n}}{1+(1/x)^{2n}} = 1$

$(1/x < 1)$

Using integration by parts, we get

$$I = \int \frac{x\log\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\,dx$$

$$= \sqrt{1+x^2}\log\left(x+\sqrt{1+x^2}\right)dx - \int dx$$

$$= \sqrt{1+x^2}\log\left(x+\sqrt{1+x^2}\right) - x + C.$$

65. Put $x^{1/3} = \theta$ so that $dx = 3\theta^2 d\theta$

$$\therefore I = 3\int \theta^2 \sin\theta\,d\theta$$

$$= -3\theta^2\cos\theta + 6\int \theta\cos\theta\,d\theta$$

$$= -3\theta^2\cos\theta + 6\theta\sin\theta - 6\int \sin\theta\,d\theta$$

$$= 3(2-\theta^2)\cos\theta + 6\theta\sin\theta + C.$$

66. First show that $f(x) = 2e^x$. Now,

$$I = \int \frac{2e^x}{3+8e^x}\,dx$$

$$= \frac{1}{4}\log(3+8e^x) + C.$$

67. Differentiating both the sides w.r.t. x, we get

$$\frac{f(x)}{x^2-x+1} = \frac{3}{2}\frac{2x-1}{x^2-x+1} + \frac{1}{\sqrt{3}}\frac{1}{1+(2x-1)^{2/3}}\left(\frac{2}{\sqrt{3}}\right)$$

$$= \frac{3}{2}\frac{2x-1}{x^2-x+1} + \frac{1}{2(x^2-x+1)}$$

$\Rightarrow f(x) = 3x - 1.$

68. Let $I = \int \frac{dx}{x^2\sqrt{1+x^2}}$

Put $x = 1/t$, to obtain

$$I = -\int \frac{t\,d\,t}{\sqrt{1+t^2}} = -\sqrt{1+t^2} + C$$

$$= C - \frac{\sqrt{1+x^2}}{x}$$

Thus, $f(x) = 1 + x^2$.

69. Write the given integral as

$$I = \frac{1}{\sqrt{2}}\int \frac{dx}{\cos^4 x\sqrt{\tan x}}$$

Put $\tan x = t$ to obtain

$$I = \frac{1}{\sqrt{2}}\int \frac{1+t^2}{\sqrt{t}}\,dt = \frac{1}{\sqrt{2}}\int [t^{-1/2} + t^{3/2}]\,dt$$

$$= \frac{1}{\sqrt{2}}\left[2\sqrt{t} + \frac{2}{5}t^{5/2}\right] + C$$

$$= \sqrt{2}\left[\sqrt{\tan x} + \frac{1}{5}(\tan x)^{5/2}\right] + C.$$

Thus, I is an irrational function of $\tan x$.

70. $I = \int \sqrt{e^x - 1}\,dx$

Put $e^x - 1 = t^2$, so that $e^x dx = 2t\,dt$

$$\therefore I = \int \frac{t\,2t}{t^2+1}\,dt = 2\int \frac{t^2+1-1}{t^2+1}\,dt$$

$$= 2[t - \tan^{-1} t] + C$$

$$= 2\left[\sqrt{e^x-1} - \tan^{-1}\sqrt{e^x-1}\right] + C$$

$= 2\,[(\text{fog})\,(x) - (\text{hofog})\,(x)\,] + C.$

71. Put $2x + 1 = t$, so that $f(x) = \sqrt{t^2-4}$, then

$$I = \frac{1}{4}\int \frac{t+5}{\sqrt{t^2-4}}\,dt$$

$$= \frac{1}{4}\sqrt{t^2-4} + \frac{5}{4}\log\left|t+\sqrt{t^2-4}\right| + C$$

$$= \frac{1}{4}f\,(x) + \frac{5}{4}\log\,|2x + 1 + f(x)| + C.$$

72. $I = \displaystyle\int \frac{\sin x\,dx}{(1-\cos^2 x)(2\cos^2 x - 1)}$

Put $\cos x = t$, so that

$$I = \int \frac{dt}{(1-t^2)(1-2t^2)}$$

$$= \int\left[\frac{2}{1-2t^2} - \frac{1}{1-t^2}\right]dt$$

$$= \frac{2}{\sqrt{2}}\ \log\left|\frac{1+\sqrt{2}t}{1-\sqrt{2}t}\right| - \log\ \left|\frac{1+t}{1-t}\right| + C$$

$= \sqrt{2}\,(\text{hof})\,(x) + (\text{hog})\,(x).$

73. $I = \displaystyle\int \frac{\sqrt{x^2+4}}{x^6}\,dx = \int \frac{\sqrt{1+4/x^2}}{x^2}\,\frac{dx}{x^3}$

Put $\dfrac{1}{x^2} = t$, so that

$$I = \int t\sqrt{1+4t}\,\frac{(-1)}{2}\,dt$$

$$= -\frac{1}{8}\int (4t+1-1)\sqrt{1+4t}\ dt$$

$$= -\frac{1}{8}\left[(4t+1)^{5/2}\left(\frac{2}{5}\right)\left(\frac{1}{4}\right) - \left(\frac{2}{3}\right)\left(\frac{1}{4}\right)(1+4t)^{3/2}\right] + C$$

$$= \frac{1}{48}\left(1+\frac{4}{x^2}\right)^{3/2} - \frac{1}{80}\left(1+\frac{4}{x^2}\right)^{5/2} + C$$

$$= \frac{1}{240x^5}\ [5x^2\,(f(x))^3 - 3(f(x))^5] + C.$$

74. $I = \displaystyle\int \frac{\log\cos x}{\cos^2 x}\,dx\ = \int \sec^2 x\log\cos x\,dx$

$$= (\tan x)\,\log\,\cos\,x - \int \tan x\left(\frac{-\sin x}{\cos x}\right)dx$$

$$= (\tan x)\,\log\,\cos\,x + \int (\sec^2 - 1)\,dx$$

$$= (\tan x)\,[\log\,\cos\,x + 1] - x + C$$

$$= \frac{g\,(x)}{f(x)}\ [\log f(x) + 1] - x + C.$$

75. $I = \displaystyle\int (2\cos^2 x - 1)\,(4\cos^3 x - 3\cos x)\,dx$

$$= \int \cos 2x\cos 3x\,dx$$

$$= \frac{1}{2}\int [\cos 5x + \cos x]\,dx$$

$$= \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C\,.$$

76. $I_n + I_{n+2} = \displaystyle\int \cot^n x\,\text{cosec}^2 x\,dx = -\frac{1}{n+1}\cot^{n+1} x$

Now,

$I_0 + I_1 + 2\,(I_2 + I_3 + \cdots + I_8) + I_9 + I_{10}$

$= (I_0 + I_2) + (I_1 + I_3) + (I_2 + I_3)$

$+ \cdots + (I_7 + I_9) + (I_8 + I_{10})$

$$= -\left(u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \cdots + \frac{1}{9}u^9\right)$$

where $u = \cot x$.

Thus, $A = -1$

77. $f(x) = \displaystyle\int (1+\cot^2 x)\,\text{cosec}^2 x\,dx$

$$= -\left(\cot x + \frac{1}{3}\cot^3 x\right) + C$$

which is a polynomial of degree 3.

78. $I = \displaystyle\int \frac{x^2}{\sqrt{x^2-a^2}}\,dx$

$$= \int \sqrt{x^2-a^2}\ dx + a^2\int \frac{dx}{\sqrt{x^2-a^2}}$$

$$= \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\,\log\,\left|x+\sqrt{x^2-a^2}\right|$$

$$+\ a^2\log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$= \frac{1}{2}x\,f\,(x) + \frac{1}{2}a^2\log|x + f(x)| + C$$

79. We have

$$(\text{fof})(x)\ = f(f(x)) = \frac{f(x)}{\left[1+(f(x))^n\right]^{1/n}}$$

$$= \frac{x}{\left(1+2x^n\right)^{1/n}}$$

Similarly, $(\text{fofof})\,(x)\ = \dfrac{x}{\left(1+3x^n\right)^{1/n}}$

Continuing in this way we get

$$g(x) = \underbrace{(\text{fof}\cdots\text{of})(x)}_{n\text{ times}} = \frac{x}{(1+nx^n)^{1/n}}$$

Thus, $I = \int x^{n-2}\frac{x}{(1+nx^n)^{1/n}}\,dx$

Put $1 + nx^n = t^n$ to obtain

$$I = \frac{1}{n}\int \frac{t^{n-1}}{t}\,dt = \frac{1}{n}\int t^{n-2}\,dt$$

$$= \frac{1}{n(n-1)}t^{n-1}+C = \frac{1}{n\left(n-1\right)}\left(1+nx^n\right)^{1-1/n}+C$$

80. Let $x^5 = t$, so $\int x^9 f(x^5)\,dx$

$$= \frac{1}{5}\int t\,f(t)\,dt$$

$$= \frac{1}{5}\left[t\int f(t)\,dt - \int \phi(t)\,dt\right]$$

$$= \frac{1}{5}\left[x^5\,\phi(x^5) - 5\int \phi(x^5)\,x^4\,dx\right] + C$$

$$= \frac{1}{5}\,x^5\,\phi(x^5) - \int x^4\,\phi(x^5)\,dx + C$$

Previous Years' AIEEE/JEE Main Questions

1. $I = \int\frac{\sin x}{\sin(x-\alpha)}dx = \int\frac{\sin\{(x-\alpha)+\alpha\}}{\sin(x-\alpha)}dx$

$$= \int[\cos\alpha+\sin\alpha\cot(x-\alpha)]dx$$

$= x\cos\alpha + \sin\alpha\log|\sin(x-\alpha)|\ C$

$\therefore\ (A, B) = (\cos\alpha, \sin\alpha)$

2. $\int\frac{dx}{\cos x-\sin x} = \int\frac{dx}{\cos x+\cos\left(\frac{\pi}{2}+x\right)}$

$$= \frac{1}{2}\int\frac{dx}{\cos(\pi/4)\cos(x+\pi/4)}$$

$$= \frac{1}{\sqrt{2}}\int\sec\left(x+\frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}+\frac{\pi}{4}\right)\right|+C$$

$$= \frac{1}{\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{3\pi}{8}\right)\right|+C$$

3. Put $\log x = t$, so that $x = e^t$ and $dx = e^t dt$. Thus

$$I = \int\frac{(t-1)^2}{(1+t^2)^2}e^t\,dt = \int\frac{t^2+1-2t^2}{(1+t^2)^2}e^t\,dt$$

$$= \int\left[\frac{1}{1+t^2}-\frac{2t}{(1+t^2)^2}\right]e^t\,dt$$

$$= \int[f(t)+f'(t)]\,e^t\,dt \quad\text{where } f(t) = \frac{1}{1+t^2}$$

$$= f(t)\ e^t + C$$

$$= x\cdot\frac{1}{1+(\log x)^2} + C$$

4. $I = \int\frac{dx}{\cos x+\sqrt{3}\sin x}$

$$= \frac{1}{2}\int\frac{dx}{\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x}$$

$$= \frac{1}{2}\int\frac{dx}{\cos\left(x-\frac{\pi}{3}\right)}$$

$$= \frac{1}{2}\int\sec\left(x-\frac{\pi}{3}\right)dx$$

$$= \frac{1}{2}\log\left|\tan\left(\frac{x}{2}-\frac{\pi}{6}+\frac{\pi}{4}\right)\right|+C$$

$$= \frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right|+C$$

5. Same as Q1. Take $\alpha = \frac{\pi}{4}$, so $\int\frac{\sin x}{\sin\left(x-\frac{\pi}{4}\right)}dx$

$$= \cos\frac{\pi}{4}x+\sin\frac{\pi}{4}\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+\text{Const}$$

$$= \frac{1}{\sqrt{2}}\left[x+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|\right]+\text{Const}$$

6. Let $I = \int\frac{5\tan x}{\tan x-2}\,dx = \int\frac{5\sin x}{\sin x-2\cos x}\,dx$

Let $5\sin x = A(\sin x - 2\cos x) + B(\cos x + 2\sin x)$. Equating coefficients of $\sin x$ and $\cos x$, we get $5 = A + 2B$ and $0 = -2A + B \Rightarrow A = 1,\ B = 2$

$$\therefore\ I = \int\frac{\sin x-2\cos x+2(\cos x+2\sin x)}{\sin x-2\cos x}\,dx$$

$$= \int\left[1+\frac{2(\cos x+2\sin x)}{\sin x-2\cos x}\right]dx$$

$= x + 2\ \text{In}\ |\sin x - 2\cos x| + k$

$\therefore\ a = 2$

7. $\int\frac{x^2-x+1}{x^2+1}e^{\cot^{-1}x}dx = -\int(\cot^2 t-\cot t+1)e^t\,dt$

$(x = \cot t)$

$$= \int(\cot t-(\cot^2 t+1))\,e^t\,dt$$

$$= \int(\cot t-(\operatorname{cosec}^2 t)\,e^t\,dt$$

$= \cot t.e^t + \int \text{cosec}^2 t.e^t dt - \int \text{cosec}^2 t.e^t dt + C$

$= \cot t.\ e^t + C$

$= x.e^{\cot^{-1} x} + C$

8. Put $\sqrt{2-x^2} = t \Rightarrow \dfrac{-xdx}{\sqrt{2-x^2}} = dt.$

$$\int \frac{xdx}{2-x^2+\sqrt{2-x^2}} = \int \frac{xdx}{\sqrt{2-x^2}\left(1+\sqrt{2-x^2}\right)}$$

$= -\int \dfrac{dt}{1+t}$

$= -\log|1+t| + C$

$= -\log\left|1+\sqrt{2-x^2}\right| + C$

9. $\int \dfrac{x^6}{x+x^7}dx = \int \dfrac{1+x^6-1}{x(1+x^6)}dx = \int \dfrac{1}{x}dx - \int \dfrac{dx}{x+x^7}$

$= \log|x| - p(x) + C.$

10. Let $I = \int x^5 f(x^3)dx$

Put $x^3 = t$ to obtain $x^2\ dx = \dfrac{1}{3}dt$

$\therefore\ I = \dfrac{1}{3}\int tf(t)\,dt$

$= \dfrac{1}{3}t\int tf(t)\,dt - \dfrac{1}{3}\int(1)\left[\int f(t)dt\right]dt$

$= \dfrac{1}{3}t\psi(t) - \dfrac{1}{3}\int \psi(t)\,dt$

$= \dfrac{1}{3}x^3\psi(x^3) - \int x^2\psi(x^3)\,dx$

11. $\int\left(1+x-\dfrac{1}{x}\right)e^{x+\frac{1}{x}}dx = \int\left(e^{x+\frac{1}{x}} + x\left(1-\dfrac{1}{x^2}\right)e^{x+\frac{1}{x}}\right)dx$

$= \int \dfrac{d}{dx}\left(xe^{x+\frac{1}{x}}\right)dx$

$= xe^{x+\frac{1}{x}} + C$

12. $\sin^8 x - \cos^8 x$

$= (\sin^2 x - \cos^2 x)\ (\sin^2 x + \cos^2 x)\ (\sin^4 x + \cos^4 x)$

$= (-\cos 2x)\ (1)\ [(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x]$

$= (-\cos 2x)\ (1 - 2\sin^2 x \cos^2 x)$

$\therefore\ I = -\int \cos 2x dx = -\dfrac{1}{2}\sin 2x + C$

13. $I = \int x\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)dx$

$= 2\int x\tan^{-1} x dx$

$= x^2\tan^{-1} x - \int \dfrac{x^2}{1+x^2}dx$

$= x^2\tan^{-1} x - \int \dfrac{x^2+1-1}{1+x^2}dx$

$= -x + (x^2 + 1)\tan^{-1} x + C$

14. Dividing numerator and denominator by $\cos^6 x$, we have $I = \int \dfrac{\tan^2 x\sec^2 x}{(\tan^3 x+1)^2}\mathrm{d}x$

Put $1 + \tan^3 x = t$

$I = \dfrac{1}{3}\int\dfrac{\mathrm{d}t}{t^2} = -\dfrac{1}{3} + C$

$= -\dfrac{1}{3}\dfrac{1}{1+\tan^3 x} + C$

15. $I = \int \dfrac{(x^{5m-1}+2x^{4m-1})}{x^{6m}(1+x^{-m}+x^{-2m})^3}\mathrm{d}x$

$= \int \dfrac{(x^{-m-1}+2x^{-2m-1})}{(1+x^{-m}+x^{-2m})^3}\mathrm{d}x$

Put $1 + x^{-m} + x^{-2m} = \mathrm{t}$,

$\Rightarrow [(-m)\ x^{-m-1} + (-2m)\ x^{-2m-1}]\ \mathrm{d}x = \mathrm{d}t$

$I = -\dfrac{1}{m}\int\dfrac{\mathrm{d}t}{t^3} = \dfrac{1}{2mt^2} + C$

$= \dfrac{1}{2m}\dfrac{x^{4m}}{(x^{2m}+x^m+1)^2} + C$

16. $I = \int \dfrac{dx}{x^2(x^4+1)^{3/4}} = \int \dfrac{dx}{x^2x^3(1+x^{-4})^{3/4}}$

$= \int \dfrac{x^{-5}}{(1+x^{-4})^{3/4}}dx$

Put $1 + x^{-4} = t$, so that

$I = \int \dfrac{(-1/4)}{t^{3/4}}dt = t^{1/4} + C$

$= -\left(1+\dfrac{1}{x^4}\right)^{1/4} + C = -\left(\dfrac{x^4+1}{x^4}\right)^{1/4} + C$

17. $I = \int \dfrac{dx}{(x+1)^2\left(\dfrac{x-2}{x+1}\right)^{5/4}}$

Put $\dfrac{x-2}{x+2} = t \Rightarrow \dfrac{dt}{dx} = \dfrac{3}{(x+1)^2}$

$$\therefore\ I = \frac{1}{3}\int \frac{dt}{t^{5/4}} = \frac{1}{3}\int t^{-5/4}dt$$

$$= -\frac{4}{3}t^{-1/4} + C$$

$$= -\frac{4}{3}\left(\frac{x+1}{x-2}\right)^{1/4} + C$$

18. Put log $\left(t+\sqrt{1+t^2}\right) = v$, so that

$$\frac{1}{t+\sqrt{1+t^2}}\left(1+\frac{t}{\sqrt{1+t^2}}\right)dt = du.$$

$$\Rightarrow \frac{1}{\sqrt{1+t^2}}dt = du$$

Thus, $I = \int u\,du = \frac{1}{2}u^2 + C$

$$\therefore\ g(t) = u = \log\left(t+\sqrt{1+t^2}\right)$$

$$\Rightarrow g(2) = \log\left(2+\sqrt{5}\right).$$

19. $I = \displaystyle\int \frac{2x^{12}+5x^{9}}{x^{15}(1+x^{-2}+x^{-5})^3}dx$

$$= \int \frac{2x^{-3}+5x^{-9}}{(1+x^{-2}+x^{-5})^3}dx$$

Put $1 + x^{-2} + x^{-5} = t$

$\Rightarrow -(2x^{-3} + 5x^{-6})dx = dt$

Thus, $I = -\displaystyle\int \frac{dt}{t^3} = \frac{1}{2t^2}+C$

$$= \frac{1}{2(1+x^{-2}+x^{-5})^2}+C$$

$$= \frac{x^{10}}{2(x^5+x^3+1)^2}+C$$

20. $I = \displaystyle\int \frac{dx/\sqrt{x}}{(1+\sqrt{x})\sqrt{1-x}}$

Put $\sqrt{x} = t$, $\dfrac{1}{2\sqrt{x}} = dt$

$$\therefore\ I = \int \frac{2dt}{(1+t)\sqrt{1-t^2}}$$

Put $1 + t = 1/u \Rightarrow dt = (-1/u^2)\ du$

Thus, $I = \displaystyle\int \frac{2(-1/u^2)du}{(1/u)\sqrt{1-(1/u-1)^2}}$

$$= -2\int \frac{du}{\sqrt{2u-1}}$$

$$= -\frac{2}{2}(2u-1)^{1/2}(2) + C$$

$$= -2\left(\frac{2}{1+t}-1\right)^{1/2} + C$$

$$= -2\left(\frac{1-t}{1+t}\right)^{1/2} + C$$

$$= -2\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}} + C$$

21. By dividing numerator and denominator by $\cos^4 x$, we can write I as

$$I = \int \frac{\sec^4 x}{\sqrt{\dfrac{4\sin x\cos x}{\cos^2 x}}}dx = \frac{1}{2}\int \frac{(1+\tan^2 x)\sec^2 x}{\sqrt{\tan x}}$$

$$= \frac{1}{2}\int\left[(\tan x)^{-1/2}+(\tan x)^{3/2}\right]\sec^2 x dx$$

$$= \sqrt{\tan x}+\frac{1}{5}(\tan x)^{5/2}+k$$

$$\therefore\ A = \frac{1}{2},\ B = \frac{5}{2},\ C = \frac{1}{5}$$

Thus, $A + B + C = \dfrac{16}{5}$

Previous Years' B-Architecture Entrance Examination Questions

1. $\displaystyle\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx = \int e^x\left(\frac{1+x^2-2x}{(1+x^2)^2}\right)dx$

$$= \int e^x\left(\frac{1}{1+x^2}-\frac{2x}{(1+x^2)^2}\right)dx$$

$$= \frac{1}{1+x^2}e^x+\int e^x\frac{2x}{(1+x^2)^2}dx-\int e^x\frac{2x}{(1+x^2)^2}dx+C$$

$$= \frac{e^x}{1+x^2}+C$$

2. $f(x) = \displaystyle\prod_{l=1}^{10}\frac{1}{x-l} = \frac{A_1}{x-1}+\frac{A_2}{x-2}+\cdots+\frac{A_{10}}{x-10}$

where $A_i = \dfrac{1}{(i-1)\cdots(i-(i+1))\cdots(i-10)}$

$$= \frac{(-1)^i}{(i-1)!(10-i)!}$$

$$\int f(x)dx = \sum_{i=1}^{10}\int\frac{A_i}{x-i}dx = \sum_{i=1}^{10}A_i\log|x-i|$$

So statement 1 is true but not statement 2.

3. $\displaystyle\int\frac{2^{x+1}-5^{x-1}}{10^x}dx = \int\left(\left(\frac{2}{10}\right)^x\cdot 2-\left(\frac{5}{10}\right)^x\frac{1}{5}\right)dx$

$$= 2\int\left(\frac{1}{5}\right)^x dx-\frac{1}{5}\int\left(\frac{1}{2}\right)^x dx$$

$$= 2\frac{5^{-x}}{-\log 5}+\frac{1}{5}\frac{2^{-x}}{\log 2}+C$$

4. Put $x = \frac{a}{b}\sin\theta \Rightarrow dx = \frac{a}{b}\cos\theta\, d\theta$

$$\int\frac{dx}{(a^2-b^2x^2)^{3/2}} = \frac{a}{b}\int\frac{\cos\theta d\theta}{a^3\cos^3\theta} = \frac{1}{a^2b}\tan\theta+C$$

$$= \frac{1}{a^2b}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}+C$$

$$= \frac{1}{a^2b}\frac{\frac{bx}{a}}{\sqrt{1-\frac{b^2}{a^2}x^2}}+C = \frac{1}{a^2}\frac{x}{\sqrt{a^2-b^2x^2}}+C$$

5. $\displaystyle\int\frac{7x^{13}+5x^{15}}{(x^7+x^2+1)^3}dx$

$$= \int\frac{7x^{13}(x^7+x^2+1)-x^{14}(7x^6+2x)}{(x^7+x^2+1)^3}dx$$

$$= \int\frac{d}{dx}\left(\frac{x^{14}}{2(x^7+x^2+1)^2}\right)dx$$

$$= \frac{x^{14}}{2(x^7+x^2+1)^2}+C$$

6. $\displaystyle\int\frac{1-5\sin^2 x}{\cos^5 x\sin^2 x}dx = \int\frac{\operatorname{cosec}^2 x}{\cos^5 x}dx-5\int\frac{1}{\cos^5 x}dx$

$$= -\frac{\cot x}{\cos^5 x}+\int\cot x(-5\cos^{-6}x)(-\sin x)dx-5\int\frac{1}{\cos^5 x}dx$$

$$= -\frac{\cot x}{\cos^5 x}+5\int\frac{dx}{\cos^5 x}-5\int\frac{dx}{\cos^5 x}+C$$

$$= -\frac{\cot x}{\cos^5 x}+C$$

So $f(x) = -\cot x$

7. $\displaystyle\int\frac{1-x^9}{x(1+x^9)}dx = A\log|x| + B\log|1+x^9| + C$

Differentiating both the sides, we get

$$\frac{1-x^9}{x(1+x^9)} = \frac{A}{x}+\frac{9Bx^8}{1+x^9}$$

$\Rightarrow 1 - x^9 = A(1 + x^9) + 9Bx^9$

$\Rightarrow A = 1,\ -1 = A + 9B$

$\Rightarrow A = 1,\ B = -\frac{2}{9}$

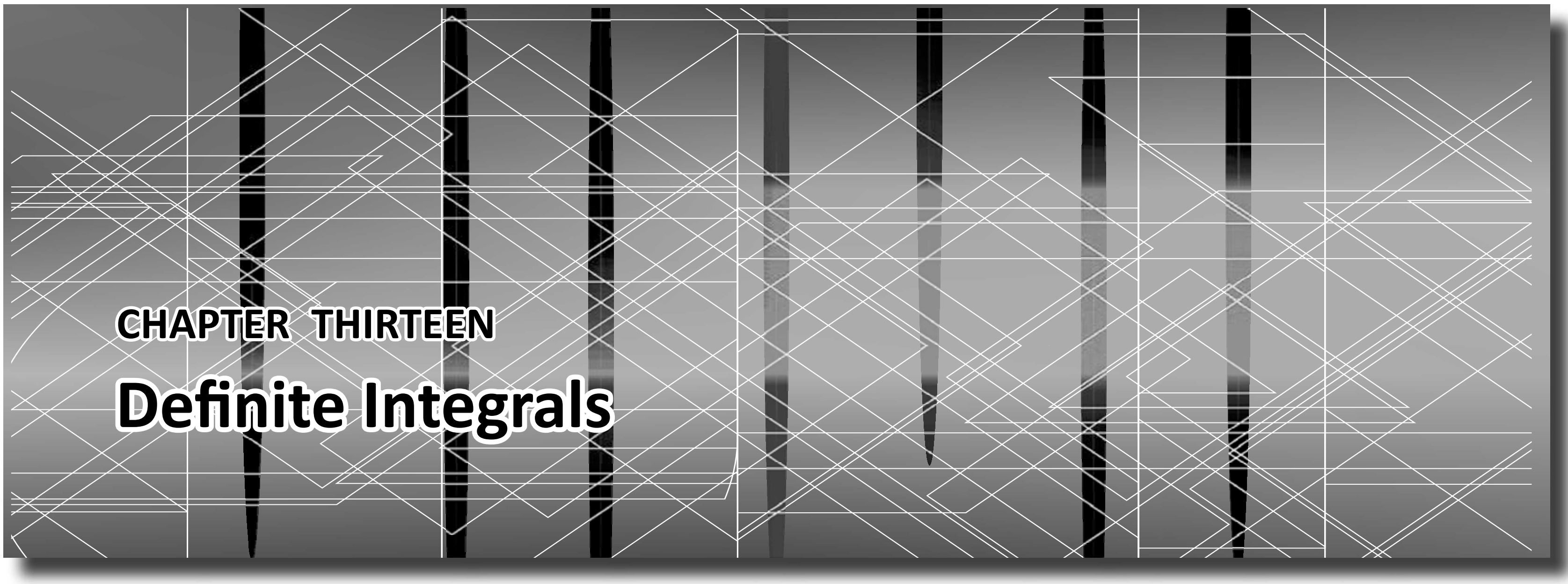
Thus, $A : B = 9 : -2$

8. $\displaystyle I = \int\frac{dx}{x^3(1+x^6)^{2/3}} = \int\frac{dx}{x^3\cdot(x^6)^{2/3}(x^{-6}+1)^{2/3}}$

$$= \int\frac{x^{-7}}{(x^{-6}+1)^{2/3}}dx$$

$= -\frac{1}{6}\cdot 3(x^{-6}+1)^{1/3}+C$ [Put $x^{-6}+1=t$]

$\therefore f(x) = -1/2$



CHAPTER THIRTEEN

Definite Integrals

THE NEWTON-LEIBNITZ FORMULA

If $F(x)$ is one of the antiderivatives of a continuous function $f(x)$ on $[a, b]$ i.e., $F'(x) = f(x)$ $(a < x < b)$, then we have the following formula due to Newton and Leibnitz: (FUNDAMENTAL THEOREM OF CALCULUS)

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_a^b = F(b) - F(a)$$

DEFINITE INTEGRAL AS THE LIMIT OF A SUM

Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$. Then $\int_a^b f(x)\ \mathrm{d}x = \lim\limits_{\substack{n\to\infty,\, h\to 0\\ nh = b-a}} h\sum\limits_{r=1}^{n} f(a+rh)$. In particular,

$$\lim_{n\to\infty}\sum_{r=1}^{r=n}\frac{1}{n}f\left(\frac{r}{n}\right) = \int_0^1 f\ (x)\ \mathrm{d}x$$

Illustration 1

$$\lim_{n\to\infty}\left[\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n}{n^2}\right]$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{r}{n} = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

Illustration 2

$$\lim_{n\to\infty}\left[\frac{1^2}{n^3}+\frac{2^2}{n^3}+\cdots+\frac{(n-1)^2}{n^3}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}\left(\frac{r}{n}\right)^2 = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

PROPERTIES OF DEFINITE INTEGRALS

1. $\int_a^a f(x)\,dx = 0$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b f(u)\,du = \int_a^b f(t)\,dt$
4. If $f(x) \geq 0$ on the interval $[a, b]$, then $\int_a^b f(x)\ \mathrm{d}x \geq 0$. If $f(x) > 0$ for all points of $[a, b]$. Then $\int_a^b f(x)\,dx > 0$.
5. If $f(x) \leq g(x)$ on $[a, b]$, then $\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b f(x)\ \mathrm{d}x$.
6. $\int_a^b (f_1(x) + f_2(x))\ dx = \int_a^b f_1(x)\ dx + \int_a^b f_2(x)\,dx$ and $\int_a^b \alpha f(x)\ dx = \alpha \int_a^b f(x)\ dx$
7. $\left|\int_a^b f(x)\mathrm{d}x\right| \leq \int_a^b |f(x)|\ \mathrm{d}x$
8. $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$
9. $\int_0^a f(x)\ \mathrm{d}x = \int_0^a f(a-x)\mathrm{d}x$
10. $\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(a+b-x)\,\mathrm{d}x$
11. $\int_0^a f(x)\ \mathrm{d}x = \int_0^{a/2} f\ (x) + \int_0^{a/2} f\ (a-x)\mathrm{d}x$. In particular,

$$\int_0^a f(x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } f(a-x) = -f(x) \\ 2\int_0^{a/2} f(x)\,\mathrm{d}x & \text{if } f(a-x) = f(x) \end{cases}$$

12. If $f(-x) = f(x)$ (i.e., f is an even function), then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$$

13. If $f(-x) = -f(x)$ (i.e., f is an odd function), then $\int_{-a}^{a} f(x)\mathrm{d}x = 0.$

14. If f is continuous on $[a, b]$, then the integral function g defined by $g(x) = \int_{a}^{x} f(t)\,\mathrm{d}t$ for $x \in [a, b]$ is derivable on $[a, b]$, and $g'(x) = f(x)$ for all $x \in [a, b]$.

15. If m and M are the smallest and greatest values of a function $f(x)$ on an interval $[a, b]$, then

$$m(b-a) \le \int_{a}^{b} f(x)\,dx \le M(b-a).$$

16. If $f(x)$ is continuous on $[a, b]$, then there exists a point $c \in (a, b)$ such that $\int_{a}^{b} f(x)\,\mathrm{d}x = f(c)\,(b-a)$. The number $f(c) = \dfrac{1}{b-a}\displaystyle\int_{a}^{b} f(x)dx$ is called the mean value of the function $f(x)$ on the interval $[a, b]$. The above result is called the first mean value theorem for integrals.

17. If $f(x)$ is periodic with period T then $\int_{a}^{b} f(x)\,dx = \int_{a+nT}^{b+nT} f(x)\,dx$, where n is an integer. In particular,

$$\int_{0}^{nT} f(x)\,\mathrm{d}x = n\int_{0}^{T} f(x)\mathrm{d}x.$$

18. If the function $\varphi(x)$ and $\psi(x)$ are defined on $[a, b]$ and differentiable at every point $x \in (a, b)$, and $f(t)$ is continuous for $\varphi(a) \le t \le \varphi(b)$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{\varphi(x)}^{\psi(x)} f(t)\,\mathrm{d}t\right) = f(\psi(x))\,\frac{\mathrm{d}\psi}{\mathrm{d}x} - f(\varphi(x))\,\frac{\mathrm{d}\varphi}{\mathrm{d}x}.$$

19. *Change of variables* If the function $f(x)$ is continuous on $[a, b]$ and the function $x = \varphi(t)$ is continuously differentiable on the interval $[t_1, t_2]$ and $a = \varphi(t_1)$, $b = \varphi(t_2)$, then

$$\int_{a}^{b} f(x)\mathrm{d}x = \int_{t_1}^{t_2} f(\varphi(\mathrm{t}))\,\varphi'(t)\mathrm{d}t$$

20. Let a function $f(x, \alpha)$ be continuous for $a \le x \le b$ and $c \le \alpha \le d$. For any $\alpha \in [c, d]$, if $I(\alpha) = \int_{a}^{b} f(x, \alpha)\mathrm{d}x$, then $I'(\alpha) = \int_{a}^{b} f'(x, \alpha)\mathrm{d}x$, where $I'(\alpha)$ is the derivative of $I(\alpha)$ w.r.t. α, and $f'(x, \alpha)$ is the derivative of $f(x, \alpha)$ w.r.t. α, keeping x constant.

21. If $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$, then

$$\left|\int_{a}^{b} f(x)g(x)\,\mathrm{d}x\right| \le \left(\int_{a}^{b} f^2(x)\,\mathrm{d}x\right)^{1/2}\left(\int_{a}^{b} g^2(x)\,\mathrm{d}x\right)^{1/2}.$$

22. Let f be continuous and non-negative at all points of $[a, b]$ and $f(x_0) > 0$ at least at one point x_0 within this interval then $\int_{a}^{b} f(x)\,\mathrm{d}x > 0.$

Illustration 3

Evaluate $\int_{0}^{2}\left(\dfrac{x^2}{4} - 7x + 5\right)dx$

The given integral is equal to

$$\frac{1}{4}\int_{0}^{2} x^2 dx - 7\int_{0}^{2} x\,dx + 5\int_{0}^{2} dx$$

$$= \frac{1}{4}\frac{x^3}{3}\bigg|_{0}^{2} - 7\frac{x^2}{2}\bigg|_{0}^{2} + 5x\bigg|_{0}^{2}$$

$$= \frac{1}{4}\left[\frac{8}{3} - 0\right] - \frac{7}{4}[4-0] + 5[2-0]$$

$$= \frac{2}{3} - 14 + 10 = -\frac{10}{3}$$

Illustration 4

Find the average value of $f(x) = 4 - x^2$ on $[0, 3]$.

$$\text{average }(f) = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$$

$$= \frac{1}{3-0}\int_{0}^{3}(4-x^2)\,dx$$

$$= \frac{1}{3}4x - \frac{x^3}{3}\bigg|_{0}^{3} = \frac{1}{3}\left[12 - \frac{27}{3}\right] = 1$$

Illustration 5

The value of $\int_{0}^{1}\sin x^2 dx$ cannot be 2

For $x \in [0, 1]$, $0 \le \sin x^2 \le 1$, so

$$0 \le \int_{0}^{1}\sin x^2 dx \le \int_{0}^{1} 1.dx = 1.$$

INTEGRALS WITH INFINITE LIMITS

If a function $f(x)$ is continuous for $a \le x < \infty$, then by definition,

$$\int_a^{\infty} f(x)\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\mathrm{d}x \qquad (1)$$

If there exists a finite limit on the right-hand side of (1), then the improper integral is said to be convergent; otherwise it is divergent.

Geometrically, the improper integral (1) for $f(x) > 0$, is the area of the figure bounded by the graph of the function $y = f(x)$, the straight line $x = a$ and the x-axis. Similarly, we can define

$$\int_{-\infty}^{b} f(x)\,\mathrm{d}x = \lim_{a\to-\infty}\int_a^b f(x)\mathrm{d}x$$

and $$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{a} f(x)\mathrm{d}x + \int_a^{\infty} f(x)\mathrm{d}x$$

REDUCTION FORMULAE FOR $\int_0^{\pi/2}\sin^n x\,dx$ and $\int_0^{\pi/2}\sin^p x\cos^q x\,dx$

$$\int_0^{\pi/2}\cos^n x\mathrm{d}x = \int_0^{\pi/2}\sin^n x\,\mathrm{d}x$$

$$= \begin{cases} \dfrac{n-1}{n}\times\dfrac{n-3}{n-2}\times\cdots\times\dfrac{1}{2}\cdot\dfrac{\pi}{2} & \text{if } n \text{ is even} \\[2ex] \dfrac{n-1}{n}\times\dfrac{n-3}{n-2}\times\cdots\dfrac{2}{3} & \text{if } n \text{ is odd} \end{cases}$$

$$I_{p,q} = \int_0^{\pi/2}\sin^p x\cos^q x\,dx$$

$$= \begin{cases} \dfrac{p-1}{p+q}\times\dfrac{p-3}{p+q-2}\times\cdots\times\dfrac{2}{3+q}\cdot I_{1,q} & \text{if } p \text{ is odd} \\[2ex] \dfrac{q-1}{p+q}\times\dfrac{p-3}{p+q-2}\times\cdots\dfrac{1}{2+q}I_{0,q} & \text{if } p \text{ is even} \end{cases}$$

where $I_{1,q} = \dfrac{1}{q+1}$ and

$$I_{0,q} = \begin{cases} \dfrac{q-1}{q}\,\dfrac{q-3}{q-2}\cdots\dfrac{2}{3} & \text{if } q \text{ is odd} \\[2ex] \dfrac{q-1}{q}\cdot\dfrac{q-3}{q-2}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2} & \text{if } q \text{ is even} \end{cases}$$

Note that if p is odd then it does not matter whether q is even or odd.

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: The value of $\displaystyle\int_0^1 \frac{x\,\mathrm{d}x}{(x^2+1)^2}$ is

(a) $\frac{1}{2}$ (b) 1

(c) $\frac{1}{3}$ (d) $\frac{1}{4}$

Ans. (d)

Solution: Substituting $x^2 + 1 = t$. When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

$$\int_0^1 \frac{x\,\mathrm{d}x}{(x^2+1)^2} = \frac{1}{2}\int_1^2 \frac{dt}{t^2} = -\frac{1}{2}\frac{1}{t}\bigg|_1^2 = -\frac{1}{2}\left[\frac{1}{2}-1\right] = \frac{1}{4}$$

Example 2: The value of $\displaystyle\int_{1/\pi}^{2/\pi}\frac{\sin 1/x}{x^2}\mathrm{d}x$ is equal

(a) 1 (b) $\frac{1}{2}$

(c) -1 (d) $\frac{1}{4}$

Ans. (a)

Solution: Substitute $\dfrac{1}{x} = t \Rightarrow -\dfrac{1}{x^2}\,dx = dt$

When $x = \dfrac{1}{\pi}$, $t = \pi$ and when $x = \dfrac{2}{\pi}$, $t = \dfrac{\pi}{2}$. Thus

$$\int_{1/\pi}^{2/\pi}\frac{\sin 1/x}{x^2}\mathrm{d}x = -\int_{\pi}^{\pi/2}\sin t\,\mathrm{d}t = \cos t\bigg|_{\pi}^{\pi/2} = -(-1) = 1$$

Example 3: If $\displaystyle\int_0^1 \frac{\sqrt{e^x}}{\sqrt{e^x+e^{-x}}}\mathrm{d}x = \log\frac{e+A}{1+\sqrt{2}}$ then A is equal to

(a) $e^2 + 1$ (b) $\sqrt{e^2+1}$

(c) e^2 (d) $e^2 - 1$

Ans. (b)

Solution: $\displaystyle\int_0^1 \frac{\sqrt{e^x}}{\sqrt{e^x+e^{-x}}}\mathrm{d}x = \int_0^1 \frac{e^x}{\sqrt{e^{2x}+1}}\mathrm{d}x$. Putting $e^x = t$ the last integral is equal to

$$\int_1^e \frac{dt}{\sqrt{t^2+1}} = \log\left|t+\sqrt{t^2+1}\right|\Big|_1^e$$

$$= \log(e+\sqrt{e^2+1})-\log(1+\sqrt{2})$$

$$= \log\frac{e+\sqrt{e^2+1}}{1+\sqrt{2}}$$

So $\quad A = \sqrt{e^2+1}$

Example 4: Value of x satisfying $\int_{\log 2}^{x}\frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$ is

(a) log 2 (b) 3 log 2
(c) 2 log 2 (d) 2

Ans. (c)

Solution: Substituting $E^x - 1 = t^2$, $e^x\,dx = 2t\,dt$

$$\Rightarrow \quad dx = \frac{2t\,dt}{t^2+1}\text{, so}$$

$$\frac{\pi}{6} = \int_{\log 2}^{x}\frac{dx}{\sqrt{e^x-1}} = \int_{1}^{\sqrt{e^x-1}}\frac{2t\,dt}{(t^2+1)t}$$

$$= 2\int_{1}^{\sqrt{e^x-1}}\frac{dt}{t^2+1} = 2\tan^{-1}t\,\Big|_{1}^{\sqrt{e^x-1}}$$

$$= 2\tan^{-1}\sqrt{e^x-1}-\frac{\pi}{2}$$

$$\Rightarrow \quad 2\tan^{-1}\sqrt{e^x-1} = \frac{2\pi}{3}$$

$$\Rightarrow \quad \sqrt{e^x-1} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$\Rightarrow \quad e^x - 1 = 3 \quad \text{i.e. } e^x = 4 \Rightarrow x = \log 4.$$

Example 5: The value of $\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} x^{10}\ \sin^9 x\,dx$ is equal to

(a) 0 (b) 1
(c) $\frac{\pi}{4}$ (d) $\left(\frac{\pi}{8}\right)^{10}$

Ans. (a)

Solution: Let $f(x) = x^{10}\sin^9 x$, $f(-x) = (-x)^{10}\sin^9(-x)$
$= -x^{10}\sin^9 x = -f(x)$

Hence f is an odd function. Now using Prop. 13 Page 13.2, we have $\int_{-a}^{a} f(x)\,dx = 0$. Thus

$$\int_{-\pi/8}^{\pi/8} x^{10}\sin^9 x\,dx = 0$$

Example 6: $\int_{0}^{\pi/2}\max(\sin x, \cos x)\,dx$ is equal to

(a) 1 (b) 2
(c) $2-\sqrt{2}$ (d) $\sqrt{2}$

Ans. (d)

Solution: max (sin x, cos x) = $x \in [0, \pi/2]$
$$\begin{cases}\cos x & 0 \le x \le \frac{\pi}{4}\\ \sin x & \frac{\pi}{4} \le x \le \frac{\pi}{4}\end{cases}$$

$$\int_0^{\pi/2}\max(\sin x, \cos x)\,dx = \int_0^{\pi/4}\cos x\,dx + \int_{\pi/4}^{\pi/2}\sin x\,dx$$

$$= \sin x\,\Big|_0^{\pi/4} - \cos x\,\Big|_{\pi/4}^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}} = \sqrt{2}\,.$$

Example 7: If $\int_0^{\pi/2}\frac{\cos x\,dx}{6-5\sin x+\sin^2 x} = \log K$ then K is equal to

(a) $\frac{2}{3}$ (b) $\frac{4}{3}$
(c) $\frac{1}{3}$ (d) $\frac{5}{3}$

Ans. (b)

Solution: Putting sin $x = t$, cos x dx = dt. When $x = 0$, sin $x = 0$ and when $x = \pi/2$, sin $x = 1$. So the given integral reduces to

$$\int_0^1\frac{dt}{6-5t+t^2} = \int_0^1\frac{dt}{\left(t-\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\frac{1}{2}}\log\left|\frac{t-\frac{5}{2}-\frac{1}{2}}{t-\frac{5}{2}+\frac{1}{2}}\right|\Bigg|_0^1$$

$$= \log\left|\frac{t-3}{t-2}\right|\Bigg|_0^1 = \log\frac{4}{3}$$

Example 8: $\int_{-1}^{1}|x|\ dx$ is equal to

(a) 1 (b) 0
(c) 2 (d) $\frac{1}{2}$

Ans. (a)

Solution: The integrand $f(x) = |x|$ is an even function so

$$\int_{-1}^{1}|x|\,dx = 2\int_0^1|x|\,dx = 2\int_0^1 x\,dx = 2\frac{x^2}{2}\Big|_0^1 = 1$$

Example 9: The value of $\lim\limits_{n\to\infty}\left[\dfrac{1}{\sqrt{4n^2-1}}+\dfrac{1}{\sqrt{4n^2-2^2}}+\cdots+\dfrac{1}{\sqrt{4n^2-n^2}}\right]$ is equal to

(a) π (b) $\dfrac{\pi}{6}$

(c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{4}$

Ans. (b)

Solution:

$$=\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+\cdots+\frac{1}{\sqrt{4n^2-n^2}}$$

$$=\frac{1}{n}\left[\frac{1}{\sqrt{4-\dfrac{1}{n^2}}}+\frac{1}{\sqrt{4-\dfrac{2^2}{n^2}}}+\cdots+\frac{1}{\sqrt{4-\left(\dfrac{n}{n}\right)^2}}\right]$$

So, required limit is equal to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\sqrt{4-\left(\dfrac{r}{n}\right)^2}}=\int_0^1\frac{dx}{\sqrt{4-x^2}}$$

$$=\sin^{-1}\frac{x}{2}\Big|_0^1=\frac{\pi}{6}.$$

Example 10: $\int_0^{10\pi}\sqrt{1-\sin^2 x}\,dx$ is equal to

(a) 0 (b) 10

(c) 10π (d) 20

Ans. (d)

Solution:

$$\int_0^{10\pi}\sqrt{1-\sin^2 x}\,dx=\int_0^{10\pi}|\cos x|\,dx$$

$$=10\int_0^{\pi}|\cos x|\,dx \quad \text{(Prop. 16)}$$

$$=20\int_0^{\pi/2}|\cos x|\,dx \quad \text{(Prop. 10)}$$

$$=20\int_0^{\pi/2}\cos x\,dx=20\,\sin x\Big|_0^{\pi/2}=20.$$

LEVEL 1

Straight Objective Type Questions

Example 11: The value $I=\int_0^{\pi/2}\dfrac{1}{1+\cot x}\,dx$ is

(a) π (b) $\pi/2$

(c) $\pi/4$ (d) $\pi/3$

Ans. (c)

Solution: Let $f(x)=\dfrac{1}{1+\cot x}=\dfrac{\sin x}{\sin x+\cos x}$, so

$$f(\pi/2-x)=\frac{\sin(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}$$

$$=\frac{\cos x}{\cos x+\sin x}$$

Considering $2I=\int_0^{\pi/2}f(x)\,dx+\int_0^{\pi/2}f(\pi/2-x)\,dx$

$$\left(\text{using}\int_0^a f(x)dx=\int_0^a f(a-x)\,dx\right)$$

$$=\int_0^{\pi/2}\frac{\sin x}{\cos x+\sin x}\,dx+\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}\,dx$$

$$=\int_0^{\pi/2}\frac{\sin x+\cos x}{\cos x+\sin x}\,dx=\int_0^{\pi/2}1\cdot dx=\frac{\pi}{2}$$

So $I=\pi/4$

Example 12: The value of $I=\int_0^{\pi}\dfrac{x\,dx}{4\cos^2 x+9\sin^2 x}$ is

(a) $\pi^2/12$ (b) $\pi^2/4$

(c) $\pi^2/6$ (d) $\pi^2/3$

Ans. (a)

Solution: Using Property 9, we have

$$I=\int_0^{\pi}\frac{(\pi-x)dx}{4\cos^2(\pi-x)+9\sin^2(\pi-x)}$$

$$=\pi\int_0^{\pi}\frac{dx}{4\cos^2 x+9\sin^2 x}-\int_0^{\pi}\frac{x\,dx}{4\cos^2 x+9\sin^2 x}$$

$$2I=\pi\int_0^{\pi}\frac{dx}{4\cos^2 x+9\sin^2 x}$$

$$=2\pi\int_0^{\pi/2}\frac{dx}{4\cos^2 x+9\sin^2 x}$$

(using Property 9)

$$= 2\pi \int_0^{\pi/2} \frac{\sec^2 x \, dx}{4 + 9\tan^2 x}$$

$$= 2\pi \int_0^{\infty} \frac{dt}{4 + 9t^2} \qquad (t = \tan x)$$

$$= \frac{2\pi}{9} \int_0^{\infty} \frac{dt}{t^2 + 4/9} = \frac{2\pi}{9} \cdot \frac{3}{2} \tan^{-1} \frac{3}{2} t \Big|_0^{\infty} = \frac{\pi^2}{6}$$

Hence $I = \pi^2/12$.

Example 13: If $\int_e^x tf(t)\, dt = \sin x - x\cos x - \frac{x^2}{2}$ for all $x \in \mathbf{R} \sim \{0\}$ then the value of $f(\pi/6)$ is

(a) 0 (b) 1

(c) – 1/2 (d) 3/2

Ans. (c)

Solution: Differentiating both the sides and using Property 17, we have

$xf(x) = \cos x - \cos x + x \sin x - x$, so $f(x) = \sin x - 1$.

Hence $f(\pi/6) = \sin \pi/6 - 1 = -1/2$.

Example 14: The value of $\int_0^2 x^{[x^2+1]}\, dx$, where $[x]$ is the greatest integer less than or equal to x is

(a) 2 (b) 8/3

(c) 4 (d) none of these

Ans. (d)

Solution: For $x \in [0, 2]$, $x^2 + 1 \in [1, 5]$, we must break $[0, 2] = [0, 1] \cup \left[1, \sqrt{2}\right] \cup \left[\sqrt{2}, \sqrt{3}\right] \cup \left[\sqrt{3}, 2\right]$.

Hence $\int_0^2 x^{[x^2+1]}\, dx$

$$= \int_0^1 x^{[x^2+1]}\, dx + \int_1^{\sqrt{2}} x^{[x^2+1]}\, dx + \int_{\sqrt{2}}^{\sqrt{3}} x^{[x^2+1]}\, dx + \int_{\sqrt{3}}^2 x^{[x^2+1]}\, dx$$

$$= \int_0^1 x\, dx + \int_1^{\sqrt{2}} x^2\, dx + \int_{\sqrt{2}}^{\sqrt{3}} x^3\, dx + \int_{\sqrt{3}}^2 x^4\, dx$$

$$= \frac{1}{2} + \frac{1}{3}\left[2^{3/2} - 1\right] + \frac{1}{4}[9 - 4] + \frac{1}{5}\left[32 - 3^{5/2}\right]$$

$$= \frac{469}{60} + \frac{1}{3} 2^{3/2} - \frac{1}{5} 3^{5/2}.$$

Example 15: Let $f(x) = \int_1^x \sqrt{2 - t^2}\, dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are

(a) ± 1 (b) $\pm 1/\sqrt{2}$

(c) ± 1/2 (d) 0 and 1

Ans. (a)

Solution: Using Property 18, we have $f'(x) = \sqrt{2 - x^2}$ and thus the given equation reduces to $x^2 - \sqrt{2 - x^2} = 0$ $\Rightarrow (x^2 + 2)(x^2 - 1) = 0$. Thus the real roots are given by $x = \pm 1$.

Example 16: The integral $\int_{-1/2}^{1/2} \left([x] + 2 \log \frac{1+x}{1-x}\right) dx$ equals

(a) – 1/2 (b) 0

(c) 1 (d) 2 log (1/2)

Ans. (a)

Solution: $\int_{-1/2}^{1/2} \left([x] + 2 \log \frac{1+x}{1-x}\right) dx = \int_{-1/2}^{1/2} [x]\, dx$

(since $\log \frac{1+x}{1-x}$ is an odd function)

$$= \int_{-1/2}^{0} [x]\, dx + \int_{0}^{1/2} [x]\, dx = \int_{-1/2}^{0} (-1)\, dx = -\frac{1}{2}.$$

Example 17: The value

$$I = \int_{-1}^{1} \left(\cos^{-1} x + \frac{x^7 - 3x^5 + 7x^3 - x}{\cos^2 x}\right) dx \text{ is}$$

(a) $\pi/2$ (b) 0

(c) 2π (d) π

Ans. (d)

Solution: Let $f(x) = \frac{x^7 - 3x^5 + 7x^3 - x}{\cos^2 x}$ then $f(-x) = -f(x)$ so $\int_{-1}^{1} f(x)\, dx = 0$ (Property 12)

Let $g(x) = \cos^{-1} x$ then $I_1 = \int_{-1}^{1} \cos^{-1} x\, dx$

$$= \int_{-1}^{1} \cos^{-1}(-x)\, dx \qquad \text{(Property 8)}$$

$$= \int_{-1}^{1} (\pi - \cos^{-1} x)\, dx = 2\pi - I_1$$

Thus $I = I_1 = \pi$.

Example 18: The value of $I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\, dx$ is

(a) 0 (b) 2/3

(c) 4/3 (d) 1/3

Ans. (c)

◎ **Solution:** $I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x}\ |\sin x|\, dx$

$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{\cos x}\,|\sin x|\, dx$$

(the integand is an even function)

$$= 2\int_{0}^{\pi/2} \sqrt{\cos x}\,\sin x\, dx$$

$$= -\frac{4}{3}(\cos x)^{3/2}\Big|_0^{\pi/2} = \frac{4}{3}.$$

◉ **Example 19:** If $n \in \mathbf{N}$, the value of $\int_0^n [x]\, dx$ (where $[x]$ is the greatest integer function) is

(a) $\dfrac{n(n+1)}{2}$ (b) $\dfrac{n(n-1)}{2}$

(c) $n(n-1)$ (d) none of these

Ans. (b)

◎ **Solution:** $\int_0^n [x]\, dx = \sum_{i=1}^{n}\int_{i-1}^{i} [x]\, dx = \sum_{i=1}^{n}\int_{i-1}^{i} (i-1)\, dx$

$$= \sum_{i=1}^{n} (i-1) = \frac{n(n-1)}{2}.$$

◉ **Example 20:** If $F(x) = \int_3^x \left(2 + \frac{d}{dt}\cos t\right) dt$ then $F'\left(\frac{\pi}{6}\right)$ is equal to

(a) 1/2 (b) 2

(c) 3/4 (d) 3/2

Ans. (d)

◎ **Solution:** $F(x) = \int_3^x (2 - \sin t)dt$ so $F'(x) = 2 - \sin x$.

Thus $F'(\pi/6) = 2 - 1/2 = 3/2$.

◉ **Example 21:** $\lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n} \frac{r}{\sqrt{n^2+r^2}}$ equals

(a) $1 + \sqrt{5}$ (b) $-1 + \sqrt{5}$

(c) $-1 + \sqrt{2}$ (d) $1 + \sqrt{2}$

Ans. (b)

◎ **Solution:** Required limit $= \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{2n}\frac{r/n}{\sqrt{1+r^2/n^2}}$

$$= \int_0^2 \frac{x}{\sqrt{1+x^2}}\, dx = \sqrt{1+x^2}\Big|_0^2 = \sqrt{5} - 1.$$

◉ **Example 22:** The value of the integral $\int_0^{n\pi+t}$ ($|\cos x|$ + $|\sin x|$) dx is

(a) n (b) $2n + \sin t + \cos t$

(c) $\cos t$ (d) $\sin t - \cos t + 4n + 1$

Ans. (d)

◎ **Solution:** Since the period of $|\sin x| + |\cos x|$ is $\pi/2$ so

$\int_0^{n\pi+t}$ ($|\sin x| + |\cos x|$) $dx = 2n \int_0^{\pi/2}$ ($|\sin x| + |\cos x|$)dx + $\int_0^t$ ($|\sin x| + |\cos x|$) dx

$$= 2n\int_0^{\pi/2} (\sin x + \cos x)\, dx + \int_0^t (\sin x + \cos x)\, dx$$

$$= (2n)(2) + \sin t - \cos t + 1$$

$$= (4n + 1) + \sin t - \cos t.$$

◉ **Example 23:** If $g(x) = \int_0^x \cos^4 t\, dt$ then $g(x + \pi)$ equals

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$

(c) $g(x)\, g(\pi)$ (d) $g(x)/g(\pi)$

Ans. (a)

◎ **Solution:** $g(x + \pi) = \int_0^{x+\pi} \cos^4 t\, dt$

$$= \int_0^{\pi} \cos^4 t\, dt + \int_{\pi}^{x+\pi} \cos^4 t\, dt$$

$$= g(\pi) + I_1$$

In I_1, put $t = \pi + u$, so that

$$I_1 = \int_0^x \cos^4(\pi + u)\, du = \int_0^x \cos^4 u\, du = g(x).$$

◉ **Example 24:** The value of $\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$ is

(a) 0 (b) 1

(c) $\pi/4$ (d) $\pi/2$

Ans. (c)

◎ **Solution:** $I = \int_0^{\pi/2}\frac{dx}{1+\tan^3 x} = \int_0^{\pi/2}\frac{dx}{1+\tan^3(\pi/2 - x)}$

$$= \int_0^{\pi/2}\frac{dx}{1+\cot^3 x}$$

$$= \int_0^{\pi/2}\frac{\tan^3 x}{1+\tan^3 x}\, dx = \int_0^{\pi/2}\left(1 - \frac{1}{1+\tan^3 x}\right)dx = \frac{\pi}{2} - I.$$

Thus $I = \pi/4$.

◉ **Example 25:** The value of $\int_0^4 3^{\sqrt{2x+1}}\, dx$ is

(a) $\dfrac{6}{\log 3}\left(13 - \dfrac{4}{\log 3}\right)$ (b) $\dfrac{66}{\log 3}$

(c) $\dfrac{6}{\log 3}\left(13 - \dfrac{5}{\log 3}\right)$ (d) none of these

Ans. (d)

◎ **Solution:** Putting $2x + 1 = t^2$, we have $dx = t\,dt$, so

$$\Rightarrow \int_0^4 3^{\sqrt{2x+1}}\,dx = \int_1^3 3^t\,t\,dt = \left.\frac{t\cdot 3^t}{\log 3}\right|_0^3 - \frac{1}{\log 3}\int_0^3 3^t\,dt$$

$$= \frac{(3)\cdot 3^3 - 3}{\log 3} - \frac{1}{(\log 3)^2}[3^3 - 3^1] = \frac{78}{\log 3} - \frac{24}{(\log 3)^2}$$

◉ **Example 26:** The value of $\int_0^1 x(1-x)^{99}\,dx$ is

(a) $\frac{1}{10100}$ (b) $\frac{11}{10100}$

(c) $\frac{1}{10010}$ (d) none of these

Ans. (a)

◎ **Solution:** $\int_0^1 x(1-x)^{99}\,dx = \int_0^1 (1-x)(1-(1-x))^{99}\,dx$ (Property 9)

$$= \int_0^1 (x^{99} - x^{100})\,dx = \left.\frac{x^{100}}{100} - \frac{x^{101}}{101}\right|_0^1 = \frac{1}{10100}.$$

◉ **Example 27:** The value of $\sum_{n=1}^{1000}\int_{n-1}^{n} e^{x-[x]}\,dx$ is ($[x]$ is the greatest integer function)

(a) $\frac{e^{1000}-1}{1000}$ (b) $\frac{e^{1000}-1}{e-1}$

(c) $1000\,(e-1)$ (d) $\frac{e-1}{1000}$

Ans. (c)

◎ **Solution:** Since the period of the function $x - [x]$ is 1 so

$$\sum_{n=1}^{1000}\int_{n-1}^{n} e^{x-[x]}\,dx = \int_0^{1000} e^{x-[x]}\,dx$$

$$= 1000\int_0^1 e^{x-[x]}\,dx = 1000\int_0^1 e^x\,dx = 1000\,(e-1).$$

◉ **Example 28:** If $f: \mathbf{R} \to \mathbf{R}$, $g: \mathbf{R} \to \mathbf{R}$ are continuous functions then the value of the integral $\int_{-\pi/2}^{\pi/2} [(f(x)+f(-x))(g(x)-g(-x))]\,dx$ is

(a) 1 (b) 0

(c) – 1 (d) π

Ans. (b)

◎ **Solution:** Let $F(x) = (f(x) + f(-x))\,(g(x) - g(-x))$ so

$$F(-x) = (f(-x) + f(x))\,(g(-x) - g(x)) = -F(x).$$

Hence $\int_{-\pi/2}^{\pi/2} f(x)\,dx = 0.$ (using Property 11)

◉ **Example 29:** If $f(x) = (1 + \tan x)\,(1 + \tan(\pi/4 - x))$ and $g(x)$ is a function with domain **R**, then $\int_0^1 x^3\, g\,o\,f(x)\,dx$ is

(a) $\frac{1}{2}\,g(\pi/4)$ (b) $\frac{1}{4}\,g(2)$

(c) $\frac{1}{4}\,g(1)$ (d) none of these

Ans. (b)

◎ **Solution:** $f(x) = (1 + \tan x)\left(1 + \frac{1-\tan x}{1+\tan x}\right)$

$$= (1+\tan x)\left(\frac{2}{1+\tan x}\right) = 2.$$

So $$\int_0^1 x^3\,g\,o\,f(x)\,dx = \int_0^1 x^3\,g(2)\,dx$$

$$= \left.\frac{x^4 g(2)}{4}\right|_0^1 = \frac{1}{4}\,g(2).$$

◉ **Example 30:** The $\lim\limits_{n\to\infty} S_n$ if

$$S_n = \frac{1}{2n} + \frac{1}{\sqrt{4n^2-1}} + \frac{1}{\sqrt{4n^2-4}} + \ldots + \frac{1}{\sqrt{3n^2+2n-1}} \text{ is}$$

(a) $\pi/2$ (b) 2

(c) 1 (d) $\pi/6$

Ans. (d)

◎ **Solution:** $\lim\limits_{n\to\infty} S_n$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{\sqrt{4-0}} + \frac{1}{\sqrt{4-1/n^2}} + \frac{1}{\sqrt{4-4/n^2}} + \ldots + \frac{1}{\sqrt{4-\left(\frac{n-1}{n}\right)^2}}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\sqrt{4-(r/n)^2}} = \int_0^1\frac{dx}{\sqrt{4-x^2}} = \left.\sin^{-1}\frac{x}{2}\right|_0^1 = \frac{\pi}{6}.$$

◉ **Example 31:** The value of $\int_{-\pi}^{3\pi} \log(\sec\theta - \tan\theta)\,d\theta$ is

(a) 1 (b) 0

(c) 2 (d) none of these

Ans. (b)

◎ **Solution:** $I = \int_{-\pi}^{3\pi} \log(\sec\theta - \tan\theta)\,d\theta$

$$= \int_{-\pi}^{3\pi} \log(\sec(2\pi-\theta) - \tan(2\pi-\theta))\,d\theta$$

$$= \int_{-\pi}^{3\pi} \log(\sec\theta + \tan\theta)\,d\theta.$$

Thus $$2I = \int_{-\pi}^{3\pi} [\log(\sec\theta - \tan\theta) + \log(\sec\theta + \tan\theta)]\,d\theta$$
$$= \int_{-\pi}^{3\pi} \log(\sec^2\theta - \tan^2\theta)d\theta$$
$$= \int_{-\pi}^{3\pi} \log 1\, d\theta = 0.$$

◉ **Example 32:** If $\int_0^{\infty} \frac{dx}{(x^2+4)(x^2+9)} = k\pi$ then the value of k is

(a) 1/60 (b) 1/80
(c) 1/40 (d) 1/20

Ans. (a)

◎ **Solution:**

$$\int_0^{\infty} \frac{dx}{(x^2+4)(x^2+9)} = \frac{1}{5}\int_0^{\infty}\left[\frac{1}{x^2+4} - \frac{1}{x^2+9}\right]dx$$
$$= \frac{1}{5}\left[\frac{1}{2}\tan^{-1}\frac{x}{2} - \frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^{\infty}$$
$$= \frac{1}{5}\left[\frac{\pi}{4} - \frac{\pi}{6}\right] = \frac{\pi}{60},\ \text{so } k = \frac{1}{60}.$$

◉ **Example 33:** If $\int_{-1}^{3/2} |x \sin \pi x|\, dx = k/\pi^2$, then the value of k is

(a) $3\pi + 1$ (b) $2\pi + 1$
(c) 1 (d) 4

Ans. (a)

◎ **Solution:**

$$I = \int_{-1}^{1} |x \sin \pi x dx| + \int_{1}^{3/2} |x \sin \pi x| dx$$
$$= 2\int_0^1 x \sin \pi x\, dx - \int_1^{3/2} x \sin \pi x\, dx$$
$$= 2\left[\frac{-x\cos\pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_0^1 - \left[\frac{-x\cos\pi x}{\pi} + \frac{\sin \pi x}{\pi^2}\right]_1^{3/2}$$
$$= 2\left[\frac{1}{\pi}\right] - \left[-\frac{1}{\pi^2} - \frac{1}{\pi}\right] = \frac{3\pi+1}{\pi^2},\ \text{so } k = 3\pi + 1.$$

◉ **Example 34:** If $f(0) = 2, f'(x) = f(x), \phi(x) = x + f(x)$ then $\int_0^1 f(x)\,\phi(x)\,dx$ is

(a) e^2 (b) $2e^2$
(c) $2e$ (d) $2e^{-3/2}$

Ans. (b)

◎ **Solution:** Since $f'(x) = f(x)$ and $f(0) = 2$ so $f(x) = 2e^x$, thus $\phi(x) = x + 2e^x$. Hence

$$\int_0^1 f(x)\,\phi(x)\,dx = 2\int_0^1 (xe^x + 2e^{2x})dx$$
$$= 2\,[xe^x - e^x|_0^1 + e^{2x}|^1_0]$$
$$= 2\,[([e - e] + 1) + e^2 - 1] = 2e^2.$$

◉ **Example 35:** The value of $\lim_{x\to 0} \frac{\int_0^x \cos t^2 dt}{x}$ is

(a) 1 (b) 0
(c) – 1 (d 2

Ans. (a)

◎ **Solution:** Let $f(x) = \int_0^x \cos t^2\, dt$ and $g(x) = x$. Then $f(0) = g(0) = 0$.

$$\therefore \quad \lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$
$$\Rightarrow \quad \lim_{x\to 0}\frac{\int_0^x \cos t^2 dt}{x} = \lim_{x\to 0}\frac{\cos x^2 \cdot 1 - \cos 0 \cdot 0}{1}$$
$$= \lim_{x\to 0}\frac{\cos x^2}{1} = \cos 0 = 1.$$

◉ **Example 36:** If $f(x) = \int_{1/x^2}^{x^2} \cos\sqrt{t}\; dt$, then $f'(1)$ is equal to

(a) cos 1 (b) 2 cos 1
(c) 4 cos 1 (d) none of these

Ans. (c)

◎ **Solution:** $f'(x) = \cos\sqrt{x^2}\,\frac{d}{dx}(x^2) - \cos\sqrt{1/x^2}\,\frac{d}{dx}\frac{1}{x^2}$

(Using Property 18)

$$= 2x\cos x + \frac{2}{x^3}\cos(1/x).$$ Hence $f'(1) = 4\cos 1$.

◉ **Example 37:** The least value of the function

$$F(x) = \int_0^x (3\sin u + 4\cos u)\,du$$

on the interval $(5\pi/4, 4\pi/3]$ is

(a) $3/2 - \sqrt{3}/2$ (b) $\frac{5-4\sqrt{3}}{2}$
(c) $\frac{7-4\sqrt{3}}{2}$ (d) $\frac{9-4\sqrt{3}}{2}$

Ans. (d)

◎ **Solution:** We have $F'(x) = 3\sin x + 4\cos x$. Since $\sin x$ and $\cos x$ assume negative values in the third quadrant, we have $F'(x) < 0$ for all $x \in (5\pi/4, 4\pi/3)$ so $F(x)$ assumes the least value at the point $x = 4\pi/3$. Thus the least value is

$$F(4\pi/3) = \int_0^{4\pi/3} (3\sin u + 4\cos u)du$$

$$= (-3\cos u + 4\sin u)\Big|_0^{4\pi/3}$$

$$= -3\cos\frac{4\pi}{3} + 4\sin\frac{4\pi}{3} - (-3)$$

$$= \frac{9}{2} - \frac{4\sqrt{3}}{2} = \frac{9-4\sqrt{3}}{2}.$$

Example 38: The integral $\int_0^{1/a} \frac{\log(1+ax)}{1+a^2x^2}\,dx$ $(a > 0)$ is equal to

(a) $a(\log 2)\frac{\pi}{8}$ (b) $\frac{1}{a}(\log 2)\frac{\pi}{4}$

(c) $\frac{1}{a}(\log 2)\frac{\pi}{8}$ (d) $\frac{1}{a^2}(\log 2)$

Ans. (c)

Solution: Put $ax = t$, the given integral reduces to

$$\frac{1}{a}\int_0^1 \frac{\log(1+t)}{1+t^2}dt = \frac{1}{a}\int_0^{\pi/4} \frac{\log(1+\tan u)}{1+\tan^2 u}\sec^2 u\,du\,(t-\tan u)$$

$$= \frac{1}{a}\int_0^{\pi/4} \log(1+\tan u)\,du$$

$$= \frac{1}{a}\int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-u\right)\right)du$$

$$= \frac{1}{a}\int_0^{\pi/4} \log\left(1+\frac{1-\tan u}{1+\tan u}\right)du$$

$$= \frac{1}{a}\int_0^{\pi/4} [\log 2 - \log(1+\tan u)]\,du$$

$$\Rightarrow \quad \frac{2}{a}\int_0^{\pi/4} \log(1+\tan u)\,du = \frac{1}{a}(\log 2)\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{1}{a}\int_0^{\pi/4} \log(1+\tan u)\,du = \frac{1}{a}(\log 2)\frac{\pi}{8}$$

Example 39: For $m > 0$, $n > 0$, let $I_{m,n} = \int_0^1 x^m (\log x)^n\,dx$, then $I_{5,5}$ is given by

(a) $-\frac{5!}{6^5}$ (b) $-\frac{5!}{5^5}$

(c) $-\frac{5!}{6^6}$ (d) $\frac{5!}{6^6}$

Ans. (c)

Solution: Integrating by parts, we obtain

$$I_{m,n} = \frac{x^{m+1}(\log x)^n}{m+1}\Bigg|_0^1 - \frac{n}{m+1}\int_0^1 x^m(\log x)^{n-1}dx$$

Since $\lim\limits_{x\to 0+} x^m(\log x)^n = \lim\limits_{x\to 0+}(x^{m/n}\log x)^n = 0$

So, $\quad I_{m,n} = -\frac{n}{m+1}\,I_{m,n-1}$

Hence $\quad I_{5,5} = -\frac{5}{6}I_{5,4}$

$$= -\frac{5}{6}\left(-\frac{4}{6}I_{5,3}\right)$$

$$= (-1)^3\frac{5.4.3}{6^3}I_{5,2} = (-1)^4\frac{5.4.3.2}{6^4}I_{5,1}$$

$$I_{5,1} = \int_0^1 x^5\log x\,dx = \frac{x^6\log x}{6}\Bigg|_0^1 - \frac{1}{6}\int_0^1 x^6\frac{1}{x}dx$$

$$= -\frac{1}{6^2}$$

So $\quad I_{5,5} = (-1)^5\frac{5!}{6^6} = -\frac{5!}{6^6}$

Example 40: If $I_n = \int_0^1 (\cos^{-1}x)^n\,dx$ then $I_6 - 360\,I_2$ is given by

(a) $6\left(\frac{\pi}{2}\right)^5 - 24\left(\frac{\pi}{2}\right)^3$ (b) $6\left(\frac{\pi}{2}\right)^5 - 120\left(\frac{\pi}{2}\right)^3$

(c) $6\left(\frac{\pi}{2}\right)^5$ (d) $6\left(\frac{\pi}{2}\right)^5 - 4\left(\frac{\pi}{2}\right)^3$

Ans. (b)

Solution: Integrating by parts, we obtain

$$I_n = \int_0^1 (\cos^{-1}x)^n dx = x\cos^{-1}x\Big|_0^1 + \int_0^1 n\frac{(\cos^{-1}x)^{n-1}x}{\sqrt{1-x^2}}dx$$

$$= n\int_0^1 \frac{x}{\sqrt{1-x^2}}(\cos^{-1}x)^{n-1}dx$$

$$= n\left[-\sqrt{1-x^2}\,(\cos^{-1}x)^{n-1}\Big|_0^1 - \int_0^1 (n-1)(\cos^{-1}x)^{n-2}dx\right]$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

$$I_6 = 6.\left(\frac{\pi}{2}\right)^5 - 6.5I_4$$

$$= 6.\left(\frac{\pi}{2}\right)^5 - 30\left[4\left(\frac{\pi}{2}\right)^3 - 12I_2\right]$$

$$I_6 - 360\,I_2 = 6.\left(\frac{\pi}{2}\right)^5 - 120\left(\frac{\pi}{2}\right)^3$$

Example 41: Let $I_n = \int\limits_0^{\pi/2} x^n \cos x \, \mathrm{d}x$ then $I_8 + 56I_6$ is equal to

(a) $\left(\dfrac{\pi}{2}\right)^6$ (b) $\left(\dfrac{\pi}{2}\right)^8$

(c) $\left(\dfrac{\pi}{2}\right)^6 - 1$ (d) $\left(\dfrac{\pi}{2}\right)^8 - 1$

Ans. (b)

Solution:

$$I_n = \int\limits_0^{\pi/2} x^n \cos x \, \mathrm{d}x$$

$$= x^n \sin x\Big|_0^{\pi/2} - n\int\limits_0^{\pi/2} x^{n-1} \sin x \, \mathrm{d}x$$

$$= \left(\frac{\pi}{2}\right)^n - n\left[-x^{n-1}\cos x\Big|_0^{\pi/2} + (n-1)\int\limits_0^{\pi/2} x^{n-2}\cos x \, \mathrm{d}x\right]$$

$$= \left(\frac{\pi}{2}\right)^n - n(n-1)\, I_{n-2}$$

$$I_8 + 8.7\, I_6 = \left(\frac{\pi}{2}\right)^8$$

Example 42: If a function $f : [0, 27] \to \mathbf{R}$ is differentiable then for some $0 < \alpha < \beta < 3$, $\int\limits_0^{27} f(x)\mathrm{d}x$ is equal to

(a) $3[\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3)]$

(b) $3[\alpha^2 f(\alpha) + \beta^2 f(\beta)]$

(c) $3[\alpha^2 f(\alpha^3) + \frac{1}{2}\beta^2 f(\beta^3)]$

(d) $3[\alpha^2 f(\alpha) + \frac{1}{2}\beta^2 f(\beta)]$

Ans. (c)

Solution: Let $g(x) = \int\limits_0^{x^3} f(t)\,dt$ so $g(0) = 0$

$$\Rightarrow \qquad g'(x) = 3x^2 f(x^3)$$

$$\int\limits_0^{27} f(t)\,dt = g(3) = \frac{g(3) - g(1)}{3 - 1} + \frac{1}{2}\,\frac{g(1) - g(0)}{1 - 0}$$

$$= g'(\alpha) + \frac{1}{2}g'(\beta)$$

for some $\alpha \in (1, 3)$, $\beta \in (0, 1) \subseteq (0, 3)$. but $g'(\alpha) = 3\alpha^2 f(\alpha^3)$ and $g'(\beta) = 3\beta^2 f(\beta)^3$

Therefore, $\int\limits_0^{27} f(t)\,dt = 3[\alpha^2 f(\alpha^3) + \frac{1}{2}\beta^2 f(\beta^3)]$

Example 43: The difference between the greatest and the least value of the function $F(x) = \int_0^x (t + 1)\,\mathrm{d}t$ on $[2, 3]$ is

(a) 3 (b) 2

(c) 7/2 (d) 3/2

Ans. (c)

Solution: Differentiating the given function, we get

$$F'(x) = t + 1]_{t = x}\,\frac{\mathrm{d}x}{\mathrm{d}x} - [t + 1]_{t = 0}\,\frac{\mathrm{d}0}{\mathrm{d}x} = x + 1.$$

This is positive for all $x \in [2, 3]$, so F is an increasing function in this interval. Therefore its greatest value is $F(3) = \int_0^3 (t + 1)\,\mathrm{d}t$ and its least value is $F(2) = \int_0^2 (t + 1)\,\mathrm{d}t$, so that the required difference between these values is $\int_0^3 (t + 1)\mathrm{d}t - \int_0^2 (t + 1)\mathrm{d}t = \int_2^3 (t + 1)\mathrm{d}t = \frac{7}{2}$.

Example 44: The value of the integral $\int_0^{\pi/4} \dfrac{\sin x + \cos x}{3 + \sin 2x}\,\mathrm{d}x$ is

(a) log 2 (b) log 3

(c) (1/4) log 3 (d) (1/8) log 3

Ans. (c)

Solution: The integral can be written

$$-\int_0^{\pi/4} \frac{\sin x + \cos x}{(\sin x - \cos x)^2 - 4}\,\mathrm{d}x.$$

Now put $t = \sin x - \cos x$. Then $\mathrm{d}t = (\cos x + \sin x)\mathrm{d}x$, and the integral becomes

$$-\int_{-1}^{0} \frac{\mathrm{d}t}{t^2 - 4} = -\frac{1}{4}\left[\log\left|\frac{t - 2}{t + 2}\right|\right]_{-1}^{0}$$

$$= -\frac{1}{4}\,(\log 1 - \log 3) = \frac{1}{4}\,\log 3.$$

Example 45: The inflection points on the graph of function $y = \int_0^x (t - 1)\,(t - 2)^2\,\mathrm{d}t$ are

(a) $x = -1$ (b) $x = 3/2$

(c) $x = 4/3$ (d) $x = 1$

Ans. (c)

Solution: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (x - 1)\,(x - 2)^2$ so $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = (x - 2)\,(3x - 4)$. The points of inflection are given by $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ so $x = 2$, $x = 4/3$ are points of inflection.

Example 46: The value of the integral $\int_0^{\pi/2} \dfrac{\mathrm{d}x}{1 + \frac{1}{6}\sin^2 x}$ is

(a) $\frac{\pi}{2}\sqrt{\frac{6}{7}}$ (b) $\frac{\pi}{3}\sqrt{\frac{2}{3}}$

(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

Ans. (a)

◎ **Solution:** $I=\int_0^{\pi/2}\frac{\mathrm{d}x}{1+\frac{1}{6}\sin^2 x}=\int_0^{\pi/2}\frac{\sec^2 x\,\mathrm{d}x}{\sec^2 x+\frac{1}{6}\tan^2 x}$

$$=\frac{6}{7}\int_0^{\infty}\frac{\mathrm{d}t}{6/7+t^2}\quad (t=\tan x)$$

$$=\frac{6}{7}\times\sqrt{\frac{7}{6}}\ \tan^{-1}t\sqrt{\frac{7}{6}}\Bigg|_0^{\infty}=\sqrt{\frac{6}{7}}\cdot\frac{\pi}{2}.$$

◉ **Example 47:** The value of $\int_{e^{-1}}^{e^2}\left|\frac{\log x}{x}\right|\mathrm{d}x$ is

(a) 3/2 (b) 5/2

(c) 3 (d) 5

Ans. (b)

◎ **Solution:** $\int_{e^{-1}}^{e^2}\left|\frac{\log x}{x}\right|\mathrm{d}x=-\int_{e^{-1}}^{1}\frac{\log x}{x}\mathrm{d}x+\int_1^{e^2}\frac{\log x}{x}\mathrm{d}x$

(since $\log x<0$ for $x\in[e^{-1},1]$ and $\log x>0$ for $x\in(1,e^2)$)

$$=-\int_{-1}^{0}t\,\mathrm{d}t+-\int_0^2 t\,\mathrm{d}t=-\frac{t^2}{2}\Bigg|_{-1}^{0}+\frac{t^2}{2}\Bigg|_0^2=\frac{1}{2}+2=\frac{5}{2}$$

◉ **Example 48:** If $f(x)=\begin{cases}e^{\cos x}\sin x & \text{for } |x|\le 2\\ 2 & \text{otherwise,}\end{cases}$ then

$\int_{-2}^{3} f(x)\,dx=$

(a) 0 (b) 1

(c) 2 (d) 3

Ans. (c)

◎ **Solution:** $\int_{-2}^{3} f(x)\,\mathrm{d}x=\int_{-2}^{2} f(x)\,\mathrm{d}x+\int_2^3 f(x)\,\mathrm{d}x$

$$=\int_{-2}^{2} e^{\cos x}\sin x\,dx+\int_2^3 2\,dx$$

$$=0+2=2\ \left(\text{since}\int_{-a}^{a} g(x)\mathrm{d}x=0\text{ if } g(-x)=-g(x)\right)$$

◉ **Example 49:** $\int_{\pi/4}^{3\pi/4}\frac{dx}{1+\cos x}$ is equal to

(a) 2 (b) – 2

(c) 1/2 (d) – 1/2

Ans. (a)

◎ **Solution:** $I=\int_{\pi/4}^{3\pi/4}\frac{\mathrm{d}x}{1+\cos x}=\int_{\pi/4}^{3\pi/4}\frac{\mathrm{d}x}{1+\cos(\pi-x)}$

(Property 10)

$$=\int_{\pi/4}^{3\pi/4}\frac{\mathrm{d}x}{1-\cos x}$$

$$\therefore\quad 2I=\int_{\pi/4}^{3\pi/4}\left(\frac{1}{1+\cos x}+\frac{1}{1-\cos x}\right)\mathrm{d}x$$

$$=\int_{\pi/4}^{3\pi/4}\frac{2}{1-\cos^2 x}\mathrm{d}x$$

$$=-2\ \cot x\Big|_{\pi/4}^{3\pi/4}=4.\ \text{Hence } I=2$$

◉ **Example 50:** The value of $\int_0^{\pi^2/4}\sin\sqrt{x}\,\mathrm{d}x$ is

(a) 0 (b) 1

(c) 2 (d) none of these

Ans. (c)

◎ **Solution:** Putting $\sqrt{x}=t$ so that $dx=2t\,\mathrm{d}t$

$$\int_0^{\pi^2/4}\sin\sqrt{x}\,dx=2\int_0^{\pi/2}t\sin t\,dt$$

$$=2\left[-t\cos t]_0^{\pi/2}+\int_0^{\pi/2}\cos t\,\mathrm{d}t\right]$$

$$=2[0+\sin t|^{\pi/2}{}_0]=2$$

◉ **Example 51:** The value of the integral $\int_0^{100\pi}\sqrt{1-\cos 2x}\,\mathrm{d}x$ is

(a) $100\sqrt{2}$ (b) $200\sqrt{2}$

(c) 0 (d) $100\,\pi$

Ans. (b)

◎ **Solution:** We have $\sqrt{1-\cos 2x}=\sqrt{2}\ |\sin x|$. Since the period of $|\sin x|$ is π, so $\int_0^{100\pi}\sqrt{1-\cos 2x}\ \mathrm{d}x=$ $\sqrt{2}\int_0^{100\pi}|\sin x|\,\mathrm{d}x=100\sqrt{2}\int_0^{\pi}\sin x\,\mathrm{d}x=200\sqrt{2}$.

◉ **Example 52:** Whenever $a<b$, the value of $\int_a^b\frac{|x|}{x}\,dx$ is

(a) $b-a$ (b) $a-b$

(c) $|b|-|a|$ (d) $|b|+|a|$

Ans. (c)

◎ **Solution:** If $0\le a<b$, then $f(x)=\frac{|x|}{x}=1$, therefore, $\int_a^b f(x)\,dx=b-a$. If $a<b\le 0$ then $f(x)=-1$ and so $\int_a^b f(x)\,dx=a-b$. Finally if $a<0<b$ then $\int_a^b f(x)\,dx=\int_a^0 f(x)\,dx+\int_0^b f(x)\,dx$

$$=-(0-a)+(b-0)=b-(-a)$$

The above three cases can be represented by

$$\int_a^b\frac{|x|}{x}\,dx=|b|-|a|.$$

Example 53: Let f be an odd function then $\int_{-1}^{1}$ $(|x| + f(x)\cos x)\,dx$ is equal to

(a) 0 (b) 1
(c) 2 (d) none of these

Ans. (b)

Solution: The function $g(x) = |x|$ is an even function and $h(x) = f(x)\cos x$ is an odd function so $\int_{-1}^{1}(|x| + f(x)\cos x)\,dx$ $= 2\int_0^1 |x|\,dx = 2\int_0^1 x\,dx = 1.$

Example 54: Let f be a periodic continuous function with period $T > 0$. If $I = \int_0^T f(x)dx$ Then the value of $I_1 = \int_4^{4+4T} f(3x)dx$ is

(a) I (b) $2I$
(c) $3I$ (d) $4I$

Ans. (d)

Solution: Put $3x = y$ in I_1

$$I_1 = \frac{1}{3}\int_{12}^{12+12T} f(y)dy$$

$$= \frac{1}{3}\left[\int_{12}^{T} f(y)dy + \sum_{K=1}^{11}\int_{KT}^{(K+1)T} f(y)dy + \int_{12T}^{12+12T} f(y)dy\right]$$

But $\int_{KT}^{(K+1)T} f(y)dy = \int_0^T f(u)du$,

(Put $KT + u = y$ or use Property 17)

and $\int_{12T}^{12+12T} f(y)dy = \int_0^{12} f(u)du$

Hence $I_1 = \frac{1}{3}\left[\int_{12}^{T} f(y)dy + 11I + \int_0^{12} f(u)du\right]$

$$= \frac{1}{3}\left[\int_0^T f(y)dy + 11I\right] = \frac{1}{3}\times 12I = 4I.$$

Example 55: Let $f(x) = \int_0^x \sqrt{6-u^2}\,du$. Then the real roots of the equation $x^2 - f'(x) = 0$ are

(a) $x = \pm\sqrt{6}$ (b) $x = \pm\sqrt{3}$
(c) $x = \pm\sqrt{2}$ (d) $x = \pm 1$

Ans. (c)

Solution: Since $f'(x) = \sqrt{6-x^2}$, the equation $x^2 - f'(x) = 0$ becomes

$$x^2 - \sqrt{6-x^2} = 0 \Rightarrow x^4 + x^2 - 6 = 0$$

$\Rightarrow (x^2 - 2)(x^2 + 3) = 0 \Rightarrow x^2 - 2 = 0\ (x^2 + 3 \neq 0)$

$\Rightarrow x = \pm\sqrt{2}.$

Example 56: If $f(x) = \int_{x^2}^{x^2+4} e^{-t^2}dt$, then the function $f(x)$ increases in

(a) $(-\infty, 0)$ (b) $(0, \infty)$
(c) $(-1, 2)$ (d) $(-2, \infty)$

Ans. (a)

Solution: $f'(x) = 2xe^{-(x^2+4)^2} - 2xe^{-x^4}$

$$= 2xe^{-(x^2+4)^2}\left[1 - e^{16+8x^2}\right]$$

Since $1 - e^{16+8x^2} < 0$ for all x so f increases for $x < 0$.

Example 57: If $\int_{\sin x}^{1} t^2 f(t)dt = 1 - \sin x$, then $f\left(1/\sqrt{3}\right)$ is equal to

(a) 1/3 (b) $1/\sqrt{3}$
(c) $\sqrt{3}$ (d) 3

Ans. (d)

Solution: Differentiating both the sides, we obtain $-\sin^2 x\, f(\sin x)\cos x = -\cos x$

$\Rightarrow f(\sin x) = \dfrac{1}{\sin^2 x}$, if $\cos x \neq 0$

If $\sin x = 1/\sqrt{3}$ then $\cos x \neq 0$ so $f\left(1/\sqrt{3}\right) = 3$.

Example 58: The integral $I =$ $\int_{-3/2}^{3/2}\left([x] + x^3 + \log_a\left(x + \sqrt{x^2+1}\right)\right)dx$ is equal

(a) 0 (b) $-3/2$
(c) 1 (d) 3/2

Ans. (b)

Solution: Let $f(x) = x^3 + \log_a\left(x + \sqrt{x^2+1}\right)$

$$f(-x) = -x^3 + \log_a\left(-x + \sqrt{x^2+1}\right)$$

$$= -x^3 + \log_a \frac{1}{x + \sqrt{x^2+1}}$$

$$= -x^3 + \log_a\left(x + \sqrt{x^2+1}\right) = -f(x)$$

$\therefore \int_{-3/2}^{3/2} f(x)dx = 0$

Thus $I = \int_{-3/2}^{3/2}[x]dx$

$$= \int_{-3/2}^{-1}(-2)dx + \int_{-1}^{0} -1\,dx + \int_0^1 0\,dx + \int_1^{3/2} 1\,dx$$

$= (-2)[-1 + 3/2] + (-1)[0 + 1] + 0 + 1[3/2 - 1]$
$= -1 - 1 + 1/2 = -3/2.$

Example 59: $\int_0^{\pi/4}\log\left(1+\tan^2\theta+2\tan\theta\right)d\theta\ =$

(a) $\pi\log 2$ (b) $(\pi\log 2)/2$
(c) $(\pi\log 2)/4$ (d) $\log 2$

Ans. (c)

Solution: Let $I = \int_0^{\pi/4}\log(1+\tan\theta)\,d\theta$

$$= \int_0^{\pi/4}\log(1+\tan(\pi/4-\theta))\,d\theta$$

$$= \int_0^{\pi/4}\log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4}[\log 2-\log(1+\tan\theta)]\,d\theta$$

$\Rightarrow$ $2I = \pi/4\log 2$.

Required integral $= 2I = \dfrac{\pi}{4}\log 2$

Example 60: $\int_a^b\sqrt{(x-a)(b-x)}\;dx\,(b>a)$ is equal to

(a) $\pi(b-a)^2/8$ (b) $\pi(b+a)^2/8$
(c) $\pi(b-a)^2$ (d) $\pi(b+a)^2$

Ans (a)

Solution: Let $t = \dfrac{x-a+x-b}{2} = x-\dfrac{1}{2}(a+b)$

and $\alpha = \dfrac{b-a}{2}$, so that

$$\int_a^b\sqrt{(x-a)(b-x)}dx = \int_\alpha^\alpha\sqrt{(t+\alpha)(\alpha-t)}dt$$

$$= 2\int_0^\alpha\sqrt{\alpha^2-t^2}\,dt$$

$$= t\sqrt{\alpha^2-t^2}+\alpha^2\sin^{-1}\left(\frac{t}{\alpha}\right)\Bigg]_0^\alpha$$

$$= \frac{\pi\alpha^2}{2} = \frac{\pi(b-a)^2}{8}.$$

Example 61: $\lim\limits_{n\to\infty}\left(\dfrac{1}{n}+\dfrac{1}{n+2}+\ldots+\dfrac{1}{3n}\right) =$

(a) $\log 2$ (b) $\log 3$
(c) $\log 5$ (d) 0

Ans. (b)

Solution: $\lim\limits_{n\to\infty}\left(\dfrac{1}{n}+\dfrac{1}{n+2}+\ldots+\dfrac{1}{3n}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\ldots+\frac{1}{n+2n}\right)$$

$$= \lim_{n\to\infty}\sum_{k=0}^{2n}\frac{1}{n+k} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{2n}\frac{1}{1+k/n}$$

$$= \int_0^2\frac{dx}{1+x} = \log(1+x)\Big|_0^2 = \log 3.$$

Example 62: Let $I = \int_0^1\dfrac{\sin x}{\sqrt{x}}dx$ and $J = \int_0^1\dfrac{\cos x}{\sqrt{x}}dx$.

Then which one of the following is true

(a) $I > \dfrac{2}{3}$ and $J > 2$ (b) $I < \dfrac{2}{3}$ and $J < 2$

(c) $I < \dfrac{2}{3}$ and $J > 2$ (d) $I > \dfrac{2}{3}$ and $J < 2$

Ans. (b)

Solution: Since $\sin x < x$ for $x > 0$, we get

$$\frac{\sin x}{\sqrt{x}} < \sqrt{x} \quad \text{for} \quad 0<x<1$$

$$\therefore \quad I = \int_0^1\frac{\sin x}{\sqrt{x}}dx < \int_0^1\sqrt{x}\,dx = \frac{2}{3}$$

$$J = \int_0^1\frac{\cos x}{\sqrt{x}}dx < \int_0^1\frac{1}{\sqrt{x}}\,dx = 2$$

Example 63: $I = \int\limits_0^\pi[\cot x]dx$, where $[\cdot]$ denotes the greatest integer function, is equal to

(a) -1 (b) $-\pi/2$
(c) $\pi/2$ (d) 1

Ans. (b)

Solution: $I = \int\limits_0^\pi[\cot x]dx = \int\limits_0^{\pi/2}([\cot x]+[\cot(\pi-x)])\,dx$

$$= \int\limits_0^{\pi/2}([\cot x]+[-\cot x])\,dx$$

Put $\cot x = t$, so that

$$I = \int\limits_0^\infty[t]+[-t]\frac{dt}{1+t^2}$$

$$= \lim_{n\to\infty}\sum_{k=1}^n\int\limits_{k-1}^k([t]+[-t])\frac{dt}{1+t^2}$$

But $[t]+[-t] = -1$ for $k-1<t<k$, therefore

$$\int\limits_{k-1}^k([t]+[-t])\frac{dt}{1+t^2}$$

$$= \int\limits_{k-1}^k(-1)\frac{dt}{1+t^2} = \left[\tan^{-1}k-\tan^{-1}(k-1)\right]$$

$$\therefore \quad I = -\lim_{n\to\infty}\sum_{k=1}^n\left(\tan^{-1}k-\tan^{-1}(k-1)\right)$$

$$= -\lim_{n\to\infty}\left[\tan^{-1} n - \tan^{-1} 0\right] = -\frac{\pi}{2}.$$

Example 64: Let $p(x)$ be a function defined on **R** such that $p'(x) = p'(1-x)$, for all $x \in [0, 1]$, $p(0) = 1$ and $p(1) = 41$.

Then $\int_0^1 p(x)\,dx$ equals

(a) 41 (b) 42

(c) $\sqrt{41}$ (d) 21

Ans. (d)

Solution:

$$\int_0^1 p(x)\,dx = \int_0^1 1 \,.\, p(x)\,dx = [xp(x)]_0^1 - I_1 = p(1) - I_1$$

where

$$I_1 = \int_0^1 xp'(x)\,dx = \int_0^1 (1-x)\,p'(1-x)\,dx$$

$$= \int_0^1 (1-x)\,p'(x)\,dx = \int_0^1 p'(x)\,dx - I_1$$

$$\Rightarrow \qquad 2I_1 = [p(x)]_0^1 = p(1) - p(0)$$

Thus,

$$\int_0^1 p(x)\,dx = p(1) - \frac{1}{2}(p(1) - p(0))$$

$$= \frac{1}{2}(p(1) + p(0)) = \frac{1}{2}(41 + 1) = 21$$

Example 65: The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}\,dx$ is

(a) log 2 (b) $\pi \log^2$

(c) $\frac{\pi}{8}\log 2$ (d) $\frac{\pi}{2}\log 2$

Ans. (b)

Solution: Let $I = 8\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

$$= 8\int_0^{\pi/4} \log\frac{(1+\tan\theta)}{\sec^2\theta}\ \sec^2\theta\, d\theta \ (x = \tan\,\theta)$$

So
$$I = 8\int_0^{\pi} \log(1+\tan\theta)\,d\theta$$

$$= 8\int_0^{\pi/4} \log(1+\tan\left(\frac{\pi}{4}-\theta\right)d\theta$$

$$\left(\int_0^a f(x)dx = \int_0^a f(a-x)\,dx\right)$$

$$= 8\int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= 8\int_0^{\pi/4} \log\frac{2}{1+\tan\theta}\,d\theta = (8\ \log 2)\,\frac{\pi}{4} - I$$

$$\Rightarrow \qquad 2I = (8\log 2)\frac{\pi}{4} \quad \Rightarrow \quad I = \pi\log 2$$

Example 66: For $x \in (0, 5\pi/2)$, define $f(x) = \int_0^x \sqrt{t}\,\sin t\,dt$ Then f has

(a) local maximum at π and local minimum at 2π.

(b) local maximum at π and 2π

(c) local minimum at π and 2π

(d) local minimum at π and local maximum at 2π

Ans. (a)

Solution: $f'(x) = \sqrt{x}\ \sin x$, so $f'(x) = 0$ $\Rightarrow x = \pi, 2\pi$ but $f''(x) = \sqrt{x}\ \cos\ x - \frac{1}{2\sqrt{x}}\ \sin x$, so $f''(\pi) = -\sqrt{\pi} < 0$ and $f''(2\pi) = \sqrt{2\pi} > 0$. Hence f has local maximum at $x = \pi$ and local minimum at $x = 2\pi$.

Example 67: Let [.] denote the greatest integer function then the value of $\int_0^{1\cdot 5} x\left[x^2\right]dx$ is

(a) 0 (b) 3/2

(c) 3/4 (d) 5/4

Ans. (c)

Solution: Put $x^2 = t$, so that $x\,dx = \frac{1}{2}dt$

$$I = \frac{1}{2}\int_0^{2\cdot 25}[t]\,dt = \frac{1}{2}\int_0^1 [t]dt + \int_1^2 [t]\,dt + \int_1^{2\cdot 25}[t]dt$$

$$= \frac{1}{2}\left[0 + \int_1^2 dt + 2\int_2^{2\cdot 25} dt\right]$$

$$= \frac{1}{2}[1 + 2\times .25] = \frac{3}{4}$$

Example 68: If $g(x) = \int_0^x \cos 4t\ dt,$, then $g(x+\pi)$ equals

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$

(c) $g(x)\ g(\pi)$ (d) $g(x)\ g(\pi)$

Ans. (a), (b)

Solution: $g(x + \pi) = \int_0^{x+\pi} \cos 4t\ dt$

$$= \int_0^{\pi} \cos 4t\ dt + \int_{\pi}^{x+\pi} \cos 4t\ dt$$

$$= g(\pi) + I$$

where $I = \int_{\pi}^{x+\pi} \cos 4t\ dt$, Put $t = \pi + \theta$, so that

$$I = \int_0^x \cos(4\pi + 4\theta)d\theta = \int_0^x \cos 4\theta d\theta = g(x)$$

So $g(x + \pi) = g(\pi) + g(x)$ but

$$g(\pi) = \int_0^{\pi} \cos 4t\ dt = \frac{1}{4}(\sin\,4t\Big|_0^{\pi} = 0.$$

$\therefore \quad g(x + \pi) = g(x) - g(\pi)$ also.

Example 69: Let $f : [-1, 2] \to [0, \infty]$ be a continuous function such that $f(x) = f(1 - x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} x\, f(x)\, dx$, , and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$ and $x = 2$ and the x-axis. Then

(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$

(c) $2R_1 = R_2$ (d) $3R_1 = R_2$

Ans. (c)

Solution:

$$R_1 = \int_{-1}^{2} x\, f(x)dx = \int_{-1}^{2} (2 + (-1) - x)\, f(2 + (-1) - x)dx$$

$$= \int_{-1}^{2} (1 - x)\, f(1 - x)dx = \int_{-1}^{2} (1 - x)\, f(x)dx$$

$$\Rightarrow 2R_1 = \int_{-1}^{2} f(x)dx \;= R_2.$$

Example 70: The value of

$$I = \int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin\left(\log 6 - x^2\right)}\, dx \text{ is}$$

(a) $\frac{1}{4}\log\frac{3}{2}$ (b) $\frac{1}{2}\log\frac{3}{2}$

(c) $\log\frac{3}{2}$ (d) $\frac{1}{6}\log\frac{3}{2}$

Ans. (a)

Solution: Put $x^2 = t$, so that

$$I = \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin t}{\sin t + \sin(\log 6 - t)}dt$$

$$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 2 + \log 3 - t)}{\sin(\log 2 + \log 3 - t) + \sin t}dt$$

$$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin t + \sin\,(\log 6 - t)}\, dt$$

$$2I = I + I$$

$$= \frac{1}{2}\int_{\log 2}^{\log 3} \frac{\sin t + \sin\,(\log 6 - t)}{\sin t + \sin\,(\log 6 - t)}dt$$

$$= \frac{1}{2}[\log 3 - \log 2]$$

$$\Rightarrow \qquad I = \frac{1}{4}\log\frac{3}{2}.$$

Example 71: Let f be a continuous function satisfying $\int_{-\pi}^{t^2} (f(x) + x^2)dx = \pi + \frac{4}{3}t^3$ for all t, then $f(\pi^2/4)$ is equal to

(a) $\pi - \frac{\pi^4}{8}$ (b) $\frac{\pi}{2} - \left(\frac{\pi}{4}\right)^4$

(c) $\frac{\pi}{2} - \left(\frac{\pi}{4}\right)^2$ (d) $\pi - \frac{\pi^4}{16}$

Ans. (d)

Solution: Differentiating both sides, we have

$$(f(t^2) + t^4)\, 2t = 4t^2$$

$$f(t^2) + t^4 = 2t$$

$$\Rightarrow \qquad f(t^2) = 2t - t^4$$

$$\text{so} \qquad f\left(\frac{\pi^2}{4}\right) = f\left(\left(\frac{\pi}{2}\right)^2\right) = 2\frac{\pi}{2} - \left(\frac{\pi}{2}\right)^4$$

$$= \pi - \frac{\pi^4}{16}.$$

Example 72: For a continuous function f, the value $\int_0^\infty f(x^n + x^{-n})\frac{\log x}{x} + \frac{1}{1 + x^2}\, dx$ is

(a) $\frac{\pi}{2}$ (b) 0

(c) $-\pi$ (d) 2π

Ans. (a)

Solution: Putting $\frac{1}{x} = t \Rightarrow dx = -\frac{1}{t^2}dt$, so

$$I = \int_0^\infty f(x^n + x^{-n})\log x \frac{dx}{x}$$

$$= -\int_\infty^0 f(t^{-n} + t^n)\left(\log\frac{1}{t}\right)t\frac{dt}{t^2}$$

$$= -\int_0^\infty f(t^{-n} + t^n)\log t\frac{dt}{t} = -I$$

$$\Rightarrow \qquad 2I = 0 \quad\Rightarrow\quad I = 0$$

The given integral is equal to

$$\int_0^\infty \frac{1}{1 + x^2}dx \;= \tan^{-1} x\Big|_0^\infty = \frac{\pi}{2}$$

Assertion-Reason Type Questions

Example 73: If $\int_0^{\infty} e^{-x^2}\,dx = \dfrac{\sqrt{\pi}}{2}$ then

Statement-1: $\int_0^{\infty} \dfrac{e^{-x}}{\sqrt{x}}\,dx = \sqrt{\pi}$

Statement-2: $\lim\limits_{x \to \infty} e^{-x^2} = 0$

Ans. (b)

Solution: Put $x = t^2$

$$\int_0^{\infty} \frac{e^{-x}}{\sqrt{x}}\,dx \;=\; 2\int_0^{\infty} \frac{e^{-t^2}}{t}\,t\,dt = 2\int_0^{\infty} e^{-t^2}\,dt$$

Example 74: If $\int_0^{\infty} \dfrac{\sin x}{x}\,dx = \dfrac{\pi}{2}$ then

Statement-1: $\int_0^{\infty} \dfrac{\sin ax \cos bx}{x}\,dx \;=\pi/2\ (a > b > 0)$

Statement-2: $\lim\limits_{x \to 0} \dfrac{\sin ax \cos bx}{x} = a$

Ans. (b)

Solution: $\sin ax \cos bx = \dfrac{1}{2}\ [\sin (a + b)\,x + \sin (a - b)x]$

$$\int_0^{\infty} \frac{\sin ax \cos bx}{x}\,dx$$

$$= \frac{1}{2}\left[\int_0^{\infty} \frac{\sin (a+b)x}{x}\,dx + \int_0^{\infty} \frac{\sin (a-b)x}{x}\,dx\right]$$

$$= \frac{1}{2}\left[\int_0^{\infty} \frac{\sin t}{t}\,dt + \int_0^{\infty} \frac{\sin u}{u}\,du\right]$$

$$= \pi/2$$

Example 75: Statement-1: $\int_{-\pi/3}^{\pi/3} x^{10}\ \sin^9 x\,dx = 0$

Statement-2: $f(x) = x^{2n}$ is an even function and $g(x) = \sin^{2m+1}x$ is an odd function, m and n are integers.

Ans. (a)

Solution: $x^{10} \sin^9 x$ is an odd function so

$\int_{-a}^{a} x^{10} \sin^9 x\,dx = 0.$

Example 76: Suppose that f is an odd function and $F(x) = \int_a^x f(t)\,dt$.

Statement-1: F is an even function

Statement-2: $\int_{-a}^{a} f(t)\,dt = 0$

Ans. (a)

Solution: $F(x) = \int_a^x f(t)\,dt = -\int_{-a}^{-x} f(u)\,du \qquad (u = -\,t)$

$$= \int_{-a}^{-x} f(u)\,du = \int_{-a}^{a} f(u)\,du + \int_a^{-x} f(u)\,du$$

$$= \int_a^{-x} f(u)\,du = F(-x)$$

Example 77: Let $I = \int_{10}^{18} \dfrac{\cos x}{\sqrt{1 + x^4}}\,dx$

Statement-1: $|\,I\,| < 0.1$

Statement-2: $\left|\dfrac{\cos x}{\sqrt{1 + x^4}}\right| < 0.1$

Ans. (b)

Solution: Since $\cos x \le 1$, the condition $x > 10$ yields $\left|\dfrac{\cos x}{\sqrt{1 + x^4}}\right| < 10^{-2} < 10^{-1}$

$$|\,I\,| < 8.10^{-2} < 10^{-1}.$$

Example 78: Statement-1: $\int\limits_0^{2\pi} \cos^{2m+1} x\ \ \mathrm{d}x = 0\ (m > 0)$

Statement-2: $\int\limits_0^{2a} f(x)\mathrm{d}x \;= 2\int\limits_0^{a} f(x)\mathrm{d}x$ if $f(2a - x) = f(x)$

Ans. (a)

Solution: $\int\limits_0^{2\pi} \cos^{2m+1} x\,\mathrm{d}x \;= 2\int\limits_0^{\pi} \cos^{2m+1} x\,\mathrm{d}x$

$$= 0 \text{ (since } \cos (\pi - x) = -\cos x)$$

Example 79:

Statement-1: $\int\limits_0^{\pi/4} \log(1 + \tan\theta)\,d\theta = \dfrac{\pi}{8}\log 2$

Statement-2: $\int\limits_0^{\pi/2} \log \sin\theta\,d\theta = -\,\pi \log 2$

Ans. (c)

Solution:

$$I = \int\limits_0^{\pi/4} \log\,(1 + \tan\theta)\,d\theta = \int\limits_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - \theta\right)\right) d\theta$$

$$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$2I = \frac{\pi}{4}\log 2 \quad \Rightarrow \quad I = \frac{\pi}{8}\log 2$$

$$I_1 = \int_0^{\pi/2} \log\sin\theta\, d\theta = \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2}-\theta\right)d\theta$$

$$= \int_0^{\pi/2} \log\cos\theta\ d\theta$$

$$2I_1 = \int_0^{\pi/2} \log(\sin\theta\cos\theta)\ d\theta = \int_0^{\pi/2} \log\left(\log\frac{\sin 2\theta}{2}\right)d\theta$$

$$= \int_0^{\pi/2} \log\sin 2\theta\ d\theta - \frac{\pi}{2}\log 2 = I_2 - \frac{\pi}{2}\log 2$$

but $$I_2 = \int_0^{\pi/2} \log\sin 2\theta\ d\theta = \frac{1}{2}\int_0^{\pi} \log\sin t\, dt$$

$$= \frac{1}{2}.2\int_0^{\pi/2} \log\sin t\, dt \ = I_1$$

Hence, $I_1 = -\frac{\pi}{2}\log 2$.

LEVEL 2

Straight Objective Type Questions

Example 80: Let f: $(0, \infty) \to \mathbf{R}$ and $F(x) = \int_0^x f(t)\ dt$.

If $F(x^2) = x^2(1 + x)$ then $f(4)$ equals

(a) 5/4 (b) 7

(c) 4 (d) 2

Ans. (c)

Solution: $F(x^2) = \int_0^{x^2} f(t)\ dt$, therefore, $x^2(1 + x) = \int_0^{x^2} f(t)\ dt$. Differentiating both sides w.r.t. x using Property 17, we have

$$2x + 3x^2 = f(x^2)\cdot 2x \Rightarrow f(x^2) = 1 + (3/2)x$$

Putting $x = 2$, we have $f(4) = 1 + 3 = 4$.

Example 81: If $I_1 = \int_{1/e}^{\tan x} \frac{t}{1+t^2}\ dt$ and $I_2 = \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$ then the value of $I_1 + I_2$ is

(a) 1/2 (b) 1

(c) $e/2$ (d) $(1/2)\,(e + 1/e)$

Ans. (b)

Solution: Putting $t = 1/u$ in I_2 we have

$$I_2 = -\int_e^{\tan x} \frac{u\, du}{1+u^2} = -\int_{1/e}^{\tan x} \frac{u\, du}{1+u^2} + \int_{1/e}^{e} \frac{u\, du}{1+u^2}$$

$$= -I_1 + \frac{1}{2}\int_{1/e}^{e} \frac{2u\, du}{1+u^2}$$

So $$I_1 + I_2 = \frac{1}{2}\ \log(u^2+1)\Big]_{1/e}^{e}$$

$$= \frac{1}{2}\left[\log\left(e^2+1\right) - \log\left(\frac{e^2+1}{e^2}\right)\right]$$

$$= \frac{1}{2}\times 2 = 1\,.$$

Example 82: If $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\ dt$ and $g(x) = \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\ dt$ then the value of $f(x) + g(x)$ is

(a) π (b) $\pi/4$

(c) $\pi/2$ (d) $\sin^2 x + \sin x + x$

Ans. (b)

Solution: $f'(x) + g'(x) = \sin^{-1}(\sin x)\ 2\sin x\cos x - \cos^{-1}(\cos x)\ 2\sin x\cos x$ (Property 17)

$$= x\sin 2x - x\sin 2x = 0$$

for all $x \in \mathbf{R}$. Hence $f(x) + g(x)$ = constant = C (say)

Putting $x = \pi/4$, $C = \int_0^{1/2} \sin^{-1}\sqrt{t}\ dt + \int_0^{1/2} \cos^{-1}\sqrt{t}\ dt$

$$= \int_0^{1/2} \frac{\pi}{2}\ dt = \frac{\pi}{4}$$

Hence $f(x) + g(x) = \pi/4$.

Example 83: Solution of the equation $\int\limits_{\log 2}^{x} \frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$ are

(a) $x = \log 6$
(b) $x = 2\log 2$
(c) $x = 3$
(d) $x = 1/2$

Ans. (b)

Solution: Putting $e^x - 1 = t^2$, we have $e^x\,dx = 2t\,dt$

so
$$\int\limits_{\log 2}^{x} \frac{dx}{\sqrt{e^x-1}} = \int\limits_{1}^{\sqrt{e^x-1}} \frac{2t\,dt}{t\left(t^2+1\right)} = 2\int\limits_{1}^{\sqrt{e^x-1}} \frac{dt}{t^2+1}$$
$$= 2\left(\tan^{-1}\sqrt{e^x-1} - \frac{\pi}{4}\right)$$

Thus the given equation reduces to
$$\tan^{-1}\sqrt{e^x-1} - \frac{\pi}{4} = \frac{\pi}{12} \Rightarrow \sqrt{e^x-1} = \tan\frac{\pi}{3} = \sqrt{3}$$

so
$$e^x = 4 \Rightarrow x = 2\log 2$$

Example 84: The value of $\lim\limits_{m\to\infty} \frac{\int_0^{\pi/2} \sin^{2m} x\,dx}{\int_0^{\pi/2} \sin^{2m+1} x\,dx}$

(a) 0
(b) 1/2
(c) 2
(d) 1

Ans. (d)

Solution: We know that $I_{2n} = \int_0^{\pi/2} \sin^{2n} x\,dx$
$$= \frac{2n-1}{2n} \times \frac{2n-3}{2n-2} \times \ldots \times \frac{1}{2} \times \frac{\pi}{2},$$
$$I_{2n+1} = \int_0^{\pi/2} \sin^{2n+1} x\,dx$$
$$= \frac{2n}{2n+1} \times \frac{2n-2}{2n-1} \times \ldots \times \frac{2}{3} \text{ and}$$

Also,
$$I_{2m+1} = \frac{2m}{2m+1} I_{2m-1}.$$

For all $x \in (0, \pi/2)$, $\sin^{2m-1} x > \sin^{2m} x > \sin^{2m+1} x$

Integrating from 0 to $\pi/2$, we get $I_{2m-1} \ge I_{2m} \ge I_{2m+1}$

hence
$$\frac{I_{2m-1}}{I_{2m+1}} \ge \frac{I_{2m}}{I_{2m+1}} \ge 1 \qquad \text{(i)}$$

Also
$$\frac{I_{2m-1}}{I_{2m+1}} = \frac{2m+1}{2m}.$$

Hence
$$\lim_{m\to\infty} \frac{I_{2m-1}}{I_{2m+1}} = \lim_{m\to\infty} \frac{2m+1}{2m} = 1.$$

From (i) and using sandwitch theorem we have
$$\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1.$$

Example 85: Let f be a positive function and
$$I_1 = \int_{1-k}^{k} x f(x(1-x))\,dx,\ I_2 = \int_{1-k}^{k} f(x(1-x))\,dx$$
where $2k - 1 > 0$ then I_1/I_2 is

(a) 2
(b) k
(c) 1/2
(d) 1

Ans. (c)

Solution: Since $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, we have
$$I_1 = \int_{1-k}^{k} (k+1-k-x) f((k+1-k-x)$$
$$(1-(k+1-k-x))\,dx.$$
$$= \int_{1-k}^{k} (1-x) f((1-x)x)\,dx$$
$$= \int_{1-k}^{k} f((1-x)x)\,dx - \int_{1-k}^{k} x f((1-x)x)\,dx$$
$$= I_2 - I_1.$$

So $2I_1 = I_2$ and $I_1/I_2 = 1/2$.

Example 86: If the equality
$$\int_0^x \frac{bt\cos 4t - a\sin 4t}{t^2}\,dt = \frac{a\sin 4x}{x} - 1$$
holds for all x such that $0 < x < \pi/4$ then a and b are given by

(a) $a = 1/4$, $b = 1$
(b) $a = 2$, $b = 2$
(c) $a = -1$, $b = 4$
(d) $a = 2$, $b = 4$

Ans. (a)

Solution: $\int_0^x \frac{bt\cos 4t - a\sin 4t}{t^2}\,dt$
$$= b\int_0^x \frac{\cos 4t}{t}\,dt - a\int_0^x \frac{\sin 4t}{t^2}\,dt$$
$$= b\int_0^x \frac{\cos 4t}{t}\,dt - a\left[-\frac{\sin 4t}{t}\Bigg|_0^x + 4\int_0^x \frac{\cos 4t}{t}\,dt\right]$$
$$= (b-4a)\int_0^x \frac{\cos 4t}{t}\,dt + \frac{a\sin 4x}{x} - 4a$$

Thus
$$(b-4a)\int_0^x \frac{\cos 4t}{t}\,dt + \frac{a\sin 4x}{x} - 4a$$
$$= \frac{a\sin 4x}{x} - 1$$

i.e.
$$(b-4a)\int_0^x \frac{\cos 4t}{t}\,dt = 4a - 1.$$

Since R.H.S. is independent of x, so we must have $b - 4a = 0$ and $4a - 1 = 0$ i.e., $a = 1/4$, $b = 1$.

Example 87: The value of $\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2}\,dx$ is

(a) $-1/2$
(b) 2
(c) 1/2
(d) none of these

Ans. (c)

◎ **Solution:** $\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2}\,dx = \int_0^{\pi/2} \frac{2(\cos x+2)-3}{(2+\cos x)^2}\,dx$

$$= 2\int_0^{\pi/2} \frac{dx}{(2+\cos x)} - 3\int_0^{\pi/2} \frac{dx}{(2+\cos x)^2}$$

$$= 4\int_0^1 \frac{dt}{3+t^2} - 6\int_0^1 \frac{1+t^2}{\left(3+t^2\right)^2}\,dt \qquad (t = \tan x/2)$$

$$= -2\int_0^1 \frac{dt}{3+t^2} + 12\int_0^1 \frac{dt}{\left(3+t^2\right)^2}$$

$$= -2\int_0^1 \frac{dt}{3+t^2} + 12\left[\frac{1}{2.3}\frac{t}{\left(t^2+3\right)}\Bigg|_0^1 + \frac{1}{2.3}\int_0^1 \frac{dt}{3+t^2}\right]$$

$$= \frac{1}{2}.$$

◉ **Example 88:** The value of $\int_{-1}^{3} \{|x-2| + [x]\}\,dx$, where $[x]$ denotes the greatest integer less than or equal to x is

(a) 5 (b) 7
(c) 4 (d) 3

Ans. (b)

◎ **Solution:** $\int_{-1}^{3} \{|x-2|+[x]\}dx = \int_{-1}^{0} \{|x-2|+[x]\}dx + \int_0^1 \{|x-2|+[x]\}dx + \int_1^2 \{|x-2|+[x]\}dx + \int_2^3 \{|x-2|+[x]\}dx$

$$= \int_{-1}^{0} (2-x-1)\,dx + \int_0^1 (2-x+0)dx + \int_1^2 (2-x+1)dx + \int_2^3 (x-2+2)\,dx$$

$$= x - \frac{x^2}{2}\Bigg|_{-1}^{0} + 2x - \frac{x^2}{2}\Bigg|_0^1 + 3x - \frac{x^2}{2}\Bigg|_1^2 + \frac{x^2}{2}\Bigg|_2^3$$

$$= -\left(-1-\frac{1}{2}\right) + \left(2-\frac{1}{2}\right) + (6-2) - \left(3-\frac{1}{2}\right) + \frac{9}{2} - 2 = 7.$$

◉ **Example 89:** If $f(x) =$

$$\begin{vmatrix} \sin x + \sin 2x + \sin 3x & \sin 2x & \sin 3x \\ 3 + 4\sin x & 3 & 4\sin x \\ 1 + \sin x & \sin x & 1 \end{vmatrix}$$

then the value of $\int_0^{\pi/2} f(x)\,dx$ is

(a) 3 (b) 0
(c) 2/3 (d) 1/3

Ans. (d)

◎ **Solution:** By applying the operation $C_1 \to C_1 - C_2 - C_3$, $f(x)$ can be written as

$$f(x) = \begin{vmatrix} \sin x & \sin 2x & \sin 3x \\ 0 & 3 & 4\sin x \\ 0 & \sin x & 1 \end{vmatrix}$$

$$= \sin x\,(3 - 4\sin^2 x) = \sin 3x$$

So $\int_0^{\pi/2} f(x)\,dx = -\frac{\cos 3x}{3}\Bigg|_0^{\pi/2} = \frac{1}{3}.$

◉ **Example 90:** If a is a positive integer, then the number of values of a satisfying

$$\int_0^{\pi/2} \left\{a^2\left(\frac{\cos 3x}{4} + \frac{3}{4}\cos x\right) + a\sin x - 20\cos x\right\} dx \le \frac{a^2}{3} \text{ are}$$

(a) one (b) two
(c) three (d) four

Ans. (d)

◎ **Solution:** The L.H.S. of the above inequality is equal to $a^2\left(\frac{\sin 3x}{12} + \frac{3}{4}\sin x\right) - a\cos x - 20\sin x\,\Big|_0^{\pi/2} =$ $a^2\left(-\frac{1}{12}+\frac{3}{4}\right) - a\,(0-1) - 20 = \frac{2a^2}{3} + a - 20.$

Thus the given inequality is $(2a^2/3) + a - 20 \le -a^2/3$

i.e., $a^2 + a - 20 \le 0 \Leftrightarrow -5 \le a \le 4$

Since a is a positive integer so $a = 1, 2, 3, 4$.

◉ **Example 91:** The area bounded by the curve $y = f(x) = x^4 - 2x^3 + x^2 + 3$, x-axis and ordinates corresponding to minimum of the function $f(x)$ is

(a) 1 (b) 91/30
(c) 30/9 (d) 4

Ans. (b)

◎ **Solution:** $f'(x) = 4x^3 - 6x^2 + 2x = 2x(2x^2 - 3x + 1) = 2x\,(2x-1)\,(x-1)$.

Since f is a differentiable function, so extremum points of $f(x)$, we must have $f'(x) = 0$ so $x = 0, 1/2, 1$. Now $f''(x) = 12x^2 - 12x + 2$, $f''(0) = 2$, $f''(1) = 2$ and $f''(1/2) = 3 - 6 + 2 = -1$. Thus the function has minimum at $x = 0$ and $x = 1$. Therefore, the required area $= \int_0^1 (x^4 - 2x^3 + x^2 + 3)\,dx$ (See Chapter 14 for area as definite integral)

$$= \left(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} + 3x\right)\Bigg|_0^1 = \frac{1}{5} - \frac{1}{2} + \frac{1}{3} + 3 = \frac{91}{30}.$$

Example 92: The value of $\int_0^{\pi} \sin^n x \cos^{2m+1} x \, dx$ is

(a) $\dfrac{(2m+1)!}{(n!)^2}$ (b) $\dfrac{(2m+1)!}{n!}$

(c) $\int_0^{\pi} \cos^{2m-1} x \, dx$ (d) none of these

Ans. (c)

Solution: $I = \int_0^{\pi} \sin^n x \cos^{2m+1} x \, dx$

$$= \int_0^{\pi} \sin^n (\pi - x) \cos^{2m+1} (\pi - x) \, dx$$

$$= -\int_0^{\pi} \sin^n x \cos^{2m+1} x \, dx = -I.$$

So $2I = 0$ which implies that $I = 0$ Also, $\int_0^{\pi} \cos^{2m-1} x \, dx = 0$ by a similar reasoning.

Example 93: The value of the integral

$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1} \frac{2x}{1+x^2} \, dx \text{ is}$$

(a) 0 (b) $\dfrac{\pi}{\sqrt{3}} + \log \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

(c) $\dfrac{1}{\sqrt{3}} + \dfrac{\pi}{2} \log \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ (d) none of these

Ans. (d)

Solution: $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1} \frac{2x}{1+x^2} \, dx$

$$= \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \left(\frac{\pi}{2} - \sin^{-1} \frac{2x}{1+x^2} \right) dx$$

$$= \frac{\pi}{2} \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \, dx$$

($\sin^{-1}(-x) = -\sin^{-1} x$, so the last integral is zero)

$$= \pi \int_0^{1/\sqrt{3}} \left(-1 + \frac{1}{1-x^4} \right) dx$$

$$= \pi \int_0^{1/\sqrt{3}} \left(-1 + \frac{1}{2} \left(\frac{1}{1-x^2} + \frac{1}{1+x^2} \right) \right) dx$$

$$= \left[-\frac{\pi}{\sqrt{3}} + \frac{\pi}{2} \left[\frac{1}{2} \log \frac{1+x}{1-x} + \tan^{-1} x \right] \Bigg|_0^{1/\sqrt{3}} \right]$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2} \left[\frac{1}{2} \log \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\pi}{6} \right].$$

Example 94: The value of $\int_0^{\pi/2} \sqrt{\sin 2\theta} \, \sin \theta \, d\theta$ is

(a) 1 (b) 0

(c) $\pi/2$ (d) $\pi/4$

Ans. (d)

Solution: $I = \int_0^{\pi/2} \sqrt{\sin 2\theta} \, \sin \theta \, d\theta$

$$= \int_0^{\pi/2} \sqrt{\sin (\pi - 2\theta)} \, \sin (\pi/2 - \theta) d\theta$$

$$= \int_0^{\pi/2} \sqrt{\sin 2\theta} \cos \theta \, d\theta$$

So $\quad 2I = \int_0^{\pi/2} \sqrt{\sin 2\theta} \, (\sin \theta + \cos \theta) d\theta$

In $2I$, put $\sin \theta - \cos \theta = t$, so that $t^2 = 1 - \sin 2\theta \Rightarrow 1 - t^2 = \sin 2\theta$. Also $dt = (\cos \theta + \sin \theta) \, d\theta$.

$$\therefore \quad 2I = \int_{-1}^{1} \sqrt{1-t^2} \, dt = 2 \left[\frac{t}{2} \sqrt{1-t^2} + \frac{1}{2} \sin^{-1} t \right]_0^1$$

$$= 2 \frac{\pi}{4} = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}.$$

Example 95: Given $I_m = \int_1^e (\log x)^m \, dx$. If $\dfrac{I_m}{K} + \dfrac{I_{m-2}}{L} = e$ then values of K and L are

(a) $1 - m, \dfrac{1}{m}$ (b) $\dfrac{1}{1-m}, m$

(c) $\dfrac{1}{1-m}, \dfrac{m(m-2)}{m-1}$ (d) $\dfrac{m}{m-1}, m-2$

Ans. (a)

Solution: $I_m = \int_1^e (\log x)^m \, dx = x (\log x)^m \Big|_1^e - m \int_1^e (\log x)^{m-1} \, dx$

$$= e - m \left[x (\log x)^{m-1} \Big|_1^e - (m-1) \int_1^e (\log x)^{m-2} \, dx \right]$$

$$= e - me + m(m-1) I_{m-2} = (1-m)e + m(m-1) I_{m-2}$$

So $\quad \dfrac{I_m}{1-m} + m I_{m-2} = e$. Thus $K = 1 - m$ and $L = \dfrac{1}{m}$.

Example 96: The numbers P, Q and R for which the function $f(x) = Pe^{2x} + Qe^x + Rx$ satisfies $f(0) = -1$, $f'(\log 2) = 31$ and $\int_0^{\log 4} [f(x) - Rx] \, dx = 39/2$ are given by

(a) $P = 2, Q = -3, R = 4$
(b) $P = -5, Q = 2, R = 3$
(c) $P = 5, Q = -2, R = 3$
(d) $P = 5, Q = -6, R = 3$

Ans. (d)

◎ **Solution:** We have $f'(x) = 2Pe^{2x} + Qe^{x} + R$, so that $31 = f'(\log 2) = 8P + 2Q + R$. Also, $-1 = f(0) = P + Q$. Further,

$$\frac{39}{2} = \int_0^{\log 4} [f(x) - Rx]\,\mathrm{d}x = \int_0^{\log 4} [Pe^{2x} + Qe^{x}]\,\mathrm{d}x$$

$$= \frac{P}{2}e^{2x} + Qe^{x}\Big|_0^{\log 4} = \frac{P}{2} \times 16 + 4Q - \frac{P}{2} - Q$$

$$= \frac{15P}{2} + 3Q$$

Solving the above equations, we get $P = 5$, $Q = -6$ and $R = 3$.

◉ **Example 97:** For $0 < \alpha < \pi$. The value of the definite integral $\int_0^1 \mathrm{d}x/(x^2 + 2x\cos\alpha + 1)$ is equal to

(a) $\alpha/(2\sin\alpha)$
(b) $\tan^{-1}(\sin\alpha)$
(c) $\alpha\sin\alpha$
(d) $(\alpha/2)(\sin\alpha)$

Ans. (a)

◎ **Solution:** $\displaystyle\int_0^1 \frac{\mathrm{d}x}{x^2 + 2x\cos\alpha + 1}$

$$= \int_0^1 \frac{\mathrm{d}x}{(x + \cos\alpha)^2 + 1 - \cos^2\alpha}$$

$$= \int_0^1 \frac{\mathrm{d}x}{(x + \cos\alpha)^2 + \sin^2\alpha}$$

$$= \frac{1}{\sin\alpha}\tan^{-1}\frac{x + \cos\alpha}{\sin\alpha}\Bigg|_0^1$$

$$= \frac{1}{\sin\alpha}\left(\tan^{-1}\left(\frac{1 + \cos\alpha}{\sin\alpha}\right) - \tan^{-1}\frac{\cos\alpha}{\sin\alpha}\right)$$

$$= \frac{1}{\sin\alpha}\left(\tan^{-1}\left(\cot\frac{\alpha}{2}\right) - \tan^{-1}\cot\alpha\right)$$

$$= \frac{1}{\sin\alpha}\left(\frac{\pi}{2} - \frac{\alpha}{2} - \frac{\pi}{2} + \alpha\right) = \frac{\alpha}{2}\sin\alpha.$$

◉ **Example 98:** The value of the integral $\displaystyle\int_0^\infty \frac{x\log x}{\left(1 + x^2\right)^2}\,\mathrm{d}x$ is

(a) 7
(b) 0
(c) 5 log 13
(d) 2 log 5

Ans. (b)

◎ **Solution:** $\displaystyle\int_0^\infty \frac{x\log x}{\left(1 + x^2\right)^2}\,\mathrm{d}x$

$$= \int_0^1 \frac{x\log x}{\left(1 + x^2\right)^2}\,\mathrm{d}x + \int_1^\infty \frac{x\log x}{\left(1 + x^2\right)^2}\,\mathrm{d}x$$

Put $x = 1/y$ in the second integral, so that $\mathrm{d}x = (-1/y^2)\mathrm{d}y$. If $x \to \infty$, then $y \to 0$, and if $x = 1$, then $y = 1$.

$$\therefore\quad \int_1^\infty \frac{x\log x}{\left(1 + x^2\right)^2}\,\mathrm{d}x = \int_1^0 \frac{1\cdot\log y^{-1}}{y\left(1 + \dfrac{1}{y^2}\right)^2}\left(-\frac{1}{y^2}\right)\mathrm{d}y$$

$$= -\int_0^1 \frac{y\log y}{\left(1 + y^2\right)^2}\,\mathrm{d}y$$

$$\Rightarrow\quad \int_0^\infty \frac{x\log x}{\left(1 + x^2\right)^2}\,\mathrm{d}x = \int_0^1 \frac{x\log x}{\left(1 + x^2\right)^2}\,\mathrm{d}x - \int_0^1 \frac{x\log x}{\left(1 + x^2\right)^2}\,\mathrm{d}x = 0.$$

◉ **Example 99:** If I is the greatest of the definite integrals

$$I_1 = \int_0^1 e^{-x}\cos^2 x\,\mathrm{d}x, \qquad I_2 = \int_0^1 e^{-x^2}\cos^2 x\,\mathrm{d}x,$$

$$I_3 = \int_0^1 e^{-x^2}\,\mathrm{d}x, \qquad I_4 = \int_0^1 e^{-x^2/2}\,\mathrm{d}x$$

then

(a) $I = I_1$
(b) $I = I_2$
(c) $I = I_3$
(d) $I = I_4$

Ans. (d)

◎ **Solution:** For $0 < x < 1$, we have $(1/2)x^2 < x^2 < x$ i.e., $-x^2 > -x$, so that $e^{-x^2} > e^{-x}$. Hence $\int_0^1 e^{-x^2}\cos^2 x\,\mathrm{d}x > \int_0^1 e^{-x}\cos^2 x\,\mathrm{d}x$. Also $\cos^2 x \le 1$, therefore

$$\int_0^1 e^{-x^2}\cos^2 x\,\mathrm{d}x \le \int_0^1 e^{-x^2}\,\mathrm{d}x < \int_0^1 e^{-x^2/2}\,\mathrm{d}x = I_4.$$

Hence I_4 is the greatest integral.

◉ **Example 100:** The value of the integral $\displaystyle\int_0^1 \frac{x^c - 1}{\log x}\,\mathrm{d}x$, $c > 0$ is

(a) $\log c$
(b) $2\log(c + 1)$
(c) $3\log c$
(d) $\log(c + 1)$

Ans. (d)

◎ **Solution:** Note that the integral is not an elementary function.

$$\text{Let } I(c) = \int_0^1 \frac{x^c - 1}{\log x}\,\mathrm{d}x \text{ so}$$

$$I'(c) = \int_0^1 \frac{x^c\log x\,\mathrm{d}x}{\log x} = \int_0^1 x^c\,\mathrm{d}x = \frac{1}{c + 1}.$$

Hence $\quad I(c) = \displaystyle\int \frac{1}{c + 1}\,\mathrm{d}c + C = \log|c + 1| + C.$

But $\quad I(0) = 0$, hence $C = 0$ so $I(c) = \log(c + 1)$.

◉ **Example 101:** The value of the integral $\displaystyle\int_{-1}^1 \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\frac{1}{x}\right)\mathrm{d}x$ is

(a) $\pi/2$
(b) $\pi/4$
(c) $-\pi/2$
(d) none of these

Ans. (c)

◎ **Solution:** $\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right) = \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) = \frac{-1}{1+x^2}$

$$\Rightarrow \quad I = \int_{-1}^{1}\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)dx = \int_{-1}^{1}\frac{-dx}{1+x^2} = -\tan^{-1}x\Big|_{-1}^{1}$$

$$= -\frac{\pi}{4}+\left(-\frac{\pi}{4}\right) = -\frac{\pi}{2}$$

Note that $I = \tan^{-1}(1/x)|_{-1}^{1} = \frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}$ is incorrect, since the function $\tan^{-1}(1/x)$ is not an antiderivative of $(d/dx)\,[\tan^{-1}(1/x)]$ on the interval $[-1, 1]$.

◉ **Example 102:** The value of $\int_0^{\pi}\frac{\sin(n+1/2)\,x}{\sin(x/2)}\,dx\,(n\in\mathbf{N})$ is

(a) π (b) 2π

(c) 3π (d) none of these

Ans. (a)

◎ **Solution:** We have,

$$2\sin\frac{x}{2}\left(\frac{1}{2}+\cos x+\cos 2x+\ldots+\cos nx\right)$$

$$= \sin\frac{x}{2} + 2\sin\frac{x}{2}\cos x + 2\sin\frac{x}{2}\cos 2x + \ldots + 2\sin\frac{x}{2}\cos nx$$

$$= \sin\frac{x}{2} + \sin\frac{3x}{2} - \sin\frac{x}{2} + \sin\frac{5x}{2} - \sin\frac{3x}{2} + \ldots$$

$$+ \sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x = \sin\left(n+\frac{1}{2}\right)x$$

$$\therefore \quad \frac{1}{2} + \cos x + \cos 2x + \ldots + \cos nx = \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin(x/2)}$$

$$\Rightarrow \quad \int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin(x/2)}\,dx$$

$$= 2\left(\int_0^{\pi}\frac{1}{2}\,dx + \int_0^{\pi}\cos x\,dx + \ldots + \int_0^{\pi}\cos nx\,dx\right)$$

$$= 2\left(\frac{\pi}{2} + \sin x\Big|_0^{\pi} + \ldots + \frac{\sin nx}{n}\Big|_0^{\pi}\right) = \pi$$

◉ **Example 103:** The equation of the tangent to the curve

$$y = \int_{x^2}^{x^3}\frac{dt}{\sqrt{1+t^2}}$$

at $x = 1$ is

(a) $\sqrt{2}\;y - 1 = x$ (b) $\sqrt{3}\;x - 1 = y$

(c) $\sqrt{3}\;x + 1 = y$ (d) $\sqrt{2}y + 1 = x$

Ans. (d)

◎ **Solution:** Differentiating the given equation, we get

$$\frac{dy}{dx} = \left[\frac{1}{\sqrt{1+t^2}}\right]_{t=x^3}\cdot\frac{d}{dx}\left(x^3\right) - \left[\frac{1}{\sqrt{1+t^2}}\right]_{t=x^2}\cdot\frac{d}{dx}\left(x^2\right)$$

(using Property 17)

$$= \frac{3x^2}{\sqrt{1+x^6}} - \frac{2x}{\sqrt{1+x^4}}$$

Therefore the slope of the required tangent is

$$\left(\frac{dy}{dx}\right)_{x=1} = \frac{3}{\sqrt{2}} - \frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

This tangent passes through the point $x = 1$ and $y(1) = \int_1^1\frac{dt}{\sqrt{1+t^2}} = 0$, so that its equation is $y - 0 = \frac{1}{\sqrt{2}}\,(x-1)$

$$\Rightarrow \quad \sqrt{2}\,y + 1 = x.$$

◉ **Example 104:** The mean value of the function $f(x) = \frac{1}{x^2+x}$ on the interval [1, 3/2] is

(a) log (6/5) (b) 2 log (6/5)

(c) 4 (d) log 3/5

Ans. (b)

◎ **Solution:** Mean value $= \frac{1}{b-a}\int_a^b f(x)\,dx$ (Property 14)

$$= \frac{1}{3/2-1}\int_1^{3/2}\frac{1}{x^2+x}\,dx = 2\int_1^{3/2}\left[\frac{1}{x}-\frac{1}{x+1}\right]dx$$

$$= 2\,(\log x - \log(x+1))\Big|_1^{3/2}$$

$$= 2[\log(3/2) - \log(5/2) - (\log 1 - \log 2)]$$

$$= 2\log(6/5).$$

◉ **Example 105:** The value of $\int_1^{16}\tan^{-1}\sqrt{\sqrt{x}-1}\,dx$ is

(a) $\frac{16\pi}{3} + 2\sqrt{3}$ (b) $\frac{4}{3}\pi - 2\sqrt{3}$

(c) $\frac{4}{3}\pi + 2\sqrt{3}$ (d) $\frac{16}{3}\pi - 2\sqrt{3}$

Ans. (d)

◎ **Solution:** Integrating by parts, the given integral is equal to

$$x\tan^{-1}\sqrt{\sqrt{x}-1}\Big|_1^{16} - \int_1^{16}\frac{x}{\sqrt{x}}\frac{1}{4\sqrt{x}\sqrt{\sqrt{x}-1}}\,dx$$

$$= \frac{16}{3}\pi - \frac{1}{4}\int_1^{16} \frac{dx}{\sqrt{\sqrt{x}-1}}$$

$$= \frac{16}{3}\pi - \frac{1}{4}\int_0^{\sqrt{3}} \frac{4t\left(1+t^2\right)}{t}\,\mathrm{d}t \ (\ \sqrt{x} = 1 + t^2)$$

$$= \frac{16}{3}\ \pi - \left(\sqrt{3}+\sqrt{3}\right) = \frac{16}{3} - 2\sqrt{3}\,.$$

Example 106: Let $g(x) = \int_0^x f(t)\,\mathrm{d}t$, where f is such that $1/2 \le f(t) \le 1$ for $0 \le t \le 1$ and $0 \le f(t) \le 1/2$ for $1 \le t \le 2$. Then $g(2)$ satisfies the inequality

(a) $-3/2 \le g(2) < 1/2$ (b) $0 \le g(2) < 2$
(c) $3/2 < g(2) \le 5/2$ (d) $2 < g(2) < 4$

Ans. (b)

Solution: $g(2) = \int_0^2 f(t)\,\mathrm{d}t = \int_0^1 f(t)\,\mathrm{d}t + \int_1^2 f(t)\,\mathrm{d}t$

Since $1/2 \le f(t) \le 1$ for $t \in [0, 1]$ so $1/2 \le \int_0^1 f(t)\,\mathrm{d}t \le 1$ by Property 13. Also $0 \le f(t) \le 1/2$ for $t \in [1, 2]$ so $0 \le \int_1^2 f(t)\,\mathrm{d}t \le 1$. Hence $0 + 1/2 \le \int_0^1 f(t)\,\mathrm{d}t + \int_1^2 f(t)\,\mathrm{d}t \le 1 + 1/2$ i.e. $1/2 \le g(2) \le 3/2 \Rightarrow 0 \le g(2) < 2$.

Example 107: A function f is defined by $f(x) = \frac{1}{2^{r-1}}$, $\frac{1}{2^r} < x \le \frac{1}{2^{r-1}}$, $r = 1, 2, 3, \ldots$ then the value of $\int_0^1 f(x)\,\mathrm{d}x$ is equal

(a) 1/3 (b) 1/4
(c) 2/3 (d) 1/3

Ans. (c)

Solution: $\int_0^1 f(x)\,\mathrm{d}x = \sum_{r=1}^{\infty} \int_{2^{-r}}^{2^{-(r-1)}} \frac{1}{2^{r-1}}\,\mathrm{d}x$

$$= \sum_{r=1}^{\infty} \frac{1}{2^{r-1}} [2^{-(r-1)} - 2^{-r}]$$

$$= \sum_{1}^{\infty} 2^{-2(r-1)} - \sum_{1}^{\infty} 2^{-2r+1}$$

$$= (2^2 - 2)\sum_{1}^{\infty} 2^{-2r} = 2 \cdot \frac{1}{4} \cdot \frac{1}{1-1/4} = \frac{2}{3}$$

Example 108: If $y = \int_1^{t^3} \sqrt[3]{z}\ \log z\,\mathrm{d}z$ and $x = \int_{\sqrt{t}}^{3} z^2 \log z\,dz$ then $\frac{\mathrm{d}y}{\mathrm{d}x}$ is

(a) $-4t^{5/2}$ (b) $32t^{5/2}$
(c) $-8t^{3/2}$ (d) $-36t^{5/2}$

Ans. (d)

Solution: As is known, we find $\frac{\mathrm{d}y}{\mathrm{d}t}$ and $\frac{\mathrm{d}x}{\mathrm{d}t}$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t \log t^3 \times 3t^2 = 9t^3 \log t \qquad \text{(Property 17)}$$

and $$\frac{\mathrm{d}x}{\mathrm{d}t} = 0 - t \log \sqrt{t} \times \frac{1}{2\sqrt{t}} = -\frac{1}{4}\sqrt{t}\ \log t.$$

So $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9t^3 \log t}{-\frac{1}{4}\sqrt{t}\log t} = -36t^2\sqrt{t}\,.$$

Example 109: Let $I_1 = \int_{-4}^{-5} e^{(x-5)^2}\ \mathrm{d}x$ and $I_2 = 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}\ \mathrm{d}x$ then the value of $I_1 + I_2$ is

(a) 0 (b) 4/3
(c) 7/4 (d) 5/4

Ans. (a)

Solution: Put $x = -t - 4$ in I_1 so

$$I_1 = \int_{-4}^{-5} e^{(x+5)^2}\ dx = -\int_0^1 e^{(-t+1)^2}\ dt = -\int_0^1 e^{(t-1)^2}\ dt$$

Applying the substituting $x = \frac{t}{3} + \frac{1}{3}$ in I_2, we have

$$I_2 = 3\int_{1/3}^{2/3} e^{9(x-2/3)^2}\ \mathrm{d}x = \int_0^1 e^{(t-1)^2}\ \mathrm{d}t. \text{ Thus } I_1 + I_2 = 0.$$

Example 110: A line tangent to the graph of the function $y = f(x)$ at the point $x = a$ forms an angle $\pi/3$ with the axis of abscissas and an angle $\pi/4$ at the point $x = b$ then $\int_a^b f''(x)\,dx$ is (f'' is assumed to be a continuous function)

(a) 0 (b) $1 - \sqrt{3}$
(c) $\sqrt{2} - 1$ (d) $\sqrt{3} - 1$

Ans. (b)

Solution: $\int_a^b f''(x)\,\mathrm{d}x = f'(b) - f'(a)$. According to the given conditions $f'(b) = \tan \pi/4 = 1$ and $f'(a) = \tan \pi/3 = \sqrt{3}$. Hence the required integral equals to $1 - \sqrt{3}$.

Example 111: The value of $\int_0^{\pi} [2 \sin x]\,\mathrm{d}x$, where [,] represents the greatest integer function is

(a) π (b) 2π
(c) $-\pi$ (d) $2\pi/3$

Ans. (d)

Solution: $\int_0^{\pi} [2\sin x]\,\mathrm{d}x$

$$= \int_0^{\pi/6} [2\sin x]\,\mathrm{d}x + \int_{\pi/6}^{\pi/2} [2\sin x]\,\mathrm{d}x + \int_{\pi/2}^{5\pi/6} [2\sin x]\,\mathrm{d}x + \int_{5\pi/6}^{\pi} [2\sin x]\,\mathrm{d}x$$

$$= \int_0^{\pi/6} 0\,dx + \int_{\pi/6}^{\pi/2} 1\,dx + \int_{\pi/2}^{5\pi/6} 1\,dx + \int_{5\pi/6}^{\pi} 0\,dx$$

$$= \frac{\pi}{2} - \frac{\pi}{6} + \frac{5\pi}{6} - \frac{\pi}{2} = \frac{2\pi}{3}.$$

Example 112: For any $t \in \mathbf{R}$ and f a continuous function, let

$$I_1 = \int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))\,\mathrm{d}x$$

and $$I_2 = \int_{\sin^2 x}^{1+\cos^2 t} f(x(2-x))\,\mathrm{d}x$$ then I_1/I_2 is equal to

(a) 2 (b) 1

(c) 4 (d) none of these

Ans. (b)

Solution: $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} (2-x)f((2-x)(2-(2-x)))\,\mathrm{d}x$

$$= \int_{\sin^2 t}^{1+\cos^2 t} (2-x)f(x(2-x))\,dx$$

$$= 2\int_{\sin^2 t}^{1+\cos^2 t} f(x(2-x))\,\mathrm{d}x -$$

$$\int_{\sin^2 t}^{1+\cos^2 t} xf(x(2-x))\,\mathrm{d}x = 2I_2 - I_1$$

Therefore, $2I_1 = 2I_2$ and so $I_1/I_2 = 1$.

EXERCISE

Concept-based Straight Objective Type Questions

1. The value of $\int_{\pi/4}^{\pi/2} \cot\theta\ \operatorname{cosec}^2\theta\ \mathrm{d}\theta$ is

(a) $-\frac{1}{2}$ (b) $\frac{1}{2}$

(c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

2. The value of $\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right)$ is

(a) π (b) $-\frac{\pi}{2}$

(c) $\frac{1}{\pi}$ (d) $\frac{2}{\pi}$

3. The value of $\int_{-\pi/2}^{\pi/2} \frac{x\sin x}{e^x+1}\,dx$ is equal to

(a) 2π (b) 1

(c) 2 (d) $\frac{\pi}{2}$

4. $\int_{-\pi/2}^{\pi/2} (x^{14}\sin^{11}x + \sin^2 x)\,\mathrm{d}x$ is equal to

(a) 0 (b) π

(c) $\frac{\pi}{2}$ (d) $-\pi$

5. $\int_0^{\pi/2} \frac{dx}{2\cos x + 3}$ is equal to

(a) $\frac{2}{\sqrt{5}}\tan^{-1}\frac{1}{\sqrt{5}}$ (b) $\frac{1}{\sqrt{5}}\tan^{-1}\frac{1}{\sqrt{5}}$

(c) 1 (d) $\frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{5}}$

6. If $\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\mathrm{d}x = \frac{\pi}{4} + A$, then A is equal to

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) $-\frac{1}{2}$

7. If $\int_0^2 \frac{dx}{\sqrt{x+1} + \sqrt{(x+1)^3}}$ is equal to $k\pi$ then k is equal to

(a) $\frac{1}{2}$ (b) 2

(c) $\frac{1}{6}$ (d) $\frac{1}{3}$

8. If f is continuously differentiable function then $\int_0^{2.5} [x^2]\ f'(x)\ \mathrm{d}x$ is equal to

(a) $6f(6.25) + \sum_{i\in\{1,\sqrt{2},\sqrt{3},2,\sqrt{5}\}} f(i)$

(b) $6f(6.25) - \sum_{i\in\{1,\sqrt{2},\sqrt{3},2,\sqrt{5},\sqrt{6}\}} f(i)$

(c) $6f(2.5) - \sum_{i \in \{1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}\}} f(i)$

(d) $6f(2.5) - \sum_{i \in \{1, \sqrt{2}, \sqrt{3}, \sqrt{5}\}} f(i)$

9. The value of $\int_0^1 x^3(1-x)^{11}\,dx$ is equal to

(a) $\frac{1}{11} + \frac{3}{12} + \frac{3}{13} + \frac{1}{14}$

(b) $-\frac{1}{11} + \frac{3}{12} - \frac{3}{13} + \frac{1}{14}$

(c) $\frac{8}{13}$

(d) $\frac{1}{11} - \frac{3}{12} + \frac{3}{14} - \frac{1}{14}$

10. The value of $\int_0^3 |2-x|\,dx$ is

(a) 1 (b) 2
(c) 4 (d) 3

LEVEL 1

Straight Objective Type Questions

11. The value of $\int_{-2}^{2} |1 - x|\,dx$ is

(a) 2 (b) 0
(c) 4 (d) 5

12. If $f(x) = \int_0^x \sin^8 t\,dt$, then $f(x + \pi)$ equals

(a) $\frac{f(x)}{f(\pi)}$ (b) $f(x)\,f(\pi)$
(c) $f(x) + f(\pi)$ (d) $f(x) - f(\pi)$

13. Suppose that the graph of $y = f(x)$, contains the points (0, 4) and (2, 7). If f' is continuous then $\int_0^2 f'(x)\,dx$ is equal to

(a) 2 (b) – 2
(c) 3 (d) none of these

14. A polynomial P is positive for $x > 0$, and the area of the region bounded by $P(x)$, the x-axis, and the vertical lines $x = 0$ and $x = K$ is $K^2(K + 3)/3$. The polynomial $P(x)$ is

(a) $x^2 + 2x$ (b) $x^2 + x + 1$
(c) $x^2 + 2x + 1$ (d) $x^3 + 1$

15. The value of $\lim_{x \to 0} \frac{\int_0^x \sin t^2\,dt}{\sin x^2}$ is

(a) 1 (b) 0
(c) 2 (d) none of these

16. Let $f(x) = \int_1^x e^{-t^2/2}(1 - t^2)\,dt$ then

(a) f has maximum at $x = 0$
(b) f has minimum at $x = -1$
(c) f has maximum at $x = -1$
(d) f has no critical point

17. A line tangent to the graph of the function $y = f(x)$ at the point $x = a$ forms an $\pi/3$ with y-axis and at $x = b$ an angle $\pi/4$ with x-axis then $\int_a^b f''(x)\,dx$ is

(a) $1 - \frac{1}{\sqrt{3}}$ (b) $-\frac{\pi}{12}$
(c) $\frac{\pi}{12}$ (d) $\sqrt{3} - 1$

18. The function $F(x) = \int_0^x \log \frac{1-t}{1+t}\,dt$ is

(a) an even function (b) an odd function
(c) a periodic function (d) none of these

19. If $f(n) = \frac{1}{n}[(n + 1)(n + 2) \ldots (n + n)]^{1/n}$, then $\lim_{n \to \infty} f(n)$ equals

(a) e (b) $1/e$
(c) $2/e$ (d) $4/e$

20. The value of the integral $\int_\alpha^\beta \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ for $\beta > \alpha$, is

(a) $\sin^{-1} \alpha/\beta$ (b) $\pi/2$
(c) $\sin^{-1} \beta/2\alpha$ (d) π.

21. The value of the integral $\int_{-\pi/3}^{\pi/3} \frac{x \sin x}{\cos^2 x}\, dx$ is

(a) $\left(\frac{\pi}{3} - \log \tan \frac{3\pi}{2}\right)$

(b) $2\left(\frac{2\pi}{3} - \log\left(2 + \sqrt{3}\right)\right)$

(c) $3\left(\frac{\pi}{2} - \log \tan \frac{\pi}{12}\right)$

(d) none of these.

22. The value of $\int_{1/e}^{\tan x} \frac{t}{1+t^2}\, dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$ is

(a) 1/2 (b) 1

(c) $\pi/4$ (d) none of these

23. The equation of the tangent to the curve

$$y = \int_{x^4}^{x^6} \frac{dt}{\sqrt{1+t^2}} \quad \text{at } x = 1 \text{ is}$$

(a) $\sqrt{2}\, y + 1 = x$ (b) $\sqrt{3}\, x + 1 = y$

(c) $\sqrt{3}\, x + 1 + \sqrt{3} = y$ (d) none of these

24. The value of $\int_{-1}^{1} x|x|\, dx$ is

(a) 2 (b) 1

(c) 0 (d) none of these

25. The difference between the greatest and least values of the function

$F(x) = \int_0^x (t + 1)\, dt$ on [1, 3] is

(a) 8 (b) 2

(c) 6 (d) 11/2

26. The value of $\lim_{x \to \infty} \frac{\left(\int_0^x e^x\, dx\right)^2}{\int_0^x e^{2x^2}\, dx}$ is

(a) 1 (b) 2

(c) 3 (d) 0

27. The absolute value of $\int_{10}^{19} \frac{\sin x}{1+x^8}\, dx$ is

(a) less than 10^{-7} (b) more than 10^{-7}

(c) less than 10^{-9} (d) none of these

28. The value of the integral $\int_0^3 \frac{dx}{\sqrt{x+1} + \sqrt{5x+1}}$ is

(a) 11/15 (b) 14/15

(c) 2/5 (d) none of these

29. Let $f(x) = \{x\}$, the fractional part of x then $\int_{-1}^{1} f(x)\, dx$ is equal to

(a) 1 (b) 2

(c) 0 (d) 1/2

30. The value of $\int_{-\pi/2}^{\pi/2} \cos t \sin (2t - \pi/4)\, dt$ is

(a) – 1/3 (b) 1/3

(c) $\sqrt{2}/3$ (d) $-\sqrt{2}/3$

31. The value of $\int_0^{\infty} \frac{x \log x}{\left(1+x^2\right)^2}\, dx$ is

(a) 0 (b) 1

(c) $\sqrt{3}/2$ (d) $\sqrt{5}/2$

32. $\int_0^1 |\sin 2\pi x|\, dx$ is equal to

(a) 0 (b) $-1/\pi$

(c) $1/\pi$ (d) $2/\pi$

33. The value of $\lim_{x \to \infty} \frac{\int_0^x \left(\tan^{-1} x\right)^2 dx}{\sqrt{x^2+1}}$ is

(a) $\pi/4$ (b) $\pi^2/2$

(c) $\pi^2/4$ (d) π

34. If $A(t) = \int_{-1}^{t} e^{-|x|}\, dx$, then $\lim_{t \to \infty} A(t)$ is equal to

(a) $2 - e^{-1}$ (b) $3 - e^{-1}$

(c) 4 (d) 0

35. If the value of $\int_{-2}^{2} |x \cos \pi x|\, dx = k/\pi$ then the value of k is

(a) 4 (b) 8

(c) 12 (d) none of these

36. The value of $\int_0^{\alpha} \frac{dx}{1 - \cos\alpha \cos x}$ $(0 < \alpha < \pi/2)$ is

(a) $\frac{\pi}{\sin\alpha}$ (b) $\frac{\pi}{2\cos\alpha}$

(c) $\frac{\pi}{\cos\alpha}$ (d) $\frac{\pi}{2\sin\alpha}$

37. The value of

$$\lim_{n\to\infty}\left(\frac{1}{n} + \frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \cdots + \frac{n}{(2n-1)^2}\right) \text{ is}$$

(a) 1 (b) 1/3

(c) 1/2 (d) 3/2

38. The value of

$$\lim_{n\to\infty}\left(\frac{1}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \cdots + \frac{n^2}{n^3+n^3}\right) \text{ is}$$

(a) $\frac{1}{3}$ (b) $\frac{1}{3}\log 2$

(c) $\frac{1}{2}\log 3$ (d) $\frac{1}{3}\log 3$

39. The value of $\int_0^{\pi} \frac{\sin 2nx}{\sin x}\, dx$ is

(a) 0
(b) 1
(c) 1/2
(d) $\pi/3$

40. If $f(x) = A \sin \frac{\pi x}{2} + B$, $f'(1/2) = \sqrt{2}$ and $\int_0^1 f(x)\, dx = \frac{2A}{\pi}$ then the constants A and B are respectively

(a) $\pi/2, \pi/2$
(b) $2/\pi, 3/\pi$
(c) $0, -4/\pi$
(d) $4/\pi, 0$

41. If $f'(x) = g(x)$ for $a \le x \le b$ then $\int_a^b f(x)\, g(x)\, dx$ is equal to

(a) $g(b) - g(a)$
(b) $f(b) - f(a)$
(c) $\frac{(f(b))^2 - (f(a))^2}{2}$
(d) $\frac{(g(b))^2 - (g(a))^2}{2}$

42. Let $I_1 = \int_a^b \left(\int_a^x f(t)\,dt\right) dx$ and $I_2 = \int_a^b (b - x) f(x)\, dx$ then

(a) $I_1 = I_2$
(b) $I_1 = (b - a) + I_2$
(c) $I_1 = (b - a)\, I_2$
(d) none of these

43. The least value of the function $F(x) = \int_x^2 \log_{1/3} t\, dt$, $x \in [1/10, 4]$ is at $x =$

(a) 1/10
(b) 4
(c) 1
(d) none of these

44. The greatest value of the function $F(x) = \int_4^x (t^2 - 8t + 16)\, dt$ on $[0, 5]$ is

(a) 5
(b) 1/3
(c) 4
(d) 2/3

45. If for a continuous function f, $\int_{-a}^{a} f(x)\, dx = K \int_0^a (f(x) + f(-x))\, dx$ then the value of K is

(a) 1
(b) 1/2
(c) 2
(d) a

46. If $J_m = \int_1^e \log^m x\, dx$ then $J_8 + 8J_7$ is equal to

(a) 2
(b) e^8
(c) $2e$
(d) e

47. Let $f_1(x) = \int_0^x f(t)\, dt$, $f_2(x) = \int_0^x f_1(t)\, dt$ and $f_3(x) = \int_0^x f_2(t)\, dt$ if $f_3(x) = A \int_0^x f(t)\,(x - t)^2\, dt$ then the value of A is

(a) 1
(b) 1/2
(c) 2
(d) none of these

48. If the variables x and y are related by the equation $x = \int_0^y \frac{dx}{\sqrt{1+9x^2}}$ and $\frac{d^2y}{dx^2}$ is proportional to y then the constant of proportionality is

(a) 4
(b) 6
(c) 5
(d) 9

49. If $y = f(x)$ be an invertible function with inverse g and $h(x) = x\, f(x)$, then $\int_{f(a)}^{f(b)} g(x)dx + \int_a^b f(x)dx$ is equal to

(a) $h(a) - h(b)$
(b) $h(b) - h(a)$
(c) $h(a) + h(b)$
(d) $b\, h(b) - a\, h(a)$

50. The value of $\int_0^{[x]} \frac{2^t}{2^{[t]}}\, dt$

(a) $[x] \log 2$
(b) $\frac{[x]}{\log 2}$
(c) $\frac{1}{2}\frac{[x]}{\log 2}$
(d) $\frac{1}{4}\frac{[x]}{\log 2}$

51. If $a_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin x} dx$ then $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$ are in

(a) A.P.
(b) G.P.
(c) H.P.
(d) Arith – Geo progression

52. $\int_0^{21} [x]^4\, dx$ is equal

(a) 44100
(b) 2100
(c) 48400
(d) 42400

53. If $x^2 f(x) + f(1/x) = 0$ for all $x \in \mathbf{R} \sim \{0\}$ then $\int_{\cos\theta}^{\sec\theta} f(x)dx =$

(a) $\sin 2\theta$
(b) 1
(c) $\sec\theta - \cos\theta$
(d) 0

Assertion-Reason Type Questions

54. Let $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

Statement-1: $I = \pi$

Statement-2: The integrand is even and $\lim\limits_{b \to \infty} F(b)$ $= \pi$, $F(b) = \int_0^b \frac{dx}{1+x^2}$

55. Let $I = \int_0^{\pi/2} \frac{dx}{5+3\cos^2 x}$

Statement-1: $\frac{\pi}{16} \le I \le \frac{\pi}{10}$

Statement-2: $\frac{1}{8} \le \frac{1}{5+3\cos^2 x} \le \frac{1}{5}$.

56. Let $I = \int_0^{\pi} \frac{\cos x}{\sqrt{1-\sin^2 x}}\, dx$

Statement-1: $I = \pi$

Statement-2: The integrand can be expressed as $X_{[0,\ \pi/2]}\ X_{[\pi/2,\ \pi]}$; X_A being the characteristic function of A.

57. Let $I = \int_0^{\pi} x^2 \cos x\, dx$ and $J = \int_0^{\pi} x \sin x\, dx$

Statement-1: $I = -2\pi$

Statement-2: $I = 2J$

58. Let $I = \int_1^2 \frac{10x^2}{\left(x^3+1\right)^2}\, dx$ and $J = \int_2^9 \frac{dx}{x^2}$

Statement-1: $I = \frac{35}{27}$

Statement-2: $3I = 10J$

LEVEL 2

Straight Objective Type Questions

59. The value of $\int_{\pi/4}^{\pi/3} \frac{\mathrm{d}x}{\sin x + \tan x}$ is
    - (a) 3/4
    - (b) 1/2
    - (c) $2/3 - \sqrt{2}/2$
    - (d) none of these

60. If f is integrable on $[a, b]$ then $\int_a^b [f(x) - f_{\text{ave}}]\, \mathrm{d}x$ is equal to
    - (a) $f(b) - f(a)$
    - (b) $\frac{1}{2}(f(b) - f(a))$
    - (c) $f(a) - f(b)$
    - (d) 0

61. If $m \neq n$, $m, n \in \mathbf{N}$ then the value of $\int_0^{2\pi} \cos mx \cos nx\, \mathrm{d}x$ is
    - (a) 0
    - (b) 2π
    - (c) π
    - (d) dependent on m and n

62. If $Q(x) = \frac{1}{2} a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \ldots + a_n \cos nx + b_n \sin nx$, then the value of $\int_0^{2\pi} Q(x) \sin kx\, \mathrm{d}x$ $(k = 1, 2, \ldots n)$
    - (a) πa_k
    - (b) πa_0
    - (c) πb_n
    - (d) πb_k

63. The value of $\int\limits_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\frac{1}{x}\right) + x^3\right) \mathrm{d}x$ is
    - (a) $\pi/2$
    - (b) $\pi/4$
    - (c) 1
    - (d) none of these

64. The value of $\int_{1/2}^{1} \left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right) \mathrm{d}x$ is
    - (a) sin 1
    - (b) cos 1
    - (c) 2 sin 1
    - (d) none of these

65. Let f be a continuous function $[a, b]$ such that $f(x) > 0$ for all $x \in [a, b]$. If $F(x) = \int_a^x f(t)\, \mathrm{d}t$ then
    - (a) F is differentiable but not increasing on $[a, b]$
    - (b) F is differentiable and increasing on $[a, b]$
    - (c) F is continuous and decreasing on $[a, b]$
    - (d) F is neither differentiable nor increasing on $[a, b]$

66. Let f be a continuous function on **R** satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$ with $f(1) = 2$ and g be a function satisfying $f(x) + g(x) = e^x$ then the value of the integral $\int_0^1 f(x)\, g(x)\, dx$ is

(a) $\frac{1}{e} - 4$ (b) $\frac{1}{4}(e-2)$

(c) 2/3 (d) $\frac{1}{2}(e-3)$

67. If $f(x) = \operatorname{cosec}(x + \pi/3)\operatorname{cosec}(x + \pi/6)$, then the value of $\int_0^{\pi/2} f(x)\, dx$ is

(a) 2 log 3 (b) −2 log 3

(c) log 3 (d) 1/4

68. The equal to integral $\int_0^1 \log\left(\sqrt{1-x} + \sqrt{1+x}\right) dx$ is

(a) $\frac{1}{2}\left(\log 2 - \frac{1}{2} + \frac{\pi}{4}\right)$ (b) $\frac{1}{2}\left(\log 2 - 1 + \frac{\pi}{2}\right)$

(c) $\frac{1}{3}\left(\log 4 - 1 + \frac{\pi}{8}\right)$ (d) $\frac{1}{4}\left(\log 3 - 1 + \frac{\pi}{2}\right)$.

69. If $I = \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2n}}}$ for $n \ge 1$, the value of I is

(a) less than 1 (b) more than 1/2

(c) more than 1 (d) less than 1/2

70. The numbers A, B and C such that a function of the form $f(x) = Ax^2 + Bx + C$ satisfies the conditions $f'(1) = 8$, $f(2) + f''(2) = 33$ and $\int_0^1 f(x)\, dx = 7/3$, are

(a) $A = 1, B = -4, C = 2$

(b) $A = 7, B = -6, C = 3$

(c) $A = 8, B = -6, C = 3$

(d) none of these.

71. The equation of the tangent to the curve

$$y = \int_x^{x^2} \log t\, dt \quad \text{at } x = 2 \text{ is}$$

(a) $y - 6 \log 2 = 7 \log 2(x-2)$

(b) $y - \log 2e^{1/3} = (\log 2)x$

(c) $y - 8 \log 2e^{-1/3} = 5 \log 2\, x$

(d) $y + 8 \log 2 + 2 = (7 \log 2)x$

72. The value of $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin\theta}{2+\sin\theta}\right) d\theta$ is

(a) 0 (b) 1

(c) 2 (d) none of these

73. The value of the integral $\int_0^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x}\, dx$ is

(a) log 2 (b) log 3

(c) (1/4) log 3 (d) (1/8) log 3.

74. If $I_1 = \int_x^1 \frac{dt}{1+t^2}$ and $I_2 = \int_1^{1/x} \frac{dt}{1+t^2}$ for $x > 0$, then

(a) $I_1 > I_2$ (b) $I_1 = I_2$

(c) $I_2 > I_1$ (d) $I_2 = (\pi/2) - \tan^{-1} x$

75. The solution of the equation

$\int_{\sqrt{2}}^{x} \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{12}$ is given by

(a) 1 (b) 2

(c) 3 (d) $\sqrt{3}$

76. The mean value of the function $f(x) = \frac{2}{e^x + 1}$ on the interval [0, 2] is

(a) $\log \frac{2}{e^2+1}$ (b) $1 + \log \frac{2}{e^2+1}$

(c) $2 + \log \frac{2}{e^2+1}$ (d) $2 + \log\left(e^2+1\right)$

77. The value of $\int_0^{\pi} x \log(\sin x)\, dx$ is (given that $\int_0^{\pi/2} \log \sin x\, dx = -\frac{\pi}{2} \log 2$)

(a) $-\frac{\pi^2}{2} \log 2$ (b) $-\frac{\pi^2}{4} \log 2$

(c) $-\frac{\pi^2}{8} \log 2$ (d) none of these

78. $\int_{-\pi}^{\pi} (\cos px - \sin qx)^2\, dx$, where p and q are integers is equal to

(a) $-\pi$ (b) 0

(c) π (d) 2π

79. Let f, g and h be continuous functions on $[0, a]$ such that $f(x) = f(a-x)$, $g(x) = -g(a-x)$ and $3h(x) - 4h(a-x) = 5$. Then $\int_0^a f(x)\, g(x)\, h(x)\, dx$ is equal to

(a) 5/4 (b) 3/4

(c) 1 (d) none of these

80. If $\int_{\pi/2}^{x} \sqrt{3 - 2\sin^2 z}\, dz + \int_0^y \cos t\, dt = 0$ then $\frac{dy}{dx}$ at $(\pi/2, \pi)$ is

(a) 1 (b) 2

(c) $\sqrt{3}$ (d) $= 0$

81. If $P(x)$ is a polynomial of least degree that has a maximum equal to 6 at $x = 1$, and a minimum equal to 2 at $x = 3$, then $\int_0^1 P(x)\,dx$ equals

(a) 17/4 (b) 13/4

(c) 19/4 (d) 5/4

82. Let $P(x)$ be a polynomial of least degree whose graph has three points of inflextion $(-1, -1)$, $(1,1)$ and a point with abscissa 0 at which the curve is inclined to the axis of abscissas at an angle of 60°. Then $\int_0^1 P(x)\,dx$ equals to

(a) 1 (b) $\sqrt{2} - 1$

(c) $2\sqrt{2} - 1$ (d) none of these

83. If $S_n = \dfrac{e^{1/n}}{n^2} + \dfrac{2}{n^2}e^{2/n} + \dfrac{3}{n^2}e^{3/n} + \cdots + \dfrac{1}{n}e$ then $\lim\limits_{n\to\infty} S_n$ is

(a) 1 (b) 0

(c) e (d) none of these

84. The value of $\int_0^1 \lim\limits_{n\to\infty} \sum\limits_{k=0}^{n} \dfrac{x^{k+2}2^k}{k!}\,\mathrm{d}x$ is

(a) $e^2 - 1$ (b) 2

(c) $\dfrac{e^2-1}{2}$ (d) $\dfrac{e^2-1}{4}$

85. The value of $\int_0^1 \dfrac{x^{2\alpha}-1}{\log x}\,\mathrm{d}x$, if $\alpha = \dfrac{2n-1}{2}$ is

(a) $\log n$ (b) $2 \log n$

(c) $\log 2n$ (d) $(1/2) \log n$

86. The value of α in $(-\pi, 0)$ satisfying $\sin\alpha + \int_\alpha^{2\alpha} \cos 2x\,\mathrm{d}x = 0$ is

(a) $-\pi/2$ (b) $-\pi$

(c) $-\pi/3$ (d) $-\pi/4$

87. Let $g(x) = \begin{cases} 1 & , \quad 0 \le x < 1 \\ x^3 & , \quad 1 \le x < 4 \\ \sqrt{x} & , \quad 4 \le x < 9 \end{cases}$ then $\int_0^9 g(x)\,\mathrm{d}x$ is

(a) 243/12 (b) $9\sqrt{3}$

(c) 929/12 (d) none of these

88. The value of $\lim\limits_{x\to 0} \dfrac{\int_{-x}^{x} f(t)\,\mathrm{d}t}{\int_0^{2x} f(t+4)\,\mathrm{d}t}$ is (where f is a continuous function and $f(x) > 0$ for all x)

(a) $f(0)$ (b) 0

(c) $f(4)/f(0)$ (d) $f(0)/f(4)$

89. If $f(x) = \begin{cases} e^{\cos x}\sin x & \text{for } |x| \le 2 \\ 2 & \text{otherwise} \end{cases}$ then $\int_{-2}^{3} f(x)\,dx =$

(a) 0 (b) 1

(c) 2 (d) 3

90. The value of $\int_{-\pi}^{\pi} \dfrac{\cos^2 x}{1 + a^x}\,dx$, $a > 0$, is

(a) $\pi/2$ (b) $a\pi$

(c) π (d) 2π

91. Let $f : (0, \infty) \to \mathbf{R}$ and let

$$F(x) = \int_0^x f(t)\,dt$$

If $F(x^2) = x^2(1 + x)$, then $f(4)$ equals

(a) $\dfrac{5}{4}$ (b) 7

(c) 4 (d) 2

92. Let $f(x) = \int_1^x \sqrt{2 - t^2}\,dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are

(a) ± 1 (b) $\pm\dfrac{1}{\sqrt{2}}$

(c) $\pm\dfrac{1}{2}$ (d) 0 and 1

93. If $l(m, n) = \int_0^1 t^m (1 + t)^n\,dt$, then the expression for $l(m, n)$ in terms of $l(m + 1, n - 1)$ is

(a) $\dfrac{2^n}{m+1} - \dfrac{n}{m+1}\,l(m+1, n-1)$

(b) $\dfrac{n}{m+1}\,l(m + 1, n - 1)$

(c) $\dfrac{2^n}{m+1} + \dfrac{n}{m+1}\,l(m + 1, n - 1)$

(d) $\dfrac{m}{n+1}\,l(m + 1, n - 1)$

94. The definite integral $\int_0^1 \sqrt{\dfrac{1-x}{1+x}}\,dx$ is equal to

(a) 1 (b) π

(c) $\dfrac{\pi}{2} - 1$ (d) $\dfrac{\pi}{2} + \dfrac{1}{2}$

95. If, for $t > 0$ the definite integral

$$\int_0^{t^2} x\,f(x)\,dx = \frac{2}{5}\,t^5, \text{ then } f\left(\frac{4}{25}\right) \text{ is equal to}$$

(a) $\frac{2}{5}$ (b) $-\frac{2}{5}$

(c) $\frac{2}{\sqrt{5}}$ (d) $-\frac{2}{\sqrt{5}}$

96. The definite integral

$$\int_{-2}^{0} (x^3 + 3x^2 + 3x + 3 + (x + 1)\cos(x + 1))\,dx$$

equals

(a) -4 (b) 0

(c) 4 (d) 6

Previous Years' AIEEE/JEE Main Questions

1. $\int_0^{\sqrt{2}} [x^2]\,dx$ is

(a) $2-\sqrt{2}$ (b) $2+\sqrt{2}$

(c) $\sqrt{2}-1$ (d) $\sqrt{2}-2$ **[2002]**

2. $I_n = \int_0^{\pi/4} \tan^n x\,dx$, then $\lim\limits_{n\to\infty} n(I_n + I_{n-2})$ equals

(a) $\frac{1}{2}$ (b) 1

(c) ∞ (d) 0 **[2002]**

3. $\int_{\pi}^{10\pi} |\sin x|\,dx$ is

(a) 20 (b) 8

(c) 10 (d) 18 **[2002]**

4. If $y = f(x)$ makes positive intercept of 2 and 1 unit on x-axis and y-axis and encloses an area of $\frac{3}{4}$ square unit with axes then $\int_0^2 x\,f'(x)\,dx$ is

(a) $\frac{3}{2}$ (b) 1

(c) $\frac{5}{4}$ (d) $-\frac{3}{4}$ **[2002]**

5. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and $g(x)$ be a function that satisfies $f(x) + g(x) = x^2$. Then the value of the integral

$$\int_0^1 f(x)\,g(x)\,dx \text{ is}$$

(a) $e+\frac{e^2}{2}-\frac{3}{2}$ (b) $e-\frac{e^2}{2}-\frac{3}{2}$

(c) $e+\frac{e^2}{2}+\frac{5}{2}$ (d) $e-\frac{e^2}{2}-\frac{5}{2}$ **[2003]**

6. If $f(y) = e^y$, $g(y) = y$, $y > 0$ and $F(t) = \int_0^t f(t-y)\,g(y)\,dy$, then

(a) $F(t) = e^t - (1 + t)$

(b) $F(t) = te^t$

(c) $F(t) = te^{-t}$

(d) $F(t) = 1 - e^t(1 + t)$ **[2003]**

7. If $f(a + b - x) = f(x)$, then $\int_a^b x\,f(x)\,dx$ is equal to

(a) $\frac{a+b}{2}\int_a^b f(x)\,dx$

(b) $\frac{b-a}{2}\int_a^b f(x)\,dx$

(c) $\frac{a+b}{2}\int_a^b f(a+b-x)\,dx$

(d) $\frac{a+b}{2}\int_a^b f(b-x)\,dx$ **[2003]**

8. The value of the integral $I = \int_0^1 x(1-x)^n\,dx$ is

(a) $\frac{1}{n+2}$ (b) $\frac{1}{n+1}-\frac{1}{n+2}$

(c) $\frac{1}{n+1}+\frac{1}{n+2}$ (d) $\frac{1}{n+1}$ **[2003]**

9. The value of $\lim\limits_{x\to 0} \dfrac{\int_0^{x^2} \sec^2 t\,dt}{x\sin x}$ is

(a) 2 (b) 1

(c) 0 (d) 3 **[2003]**

10. The value of $\int_{-2}^{3}\left|1-x^2\right|dx$ is

(a) $\frac{7}{3}$ (b) $\frac{14}{3}$

(c) $\frac{28}{3}$ (d) $\frac{1}{3}$ **[2004]**

11. The value of $I = \int_0^{\pi/2} \frac{(\sin x+\cos x)^2}{\sqrt{1+\sin 2x}}\,dx$ is

(a) 2 (b) 1

(c) 0 (d) 3 **[2004]**

12. If $\int_0^{\pi} x\, f(\sin x)\,dx = A\int_0^{\pi/2} f(\sin x)\,dx$, then A is

(a) $\frac{\pi}{4}$ (b) π

(c) 0 (d) 2π **[2004]**

13. If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int\limits_{f(-a)}^{f(a)} x\, g(x(1-x))\,dx$, and $I_2 = \int\limits_{f(-a)}^{f(a)} g(x(1-x))\,dx$, then the value of $\frac{I_2}{I_1}$ is

(a) – 1 (b) – 3

(c) 2 (d) 1 **[2004]**

14. $\lim\limits_{n\to\infty}\left[\frac{1}{n^2}\sec^2\frac{1}{n^2}+\frac{2}{n^2}\sec^2\frac{4}{n^2}+\cdots\frac{1}{n^2}\sec^2 1\right]$ equals

(a) tan 1 (b) tan 1

(c) $\frac{1}{2}$ sec 1 (d) $\frac{1}{2}$ cosec 1 **[2005]**

15. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function having $f(2) = 6$, $f'(2) = \frac{1}{48}$. Then $\lim\limits_{x\to 0}\int\limits_{6}^{f(x)} \frac{4t^3}{x-2}\,dt$ equals

(a) 12 (b) 18

(c) 24 (d) 36 **[2005]**

16. If $I_1 = \int_0^1 2^{x^2}\,dx$, $I_2 = \int_0^1 2^{x^3}\,dx$, $I_3 = \int_1^2 2^{x^2}\,dx$ and $I_4 = \int_1^2 2^{x^3}\,dx$ then

(a) $I_3 = I_4$ (b) $I_3 > I_4$

(c) $I_2 > I_1$ (d) $I_1 > I_2$ **[2005]**

17. The value of $\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}$, $a > 0$ is

(a) $\frac{\pi}{a}$ (b) 2π

(c) $a\pi$ (d) $\frac{\pi}{2}$ **[2005]**

18. The value of the integral, $\int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}}\,dx$ is

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{3}{2}$ (d) 2 **[2006]**

19. $\int_{-3\pi/2}^{-\pi/2}\left[(x+\pi)^3+\cos^2(x+3\pi)\right]dx$ is equal to

(a) $\left(\frac{\pi}{4}\right)-1$ (b) $\frac{\pi^4}{32}$

(c) $\left(\frac{\pi^4}{32}\right)+\left(\frac{\pi}{2}\right)$ (d) $\frac{\pi}{2}$ **[2006]**

20. $\int_0^{\pi} x\, f(\sin x)\,dx$ is equal to

(a) $\pi\int_0^{\pi/2} f(\cos x)\,dx$ (b) $\pi\int_0^{\pi} f(\cos x)\,dx$

(c) $\pi\int_0^{\pi} f(\sin x)\,dx$ (d) $\frac{\pi}{2}\int_0^{\pi/2} f(\sin x)\,dx$

[2006]

21. The value of $\int_1^a [x]\, f'(x)\,dx$, $a > 1$, where $[x]$ denotes the greatest integer not exceeding x is

(a) $af([a]) - \{f(1)+f(2)+\ldots+f(a)]$

(b) $af(a) - \{f(1)+f(2)+\ldots+f([a])\}$

(c) $[a]\, f(a) - \{f(1)+f(2)+\ldots+f([a])\}$

(d) $[a]\, f([a]) - \{f(1)+f(2)+\ldots+f(a)\}$

[2006]

22. Let $F(x) = f(x) + f\left(\frac{1}{x}\right)$ where $f(x) = \int_1^x \frac{\log t}{1+t}\,dt.$ Then $F(e)$ equals

(a) $\frac{1}{2}$ (b) 0

(c) 1 (d) 2 **[2007]**

23. The solution for x of the equation

$$\int_{\sqrt{2}}^{x}\frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2} \text{ is}$$

(a) 2 (b) $-\sqrt{2}$

(c) $\frac{\sqrt{3}}{2}$ (d) $2\sqrt{2}$ **[2007]**

24. Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$. Then which one of the following is true?

(a) $I > 2/3, J > 2$ (b) $I < 2/3, J < 2$

(c) $I < 2/3, J > 2$ (d) $I > 2/3, J < 2$ **[2008]**

25. $\int_0^{\pi}[\cot x]\,dx$, where [ ] denotes the greatest integer function, is equal to

(a) -1 (b) $-\pi/2$
(c) $\pi/2$ (d) 1 **[2009]**

26. Let $p(x)$ be a function defined on R such that $p'(x) = p'(1-x)$, for all $x \in [0, 1]$, $p(0) = 1$ and $p(1) = 41$. Then $\int_0^1 p(x)\,dx$ equals

(a) 41 (b) 42
(c) $\sqrt{41}$ (d) 21 **[2010]**

27. The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$ is

(a) $\log 2$ (b) $\pi\log 2$
(c) $\frac{\pi}{8}\log 2$ (d) $\frac{\pi}{2}\log 2$ **[2011]**

28. For $x \in (0, 5\pi/2)$, define $F(x) = \int_0^x \sqrt{t}\sin t\,dt$ then f has

(a) local maximum at π and local minimum at 2π.
(b) local maximum at π and 2π
(c) local minimum at π and 2π
(d) local minimum at π and local maximum at 2π **[2011]**

29. Let [.] denote the greatest integer function then the value of $\int_0^{1.5} x[x^2]\,dx$ is

(a) 0 (b) $\frac{3}{2}$
(c) $\frac{3}{4}$ (d) $\frac{5}{4}$ **[2011]**

30. If $g(x) = \int_0^x \cos 4t\,dt$, then $g(x+\pi)$ equals

(a) $g(x) + g(\pi)$ (b) $g(x) - g(\pi)$
(c) $g(x)\,g(\pi)$ (d) $\frac{g(x)}{g(\pi)}$ **[2012]**

31. The intercepts on x-axis by the tangents to the curve, $y = \int_0^x |t|\,dt$, $x \in \mathbf{R}$, which are parallel to the line $y = 2x$, are equal to

(a) ± 2 (b) ± 3
(c) ± 4 (d) ± 1 **[2013]**

32. **Statement-1:** The value of the integral

$$\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{6}$$

Statement-2: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ **[2013]**

33. The value of $\int_{-\pi/2}^{\pi/2}\frac{\sin^2 x}{1+2^x}dx$ is

(a) π (b) $\frac{\pi}{2}$
(c) 4π (d) $\frac{\pi}{4}$ **[2013, online]**

34. If $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$, then $\frac{d^2y}{dx^2}$ is equal to

(a) y (b) $\sqrt{1+y^2}$
(c) $\frac{y}{\sqrt{1+y^2}}$ (d) y^2 **[2013, online]**

35. The integral $\int_{7\pi/4}^{7\pi/3}\sqrt{\tan^2 x}\,dx$ is equal to

(a) $\log 2\sqrt{2}$ (b) $\log 2$
(c) $2\log 2$ (d) $\log\sqrt{2}$ **[2013, online]**

36. The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals

(a) $\pi - 4$ (b) $\frac{2\pi}{3} - 4$
(c) $4\sqrt{3} - 4$ (d) $4\sqrt{3} - 4 - \frac{\pi}{3}$ **[2014]**

37. The integral $\int_0^{1/2}\frac{\log(1+2x)}{1+4x^2}dx$ equals

(a) $\frac{\pi}{4}\log 2$ (b) $\frac{\pi}{8}\log 2$
(c) $\frac{\pi}{16}\log 2$ (d) $\frac{\pi}{32}\log 2$ **[2014, online]**

38. If for $n \geq 1$, $P_n = \int_1^e (\log x)^n dx$, then $P_{10} - 90P_8$ is equal to

(a) -9 (b) $10e$
(c) $-9e$ (d) 10 **[2014, online]**

39. If [,] denotes the greatest integer function, then the integral $\int_0^{\pi}[\cos x]\,dx$ is equal to

(a) $\frac{\pi}{2}$ (b) 0
(c) -1 (d) $-\frac{\pi}{2}$ **[2014, online]**

40. If for a continuous function $f(x)$, $\int_{-\pi}^{t} (f(x)+x)\,dx$ $= \pi^2 - t^2$, for all $t \geq -\pi$, then $f\left(-\frac{\pi}{3}\right)$ is equal to:

(a) π (b) $\frac{\pi}{2}$

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$ **[2014, online]**

41. Let function F be defined as $F(x) = \int_{1}^{x} \frac{e^t}{t}\,dt$ $x > 0$ then the value of the integral $\int_{1}^{x} \frac{e^t}{t+a}\,dt$ where $a > 0$ is

(a) $e^a\,[F(x) - F(1+a)]$

(b) $e^{-a}\,[F(x+a) - F(a)]$

(c) $e^a\,[F(x+a) - F(1+a)]$

(d) $e^{-a}\,[F(x+a) - F(1+a)]$ **[2014, online]**

42. The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)}\,dx$ is equal to

(a) 2 (b) 4

(c) 1 (d) 6 **[2015]**

43. Let $f : \mathbf{R} \to \mathbf{R}$ be a function such that $f(2-x) = f(2+x)$ and $f(4-x) = f(4+x)$ for all $x \in \mathbf{R}$ and $\int_0^2 f(x)\,dx = 5$. Then the value of $\int_{10}^{50} f(x)\,dx$ is:

(a) 80 (b) 100

(c) 125 (d) 200 **[2015, online]**

44. Let f: $(-1, 1) \to \mathbf{R}$ be a continuous function. If $\int_0^{\sin x} f(t)\,dt = \frac{\sqrt{3}}{2}x$, then $f\left(\frac{\sqrt{3}}{2}\right)$ is equal to

(a) $\frac{\sqrt{3}}{2}$ (b) $\sqrt{3}$

(c) $\sqrt{\frac{3}{2}}$ (d) $\frac{1}{2}$ **[2015, online]**

45. For $x > 0$, let $f(x) = \int_1^x \frac{\log t}{1+t}\,dt$. Then $f(x) + f\left(\frac{1}{x}\right)$ is equal to

(a) $\frac{1}{4}(\log x)^2$ (b) $\frac{1}{2}(\log x)^2$

(c) $\log x$ (d) $\frac{1}{4}\log x^2$ **[2015 online]**

46. If $2\int_0^1 \tan^{-1} x\,dx = \int_0^1 \cot^{-1}(1-x+x^2)dx$, then $\int_0^1 \tan^{-1}(1-x+x^2)dx$ is equal to

(a) $\frac{\pi}{2} + \log 2$ (b) $\log 2$

(c) $\frac{\pi}{2} - \log 4$ (d) $\log 4$ **[2016, online]**

47. $\lim\limits_{n\to\infty}\left(\frac{(n+1)(n+2)\cdots(3n)}{n^{2n}}\right)^{1/n}$ is equal to:

(a) $\frac{18}{e^4}$ (b) $\frac{27}{e^2}$

(c) $\frac{9}{e^2}$ (d) $3\log 3 - 2$ **[2016]**

48. The value of the integral $\int_4^{10} \frac{[x^2]}{[x^2-28x+196]+[x^2]}\,dx$, where $[x]$ denotes the greatest integer less than or equal to x, is

(a) $\frac{1}{3}$ (b) 6

(c) 7 (d) 3 **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. $\int_{-4}^{-5} e^{(x+5)^2}\,dx + 3\int_{1/3}^{2/3} e^{9\left(x-\frac{2}{3}\right)^2}\,dx$ is

(a) 0 (b) -2

(c) 1 (d) 2 **[2006]**

2. If f is continuously differentiable function then $\int_0^{1.5} [x^2]\,f'(x)\,dx$ is

(a) $f(1.5) - f(\sqrt{2}) - f(1)$

(b) $f(1.5) + f(\sqrt{2}) + f(1)$

(c) $2f(1.5) + f(\sqrt{2}) + f(1)$

(d) $2f(1.5) - f(\sqrt{2}) - f(1)$ **[2006]**

3. $\int_{-a}^{a} \log(x+\sqrt{x^2+1})\,dx$

(a) $2\log(a^2+1)$ (b) $2\log(\sqrt{a^2+1}-a)$

(c) 0 (d) $2\log\left(a+\sqrt{a^2+1}\right)$

[2007]

4. If $f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, $n = 0, 1, 2\ldots$ then $\underset{n\to\infty}{lt}\int_{1/2^n}^{1} f(x)\,\mathrm{d}x$ equals

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) 0 (d) $\frac{1}{2}$ **[2007]**

5. The value of $\int_0^1 x^2(1-x)^9\,\mathrm{d}x$ is

(a) $\frac{1}{610}$ (b) $\frac{1}{630}$

(c) $\frac{1}{640}$ (d) $\frac{1}{660}$ **[2008]**

6. The value of $\int_0^1 \max\left(e^x, e^{1-x}\right)\,\mathrm{d}x$ equals

(a) $2(e-1)$ (b) $2(e-\sqrt{e})$

(c) $2(e+\sqrt{e})$ (d) $2(e+1)$ **[2010]**

7. $\int_0^{\pi/2} \min(\sin x, \cos x)\,\mathrm{d}x$ equals to

(a) $2\sqrt{2}$ (b) $\sqrt{2}$

(c) $2-\sqrt{2}$ (d) $2+\sqrt{2}$ **[2011]**

8. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \int_0^1 \frac{x^2+t^2}{2-t}\,dt$. Then the curve $y = f(x)$ is

(a) an ellipse (b) a straight line

(c) a parabola (d) a hyperbola **[2011]**

9. The equation of a curve is given by $y = f(x)$, where $f'(x)$ is a continuous function. The tangent at points $(1, f(1))$, $(2, f(2))$ and $(3, f(3))$ make angles $\frac{\pi}{6}, \frac{\pi}{3}$ and $\frac{\pi}{4}$ respectively with positive x-axis. Then $\int_2^3 f'(x)f''(x)\,\mathrm{d}x + \int_1^3 f''(x)\,\mathrm{d}x$ is:

(a) 1 (b) $-\frac{1}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{3}}$ (d) 0 **[2011]**

10. Using the fact that $0 \le f(x) \le g(x)$, $c < x < d$ $\Rightarrow \int_c^d f(x)\,\mathrm{d}x \le \int_c^d g(x)\,\mathrm{d}x$, we can conclude that $\int_1^3 \sqrt{3+x^3}$ lies in the interval

(a) $\left(\frac{1}{2}, 3\right)$ (b) $\left(2, \sqrt{30}\right)$

(c) $\left(\frac{3}{2}, 5\right)$ (d) $\left(4, 2\sqrt{30}\right)$ **[2012]**

11. $\lim_{n\to\infty} \frac{1}{n}\left[1 + \frac{n^2}{n^2+1^2} + \frac{n^2}{n^2+2^2} + \cdots + \frac{n^2}{n^2+(n-1)^2}\right]$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$ **[2012]**

12. If $f(x) = x|x|$, then for any real number a and b with $a < b$, the value of $\int_a^b f(x)\,\mathrm{d}x$ equals

(a) $\frac{1}{3}(|b|^3 - |a|^3)$ (b) $\frac{1}{3}|b^3 - a^3|$

(c) $\frac{1}{3}(a^3 + b^3)$ (d) $\frac{1}{3}(a^3 - b^3)$ **[2013]**

13. The integral $\int_{\sqrt{\log 5}}^{\sqrt{\log 7}} \frac{x\cos x^2}{\cos(\log 35 - x^2) + \cos x^2}\,\mathrm{d}x$ is equal to

(a) $\frac{1}{4}\log\frac{5}{7}$ (b) $\frac{1}{2}\log\frac{5}{7}$

(c) $\frac{1}{4}\log\frac{7}{5}$ (d) $\frac{1}{2}\log\frac{7}{5}$ **[2013]**

14. If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} x\,g\{x(1-x)\}\,\mathrm{d}x$ and $I_2 = \int_{f(-a)}^{f(a)} g\{x(1-x)\}\,\mathrm{d}x$ where g is not an identity function. Then the value of $\frac{I_2}{I_1}$ is

(a) $\frac{1}{2}$ (b) 2

(c) 1 (d) -1 **[2014]**

15. The integral $\int_0^{1/2} \frac{e^x(2-x^2)}{(1-x)^{3/2}(1+x)^{1/2}}dx$ is equal to

(a) $\sqrt{3e}$ (b) $\sqrt{3e}-1$

(c) $\sqrt{\frac{e}{3}}$ (d) $\sqrt{\frac{e}{3}}-1$ **[2015]**

16. The integral $I = \int_0^2 [x^2]dx$ ([t] denotes the greatest integer less than or equal to t) is equal to:

(a) $5-2\sqrt{3}$ (b) $5-\sqrt{2}-\sqrt{3}$

(c) $6-\sqrt{2}-\sqrt{3}$ (d) $3-\sqrt{2}$

Answers

Concept-based

1. (b) **2.** (d) **3.** (b) **4.** (c)
5. (a) **6.** (d) **7.** (c) **8.** (c)
9. (d) **10.** (b)

Level 1

11. (d) **12.** (c) **13.** (c) **14.** (a)
15. (b) **16.** (b) **17.** (a) **18.** (a)
19. (d) **20.** (d) **21.** (b) **22.** (b)
23. (d) **24.** (c) **25.** (c) **26.** (d)
27. (a) **28.** (d) **29.** (a) **30.** (d)
31. (a) **32.** (d) **33.** (c) **34.** (a)
35. (b) **36.** (a) **37.** (c) **39.** (b)
39. (a) **40.** (d) **41.** (c) **42.** (a)
43. (c) **44.** (b) **45.** (a) **46.** (d)
47. (b) **48.** (d) **49.** (b) **50.** (b)
51. (c) **52.** (a) **53.** (d) **54.** (c)
55. (a) **56.** (d) **57.** (c) **58.** (a)

Level 2

59. (d) **60.** (d) **61.** (a) **62.** (d)
63. (d) **64.** (d) **65.** (b) **66.** (c)
67. (b) **68.** (b) **69.** (a) **70.** (b)
71. (d) **72.** (a) **73.** (c) **74.** (a)
75. (b) **76.** (c) **77.** (a) **78.** (d)
79. (d) **80.** (a) **81.** (c) **82.** (d)
83. (a) **84.** (d) **85.** (c) **86.** (c)
87. (c) **88.** (d) **89.** (c) **90.** (a)
91. (c) **92.** (a) **93.** (a) **94.** (c)
95. (a) **96.** (c)

Previous Years' AIEEE/JEE Main Questions

1. (c) **2.** (b) **3.** (d) **4.** (d)
5. (b) **6.** (a) **7.** (a) **8.** (b)
9. (b) **10.** (c) **11.** (a) **12.** (b)
13. (c) **14.** (b) **15.** (b) **16.** (d)
17. (d) **18.** (c) **19.** (d) **20.** (a)
21. (c) **22.** (a) **23.** (b) **24.** (b)
25. (b) **26.** (d) **27.** (b) **28.** (a)
29. (c) **30.** (a)(b) **31.** (d) **32.** (c)
33. (b) **34.** (a) **35.** (a) **36.** (d)
37. (c) **38.** (c) **39.** (d) **40.** (a)
41. (d) **42.** (a) **43.** (b) **44.** (b)
45. (b) **46.** (b) **47.** (b) **48.** (d)

Previous Years' B-Architecture Entrance Examination Questions

1. (a) **2.** (d) **3.** (c) **4.** (b)
5. (d) **6.** (b) **7.** (c) **8.** (c)
9. (b) **10.** (d) **11.** (c) **12.** (b)
13. (c) **14.** (b) **15.** (b) **16.** (b)

Hints and Solutions

Concept-based

1. $$\int_{\pi/4}^{\pi/2} \cot\theta\,\mathrm{cosec}^2\theta\,\mathrm{d}\theta = -\int_1^0 u\,\mathrm{d}u \quad (u=\cot\theta)$$

$$= \int_0^1 u\,\mathrm{d}u = \left.\frac{u^2}{2}\right|_0^1 = \frac{1}{2}$$

2. Reqd. limit $= \lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{n=1}^{n} \sin\frac{r\pi}{n}$

$$= \int_0^1 \sin \pi x\,\mathrm{d}x = -\left.\frac{\cos\pi x}{\pi}\right|_0^1$$

$$= \frac{-1}{\pi}[-1-1] = \frac{2}{\pi}$$

3. $I = \int_{-\pi/2}^{\pi/2} \frac{x\sin x}{e^x+1}dx = \int_{-\pi/2}^{\pi/2} \frac{(-x)\sin(-x)}{e^{-x}+1}dx$ (Prop. 11)

$$= \int_{-\pi/2}^{\pi/2} \frac{(x\sin x)e^x}{e^x+1}dx$$

$$2I = \int_{-\pi/2}^{\pi/2} \frac{x\sin x}{e^x+1} + \int_{-\pi/2}^{\pi/2} \frac{e^x(x\sin x)}{e^x+1}\,dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{(e^x+1)(x\sin x)}{e^x+1}\,dx$$

$$= \int_{-\pi/2}^{\pi/2} x\sin x\,dx \;=\; 2\int_0^{\pi/2} x\sin x\,dx$$

(Prop. 12)

$$= 2\left[-x\cos x\Big|_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx\right]$$

$$= 2\;\sin x\Big|_0^{\pi/2} = 2$$

$$\Rightarrow \qquad I = 1.$$

4. The function $f(x) = x^{14}\sin^{11}x$ is an odd function, so $\int\limits_{-\pi/2}^{\pi/2} x^{14}\sin^{11}x\,\mathrm{d}x = 0$. Thus the given integral is equal to $\int\limits_{-\pi/2}^{\pi/2}\sin^2 x\,\mathrm{d}x = 2\int\limits_0^{\pi/2}\sin^2 x\,\mathrm{d}x = 2.\frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{2}$

5. Put $\tan x/2 = t$. The given integral reduces to

$$\int\limits_0^1 \frac{2dt}{(1+t^2)\left(2\dfrac{(1-t^2)}{1+t^2}+3\right)} = 2\int\limits_0^1 \frac{dt}{2-2t^2+3+3t^2}$$

$$= 2\int\limits_0^1 \frac{dt}{t^2+5}$$

$$= \frac{2}{\sqrt{5}}\tan^{-1}\frac{t}{\sqrt{5}}\Big|_0^1 = \frac{2}{\sqrt{5}}\tan^{-1}\frac{1}{\sqrt{5}}$$

6. $$\int\limits_0^{\pi/4}\frac{x\sin x}{\cos^3 x}\mathrm{d}x \;=\; \int\limits_0^{\pi/4} x\tan x\sec^2 x\,\mathrm{d}x$$

$$= x\frac{\tan^2 x}{2}\Big|_0^{\pi/4} - \frac{1}{2}\int\limits_0^{\pi/4}\tan^2 x\,\mathrm{d}x$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\int\limits_0^{\pi/4}(\sec^2 x - 1)\,\mathrm{d}x$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}[\tan x - x]^{\pi/4}$$

$$= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\left[1-\frac{\pi}{4}\right] = \frac{\pi}{4} - \frac{1}{2}$$

7. The given integral in equal to

$$\int\limits_0^2 \frac{dx}{\sqrt{x+1}(2+x)} \;=\; \int\limits_1^{\sqrt{3}}\frac{2t\,dt}{t(t^2+1)},\; t^2 = x+1$$

$$= 2\int\limits_1^{\sqrt{3}}\frac{dt}{t^2+1} = 2\tan^{-1}t\Big|_1^{\sqrt{3}}$$

$$= 2\left[\frac{\pi}{3}-\frac{\pi}{4}\right] = \frac{\pi}{6}.$$

8. $$\int\limits_0^{2.5}[x^2]f'(x)\mathrm{d}x$$

$$= \int\limits_0^1[x^2]f'(x)\mathrm{d}x + \int\limits_1^{\sqrt{2}}[x^2]f'(x) + \int\limits_{\sqrt{2}}^{\sqrt{3}}[x^2]f'(x)\mathrm{d}x$$
$$+ \int\limits_{\sqrt{3}}^{2}[x^2]f'(x)\mathrm{d}x + \int\limits_2^{\sqrt{5}}[x^2]f'(x)\,\mathrm{d}x$$
$$+ \int\limits_{\sqrt{5}}^{\sqrt{6}}[x^2]f'(x)\,\mathrm{d}x + \int\limits_{\sqrt{6}}^{2.5}[x^2]f'(x)\mathrm{d}x$$

$$= 0 + \int\limits_1^{\sqrt{2}}f'(x)\mathrm{d}x + 2\int\limits_{\sqrt{2}}^{\sqrt{3}}f'(x)\mathrm{d}x + 3\int\limits_{\sqrt{3}}^{2}f'(x)\mathrm{d}x$$
$$+4\int\limits_2^{\sqrt{5}}f'(x)\mathrm{d}x + 5\int\limits_{\sqrt{5}}^{\sqrt{6}}f'(x)\,\mathrm{d}x + 6\int\limits_{\sqrt{6}}^{2.5}f'(x)\,\mathrm{d}x$$

$$= \left(f(\sqrt{2})-f(1)\right) + 2\left(f(\sqrt{3})-f(\sqrt{2})\right)$$
$$+3\left(f(2)-f(\sqrt{3})\right) + 4\left(f(\sqrt{5})-f(2)\right)$$
$$+5\left[f(\sqrt{6})-f(\sqrt{5})\right] + 6\left[f(2.5)-f(\sqrt{6})\right]$$

$$= 6f(2.5) - (f(1) + f(\sqrt{2}) + f(\sqrt{3}) + f(2)$$
$$+ (f(\sqrt{5}) + f(\sqrt{6}))$$

9. Given integral is equal to

$$\int\limits_0^1 (1-x)^3(1-(1-x))^{11}\mathrm{d}x \;=\; \int\limits_0^1 x^{11}(1-x)^3\mathrm{d}x$$

$$= \int\limits_0^1 x^{11}(1-3x+3x^2-x^3)\mathrm{d}x$$

$$= \int\limits_0^1 (x^{11}-3x^{12}+3x^{13}-x^{14})\mathrm{d}x$$

$$= \frac{1}{11}-\frac{3}{12}+\frac{3}{13}-\frac{1}{14}$$

10. $$|2-x| = \begin{cases} 2-x\,, & 0 \le x \le 2 \\ x-2\,, & 2 < x \le 3 \end{cases}$$

$$\int\limits_0^3 |2-x|\mathrm{d}x \;=\; \int\limits_0^2(2-x)\mathrm{d}x + \int\limits_2^2(x-2)\mathrm{d}x$$

$$= 2x - x^2\Big|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Big|_2^3$$

$$= 4 - 4 + \frac{8}{2} - 4 - \left(\frac{4}{2} - 4\right)$$

$$= 2$$

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11. $|1 - x| = \begin{cases} 1 - x & , \quad x \le 1 \\ x - 1 & , \quad x > 1 \end{cases}$ so

$$\int_{-2}^{2} |1 - x|\,dx = \int_{-2}^{1} |1 - x|\,dx + \int_{1}^{2} |1 - x|\,dx$$

$$= \int_{-2}^{1} (1 - x)\,dx + \int_{1}^{2} (x - 1)\,dx$$

$$= 5.$$

12. $f\,(x + \pi) = \int_0^{x+\pi} \sin^8 t\,dt$

$$= \int_0^{\pi} \sin^8 t\,dt\; t\int_{\pi}^{x+\pi} \sin^8 t\,dt$$

$$= f\,(x) + \int_0^{x} \sin^8 (\pi - u)\,du = f\,(\pi) + f\,(x)\,.$$

13. $f\,(0) = 4$, $f\,(2) = 7$ and $\int_0^2 f'(x)\,dx = f(2) - f(0) = 7 - 4 = 3$.

14. We have $\int_0^K P(t)dt = \dfrac{K^2(K+3)}{3}$. Differentiating w.r.t K, we get $P(K) = \dfrac{K^2}{3} + \dfrac{2K}{3}(K+3) = K^2 + 2K$.

15. $\displaystyle\lim_{x \to 0} \frac{\int_0^x \sin t^2 dt}{\sin x^2} = \lim_{x \to 0} \frac{\sin x^2}{\cos x^2 . 2x}$

$$= \frac{1}{2} \lim_{x \to 0} \frac{\sin x^2}{x^2} \,.\, x \,.\, \cos x = 0$$

16. $f'(x) = e^{-x^2/2}(1 - x^2) > 0$ for $-1 < x < 0$ and is negative for $x < -1$. $f(x)$ has minimum at $x = -1$.

17. $f'(a\,) = \tan\left(\dfrac{\pi}{2} - \dfrac{\pi}{3}\right) = \tan\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$ and $f'(b) = \tan\dfrac{\pi}{4} = 1$, so

$$\int_a^b f''(x)dx = f'(b) - f'(a) = 1 - \frac{1}{\sqrt{3}}\,.$$

18. $F(-x) = \displaystyle\int_0^{-x} \log\frac{1-t}{1+t}\,dt = -\int_0^{x} \log\frac{1+u}{1-u}\,du\;(u = -t)$

$$= \int_0^{x} \log\frac{1-u}{1+u}\,du = F(x)$$

19. $A = \displaystyle\lim_{n \to 0} f(n)$

$$= \lim_{n \to \infty}\left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\cdots\left(\frac{n+n}{n}\right)\right]^{\frac{1}{n}}$$

$$= \lim_{n \to \infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$$

$$\log A = \lim_{n \to \infty} \frac{1}{n}\sum_{r=1}^{n} \log\left(1 + \frac{r}{n}\right) = \int_0^1 \log(1+x)\,dx$$

$$= \log\frac{4}{e}$$

$$\Rightarrow \quad A = \frac{4}{e}$$

20. The integrand can be written as

$$\frac{1}{\sqrt{\left(\dfrac{B-\alpha}{2}\right)^2 - \left(x - \dfrac{\alpha+\beta}{2}\right)^2}}$$

So the value of the required integral is

$$\sin^{-1}\frac{x - \dfrac{\alpha+\beta}{2}}{\dfrac{\beta-\alpha}{2}}\Bigg|_{\alpha}^{\beta} = \sin^{-1}(1) - \sin^{-1}(-1) = \pi$$

21. The integrand is an even function, so the given integral is equal to

$$2\int_0^{\pi/3} \frac{x \sin x}{\cos^2 x}\,dx = 2[x \sec x]_0^{\pi/3} - \int_0^{\pi/3} \sec x\,dx]$$

$$= 2.\frac{\pi}{3}.2 - 2[\log(\sec x + \tan x)]_0^{\pi/3}$$

$$= \frac{4\pi}{3} - 2\log\left(2 + \sqrt{3}\right)$$

22. $\dfrac{1}{t\left(1+t^2\right)} = \dfrac{1}{t} - \dfrac{t}{1+t^2}$. So the required integral is equal to

$$\int_{1/e}^{\tan x} \frac{t\,dx}{1+t^2} + \int_{1/e}^{\cot x} \frac{dt}{t} - \int_{1/e}^{\cot x} \frac{t\,dt}{1+t^2}$$

$$= \frac{1}{2}\log\left(1+t^2\right)\Big|_{1/e}^{\tan x} + \log t\Big|_{1/e}^{\cot x} - 2\log\left(1+t^2\right)\Big|_{1/e}^{\cot x}$$

$= \log |\sec x| + \log |\cot x| + 1 - \log |\text{cosec } x| = 1.$

23. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+t^2}}\Bigg|_{t=x^3} \dfrac{d}{dx}(x^3) - \dfrac{1}{\sqrt{1+t^2}}\Bigg|_{t=x^2} \dfrac{d}{dx}(x^2)$

$$= \frac{3x^2}{\sqrt{1+x^6}} - \frac{2x}{\sqrt{1+x^4}}$$

$$\frac{dy}{dx}\Big|_{x=1} = \frac{1}{\sqrt{2}}\,.$$

Also $y(1) = 0$. So the equation of tangent is $y - 0 = \frac{1}{\sqrt{2}}\ (x - 1)$.

24. The integrand is an odd function, so the given integral is zero.

25. $F'(x) = x + 1 > 0$ for $x \in [1, 3]$. Therefore, the greatest value is $F(3)$ and the least value is $F(1)$. The required difference is

$$\int_0^3 (t+1)\,dt - \int_0^1 (t+1)\,dt \;=\; \int_1^3 (t+1)\,dt \;= 6.$$

26. $$\lim_{x \to \infty} \frac{\left(\int_0^x e^x dx\right)^2}{\int_0^x e^2 x^2 dx} \;=\; \lim_{x \to \infty} \frac{\left(e^x - 1\right)^2}{\int_0^x e^{2x^2}\, dx}$$

$$= \;\lim_{x \to \infty} \frac{2\left(e^x - 1\right)e^x}{e^{2x^2}} = 2\,\lim_{x \to \infty} \frac{1 - e^{-x}}{e^{2x^2 - 2x}} \;= 0.$$

27. For $x \geq 10$, we have $|\sin x| \leq 1$ and $1 + x^8 \geq 10^8$

$$\Rightarrow \quad \frac{1}{1 + x^8} \;\leq\; 10^{-8}$$

Therefore,

$$1\int_{10}^{19} \frac{\sin x}{1 + x^8}\, dx\Bigg| \;\leq\; \int_{10}^{19} \frac{|\sin x|}{1 + x^8}\, dx \leq \int_{10}^{19} 10^{-8}\ dx$$

$$= 9(10^{-8}) < 10^{-7}$$

28. The integrand can be written as $\dfrac{\sqrt{x+1} - \sqrt{5x+1}}{-\,4x}$.

So the given integral is of the form $I_1 + I_2$

$$I_1 = \;\frac{-1}{4}\int_0^3 \frac{\sqrt{x+1}}{x}\, dx = -\frac{1}{2}\int_1^2 \frac{t^2}{t^2 - 1}\, dt$$

$$= \;-\frac{1}{2}\int_1^2 \left(1 + \frac{1}{t^2 - 1}\right) dt$$

$$= \;-\frac{1}{2}\left(1 - \frac{1}{2}\lim_{t \to 1} \log\frac{|t-1|}{t+1}\right)$$

$$(x + 1 = t^2)$$

and $$I_2 = \;\frac{1}{4}\int_0^3 \frac{\sqrt{5x+1}}{x}\, dx$$

$$= \;\frac{1}{4}\int_1^4 \frac{2u^2}{u^2 - 1}\, du \left(u^2 = 5x + 1\right)$$

$$= \;\frac{1}{2}\left(3 + \frac{1}{2}\log\frac{3}{5} - \frac{1}{2}\lim_{u \to 1} \log\frac{|u-1|}{u+1}\right)$$

Thus $\;I_1 + I_2 = \;1 + \frac{1}{4}\log\frac{3}{5}\;$.

29. $f(x) = x - [x]$ so $\int_{-1}^{1} f(x)\,dx \;=\; \int_{-1}^{0} (x+1)\,dx + \int_0^1 x\,dx$

$$= \;0 - \left(\frac{1}{2} - 1\right) + \frac{1}{2} = 1$$

30. $$\int_{-\pi/2}^{\pi/2} \cos t\, \sin\left(2t - \frac{\pi}{4}\right) dt$$

$$= \;\frac{1}{\sqrt{2}}\int_{-\pi/2}^{\pi/2} \cos t\,(\sin 2t - \cos 2t)\,dt\,.$$

$$= \;-\frac{1}{\sqrt{2}}\int_{-\pi/2}^{\pi/2}\ \cos t \cos 2t\ dt$$

$$= \;-\frac{\sqrt{2}}{2}\int_0^{\pi/2}\ (\cos 3t + \cos t)\ dt$$

$$= \;-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{3}} + 1\right) = -\frac{1}{\sqrt{2}}.\frac{2}{3} = -\frac{\sqrt{2}}{3}\,.$$

31. Putting $u = 1/x$

$$I = \int_0^\infty \frac{x \log x}{\left(1 + x^2\right)}\, dx \;= \int_\infty^0 \frac{u \log u}{\left(1 + u^2\right)^2}\, du \;= -\,I$$

So $I = 0$.

32. $$\int_0^1 |\sin\, 2\pi x|\,dx \;= \int_0^{1/2} |\sin\, 2\pi x|\,dx \;+ \int_{1/2}^{1} |\sin\, 2\pi x|\,dx$$

$$= \;\int_0^{1/2}\ \sin 2\pi x\ dx - \int_{1/2}^{1}\ \sin 2\pi x\ dx$$

$$= \;-\frac{\cos 2\pi x}{2\pi}\Bigg|_0^{1/2} + \frac{\cos 2\pi x}{2\pi}\Bigg|_{1/2}^{1} = \frac{2}{\pi}\,.$$

33. $$\lim_{x \to \infty} \frac{\int_0^x \left(\tan^{-1} x\right)^2\, dx}{\sqrt{x^2 + 1}} \;=\; \lim_{x \to \infty} \frac{\left(\tan^{-1} x\right)^2 \sqrt{x^2 + 1}}{x}$$

(L' Hôpital Rule)

$$= \;\lim_{x \to \infty} \left(\tan^{-1} x\right)^2 \sqrt{1 + \frac{1}{x^2}} = \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}\,.$$

34. $$A(t) \;=\; \int_{-1}^{0} e^x\ dx + \int_0^t e^{-x}\ dx \;=\; 1 - e^{-1} - (e^{-t} - 1)$$

$$\lim_{t \to \infty} A(t) \;= 2 - e^{-1}.$$

35. Since the integrand is an even function so required integral is equal to $2\int_0^2 |x \cos \pi x|\,dx$

$$= \;2\left[\int_0^{1/2} x \cos \pi x\ dx - \int_{1/2}^{3/2} x \cos \pi x\ dx + \int_{3/2}^{2} x \cos \pi x\ dx\right]$$

$$= \;2\left[x\,\frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\Bigg|_0^{1/2} - \left(x\,\frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right)\Bigg|_{1/2}^{3/2}\right.$$

$$\left. + \left(\frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right)\Bigg|_{3/2}^{2}\ \right]$$

$$= 2\left[\frac{1}{2\pi} - \frac{1}{\pi^2} - \left(-\frac{3}{2\pi} - \frac{1}{2\pi}\right) + \frac{1}{\pi^2} - \left(-\frac{3}{2\pi}\right)\right] = \frac{8}{\pi}$$

36. Putting $\tan\frac{x}{2} = t$, the given integral reduces to

$$= 2\int_0^{\tan\alpha/2} \frac{dt}{(1-\cos\alpha) + t^2(1+\cos\alpha)}$$

$$= \frac{1}{\cos^2\frac{\alpha}{2}}\int_0^{\tan\alpha/2} \frac{dt}{t^2 + \tan^2\frac{\alpha}{2}}.$$

$$= \frac{1}{\cos^2\frac{\alpha}{2}}\,\frac{1}{\tan\frac{\alpha}{2}}\tan^{-1}\frac{t}{\tan\frac{\alpha}{2}}\Bigg|_0^{\tan\alpha/2}$$

$$= \frac{1}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}.\frac{\pi}{4} = \frac{\pi}{2\sin\alpha}$$

37. Required limit $= \lim\limits_{n\to\infty}\sum\limits_{r=0}^{n-1}\frac{n}{(n+r)^2}$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{\left(1+\frac{r}{n}\right)^2}$$

$$= \int_0^1 \frac{dx}{(1+x)^2} = -\frac{1}{1+x}\Bigg|_0^1.$$

$$= -\frac{1}{2} + 1 = \frac{1}{2}$$

38. Required limit $= \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{r^2}{n^3+r^3}$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{n=1}^{n}\frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3}$$

$$= \int_0^1 \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\log 2.$$

39. For the integrand $f(\pi - x) = -f(x)$.

40. $f'(x) = A\,\frac{\pi}{2}\cos\frac{\pi}{2}x \;\Rightarrow\; \sqrt{2} = f'\left(\frac{1}{2}\right) = \frac{A\pi}{2\sqrt{2}}$

so $A = \frac{4}{\pi}$. Also

$$\frac{2A}{\pi} = \int_0^1 f(x)\,dx = -A\frac{2}{\pi}\cos\frac{\pi}{2}x + Bx\Big|_0^1$$

$$= B + \frac{2A}{\pi} \quad\Rightarrow\quad B = 0.$$

41. $\int_a^b f(x)\,g(x)\,dx = \int_a^b g'(x)\,g(x)\,dx$

$$= \frac{1}{2}\left((g(b))^2 - (g(a))^2\right)$$

42. Let $F(x) = \int_\alpha^x f(t)\,dt$ then $F'(x) = f(x)$.

$$I_1 = \int_a^b F(x)\,dx = x\,F(x)\Big|_a^b - \int_a^b x\,f(x)\,dx$$

$$= bF(b) - \int_a^b x\,f(x)\,dx$$

$$= \int_a^b (b-x)\,f(x)\,dx = I_2.$$

43. $f'(x) = -\log_{1/3} x = \frac{\log x}{\log 3}$. So $F'(x) < 0$ for $\frac{1}{10} < x < 1$ and $F'(x) > 0$ for $x > 1$. Hence f has least value at $x = 1$.

44. $F'(x) = x^2 - 8x + 16 = (x-4)^2 > 0$ for $x \in [0, 5] \sim \{4\}$
So the greatest value of F is $F(5)$

$$= \int_4^5 (t-4)^2\,dt = \frac{1}{3}.$$

45. $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$

$$= -\int_a^0 f(-x)\,dx + \int_0^a f(x)\,dx = \int_0^a \left(f(-x) + f(x)\right)dx.$$

46. $J_m = \int_1^e \log^m x\,dx = x\log^m x\Big|_1^e - \int_1^e m\,x\frac{\log^{m-1}x}{x}\,dx$

$$= e - m\,J_{m-1}$$

So $J_8 + 8\,J_7 = e$.

47. Note that $\frac{d}{dx}f_3(x) = f_2(x)$, $\frac{d}{dx}f_2(x) = f(x)$ and $\frac{d}{dx}f_1(x) = f(x)$. Integrating by parts

$$f_3(x) = t\,f_2(t)\Big|_0^x - \int_0^x t\,f_1(t)\,dt$$

$$= x\,f_2(x) - \int_0^x t\,f_1(t)\,dt = \int_0^x (x-t)\,f_1(t)\,dt$$

$$= -\frac{(x-t)^2}{2}f_1(t)\Big|_0^x + \frac{1}{2}\int_0^x (x-t)^2 f(t)\,dt$$

$$= \frac{1}{2}\int_0^x (x-t)^2 f(t)\,dt. \text{ Thus } A = \frac{1}{2}.$$

48. $\frac{dx}{dy} = \frac{1}{\sqrt{1+9y^2}} \;\Rightarrow\; \frac{dy}{dx} = \sqrt{1+9y^2}$

$\dfrac{d^2y}{dx^2} = \dfrac{1}{2}\dfrac{18y}{\sqrt{1+9y^2}}\dfrac{dy}{dx} = 9y$. Thus the constant of proportionality is equal to 9.

49. Put $g(x) = t \Rightarrow x = f(t) \Rightarrow dx = f'(t)\ dt$

$$I = \int_a^b t\, f'(t)\, dt + \int_a^b f(x)\, dx$$

$$= [t\, f(t)]_a^b - \int_a^b 1 \,.\, f(t)\, dt + \int_a^b f(x)\, dx$$

$$= h(t)]_a^b = h(b) - h(a)$$

50. $I = \displaystyle\int_0^{[x]} \frac{2^t}{2^{[t]}}\, dt = \int_0^{[x]} 2^{t-[t]} dt$

The function $g(t) = 2^{t-[t]}$ is periodic with period 1, Therefore

$$I = [x]\int_0^1 2^{t-[t]} dt = [x]\int_0^1 2^t\, dt = \frac{[x]}{\log 2}[2t]_0^1$$

$$= \frac{[x]}{\log 2}\,.$$

51. $n > 1$, $a_n - a_{n-1} = \displaystyle\int_0^{\pi/2} \frac{\sin^{-2} nx - \sin^2 (n-1)x}{\sin x}\, dx$

$$= \int_0^{\pi/2} \frac{\sin(2n-1)x \sin x}{\sin x}\, dx$$

$$= \int_0^{\pi/2} \sin(2n-1)x\, dx = -\left.\frac{\cos(2n-1)x}{2n-1}\right|_0^{\pi/2}$$

$$= \frac{1}{2n-1}$$

which is a Harmonic Progression.

52. $\displaystyle\int_0^{21} [x]^4\, dx = \sum_{r=0}^{20} \int_r^{r+1} [x]^4\, dx$

$$= \sum_{r=0}^{20} r^4 = \left(\frac{20 \times 21}{2}\right)^2 = 44100$$

53. $I = \displaystyle\int_{\cos\theta}^{\sec\theta} f(x)\, dx = -\int_{\cos\theta}^{\sec\theta} \frac{1}{x^2} f\left(\frac{1}{x}\right) dx \left(\text{Put}\ \frac{1}{x} = t\right)$

$$= \int_{\sec\theta}^{\cos\theta} f(t)\, dt = -\int_{\cos\theta}^{\sec\theta} f(t)\, dt = -I$$

Hence $2I = 0 \Rightarrow I = 0$

54. The integrand is even, so

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\int_0^{\infty} \frac{dx}{1+x^2}$$

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \left(\tan^{-1} b - 0\right) = \frac{\pi}{2}$$

55. Since $0 \le \cos^2 x \le 1$, so

$$\frac{1}{8} \le \frac{1}{5 + 3\cos^2 x} \le \frac{1}{5}$$

$$\Rightarrow \frac{\pi}{16} \le \int_0^{\pi/2} \frac{dx}{5 + 3\cos^2 x} \le \frac{\pi}{10}$$

56. $\dfrac{\cos x}{\sqrt{1-\sin^2 x}} = \dfrac{\cos}{|\cos x|} = \begin{cases} 1 & x \in [0, \pi/2] \\ -1 & , x \in (\pi/2, \pi) \end{cases}$

$$\int_0^{\pi} \frac{\cos x}{\sqrt{1-\sin^2 x}} = \int_0^{\pi/2} dx - \int_{\pi/2}^{\pi} dx = 0$$

57. Integrating by parts,

$$\int_0^{\pi} x^2 \cos x = x^2 \sin x\Big|_0^{\pi} - \int_0^{\pi} 2x \sin x\, dx = -2J$$

$$J = \int_0^{\pi} x \sin x\ dx = \int_0^{\pi} \pi \sin x\ dx - \int_0^{\pi} x \sin x\ dx$$

$$\Rightarrow 2J = \pi\left(-\cos x\right)\Big|_0^{\pi} = 2\pi \Rightarrow J = \pi.$$

58. Set $x^3 + 1 = u$

$$\Rightarrow \int_1^2 \frac{10x^2}{\left(x^3+1\right)^2}\, dx = \frac{10}{3}\int_2^9 \frac{du}{u^2} = \frac{10}{3}\left[-\frac{1}{u}\right]_2^9 = \frac{35}{27}\,.$$

Level 2

59. $I = \displaystyle\int_{\pi/4}^{\pi/3} \frac{\cos x}{\sin x(\cos x + 1)}\, dx$

$$= \int_{\pi/4}^{\pi/3} \frac{\cos x \sin x}{\left(1-\cos^2 x\right)(1+\cos x)}\, dx$$

Put $\cos x = t$, so that

$$I = \int_{1/2}^{1/\sqrt{2}} \frac{t\ dt}{(1+t)^2\,(1-t)}$$

$$= \int_{1/2}^{1/\sqrt{2}} \left[\frac{1}{4}\left(\frac{1}{1+t} + \frac{1}{1-t}\right) - \frac{1}{2(1+t)^2}\right] dt$$

$$= \left[\frac{1}{4} \log \frac{1+t}{1-t} + \frac{1}{2(1+t)}\right]_{1/2}^{1/\sqrt{2}}$$

$$= \frac{1}{4} \log\left[\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\left(\frac{2}{3}\right)\right] + \frac{1}{2+\sqrt{2}} - \frac{1}{3}\,.$$

60. $\displaystyle\int_a^b [f(x) - fave]\, dx$

$$= \int_a^b f(x)\, dx - (fave)\int_a^b dx$$

$$= \int_a^b f(x)\, dx - \left[\frac{1}{b-a}\int_a^b f(x)\, dx\right](b-a)$$

$= 0.$

61. $\int_0^{2\pi} \cos mx \cos nx \, dx$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x] dx$$

$$= \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x + \frac{1}{m-n}\sin(m-n)x\right]_0^{2\pi}$$

$= 0.$

62. For $r \neq k,\ r,\ k \in \mathbf{N}$,

$$\int_0^{2\pi} \sin k\,x \sin r\,x$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(k+r)x + \cos(k-r)x]\, dx \ = 0$$

and $\int_0^{2\pi} \sin kx \sin kx \, dx \ = \ \frac{1}{2}\int_0^{2\pi}[1 - \cos(2kx)]\, dx$

$$= \frac{1}{2}\ (2\pi) = \pi$$

Also, $\int_0^{2\pi} \sin kx \cos(rx)\, dx \ = 0 \ \forall\ r,\ k \in \mathbf{N}$

Thus, $\int_0^{2\pi} Q(x) \sin(kx)\, dx \ = \pi a_k.$

63. Imitate Example 101.

64. $\int_{1/2}^{1}\left[2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right]$

$$= x^2 \sin\left(\frac{1}{x}\right)\Big]_{1/2}^{1} - \int_{1/2}^{1} x^2\left(\cos\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right) dx$$

$$- \int_{1/2}^{1}\cos\left(\frac{1}{x}\right) dx$$

$= \sin(1) - \frac{1}{4}\sin(2).$

65. $F'(x) = f(x) > 0$

$\Rightarrow$ F is differentiable and increasing on $[a, b]$.

66. First show that $f(x) = 2x,\ x \in \mathbf{R}$

Now, $\int_0^1 f(x)\, g(x)\, dx$

$$= \int_0^1 2x(e^x - 2x)\, dx$$

$$= 2xe^x\Big]_0^1 - 2\int_0^1 e^x dx - \frac{4}{3}x^3\Big]_0^1$$

$= 2e - 2\,(e - 1) - \frac{4}{3} \ = \ \frac{2}{3}.$

67. $I = \int_0^{\pi/2} \operatorname{cosec}(x + \pi/3) \operatorname{cosec}(x + \pi/6)\, dx$

$$= \frac{1}{\sin(\pi/6)}\int_0^{\pi/2} \frac{\sin[(x+\pi/3) - (x+\pi/6)]}{\sin(x+\pi/3)\sin(x+\pi/6)}\, dx$$

$$= 2\int_0^{\pi/2}[\cot(x+\pi/6) - \cot(x+\pi/3)]\, dx$$

$$= 2\log\left|\frac{\sin(x+\pi/6)}{\sin(x+\pi/3)}\right|\Bigg]_0^{\pi/2}$$

$$= 2\left[\log\left(\frac{\sqrt{3}/2}{1/2}\right) - \log\left(\frac{1/2}{\sqrt{3}/3}\right)\right]$$

$= 2 \log 3.$

68. $I = \int_0^1 \log\left(\sqrt{1-x} + \sqrt{1+x}\right) dx$

$$= x \log\left(\sqrt{1-x} + \sqrt{1+x}\right)\Big]_0^1$$

$$- \int_0^1 x\left(\frac{1}{\sqrt{1-x}+\sqrt{1+x}}\right)\left(\frac{-1}{2\sqrt{1-x}} + \frac{1}{2\sqrt{1+x}}\right) dx$$

$= \log\ (\sqrt{2})$

$$+ \ \frac{1}{2}\int_0^1 \frac{x}{\sqrt{1-x}+\sqrt{1+x}}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1-x^2}}\right) dx$$

$$= \frac{1}{2}\log 2$$

$$+ \frac{1}{2}\int_0^1 \frac{x}{(1+x)-(1-x)}\ \frac{1+x+1-x-2\sqrt{1-x^2}}{\sqrt{1-x^2}}\, dx$$

$$= \frac{1}{2}\log 2 + \frac{1}{2}\int_0^1\left(\frac{1}{\sqrt{1-x^2}} - 1\right) dx$$

$$= \frac{1}{2}\left(\log 2 + \frac{\pi}{2} - 1\right).$$

69. For $n \geq 1,\ 0 \leq x \leq 1/2,\ x^{2n} \leq x^2$,

$$\Rightarrow 0 \leq 1 - x^2 \leq 1 - x^{2n} \Rightarrow \frac{1}{\sqrt{1-x^{2n}}} \leq \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \int_0^{1/2}\frac{dx}{\sqrt{1-x^{2n}}} \leq \int_0^{1/2}\frac{dx}{\sqrt{1-x^2}} \ = \ \sin^{-1} x\Big]_0^{1/2}$$

$$= \frac{\pi}{6} < 1.$$

70. $8 = f'(1) = 2A + B$

$33 = f(2) + f''(2) = 6A + 2B + C$

Also, $\frac{7}{3} \ = \ \frac{1}{3}A + \frac{1}{2}B + C$

$\Rightarrow 14 = 2A + 3B + 6C$

Solving we obtain

$A = 7,\ B = -6,\ C = 3$

71. $y = \int_x^{x^2} \log t\, dt = (t \log t - t)\Big]_x^{x^2}$

$= (2x^2 - x) \log x - x^2 + x$

$\Rightarrow y(2) = 6 \log 2 - 2$

Also, $y'(2) = 7 \log 2$

Thus, equation of tangent at $x = 2$ is

$y - (6 \log 2 - 2) = (7 \log 2)(x - 2)$

or $y + 8 \log 2 + 2 = (7 \log 2)(x)$

72. $\int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin\theta}{2+\sin\theta}\right) d\theta = 0$, as the integrand is an odd function.

73. Put $\sin x - \cos x = t$, so that

$$I = \int_{-1}^{0} \frac{dt}{3+1-t^2}$$

$$= \frac{1}{2(2)} \log\left|\frac{2+t}{2-t}\right|\Bigg]_{-1}^{0} = \frac{1}{4} \log 3 .$$

74. In I_2, put $t = 1/u$ to obtain

$$I_2 = \int_1^x \frac{\left(-1/u^2\right)du}{1+1/u^2} = \int_x^1 \frac{dt}{1+t^2} = I_1$$

75. $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{12}$

$\Rightarrow \sec^{-1} x - \sec^{-1} \sqrt{2} = \pi/12$

$\Rightarrow \sec^{-1} x = \pi/3$

$\Rightarrow x = \sec(\pi/3) = 2.$

76. Average $= \frac{1}{2}\int_0^2 \frac{2}{e^x+1} dx = \int_0^2 \frac{e^{-x}}{1+e^{-x}} dx$

$$= -\log\left(1+e^{-x}\right)\Big]_0^2$$

$= \log 2 - \log(1 + e^{-2})$

$= 2 + \log[2/(1 + e^2)].$

77. $I = \int_0^{\pi} x \log(\sin x)\, dx$

$$= \int_0^{\pi} (\pi - x) \log[\sin(\pi - x)]\, dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \log \sin x\, dx$$

$$= \pi\left[\int_0^{\pi/2} \log \sin x\, dx + \int_0^{\pi/2} \log \sin(\pi - x)\, dx\right]$$

$$\Rightarrow I = \pi\int_0^{\pi/2} \log(\sin)\, dx = \frac{\pi^2}{2} \log 2.$$

78. $I = \int_{-\pi}^{\pi}\left[\cos^2 px + \sin^2 qx - 2 \cos px \sin qx\right] dx$

$$= \int_0^{\pi} [2 + \cos 2px - \cos 2qx]\, dx = 2\pi.$$

[∵ third term in the integrand is odd]

79. $I = \int_0^a f(x)g(x)h(x)\, dx$

$$= \int_0^a f(a-x)g(a-x)h(a-x)\, dx$$

$$= \int_0^a f(x)\{-g(x)\}\left[\frac{3}{4} h(x) - \frac{5}{4}\right] dx$$

$$\frac{7}{4} I = \frac{5}{4}\int_0^a f(x)g(x)\, dx$$

$$\Rightarrow I = \frac{5}{7} I_1$$

where $I_1 = \int_0^a f(x)g(x)\, dx$

$$= \int_0^a f(a-x)g(a-x)\, dx$$

$$= \int_0^a f(x)[-g(x)]\, dx = -I_1$$

$\Rightarrow 2I_1 = 0 \Rightarrow I_1 = 0$

Thus, $I = 0$.

80. Differentiating w·r·t· x, we get

$$\sqrt{3 - 2\sin^2 x} + \frac{dy}{dx} \cos y = 0$$

Putting $x = \pi/2$ and $y = \pi$, we get

$$\frac{dy}{dx}\Bigg]_{(\pi/2,\, \pi)} = 1.$$

81. We have

$P'(x) = a(x - 1)(x - 3)$

$= a[(x - 2)^2 - 1]$

$$\Rightarrow P(x) = a\left[\frac{1}{3}(x-2)^3 - x\right] + b$$

As $6 = P(1) = -\frac{4}{3} a + b$

$2 = P(3) = -\frac{8}{3} a + b$

$\Rightarrow a = 3,\ b = 10$

Thus

$P(x) = (x - 2)^3 - 3x + 10$

$$\Rightarrow \int_0^1 P(x)\, dx = \frac{19}{4}.$$

82. We have $P''(x) = a(x + 1)x(x - 1)$

$$= a\,(x^3 - x)$$

$$\Rightarrow P'(x) = a\left(\frac{1}{4}x^4 - \frac{1}{2}x^2\right) + b$$

$$\Rightarrow P(x) = a\left(\frac{1}{20}x^5 - \frac{1}{6}x^3\right) + bx + c$$

As

$$-1 = a\left(-\frac{1}{20} + \frac{1}{6}\right) - b + c$$

$$1 = a\left(\frac{1}{20} - \frac{1}{6}\right) + b + c$$

$$\Rightarrow c = 0$$

and $1 = -\frac{7}{60}a - a + b$

Also, $\sqrt{3} = b$

$$\Rightarrow a = 60\left(\sqrt{3} - 1\right)/7$$

We have

$$\int_0^1 P(x)\,dx = -\frac{1}{30}a + \frac{1}{2}b$$

$$= \frac{2}{7}\left(1 - \sqrt{3}\right) + \frac{\sqrt{3}}{2}.$$

83. $S_n = \frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}e^{k/n}$

$$\lim_{n\to\infty} S_n = \int_0^1 x\,e^x\,dx = 1.$$

84. As $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^{k+2}\,2^k}{k!} = x^2\sum_{k=0}^{\infty}\frac{(2x)^k}{k!} = x^2\,e^{2x}$

Thus, $\int_0^1 \lim_{n\to\infty}\left(\sum_{k=0}^{n}\frac{x^{k+2}\,2^k}{k!}\right)dx$

$$= \int_0^1 x^2\,e^{2x}\,dx = \frac{1}{4}\left(2x^2 - 2x + 1\right)e^{2x}\Big]_0^1$$

$$= \frac{1}{4}(e^2 - 1).$$

85. Let $I = \int_0^1 \frac{x^{2\alpha} - 1}{\log x}\,dx$

$$\Rightarrow \frac{dI}{d\alpha} = \int_0^1 \frac{2x^{2\alpha}\log x}{\log x}\,dx = \frac{2}{2\alpha + 1}\,x^{2\alpha+1}\Big]_0^1$$

$$= \frac{2}{2\alpha + 1}$$

$$\Rightarrow I = \log\,(2\alpha + 1) + C$$

When $\alpha = 0$, $I = 0 \Rightarrow C = 0$

Thus, $I = \log\,(2\alpha + 1) = \log\,(2n)$.

86. $\sin\alpha + \int_{\alpha}^{2\alpha}\cos 2\,x\,dx = 0$

$$\Rightarrow \sin\,\alpha\ \frac{1}{2}\ (\sin 4\alpha - \sin 2\alpha) = 0$$

$$\Rightarrow \sin\,\alpha\,[1 + \cos(3\alpha)] = 0$$

As $-\pi < \alpha < 0$, $\sin\,\alpha \neq 0$, therefore

$\cos\,(3\alpha) = -1 \Rightarrow \alpha = -\pi/3$.

87. $\int_0^9 g(x)\,dx = \int_0^1 dx + \int_1^4 x^3 dx + \int_4^9 \sqrt{x}\,dx$

$$= 1 + \frac{1}{4}(4^4 - 1) + \frac{2}{3}(9^{3/2} - 4^{3/2})$$

$$= \frac{929}{12}.$$

88. $\lim_{x\to 0}\frac{\int_{-x}^{x} f(2t)\,dt}{\int_0^{2x} f(t+4)\,dt}$ $\qquad\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0}\frac{(1)f(2x) - (f(-2x))(-1)}{2f(2x+4)}$$

$$= \lim_{x\to 0}\frac{f(2x) + f(-2x)}{2f(2x+4)}$$

$$= \frac{f(0) + f(0)}{2f(4)} \qquad [\because f \text{ is continuous}]$$

$$= \frac{f(0)}{f(4)}$$

89. $\int_{-2}^{3} f(x)\,dx = \int_{-2}^{2} e^{\cos x}\sin x\,dx + \int_2^3 2\,dx$

$$= 0 + 2 = 2.$$

[integrand of first integral is odd]

90. Let $I = \int_{-\pi}^{\pi}\frac{\cos^2 x}{1 + a^x}\,dx$ $\qquad (1)$

Put $x = -\theta$, to obtain

$$I = \int_{\pi}^{-\pi}\frac{\cos^2\theta}{1 + a^{-\theta}}(-1)d\theta = \int_{-\pi}^{\pi}\frac{a^x\cos^2 x}{1 + a^x}\,dx \qquad (2)$$

Adding (1) and (2), we get

$$2\,I = \int_{-\pi}^{\pi}\cos^2 x\,dx = 2\int_0^{\pi}\cos^2 x\,dx = \pi$$

$$\Rightarrow I = \pi/2.$$

91. $F(x) = \int_0^{\pi} f(t)\,dt$

$$\Rightarrow F'(x) = f\,(x)$$

$$\therefore f(4) = F'(2)$$

Also, $2xF'(x^2) = 2x\,(1 + x) + x^2$

$\Rightarrow F'(x^2) = 1 + 3x/2$

$\therefore f(4) = F'(2^2) = 1 + 3 = 4.$

92. $f'(x) = \sqrt{2-x^2}$

Now, $x^2 - f'(x) = 0$

$\Rightarrow x^2 = \sqrt{2-x^2}$

$\Rightarrow x = \pm 1$

93. $I(m, n) = \frac{1}{m+1} t^{m+1}(1+t)^n \Big]_0^1$

$- \frac{n}{m+1} I(m + 1, n - 1)$

$\Rightarrow I(m, n) = \frac{2^n}{m+1} - \frac{n}{m+1}\; I(m + 1, n - 1)$

94. $I = \int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx = \int_0^1 \frac{1-x}{\sqrt{1-x^2}}\, dx$

$= \left(\sin^{-1}x + \sqrt{1-x^2}\right)\Big]_0^1 = \frac{\pi}{2} - 1.$

95. Differentiating both the sides of

$\int_0^{t^2} x\, f(x) = \frac{2}{5}t^5$

we get

$(2t)\,(t^2)\, f(t^2) = 2t^4$

$\Rightarrow f(t^2) = t$

$\Rightarrow f\left(\frac{4}{25}\right) = \frac{2}{5}.$

96. $I = \int_{-2}^{0} [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)]\, dx$

$= \int_{-2}^{0} \left[(x+1)^3 + 2 + (x+1)\cos(x+1)\right] dx$

Put $x + 1 = t$, therefore,

$I = \int_{-1}^{1} [t^3 + 2 + t\cos t]\, dt$

$= 4$ [∵ $t^3 + t\cos t$ is an odd function].

Previous Years' AIEEE/JEE Main Questions

1. $\int_0^{\sqrt{2}} [x^2]dx = \int_0^1 [x^2]dx + \int_1^{\sqrt{2}} [x^2]dx$

$= \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx$

$= \sqrt{2} - 1$

2. $I_n = \int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1)\, dx$

$= \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\, dx - I_{n-2}$

$\Rightarrow I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}\Bigg|_0^{\pi/4} = \frac{1}{n-1}$

$\Rightarrow n(I_n + I_{n-2}) = \frac{n}{n-1} = \frac{1}{1-\frac{1}{n}}$

$\Rightarrow \lim\limits_{n\to\infty} n(I_n + I_{n-2}) = 1.$

3. $\int_\pi^{10\pi} |\sin x|\, dx = \int_0^{9\pi} |\sin(\pi + u)|\, du = \int_0^{9\pi} |\sin u|\, du$

$= 9\int_0^\pi |\sin u|\, du$ ($|\sin u|$ has period π)

$= 9 \int_0^\pi \sin u\, du = -9 \cos u\big|_0^\pi$

$= 18$

4. $\int_0^2 x f'(x)dx = x f(x)\big|_0^2 - \int_0^2 f(x)dx$

$= 2 f(2) - \int_0^2 f(x)dx$

$= 0 - \frac{3}{4} = -\frac{3}{4}$

5. $f'(x) = f(x) \Rightarrow \frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx$

$\Rightarrow \log y = x +$ const.

$\Rightarrow y = Ce^x$. Since $y(0) = 1$ so $C = 1$

Hence $f(x) = e^x$,

$\int_0^1 f(x)g(x)\, dx = \int_0^1 e^x(x^2 - e^x)\, dx$

$= \int_0^1 x^2 e^x\, dx - \int_0^1 e^{2x}\, dx$

$= \left[x^2 e^x\Big|_0^1 - 2\int_0^1 x e^x\, dx\right] - \frac{e^{2x}}{2}\Bigg|_0^1$

$= \left[e - 2\left[xe^x\Big|_0^1 - \int_0^1 e^x\, dx\right]\right] - \frac{1}{2}\,(e^2 - 1)$

$= [e - 2e + 2e - 2] - \frac{1}{2}\,(e^2 - 1)$

$= e - 2 - \frac{1}{2}e^2 + \frac{1}{2} = e - \frac{1}{2}e^2 - \frac{3}{2}$

6. $F(t) = \int_0^t f(t-y)g(y)\, dy = \int_0^t e^{t-y} y\, dy$

$= e^t \int_0^t y\, e^{-y}\, dy$

$= e^t \left[-y e^{-y}\Big|_0^t + \int_0^t e^{-y}\, dy \right]$

$= e^t\, [- t\, e^{-t} - [e^{-t} - 1]$

$= - t - 1 + e^t = e^t - (t + 1).$

7. $\int_a^b x\, f(x)\, dx = \int_a^b (a+b-x) f(a+b-x)\, dx$

$= \int_a^b (a+b-x) f(x)\, dx$

$= (a + b) \int_a^b f(x)\, dx - \int_a^b x\, f(x)\, dx$

$\Rightarrow 2 \int_a^b x\, f(x)\, dx = (a + b) \int_a^b f(x)\, dx$

$\Rightarrow \int_a^b x\, f(x)\, dx = \frac{a+b}{2} \int_a^b f(x)\, dx$

8. $I = \int_0^1 x(1-x)^n\, dx = \int_0^1 (1-x)(1-(1-x))^n\, dx$

$= \int_0^1 (1-x) x^n\, dx$

$= \int_0^1 x^n\, dx - \int_0^1 x^{n+1}\, dx$

$= \frac{1}{n+1} - \frac{1}{n+2}.$

9. $\lim_{x\to 0} \frac{\int_0^{x^2} \sec^2 t\, dt}{x \sin x}$

$= \lim_{x\to 0} \frac{\tan t\big|_0^{x^2}}{x \sin x} = \lim_{x\to 0} \frac{\tan x^2}{x^2} \cdot \frac{x}{\sin x} = 1$

10. $\int_{-2}^3 |1-x^2|\, dx$

$= \int_{-2}^{-1} |1-x^2|\, dx + \int_{-1}^1 |1-x^2|\, dx + \int_1^3 |1-x^2|\, dx$

$= -\int_{-2}^{-1} (1-x^2)\, dx + \int_{-1}^1 (1-x^2)\, dx - \int_1^3 (1-x^2)\, dx$

$= -x\Big|_{-2}^{-1} + \frac{x^3}{3}\Bigg|_{-2}^{-1} + x\Big|_{-1}^1 - \frac{x^3}{3}\Bigg|_{-1}^1 - x\Big|_1^3 + \frac{x^3}{3}\Bigg|_1^3$

$= - 1 + \frac{7}{3} + 2 - \frac{2}{3} - 2 + \frac{26}{3} = \frac{28}{3}.$

11. We have $1 + \sin 2x = (\sin x + \cos x)^2$. Thus,

$I = \int_0^{\pi/2} (\sin x + \cos x)\, dx$

$= (-\cos x + \sin x)\big]_0^{\pi/2}$

$= - \cos \frac{\pi}{2} + \sin \frac{\pi}{2} - (- \cos 0 + \sin 0) = 2$

12. Let $I = \int_0^\pi x f(\sin x) dx = \int_0^\pi (\pi - x) f\{\sin(\pi - x)\} dx$

$= \pi \int_0^\pi f(\sin x) dx - I$

$2I = \pi \int_0^{\pi/2} [f(\sin x) + f\{\sin(\pi - x)\}] dx$

$I = \pi \int_0^{\pi/2} f(\sin x)\, dx$

13. $f(a) + f(- a) = \frac{e^a}{1+e^a} + \frac{e^{-a}}{1+e^{-a}} = 1$

Let $f(- a) = b$, so $f(a) = 1 - b$

$I_1 = \int_b^{1-b} x g(x(1-x))\, dx$

$= \int_b^{1-b} (1-b+b-x)\ g((1 - b + b - x)\ (1 - 1 + b - b + x))dx$

$= \int_b^{1-b} (1-x)\ g((1 - x)x)dx$

$= I_2 - I_1$

$\Rightarrow 2I_1 = I_2 \Rightarrow \frac{I_2}{I_1} = 2.$

14. The given limit can be written as

$\lim_{n\to\infty} \frac{1}{n} \sum \frac{k}{n} \sec^2 \left(\frac{k}{n}\right)^2 = \int_0^1 x \sec^2 x^2\, dx$

$= \frac{1}{2} \tan x^2 \Big|_0^1 = \frac{1}{2}\ \tan 1.$

15. $\lim_{x\to 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt$

$= \lim_{x\to 2} \frac{1}{x-2} t^4 \Big]_6^{f(x)}$

$= \lim_{x\to 2} \frac{[f(x)]^4 - 6^4}{x-2}$

$= \lim_{x\to 2} \left(\frac{f(x)-6}{x-2}\right)\ (f(x) + 6)\ ([f(x)]^2 + 36)$

$= f'(2)\ (f(2) + 6)\ ([f(2)]^2 + 36)$

$= \frac{1}{48}\ (6 + 6)\ [36 + 36) = 18.$

16. For $0 < x < 1$, $x^2 > x^3$

$\Rightarrow 2^{x^2} > 2^{x^3} \Rightarrow \int_0^1 2^{x^2} dx > \int_0^1 2^{x^3} dx$ or $I_1 > I_2$

For $1 < x < 2$, $x^2 < x^3$

$\Rightarrow \int_1^2 2^{x^2} dx < \int_1^2 2^{x^3} dx$ or $I_3 < I_4$

17. Putting $x = -y$

$$I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$$

we get

$$I = \int_{\pi}^{-\pi} \frac{[\cos(y)]^2}{1+a^{-y}} (-1)dy = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_{-\pi}^{\pi} \frac{(1+a^x)\cos^2 x}{1+a^x} dx = \int_{-\pi}^{\pi} \cos^2 x\, dx$$

$$= 2\int_0^{\pi} \cos^2 x\, dx = \int_0^{\pi} (1+\cos 2x)\, dx$$

$$= \left(x + \frac{1}{2}\sin 2x\right)\Big]_0^{\pi} = \pi$$

$\Rightarrow I = \pi/2$.

18. Let $I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x}+\sqrt{x}} dx \quad (1)$

$$= \int_3^6 \frac{\sqrt{3+6-x}}{\sqrt{9-(3+6-x)}+\sqrt{3+6-x}} dx$$

$$= \int_3^6 \frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}} dx \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_3^6 dx = 6 - 3 = 3$$

$\Rightarrow I = \frac{3}{2}$.

19. Put $x + \pi = \theta$, so that

$$I = \int_{-\pi/2}^{\pi/2} [\theta^3 + \cos^2(\theta + 2\pi)]d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = 2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

20. Let $I = \int_0^{\pi} xf(\sin x)\, dx$

$$= \int_0^{\pi} (\pi - x) f\{\sin(\pi - x)\}\, dx$$

$$= \pi\int_0^{\pi} f(\sin x)\, dx - I$$

$$\Rightarrow 2I = \pi \int_0^{\pi/2} [f(\sin x) + f(\sin(\pi - x))]\, dx$$

$$= 2\pi \int_0^{\pi/2} f(\sin x) dx$$

$$\Rightarrow I = \pi \int_0^{\pi/2} f\{\sin(\pi/2 - x)\} dx$$

$$= \pi \int_0^{\pi/2} f(\cos x) dx$$

21. Let $m \le a < m + 1$, $m \in \mathbf{N}$, so that $[a] = m$

$$\int_1^a [x] f'(x) dx$$

$$= \int_1^2 f'(x)dx + \int_2^3 2f'(x)dx + \cdots + \int_{m-1}^{m} (m-1)f'(x)dx$$

$$+ \int_m^a m\, f'(x)dx$$

$= (f(2) - f(1)) + 2\,(f(3) - f(2)) + \ldots$

$+ (m - 1)\,(f(m) - f(m - 1)) + m(f(a) - f(m))$

$= mf(a) - [f(1) + f(2) + \ldots + f(m)]$

$= [a]\, f(a) - (f(1) + \ldots + f([a]))$.

22. $F(e) = f(e) + f\left(\frac{1}{e}\right)$

$$= \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{1+t} dt$$

In the second integral, put $t = 1/u$ to obtain

$$\int_1^{1/e} \frac{\log t}{1+t} dt = \int_1^e \frac{\log(1/u)}{1+1/u}\left(-\frac{1}{u^2}\right) du$$

$$= \int_1^e \frac{\log u}{u(1+u)} du$$

$$\therefore F(e) = \int_1^e \left[\frac{\log t}{1+t} + \frac{\log t}{t(1+t)}\right] dt$$

$$= \int_1^e \frac{\log t}{1+t}\left(1 + \frac{1}{t}\right) dt$$

$$= \int_1^e \frac{\log t}{t} dt = \frac{1}{2}(\log t)^2 \Big]_1^e$$

$= \frac{1}{2}$.

23. $\int_{\sqrt{2}}^{x} \frac{dt}{t\sqrt{t^2-1}} = \frac{\pi}{2}$

$$\Rightarrow \sec^{-1} t \Big]_{\sqrt{2}}^{x} = \frac{\pi}{2}$$

$$\Rightarrow \sec^{-1} x - \sec^{-1} \sqrt{2} = \frac{\pi}{2}$$

$$\Rightarrow \sec^{-1} x = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\Rightarrow x = \sec\left(\frac{3\pi}{4}\right) = -\sqrt{2}$$

24. For $0 < x < 1$, $\sin x < x$, so

$$I = \int_0^1 \frac{\sin x}{\sqrt{x}}\, dx < \int_0^1 \frac{x}{\sqrt{x}} dx = \int_0^1 \sqrt{x}\, dx = \left.\frac{2x^{3/2}}{3}\right|_0^1 = \frac{2}{3}$$

$$I = \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 \frac{1}{\sqrt{x}} dx = 2\sqrt{x}\Big|_0^1 = 2$$

25. Using $\int\limits_0^{2a} f(x)dx = \int\limits_0^{a} [f(x) + f(2a - x)]dx$, we can write

$$I = \int\limits_0^{\pi} [\cot x]dx = \int\limits_0^{\pi/2} ([\cot x] + [\cot(\pi - x)])dx$$

$$= \int\limits_0^{\pi/2} ([\cot x] + [-\cot x])dx$$

Put $\cot x = t$, so that

$$I = \int\limits_0^{\infty} ([t] + [-t]) \frac{dt}{1+t^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \int\limits_{k-1}^{k} ([t] + [-t]) \frac{dt}{1+t^2}$$

But $[t] + [-t] = -1$ for $k - 1 < t < k$, therefore

$$\int\limits_{k-1}^{k} ([t] + [-t]) \frac{dt}{1+t^2} = \int\limits_{k-1}^{k} (-1) \frac{dt}{1+t^2} =$$

$$-[\tan^{-1} k - \tan^{-1}(k - 1)]$$

$$\therefore\ I = -\lim_{n\to\infty} \sum_{k=1}^{n} (\tan^{-1} k - \tan^{-1}(k-1))$$

$$= -\lim_{n\to\infty} [\tan^{-1} n - \tan^{-1} 0] = -\frac{\pi}{2}.$$

26. $\int_0^1 p(x)dx = \int_0^1 1.p(x)dx = [xp(x)]_0^1 - I_1 = p(1) - I_1$

where

$$I_1 = \int_0^1 xp'(x)dx = \int_0^1 (1-x)p'(1-x)dx$$

$$= \int_0^1 (1-x)p'(x)dx = \int_0^1 p'(x)dx - I_1$$

$$\Rightarrow 2I_1 = [p(x(x)]_0^1 = p(1) - p(0)$$

Thus,

$$\int_0^1 p(x)dx = p(1) - \frac{1}{2}(p(1) - p(0))$$

$$= \frac{1}{2}(p(1) + p(0)) = \frac{1}{2}(41 + 1) = 21$$

27. Let $I = 8\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

Put $x = \tan\theta$, so that

$$I = 8\int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \sec^2\theta\, d\theta$$

$$= 8\int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - \theta\right)\right] d\theta$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= 8\int_0^{\pi/4} \log\left[1 + \frac{1 - \tan\theta}{1 + \tan\theta}\right] d\theta$$

$$= 8\int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= 8\left[\int_0^{\pi/4} (\log 2)dx - \int_0^{\pi/4} \log(1 + \tan\theta)d\theta\right]$$

$$= 8\left[\frac{\pi}{4}\log 2\right] - I$$

$$\Rightarrow 2I = 2\pi \log 2 \Rightarrow I = \pi \log 2.$$

28. $F'(x) = \sqrt{x}\, \sin x$

For local maximum or local minimum, set $f'(x) = 0$

$\Rightarrow x = \pi, 2\pi$ as $x \in (0, 5\pi/2)$

As $\sqrt{x} > 0$ for $0 < x < 5\pi/2$

$f'(x) > 0$ for $0 < x < \pi$

< 0 for $\pi < x < 2\pi$

> 0 for $2\pi < x < 5\pi/2$

By the first derivative test, f has a local maximum at $x = \pi$ and a local minimum at $x = 2\pi$.

29. $\int_0^{1.5} x[x^2]dx = \int_0^1 x[x^2]dx + \int_1^{\sqrt{2}} x[x^2]dx + \int_{\sqrt{2}}^{1.5} x[x^2]dx$

$$= 0 + \int_1^{\sqrt{2}} x\, dx + 2\int_{\sqrt{2}}^{1.5} x\, dx$$

$$= 0 + \left.\frac{x^2}{2}\right|_1^{\sqrt{2}} + 2\left.\frac{x^2}{2}\right|_{\sqrt{2}}^{1.5} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

30. $g(x+\pi) = \int_0^{x+\pi}\cos(4t)dt = I_1 + I_2$

Where $I_1 = \int_0^{\pi}\cos(4t)dt = g(\pi)$ and

$I_2 = \int_{\pi}^{x+\pi}\cos(4t)dt$

In I_2, put $t = \pi + \theta$, so that

$I_2 = \int_0^x \cos(4\pi + 4\theta)d\theta = \int_0^x \cos(4\theta)d\theta$

$= g(x)$.

However, $g(\pi) = \int_0^{\pi}\cos 4t dt = \frac{1}{4}(\sin 4t)\Big]_0^{\pi}$

$= \frac{1}{4}$ $(\sin 4\pi - \sin 0) = 0$.

Therefore, $g(x+\pi) = g(x) + g(\pi)$ and $g(x+\pi) = g(x) - g(\pi)$ both are correct options.

31. $\frac{dy}{dx} = |x|$, so $|x| = 2 \Rightarrow x = \pm 2$. If $x = 2$, then y

$= \int_0^2 |t|\,dt = \int_0^2 t dt = \frac{t^2}{2}\Big|_0^2$ $= 2$. Hence the equation tangent is $Y - 2 = 2(X - 2)$. The intercept on x-axis is given by $0 - 2 = 2(X - 2) \Rightarrow X = 1$.

If $x = -2$, then $y = \int_0^{-2}|t|\,dt = -\int_0^{-2} t dt = -\frac{t^2}{2}\Big|_0^{-2}$

$= -2$

Hence the equation of tangent is

$Y + 2 = 2(X + 2)$

The x-axis intercept is given by $0 + 2 = 2(X + 2)$

$\Rightarrow X = -1$.

32. $I = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}} = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}$

$= \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$

$= \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}} = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}dx$

$2I = I + I$

$= \int_{\pi/6}^{\pi/3}\left(\frac{1}{1+\sqrt{\tan x}} + \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\right)dx$

$= \int_{\pi/6}^{\pi/3} dx = \frac{\pi}{6}$

33. $I = \int_{-\pi/2}^{\pi/2}\frac{\sin^2 x}{1+2^x}dx = \int_{-\pi/2}^{\pi/2}\frac{\sin^2 x}{1+2^{-x}}dx$

$= \int_{-\pi/2}^{\pi/2}\frac{2^x\sin^2 x}{1+2^x}dx$

$2I = I + I = \int_{-\pi/2}^{\pi/2}\sin^2 x\,dx = 2\int_0^{\pi/2}\sin^2 x\,dx$

$= \int_0^{\pi/2}(1-\cos 2x)\,dx = \frac{\pi}{2} - \frac{\sin 2x}{2}\Big|_0^{\pi/2}$

$= \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}$

34. $\frac{dx}{dy} = \frac{1}{\sqrt{1+y^2}} \Rightarrow \frac{dy}{dx} = \sqrt{1+y^2}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{1+y^2}}\cdot 2y\frac{dy}{dx}$ $= y$.

35. $\int_{7\pi/4}^{7\pi/3}\sqrt{\tan^2 x}\,dx = \int_{7\pi/4}^{7\pi/3}|\tan x|dx$

$= \int_{7\pi/4}^{2\pi}|\tan x|dx + \int_{2\pi}^{7\pi/3}|\tan x|dx$

$= -\int_{7\pi/4}^{2\pi}\tan x\,dx + \int_{2\pi}^{7\pi/3}\tan x\,dx$

$= -\log|\sec x|\Big|_{7\pi/4}^{2\pi} + \log|\sec x|\Big|_{2\pi}^{7\pi/3}$

$= -\left[\log 1 - \log\sqrt{2}\right] + (\log 2 - \log 1)$

$= \log 2\sqrt{2}$

36. $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\,dx = \int_0^{\pi}\left|2\sin\frac{x}{2} - 1\right|dx$

$= 2\int_0^{\pi/2}|2\sin t - 1|dt$

$= 2\left[\int_0^{\pi/6}|2\sin t - 1|dt + \int_{\pi/6}^{\pi/2}|2\sin t - 1|\,dt\right]$

$= 2\left[-\int_0^{\pi/6}(2\sin t - 1)\,dt + \int_{\pi/6}^{\pi/2}(2\sin t - 1)dt\right]$

$= 2\left[2\cos t\Big|_0^{\pi/6} + \frac{\pi}{6} + -2\cos t\Big|_{\pi/6}^{\pi/2} - \frac{\pi}{3}\right]$

$= 2\left[\sqrt{3} - 2 + \frac{\pi}{6} + \sqrt{3} - \frac{\pi}{3}\right]$

$$= 2\left[2\sqrt{3}-2-\frac{\pi}{6}\right]$$

$$= 4\sqrt{3}-4-\frac{\pi}{3}$$

37. $\int_0^{1/2}\frac{\log(1+2x)}{1+4x^2}dx = \frac{1}{2}\int_0^{\pi/4}\log(1+\tan t)dt$

$(2x = \tan t)$

$$I = \frac{1}{2}\int_0^{\pi/4}\log(1+\tan t)dt = \frac{1}{2}\int_0^{\pi/4}\log\left(1+\tan\left(\frac{\pi}{4}-t\right)\right)dt$$

$$= \frac{1}{2}\int_0^{\pi/4}\log\left(1+\frac{1-\tan t}{1+\tan t}\right)dt$$

$$= \frac{1}{2}\int_0^{\pi/4}(\log 2-\log(1+\tan t))dt$$

$$= \frac{1}{2}\frac{\pi}{4}\ \log 2 - I$$

$$I = \frac{\pi}{16}\ \log 2$$

38. $P_n = \int_1^e (\log x)^n dx = x(\log x)^n \Big|_1^e - n\int_1^e (\log x)^{n-1}dx$

$= e - n\,P_{n-1}$

So $P_n = e - n[e - (n-1)\,P_{n-2}]$

Putting $n = 10$, $P_{10} = e - 10e + 10.9\,P_8$
$\Rightarrow P_{10} - 90\,P_8 = -\,9e$

39. $\int_0^{\pi}[\cos x]dx = \int_0^{\pi/2}[\cos x]dx + \int_{\pi/2}^{\pi}[\cos x]dx$

$= 0 + \int_{\pi/2}^{\pi}(-1)dx = -\frac{\pi}{2}$

40. Differentiating both sides, we have

$f(t) + t = -\,2t \Rightarrow f(t) = -\,3t,$

so $f\left(-\frac{\pi}{3}\right) = \pi.$

41. Let $I = \int_1^x \frac{e^t}{t+a}dt$

Put $t + a = u$, then

$$I = \int_{1+a}^{x+a}\frac{e^{u-a}}{u}du$$

$$= e^{-\,a}\left[\int_1^{x+a}\frac{e^u}{u}du - \int_1^{1+a}\frac{e^u}{u}du\right]$$

$$= e^{-\,a}\ [F(x+a) - F(1+a)]$$

42. $I = \int_2^4 \frac{\log(x^2)}{\log(x^2)+\log((6-x)^2)}dx \qquad (1)$

$$= \int_2^4 \frac{\log((2+4-x)^2)}{\log((2+4-x)^2)+\log((6-(2+4-x))^2)}$$

$$= \int_2^4 \frac{\log((6-x)^2)}{\log((6-x)^2)+\log(x^2)}dx \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_2^4 \frac{\log(x^2)+\log((6-x)^2)}{\log(x^2)+\log((6-x)^2)}dx$$

$2I = 2 \Rightarrow I = 1$

43. $f(4+x) = f(4-x)$

$= f(2 + (2-x))$

$= f(2 - (2-x))$

$= f(x)\ \forall\ x \in \mathbf{R}$

Thus, f is a periodic function with period 4.

Now,

$$5 = \int_0^2 f(x)dx = \int_0^2 f(2-x)dx = \int_0^2 f(2+x)dx$$

$$= \int_2^4 f(u)du = \int_2^4 f(x)dx$$

$$\therefore\ \int_0^4 f(x)dx = \int_0^2 f(x)dx + \int_2^4 f(x)dx = 10$$

As f is periodic with period 4,

$$\int_{10}^{50} f(x)dx = \int_{10}^{10+4(10)} f(x)dx = 10\int_0^4 f(x)dx$$

$= 10(10) = 100.$

44. Differentiating both the sides, we get

$$(\cos x)\,f(\sin x) = \frac{\sqrt{3}}{2}$$

To obtain $f\left(\frac{\sqrt{3}}{2}\right)$, put $x = \frac{\pi}{3}$, so that

$$\frac{1}{2}f\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow\ f\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}\,.$$

45. $f\left(\frac{1}{x}\right) = \int_1^{1/x}\frac{\log t}{1+t}dt$

Put $t = 1/u$, so that $dt = (-1/u^2)\ du$, then

$$f\left(\frac{1}{x}\right)=\int_1^x \frac{\log(1/u)}{1+1/u}\left(-\frac{1}{u^2}\right)du$$

$$= \int_1^x \frac{\log u}{u(1+u)}du = \int_1^x \frac{\log t}{t(1+t)}dt$$

$$\therefore f(x) + f\left(\frac{1}{x}\right) = \int_1^x \left(1+\frac{1}{t}\right)\frac{\log t}{1+t}dt$$

$$= \int_1^x \frac{\log t}{t}dt$$

$$= \frac{1}{2}(\log t)^2\Big]_1^x = \frac{1}{2}(\log x)^2.$$

46. Let $I = \int_0^1 \tan^{-1}(1-x+x^2)dx$

$$= \int_0^1\left(\frac{\pi}{2} - \cot^{-1}(1-x+x^2)\right)dx$$

$$= \frac{\pi}{2} - 2\int_0^1 \tan^{-1} x\,dx$$

$$= \frac{\pi}{2} - 2x\tan^{-1}x\Big]_0^1 + \int_0^1 \frac{2x}{1+x^2}dx$$

$$= \frac{\pi}{2} - 2\tan^{-1}(1) + \text{In}(1+x^2)\Big]_0^1$$

$$= \frac{\pi}{2} - 2\left(\frac{\pi}{4}\right) + \log 2$$

$= \log 2$

47. Let $y = \lim_{x\to\infty}\left(\frac{(n+1)(n+2)\ldots(n+2n)}{n^{2n}}\right)^{1/n}$

$$\log y = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{2n}\log\left(1+\frac{k}{h}\right)$$

$$= \int_0^2 \log(1+x)dx$$

$$= x\log(1+x)\Big]_0^2 - \int_0^2 \frac{x}{1+x}dx$$

$$= 2\log 3 - \int_0^2 \frac{x+1-1}{x+1}dx$$

$$= 2\log 3 - 2 + \log(x+1)\Big|_0^2$$

$$= 2\log 3 - 2 + \log 3$$

$$= \log 27 - \log e^2 = \log\frac{27}{e^2} \Rightarrow y = \frac{27}{e^2}.$$

48. Let $I = \int_4^{10} \frac{[x^2]dx}{[(14-x)^2]+[x^2]}$ (1)

$$\int_4^{10}\frac{[(10+4-x)^2]}{[(14-(10+4-x))^2]+[(10+4+x)^2]}dx$$

$$I = \int_4^{10}\frac{[(14-x)^2]}{[x^2]+[(14+x)^2]}dx \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_4^{10} dx = 6 \Rightarrow I = 3$$

Previous Years' B-Architecture Entrance Examination Questions

1. $I_1 = \int_{-4}^{-5} e^{(x+5)^2}dx = -\int_0^1 e^{t^2}dt \quad (t = x+5)$

$$I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2}dx = \int_{1/3}^{2/3} e^{(3x-2)^2}dx \quad (3x-2=t)$$

$$= \frac{1}{3}\int_{-1}^0 e^{t^2}dt = -\frac{1}{3}\int_1^0 e^{t^2}dt$$

$$= \frac{1}{3}\int_0^1 e^{t^2}dt$$

$$I_1 + 3I_2 = 0$$

2. $\int_0^{1.5}[x^2]f'(x)dx$

$$= \int_0^1[x^2]f'(x)dx + \int_1^{\sqrt{2}}[x^2]f'(x)dx + \int_{\sqrt{2}}^{1.5}[x^2]f'(x)dx$$

$$= 0 + \int_1^{\sqrt{2}} f'(x)dx + 2\int_{\sqrt{2}}^{1.5} f'(x)dx$$

$$= f(\sqrt{2}) - f(1) + 2\,[f(1.5) - f(\sqrt{2})]$$

$$= 2f(1.5) - f(\sqrt{2}) - f(1)$$

3. $I = \int_{-a}^{a}\log(x+\sqrt{x^2+1})dx = \int_{-a}^{a}\log(-x+\sqrt{x^2+1})dx$

$$\left(\int_{-a}^{a} f(x)dx = \int_{-a}^{a} f(-x)dx\right)$$

$$\int_{-a}^{a}\log\left(\frac{-x^2+x^2+1}{x+\sqrt{x^2+1}}\right)dx = -\int_{-a}^{a}\log(x+\sqrt{x^2+1})dx$$

$= -I$

$\Rightarrow I = 0$

4. $\int_{1/2^n}^{1} f(x)dx$

$$= \int_{1/2^n}^{1/2^{n-1}} f(x)dx + \int_{1/2^{n-1}}^{1/2^{n-2}} f(x)dx + \cdots + \int_{1/2}^{1} f(x)dx$$

$$= \frac{1}{2^{n-1}}\left[\frac{1}{2^{n-1}} - \frac{1}{2^n}\right] + \frac{1}{2^{n-2}}\left[\frac{1}{2^{n-2}} - \frac{1}{2^{n-1}}\right] + \cdots + 1\left[1 - \frac{1}{2}\right]$$

$$= \frac{1}{2^{2n-1}} + \frac{1}{2^{2n-3}} + \cdots + \frac{1}{2} = \frac{1}{2}\left[\frac{1-(1/2^2)^n}{1-1/2^2}\right] = \frac{2}{3}\left[1-\left(\frac{1}{4}\right)^n\right]$$

$$\lim_{n\to\infty}\int_{1/2^n}^{1} f(x)dx = \frac{2}{3}$$

5. $\int_0^1 x^2(1-x)^9\,dx = -\int_1^0 (1-t)^2 t^9 dt \quad (1 - x = t)$

$$= \int_0^1 (1-t^2-2t)t^9 dt$$

$$= \left.\frac{t^{10}}{10} + \frac{t^{12}}{12} - \frac{2t^{11}}{11}\right|_0^1$$

$$= \frac{1}{10} + \frac{1}{12} - \frac{2}{11} = \frac{1}{660}$$

6. $\int_0^1 \max(e^x, e^{1-x})dx$

$$= \int_0^{1/2} \max(e^x, e^{1-x})dx + \int_{1/2}^1 \max(e^x, e^{1-x})dx$$

$$= \int_0^{1/2} e^{1-x}dx + \int_{1/2}^1 e^x dx$$

$$= -e^{1-x}\Big|_0^{1/2} + e - e^{1/2}$$

$$= - e^{1/2} + e + e - e^{1/2}$$

$$= 2(e - e^{1/2}).$$

7. $\int_0^{\pi/2} \min(\sin x, \cos x)dx$

$$= \int_0^{\pi/4} \min(\sin x, \cos x)dx + \int_{\pi/4}^{\pi/2} \min(\sin x, \cos x)dx$$

$$= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx$$

$$= -\cos x\Big|_0^{\pi/4} + \sin x\Big|_{\pi/4}^{\pi/2}$$

$$= -\left[\frac{1}{\sqrt{2}} - 1\right] + \left[1 - \frac{1}{\sqrt{2}}\right] = 2 - \sqrt{2}$$

8. $f(x) = x^2\int_0^1 \frac{dt}{2-t} + \int_0^1 \frac{t^2}{2-t}dt$

$$= - x^2 \log\,|2-t|\,\Big|_0^1 + \int_0^1\left(-t-2+\frac{4}{2-t}\right)dt$$

$$y = x^2 \log 2 + \left(-\frac{1}{2} - 2 + 4\log 2\right)$$

which represent a parabola.

9. $\int_2^3 f'(x)f''(x)dx + \int_1^3 f''(x)dx$

$$= \left.\frac{(f'(x))^2}{2}\right|_2^3 + f'(x)\Big|_1^3$$

$$= \frac{1}{2}\,[(f'(3))^2 - (f'(2))^2] + f'(3) - f'(1)$$

$$= \frac{1}{2}\left[\left(\tan\frac{\pi}{4}\right)^2 - \left(\tan\frac{\pi}{3}\right)^2\right] + \tan\frac{\pi}{4} - \tan\frac{\pi}{6}$$

$$= \frac{1}{2}\,[1 - 3] + 1 - \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

10. For $x \in (1, 3)$, $2 < \sqrt{3+x^3} < \sqrt{30}$ so

$$2\int_1^3 dx < \int_1^3 \sqrt{3+x^3}\,dx < \sqrt{30}\int_1^3 dx$$

$$\Rightarrow 4 < \int_1^3 \sqrt{3+x^3}\,dx < 2\sqrt{30}$$

11. $\lim_{n\to\infty}\frac{1}{n}\left[1 + \frac{n^2}{n^2+1^2} + \frac{n^2}{n^2+2^2} + \cdots + \frac{n^2}{n^2+(n-1)^2}\right]$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{n^2}{n^2+r^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{1+\left(\frac{r}{n}\right)^2}$$

$$= \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^1 = \frac{\pi}{4}$$

12. $\int_a^b f(x)dx = \int_a^b x\,|\,x\,|\,dx$

$$= \begin{cases} \int_a^b (-x^2)dx & b < 0 \\ \int_a^0 -x^2dx + \int_0^b x^2dx, & a < 0 < b \\ \int_a^b x^2dx, & 0 < a \end{cases}$$

$$= \begin{cases} -\frac{1}{3}(b^3 - a^3), & b < 0 \\ \frac{1}{3}(b^3 + a^3), & a < 0 < b \\ \frac{1}{3}(b^3 - a^3), & 0 < a \end{cases}$$

$$= \frac{1}{3}\,(|b|^3 - |a|^3)$$

13. $\int_{\sqrt{\log 5}}^{\sqrt{\log 7}} \frac{x\cos x^2}{\cos(\log 35 - x^2) + \cos x^2}dx$

$$= \frac{1}{2}\int_{\log 5}^{\log 7}\frac{\cos t}{\cos(\log 35 - t) + \cos t}dt \quad (x^2 = t)$$

$$= \frac{1}{2}\,I'$$

$$I' = \int_{\log 5}^{\log 7} \frac{\cos(\log 7 + \log 5 - t)}{\cos t + \cos(\log 7 + \log 5 - t)}\, dt$$

$$= \int_{\log 5}^{\log 7} \frac{\cos(\log 35 - t)}{\cos t + \cos(\log 35 - t)}\, dt$$

$$2I' = I' + I' = \int_{\log 5}^{\log 7} \frac{\cos t + \cos(\log 35 - t)}{\cos t + \cos(\log 35 - t)}\, dt$$

$$= \log \frac{7}{5}$$

$$\Rightarrow I = \frac{1}{4} \log \frac{7}{5}$$

14. Same as *Q* 13 of AIEEE.

15. Let $I = \int\limits_0^{1/2} \frac{e^x (2 - x^2)}{(1-x)^{3/2}(1+x)^{1/2}}\, \mathrm{d}x$

$$= \int\limits_0^{1/2} e^x \left[\frac{(1-x^2)+1}{(1-x)\sqrt{1+x^2}} \right] \mathrm{d}x$$

$$= \int\limits_0^{1/2} e^x \left[\frac{\sqrt{1-x^2}}{1-x} + \frac{1}{(1-x)\sqrt{1-x^2}} \right] \mathrm{d}x$$

As $\frac{d}{dx}\left(\frac{\sqrt{1-x^2}}{1-x} \right) = \frac{(1-x)\left(\frac{-x}{\sqrt{1-x^2}} \right) + \sqrt{1-x^2}}{(1-x)^2}$

$$= \frac{-x + x^2 + 1 - x^2}{(1-x)^2 \sqrt{1-x^2}} = \frac{1}{(1-x)\sqrt{1-x^2}}$$

$$\therefore I = e^x \left(\frac{\sqrt{1-x^2}}{1-x} \right) \Bigg]_0^{1/2}$$

$$= \sqrt{e}\, \frac{\sqrt{3/4}}{2/2} - 1 = \sqrt{3e} - 1$$

16. $\int_0^2 [x^2] dx$

$$= \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2] dx + \int_{\sqrt{3}}^{2} [x^2] dx$$

$$= \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\, dx + \int_{\sqrt{3}}^{2} 3\, dx$$

$$= 0 + (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3})$$

$$= -\sqrt{2} - \sqrt{3} + 6$$

CHAPTER FOURTEEN

Areas by Integration

Definite integral can be used to calculate areas. In simple cases, the area is given by a single definite integral.

14.1 Area between a curve and the x-axis

For the curve $y = f(x) \geq 0$, $\int_a^b f(x)dx$ is the area bounded by the curve $y = f(x)$, the x-axis and the straight line $x = a$, $x = b$.

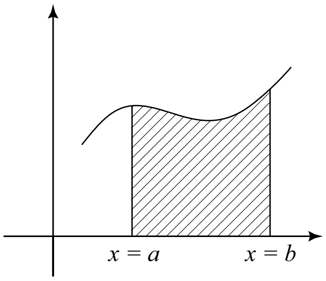


Fig. 14.1

Illustration 1

Find the area between the curve $y = x(x - 3)$ and the ordinates $x = 3$ and $x = 5$.

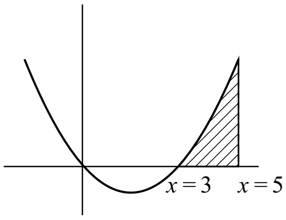


Fig. 14.2

$$\text{Required area } = \int_3^5 y\,dx = \int_3^5 x(x-3)dx$$

$$= \left.\frac{x^3}{3}\right|_3^5 - 3\left.\frac{x^2}{2}\right|_3^5$$

$$= 8\frac{2}{3}.$$

14.2 Area between two curves

The area between $y = f(x)$ and $y = g(x)$ on the interval $[a, b]$ provided $f(x) \geq g(x)$ on $[a, b]$ is

$$\int_a^b [f(x) - g(x)\,dx = \int_a^b \left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right) - \left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right) dx$$

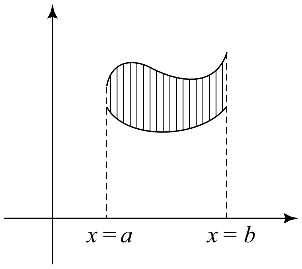


Fig. 14.3

Illustration 2

Determine the area between $y = 2x^2 + 10$ and $y = 4x + 16$

We first find the points of intersection $2x^2 + 10 = 4x + 16$

$\Rightarrow \quad 2x^2 - 4x - 6 = 0$

$\Rightarrow \quad 2(x + 1)(x - 3) = 0$

The first curve is a parabola and the second curve is a straight line

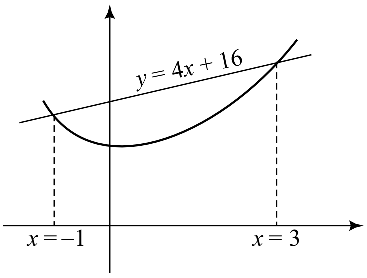


Fig. 14.4

$$\text{Required area } = \int_1^3 ((4x+16) - (2x^2+10))\,dx$$

$$= \int_{-1}^3 (-2x^2 + 4x + 6)\,dx$$

$$= -\frac{2}{3}x^3 + 2x^2 + 6x\Big|_{-1}^{3} = \frac{64}{3}.$$

14.3 If the curve crosses x-axis at $x = x_0$, then the area bounded by $y = f(x)$ and the ordinates $x = a$, $x = b$ is given by

$$\left|\int_a^{x_0} f(x)dx\right| + \left|\int_{x_0}^{b} f(x)dx\right|$$

$$= \int_a^{x_0} f(x)dx - \int_{x_0}^{b} f(x)dx$$

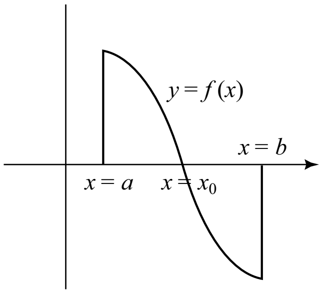


Fig. 14.5

Illustration 3

Find the area between the curve $y = x(x - 3)$ and the ordinates $x = 0$ and $x = 5$.

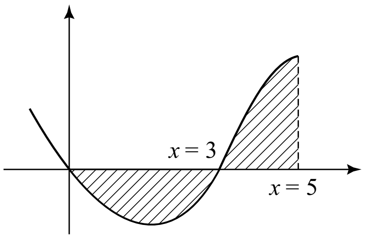


Fig. 14.6

Area between $y = x(x - 3)$ between $x = 0$ and $x = 3$ is

$$\int_0^3 (x^2 - 3x)dx = -\frac{9}{2}$$

Area between $y = x(x - 3)$ between $x = 3$ and $x = 5$ is

$$\int_3^5 (x^2 - 3x)dx = \frac{26}{3}$$

Required area = $\frac{26}{3} - \left(-\frac{9}{2}\right) = \frac{79}{6}$.

14.4 (Integration along the *y*-axis) If $f(y)$ and $g(y)$ are continuous function on $[c, d]$ and $f(y) \geq g(y)$ for all y in $[c, d]$, then the area of the region bounded by $x = f(y)$ and $x = g(y)$ and horizontal lines $y = c$ and $y = d$ is

$$\int_c^d (f(y) - g(y))dy$$

Illustration 4

Find the area of the region in the first quadrant enclosed by $x = y^2$ and $x = y + 2$

$$y^2 = y + 2 \Rightarrow y = -1 \text{ or } y = 2$$

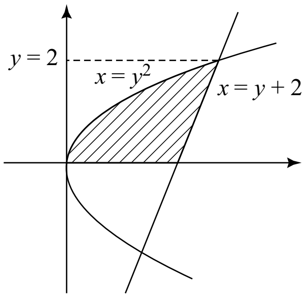


Fig. 14.7

Required area

$$= \int_0^2 ((y+2) - y^2)dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big|_0^2$$

$$= \frac{10}{3}.$$

14.5 When calculating the area under a curve $y = f(x)$, follow the steps below:

1. Sketch the curve
2. Determine the boundaries $x = a$ and $x = b$ or $y = c$ and $y = d$.
3. Set up the definite integral
4. Integrate

Following steps will be useful for tracing the curve.

1. **Symmetry:** Find out whether the curve is symmetrical about any line with the help of the following points:
   (i) The curve is symmetrical about the x-axis if the equation of the curve remain unchanged when y is replaced by $-y$ i.e. if the equation of the curve contains only even powers of y, e.g. $x^2 + y^2 = 4$, $y^2 = 8x$.
   (ii) The curve is symmetrical about y-axis if the equation of the curve remain unchanged when x is replaced by $-x$ i.e. if the equation of the curve contains only even powers of x e.g. $x^2 + y^2 = 4$, $x^2 - 4y$.
   (iii) The curve is symmetrical in opposite quadrants if the equation of curve remain unchanged when both x and y are replaced by $-x$ and $-y$, e.g. $x^2 + y^2 = 2$.
   (iv) The curve is symmetrical about the line $y = x$ if the equation of the curve remains unchanged when x and y are interchanged e.g. $x^2 + y^2 = 8$.
2. **Origin:** Find out whether the origin lies on the curve i.e. (0, 0) satisfies the equation of curve e.g. $y^2 = 4x$ passes through origin. If it does, find the equation of tangents at origin by equating to zero the lowest degree terms. For example, $x = 0$ is tangent at origin to the curve $y^2 = 4x$.

3. **Intersection with origin:** Find out the points of intersection of the curve with the coordinate axes e.g.

   $y^2 = 2x + 1$, put $x = 0$, $y^2 = 1$, $y = \pm 1$ put $y = 0$, $x = -\frac{1}{2}$.

   Thus points of intersection with coordinate axes are (0, 1) (0, –1) and $\left(-\frac{1}{2}, 0\right)$.

4. **Regions where no part of the curve lies:** Find the regions of the plane where no part of the curve lies. For example, $y^2 = 4x$. $x < 0$ no part of the curve lies.

The curve like straight line, circle parabola, hyperbola have been already studied, however we briefly describe them here.

(a) Straight Line:

General equation of straight line is of the form $ax + by + c = 0$.

To plot a straight line we put $y = 0$ and find x. Also we put $x = 0$ and find y. We plot these two points and join them to get the required line.

Some particular cases of straight lines are as follows.

(i) Lines parallel to the coordinate axes.

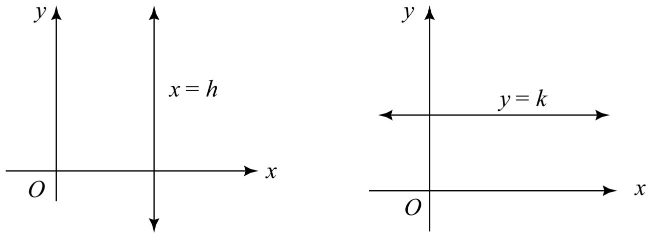


Fig. 14.8

(ii) Lines passes through origin

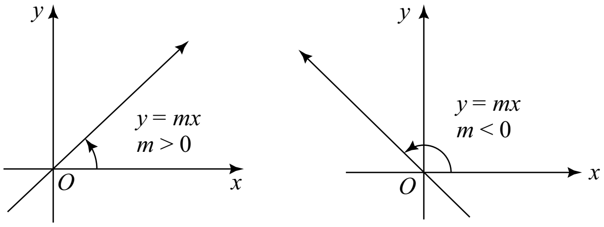


Fig. 14.9

(iii) Lines making given intercept on coordinate axes.

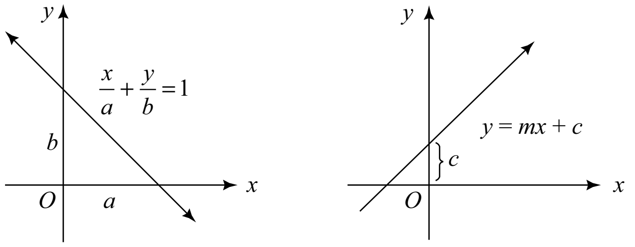


Fig. 14.10

(b) Circle:

General equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$. Its centre is $(-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$.

(i) Circle with centre at origin and radius a and

(ii) Circle with centre at $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$ are shown below.

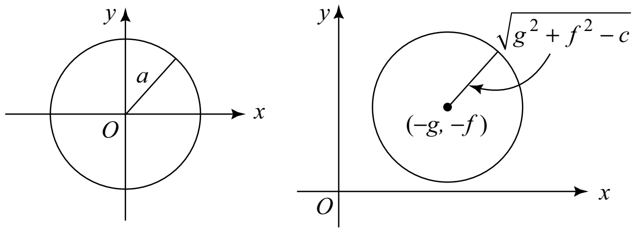


Fig. 14.11

(iii) Circle with centre on the x-axis and passing through origin. Its equation is of the form $x^2 + y^2 \pm 2ax = 0$.

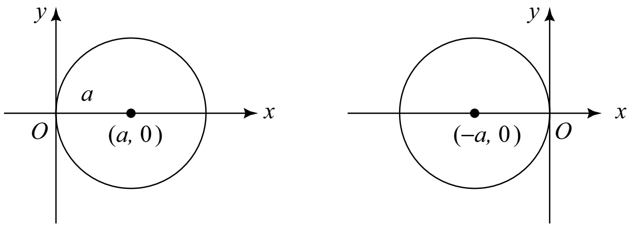


Fig. 14.12

(iv) Circle with centre on the y-axis and passing through origin. Its equation is of the form $x^2 + y^2 \pm 2by = 0$.

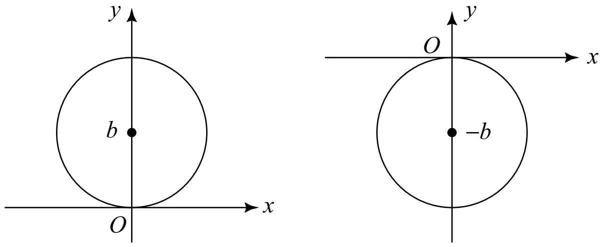


Fig. 14.13

(v) Circle with centre on the axes but not passing through origin. Its equation is of the form $x^2 + y^2 \pm 2gx + c = 0$, $x^2 + y^2 \pm 2fy + c = 0$.

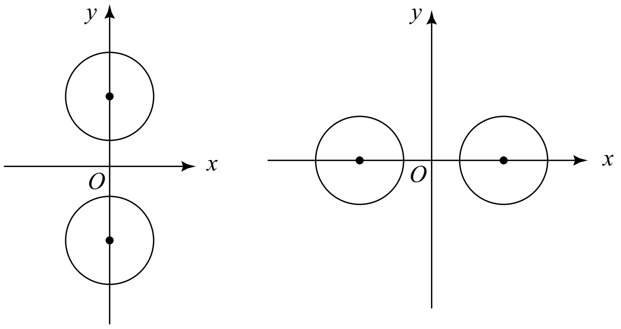


Fig. 14.14

(c) Parabola:

General equation of parabola is

$$y = ax^2 + bx + c \quad \text{or} \quad x = ay^2 + by + c.$$

By completing the square on x or on y and by shifting the origin the equation can be written in standard form as $y^2 = 4ax$ or $x^2 = 4\,by$.

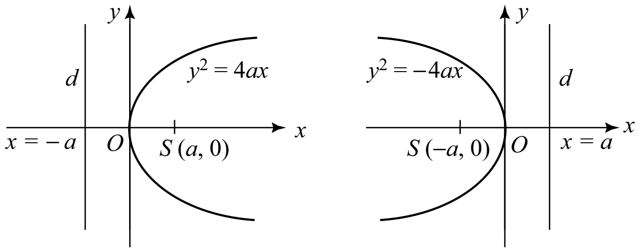


Fig. 14.15

The equation of the parabola with vertex at (h, k) and

(i) the axis parallel to the x-axis, opening to the right is $(y - k)^2 = 4a\,(x - h)$

(ii) the axis parallel to the x-axis, opening to the left is $(y - k)^2 = -\,4a\,(x - h)$

(iii) the axis parallel to the y-axis, opening upwards is $(x - h)^2 = 4a\,(y - k)$

(iv) the axis parallel to the y-axis, opening downwards is $(x - h)^2 = -\,4a\,(y - k)$

(d) Hyperbola:

Equation of hyperbola in standard form is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ or } -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

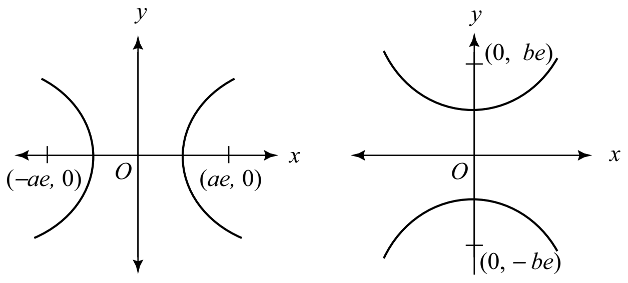


Fig. 14.16

Rectangular hyperbola is $xy = c$ or $xy = -\,c$

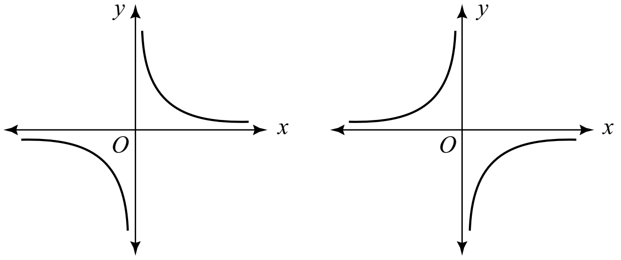


Fig. 14.17

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: The area of the region between $y = 4 - x^2$, $0 \le x \le 2$ and the x-axis is

(a) $\frac{4}{3}$ (b) $\frac{7}{3}$

(c) $\frac{16}{3}$ (d) $\frac{5}{3}$

Ans. (c)

Solution: The required are

$$= \int_0^2 (4 - x^2)\mathrm{d}x$$

$$= \int_0^2 4\mathrm{d}x - \int_0^2 x^2\mathrm{d}x$$

$$= 8 - \frac{8}{3} = \frac{16}{3}$$

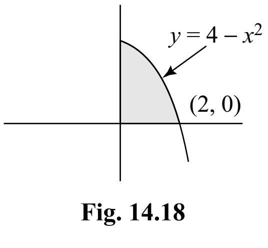


Fig. 14.18

Example 2: The area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$ is

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{3}$ (d) 2

Ans. (c)

Solution: Both the curves represent parabolas and intersects at (0, 0) and (1, 1).

$$\text{Required area} = \int_0^1 (\sqrt{x} - x^2)dx$$

$$= \left.\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right|_0^1$$

$$= \frac{1}{3}.$$

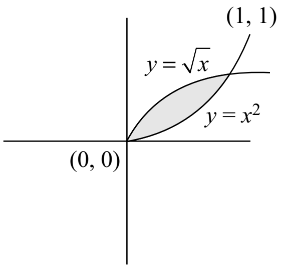


Fig. 14.19

Example 3: The area of the region bounded by the curves $y = x^2$ and $y = x$ is

(a) $\frac{1}{6}$ (b) $\frac{1}{2}$

(c) $\frac{1}{3}$ (d) 1

Ans. (a)

Solution: $y = x^2$ represent a parabola opening upward and $y = x$ is a straight line passing through origin. The two curves are meeting at (0, 0) and (1, 1).

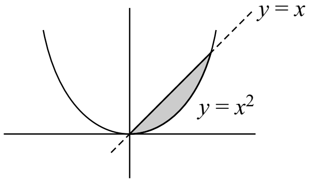


Fig. 14.20

$$\text{Required area} = \int_0^1 (x - x^2)dx$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Example 4: The area of the region bounded by $y = \log x$, $y = 2$ and the coordinate axes is

(a) $e^2 - 2$ (b) $e^2 - 1$

(c) $e^2 - e$ (d) $e^2 + 1$

Ans. (b)

Solution: Required area = area of rectangle *OABC* – area under $y = \log x$, $x = 1$, $x = e^2$

$$= AO \times OC - \int_1^{e^2} \log x \, dx$$

$$= 2e^2 - [x \log x - x]_1^{e^2}$$

$$= 2e^2 - [2e^2 - e^2] - 1 = e^2 - 1.$$

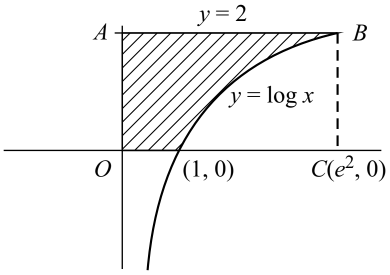


Fig. 14.21

Example 5: The area bounded by $x = 2y - y^2$ and y-axis is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) 1 (d) $\frac{4}{3}$

Ans. (d)

Solution: $x = -(y^2 - 2y) = -(y - 1)^2 + 1$. So the curve represent a parabola open to left and vertex (1, 1).

$$\text{Required area} = \int_0^2 x \, dy$$

$$= \int_0^2 (2y - y^2)\, dy = \frac{4}{3}$$

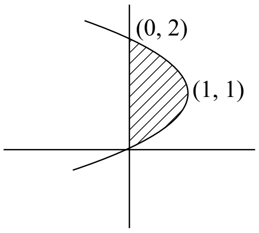


Fig. 14.22

LEVEL 1

Straight Objective Type Questions

Example 6: The area of the figure bounded by the lines $x = 0$, $x = \pi/2$, $f(x) = \sin x$ and $g(x) = \cos x$ is

(a) $2(\sqrt{2} - 1)$ (b) $\sqrt{3} - 1$

(c) $2(\sqrt{3} - 1)$ (d) $2(\sqrt{2} + 1)$

Ans. (a)

Solution: We can divide the interval [0, π/2] into the parts, [0, π/4], where $\sin x \le \cos x$, and [π/4, π/2], where $\sin x \ge \cos x$. so the required area is

$$\int_0^{\pi/4} (\cos x - \sin x) \mathrm{d}x + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \mathrm{d}x$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= (\sqrt{2} - 1) + (-1 + \sqrt{2}) = 2(\sqrt{2} - 1)$$

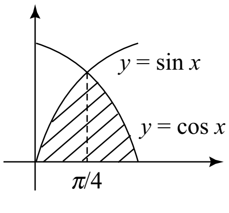


Fig. 14.23

Example 7: The area bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ is (in square units)

(a) $\left(\frac{5}{4}\pi - 2\right)$ (b) $\left(\frac{5\pi - 2}{4}\right)$

(c) $(5\pi - 2)/2$ (d) $(\pi/2 - 5)$

Ans. (b)

◎ **Solution:** The curves $y = \sqrt{5-x^2}$ and $y = |x - 1|$ intersect at (2, 1) and (– 1, 2).

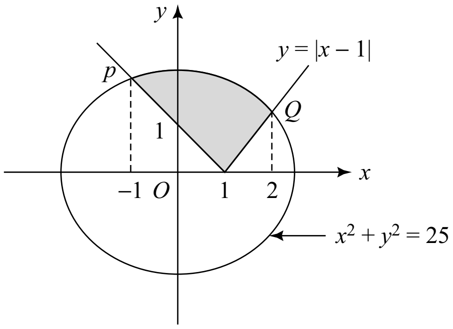


Fig. 14.24

Required area

$$= \int_{-1}^{2}\sqrt{5-x^2}\,dx - \int_{-1}^{2}|x-1|\,dx$$

But $\int_{-1}^{2}\sqrt{5-x^2}\,dx = \left(\frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right)\Bigg]_{-1}^{2}$

$$= 1 + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)+1+\frac{5}{2}\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$= 2 + \frac{5}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\sqrt{1-\frac{5}{2}}+\frac{1}{\sqrt{5}}\sqrt{1-\frac{4}{5}}\right)$$

$$= 2 + \frac{5}{2}\sin^{-1}\left(\frac{5}{5}\right) = 2+\frac{5\pi}{4}$$

and $\int_{-1}^{2}|x-1|\,dx = \int_{-2}^{1}|t|\,dt$ [Put $t = x - 1$]

$$= \int_{-2}^{0}(-t)dt + \int_{0}^{1}t\,dt = \frac{t^2}{2}\Bigg]_{2}^{0} + \frac{t^2}{2}\Bigg]_{0}^{1}$$

$$= -\,0 + 2 + \frac{1}{2} = \frac{5}{2}$$

Thus, required area $= \left(\frac{5\pi}{4}-\frac{1}{2}\right) = \frac{5\pi-2}{4}$.

◉ **Example 8:** The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ is

(a) 32/3 (b) 16/3

(c) 8/3 (d) 0

Ans. (b)

◎ **Solution:** The two curves intersect in (0, 0) and (4, 4). Required area

$$= \int_0^4 2\sqrt{x}\,dx - \int_0^4 \frac{1}{4}x^2 dx$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^4 - \left[\frac{1}{12}x^3\right]_0^4$$

$$= \frac{32}{3}-\frac{16}{3}=\frac{16}{3}$$

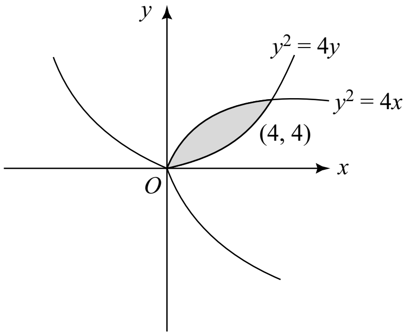


Fig. 14.25

◉ **Example 9:** The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ above the x-axis is equal to

(a) 5/3 (b) 1/3

(c) 2/3 (d) 4/3

Ans. (d)

◎ **Solution:** We first find the points of intersection of the given curves. Put $x = -2y^2$ in $x + 3y^2 = 1$ to obtain $y^2 = 1 \Rightarrow y = \pm 1 \Rightarrow x = -2$.

Thus the points of intersection are (– 2, 1) and (– 2, – 1). Both the curves are parabola with $x + 2y^2 = 0$ having (0, 0) as vertex and $x + 3y^2 = 1$ has vertex as (1, 0).

Required area $= \left[\int_{-2}^{1}\sqrt{\frac{1-x}{3}}dx - \int_{-2}^{0}\sqrt{\frac{-x}{2}}dx\right]$

$$= \frac{1}{\sqrt{3}}(1-x)^{3/2}\left(\frac{2}{3}\right)(-1)\Big]_{-2}^{1} - \frac{1}{\sqrt{2}}(-x)^{3/2}\left(\frac{2}{3}\right)(-1)\Big]_{-2}^{0}$$

$$= \frac{4}{3}\left[\frac{1}{\sqrt{2}}(0-2^{3/2}) - \frac{1}{\sqrt{3}}(0-3^{3/2})\right]$$

$$= \frac{4}{3}\,(3-2) = \frac{4}{3}.$$

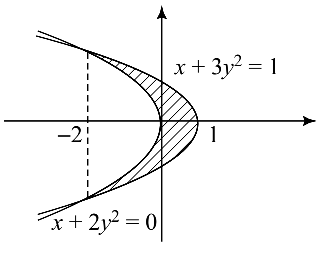


Fig. 14.26

◉ **Example 10.** The area of the region enclosed by the curves $y = x$, $x = e$, $y = 1/x$ and the positive axis is

(a) 5/2 square unit

(b) $\frac{1}{2}$ square units

(c) 1 square unit

(d) $\frac{3}{2}$ square units

Ans. (d)

Solution: The required area

$$= R_1 + R_2$$

$$= \int_0^1 x\,dx + \int_1^e \frac{1}{x}\,dx = \left.\frac{x^2}{2}\right] + \ln|x|]_1^e$$

$$= \frac{1}{2} + 1 = \frac{3}{2}$$

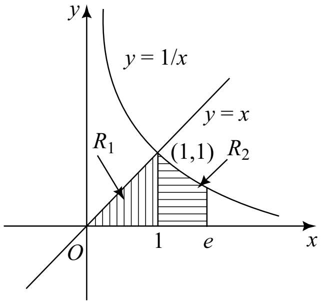


Fig. 14.27

Example 11: The area between the parabola $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line $y = 2$ is

(a) $\frac{10\sqrt{2}}{3}$ (b) $\frac{20\sqrt{2}}{3}$

(c) $10\sqrt{2}$ (d) $20\sqrt{2}$

Ans. (b)

Solution: Required area

$$= 2\left[\int_0^2 \left(3\sqrt{y} - \frac{1}{2}\sqrt{y}\right)dy\right]$$

$$= 5\left(\frac{2}{3}\right)y^{3/2}\Big]_0^2 = \frac{20}{3}\sqrt{2}$$

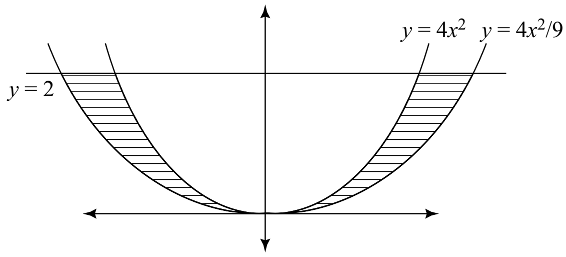


Fig. 14.28

Example 12: Let $F(x)$ be the area bounded by the curve $f(t) = \frac{e^t}{t}$ between $t = a$ $(a > 1)$, $t = x$ and axis of abscissa then the area bounded by $g(t) = \frac{e^t}{t + t_0}$ $(t_0 > 0)$ between $t = a$, $t = x$ and axis of abscissa is

(a) $e^{t_0}\ [F(x + t_0) - 2F(a + t_0)]$

(b) $e^{t_0}\ [F(x + t_0) - F(a + t_0)]$

(c) $e^{-t_0}\ [F(x + t_0) - F(a + t_0)]$

(d) $2e^{-t_0}\ [F(x + t_0) - F(a + t_0)]$

Ans. (c)

Solution: $F(x) = \int_a^x \frac{e^t}{t}\,dt$, we need to find

$g(x) = \int_a^x \frac{e^t}{t + t_0}\,dt$. Putting $t + t_0 = u$ in $g(x)$, we have,

$$g(x) = \int_{a+t_0}^{x+t_0} \frac{e^{u-t_0}}{u}\,du = e^{-t_0}\left[\int_a^{x+t_0} \frac{e^u}{u} - \int_a^{a+t_0} \frac{e^u}{u}\,du\right]$$

$$= e^{-t_0}\ [F(x + t_0) - F(a + t_0)]$$

Example 13: The area, lying above x-axis and included between $x^2 + y^2 = 8x$ and $y^2 = 4x$ is

(a) $\frac{32}{3} + 2\pi$ (b) $\frac{32}{3} + 6\pi$

(c) $\frac{16}{3} + 4\pi$ (d) $\frac{32}{3} + 4\pi$

Ans. (d)

Solution: $x^2 + y^2 = 8x$ represent circle with centre (4, 0) and radius 4. The parabola $y^2 = 4x$ intersects the given circle at (0, 0) and (4, 4) above x-axis.

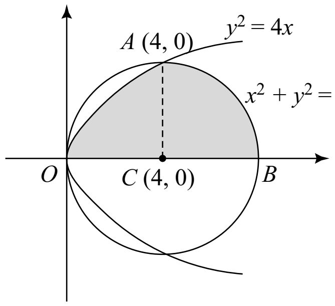


Fig. 14.29

Required area = OAB

$$= OAC + ACB$$

$$= \int_0^4 \sqrt{4x}\,dx + \frac{1}{4} \text{ area of a circle of radius 4}$$

$$= \frac{4}{3}x^{3/2}\Big|_0^4 + \frac{1}{4}\pi \cdot 16 = \frac{32}{3} + 4\pi$$

Example 14: The ratio in which the area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$ is

(a) 15 : 49 (b) 13 : 48

(c) 12 : 37 (d) 16 : 35

Ans. (a)

Solution: The point of intersection of $y^2 = 12x$ and $x^2 = 12y$ is given by (12, 12)

$$A_1 = \text{Area } OAB = \int_0^3 \sqrt{12x}\,dx - \int_0^3 \frac{x^2}{12}\,dx$$

$$= \frac{2}{3}\sqrt{12}\,x^{3/2}\Big|_0^3 - \frac{x^3}{36}\Big|_0^3 = \frac{45}{4}$$

$$A_2 = \text{Area } ABC = \int_3^{12} \sqrt{12x}\,dx - \int_3^{12} \frac{x^2}{12}\,dx$$

$$= \frac{147}{4}$$

$$A_1 : A_2 = 45 : 147 = 15 : 49$$

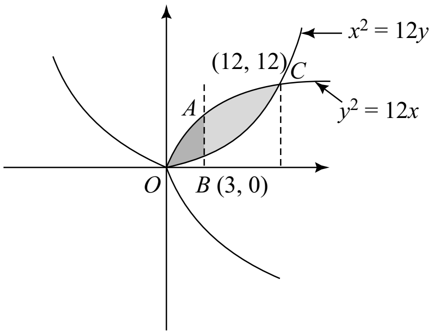


Fig. 14.30

Example 15: The area of the region bounded by the curves $y + |x| = 3$ and $y = |x - 1|$ is

(a) 6 (b) 2

(c) 4 (d) 3

Ans. (c)

Solution: $y = 3 - |x| = \begin{cases} 3 + x, & x < 0 \\ 3 - x, & x \ge 0 \end{cases}$

$$y = |x - 1| = \begin{cases} x - 1, & x \ge 0 \\ 1 - x, & x < 0 \end{cases}$$

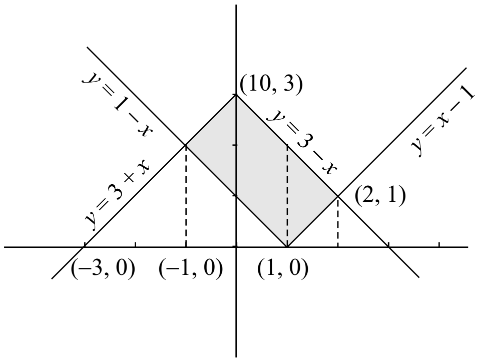


Fig. 14.31

Required area

$$= \int_{-1}^{0}((3+x)-(1-x))dx + \int_0^1((3-x)-(1-x))dx$$

$$+\int_1^2((3-x)-(x-1))dx$$

$$= \int_{-1}^{0}(2+2x)dx + \int_0^1 2dx + \int_1^2(4-2x)dx$$

$$= 2x + x^2\Big|_{-1}^{0} + 2x\Big|_0^1 + 4x - x^2\Big|_1^2$$

$= -(-2 + 1) + 2 + (4 - 3)$

$= 1 + 3 = 4.$

Example 16: Area bounded by $y = f(x) = \min\left(x+1, \sqrt{1-x}\right)$ for all $x \le 1$ is

(a) $\frac{7}{3}$ (b) $\frac{7}{6}$

(c) $\frac{11}{6}$ (d) $\frac{1}{6}$

Ans. (b)

$$f(x) = \min\left(x+1, \sqrt{1-x}\right) = \begin{cases} x+1, & x \le -1 \\ x+1, & -1 < x < 0 \\ \sqrt{1-x}, & 0 \le x \le 1 \end{cases}$$

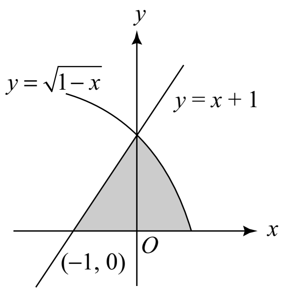


Fig. 14.32

$$\text{Required area} = \int_{-1}^{0}(x+1)dx + \int_0^1 \sqrt{1-x}\,dx$$

$$= \frac{x^2}{2} + x\Bigg|_{-1}^{0} - \frac{2}{3}(1-x)^{3/2}\Bigg|_0^1$$

$$= -\left(-1 + \frac{1}{2}\right) + \frac{2}{3} = \frac{7}{6}$$

Example 17: The area of the region bounded by $x = 1$, $x = 2$, $y = 4x + 1$, $y = e^x$ is

(a) $7 + e - e^2$ (b) $5 + e - e^2$

(c) $7 - e + e^2$ (d) $7 - 2e + e^2$

Ans. (a)

Solution: For $1 \le x \le 2$, $4x + 1$ is always greater than e^x. So the required area

$$= \int_1^2((4x+1) - e^x)dx = 2x^2 + x - e^x\Big|_1^2$$

$$= 7 + e - e^2$$

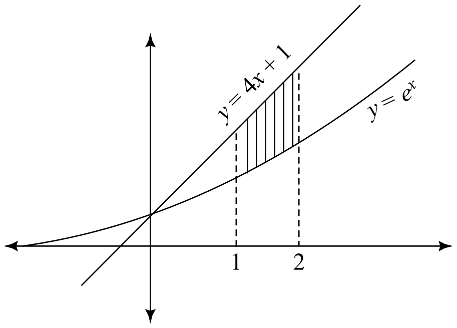


Fig. 14.33

Assertion-Reason Type Questions

Example 18: Let $y = f(x)$ be symmetrical about y-axis.
Statement-1 : Area of the region bounded by $x = -a$, $y = f(x)$, $x = a$ and x-axis is twice the area of the region bounded by $x = 0$, $y = f(x)$, $x = a$ and x-axis.
Statement-2 : f is an even function.
Ans. (a)

Solution: $y = f(x)$ is symmetrical about y-axis so $y = f(x) = f(-x)$. In this case

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

Example 19: Let A_n be the area bounded by $y = x$ and $y = x^n$.

Statement-1 : $A_2 A_3 \dots A_n = \dfrac{1}{2^{n-2} n(n+1)}$

Statement-2 : $A_n = \dfrac{n}{2(n+1)}$

Ans. (c)

Solution: The straight line $y = x$ and the curve $y = x^n$ meet at (0, 0) and (1, 1), so

$$A_n = \int_0^1 (x - x^n)dx$$

$$= \frac{1}{2} - \frac{1}{n+1} = \frac{n-1}{2(n+1)}$$

$$A_2 A_3 \dots A_n = \frac{1}{2^{n-1}}\left[\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdots\cdot\frac{n-1}{n+1}\right]$$

$$= \frac{1}{2^{n-2}}\frac{1}{n(n+1)}$$

Example 20: Statement-1 : The area between $y = 2 - x^2$ and $y = \dfrac{1}{2}x$ from $x = 0$ to $x = 1$ is $\dfrac{17}{6}$.

Statement-2 : The area between $y = 2 - x^2$ and $y = \dfrac{1}{2}x$ from $x = 0$ to $x = 1$ is given by $\int_0^1\left(2 - x^2 - \dfrac{1}{2}x\right)dx$.

Ans. (d)

Solution: $y = 2 - x^2$ is parabola with vertex (0, 2) opening downwards.

$$\text{Required area} = \int_0^1\left(2 - x^2 - \frac{1}{2}x\right)dx$$

$$= 2x - \frac{x^3}{3} - \frac{x^2}{4}\Bigg|_0^1 = 2 - \frac{1}{3} - \frac{1}{4} = \frac{17}{12}$$

LEVEL 2

Straight Objective Type Questions

Example 21: The area between the curve $y = -x^2 + 4x$ and $y = x^2 - 6x + 5$ over the interval $0 \le x \le 1$ is

(a) $\dfrac{52}{3} + 5\sqrt{15}$ (b) $-\dfrac{52}{3} + 5\sqrt{15}$

(c) $-\dfrac{52}{3} + 3\sqrt{15}$ (d) $\dfrac{47}{3} + 5\sqrt{15}$

Ans. (b)

Solution: We find the intersection of the two curve by equating

$$-x^2 + 4x = x^2 - 6x + 5$$

$$\Rightarrow \quad 2x^2 - 10x + 5 = 0$$

$$\Rightarrow \quad x = \frac{10 \pm \sqrt{100 - 40}}{2} = \frac{5 \pm \sqrt{15}}{2}$$

Since $0 \le x \le 1$ so required point of intersection is $a = \dfrac{5 - \sqrt{15}}{2}$.

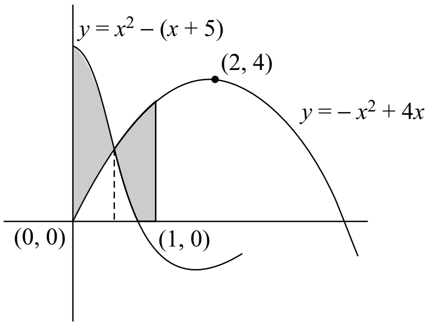


Fig. 14.34

The required area

$$= \int_0^a [x^2 - 6x + 5 - (-x^2 + 4x)]dx$$

$$+\int_a^1 [-x^2 + 4x - (x^2 - 6x + 5)]dx$$

$$= \int_0^a [2x^2 - 10x + 5]dx + \int_a^1 (-2x^2 + 10x - 5)dx$$

$$= \left.\frac{2x^3}{3} - 5x^2 + 5x\right|_0^a + \left.\left(-\frac{2}{3}x^3 + 5x^2 - 5x\right)\right|_a^1$$

$$= -\frac{52}{3} + 5\sqrt{15} \text{ (on simplification)}$$

Example 22: The area of the figure bounded by $y^2 = 9x$ and $y = 3x$ is

(a) 1 (b) 1/4

(c) 1/2 (d) 2

Ans. (c)

Solution: Required area

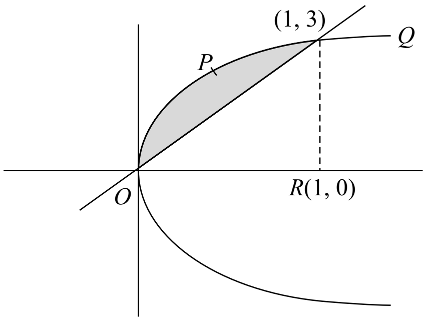


Fig. 14.35

= area *OPQRO* – area Δ*OQR*

$$= \int_0^1 \sqrt{9x}\, dx - \frac{1}{2} \times 1 \times 3 = 3\left.\frac{2}{3}x^{3/2}\right|_0^1 - \frac{3}{2} = \frac{1}{2}$$

Example 23: The area of the plane figure bounded by lines $y = \sqrt{x}$, $x \in [0, 1]$, $y = x^2$, $x \in [1, 2]$ and $y = -x^2 + 2x + 4$, $x \in [0, 2]$ is

(a) 10/7 (b) 10/3

(c) 3/5 (d) 4/3

Ans. (b)

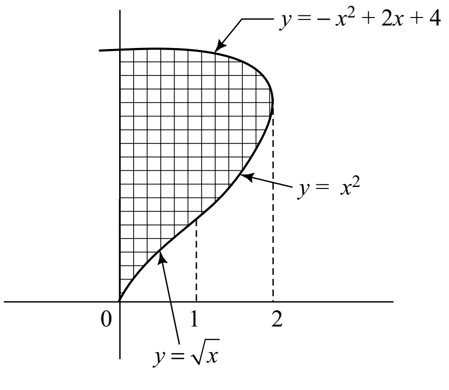


Fig. 14.36

Solution: Required area

$$= \int_0^2 (-x^2 + 2x + 4)\mathrm{d}x - \int_0^1 \sqrt{x}\,\mathrm{d}x - \int_1^2 x^2\mathrm{d}x$$

$$= -\left.\frac{x^3}{3}\right|_0^1 + x^2\Big|_0^2 + 4x\Big|_0^2 - \left.\frac{2}{3}x^{3/2}\right|_0^1 - \left.\frac{x^3}{3}\right|_1^2 = \frac{19}{3}.$$

Example 24: The value of 'b' ($b > 0$) for which the area bounded by the curves $y = \dfrac{x}{6} + \dfrac{1}{x^2}$, x-axis, $x = b$ and $x = 2b$ has the least value is

(a) 1 (b) $\sqrt{2}$

(c) 2 (d) $2^{1/3}$

Ans. (d)

Solution: The area bounded by the curves $y = \dfrac{x}{6} + \dfrac{1}{x^2}$, x-axis, $x = b$ and $x = 2b$ is given by

$$f(b) = \int_b^{2b}\left(\frac{x}{6} + \frac{1}{x^2}\right)dx$$

$$= \left(\frac{x^2}{12} - \frac{1}{x}\right)_b^{2b} = \frac{b^2}{4} + \frac{1}{b}$$

$$f'(b) = \frac{2b}{4} - \frac{1}{b^2} = 0 \Rightarrow b = 2^{1/3}$$

$$f''(b) = \frac{1}{2} + \frac{2}{b^3} > 0$$

so $f(b)$ is minimum for $b = 2^{1/3}$.

Example 25: The area bounded by the curve $y = x(x^2 - 1)$ and x-axis is

(a) 0 (b) $\dfrac{1}{2}$

(c) 1 (d) $\dfrac{1}{4}$

Ans. (b)

Solution: The given curve is a cubic curve and cuts x-axis at – 1, 0 and 1.

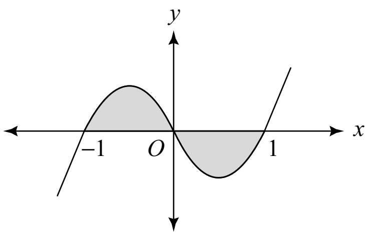


Fig. 14.37

Required area

$$= \int_{-1}^0 x(x^2 - 1)dx + \left|\int_0^1 x(x^2 - 1)dx\right|$$

$$= \left[\frac{1}{4}x^4 - \frac{x^2}{2}\right]_{-1}^{0} + \left|\left[\frac{1}{4}x^4 - \frac{x^2}{2}\right]_{0}^{1}\right|$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Example 26: The area between the curves $x = 1 - y^2$ and $x = y^2 - 1$ is

(a) $\frac{4}{3}$ (b) $\frac{1}{3}$

(c) $\frac{8}{3}$ (d) $\frac{5}{3}$

Ans. (c)

Solution: The two curves are parabolas with $x = 1 - y^2$ opening to the left and $x = y^2 - 1$ opening to the right.

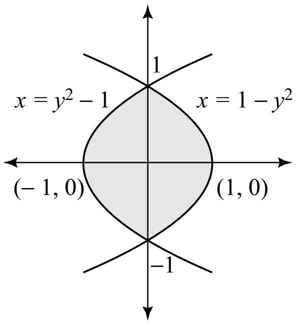


Fig. 14.38

Required area

$$= \int_{-1}^{1}(\text{Right} - \text{left})dy$$

$$= \int_{-1}^{1}(1 - y^2 - (y^2 - 1))dy = 2\int_{-1}^{1}(1 - y^2)dy$$

$$= 2\left[y - \frac{y^3}{3}\right]_{-1}^{1} = 2\left[\frac{2}{3} - \left(-1 + \frac{1}{3}\right)\right] = \frac{8}{3}.$$

Example 27: The area bounded by $y = x^2$, $y = \sqrt{x}$, $0 \leq x \leq 4$ is

(a) $\frac{20}{3}$ (b) $\frac{10}{3}$

(c) 10 (d) $\frac{50}{3}$

Ans. (d)

Solution: On [0, 1], $x^2 \leq \sqrt{x}$, so the area between two curves is

$$\int_0^1\left(\sqrt{x} - x^2\right)dx = \frac{1}{3}$$

On [1, 4], $x^2 \geq \sqrt{x}$, so the area is

$$\int_1^4 (x^2 - x)dx = \frac{49}{3}$$

So the total area between $y = x^2$, $y = \sqrt{x}$ on the interval [0, 4] is $\frac{1}{3} + \frac{49}{3} = \frac{50}{3}$.

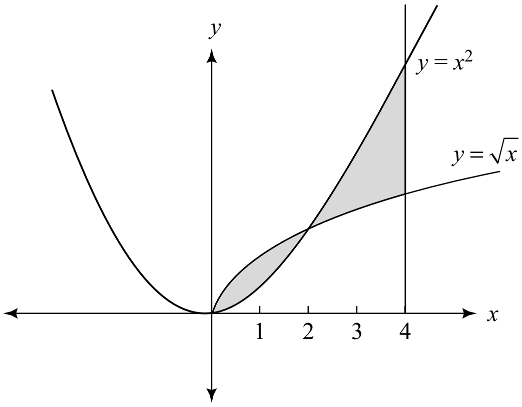


Fig. 14.39

Example 28: The area of the region enclosed by $y = x^3 - 2x^2 + 2$ and $y = 3x + 2$ is

(a) $\frac{71}{6}$ (b) 14

(c) $\frac{29}{3}$ (d) $\frac{71}{3}$

Ans. (a)

Solution: The intersection of two curves is given by

$$x^3 - 2x^2 + 2 = 3x + 2$$

$$\Rightarrow \quad x^3 - 2x^2 - 3x = 0$$

$$\Rightarrow \quad x(x + 1)(x - 3) = 0$$

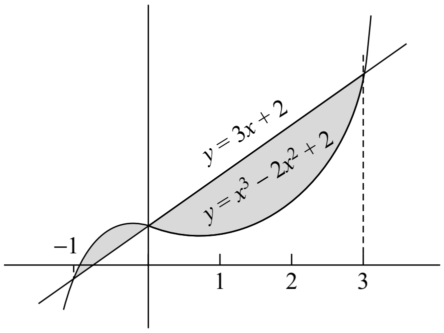


Fig. 14.40

$$\text{Required area} = \int_{-1}^{0}[(x^3 - 2x^2 + 2) - (3x + 2)]dx + \int_0^3[(x + 2) - (x^3 - 2x^2 + 2)]dx$$

$$= \frac{7}{12} + \frac{45}{4} = \frac{71}{6}.$$

EXERCISE

Concept-based Straight Objective Type Questions

1. The area below $y = e^x$ and above $y = x$ between $x = 0$ and $x = 2$ is

(a) $e^2 - 1$ (b) $e^2 + e$

(c) $e^2 - 3$ (d) $e + 2$

2. The area between $y = x$ and $y = x^2$, $0 \le x \le 2$ is

(a) 1 (b) $\frac{2}{3}$

(c) $\frac{1}{3}$ (d) $\frac{4}{3}$

3. The area of the region bounded by $y = x^2 - x$ and $y = 2x + 4$ is

(a) $\frac{175}{6}$ (b) $\frac{31}{6}$

(c) 12 (d) $\frac{125}{6}$

4. The area of the region enclosed by the parabola $y = 2 - x^2$, and line $y = -x$ and x-axis is

(a) $\frac{8\sqrt{2}+3}{7}$ (b) $\frac{8\sqrt{2}+6}{7}$

(c) $\frac{7\sqrt{2}-1}{5}$ (d) $\frac{7\sqrt{2}+1}{3}$

5. The area of the region bounded by $y = x^2$ and $y = -x^2 + 2$ is

(a) $\frac{5}{3}$ (b) $\frac{4}{3}$

(c) $\frac{8}{3}$ (d) 1

LEVEL 1

Straight Objective Type Questions

6. The total area between $y = 4x - x^2$ and $y = x$ from $x = 0$ and $x = 4$ is

(a) 17/3 (b) 37/6

(c) 16/3 (d) 4

7. The area between the curves $y = x^2$ and $y = x^{1/3}$, $-1 \le x \le 1$ is

(a) $\frac{1}{2}$ (b) 2

(c) $\frac{3}{4}$ (d) $\frac{3}{2}$

8. The area of the region bounded by the parabola $(y - 2)^2 = (x - 1)$, the tangent to the parabola at the point (2, 3) and the x-axis is

(a) 9 (b) 12

(c) 3 (d) 6

9. The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3\pi}{2}$ is

(a) $4\sqrt{2}-1$ (b) $4\sqrt{2}+1$

(c) $4\sqrt{2}-2$ (d) $4\sqrt{2}+2$

10. The area included between $x^2 + y^2 = 4x$ and $y^2 = x$ above x-axis is

(a) $\frac{4}{3}(3\pi - 8)$ (b) $\frac{2}{3}(8 + 3\pi)$

(c) $\frac{1}{3}(4 + 3\pi)$ (d) $\frac{5\sqrt{3}}{2}+\frac{2\pi}{3}$

11. The area of the region bounded by $y = 2\cos x$, $y = 3\tan x$ and the y-axis is

(a) $1 + \frac{3}{2}\log 3 - \log 2$ (b) $\log\frac{3}{2}$

(c) $1 + 3\log\frac{2}{\sqrt{3}}$ (d) $1 + \frac{3}{2}\log 3 - \log 8.$

12. The area bounded by $y = \sin^{-1} x$, y-axis and $|y| = \frac{\pi}{2}$ is

(a) 2 (b) π

(c) 2π (d) 1

13. The area of the region bounded by $x = 1$, $x = 2$, $y = \log x$ and $y = 3^x$ is

(a) $\frac{9}{\log 3}+\log\frac{e}{4}$ (b) $\frac{6}{\log 3} - 2\log 2 + 1$

(c) $\frac{3}{\log 3} - 2\log 2$ (d) $\frac{9}{\log 3} - 2\log 2 + 1$

14. The area inside the parabola $y = 5x^2$ but outside $y = 2x^2 + 9$ is

(a) $4\sqrt{3}$ (b) $6\sqrt{3}$

(c) $12\sqrt{3}$ (d) $8\sqrt{3}$

15. The area bounded by the curve $y = 2 \log x$, x-axis, y-axis and $y = \log 5$ is equal to

(a) $3(\sqrt{5}-1)$ (b) 4

(c) $2(\sqrt{5}-1)$ (d) $\sqrt{5}-1$

16. Let $A = \{(x, y): y^2 \leq 4x,\ y - 2x \geq -2,\ y \geq 0\}$. The area of the region A is

(a) $\frac{2}{3}(1+\sqrt{5})^{3/2}-2$

(b) $\frac{4}{3}(3+\sqrt{5})^{3/2}$

(c) $\frac{2}{3\sqrt{3}}(3+\sqrt{5})^{3/2}-\frac{1}{2}(7+3\sqrt{5})$

(d) $\frac{4}{3}(3+\sqrt{5})^{3/2}-(5+4\sqrt{5})$

17. The area of the region above x-axis bounded by $y = \cot x$, $\frac{\pi}{12} \leq x \leq \frac{\pi}{4}$ is

(a) 1 (b) $\log(\sqrt{2}+1)$

(c) $\log(\sqrt{5}+1)$ (d) $\log(\sqrt{3}+1)$

Assertion–Type Questions

18. Let $f(x) = \min\left(|x|, 1-|x|, \frac{1}{4}\right)$

Statement-1 : The area of the curve $y = f(x)$, $-1 \leq x \leq 1$ is equal $\frac{3}{8}$.

Statement-2 : The area of the curve $y = f(x)$ is twice the area of a trapezium of parallel sides 1 and $\frac{1}{4}$.

19. Let (a, b) and (c, d) be two points on the curve $y = f(x)$.

Statement-1 : Area bounded by the $y = f'(x)$, x-axis and $x = a$, $x = c$ is $d - b$.

Statement-2 : $\int_{x_0}^{x'_0} \phi'(x)dx = \phi(x'_0) - \phi(x_0)$

20. **Statement-1 :** The area bounded by $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis lying in the first quadrant is 9.

Statement-2 : The area in statement 1 is given by

$$\int_0^9\left(\sqrt{x}-\left(\frac{x-3}{2}\right)\right)dx$$

LEVEL 2

21. The area between $x = y + 3$ and $x = y^2$ from $y = -1$ to $y = 1$ is

(a) $\frac{4}{3}$ (b) $\frac{16}{3}$

(c) $\frac{5}{3}$ (d) 4

22. The area of the region between $y = 3x^3 - x^2 - 10x$ and $y = -x^2 + 2x$ is

(a) 10 (b) 12

(c) 18 (d) 24

23. The area of the region bounded by $y = x - 1$ and $x = 3 - y^2$ is

(a) $\frac{5}{2}$ (b) $\frac{7}{2}$

(c) $\frac{9}{2}$ (d) 4

24. The area between the curves $y = x^2 + 3$ and $y = -x^2 - 1$ on $[-2, 2]$ is

(a) $\frac{80}{3}$ (b) $\frac{29}{3}$

(c) 21 (d) $\frac{47}{3}$

25. The area the region enclosed by $f(x) = x^3 - 10x$ and $g(x) = 6x$, $x \geq 0$, $y \geq 0$ is

(a) 24 (b) 39

(c) 42 (d) 84

26. The area bounded by $y = x^2$ and $y = 1 - x^2$ is

(a) $\sqrt{8}/3$ (b) 16/3

(c) 32/3 (d) 16/3

27. The area of the figure bounded by the lines $x = 0$, $x = \pi/2$, $f(x) = \sin x$ and $g(x) = \cos x$ is

(a) $2(\sqrt{2}-1)$ (b) $\sqrt{3}-1$

(c) $2(\sqrt{3}-1)$ (d) $2(\sqrt{2}+1)$

28. The area of the figure bounded by the lines $y^2 = 2x + 1$ and $x - y - 1 = 0$ is

(a) 16/3 (b) 4/3 (c) 8/3 (d) 11/3

Previous Years' AIEEE/JEE Main Questions

1. The area of the region bounded by the curves $y = |x - 2|$, $x = 1$, $x = 3$ and the x-axis is

(a) 3 (b) 2

(c) 1 (d) 3 **[2004]**

2. The area enclosed between the curve $y = \log_e (x + e)$ and the coordinate axes is

(a) 3 (b) 4

(c) 1 (d) 2 **[2005]**

3. The area enclosed between the curves $y^2 = x$ and $y = |x|$ is

(a) $\frac{2}{3}$ (b) 1

(c) $\frac{1}{6}$ (d) $\frac{1}{3}$ **[2007]**

4. The area of the plane region bounded by the curves $x + 2y^2 = 0$, $x + 3y^2 = 1$ is equal to

(a) 5/3 (b) 1/3

(c) 2/3 (d) 4/3 **[2008]**

5. The area of the region bounded by the parabola $(y - 2)^2 = (x - 1)$, the tangent to the parabola at the point (2, 3) and the x-axis is

(a) 9 (b) 12

(c) 3 (d) 6 **[2009]**

6. The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3\pi}{2}$ is

(a) $4\sqrt{2} - 1$ (b) $4\sqrt{2} + 1$

(c) $4\sqrt{2} - 2$ (d) $4\sqrt{2} + 2$ **[2010]**

7. The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ is

(a) $\frac{32}{3}$ (b) $\frac{16}{3}$

(c) $\frac{8}{3}$ (d) 0 **[2011]**

8. The area bounded between the parabola $x^2 = \frac{y}{4}$ and $x^2 = 9y$, and the straight line $y = 2$ is

(a) $\frac{10\sqrt{2}}{3}$ (b) $\frac{20\sqrt{3}}{3}$

(c) $10\sqrt{2}$ (d) $20\sqrt{2}$ **[2012]**

9. The area bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis and lying in first quadrant is

(a) 36 (b) 18

(c) $\frac{27}{4}$ (d) 9 **[2013]**

10. The area under the curve $y = |\cos x - \sin x|$, $0 \le x \le \frac{\pi}{2}$, and above x-axis is

(a) $2\sqrt{2}$ (b) $2\sqrt{2} - 2$

(c) $2\sqrt{2} + 2$ (d) 0 **[2013, online]**

11. The area bounded by the curve $y = \log x$ and the lines $y = 0$, $y = \log 3$ and $x = 0$ is equal to

(a) 3 (b) 3 log – 2

(c) 3 log 3 + 2 (d) 2 **[2013, online]**

12. The area of the region described by $A = \{(x, y): x^2 + y^2 \le 1 \text{ and } y^2 \ge 1 - x\}$ is

(a) $\frac{\pi}{2} + \frac{4}{3}$ (b) $\frac{\pi}{2} - \frac{4}{3}$

(c) $\frac{\pi}{2} - \frac{2}{3}$ (d) $\frac{\pi}{2} + \frac{2}{3}$ **[2014]**

13. Let $A = \{(x, y): y^2 \le 4x, y - 2x \ge -4\}$. The area (in square units) of the region A is

(a) 8 (b) 9

(c) 10 (d) 11 **[2014, online]**

14. The area of the region above x-axis bounded by the curve $y = \tan x$, $0 \le x \le \frac{\pi}{4}$ and the tangent to the curve at $x = \frac{\pi}{4}$ is:

(a) $\frac{1}{2}\left(\log 2 - \frac{1}{2}\right)$

(b) $\frac{1}{2}\left(\log 2 + \frac{1}{2}\right)$

(c) $\frac{1}{2}(1 - \log 2)$

(d) $\frac{1}{2}(1 + \log 2)$ **[2014, online]**

15. The area (in square units) of the region bounded by the curves $y + 2x^2 = 0$ $y + 3x^2 = 1$ is equal to

(a) $\frac{3}{5}$ (b) $\frac{3}{4}$

(c) $\frac{1}{3}$ (d) $\frac{4}{3}$ **[2015, online]**

16. The area (in square units) of the region described by $\{(x, y): y^2 \le 2x \text{ and } y \ge 4x - 1\}$ is

(a) $\frac{7}{32}$ (b) $\frac{5}{64}$

(c) $\frac{15}{64}$ (d) $\frac{9}{32}$ **[2015]**

17. The area (in square units) of the region $\{(x, y): y^2 \geq 2x \text{ and } x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is

(a) $\pi - \frac{4}{3}$ (b) $\pi - \frac{8}{3}$

(c) $\pi - \frac{4\sqrt{2}}{3}$ (d) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$ **[2016]**

18. The area (in square units) of the region described by $A = \{(x, y): y \geq x^2 - 5x + 4, x + y \geq 1, y \leq 0\}$ is

(a) $\frac{19}{6}$ (b) $\frac{17}{6}$

(c) $\frac{7}{2}$ (d) $\frac{13}{6}$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. The area enclosed by the parabola $y = 3(1 - x^2)$ and the x-axis is

(a) 4 (b) 3

(c) 9 (d) 2 **[2006]**

2. The line $y = x + 1$ divided the area the curves $y = \cos x$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the x-axis into two regions which are in the ratio

(a) 2 : 1 (b) 1 : 3

(c) 2 : 3 (d) 1 : 1 **[2006]**

3. The area enclosed by the curves $x^2 = y$, $y = x + 2$ and x-axis is

(a) $\frac{5}{4}$ (b) $\frac{5}{2}$

(c) $\frac{15}{4}$ (d) $\frac{5}{6}$ **[2007]**

4. The area of the region described by the set $\{(x, y): x^2 \leq y \leq |x|\}$ is

(a) $2\frac{1}{6}$ (b) $1\frac{1}{3}$

(c) $\frac{1}{3}$ (d) $2\frac{5}{6}$ **[2009]**

5. The area of the region $\{(x, y): 0 \leq y \leq x^2 + 1, 0 \leq y \leq x + 1, 0 \leq x \leq 2\}$ is

(a) $\frac{23}{3}$ (b) $\frac{21}{6}$

(c) $\frac{23}{6}$ (d) $\frac{15}{2}$ **[2010]**

6. The area of the region bounded by curves $y = 1 - x^2$, $x + y + 1 = 0$ and $x - y - 1 = 0$ is

(a) $\frac{10}{3}$ (b) $\frac{7}{3}$

(c) $\frac{8}{3}$ (d) 3 **[2011]**

7. The area bounded by the curves $y^2 = 12x$ and $x^2 = 12y$ is divided by the line $x = 3$ in two parts. The area (in square units) of the larger part is:

(a) $\frac{147}{4}$ (b) $\frac{45}{4}$

(c) $\frac{137}{4}$ (d) $\frac{245}{4}$ **[2014]**

8. The area (in sq. units) of region bounded by the curve $y = \sqrt{x}$ and the lines $y = 0$, $y = x - 2$ is

(a) $\frac{10}{3}$ (b) $\frac{8}{3}$

(c) $\frac{4}{3}$ (d) $\frac{16}{3}$ **[2015]**

9. If the line $x = a$ bisects the area under the curve $y = \frac{1}{x^2}$, $1 \leq x \leq 9$ then a is equal to

(a) $\frac{9}{5}$ (b) $\frac{5}{9}$

(c) $\frac{9}{4}$ (d) $\frac{4}{9}$ **[2016]**

Answers

Concept-based

1. (c) 2. (a) 3. (d) 4. (b)
5. (c)

Level 1

6. (b) 7. (d) 8. (a) 9. (c)
10. (d) 11. (d) 12. (a) 13. (b)
14. (c) 15. (c) 16. (c) 17. (d)

Assertion Type

18. (a) 19. (a) 20. (c)

Level 2

21. (b) 22. (d) 23. (c) 24. (a)
25. (b) 26. (a) 27. (a) 28. (a)

Previous Years' AIEEE/JEE Main Questions

1. (c) 2. (c) 3. (c) 4. (d)
5. (a) 6. (c) 7. (b) 8. (b)
9. (d) 10. (b) 11. (d) 12. (b)
13. (b) 14. (a) 15. (d) 16. (d)
17. (b) 18. (a)

Previous years' B-Architecture Entrance Examination Questions

1. (a) 2. (b) 3. (d) 4. (c)
5. (c) 6. (b) 7. (a) 8. (a)
9. (a)

Hints and Solutions

Concept-based

1. Since $e^x \geq x$ on [0, 2], so required area is

$$\int_0^2 (e^x - x)dx = e^x - \frac{x^2}{2}\Big|_0^2 = e^2 - 2 - 1 = e^2 - 3.$$

2. For $0 \leq x \leq 1$,
$x \geq x^2$ and for $1 \leq x \leq 2$, $x^2 \geq x$. So the required area is

$$\int_0^1 (x - x^2)dx + \int_1^2 (x^2 - x)dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 + \left(\frac{x^3}{3} - \frac{x^2}{2}\right)\Big|_1^2$$

$$\frac{1}{6} + \left(\frac{8}{3} - 2\right) + \frac{1}{6} = 1.$$

3. The intersection of $y = x^2 - x = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4}$ is $x^2 - x$ $= 2x + 4 \Rightarrow x = 4, -1$, so the required area is

$$= \int_{-1}^4 ((2x+4) - (x^2 - x))dx = \int_{-1}^4 (-x^2 + 3x + 4)dx.$$

$$= \frac{125}{6}$$

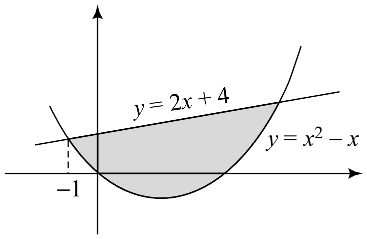


Fig. 14.41

4. Reqd. area $= \int_{-1}^{\sqrt{2}} (2 - x^2)\mathrm{d}x - \frac{1}{2} \times 1 \times 1$

$$= 2x - \frac{x^3}{3}\Big|_{-1}^{\sqrt{2}} - \frac{1}{2}$$

$$= 2\sqrt{2} - \frac{2\sqrt{2}}{3} - \left(-2 + \frac{1}{3}\right) - \frac{1}{2}$$

$$= \frac{4\sqrt{2}}{3} + \frac{5}{3} - \frac{1}{2} = \frac{8\sqrt{2} + 7}{6}$$

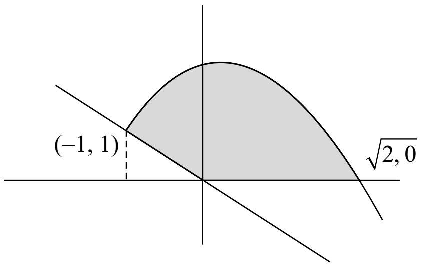


Fig. 14.42

5. Intersection of $y = x^2$ and $y = -x^2 + 2$ is $x = -1, 1$.

Reqd. area $= \int_{-1}^1 ((-x^2 + 2) - x^2)dx = \left(-\frac{2}{3}x^3 + 2x\right)\Big|_{-1}^1$
$= \frac{8}{3}$

Level 1

6. Required area $= \frac{1}{2} \times 3 \times 3 + \int_3^4 (4x - x^2)dx$

$$= \frac{9}{2} + \left(2x^2 - \frac{x^3}{3}\right)\Big|_3^4 = \frac{9}{2} + \frac{5}{3} = \frac{37}{6}$$

7. Required area $= \int_{-1}^1 |x^{1/3} - x^2|\, dx$

$$= \int_{-1}^0 \left|x^{\frac{1}{3}} - x^2\right| dx + \int_0^1 \left|x^{\frac{1}{3}} - x^2\right| dx$$

$$= \int_{-1}^0 \left(x^2 - x^{\frac{1}{3}}\right) dx + \int_0^1 \left(x^{\frac{1}{3}} - x^2\right) dx$$

$$= \left.\frac{x^3}{3}-\frac{3}{4}x^{\frac{4}{3}}\right|_{-1}^{0} + \left.\frac{3}{4}x^{\frac{4}{3}}-\frac{x^3}{3}\right|_{0}^{1}$$

$$= 0-\left(\frac{-1}{3}-\frac{3}{4}\right)+\left(\frac{3}{4}-\frac{1}{3}\right)=\frac{3}{2}$$

8. Equation of tangent at (2, 3) to the parabola $y^2 - 4y - x + 5 = 0$ is $x - 2y + 4 = 0$

   Required area = $\int_0^3 [(y-2)^2+1-(2y-4)]dy = 9$

9. Required area = $\int_0^{3\pi/2} |\sin x - \cos x|\, dx$

$$= \sqrt{2}\int_0^{3\pi/2}|\sin(x-\pi/4)|\,dx = \sqrt{2}\int_{-\pi/4}^{5\pi/4}|\sin\theta|\,d\theta$$

$$= \sqrt{2}\left[\int_{-\pi/4}^{0}(-\sin\theta)d\theta+\int_0^{\pi}\sin\theta d\theta+\int_{\pi}^{5\pi/4}(-\sin\theta)d\theta\right]$$

$$= 4\sqrt{2}-2.$$

10. The intersection $x^2 + y^2 = 4x$ and $y^2 = x$ is $x^2 = 3x \Rightarrow x = 0, 3$

    Reqd. area = $\int_0^3 \sqrt{x}\,dx + \int_3^4 \sqrt{4x-x^2}\,dx$

$$= \left. x^{3/2}\right|_0^3 + \int_3^4 \sqrt{4-(x-2)^2}\,dx$$

$$= 3^{3/2} + \left[\frac{(x-2)}{2}\sqrt{4-(x-2)^2}+\frac{1}{2}4\sin^{-1}\left(\frac{x-2}{2}\right)\right]_3^4$$

$$= 3^{3/2} + 2\sin^{-1}1 - \frac{1}{2}\sqrt{3} - 2\sin^{-1}\left(\frac{1}{2}\right)$$

$$= \sqrt{3}\left(3-\frac{1}{2}\right)+2\frac{\pi}{2}-2\frac{\pi}{6}$$

$$= \frac{5\sqrt{3}}{2}+\frac{2\pi}{3}.$$

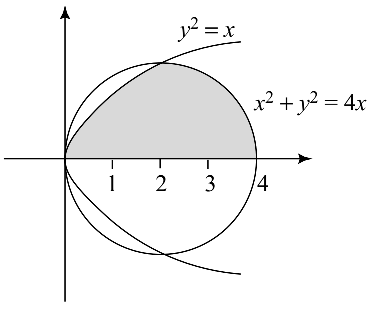


Fig. 14.43

11. The intersection of two curves is given by 2 cos x = 3 tan $x \Rightarrow 2(1 - \sin^2 x) = 3\sin x$

$$\Rightarrow 2\sin^2 x + 3\sin x - 2 = 0$$

$$\Rightarrow \sin x = \frac{-3\pm\sqrt{9+16}}{4}=\frac{-3\pm 5}{4} \quad -2,\ \frac{1}{2}$$

   So $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$

   Reqd. area = $\int_0^{\pi/6}(2\cos x - 3\tan x)dx$

$$= [2\sin x - 3\log|\sec x|]_0^{\pi/6}$$

$$= 1 - 3\log\frac{2}{\sqrt{3}} + 3\log 1 = 1 + \frac{3}{2}\log 3 - \log 8.$$

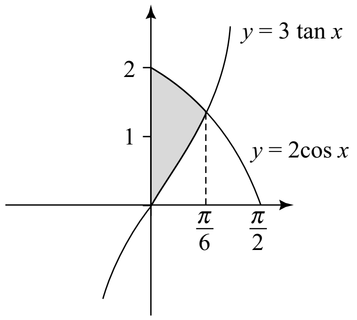


Fig. 14.44

12. Reqd. area = $2\int_0^{\pi/2}|\sin y|\,dy$

$$= 2\int_0^{\pi/2}\sin y\,dy = 2.$$

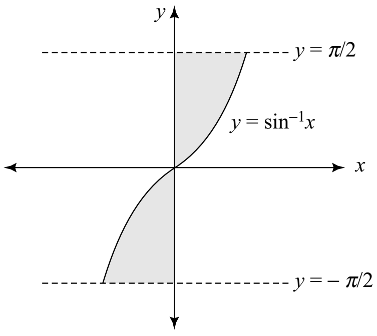


Fig. 14.45

13. Reqd. area = $\int_1^2 (3^x - \log x)dx$

$$= \left.\frac{3^x}{\log 3}\right|_1^2 - \left.(x\log x - x)\right|_1^2$$

$$= \frac{1}{\log 3}[9-3]-[2(\log 2-1)+1]$$

$$= \frac{6}{\log 3} - 2\log 2 + 1$$

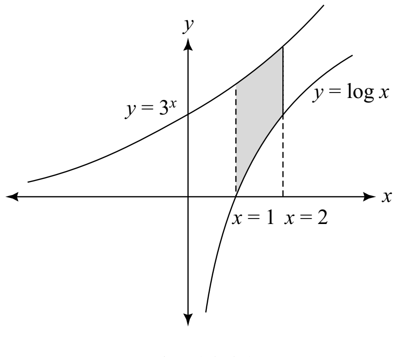
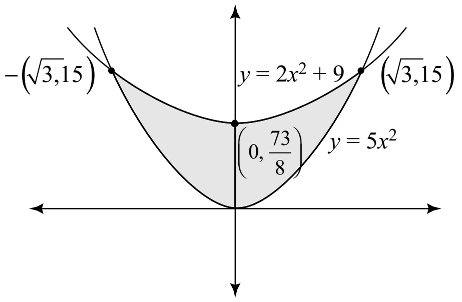


Fig. 14.46

14. The intersection of two curves is given by $5x^2 = 2x^2 + 9 \Rightarrow x^2 = 3 \Rightarrow x = \sqrt{3}, -\sqrt{3}$



$-(\sqrt{3},15)$
$y = 2x^2 + 9$
$(\sqrt{3},15)$
$\left(0, \frac{73}{8}\right)$
$y = 5x^2$

Fig. 14.47

$y = 2x^2 + 9$ is a parabola with focus at $\left(0, \frac{73}{8}\right)$. Both the curves are symmetrical about y-axis.

Reqd. area $= 2\int_0^{\sqrt{3}} ((2x^2 + 9) - 5x^2)dx$

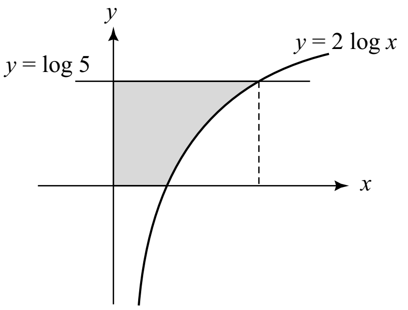
$= 2\int_0^{\sqrt{3}} (9 - 3x^2)dx = 2[9\sqrt{3} - x^3\Big|_0^{\sqrt{3}}\Big)$

$= 12\sqrt{3}$

15. Reqd. area $= \int_0^{\log 5} e^{y/2}dy$

$= 2e^{y/2}\Big|_0^{\log 5}$

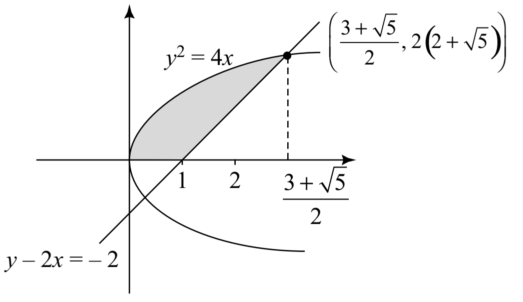
$= 2(\sqrt{5} - 1)$.



y
$y = 2 \log x$
$y = \log 5$
x

Fig. 14.48

16. The intersection of $y^2 = 4x$ and $y - 2x = -2$ is given by



$y^2 = 4x$
$\left(\frac{3+\sqrt{5}}{2}, 2\left(2+\sqrt{5}\right)\right)$
1
2
$\frac{3+\sqrt{5}}{2}$
$y - 2x = -2$

Fig. 14.49

$\Rightarrow (2x - 2)^2 = 4x$

$\Rightarrow x^2 - 2x + 1 = x$

$\Rightarrow x^2 - 3x + 1 = 0$

$= x = \dfrac{3 \pm \sqrt{5}}{2}$

Reqd. area $= \int_0^{\frac{3+\sqrt{5}}{2}} \sqrt{4x}\,dx - \frac{1}{2}\frac{1+\sqrt{5}}{2} \times 2(2+\sqrt{5})$

$= \frac{4}{3}x^{3/2}\Big|_0^{\frac{3+\sqrt{5}}{2}} - \frac{1}{2}\left(2+\sqrt{5}\right)\left(1+\sqrt{5}\right)$

$= \frac{4}{6\sqrt{2}}(3+\sqrt{5})^{3/2} - \frac{1}{2}\left(7+3\sqrt{5}\right)$

$= \frac{2}{3\sqrt{3}}(3+\sqrt{5})^{3/2} - \frac{1}{2}\left(7+3\sqrt{5}\right)$

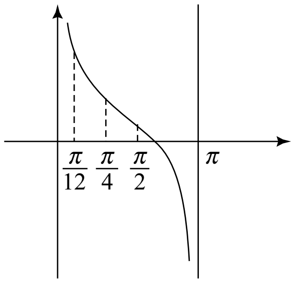
17. Reqd. area $= \int_{\pi/12}^{\pi/4} \cot x\,dx$

$= \log\sin x\Big|_{\pi/12}^{\pi/4}$

$= \log\frac{1}{\sqrt{2}} - \log\frac{\sqrt{6}-\sqrt{2}}{4}$

$= \log\frac{2\sqrt{2}}{(\sqrt{6}-\sqrt{2})} = \log\frac{(\sqrt{6}+\sqrt{2})}{\sqrt{2}}$

$= \log(\sqrt{3}+1)$



$\frac{\pi}{12}$
$\frac{\pi}{4}$
$\frac{\pi}{2}$
π

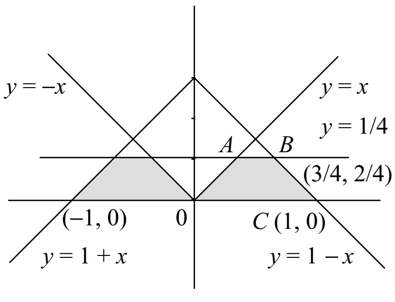
Fig. 14.50

Assertion-based Questions

18. Reqd. area $= \int_{-1}^{1} f(x)dx$

= 2 area of trapezium *OABC*

$= 2\left(\frac{1}{2}\left(1+\frac{1}{2}\right)\frac{1}{4}\right) = \frac{3}{8}$



$y = -x$
$y = x$
$y = 1/4$
A
B
(3/4, 2/4)
(–1, 0)
0
C (1, 0)
$y = 1 + x$
$y = 1 - x$

Fig. 14.51

19. Reqd. area = $\int_a^c f'(x)dx = f(c) - f(a) = d - b$.

20. The intersection two curves is given by $2\sqrt{x} = x - 3$
$\Rightarrow \left(\sqrt{x}-3\right)\left(\sqrt{x}+1\right) = 0 \Rightarrow \sqrt{x} = 3 \Rightarrow x = 9$

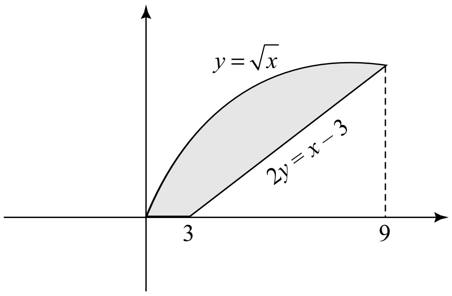


Fig. 14.52

Reqd. area =

$= \int_1^9 \sqrt{x}\,dx - \frac{1}{2}\times 6\times 3$

$= 9$

Level 2

21. Reqd. area = $\int_{-1}^{1}(y+3-y^2)dy = \frac{16}{3}$.

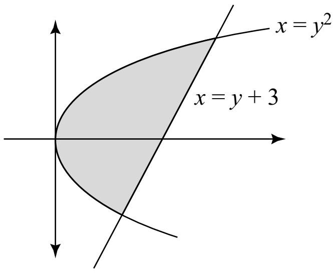


Fig. 14.53

22. Intersection is given by
$3x^3 - x^2 - 10x = -x^2 + 2x$
$\Rightarrow 3x^3 - 12x = 0$
$\Rightarrow x = -2, 0, 2$
Reqd. area = $\int_{-2}^{0}[3x^3 - x^2 - 10x] - (-x^2 + 2x)]dx$
$+\int_0^2((-x^2+2x)-(3x^3-x^2-10))dx$
$= 24$

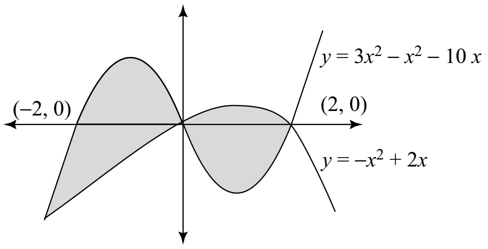


Fig. 14.54

23. Intersection is given by $y = -2$ and 1
Reqd. are = $\int_{-2}^{1}[(3-y^2)-(y+1)]dy$
$= \frac{9}{2}$.

24. Reqd. area = $\int_{-2}^{2}(x^2+3-(-x^2-1))dx = \frac{80}{3}$

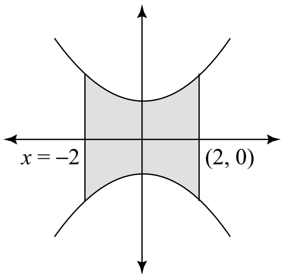


Fig. 14.55

25. Reqd. area = $\frac{1}{2}\times 4\times 24 \; -\int_{\sqrt{10}}^{4}\left(x^3 - 10x\right)dx$

$= 128$

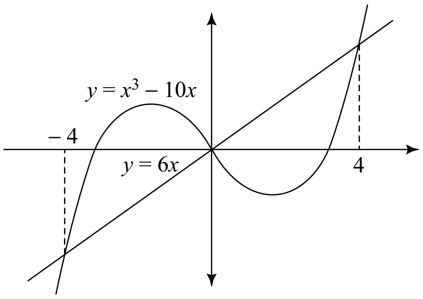


Fig. 14.56

26. The two curves $y = x^2$ and $y = 1 - x^2$ intersect at
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

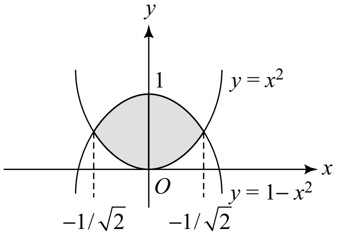


Fig. 14.57

Thus, required area

$= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}[(1-x^2)-x^2]dx$

$= 2\int_0^{1/\sqrt{2}}(1-2x^2)dx = \frac{\sqrt{8}}{3}$

27. Required area
$= \int_0^{\pi/4}[\cos x - \sin x]dx + \int_{\pi/4}^{\pi/2}[\sin x - \cos x]dx$

$$= (\sin x + \cos x)\Big]_0^{\pi/4} + (-\cos x - \sin x)\Big]_{\pi/4}^{\pi/2}$$

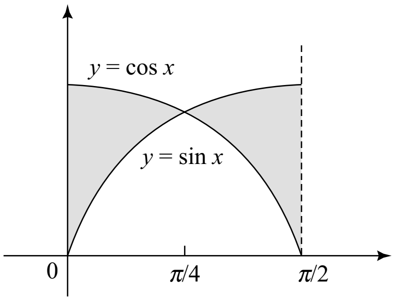


Fig. 14.58

$$= 2(\sqrt{2} - 1)$$

28. Two curves intersect at (0, – 1) and (4, 3)

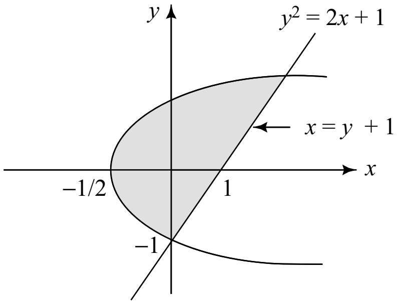


Fig. 14.59

Required area

$$= \int_{-1}^{3}\left[y + 1 - \left(\frac{y^2 - 1}{2}\right)\right]dy$$

$$= \frac{1}{2}\int_{-1}^{3}[4 - (y-1)^2]dy$$

$$= \frac{1}{2}\left[4y - \frac{1}{3}(y-1)^3\right]_{-1}^{3} = 16/3.$$

Previous Years' AIEEE/JEE Questions

1. Required area

$$= \int_1^3 |x - 2|\,dx$$

$$= \int_{-1}^{1} |t|\,dt \qquad [\text{put } x - 2 = t]$$

$$= 2\int_0^1 |t|\,dt = 2\int_0^1 t\,dt = t^2\Big]_0^1 = 1.$$

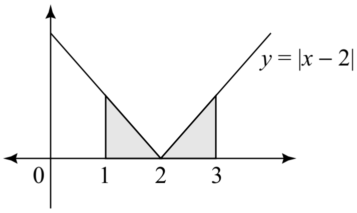


Fig. 14.60

2. Required area: $A = \int_{1-e}^{0} \log(x + e)dx$

Put $x + e = t$, so that

$$A = \int_1^e \log t\,dt$$

$$= t(\log t - 1)]_1^c$$

$$= e(\log e - 1) - 1(\log 1 - 1) = 1.$$

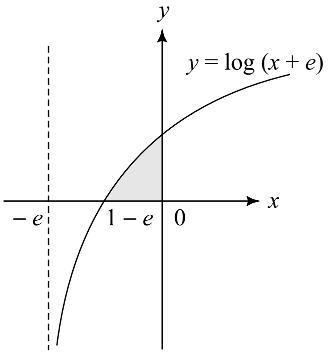


Fig. 14.61

3. Required area

$$= \int_0^1 (\sqrt{x} - x)dx$$

$$= \left(\frac{2}{3}x^{3/2} - \frac{1}{2}x^2\right)\Bigg]_0^1$$

$$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

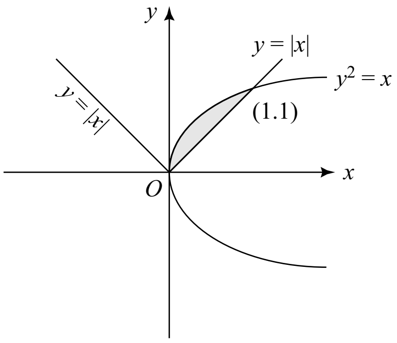


Fig. 14.62

4. The two curves intersect at $x = -2$, $y = \pm 1$

$$\text{Reqd. area} = 2\left[\int_{-2}^{1}\sqrt{\frac{1}{3}(1-x)}\,dx - \int_{-2}^{0}\sqrt{-\frac{1}{2}x}\,dx\right]$$

$$= 2\left[-\frac{2}{3\sqrt{3}}(1-x)^{3/2}\Big|_{-2}^{1} - \frac{1}{\sqrt{2}}\int_0^2 \sqrt{u}\,du\right]$$

$$= 2\left[2 - \frac{4}{3}\right] = \frac{4}{3}.$$

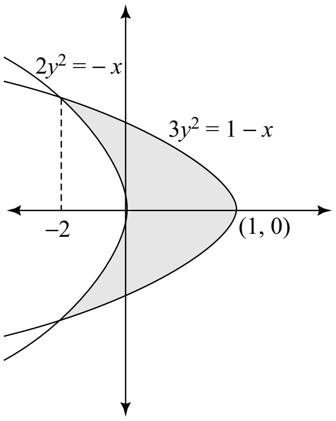


Fig. 14.63

5. $(y-2)^2 = x - 1$

$\Rightarrow y^2 - 4y - x + 5 = 0$

Equation of tangent at (2, 3) is

$3y - 2(y + 3) - \frac{1}{2}(x + 2) + 5 = 0$

$\Rightarrow x - 2y + 4 = 0$

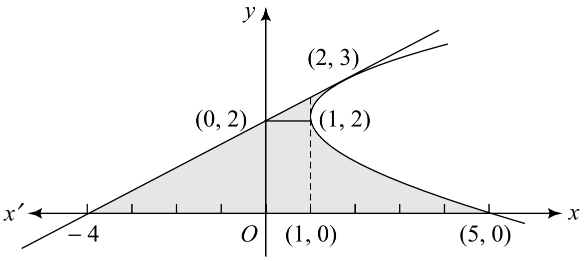


Fig. 14.64

Area of shaded region

$$= \int_0^3 [(y-2)^2 + 1 - (2y-4)]dy$$

$$= \int_0^3 [(y-2)^2 - 2y - 5]dy$$

$$= \left(\frac{1}{3}(y-2)^3 - y^2 + 5y\right]_0^3$$

$$= \frac{1}{3}\ \{1 - (-2)^3\} - 9 + 15 = 9$$

6. Required area

$A = \int_0^{3\pi/2} |\sin x - \cos x|\, dx = \int_0^{3\pi/2} |\sqrt{2}\sin(x - \pi/4)|\, dx$

Put $x - \pi/4 = \theta$ so that

$A = \int_{-\pi/4}^{5\pi/4} \sqrt{2}\,|\sin\theta|\, d\theta$

$= \sqrt{2}\int_{-\pi/4}^{0}(-\sin\theta)d\theta + \sqrt{2}\int_0^{\pi}\sin\theta\, d\theta + \sqrt{2}\int_{\pi}^{5\pi/4}$ $(-\sin\theta)\, d\theta$

$= \sqrt{2}[\cos\theta]_{-\pi/4}^{0} + \sqrt{2}[-\cos\theta]_0^{\pi} + \sqrt{2}[-\cos\theta]_{\pi}^{5\pi/4}$

$= \sqrt{2}\left[1 - \frac{1}{\sqrt{2}}\right] + \sqrt{2}[1+1] + \sqrt{2}\left[-\frac{1}{\sqrt{2}} + 1\right]$

$= 4\sqrt{2} - 2$

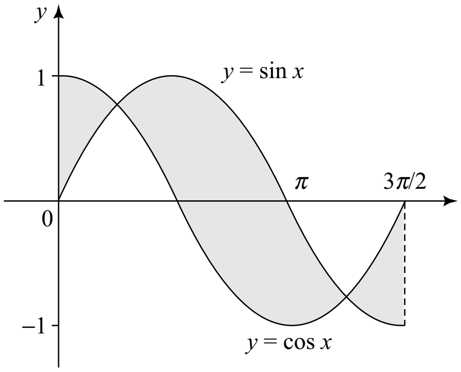


Fig. 14.65

7. Reqd. area $= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx$

$$= \frac{4}{3}x^{3/2}\Big|_0^4 - \frac{x^3}{12}\Big|_0^4$$

$$= \frac{4.8}{3} - \frac{64}{12} = \frac{64}{12} = \frac{16}{3}$$

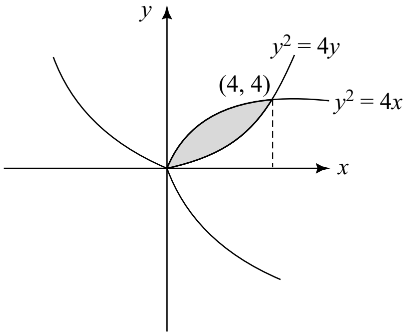


Fig. 14.66

8. The required region is shaded below. Area

$= 2\left[\int_0^2 3\sqrt{y} - \frac{1}{2}\sqrt{y}\right]dy$

$= 5\left(\frac{2}{3}\right)y^{3/2}\Big]_0^2 = \frac{20}{3}\sqrt{2}$

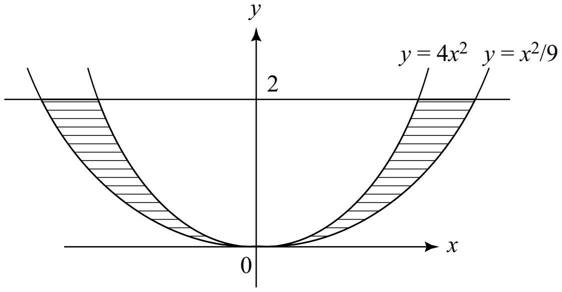

y
$y = 4x^2$
$y = x^2/9$
2
0
x

Fig. 14.67

9. Same as Q20
10. Reqd. area

$$= \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)dx$$

$$= \sin x + \cos x\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= \sqrt{2} - 1 + (-1 + \sqrt{2})$$

$$= 2(\sqrt{2} - 1)$$

11. Reqd. area = $OABC - \int_1^3 \log x\, dx$

$$= 3 \log 3 - (x \log x - x)\Big|_1^3$$

$$= 3 \log 3 - (3 \log 3 - 3 + 1)$$

$$= 2$$

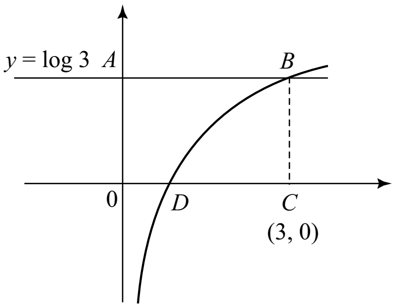

y = log 3
A
B
0
D
C
(3, 0)

Fig. 14.68

12. Reqd. area

$$= 2\left[\frac{\pi}{4} - \int_0^1 \sqrt{1-x}\, dx\right]$$

$$= 2\left[\frac{\pi}{4} + \frac{2}{3}(1-x)^{3/2}\Big|_0^1\right]$$

$$= 2\left[\frac{\pi}{4} - \frac{2}{3}\right] = \frac{\pi}{2} - \frac{4}{3}.$$

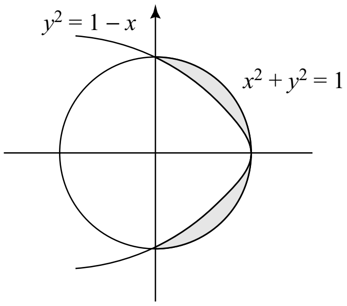


Fig. 14.69

13. We first solve

$y^2 = 4x,\ y - 2x = -4$

$\Rightarrow y^2 = 2(y + 4)$

$\Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow (y - 4)(y + 2) = 0$

Required area

$$= \int_{-2}^{4} \left(\frac{y+4}{2} - \frac{y^2}{4}\right) dy = 9(\text{unit})^2$$

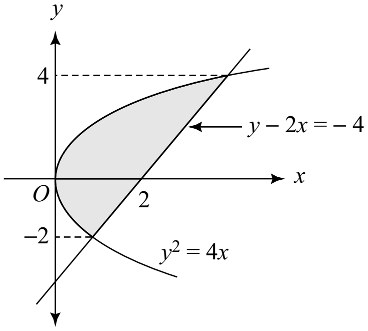

y
4
$y - 2x = -4$
x
O
2
−2
$y^2 = 4x$

Fig. 14.70

14. Area of shaded region

$$= \int_0^{\pi/4} \tan x\, dx - \frac{1}{2}(1)\left(\frac{1}{2}\right)$$

$$= \log(\sec x)\Big]_0^{\pi/4} - \frac{1}{4}$$

$$= \frac{1}{2}\left(\log 2 - \frac{1}{2}\right)$$

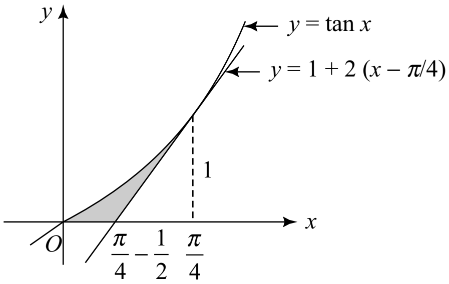

y
$y = \tan x$
$y = 1 + 2(x - \pi/4)$
1
x
O
$\frac{\pi}{4} - \frac{1}{2}$
$\frac{\pi}{4}$

Fig. 14.71

15. The curve $y = -2x^2$ and $y = 1 - 3x^2$ are shown in the figures. Also, the required area is shaded.

For point of intersection of two curves, we set

$-2x^2 = 1 - 3x^2$

$\Rightarrow x = \pm 1$

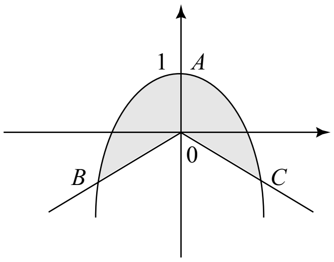


Fig. 14.72

As the two curves are symmetrical about the *y*-axis required area = area (*ABOCA*)

= 2 area (*AOCA*)

$= 2\int_0^1 [(1-3x^2)-(-2x^2)]dx$

$= 2\left(x - \frac{1}{3}x^3\right)\Bigg]_0^1 = 2\left(1-\frac{1}{3}\right) = \frac{4}{3}.$

16. For point of intersection of two curves, we set

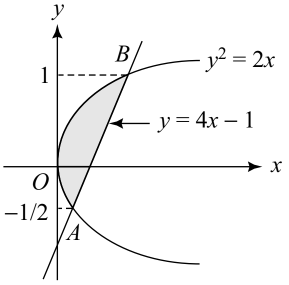


Fig. 14.73

$y^2 = 2x = \frac{1}{2}\ (y+1)$

$\Rightarrow 2y^2 - y - 1 = 0$

$\Rightarrow y = 1, -1/2$

Required area $= \int_{-1/2}^{1}\frac{1}{4}(y+1)dy - \int_{-1/2}^{1}\frac{1}{2}y^2 dy$

$= \frac{1}{8}(y+1)^2\Big]_{-1/2}^{1} - \frac{1}{6}y^3\Big]_{-1/2}^{1}$

$= \frac{1}{8}\left[2^2 - \frac{1}{2^2}\right] - \frac{1}{6}\left[1+\frac{1}{8}\right] = \frac{9}{32}$

17. Required area = area (*OAPBO*)

$= \int_0^2\sqrt{4x-x^2}\,dx - \int_0^2\sqrt{2x}\,dx$

$= \int_0^2\sqrt{4-(2-x)^2}\,dx - \sqrt{2}\int_0^2 x^{1/2}dx$

$= -\left[\frac{(2-x)\sqrt{4x-x^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{2-x}{2}\right)\right]_0^2 - 2\sqrt{2}\,\frac{x^{3/2}}{3/2}\Bigg]_0^2$

$= -2\sin^{-1}(-1) - \frac{8}{3} = \pi - \frac{8}{3}$

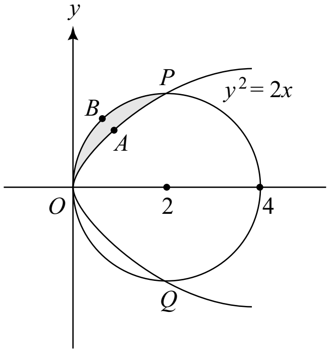


Fig. 14.74

18. For point of intersection of $y = x^2 - 5x + 4$ and $x + y = 1$ is given by

$1 - x = x^2 - 5x + 4$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow x = 1, x = 3.$

Required area is shaded in the figure

$A = A_1 + A_2 = \int_1^3 -(1-x)dx + \int_3^4(-x^2+5x-4)dx$

$= \frac{1}{2}(1-x)^2\Big]_1^3 - \int_3^4\left[\left(x-\frac{5}{2}\right)^2 - \frac{9}{4}\right]dx$

$= \frac{1}{2}(4-0) - \frac{1}{3}\left(x-\frac{5}{2}\right)^3\Bigg]_3^4 + \frac{9}{4}$

$= \frac{17}{4} - \frac{1}{3}\left\{\left(\frac{3}{2}\right)^3 - \left(\frac{1}{2}\right)^3\right\}$

$= \frac{19}{6}.$

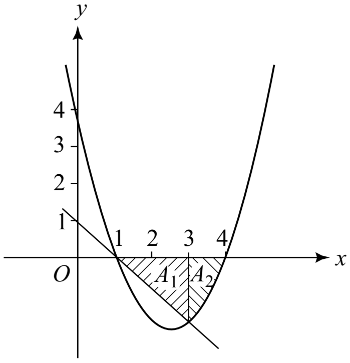


Fig. 14.75

Previous Years' B-Architecture Entrance Examination Questions

1. Reqd. area $= 2\int_0^1 3(1-x^2)\,dx$

$$= 6\left[x-\frac{x^3}{3}\right]_0^1$$

$$= 6\cdot\frac{2}{3} = 4.$$

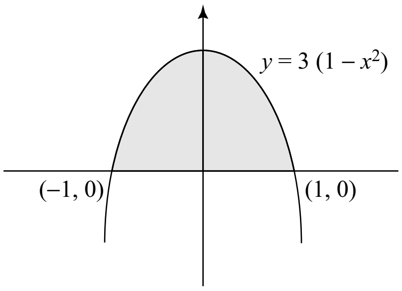


Fig. 14.76

2. Let $A_1 = ABDA$

$$= \int_{\frac{-\pi}{2}}^{0}\cos x\,dx \;-\text{ area of }\Delta BDO$$

$$= \sin x\Big|_{-\frac{\pi}{2}}^{0} - \frac{1}{2}\times1\times1 = \frac{1}{2}$$

$A_2 = DBCD$

$$= \frac{1}{2}+\int_0^{\pi/2}\cos x\,dx = \frac{3}{2}$$

$A_1 : A_2 = 1 : 3$

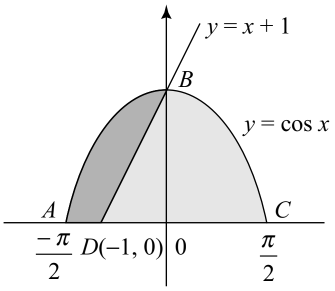


Fig. 14.77

3. Required area

$$= \text{area of }\Delta BEA + \int_{-1}^{0}x^2\,dx$$

$$= \frac{1}{2}+\frac{x^3}{3}\Bigg|_{-1}^{0} = \frac{1}{2}+\frac{1}{3} = \frac{5}{6}$$

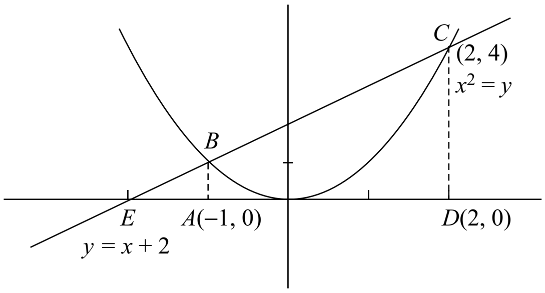


Fig. 14.78

4. Reqd. area $= 2\int_0^1 (x-x^2)\,dx$

$$= 2\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{3-2}{6}\right] = \frac{1}{3}$$

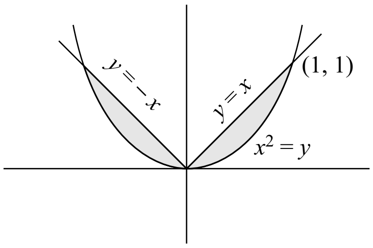


Fig. 14.79

5. Reqd. area

$$= \int_0^1 (x^2+1)\,dx + \int_1^2 (x+1)\,dx$$

$$= \frac{1}{3}+1+\frac{1}{2}\;[4-1] + [2-1]$$

$$= \frac{4}{3}+\frac{3}{2}+1 = \frac{23}{6}$$

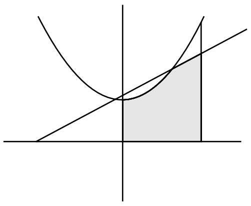


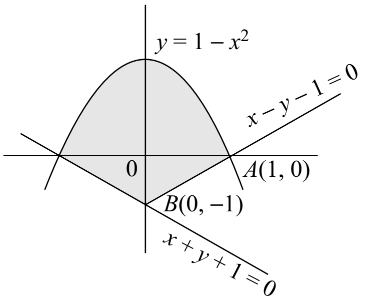
Fig. 14.80

6. Reqd. area

$$= 2\left[\int_0^1 (1-x^2)\,dx + \text{area of }\Delta OAB\right]$$

$$= 2\left[x-\frac{x^3}{3}\Bigg|_0^1 + \frac{1}{2}\times1\times1\right]$$

$$= 2\left[\frac{2}{3}+\frac{1}{2}\right] = \frac{14}{6} = \frac{7}{3}$$



$y = 1 - x^2$
$x - y - 1 = 0$
0
$A(1, 0)$
$B(0, -1)$
$x + y + 1 = 0$

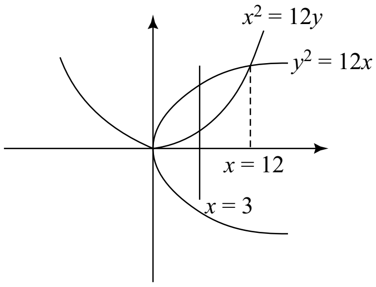
Fig. 14.81

7. Reqd. area

$$= \int_3^{12}\left(\sqrt{12x}-\frac{x^2}{12}\right)dx$$

$$= \frac{2}{3}\sqrt{12}\,x^{3/2}\Big|_3^{12} - \frac{1}{36}[12^3 - 3^3]$$

$$= 84 - \frac{567}{12} = \frac{441}{12} = \frac{147}{4}.$$



$x^2 = 12y$
$y^2 = 12x$
$x = 12$
$x = 3$

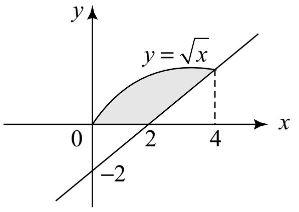
Fig. 14.82

8. For point of intersection of curves, we solve

$$\sqrt{x} = y = x - 2$$

$$\Rightarrow x - \sqrt{x} - 2 = 0$$

$$\Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$$



y
$y = \sqrt{x}$
x
0
2
4
−2

Fig. 14.83

Required area

$$= \int_0^4 \sqrt{x}\,dx - \int_0^4 (x-2)dx$$

$$= \frac{2}{3}x^{3/2}\Big]_0^4 - \frac{1}{2}(x-2)^2\Big]_2^4$$

$$= \frac{2}{3}\,[4^{3/2} - 0] - \frac{1}{2}\,[(2^2) - 0]$$

$$= \frac{16}{3} - 2 = \frac{10}{3}\ (\text{unit})^2$$

9. $\int_1^a \frac{1}{x^2}dx = \int_a^9 \frac{1}{x^2}dx$

$$\Rightarrow \frac{-1}{x}\Big]_1^a = \frac{-1}{x}\Big]_a^9$$

$$\Rightarrow 1 - \frac{1}{a} = +\frac{1}{a} - \frac{1}{9}$$

$$\Rightarrow \frac{10}{9} = \frac{2}{a} \Rightarrow a = \frac{9}{5}$$

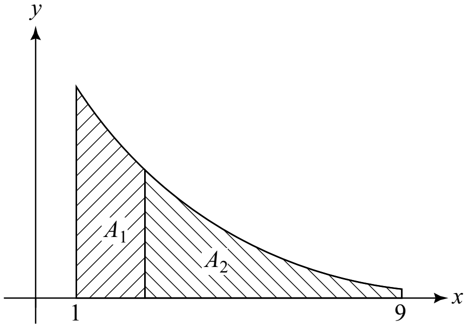


Fig. 14.84

CHAPTER FIFTEEN

Differential Equations

An equation involving the derivatives of a dependent variable with respect to one or more independent variables is called a differential equation. *The order* of a *differential equation* is the order of the highest differential coefficient involved. When an equation is a polynomial in all the differential coefficients, the power to which the highest differential coefficient is raised is known as *the degree of the equation*.

Illustration 1

Any differential equation of order 1 is of the form $f(x, y, y') = 0$ e. g.
$y' = y$, $xy' = 2y + x$ are 1 order differential equations. The order of the equation
$y'' + 5y = 0$ is 2.
The order of $x^2 y''' y' + e^x y'' = (x^2 + 3) y^3$
is order 3. The degree of the equations

$$\frac{dy}{dx} + y^2 = 0,\quad \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$$

and $\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + y = x^2$ are

1, 1 and 2 respectively.

FORMATION OF A DIFFERENTIAL EQUATION

To obtain the differential equation whose solution is the equation $f(x, y, c_1, c_2, \cdots, c_n) = 0$, where x and y are variables and $c_1, c_2, \cdots, c_n$ are arbitrary constants we differentiate the above equation n times successively, so that $n + 1$ equations are obtained. From these $n + 1$ equations, eliminate the constants $c_1, c_2, \cdots c_n$.

Illustration 2

Find the differential equation of family of circles of radius 5cm having centre on x-axis.
The equation of such a circle is

$$(x - a)^2 + y^2 = 25 \tag{1}$$

Differentiating this equation with respect to x, we get

$$2(x - a) + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow x - a = -y\frac{dy}{dx}$$

Substituting this value in (1), we get

$$y^2\left(\frac{dy}{dx}\right)^2 + y^2 = 25$$

which is the required differential equation.

Illustration 3

Find the differential equation corresponding to the equation $y = ae^x + be^{2x} + ce^{-3x}$ (i)
where a, b, c are arbitrary constants.

$$y' = ae^x + 2be^{2x} - 3ce^{-3x} \tag{ii}$$
$$y'' = ae^x + 4be^{2x} + 9ce^{-3x} \tag{iii}$$
$$y''' = ae^x + 8be^{2x} - 27ce^{-3x} \tag{iv}$$
$$y' - y = be^{2x} - 4ce^{-3x} \tag{v}$$
$$y'' - y' = 2be^{2x} + 12ce^{-3x} \tag{vi}$$
$$y''' - y'' = 4be^{2x} - 36ce^{-3x} \tag{vii}$$

Putting (v) in (vi)

$$y'' - y' = 2(y' - y) + 8ce^{-3x} + 12ce^{-3x}$$
$$\Rightarrow y'' - 3y' + 2y = 20ce^{-3x}$$

Putting this value in (v),

$$be^{2x} = (y' - y) + 4ce^{-3x}$$

$$= (y' - y) + \frac{1}{5}(y'' - 3y' + 2y)$$

$$= \frac{1}{5}y'' + \frac{2}{5}y' - \frac{3}{5}y$$

Substituting all these values in (vii), we have

$$y''' - y'' = \frac{4}{5}(y'' + 2y' - 3y) - \frac{36}{20}(y'' - 3y' + 2y)$$

$$y''' + \left(-1 - \frac{4}{5} + \frac{9}{5}\right)y'' + \left(-\frac{8}{5} - \frac{27}{5}\right)y' + \left(\frac{12}{5} + \frac{18}{5}\right)y = 0$$

$$y''' - 7y' + 6y = 0$$

METHODS OF SOLVING DIFFERENTIAL EQUATIONS SOLUTION OR INTEGRAL OF A DIFFERENTIAL EQUATION

It is a relation between the variables, not involving the differential coefficients such that this relation and the derivatives obtained from it satisfy the given differential equation. The solution of a differential equation is also called its *primitive*.

A solution which involves a number of essentially distinct arbitrary constants equal to the order of the equation is known as the *general solution.*

A solution of a differential equation obtained from a general solution by giving particular values to one or more arbitary constants is called a *particular solution.*

A solution which cannot be obtained from any general solution by any choice of the arbitrary constants is called a *singular solution.*

EQUATIONS OF FIRST ORDER AND FIRST DEGREE

A differential equation is said to be linear if the dependent variable and its differential coefficients occur in it in the first degree only and are not multiplied together.

Equations in which the Variables are Separable

These are the equations which can be so expressed that the coefficients of dx is only a function of x and that of dy is only a function of y. The general form of such an equation is $f(x)\,dx = g(y)dy$. Integrating both the sides, we get the solution.

Illustration 4

Solve $9yy' + 4x = 0$.

Separating variables, we have

$$9y\,dy = -4x\,dx$$

Integrating both sides, we obtain

$$9\frac{y^2}{2} = -2x^2 + \text{Const}$$

$$\Rightarrow \quad \frac{x^2}{9} + \frac{y^2}{4} = \text{Const}$$

Linear Equations

A linear equation of first order and first degree is either of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{(i)}$$

where P and Q are functions of x, or

$$\frac{dx}{dy} + P_1x = Q_1 \qquad \text{(ii)}$$

where P_1 and Q_1 are functions of y. In order to solve equation (i), we multiply both sides by the integrating factor is $e^{\int P dx}$. After the multiplication, left hand side becomes the differential coefficient of $ye^{\int P dx}$ and now integrating both the sides, we have

$$ye^{\int P dx} = \int Qe^{\int P dx}\,dx + C$$

as the solution. Similarly we solve equation (ii)

Equations reducible to linear form. If the given equation is of the form $dy/dx + Py = Qy^n$ where P, Q are functions of x, we can reduce it to a linear equation by dividing both the sides by y^n and then substituting $y^{-n+1} = z$. The given equation will be linear in z.

Illustration 5

Solve $y' - y = e^{2x}$

This is a linear equation with $P = -1$, $Q = e^{2x}$.

The I.F. $= e^{-\int 1 dx} = e^{-x}$. Multiplying with I.F., we have

$$\frac{d}{dx}(ye^{-x}) = e^{-x}y' - ye^{-x} = e^x$$

Integrating both sides

$$ye^{-x} = \int e^x\,dx + C$$
$$= e^x + C$$
$$\Rightarrow y = e^{2x} + ce^x$$

Illustration 6

$y' - Ay = -By^2$, A, B are constants. This is a differential equation reducible to linear equation with $P = -A$, $Q = -B$, $n = 2$. Dividing by y^2, we have

$$\frac{1}{y^2}y' - A\frac{1}{y} = -B$$

$$\text{Put } -\frac{1}{y} = Z \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dZ}{dx}$$

$$\frac{dZ}{dx} + AZ = -B$$

This is a linear equation with $P = A$, $Q = -B$

I.F. $= e^{Ax}$, so

$$\frac{d}{dx}(Ze^{Ax}) = -Be^{Ax}$$

$$\Rightarrow Ze^{Ax} = -\frac{B}{A}e^{Ax} + C$$

$$\Rightarrow -\frac{1}{y} = -\frac{B}{A} + Ce^{-Ax}$$

Homogeneous Equations

It is a differential equation of the form $\frac{dy}{dx} = \frac{f(x, y)}{\phi(x, y)}$, where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of x and y of the same degree. A function $f(x, y)$ is said to be homogeneous of degree n if it can be written as $x^n f_1(y/x)$. Such an equation can be solved by putting $y = Vx$. After substituting $y = Vx$, the given equation will have variables separable in V and x.

The equations of the form $dy/dx = (ax + by + c)/(Ax + By + C)$ can be reduced to a homogeneous equation by changing $x = X + h$ and $y = Y + k$, where h and k are the constants to be chosen so that it makes the given equation homogeneous provided $aB - bA \neq 0$.

If $aB - bA = 0$ then the given equation can be written as

$$\frac{dy}{dx} = \frac{b\left(\frac{a}{b}x + y\right) + c}{B\left(\frac{A}{B}x + y\right) + C}$$

Now put $(a/b)\,x + y = z$ so that the given equation reduces to an equation whose variables are seperable in z and x.

Illustration 7

Solve $y' = \dfrac{x + y}{x - y}$

$y' = f(x, y)$, $f(x, y)$ is a homogeneous function of degree 1. Putting $y = Vx$, we have $\dfrac{dy}{dx} = V + x\dfrac{dv}{dx}$

The given equation reduces to

$$V + x\frac{dV}{dx} = \frac{x(1+V)}{x(1-V)} = \frac{1+V}{1-V}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1+V}{1-V} - V = \frac{1 + V - V + V^2}{1-V} = \frac{1+V^2}{1-V}$$

$$\Rightarrow \frac{1-V}{1+V^2}\,dV = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1+V^2} - \frac{1}{2}\frac{2V}{1+V^2}\right)dV = \frac{dx}{x}$$

$$\Rightarrow \tan^{-1}V - \frac{1}{2}\log(1 + V^2) = \log x + C$$

$$\Rightarrow \tan^{-1}\frac{y}{x} = \log x\,(1 + V^2)^{1/2} + C$$

$$\Rightarrow \tan^{-1}\frac{y}{x} = \log\,(x^2 + y^2)^{1/2} + C$$

ORTHOGONAL TRAJECTORY

Any curve which cuts every member of a given family of curve at right angle is called an orthogonal trajectory of the family. For example, each straight line $y = mx$ passing through the origin, is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$.

Procedure for finding the orthogonal trajectory

(i) Let $f(x, y, c) = 0$ be the equation, where c is an arbitrary parameter.

(ii) Differentiate the given equation w.r.t. x and then eliminate c.

(iii) Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$ in the equation obtained in (ii)

(iv) Solve the differential equation in (iii).

DIFFERENTIAL EQUATIONS OF FIRST ORDER BUT NOT OF FIRST DEGREE

The equations which are of first order but not of the first degree, the following types of equations are discussed.

(i) Equations solvable for $p = dy/dx$
(ii) Equations solvable for y
(iii) Equations solvable for x
(iv) Clairut's Equations

(i) If there is a quadratic (or third degree) equation in p, solve the equation for p to get two (or three) first order and first degree equations in x, y and p. These equations can be solved as above.

(ii) Suppose that the given differential equation on solving for y, gives

$$y = f(x, p) \qquad (1)$$

Differentiating w.r.t. x, we obtain $p = \dfrac{dy}{dx} = Q\left(x, p, \dfrac{dp}{dx}\right)$, so that we obtain a new differential equation with variables x and p. Suppose that it is possible to solve this equation. Let the solution be

$$f(x, p, c) = 0 \qquad (2)$$

where, c is the arbitrary constant.

We may either eliminate p between (1) and (2) or we may solve (1) and (2) for x, y.

(iii) In this case differentiate w.r.t. y and proceed as in case (ii).

(iv) The equation of the form $y = px + f(p)$ is known as Clairut's equation.

Differentiating w.r.t. x, we get $p = p + x\dfrac{dp}{dx} + f'(p)\dfrac{dp}{dx}$

$$[x + f'(p)]\frac{dp}{dx} = 0 \quad \Rightarrow \quad \frac{dp}{dx} = 0,\ x + f'(p) = 0$$

If $\dfrac{dp}{dx} = 0$, we have $p = \text{constant} = c$ (say). Eliminating p we have $y = cx + f(c)$ as a solution. If $x + f'(p) = 0$, then by eliminating p, we will obtain another solution. This solution is called singular solution.

Remark If $u = rf(\theta)$, then $du = rf'(\theta)d\theta + f(\theta)\,dr$

SECOND ORDER BUT OF DEGREE ONE

If there is a linear second order equations with constant coefficients say $\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0$ then we solve the quadratic equation $m^2 + a_1 m + a_2 = 0$. There will be three cases

Case I Discriminant of the quadratic equation above is positive say roots are α_1 and α_2 then $y = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x}$ is a solution.

Case II If $\alpha_1 = \alpha_2 = \alpha$(say) then a solution is $y = (C_1 x + C_2)e^{\alpha x}$.

Case III $\alpha_1 = \alpha + i\beta$ (complex roots) then a solution is $y = C_1 \cos \alpha x + C_2 \sin \beta x$.

Illustration 8

Solve $y'' + y' - 2y = 0$

Consider the equation $m^2 + m - 2 = 0$

$\Rightarrow (m + 2)(m - 1) = 0$

$\Rightarrow m = -2, 1$

Hence the required solution $y = C_1e^{-2x} + C_2e^x$

Illustration 9

Solve $y'' + 2y' + y = 0$

Consider the equation $m^2 + 2m + 1 = 0$

$(m + 1)^2 = 0$, so

$m = -1$ are coincident roots. Hence required solution $y = (C_1 + C_2x)e^{-x}$.

Illustration 10

Solve $y'' + y = 0$

Consider the equation $m^2 + 1 = 0 \Rightarrow m = \pm i$ are two complex roots with $\alpha = 0$, $\beta = 1$.

Hence required solution is

$y = C_1 \cos 0.x + C_2 \sin x = C_1 + C_2 \sin x$.

Equation of the form $\frac{d^2y}{dx^2} = f(x)$

If the given differential equation is $\frac{d^2y}{dx^2} = f(x)$ then integrating both the sides, we get $\frac{dy}{dx} = F(x) + C_1$, where $F(x) = \int f(x)\, dx$ and C_1 is a parameter.

Integrating again, we have

$$y = G(x) + C_1 x + C_2$$

where $G(x) = \int F(x)\, dx$ and C_1, C_2 are parameter.

Illustration 11

$$\frac{d^2y}{dx^2} = \sin^2 x$$

Integrating, we get

$$\frac{dy}{dx} = \int \sin^2 x\, dx + C_1 = \frac{1}{2}\int (1 - \cos 2x)\, dx + C_1$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C_1$$

Integrating again, we have

$$y = \frac{x^2}{4} + \frac{1}{8}\cos 2x + C_1 x + C_2.$$

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: The solution of the equation $y' = \frac{y+1}{x}$ is given by

(a) $y = Cx + 1$ (b) $y = Cx - 1$

(c) $y = Cx^2 + 1$ (d) $y = Cx - 2$

Ans. (b)

Solution: The equation is with separable variables, so can be written as

$$\frac{dy}{y+1} = \frac{dx}{x}$$

Integrating, we have

$\log(y + 1) = \log x + \text{Conts}$

$\Rightarrow y + 1 = Cx \Rightarrow y = Cx - 1.$

Example 2: The solution of the equation

$$\frac{xy' - y}{x} = \tan\frac{y}{x} \text{ is given by}$$

(a) $\sin \dfrac{y}{x} = Cx$ (b) $\cos \dfrac{y}{x} = Cx$

(c) $\tan \dfrac{y}{x} = Cx^2$ (d) $\tan \dfrac{y}{x} = Cx^2 + x$

Ans. (a)

◎ **Solution:** The given equation can be written as $y' = \tan \dfrac{y}{x} + \dfrac{y}{x}$

This is a homogeneous equation of degree 1 so put $y = Vx$.

$$V + x\frac{dV}{dx} = \tan V + V$$

$$x\frac{dV}{dx} = \tan V \Rightarrow \cot V\, dV = \frac{dx}{x}$$

$$\Rightarrow \quad \log \sin V = \log x + \text{Const} \Rightarrow \sin y/x = Cx.$$

● **Example 3:** The solution of $y' \dfrac{2xy}{1-x^2} = 1 + x$

$y|_{x=0} = 1$ is given by

(a) $y = \sqrt{\dfrac{1+x}{1-x}} + \sin^{-1} x$

(b) $y = \dfrac{1}{2}\sqrt{\dfrac{1-x}{1+x}}\left[2 + x\sqrt{1-x^2}\right]$

(c) $y = \dfrac{1}{2}\sqrt{\dfrac{1+x}{1-x}}\left[2 + x\sqrt{1-x^2} + \sin^{-1} x\right]$

(d) $y = \dfrac{1}{1-x^2}\left(1 + x + \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4}\right)$

Ans. (d)

◎ **Solution:** The equation is a linear equation with integrating factor $e^{-\int \frac{dx}{1-x^2}} = e^{-\int \frac{2x}{1-x^2} dx} = e^{\log(1-x^2)} = (1 - x^2)$

Multiplying with I.F., we have

$$\frac{d}{dx}(y(1 - x^2)) = (1 - x^2)(1 + x)$$

$$y(1 - x^2) = \int (1 - x^2 + x - x^3)dx + C$$

$$= x - \frac{x^3}{3} + \frac{x^2}{2} - \frac{x^4}{4} + C$$

$$y = \frac{1}{1-x^2}\left(x - \frac{x^3}{3} + \frac{x^2}{2} - \frac{x^4}{4}\right) + \frac{C}{1-x^2}$$

since $y(0) = 1$ so $C = 1$. Hence

$$y = \frac{1}{(1-x^2)}\left(1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right)$$

● **Example 4:** The solution of $y' + y = -\dfrac{x}{y}$ is given by

(a) $\dfrac{1}{y} = -x + e^x$

(b) $y^2 = -x^2 + \dfrac{1}{2}(1 + e^{-x})$

(c) $y^2 = -x + \dfrac{1}{2}(1 + e^{-2x})$

(d) $\dfrac{1}{y} = -\dfrac{1}{2}x + \dfrac{3}{2}e^{-x}$

Ans. (c)

◎ **Solution:** The given equation is reducible to linear equation $yy' + y^2 = -x$

Put $y^2 = Z \Rightarrow 2y\dfrac{dy}{dx} = \dfrac{dZ}{dx}$. Thus

$$\frac{1}{2}\frac{dZ}{dx} + Z = -x$$

$$\Rightarrow \quad \frac{dZ}{dx} + 2Z = -2x$$

which is a linear equation with I.F. $= e^{2x}$, so

$$\frac{d}{dx}(Ze^{2x}) = -2xe^{2x}$$

$$\Rightarrow \quad Ze^{2x} = -2\int xe^{2x}\,dx + C = -2\left[\frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx\right] + C$$

$$= -xe^{2x} + \frac{e^{2x}}{2} + C$$

$$\Rightarrow \quad y^2 = -x + \frac{1}{2} + C\,e^{-2x}$$

For $x = 0, y(0) = 1$ so $1 = \dfrac{1}{2} + C \Rightarrow C = \dfrac{1}{2}$.

Thus $y^2 = -x + \dfrac{1}{2}(1 + e^{-2x})$

● **Example 5:** The solution of $y' = \dfrac{2x - y + 1}{x - 2y + 1}$ is given by

(a) $x^2 - xy + y^2 + x + y = C$
(b) $x^2 - xy + y^2 + x - y = C$
(c) $x^2 - xy + y^2 - x + y = C$
(d) $-x^2 + 2xy - y^2 - x + 2y = C$

Ans. (b)

◎ **Solution:** The given equation is reducible to homogeneous equation. Put $y = Y + k$, $x = X + h$

$$\frac{dy}{dx} = \frac{dy}{dY}\frac{dY}{dX}\cdot\frac{dX}{dx} = \frac{dY}{dX}$$

Select h, k such that $2h - k + 1 = 0$, $h - 2k + 1 = 0$ so $h = -\frac{1}{3}$ and $k = \frac{1}{3}$. Thus

$$\frac{dY}{dX} = \frac{2X - Y}{X - 2Y}$$ which is a homogeneous equation

Putting $Y = vX$, we have

$$v + X\frac{dv}{dX} = \frac{2 - v}{1 - 2v}$$

$$\Rightarrow \quad X\frac{dv}{dx} = \frac{2 - v - v + 2v^2}{1 - 2v} = -\frac{2(1 - v + v^2)}{2v - 1}$$

$$\Rightarrow \frac{2v - 1}{v^2 - v + 1}dV = -2\frac{dX}{X}$$

$$\Rightarrow \quad \log(v^2 - v + 1) + \log X^2 = \text{Const}$$

$$\Rightarrow \quad \left(\frac{y^2}{x^2} - \frac{y}{x} + 1\right)X^2 = \text{Const}$$

$$\Rightarrow \quad Y^2 - YX + X^2 = \text{Const}$$

$$\Rightarrow \quad \left(y - \frac{1}{3}\right)^2 - \left(y - \frac{1}{3}\right)\left(x + \frac{1}{3}\right) + \left(x + \frac{1}{3}\right)^2 = \text{Const}$$

$$\Rightarrow \quad y^2 - \frac{2}{3}y + \frac{1}{9} - \left(xy - \frac{1}{3}x + \frac{1}{3}y - \frac{1}{9}\right) + x^2 + \frac{2}{3}x + \frac{1}{9} = \text{Const}$$

$$\Rightarrow \quad y^2 + x^2 - xy - y + x = C$$

Example 6: A particular solution of

$\log\frac{dy}{dx} = 4x + 5y$, $\quad y(0) = 0$ is given by

(a) $5e^{4x} + 4e^{-5y} = 9$ (b) $5e^{4x} + 4e^{5y} = 9$
(c) $4e^{4x} + 5e^{-4x} = 20$ (d) $4e^{4x} + 5e^{-4y} = 9$

Ans. (a)

Solution: $\frac{dy}{dx} = e^{4x + 5y} = e^{4x} . e^{5y}$

$$\Rightarrow \quad e^{-5y}dy = e^{4x}\,dx$$

Integrating, we have

$$\frac{e^{-5y}}{-5} = \frac{e^{4x}}{4} + C$$

$$\Rightarrow \quad C = \frac{e^{4x}}{4} + \frac{e^{-5y}}{5}$$

Putting $x = 0$ and $y(0) = 0$, $C = \frac{1}{4} + \frac{1}{5} = \frac{9}{20}$

Thus $9 = 5e^{4x} + 4e^{-5y}$

Example 7: The motion of a particle is given by $\frac{dx}{dt} = -3a\cos^2 t \sin t$, $\frac{dy}{dt} = 3a\sin^2 t \cos t$ and passes through at $t = 0$, then the path is given by

(a) $(x + a)^{2/3} + (y - a)^{2/3} = 1$
(b) $(x - a)^{2/3} + (y + a)^{2/3} = 2a^{2/3}$
(c) $x^{2/3} + y^{2/3} = a^{2/3}$
(d) $(x + a)^{2/3} + y^{2/3} = a^{2/3}$

Ans. (d)

Solution: $x = -3a\int\cos^2 t \sin t\,dt + C = a\cos^3 t + C$

For $t = 0$, $C = -a$

$y = a\int\sin^2 t \cos t\,dt = a\sin^3 t + G$, at $t = 0$, $y = 0$ so $C_1 = 0$

Thus $x = a(\cos^3 t - 1)$, $y = a\sin^3 t$

$$\Rightarrow \left(\frac{x}{a} + 1\right)^{1/3} = \cos t,\ \left(\frac{y}{a}\right)^{1/3} = \sin t$$

$$\Rightarrow \left(\frac{x}{a} + 1\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1. \Rightarrow (x + a)^{2/3} + y^{2/3} = a^{2/3}$$

Example 8: The solution of initial value problem $xy' + y = 0$, $y(2) = -2$ is given by

(a) $xy + 4 = 0$ (b) $(y + 2) = 0$
(c) $x - 2 = y + 2$ (d) $(x - 2)(y + 2)^2 = 4$

Ans. (a)

Solution: $x\frac{dy}{dx} = -y \Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0$

$$\Rightarrow \quad \log y + \log x = \text{Const}$$

$$\Rightarrow \quad xy = \text{Const}$$

Since $y(2) = -2$ so Const $= 2(-2) = -4$

Thus $xy + 4 = 0$ is the required solution

Example 9: The solution of $y' + xy = xy^{-1}$, $y(0) = 2$ is given by

(a) $y^2 = 2 + 2e^{-x^2}$ (b) $y^2 = 3 + e^{x^2}$
(c) $y^2 = 3 + e^{-x^2}$ (d) $y^2 = 2 + 2^{e^{-x}}$

Ans. (c)

Solution: $y' + xy = xy^{-1} \Rightarrow yy' + xy^2 = x$

This is reducible to linear equation, so put $y^2 = z \Rightarrow y\frac{dy}{dx} = \frac{1}{2}\frac{dz}{dx}$

$$\frac{1}{2}\frac{dz}{dx} + xz = x$$

$$\Rightarrow \quad \frac{dz}{dx} + 2xz = 2x$$

I.F. $= e^{2\int x dx} = e^{x^2}$. Multiplying with I.F., we have

$$\frac{d}{dx}(ze^{x^2}) = 2x\,e^{x^2}$$

$$\Rightarrow \quad z\,e^{x^2} = \int 2x\,e^{x^2}\,dx + C = e^{x^2} + C$$

$$\Rightarrow \quad y^2\,e^{x^2} = e^{x^2} + C$$

Since $y(0) = 1$ so $C = 3$

$$\Rightarrow \quad y^2 = 3 + e^{-x^2}$$

Example 10: The solution of $y' + \cos(x + y) = \cos(x - y)$ and $y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$ is given by

(a) $\sin y = 2\sin x\cos^2 y$ (b) $\cos y = 2\sin x\sin^2 y$
(c) $\cos y = 2\cos x\sin^2 y$ (d) $\sin x = 2\cos y\sin^2 y$

Ans. (b)

Solution: $\dfrac{dy}{dx} = \cos(x - y) - \cos(x + y) = 2\sin x\sin y$

$$\Rightarrow \quad \frac{dy}{\sin y} = 2\sin x\,dx$$

$$\Rightarrow \quad -\operatorname{cosec} y\cot y = -2\cos x + C$$

Since $y\left(\dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$ so $C = 0$. Thus

$$\operatorname{cosec} y\cot y = 2\cos x$$

$$\Rightarrow \quad \cos y = \cos x\sin^2 y$$

LEVEL 1

Straight Objective Type Questions

Example 11: The degree of the differential equation

$$\frac{d^3y}{dx^2} + 5\left(\frac{d^2y}{dx^2}\right)^2 = x^3\log\frac{d^2y}{dx^2} \text{ is}$$

(a) 1 (b) 2
(c) 3 (d) none of these

Ans. (d)

Solution: Since the equation is not a polynomial in all the differential coefficients so the degree is not defined.

Example 12: The degree of the differential equation $y_2^{3/2} - y_1^{1/2} - 4 = 0$ is

(a) 6 (b) 3
(c) 2 (d) 4

Ans. (a)

Solution: $y_2^{3/2} = y_1^{1/2} + 4$. Squaring both sides, we have $y_2^3 = y_1 + 16 + 8y_1^{1/2}$
$\Rightarrow (y_2^3 - y_1 - 16)^2 = 64y_1 \Rightarrow y_2^6 - 32y_2^3 - 2y_2^3y_1 + y_1^2 - 32y_1 + 256 = 0$. Hence the degree of the given equation is 6.

Example 13: Which of the following equations is a linear equation of order 3 ?

(a) $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2}\dfrac{dy}{dx} + y = x$

(b) $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + y^2 = x^2$

(c) $x\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = e^x$

(d) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = \log x$

Ans. (c)

Solution: The equations in (a), (b) and (c) are of order 3 and in (d) is of order 2. The equations in (c) and (d) are linear, see theory for definition.

Example 14: The order and degree of the differential equation $x^2 = \dfrac{\left(1 + \left(\dfrac{dy}{dx}\right)^2\right)^{3/2}}{d^2y/dx^2}$ are (respectively)

(a) 2, 1 (b) 2, 3
(c) 2, 2 (d) 2, 6

Ans. (c)

Solution: The given equation can be written as

$$x^4\left(\frac{d^2y}{dx^2}\right)^2 = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^3$$

which is clearly of order 2 and degree 2.

Example 15: An equation of the curve in which subnormal varies as the square of the ordinate is (k is constant of proportionality)

(a) $y = Ae^{kx}$ (b) $y = e^{kx}$
(c) $y^2/2 + kx = A$ (d) $y^2 + kx^2 = A$

Ans. (a)

Solution: According to the given condition $y\dfrac{dy}{dx} = ky^2$

$\Rightarrow \quad \dfrac{dy}{y} = k\,dx$ (variables separable equation)

$\Rightarrow \log|y| = kx + C \Rightarrow |y| = Be^{kx} \Rightarrow y = Ae^{kx}$ where $A = \pm B$ and k is the constant of proportionality.

Example 16: A solution of the differential equations $\left(\dfrac{dy}{dx}\right)^2 - x\dfrac{dy}{dx} + y = 0$ is

(a) $y = 2$ (b) $y = 2x$
(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

Ans. (c)

Solution: Let $p = \dfrac{dy}{dx}$ so that the given equation can be written as $p^2 - xp + y = 0$. Therefore $y = xp - p^2$ which is a

Clairut's equation see theory. Differentiating with respect to x, we have

$$p = \frac{dy}{dx} = p + x\frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$(x - 2p)\frac{dp}{dx} = 0 \Rightarrow \frac{dp}{dx} = 0 \text{ so } p = \text{const} = C \text{ (say)}$$

So $y = Cx - C^2$ is a solution, in particular $C = 2$.

◉ **Example 17:** The solution of $\frac{dy}{dx} = \frac{1}{2x - y^2}$ is given by

(a) $y = Ce^{-2x} + \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4}$

(b) $x = Ce^{-y} + \frac{1}{4}y^2 + \frac{1}{4}y + \frac{1}{2}$

(c) $x = Ce^{y} + \frac{1}{4}y^2 + y + \frac{1}{2}$

(d) $x = Ce^{2y} + \frac{1}{2}y^2 + \frac{1}{2}y + \frac{1}{4}$

Ans. (d)

◎ **Solution:** $\frac{dx}{dy} = 2x - y^2$ (This is a linear equation in x).

The integrating factor is $e^{-\int 2dy} = e^{-2y}$.

So $$\frac{d}{dy}(xe^{-2y}) = -y^2 e^{-2y}$$

Integrating, we have

$$xe^{-2y} = \frac{-y^2e^{-2y}}{-2} - \int ye^{-2y}\,dy + \text{Const}$$

$$= \frac{y^2}{2}e^{-2y} + \frac{ye^{-2y}}{2} - \frac{1}{2}\int e^{-2y}\,dy + \text{Const}$$

$$= \frac{y^2}{2}e^{-2y} + \frac{y}{2}e^{-2y} + \frac{e^{-2y}}{4} + C_1$$

$$\therefore \quad x = C_1 e^{2y} + \frac{y^2}{2} + \frac{y}{2} + \frac{1}{4}$$

◉ **Example 18:** The curves whose subtangents are proportional to the abscissas of the point of tangency (the proportionality factor is equal to k) is

(a) $y^k = Cx$ (b) $y^k = Cx^2$

(c) $y^{k/2} = Cx^3$ (d) none of these

Ans. (a)

◎ **Solution:** Subtangent $= y\frac{dx}{dy}$ so according to the given condition $y\frac{dx}{dy} = kx$

$$\Rightarrow \frac{dx}{x} = k\frac{dy}{y} \Rightarrow \log A + \log|x| = k\log|y|$$

$$\Rightarrow A|x| = |y|^k \Rightarrow y^k = \pm Ax = Cx$$

◉ **Example 19:** The degree of the differential equation of all curves having normal of constant length C is

(a) 1 (b) 3

(c) 4 (d) none of these

Ans. (d)

◎ **Solution:** According to the given condition

$$y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = C \Rightarrow y^2 + y^2\left(\frac{dx}{dy}\right)^2 = C^2$$

The degree of this equation is 2.

◉ **Example 20:** Which of the following transformation reduce the differential equation $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$ into the form

$$\frac{du}{dx} + uP(x) = Q(x)$$

(a) $u = \log z$ (b) $u = e^z$

(c) $u = (\log z)^{-1}$ (d) $u = (\log z)^2$

Ans. (c)

◎ **Solution:** Dividing the given equation by $z(\log z)^2$, we have

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{\log z}\frac{1}{x} = \frac{1}{x^2} \qquad \text{(i)}$$

Putting $\frac{1}{\log z} = u$, we have $\frac{du}{dx} = -(\log z)^{-2}\frac{1}{z}\frac{dz}{dx}$

So (i) can be written as

$$-\frac{du}{dx} + \frac{u}{x} = \frac{1}{x^2} \Rightarrow \frac{du}{dx} - \frac{u}{x} = \frac{-1}{x^2}$$

which is in the required form with $P(x) = -1/x$ and $Q(x) = -1/x^2$.

◉ **Example 21:** A particular solution of $\log\frac{dy}{dx} = 3x + 4y$, $y(0) = 0$ is

(a) $e^{3x} + 3e^{-4y} = 4$ (b) $4e^{3x} - e^{-4y} = 3$

(c) $3e^{3x} + 4e^{4y} = 7$ (d) $4e^{3x} + 3e^{-4y} = 7$

Ans. (d)

◎ **Solution:** $\frac{dy}{dx} = e^{3x+4y} = e^{3x}e^{4y} \Rightarrow e^{-4y}\,dy = e^{3x}\,dx$

Thus $\frac{e^{-4y}}{-4} - \frac{e^{3x}}{3} = \text{Const}$. But $y(0) = 0$, so, $-\frac{1}{4} - \frac{1}{3} = C$.

Hence $(e^{-4y}/(-4)) - (e^{3x}/3) = -7/12 \Rightarrow 3e^{-4y} + 4e^{3x} = 7$.

Example 22: The solution of $\frac{dy}{dx} = \frac{ax+b}{cy+d}$ represents a parabola if

(a) $a = 0, c = 0$ (b) $a = 1, b = 2$
(c) $a = 0, c \neq 0$ (d) $a = 1, c = 1$

Ans. (c)

Solution: The given equation has separable variables so $(cy + d)dy = (ax + b)\,dx$. Integrating we have $\frac{cy^2}{2} + dy + K = \frac{ax^2}{2} + bx$, K being the constant of integration. The last equation represents a parabola if $c = 0, a \neq 0$ or $a = 0, c \neq 0$.

Example 23: The differential equation corresponding to the family of curves $y = e^x (a \cos x + b \sin x)$, a and b being arbitrary constants is

(a) $2y_2 + y_1 - 2y = 0$ (b) $y_2 - 2y_1 + 2y = 0$
(c) $2y_2 - y_1 + 2y = 0$ (d) none of these

Ans. (b)

Solution: $y_1 = e^x(-a \sin x + b \cos x) + e^x(a \cos x + b \sin x)$

$$\Rightarrow \quad y_1 = e^x(-a \sin x + b \cos x) + y$$

$$\Rightarrow \quad y_2 = y_1 + e^x(-a \cos x - b \sin x) + e^x(-a \sin x + b \cos x)$$

$$\Rightarrow \quad y_2 - 2y_1 + 2y = 0.$$

Example 24: The solution of $y^5 x + y - x\frac{dy}{dx} = 0$ is

(a) $x^4/4 + (1/5)(x/y)^5 = C$
(b) $x^5/5 + (1/4)(x/y)^4 = C$
(c) $(x/y)^5 + x^4/4 = C$
(d) $(x\,y)^4 + x^5/5 = C$

Ans. (b)

Solution: The given differential equation can be written as $y^5 x\,dx + y\,dx - x\,dy = 0$. Multiplying by x^3/y^5, we have

$$x^4\,dx + \frac{x^3}{y^3}\left(\frac{y\,dx - x\,dy}{y^2}\right) = 0$$

Integrating, we get $x^5/5 + (1/4)(x/y)^4 = C$,

since $\frac{x^3}{y^3}\left(\frac{y\,dx - x\,dy}{y^2}\right) = u^3\,du$ where $u = \frac{x}{y}$.

Example 25: The equation of the curve passing through (3, 9) which satisfies $dy/dx = x + 1/x^2$ is

(a) $6xy = 3x^2 - 6x + 29$
(b) $6xy = 3x^2 - 29x + 6$
(c) $6xy = 3x^3 + 29x - 6$
(d) none of these

Ans. (c)

Solution: The given differential equation has variable separable so integrating, we have $y = x^2/2 - 1/x + C$. This will pass through (3, 9) if $9 = 9/2 - 1/3 + C \Rightarrow C = 29/6$. Hence the required equation is $6xy = 3x^3 + 29x - 6$.

Example 26: The solution of $\frac{x\,dy}{x^2+y^2} = \left(\frac{y}{x^2+y^2} - 1\right)dx$ is

(a) $y = x \cot(c - x)$ (b) $\cos^{-1}(y/x) = -x + c$
(c) $y = x \tan(c - x)$ (d) $y^2/x^2 = x \tan(c - x)$

Ans. (c)

Solution: The given equation can be written as

$$\frac{x\,dy - y\,dx}{x^2+y^2} = -\,dx \Rightarrow \frac{x\,dy - y\,dx}{x^2} \times \frac{1}{1 + y^2/x^2} = -\,dx$$

$\Rightarrow \frac{1}{1 + y^2/x^2}\frac{d}{dx}\left(\frac{y}{x}\right) = -\,dx$. Integrating we have

$$\tan^{-1}(y/x) = -x + c$$

$$\Rightarrow \quad y = x \tan(c - x).$$

Example 27: The solution of the equation $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0$ is

(a) $\log|2x + y - 1| = C + x + y$
(b) $\log(4x + 2y - 1) = C + 2x + y$
(c) $\log(2x + y + 1) + x + 2y = C$
(d) $\log|2x + y - 1| + x + 2y = C$

Ans. (d)

Solution: Put $2x + y = X \Rightarrow 2 + \frac{dy}{dx} = \frac{dX}{dx}$. Therefore, the given equation is reduced to (see theory the case when $aB - bA = 0$)

$$\frac{dX}{dx} - 2 = -\frac{X+1}{2X-1} \quad \Rightarrow \quad \frac{dX}{dx} = \frac{3(X-1)}{2X-1}$$

$$\Rightarrow \quad \frac{2X-1}{3(X-1)}\,dX = dx \quad \Rightarrow \quad \frac{1}{3}\left[2 + \frac{1}{X-1}\right]dX = dx$$

$$\Rightarrow \quad \frac{1}{3}[2X + \log|X - 1|] = x + \text{Const}$$

$$\Rightarrow \quad 2(2x + y) + \log|2x + y - 1| = 3x + \text{Const}$$

$$\Rightarrow \quad x + 2y + \log|2x + y - 1| = C.$$

Example 28: Solution of the differential equation $2y \sin x\frac{dy}{dx} = 2 \sin x \cos x - y^2 \cos x$ satisfying $y(\pi/2) = 1$ is given by

(a) $y^2 = \sin x$ (b) $y = \sin^2 x$
(c) $y^2 = \cos x + 1$ (d) $y^2 \sin x = 4 \cos^2 x$

Ans. (a)

Solution: The given equation can be written as

$$2y \sin x\frac{dy}{dx} + y^2 \cos x = \sin 2x$$

$$\Rightarrow \quad \frac{d}{dx}(y^2 \sin x) = \sin 2x$$

$$\Rightarrow \quad y^2 \sin x = (-1/2)\cos 2x + C.$$

so $\quad (y(\pi/2))^2 \sin(\pi/2) = (-1/2)\cos(2\pi/2) + C$

$$\Rightarrow \quad C = 1/2.$$

Hence $\quad y^2 \sin x = (1/2)(1 - \cos 2x) = \sin^2 x$

$$\Rightarrow \quad y^2 = \sin x.$$

Example 29: Solution of the differential equation $x\mathrm{d}y - y\,\mathrm{d}x - \sqrt{x^2 + y^2}\,\mathrm{d}x = 0$ is

(a) $y - \sqrt{x^2 + y^2} = Cx^2$ (b) $y + \sqrt{x^2 + y^2} = Cx^2$

(c) $x + \sqrt{x^2 + y^2} = Cy^2$ (d) $x - \sqrt{x^2 + y^2} = Cy^2$

Ans. (b)

Solution: Writing the given equation as $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$ and putting $y = vx$, we have $v + x\frac{dv}{dx} = v + \sqrt{1 + v^2} \Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

Integrating, we have $\log(v + \sqrt{1 + v^2}) = \log x + \text{Const.}$

$$\Rightarrow \quad y/x + \sqrt{1 + y^2/x^2} = Cx \quad \Rightarrow \quad y + \sqrt{x^2 + y^2} = Cx^2.$$

Example 30: An equation of the curve satisfying $x\mathrm{d}y - y\,\mathrm{d}x = \sqrt{x^2 - y^2}\,\mathrm{d}x$ and $y(1) = 0$ is

(a) $y = x^2 \log|\sin x|$ (b) $y = x\sin(\log|x|)$

(c) $y^2 = x(x - 1)^2$ (d) $y = 2x^2(x - 1)$

Ans. (b)

Solution: The equation can be written as

$$x^2\,\frac{x\,dy - y\,dx}{x^2} = x\sqrt{1 - (y/x)^2}\;\mathrm{d}x$$

$$\Rightarrow \quad \frac{d(y/x)}{\sqrt{1 - (y/x)^2}} = \frac{dx}{x} \quad \Rightarrow \quad \sin^{-1} y/x = \log|x| + \text{Const}$$

Since $y(1) = 0$ so Const $= 0$. Hence $y = x\sin(\log|x|)$.

Note that one can also solve the given equation as a homogenous equation also, i.e. by putting $y = vx$

Example 31: A solution of the equation

$$x\frac{dy}{dx} = y(\log y - \log x + 1) \text{ is}$$

(a) $y = xe^{cx}$ (b) $y^2/x = cx$

(c) $y^2 = cx\log x$ (d) $\log y = cx$

Ans. (a)

Solution: Putting $y = vx$ in the given equation, we have

$$v + x\frac{dv}{dx} = v(\log v + 1) \quad \Rightarrow \quad x\frac{dv}{dx} = v\log v$$

$$\Rightarrow \quad \frac{dv}{v\log v} = \frac{dx}{x} \quad \Rightarrow \quad \log|\log v| = \log|x| + \text{Const}$$

$$\Rightarrow \quad \log v = \pm Ax = cx \quad \Rightarrow \quad y = xe^{cx}.$$

Example 32: The solution of $x^3\frac{dy}{dx} + 4x^2\tan y = e^x \sec y$ satisfying $y(1) = 0$ is

(a) $\tan y = (x - 2)e^x \log x$

(b) $\sin y = e^x(x - 1)x^{-4}$

(c) $\tan y = (x - 1)e^x x^{-3}$

(d) $\sin y = e^x(x - 1)x^{-3}$

Ans. (b)

Solution: Rewriting the given equation in the form

$$x^4\cos y\frac{dy}{dx} + 4x^3\sin y = xe^x \quad \Rightarrow \quad \frac{d}{dx}(x^4\sin y) = xe^x$$

$$\Rightarrow \quad x^4\sin y = \int xe^x\,\mathrm{d}x$$

$$= (x - 1)e^x + C$$

Since $y(1) = 0$, so $C = 0$.

Thus $\sin y = x^{-4}(x - 1)e^x$.

Example 33: If for the differential equation $y' = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$ the general solution is $y = \frac{x}{\log|Cx|}$ then $\phi(x/y)$ is given by

(a) $-x^2/y^2$ (b) y^2/x^2

(c) x^2/y^2 (d) $-y^2/x^2$

Ans. (d)

Solution: Putting $v = y/x$ so that $\frac{x\,dv}{dx} + v = \frac{dy}{dx}$, we have

$$\frac{x\,dv}{dx} + v = v + \phi(1/v) \quad \Rightarrow \quad \frac{dv}{\phi(1/v)} = \frac{dx}{x}$$

$\Rightarrow \quad \log|Cx| = \int \frac{dv}{\phi(1/v)}$ (C being constant of integration.)

But $y = \frac{x}{\log|(Cx)|}$ is the general solution so

$$\frac{x}{y} = \frac{1}{v} = \int \frac{dv}{\phi(1/v)} \quad \Rightarrow \phi(1/v) = -v^2$$

$$\Rightarrow \quad \phi(x/y) = -y^2/x^2.$$

Example 34: The solution $y(x)$ of the differential equation $\frac{d^2y}{dx^2} = \sin 3x + e^x + x^2$ when $y_1(0) = 1$ and $y(0) = 0$ is

(a) $-\frac{\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x - 1$

(b) $-\dfrac{\sin 3x}{9}+e^x+\dfrac{x^4}{12}+\dfrac{1}{3}x$

(c) $-\dfrac{\cos 3x}{3}+e^x+\dfrac{x^4}{12}+\dfrac{1}{3}x+1$

(d) none of these

Ans. (a)

Solution: Integrating the given differential equation, we have

$$\frac{dy}{dx}=-\frac{\cos 3x}{3}+e^x+\frac{x^3}{3}+C_1$$

but $y_1(0)=1$ so $1=-\dfrac{1}{3}+1+C_1 \Rightarrow C_1=\dfrac{1}{3}$

Again integrating, we get

$$y=-\frac{\sin 3x}{9}+e^x+\frac{x^4}{12}+\frac{1}{3}x+C_2$$

but $\quad y(0)=0$ so $0=1+C_2 \Rightarrow C_2=-1$. Thus

$$y=-\frac{\sin 3x}{9}+e^x+\frac{x^4}{12}+\frac{1}{3}x-1.$$

Example 35: The differential equation $\dfrac{dy}{dx}=\dfrac{\sqrt{1-y^2}}{y}$ determines a family of circles with

(a) variable radii and a fixed centre (0, 1)

(b) variable radii and a fixed centre (0, –1)

(c) fixed radius 1 and a variable centres along the x-axis

(d) fixed radius 1 and variable centres along the y-axis

Ans. (c)

Solution: $dx=\dfrac{y}{\sqrt{1-y^2}}\,dy$

$$\Rightarrow \quad c+x=-\frac{1}{2}\int\frac{2y}{\sqrt{1-y^2}}\,dy=-\sqrt{1-y^2}$$

$$\Rightarrow \quad (x+c)^2+y^2=1$$

which represents a family of circles of fixed radius 1 and variable centre on the x-axis.

Example 36: Let $f(x)$ be a differentiable on the interval $(0,\infty)$ such that $f(1)=1$, and $\displaystyle\lim_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1$ for each $x>0$. Then $f(x)$ is

(a) $\dfrac{1}{3x}+\dfrac{2x^2}{3}$ (b) $-\dfrac{1}{3x}+\dfrac{4x^2}{3}$

(c) $-\dfrac{1}{x}+\dfrac{2}{x^2}$ (d) $\dfrac{1}{x}$

Ans. (a)

Solution: $\displaystyle 1=\lim_{t\to x}\frac{t^2f(x)-t^2f(t)+t^2f(t)-x^2f(t)}{t-x}$

$$=\lim_{t\to x}\frac{t^2(f(x)-f(t))}{t-x}+\lim_{t\to x}(t+x)f(t)$$

$$=-x^2f'(x)+2x\,f(x)$$

Thus $f(x)$ is a solution of the differential equation

$$x^2\frac{dy}{dx}-2xy=-1 \Rightarrow \frac{dy}{dx}-\frac{2}{x}y=-\frac{1}{x^2} \qquad (1)$$

This is a linear equation with I.F. $=e^{-\int\frac{2}{x}dx}=\dfrac{1}{x^2}$.

Multiplying (1) with $\left(1/x^2\right)$ we get

$$\frac{d}{dx}\left(\frac{y}{x^2}\right)=-\frac{1}{x^4} \Rightarrow \frac{y}{x^2}=\frac{1}{3x^3}+C$$

Since $f(1)=1$ so $C=\dfrac{2}{3}$ $\therefore$ $y=\dfrac{2}{3}x^2+\dfrac{1}{3x}$.

Example 37: Suppose $y=y(x)$ satisfies the differential equation $ydx+y^2dy=xdy$. If $y(x)>0\ \forall\ x\in\mathbf{R}$ and $y(1)=1$ then $y(-3)$ equals

(a) 1 (b) 2

(c) 3 (d) 5

Ans. (c)

Solution: $\dfrac{ydx-xdy}{y^2}=-dy$

$$\Rightarrow \quad d\left(\frac{x}{y}\right)=-dy \Rightarrow x/y=-y+c$$

since $\quad y(1)=1$ so $c=2$ Thus $x/y=-y+2$

when $\quad x=-3,\ -\dfrac{3}{y}=-y+2$

$$\Rightarrow \quad -3=-y^2+2y \Rightarrow (y-1)^2=4$$

$$\Rightarrow \quad y=1\pm 2=3 \text{ or } -1$$

As $y(x)>0$ for all $x\in\mathbf{R}$ so $y(-3)=3$.

Example 38: The equation of the curve whose tangent at any point (x, y) makes an angle $\tan^{-1}(2x+3y)$ with x-axis and which passes through (1, 2) is:

(a) $6x+9y+2=26e^{3(x-1)}$

(b) $6x-9y+2=26e^{3(x-1)}$

(c) $6x+9y-2=26e^{3(x-1)}$

(d) $6x-9y-2=26e^{3(x-1)}$

Ans. (a)

Solution: $\dfrac{dy}{dx}=\tan[\tan^{-1}(2x+3y)]=2x+3y$

$$\Rightarrow \quad \frac{dy}{dx}-3y=2x$$

I.F. $=e^{-3x}$ Multilplying (1) by e^{-3x}, we get

$$e^{-3x}\frac{dy}{dx}-3e^{-3x}y=2xe^{-3x} \Rightarrow \frac{d}{dx}[ye^{-3x}]=2xe^{-3x}$$

$$\Rightarrow \quad ye^{-3x} = \int 2xe^{-3x}\,dx = -\frac{2}{3}xe^{-3x} + \frac{2}{3}\int (1)e^{-3x}\,dx$$

$$= -\frac{2}{3}xe^{-3x} - \frac{2}{9}e^{-3x} + C$$

As this curve passes through (1, 2), we get

$$2 = -\frac{2}{9}(3+1) + Ce^{3} \quad \Rightarrow C = \frac{26}{9}e^{-3}$$

Thus, required curve is

$$y = -\frac{2}{9}(3x+1) + \frac{26}{9}e^{3(x-1)} \quad \Rightarrow \; 6x + 9y + 2 = 26e^{3(x-1)}$$

Example 39: The degree and order respectively of the differential equation of all parabolas whose axis is x-axis, are:

(a) 2, 1 (b) 1, 2
(c) 2, 2 (d) 1, 1

Ans. (b)

Solution: Equation of any parabola whose axis is x-axis is

$$y^2 = 4a(x+b)$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a \quad \Rightarrow \quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

which is a differential equation of order 2 and degree 1.

Example 40: The solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$ satisfying the condition $y(1) = 1$ is

(a) $y = \log x + x$ (b) $y = x\log x + x^2$
(c) $y = xe^{(x-1)}$ (d) $y = x\log x + x$

Ans. (d)

Solution: $xdy = xdx + ydx$

$$\frac{xdy - ydx}{x^2} = \frac{dx}{x}$$

$$\Rightarrow \quad d(y/x) = d(\log x) \quad \Rightarrow \quad y = x(\log x + C)$$

Putting $x = 1$, we get $C = 1$, so

$$y = x(\log x + 1)$$

Example 41: The differential equation which represents the family of curves $y = c_1e^{c_2x}$, where c_1 and c_2 are arbitrary constants, is

(a) $yy'' = y'$ (b) $yy'' = y'^2$
(c) $y'' = y^2$ (d) $y'' = y'y$

Ans. (b)

Solution: $y = c_1e^{c_2x} \quad \Rightarrow \quad y' = c_1c_2\,e^{c_2x}$

$$\Rightarrow \quad y'' = c_1\,c_2^2e^{c_2x}$$

$$\Rightarrow \quad yy'' = c_1^2c_2^2e^{c_2x}e^{c_2x} = \left(c_1c_2e^{c_2x}\right)^2 = y'^2$$

Example 42: Solution of the differential equation $\cos x\,dy = y(\sin x - y)dx$, $0 < x < \pi/2$ is

(a) $y\tan x = \sec x + c$ (b) $\tan x = (\sec x + c)y$
(c) $\sec x = (\tan x + c)y$ (d) $y\sec x = \tan x + c$

Ans. (c)

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{y(\sin x - y)}{\cos x} \quad \Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y}\tan x = -\sec x$$

Put $-\frac{1}{y} = z$, we have

$$\frac{dz}{dx}\;(\tan x)z = -\sec x \qquad (1)$$

$$\text{I.F.} = e^{\int \tan x\,dx} = e^{\log\sec x} = \sec x$$

(1) reduces to $\frac{d}{dx}(z\sec x) = -\sec^2 x$

$$\Rightarrow \quad z\sec x = -\tan x + c_1$$

$$\Rightarrow \quad \sec x = (c + \tan x)y,\; c = -c_1.$$

Example 43: If $\frac{dy}{dx} = y + 3 > 0$ and $y(0)$ and $y(0) = 2$, $y(\log 2)$ is equal to

(a) – 2 (b) 7
(c) 5 (d) 13

Ans. (b)

Solution: $\frac{dy}{y+3} = dx \Rightarrow \log(y+3) = x + c$ when $x = 0$, $y = 2$, therefore $\log 5 = C$

Thus $\log(y+3) = x + \log 5$

When $x = \log 2$, we have

$$\log(y+3) = \log 2 + \log 5 = \log 10$$

$$\Rightarrow \quad y + 3 = 10 \Rightarrow y = 7.$$

Example 44: Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T-t)$, where k > 0 is a constant and T is the total life in years of the equipment. Then the scrap value $V(T)$ of the equipment is:

(a) e^{-kT} (b) $T^2 - \frac{I}{k}$
(c) $I - \frac{kT^2}{2}$ (d) I – K(T)

Ans. (c)

Solution: $\frac{dv}{dt} = -K(T-t),\; K > 0$

$$\Rightarrow \quad dV = -K(T-t)\,dt$$

Integrating, we have $V(t) = \frac{k(T-t)^2}{2} + C$

We have $V(0) = I$, therefore

$$I = \frac{KT^2}{2} + C \quad \Rightarrow \quad C = I - \frac{k}{2}T^2$$

Scrap value $V(T) = C = I - \frac{K}{2}T^2$.

Example 45: The population $p(t)$ at time t of a certain mouse species follows the differential equation $\frac{dp(t)}{dt} = 0.5$ $p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is

(a) log 9 (b) $\frac{1}{2}\log 18$

(c) log 18 (d) 2 log 18

Ans. (d)

Solution: The given differential equation in a linear equation with I.F. = $e^{-\int(0.5)dt} = e^{-t/2}$. Multiplying with I.F. we have

$$\frac{d}{dt}\left(e^{-t/2}p\right) = -450\,e^{-t/2} \quad \Rightarrow \quad e^{-t/2}\,p(t) = 900\,e^{-t/2} + C$$

When $t = 0$, $p = 850$,

$$850 = 900 + C \quad \Rightarrow \quad C = -50$$

Thus $\quad p(t) = 900 - 50\,e^{t/2}$

When $\quad p(t) = 0$, we get $900 - 50e^{t/2} = 0$

$\Rightarrow \quad e^{t/2} = 18 \quad \Rightarrow \quad t/2 = \log 18$ i.e. $t = 2\log 18$

Example 46: If the differential equation representing the family of all circles touching y-axis at the origin is $x^2 - y^2 = f(x)y\,\frac{dx}{dy}$ then $f(x)$ is equal to

(a) $2x$ (b) x

(c) x^2 (d) $3x$

Ans. (a)

Solution: Equation of circles touching y-axis at origin is

$$x^2 + (y - a)^2 = a^2 \qquad \text{(i)}$$

Differentiating, we get

$$2x + 2(y - a)\,\frac{dy}{dx} = 0$$

$$\Rightarrow \quad y - a = -\frac{x}{\frac{dy}{dx}} \Rightarrow a = y + x\frac{dx}{dy}$$

Substituting this value in (i), we have

$$x^2 + x^2\left(\frac{dx}{dy}\right)^2 = \left(y + x\frac{dx}{dy}\right)^2$$

$$\Rightarrow x^2\left(1 + \left(\frac{dx}{dy}\right)^2\right) = y^2 + x^2\left(\frac{dx}{dy}\right)^2 + 2xy\,\frac{dx}{dy}$$

$\Rightarrow \quad x^2 - y^2 = 2xy\frac{dx}{dy}$. So $f(x) = 2x$

Example 47: The general solution of the differential equation $\cos 2x\left(\frac{dy}{dx} - \sqrt{\frac{1 + \tan x}{1 - \tan x}}\right) - y = 0$ is given by

(a) $y\sqrt{\tan\left(x + \frac{\pi}{4}\right)} = x + C$

(b) $y\sqrt{\cot\left(x + \frac{\pi}{4}\right)} = x + C$

(c) $y\sqrt{\tan\left(x + \frac{\pi}{4}\right)} = \tan x + C$

(d) $y\sqrt{\cot\left(x + \frac{\pi}{4}\right)} = \cot x + C$

Ans. (b)

Solution: The given equation can be written as

$$\frac{dy}{dx} - \sec 2x\,y = \sqrt{\tan\left(x + \frac{\pi}{4}\right)}$$

This is a linear equation whose I.F. = $e^{-\int \sec 2x\,dx}$

$$= e^{-\frac{1}{2}\log\tan\left(x + \frac{\pi}{4}\right)}$$

$$= \left(\tan\left(x + \frac{\pi}{4}\right)\right)^{-1/2}$$

Multiplying with I.F., we have

$$\frac{d}{dx}\left(y\,\frac{1}{\sqrt{\tan\left(x + \frac{\pi}{4}\right)}}\right) = 1$$

$$\Rightarrow \; y\sqrt{\cot\left(x + \frac{\pi}{4}\right)} = x + C$$

Example 48: The curve $y = f(x)$ ($f(x) \ge 0$, $f(0) = 0$) bounding a curvilinear trapezoid with the base $[0, x]$, whose area is proportional to the $(n + 1)$th power of $f(x)$ $f(1) = 1$ is given by

(a) $x = y^n$ (b) $y = x^n$

(c) $y + 1 = (x + 1)^n$ (d) $x = y^{n+1}$

Ans. (a)

Solution: According to the given condition

$\int_0^x f(x)\,dx = k(f(x))^{n+1}$, K being the constant of proportionality. Differentiating we have

$$f(x) = k(n + 1)\,(f(x))^n f'(x)$$

$$y = k(n + 1)\,y^n\,\frac{dy}{dx}$$

$$dx = k(n + 1)y^{n-1}\,dy$$

$$\Rightarrow \quad x + C = \frac{k(n+1)y^n}{n}$$

since $y(0) = 0$ so $C = 0$. Also $y(1) = 1$, so $k = \frac{n}{n+1}$. Thus $x = y^n$.

Assertion-Reason Type Questions

Example 49: Consider the differential equation

$$\frac{dy}{dx} = \frac{y}{2y\log y + y - x}.$$

Statement-1: $xy = y^2 \log y + C$ is a solution of the given differential equation

Statement-2: The differential equation is a linear equation in y and x.

Ans. (a)

Solution: $\dfrac{dx}{dy} = \dfrac{2y\log y + y - x}{y} = 2\log y + 1 - \dfrac{x}{y}$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{y} = 2 \log y + 1 \text{ which is linear in } x \text{ and } y.$$

$$\text{I.F.} = e^{\int \frac{1}{y} dy} = y.$$

So $\quad \dfrac{d}{dy}(xy) = 2\, y \log y + y$

$$\Rightarrow \quad xy = 2\int y \log y \, dy + \frac{y^2}{2} + C$$

$$= 2\left[\frac{y^2}{2}\log y - \frac{y^2}{4}\right] + \frac{y^2}{2} + C$$

$$= y^2 \log y + C.$$

Example 50: Statement-1: The differential equation of all circles in a plane must be of order 3.

Statement-2: There is only one circle passing through three non-collinear points.

Ans. (a)

Solution: The equation of a circle contains three independent constants if it passes through three non-collinear points.

Example 51: Statement-1: Curve satisfying the differential equation $y' = y/2x$ passing through (2, 1) is a parabola with focus (1/4, 0)

Statement-2: The differential equation $y' = y/2x$ is of variable separable.

Ans. (d)

Solution: $\dfrac{dy}{dx} = \dfrac{y}{2x} \quad \Rightarrow \quad \dfrac{2dy}{y} = \dfrac{dx}{x}$

$\Rightarrow \quad \log y^2 = \log x + \text{const} \quad \Rightarrow \quad y^2 = Cx$, this passes through (2, 1) if $C = 1/2$. Thus $y^2 = 1/2\,x$ which represents a parabola with focus (1/8, 0).

Example 52: Let $y' + \sin\dfrac{x+y}{2} = \sin\dfrac{x-y}{2}$

Statement-1: A solution satisfying $y(0) = \pi$ is a periodic function with period 4π.

Statement-1: y can be explicitly represented in terms of x.

Ans. (b)

Solution: $y' = \sin\dfrac{x-y}{2} - \sin\dfrac{x+y}{2} = -2\sin\dfrac{y}{2}\cos\dfrac{x}{2}.$

$\Rightarrow \quad \operatorname{cosec} y/2 \; dy = -2 \cos (x/2)\, dx$

$\Rightarrow \quad 2 \log |\tan (y/4)| = -4 \sin (x/2) + \text{Const}$

$\Rightarrow \quad \log |\tan (y/4)| = -2 \sin (x/2) + \text{Const}$

$\Rightarrow \quad |\tan (y/4)| = \text{Const}\; e^{-2 \sin (x/2)}$

Since $y(0) = \pi$ so Const = 1. Thus $|\tan (y/4)| = e^{-2 \sin x/2}$ so $y = 4 \tan^{-1} (\pm e^{-2 \sin x/2})$ which is periodic with period 4π.

Example 53: Let $(xy^2 + x)dx + (y - x^2y)dy = 0$ satisfy $y(0) = 0$.

Statement-1: The curve represented by the solution of the given differential equation is a circle.

Statement-2: It is circle with radius 1 and centre (0, 0).

Ans. (c)

Solution: $x(1 + y^2)dx + y(1 - x^2)dx = 0$

$$\Rightarrow \quad \frac{x}{1-x^2}\,dx + \frac{y}{1+y^2}\,dy = 0$$

$\Rightarrow \quad -\log (1 - x^2) + \log (1 + y^2) = \text{Const}$

$\Rightarrow \quad 1 + y^2 = C(1 - x^2)$

Since $y(0) = 0$ so $C = 1$

$\Rightarrow \quad x^2 + y^2 = 0$ which is a point circle.

Example 54: Let a solution $y = y(x)$ of the differential equation $x\sqrt{x^2-1}\; dy - y\sqrt{y^2-1}\; dx = 0$ satisfy $y(2) = 2\sqrt{3}$

Statement 1: $y(x) = \sec(\sec^{-1} x - \pi/6)$

Statement 2: $y(x)$ is given by $\dfrac{1}{y} = \dfrac{2\sqrt{3}}{x} - \sqrt{1 - \dfrac{1}{x^2}}$

Ans. (c)

Solution: The given equation can be written as

$$\frac{dy}{y\sqrt{y^2-1}} - \frac{dx}{x\sqrt{x^2-1}} = 0$$

Integrating $\sec^{-1}y - \sec^{-1}x = \text{Const}$

Putting $x = 2$, Const $= \sec^{-1} 2/\sqrt{3} - \sec^{-1} 2 = \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6}$

so $\quad y = \sec(\sec^{-1}x - \pi/6)$.

Example 55: Statement 1: The differential equation of all circles passing through one fixed point is of order 2.
Statement 2: General equation of a circle involves three arbitrary constants.
Ans. (a)

Solution: General equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + C = 0$$

where g, f, c are arbitrary constant. Since the circle passes through a fixed point so the arbitrary constants are two. Hence required differential equation is of order 2.

Example 56: Statement 1: The solution of $xy' + y = y^2 \log x$ is given by $y(1 + \log x + Cx) = 1$.
Statement 2: The given differential equation is a linear equation whose I.F. is x.
Ans. (c)

Solution: $xy' + y = y^2 \log x \Rightarrow \frac{1}{y^2} y' + \frac{1}{x}\frac{1}{y} = \frac{\log x}{x}$

This is reducible to linear equation but not a linear equation. Put $\frac{1}{y} = Z \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dZ}{dx}$, so

$$-\frac{dZ}{dx} + \frac{1}{x} Z = \frac{\log x}{x} \Rightarrow \frac{dZ}{dx} - \frac{1}{x} Z = -\frac{\log x}{x}$$

The I.F. is $e^{-\int \frac{1}{x} dx} = \frac{1}{x}$. Multiplying with I.F.

$$\frac{d}{dx}\left(Z \cdot \frac{1}{x}\right) = -\frac{\log x}{x^2}$$

Integrating, we have

$$Z\frac{1}{x} = -\int \frac{\log x}{x^2} dx + C$$

$$\Rightarrow \quad \frac{1}{y} \cdot \frac{1}{x} = -\left[-\frac{1}{x}\log x + \int \frac{1}{x^2} dx\right] + C$$

$$= -\left[-\frac{1}{x}\log x - \frac{1}{x}\right] + C$$

$$\Rightarrow \quad \frac{1}{y} = [1 + \log x] + Cx$$

$$\Rightarrow \quad 1 = y(1 + \log x + Cx)$$

LEVEL 2

Straight Objective Type Questions

Example 57: The degree of the differential equation satisfying

$$\sqrt{1+x^2} + \sqrt{1+y^2} = A\left(x\sqrt{1+y^2} - y\sqrt{1+x^2}\right) \text{ is}$$

(a) 2 (b) 3
(c) 4 (d) none of these

Ans. (d)

Solution: Put $x = \tan\theta$ and $y = \tan\phi$. Then $\sqrt{1+x^2}$ $= \sec\theta$, $\sqrt{1+y^2} = \sec\phi$, and the equation becomes

$$\sec\theta + \sec\phi = A\,(\tan\theta\sec\phi - \tan\phi\sec\theta)$$

$$\Rightarrow \quad \frac{\cos\phi + \cos\theta}{\cos\theta\cos\phi} = A\left(\frac{\sin\theta - \sin\phi}{\cos\theta\cos\phi}\right)$$

$$\Rightarrow \quad \cos\phi + \cos\theta = A\,(\sin\theta - \sin\phi)$$

$$\Rightarrow \quad 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = 2A\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}$$

$$\Rightarrow \quad \cot\frac{\theta-\phi}{2} = A \quad \Rightarrow \quad \theta - \phi = 2\cot^{-1}A$$

$$\Rightarrow \quad \tan^{-1}x - \tan^{-1}y = 2\cot^{-1}A.$$

Differentiating this, we get $\frac{1}{1+x^2} - \left(\frac{1}{1+y^2}\right)\frac{dy}{dx} = 0$,

which is a differential equation of degree 1.

Example 58: The solution of differential equation

$$\frac{dy}{dx} = \frac{y}{x} + 2\frac{\phi(y/x)}{\phi'(y/x)} \text{ is}$$

(a) $x^2\phi(y/x) = k$ (b) $y^2\phi(y/x) = k$
(c) $\phi(y/x) = kx^2$ (d) $\phi(y/x) = ky^2$

Ans. (c)

Solution: Putting $\frac{y}{x} = u$ we have $\frac{dy}{dx} = u + x\frac{du}{dx}$. The given differential equation can be written as $u + x\frac{du}{dx} = u$

$+ 2\dfrac{\phi(u)}{\phi'(u)}$

$$\Rightarrow \quad x\frac{du}{dx} = 2\frac{\phi(u)}{\phi'(u)} \quad \Rightarrow \quad \frac{\phi'(u)}{\phi(u)}\,du = 2\frac{dx}{x}$$

Integrating, we get $\log \phi(u) = \log x^2 + \log k$, so $\phi(u) = kx^2$ i.e. $\phi(y/x) = kx^2$, k being an arbitrary constant.

Example 59: The orthogonal trajectories of the family of curves $a^{n-1}y = x^n$ are given by

(a) $x^n + n^2y = \text{const}$ (b) $ny^2 + x^2 = \text{const}$
(c) $n^2x + y^n = \text{const}$ (d) $n^2x - y^n = \text{const}$

Ans. (b)

Solution: Differentiating, we have (see theory)

$$a^{n-1}\frac{dy}{dx} = nx^{n-1} \quad \Rightarrow \quad a^{n-1} = nx^{n-1}\frac{dx}{dy}$$

Putting this value in the given equation, we have

$$nx^{n-1}\frac{dx}{dy}y = x^n$$

Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$, we have $ny = -x\dfrac{dx}{dy}$

$$\Rightarrow \quad ny\,dy + x\,dx = 0 \quad \Rightarrow \quad ny^2 + x^2 = \text{const.}$$

which is the required family of orthogonal trajectories.

Example 60: The solution of the differential equation $\dfrac{d^2x}{dt^2} + x = 0;\ x(0) = 1,\ x'(0) = 0$

(a) approaches infinity as $t \to \infty$
(b) is a periodic function
(c) is always greater than or equal to unity
(d) does not exist

Ans. (b)

Solution: Consider the equation $m^2 + 1 = 0$, $m = \pm i$. Hence $x = C_1\cos t + C_2\sin t$ (see theory)

Now $x(0) = 1 \Rightarrow C_1 = 1$ and $x' = -C_1\sin t + C_2\cos t$ so $C_2 = 0$. Hence $x = \cos t$ is the required solution which is a periodic function.

Alternate Solution

We have $2\dfrac{dx}{dt}\dfrac{d^2x}{dt^2} = -2x\dfrac{dx}{dt} \quad \Rightarrow \quad \left(\dfrac{dx}{dt}\right)^2 = -x^2 + C$

Since $x'(0) = 0$ and $x(0) = 1$ so $C = 1$. Hence $\dfrac{dx}{dt} = \pm\sqrt{1-x^2}$

$$\Rightarrow \quad \frac{dx}{\sqrt{1-x^2}} = \pm t$$

$$\Rightarrow \quad \sin^{-1}x = t + C_1 \text{ or } \cos^{-1}x = t + C_2$$

$$\Rightarrow \quad x = \sin(t + C_1) \text{ or } x = \cos(t + C_2)$$

When $t = 0$, $x = 1$ so $C_1 = \pi/2$ and $C_2 = 0$

Hence $x = \cos t$.

Example 61: The solution of the equation $\dfrac{dy}{dx} = \cos(x - y)$ is

(a) $y + \cot\left(\dfrac{x-y}{2}\right) = c$ (b) $x + \cot\left(\dfrac{x-y}{2}\right) = c$
(c) $x + \tan\left(\dfrac{x-y}{2}\right) = c$ (d) none of these

Ans. (b)

Solution: Putting $u = x - y$, we have $\dfrac{du}{dx} = 1 - \dfrac{dy}{dx}$. The given equation can be written as $1 - \dfrac{du}{dx} = \cos u$

$$\Rightarrow \quad \int\frac{du}{1-\cos u} = dx$$

$$\Rightarrow \quad \frac{1}{2}\int \operatorname{cosec}^2\frac{u}{2}\,du = \int dx + \text{Const}$$

$$\Rightarrow \quad x + \cot\frac{u}{2} = \text{Const, i.e. } x + \cot\frac{x-y}{2} = c.$$

Example 62: A solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{xy[x^2\sin y^2 + 1]} \text{ is}$$

(C is an arbitrary constant)

(a) $x^2(\cos y^2 - \sin y^2 - 2Ce^{-y^2}) = 2$
(b) $y^2(\cos x^2 - (\sin y^2 - 2Ce^{-y^2}) = 2$
(c) $x^2(\cos y^2 - \sin y^2 - e^{-y^2}) = 4$
(d) none of these

Ans. (a)

Solution: The given differential equation can be written as $\dfrac{dx}{dy} = xy[x^2\sin y^2 + 1]$

$\Rightarrow \quad \dfrac{1}{x^3}\dfrac{dx}{dy} - \dfrac{1}{x^2}y = y\sin y^2$. This equation is reducible to linear equation, so putting $-1/x^2 = u$, the last equation can be written as

$$\frac{du}{dy} + 2uy = 2y\sin y^2$$

The integrating factor of this equation is e^{y^2}. So required solution is

$$ue^{y^2} = \int 2y\sin y^2.e^{y^2}\,dy + C$$

$$= \int(\sin t).e^t\,dt + C \qquad (t = y^2)$$

$$= (1/2)\,e^{y^2}(\sin y^2 - \cos y^2) + C$$

$$\Rightarrow \quad 2u = (\sin y^2 - \cos y^2) + Ce^{-y^2}$$

$$\Rightarrow \quad 2 = x^2[\cos y^2 - \sin y^2 - 2\,Ce^{-y^2}].$$

Example 63: The solution of $(y\,(1 + x^{-1}) + \sin y)\,\mathrm{d}x + (x + \log x + x\cos y)\,\mathrm{d}y = 0$ is

(a) $(1 + y^{-1}\sin y) + x^{-1}\log x = C$
(b) $(y + \sin y) + xy\log x = C$
(c) $xy + y\log x + x\sin y = C$
(d) none of these

Ans. (c)

Solution: The given equation can be written as

$$y(1 + x^{-1})\mathrm{d}x + (x + \log x)\mathrm{d}y + \sin y\,\mathrm{d}x + x\cos y\,\mathrm{d}y = 0$$

$$\Rightarrow \quad \mathrm{d}(y(x + \log x)) + \mathrm{d}(x\sin y) = 0$$

$$\Rightarrow \quad y(x + \log x) + x\sin y = C$$

Example 64: If $\phi(x)$ is a differentiable function then the solution of $\mathrm{d}y + (y\,\phi'(x) - \phi(x)\,\phi'(x))\,\mathrm{d}x = 0$ is

(a) $y = (\phi(x) - 1) + C\,e^{-\phi(x)}$
(b) $y\phi(x) = (\phi(x))^2 + C$
(c) $ye^{\phi(x)} = \phi(x)\,e^{\phi(x)} + C$
(d) $(y - \phi(x)) = (\phi(x))\,e^{-\phi(x)}$.

Ans. (a)

Solution: The given equation can be written in the linear form as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\phi'(x) = \phi(x)\,\phi'(x)$$

The integrating factor of this equation is $e^{\int\phi'(x)\,\mathrm{d}x} = e^{\phi(x)}$.

Hence $\qquad \dfrac{\mathrm{d}}{\mathrm{d}x}\,(ye^{\phi(x)}) = \phi(x)\,\phi'(x)\,e^{\phi(x)}$

Integrating, we have $ye^{\phi(x)} = \int te^t\,\mathrm{d}t + C$, (where $t = \phi(x)$)

$= te^t - e^t + C$ Hence $y = (\phi(x) - 1) + C\,e^{-\phi(x)}$.

Example 65: The solution of $y_2 - 2y_1 + y = 0$ is

(a) $y = x^2 e^x + c_1 xe^x + c_2$
(b) $y = (c_1 + c_2 x)\,e^x$
(c) $y = c_1 x^2 e^x + e^x + c_2$
(d) none of these

Ans. (b)

Solution: The given equation can be written as

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - 1\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - y\right) = 0 \qquad \text{(i)}$$

If $\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = u$ then (i) reduces to $\dfrac{\mathrm{d}u}{\mathrm{d}x} - u = 0 \;\Rightarrow\; u = c_1\,e^x$. Therefore, we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} - y = c_1 e^x$ which is a linear equation whose I.F. is e^{-x}. So $\dfrac{\mathrm{d}}{\mathrm{d}x}\,(ye^{-x}) = c_1 \;\Rightarrow\; ye^{-x} = c_1 x + c_2 \;\Rightarrow\; y = (c_1 x + c_2)e^x$. (For a short cut see theory).

Example 66: The solution of $y_2 - 7\,y_1 + 12y = 0$ is

(a) $y = C_1\,e^{3x} + C_2 e^{4x}$ (b) $y = C_1\,x\,e^{3x} + C_2\,e^{4x}$
(c) $y = C_1 e^{3x} + C_2\,x\,e^{4x}$ (d) none of these.

Ans. (a)

Solution: The given equation can be written as

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - 3\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - 4y\right) = 0 \qquad \text{(i)}$$

If $\dfrac{\mathrm{d}y}{\mathrm{d}x} - 4y = u$ then (i) reduces to $\dfrac{\mathrm{d}u}{\mathrm{d}x} - 3u = 0$

$\Rightarrow \quad \dfrac{\mathrm{d}u}{u} = 3\mathrm{d}x \;\Rightarrow\; u = C_1 e^{3x}$. Therefore, we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} - 4y = C_1 e^{3x}$ which is a linear equation whose I.F. is e^{-4x}. So $\dfrac{\mathrm{d}}{\mathrm{d}x}\,(ye^{-4x}) = C_1\,e^{-x} \;\Rightarrow\; ye^{-4x} = -\,C_1\,e^{-x} + C_2$

$\Rightarrow y = C_1 e^{3x} + C_2 e^{4x}$ (For a short cut see theory).

Example 67: A solution of $y = 2x\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) + x^2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^4$ is

(a) $y = 2\,c^{1/2}\,x^{1/4} + c$ (b) $y = 2\sqrt{c}\;x^2 + c^2$
(c) $y = 2\sqrt{c}\;(x + 1)$ (d) $y = 2\sqrt{cx} + c^2$

Ans. (d)

Solution: Writing $p = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ and differentiating w.r.t. x, we have

$$p = 2p + 2x\,\frac{\mathrm{d}p}{\mathrm{d}x} + 2x\,p^4 + 4p^3\,x^2\,\frac{\mathrm{d}p}{\mathrm{d}x}$$

$$\Rightarrow \quad 0 = p(1 + 2x\,p^3) + 2x\,\frac{\mathrm{d}p}{\mathrm{d}x}\,(1 + 2p^3\,x)$$

$$\Rightarrow \quad p + 2x\,\frac{\mathrm{d}p}{\mathrm{d}x} = 0 \;\Rightarrow\; 2\,\frac{\mathrm{d}p}{p} = -\frac{\mathrm{d}x}{x}$$

$$\Rightarrow \quad 2\log p + \log x = \text{const} \;\Rightarrow\; p^2 x = c \text{ or } p = \sqrt{\frac{c}{x}}$$

Substituting this value in the given equation, we get

$$y = 2\sqrt{cx} + c^2.$$

Example 68: The equation of the curve not passing through origin and having the portion of the tangent included between the coordinate axes is bisected at the point of contact is

(a) a parabola
(b) an ellipse or a straight line
(c) a circle or an ellipse
(d) a hyperbola

Ans. (d)

◎ **Solution**: The equation of tangent at any point $P(x, y)$ is $Y - y = \frac{dy}{dx}(X - x)$

This intersects cut the X-axis at $A\left(x - y\frac{dx}{dy}, 0\right)$ and the Y-axis at $B\left(0, y - x\frac{dy}{dx}\right)$. According to the given condition

mid point of $AB = (x, y)$

$$\Rightarrow \quad x - y\frac{dx}{dy} = 2x \quad \text{and} \quad y - x\frac{dy}{dx} = 2y$$

$$\Rightarrow \quad x + y\frac{dx}{dy} = 0 \quad \text{and} \quad y + x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} = 0$$

$$\Rightarrow \quad \log(xy) = \log c$$

$\Rightarrow \quad xy = c$, which is a hyperbola.

◉ **Example 69:** An equation of the curve for which the portion of y-axis cut off between the origin and the tangent varies as the cube of the abscissa of the point of contact is

(a) $y = Kx^3/3 + Cx$ (b) $y = -Kx^2/2 + C$

(c) $y = -Kx^3/2 + Cx$ (d) $y = Kx^3/3 + Cx^2/2$

(K is constant of proportionality)

Ans. (c)

◎ **Solution**: The portion of y-axis cut off between the origin and the tangent is $y - x\frac{dy}{dx}$. According to the given condition $y - x\frac{dy}{dx} = Kx^3$ (K is constant of proportionality)

$\Rightarrow \quad \frac{dy}{dx} - \frac{1}{x}y = -Kx^2$. This is a linear equation whose I.F. is $1/x$.

Hence $\frac{d}{dx}(y/x) = -Kx \quad \Rightarrow \quad y = -Kx^3/2 + Cx.$

◉ **Example 70:** Through any point (x, y) of a curve which passes through the origin, lines are drawn parallel to the co-ordiante axes. The curve, given that it divides the rectangle formed by the two lines and the axes into two areas, one of which is twice the other, represents a family of

(a) circles (b) parabolas

(c) hyperbolas (d) straight lines

Ans. (d)

◎ **Solution**: Let $P(x, y)$ be the point on the curve passing through the origin $O(0, 0)$, and let PN and PM be the lines parallel to the x- and y-axes, respectively (Fig. 15.1). If the equation of the curve is $y = y(x)$, the area $POM = \int_0^x y\,dx$ and the area $PON = xy - \int_0^x y\,dx$ Assuming that 2 (POM) = PON, we therefore have

$$2\int_0^x y\,dx = xy - \int_0^x y\,dx \quad \Rightarrow \quad 3\int_0^x y\,dx = xy.$$

Differentiating both sides of this gives

$$3y = x\frac{dy}{dx} + y \quad \Rightarrow \quad 2y = x\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{y} = 2\frac{dx}{x}$$

$\Rightarrow \quad \log|y| = 2\log|x| + C$

$\Rightarrow \quad y = Cx^2$, with C being a constant.

This solution represents a parabola. We will get a similar result if we had started instead with 2(PON) = POM

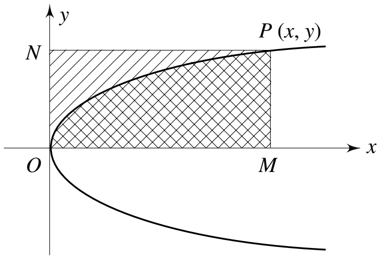


Fig. 15.1

◉ **Example 71:** The solution of $(y + x + 5)dy = (y - x + 1)\,dx$ is

(a) $\log((y+3)^2 + (x+2)^2) + \tan^{-1}\frac{y+3}{x+2} = C$

(b) $\log((y+3)^2 + (x-2)^2) + \tan^{-1}\frac{y-3}{x-2} = C$

(c) $\log((y+3)^2 + (x+2)^2) + 2\tan^{-1}\frac{y+3}{x+2} = C$

(d) $\log((y+3)^2 + (x+2)^2) - 2\tan^{-1}\frac{y+3}{x+2} = C$

Ans. (c)

◎ **Solution**: The intersection of $y - x + 1 = 0$ and $y + x + 5 = 0$ is $(-2, -3)$. Put $x = X - 2$, $y = Y - 3$. The given equation reduces to $\frac{dY}{dX} = \frac{Y - X}{Y + X}$. This is a homogeneous equation, so putting $Y = vX$, we get

$$X\frac{dv}{dX} = -\frac{v^2+1}{v+1} \quad \Rightarrow \quad \left(-\frac{v}{v^2+1} - \frac{1}{v^2+1}\right)dv = \frac{dX}{X}$$

$$\Rightarrow \quad -\frac{1}{2}\log(v^2+1) - \tan^{-1}v = \log|X| + \text{Const}$$

$$\Rightarrow \quad \log(Y^2 + X^2) + 2\tan^{-1}\frac{Y}{X} = \text{Const}$$

$$\Rightarrow \quad \log((y+3)^2 + (x+2)^2) + 2\tan^{-1}\frac{y+3}{x+2} = C.$$

◉ **Example 72:** General solution of the differential equation $y = x\frac{dy}{dx} + \frac{dx}{dy}$ represent

(a) a straight line or a hyperbola
(b) a straight line or a parabola
(c) a parabola or a hyperbola
(d) circles

Ans. (b)

◎ **Solution**: Putting $p = \frac{dy}{dx}$, the given equation can be written as $y = px + 1/p$. Differentiating w.r.t. x, we have

$$p = \frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \qquad (x-(1/p^2))\,\frac{dp}{dx} = 0 \Rightarrow p^2 = \frac{1}{x} \text{ or } \frac{dp}{dx} = 0$$

If $\frac{dp}{dx} = 0$ then p = constant = c putting this value in given equation, we get $y = cx + 1/c$ which represents a straight line. If $p^2 = 1/x$ then $y^2 = (px + 1/p)^2 = p^2x^2 + 1/p^2 + 2x = \frac{1}{x}x^2 + x + 2x = 4x$, which represents a parabola.

◉ **Example 73:** The curves satisfying the differential equation $(1 - x^2)\,y' + xy = ax$ are

(a) ellipses and hyperbolas
(b) ellipses and parabola
(c) ellipses and straight lines
(d) circles and ellipses

Ans. (a)

◎ **Solution**: The given equation is linear in y and can be written as

$$\frac{dy}{dx} + \frac{x}{1-x^2}y = \frac{ax}{1-x^2}$$

Its integrating factor is $e^{\int \frac{x}{1-x^2}dx} = e^{-(1/2)\log\left(1-x^2\right)}$

$= \frac{1}{\sqrt{1-x^2}}$ if $-1 < x < 1$ and if $x^2 > 1$ then I.F. $= \frac{1}{\sqrt{x^2-1}}$

$$\frac{d}{dx}\left(y\frac{1}{\sqrt{1-x^2}}\right) = \frac{ax}{\left(1-x^2\right)^{3/2}} = -\frac{1}{2}a\frac{-2x}{\left(1-x^2\right)^{3/2}}$$

$$\Rightarrow \qquad y\frac{1}{\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + C \Rightarrow y = a + C\sqrt{1-x^2}$$

$$\Rightarrow \qquad (y-a)^2 = C^2(1-x^2)$$

$$\Rightarrow \qquad (y-a)^2 + C^2x^2 = C^2$$

Thus if $-1 < x < 1$ the given equation represents an ellipse. If $x^2 > 1$ then the solution is of the form $-(y-a)^2 + C^2x^2 = C^2$ which represents a hyperbola.

◉ **Example 74:** Consider the differential equation $y^2\,dx + \left(x - \frac{1}{y}\right)dy = 0$. If $y\,(1) = 1$, then x is given by:

(a) $4 - \frac{2}{y} - \frac{e^{1/y}}{e}$ (b) $3 - \frac{1}{y} + \frac{e^{1/y}}{e}$
(c) $1 + \frac{1}{y} - \frac{e^{1/y}}{e}$ (d) $1 - \frac{1}{y} + \frac{e^{1/y}}{e}$

Ans. (c)

◎ **Solution**: The given differential equation can be written as $\frac{dx}{dy} + \frac{1}{y^2}x = \frac{1}{y^3}$ which is a linear equation with I.F. = $e^{\int\left(1/y^2\right)dy} = e^{-1/y}$. Multiplying with the integrating factor

$$\frac{d}{dx}\left(x\,e^{-1/y}\right) = \frac{1}{y^3}e^{-1/y}$$

$$\Rightarrow \qquad x\,e^{-1/y} = \int \frac{1}{y^3}e^{-1/y}dy + C$$

$$= \int -u\,e^u du + C \quad (u = -1/y)$$

$$= -u\,e^u + \int e^u\,du + C$$

$$= e^{-1/y}\,(1 + 1/y) + C$$

Thus, $\qquad x = 1 + \frac{1}{y} + C\,e^{1/y}$

When $\qquad x = 1, y = 1$

$1 = 1 + 1 + C\,e = \Rightarrow C = -1/e$.

∴ $\qquad x = 1 + \frac{1}{y} - \frac{e^{1/y}}{e}$.

EXERCISE

Concept-based
Straight Objective Type Questions

1. The general solution of $y' + xy = 4x$ is given by

(a) $y = e^{(-1/2)x^2} + Cx$ (b) $y = e^{(1/2)x^2} + Cx$
(c) $y = Ce^{(-1/2)x^2} - 4$ (d) $y = e^{(1/2)x^2} + Cx + 4$

2. The solution of $x' + x\tan t = \sec t$, $x(0) = 1$ is

(a) $x = \cot t + \sin t$ (b) $x = \cos t + 2\sin t$
(c) $x = \cos t + \sin^2 t$ (d) $x = \cos^2 t + \sin t$

3. A particular solution of the initial value differential equation

$\log \dfrac{dy}{dx} = 7x + 8y$, $y(0) = 0$ is

(a) $15\,y = 8e^{-8y} + 7e^{-7y}$
(b) $15x = 7e^{-8y} + 8e^{7x}$
(c) $8e^{-8y} + 7e^{7x} = 15$
(d) $7e^{-8y} + 8e^{7x} = 15$

4. If $x(t)$ is a solution of $(t + 1)dx = (2x + (t + 1)^4)\,dt$, $x(0) = 2$ then $\lim\limits_{t\to 1} x(t)$ is

(a) 8	(b) 10
(c) 14	(d) 12

5. The degree of the differential equation $\dfrac{dy}{dx} + x = \left(y + x\dfrac{dy}{dx}\right)^{-4}$ is

(a) 2	(b) 3
(c) 4	(d) 5

6. The solution $y' = \dfrac{1 - 2y - 4x}{1 + y + 2x}$ is given by

(a) $(y + 2x) + \dfrac{1}{2}(y + 2x)^2 = 2x + C$
(b) $(y + \dfrac{1}{2}(y + 2x)^2) = x + C$
(c) $x + \dfrac{1}{2}(2x + y)^2 = y + C$
(d) $2x + \dfrac{1}{2}(2x + y)^2 = y + C$

7. The solution of $\left(x - y\cos\dfrac{y}{x}\right)dx + x\cos\dfrac{y}{x}\,dy = 0$, $y(1) = 0$ is given by

(a) $\sin\dfrac{y}{x} + \log\lvert x\rvert = 0$	(b) $y\cos\dfrac{y}{x} + \log\lvert x\rvert = 0$
(c) $\tan\dfrac{y}{x} + \log\lvert x\rvert = 0$	(d) $y\sin\dfrac{y}{x} + \log\lvert x\rvert = 0$

8. The solution $y^{n-1}\,(ay' + y) = x$ is

(a) $(n - 1)y^{n-1} = Ce^{-\frac{nx}{a}} + nx + a$
(b) $ny^{n-1} = Ce^{-\frac{nx}{a}} + nx$
(c) $ny^{n} = Ce^{-\frac{nx}{a}} + nx - a$
(d) $ny^{n-1} = Ce^{-\frac{nx}{a}} + nx + a$

9. A curve passing through origin, all the normals to which pass through (1, 2) is
(a) a circle with centre origin
(b) a straight line
(c) a parabola
(d) a circle with centre (1, 2)

10. The solution of $y' = (x + y)^2$, $y(0) = 0$ is given by

(a) $\tan x = x + y$	(b) $\sin^2 y = x + y$
(c) $\tan y = x + y$	(d) $\sin y = x + y$

LEVEL 1

Straight Objective Type Questions

11. $y = ae^{-1/x} + b$ is a solution of $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y}{x^2}$ when

(a) $a = 1, b = 0$	(b) $a = 3, b = 0$
(c) $a = 1, b = 1$	(d) $a = 2, b = 2$

12. The solution of $y - xy' = 2(x + yy')$, $y(1) = 1$ is
(a) $\tan^{-1}(y/x) + \log(x^2 + y^2) = \log 2$
(b) $\tan^{-1}(y/x) + \log(x^2 + y^2) = \pi/4 + \log 2$
(c) $\tan^{-1}(y/x) + \log(x^2 + y^2) = \log 3$
(d) none of these

13. The solution of $\dfrac{x^3\,\mathrm{d}x + yx^2\,\mathrm{d}y}{\sqrt{x^2 + y^2}} = y\mathrm{d}x - x\mathrm{d}y$ is

(a) $\sqrt{x^2 + y^2} = Cx$
(b) $\sqrt{x^2 + y^2} + y/x = C$
(c) $\sqrt{x^2 + y^2} + y/x^2 = C$
(d) $(x^2 + y^2)^2 + xy^2 = C$

14. Let $f(x)$, $g(x)$ be twice differentiable function on [0, 2] satisfying $f''(x) = g''(x)$, $f'(1) = 2g'(1) = 4$ and $f(2) = 3$, $g(2) = 9$, then the value of $f(4) - g(4)$ is

(a) 0	(b) 2
(c) 8	(d) – 2

15. Which of the following differential equation is not degree 1

(a) $x^3\,y_2 + (x + x^2)\,y_1^{\,2} + e^x\,y^3 = \sin x$
(b) $y_2^{1/2} + \sin xy_1 + xy = x$
(c) $\sqrt{y_1 + y} = x + 1$
(d) $y = 2y_1 + \sqrt{y_1 + y}$

16. The general solution of

$$y' = \frac{y}{x} + \frac{x}{y}\,\frac{\phi\left(y^2/x^2\right)}{\phi'\left(y^2/x^2\right)} \text{ is}$$

(a) $y^2 = K\,\phi\,(y^2/x^2)$ (b) $x^2 = K\,\phi(y^2/x^2)$
(c) $x = K\,\phi(y^2/x^2)$ (d) $y = Kx\,\phi(y^2/x^2)$

17. The solution of $\log \dfrac{dy}{dx} = 3x + 4y$, $y(0) = 0$ is

(a) $e^{3x} + {}^{-4y} = 4$ (b) $4e^{3x} - e^{-4y} = 3$
(c) $3e^{3x} + 4e^{4y} = 7$ (d) $4e^{3x} + 3e^{-3x} = 7$

18. The solution of the equation $(x + y)\,\mathrm{d}y - (x - y)\,\mathrm{d}x = 0$ is

(a) $y^2 + 2xy + x^2 = K$ (b) $y^2 + 2xy - x^2 = K$
(c) $x^3 + 2xy - y^2 = K$ (d) $y^2 - 2xy + x^2 = K$

19. The solution of $(1 + y + x^2y)\,\mathrm{d}x + (x + x^3)\,\mathrm{d}y = 0$ is

(a) $y + \tan^{-1} x = C$ (b) $xy + \tan^{-1} x = C$
(c) $y^2 + \tan^{-1} x = C$ (d) $x^2 + \tan^{-1} y/x = C$

20. The equation of curves which intersect the hyperbola $xy = 4$ at an angle $\pi/2$ is

(a) $y = \dfrac{x^2}{\sqrt{2}} + C$ (b) $y^2 = \dfrac{x^3}{3} + C$
(c) $y^2 - x^2 = C$ (d) $xy = x^2 + C$

21. The differential equation of a curve such that the initial ordinate of any tangent at the point of contact is equal to the corresponding subnormal is

(a) a linear equation
(b) not a homogeneous equation
(c) an equation with separable variables
(d) none of these

22. The curve satisfying $y\,\mathrm{d}x - x\,\mathrm{d}y + \log x\,\mathrm{d}x = 0$ $(x > 0)$ passing through $(1, -1)$ is

(a) $y + \log x + 1 = 0$ (b) $-y^2 + \log x + 1 = 0$
(c) $y^3 + (\log x)^2 + 1 = 0$ (d) none of these

23. A differential equation associated with the primitive

$$y = a + be^{5x} + ce^{-7x} \text{ is}$$

(a) $y_3 + 2y_2 - y_1 = 0$
(b) $4y_3 + 5y_2 - 20y_1 = 0$
(c) $y_3 + 2y_2 - 35y_1 = 0$
(d) none of these

24. The differential equation of the family of circles passing through the fixed points $(a, 0)$ and $(-a, 0)$ is

(a) $y_1\,(y^2 - x^2) + 2xy + a^2 = 0$
(b) $y_1\,y^2 + xy + a^2x^2 = 0$
(c) $y_1(y^2 - x^2 + a^2) + 2xy = 0$
(d) none of these

25. Solution of the differential equation $(x - y)^2 \dfrac{\mathrm{d}y}{\mathrm{d}x} = a^2$ is

(a) $y = \dfrac{a}{2}\log\left|\dfrac{x - y - a}{x - y + a}\right| + c$

(b) $x = \dfrac{a}{2}\log\left|\dfrac{x - y + a}{x - y - a}\right| + c$

(c) $y^2 = a\log\left|\dfrac{x - y + a}{x - y - a}\right| + c$

(d) none of these

26. Solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sin(x + y) + \cos(x + y)$ is

(a) $\log\left|1 + \tan\left(\dfrac{x + y}{2}\right)\right| = y + c$

(b) $\log\left|2 + \sec\left(\dfrac{x + y}{2}\right)\right| = x + c$

(c) $\log|1 + \tan(x + y)| = y + c$
(d) none of these

27. Equation of the curve through the origin satisfying $\mathrm{d}y = (\sec x + y\tan x)\,\mathrm{d}x$ is

(a) $y\sin x = x$ (b) $y\cos x = x$
(c) $y\tan x = x$ (d) none of these

28. The solution of $(x\,\mathrm{d}x + y\,\mathrm{d}y)(x^2 + y^2) + (x\mathrm{d}y - y\mathrm{d}x) = 0$ is

(a) $x^2 + y^2 + \dfrac{y}{x} = C$

(b) $x^2 + y^2 + 2\tan^{-1}\left(\dfrac{y}{x}\right) = C$

(c) $\dfrac{x^2 + y}{2} + \tan^{-1}\left(\dfrac{x}{y}\right) = C$

(d) $\dfrac{x^2 + y^2}{2} + \tan^{-1}\left(\dfrac{x}{y}\right) = C$

29. General solutions of $y' = e^{y/x} + y/x$ is

(a) $Cx = e^{-y/x}$ (b) $\log|Cx| = -e^{-y/x}$
(c) $\log|Cx| = e^{y/x}$ (d) $\log|Cx| = e^{-y/x}$

30. The general solution of $y' + \sin\dfrac{x + y}{2} = \sin\dfrac{x - y}{2}$ is

(a) $\log|\tan(y/2)| = \mathrm{C} + \sin x/2$
(b) $\log|\tan(y/2)| = C - 2\sin x$
(c) $\log|\tan(y/4)| = C - 2\sin x/2$
(d) none of these

31. The order and the degree (respectively) of the differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, $c > 0$ is a parameter are

(a) 3, 1 (b) 1, 3
(c) 2, 4 (d) 4, 1

32. Let f be a real-valued on **R** such that $f(1) = 1$. If the y-intercept of the tangent at any point {(x, y) on the curve $y = f(x)$ is equal to cube of the abscissa of P, then the value of $f(-3)$ is equal to
(a) 3 (b) 8
(c) 12 (d) 9

33. The curves given by $2xyy' = y^2 - x^2$ represent
(a) family of circles with centre on y-axis
(b) family of parabolas passing through origin
(c) family of circles with centre on x-axis
(d) family of hyperbola

34. Solution to $9yy' + 4x = 0$ represent
(a) family of circles
(b) family of ellipses
(c) family of straight lines
(d) family of hyperbolas

35. The solution of initial value problem $y' = -2xy$, $y(0) = 1$ represents
(a) a circle with centre on x-axis
(b) an ellipse
(c) bell-shaped curve
(d) a circle with centre on y-axis

Assertion-Reason Type Questions

36. Let $y' = \cos(x - y)$ such that $y(0) = -\pi$ then
Statement-1: y can be expressed explicitly in terms of x
Statement-2: $x + \sec(x - y) = -1$

37. Let $y = f(x)$ be a curve having the following property: The segment of the tangent between the point of tangency and the x-axis is bisected at the point of intersection with the y-axis.
Statement-1: Such curves represent hyperbola
Statement-2: Curves have latus rectum parallel to y-axis.

38. Let $y' = 3x - 2y + 5$
Statement-1: The solution of the above equation is $4y - 6x - 7 = Ce^{-2x}$
Statement-2: The given equation is linear in y and x with I.F. e^{2x}

39. Let a solution $y = y(x)$ of the differential equation $y \sin x + y' \cos x = 1$ satisfy $y(0) = 1$
Statement-1: $y(x) = \sin(x + \pi/4)$
Statement-2: The integrating factor of the given differentiable equation is $\sec x$.

40. Let $xy' + y - e^x = 0$, $y(a) = b$
Statement-1: The solution is given by $yx = e^x + ab - e^a$
Statement-2: The given equation is a linear function with I.F. x.

LEVEL 2

Straight Objective Type Questions

41. A curve such that the area of the trapezoid formed by coordinate axes, ordinate at an arbitrary point and the tangent at this point equals half the square of its abscissa is
(a) $y = Cx^2$ (b) $y = (1/2)x + Cx^2$
(c) $3y = x + Cx^2$ (d) $y = x + Cx^2$

42. A curve passing through origin, all the normals to which pass through (x_0, y_0) is
(a) $yy_0 = x^2 + y^2$ (b) $x^2 + y^2 = 2(xx_0 + yy_0)$
(c) $x^2 + y^2 = x_0^2 + y_0^2$ (d) none of these

43. The trajectories orthogonal to $(2a - x)y^2 = x^3$ is
(a) $x^2 + y^2 = C(y^2 + 2x^2)$
(b) $(x^2 + y^2)^2 = Cxy$
(c) $(x^2 + y^2)^2 = C(y^2 + 2x^2)$
(d) $(x^2 + y^2) = x^3 + xy^2$

44. The general solution of $xy_5 = y_4$ is given by
(a) $y = C_1 x^5 + C_2 x^3 + C_3 x^2 + C_4 x + C_5$
(b) $y = C_1 x^5 + C_2 x^4 + C_3 x^2 + C_4 x + C_5$
(c) $y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4$
(d) none of these
(C_1, C_2, C_3, C_4, C_5 being arbitrary constants)

45. The general solution of $yy'' = (y')^2$ is
(a) $y = C_1 x + C_2$ (b) $y = C_2 e^{C_1 x}$
(c) $y = C_2 + e^{C_1 x}$ (d) $y = e^{C_2 x} + e^{C_1 x}$

46. The curve such that the ratio of the subnormal at any point to the sum of its abscissa and ordinate is equal to the ratio of the ordinate of this point to its abscissa is

(a) $y = \log |Cx|$ (b) $y = x^2 + Cx$
(c) $y = x \log C(x^2 + y^2)$ (d) $y = x \log |Cx|$

47. The curve $y = f(x)$ ($f(x) \geq 0$, $f(0)$, $= 0$, $f(1) = 1$) bounding a curvilinear trapezoid with the base $[0, x]$, whose area is proportional to the $(n + 1)$th power of $f(x)$ is

(a) $y' = x$ (b) $y^{n+1} = x$
(c) $x^n = y$ (d) $x^{n+1} = y$

48. The curve satisfying the equation

$$y_1 = \frac{y^2 - 2xy - x^2}{y^2 + 2xy - x^2}$$

passing through $(1, -1)$ is a

(a) straight line (b) circle
(c) ellipse (d) none of these

49. The curve whose subtangent is n times the abscissa of the point of contact and passes through (2, 3) represent

(a) a straight line for $n = 2$
(b) a parabola with vertex (2, 3) for $n = 1$
(c) a circle with centre (2, 3) for $n = 2$
(d) a parabola with vertex origin and axes coincide with x-axis for $n = 2$

50. Solution of $y\left(\dfrac{dy}{dx}\right)^2 + 2x\dfrac{dy}{dx} = y$ is

(a) $y = 2Cx + C^2$ (b) $x^2 = 2Cy + C^2$
(c) $y^2 = 2Cx + C^2$ (d) $xy = Cx^2 + C^2$

51. If the area of the figure bounded by a curve, the x-axis, and two ordinates, one of which is constant, the other variable is equal to the ratio of the cube of the variable ordinate to the variable abscissa, then the curve is

(a) $(2y^2 - x^2)^3 = Cx^2$ (b) $(y^2 - x^2)^3 = Cx^2$
(c) $(2y - x^2)^2 = Cx^3$ (d) $(2y + x^2)^2 = Cx^3$

52. The curve such that the initial ordinate of any tangent is less than the abscissa of the point of tangency by two unit is

(a) $y = Cx + \log |x| - 2$
(b) $y = Cx - \dfrac{x^2}{2} \log |x| - 2$
(c) $y = Cx^2 + \log |x| - 2$
(d) $y = Cx - x \log |x| - 2$

53. The solution of $y'(y - x - 4) = x + y - 2$ is given by

(a) $x^2 + 2xy + y^2 - 4x + 16y = C$
(b) $x^2 + 2xy - y^2 - 4x + 8y = C$
(c) $x^2 - 2xy + y^2 - 2x + 4y = C$
(d) $x^2 - 2xy + y^2 - 4x + 8y = C$

54. A solution of $y = xy' - 3y'^3$ is

(a) $y = 2x - 3$ (b) $y = 3x - 9$
(c) $y = x - 3$ (d) $y = 2x - 6$

55. The trajectories orthogonal to $x^2 + y^2 = 2ax$ is

(a) $y = x^2$ (b) $y = C(x^2 + y^2)$
(c) $y = C(x^2 + 2y^2)$ (d) $y^2 = Cx$

56. The solution of $y'' - 4y' + 3y = 0$, $y(0) = 6$ and $y'(0) = 10$ is

(a) $y = 4e^x + 2e^{3x}$ (b) $y = e^x + 6e^{3x}$
(c) $y = (2x + 6)e^x$ (d) $y = 2e^x + 4e^{3x}$

57. The order of the differential equation whose general solution is given by $y = (C_1 + C_2) \cos (x + C_3) - C_4 e^{x+C_5}$, where C_1, C_2, C_3, C_4, C_5 are arbitrary constant is

(a) 5 (b) 4
(c) 3 (d) 2

58. The solution of the equation $\dfrac{dy}{dx} = \sec (x + y)$ is given by

(a) $y - \tan (x + y) = C$
(b) $y - \dfrac{1}{2} \tan (x + y) = C$
(c) $y - \tan \dfrac{1}{2} (x + y) = C$
(d) $y + \dfrac{1}{2} \tan (x + y) = C$

59. The differential equation corresponding to $y = C_1e^{m_1x} + C_2e^{m_2x} + C_3e^{m_3x}$, where C_i are arbitrary constants and m_1, m_2, m_3 are roots of the equation $m^3 - 7m + 6 = 0$ is

(a) $y_3 - 7y_1 + 6y = 0$
(b) $y_3 - 7y_2 + 6y_1 + y = 0$
(c) $y_3 - 6y_2 + 7y_1 + y = 0$
(d) $y_3 + 7y_2 - 6y = 0$

60. The solution of the differential equation

$$\frac{d^2y}{dx^2} = x + e^{3x} \text{ when } y_1(0) = \frac{1}{3} \text{ and } y(0) = 0 \text{ is}$$

(a) $y = \dfrac{x^3}{6} + \dfrac{e^{3x}}{9} + \dfrac{1}{3}x - \dfrac{1}{9}$
(b) $y = \dfrac{1}{18}(3x^3 + 2e^{3x} - 2)$
(c) $y = \dfrac{x^3}{6} + \dfrac{e^{3x}}{9} + \dfrac{2x}{9} + \dfrac{1}{3}$
(d) none of these

61. The solution of the differential equation $\dfrac{d^2y}{dx^2} = e^x \sin x$ when $y_1\left(\dfrac{\pi}{4}\right) = 0,\ y\left(\dfrac{\pi}{2}\right) = 0$ is

(a) $y = e^x \sin(x - \pi/4) + x + \dfrac{\pi}{2}$

(b) $y = (1/2)\, e^x \sin(x + \pi/2)$

(c) $y = (1/\sqrt{2})\, e^x \sin(x - \pi/2)$

(d) $y = (1/2)\, e^x \sin(x - \pi/2)$

Previous Years' AIEEE/JEE Main Questions

1. The solution of the differential equations

$$(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0 \text{ is}$$

(a) $2x\, e^{\tan^{-1} y} = e^{2\tan^{-1} y} + k$

(b) $x\, e^{\tan^{-1} y} = \tan^{-1} y + k$

(c) $x\, e^{2\tan^{-1} y} = e^{\tan^{-1} y} + k$

(d) $(x - 2) = k\, e^{-\tan^{-1} y}$ **[2003]**

2. The differential equation for the family of curves $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant is

(a) $(x^2 - y^2)y' = 2xy$ (b) $2(x^2 + y^2)y' = xy$

(c) $2(x^2 - y^2)y' = xy$ (d) $(x^2 + y^2)y' = 2xy$

[2004]

3. The solution of the differential equation $y\,dx + (x + x^2y)\,dy = 0$ is

(a) $\dfrac{1}{xy} + \log y = C$ (b) $-\dfrac{1}{xy} + \log y = C$

(c) $-\dfrac{1}{xy} = C$ (d) $\log y = Cx$ **[2004]**

4. If $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is

(a) $\log\left(\dfrac{y}{x}\right) = Cx$ (b) $\log\left(\dfrac{x}{y}\right) = Cy$

(c) $y\log\left(\dfrac{x}{y}\right) = Cx$ (d) $x\log\left(\dfrac{y}{x}\right) = Cy$ **[2005]**

5. The differential equation whose solution is $Ax^2 + By^2 = 1$, where A and B are arbitrary constants is of

(a) second order and first degree

(b) second order and second degree

(c) first order and second degree

(d) first order and first degree **[2006]**

6. The differential equation of all circles passing through the origin and having their centres on the x-axis is

(a) $x^2 = y^2 + xy\dfrac{dy}{dx}$ (b) $x^2 = y^2 + 3xy\dfrac{dy}{dx}$

(c) $y^2 = x^2 + 2xy\dfrac{dy}{dx}$ (d) $y^2 = x^2 - 2xy\dfrac{dy}{dx}$

[2007]

7. The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$ is

(a) $(x - 2)y'^2 = 25 - (y - 2)^2$

(b) $(y - 2)y'^2 = 25 - (y - 2)^2$

(c) $(y - 2)^2y'^2 = 25 - (y - 2)^2$

(d) $(x - y)^2\, y'^2 = 25 - (y - 2)^2$ **[2008]**

8. The differential equation which represents the family of curves $y = c_1e^{c_2x}$, where c_1 and c_2 are arbitrary constants is

(a) $yy'' = y'$ (b) $yy'' = y'^2$

(c) $y'' = y^2$ (d) $y'' = y'y$ **[2009]**

9. Solution of the differential equation

$\cos x\, dy = y(\sin x - y)dx, \quad 0 < x < \pi/2$ is

(a) $y\tan x = \sec x + C$ (b) $\tan x = (\sec x + C)y$

(c) $\sec x = (\tan x + C)y$ (d) $y\sec x = \tan x + C$

[2010]

10. If $\dfrac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, $y(\log 2)$ is equal to:

(a) –2 (b) 7

(c) 5 (d) 13 **[2011]**

11. Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t years. The value $V(t)$ depreciates at a rate given by differential equation $\dfrac{dV(t)}{dt} = -k(T - t)$, where $k > 0$ is a constant and T is the total life in years of the

equipment. Then the scrap value $V(T)$ of the equipment is:

(a) e^{-kT} (b) $T^2 - \frac{I}{K}$

(c) $I - \frac{kT^2}{2}$ (d) $I - \frac{K(T-t)^2}{2}$

[2011]

12. Consider the differential equation $y^2\, dx + \left(x - \frac{1}{y}\right) dy = 0$. If $y(1) = 1$, then x is given by:

(a) $4 - \frac{2}{y} - \frac{e^{1/y}}{e}$ (b) $3 - \frac{1}{y} + \frac{e^{1/y}}{y}$

(c) $1 + \frac{1}{y} - \frac{e^{1/y}}{e}$ (d) $1 - \frac{1}{y} + \frac{e^{1/y}}{y}$ **[2011]**

13. The population $p(t)$ dt time t of a certain mouse species satisfies the differential equation $\frac{d}{dt}\, p(t) = 0.5\, p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is :

(a) log 9 (b) $\frac{1}{2}\log 18$

(c) log 18 (d) 2 log 18 **[2012]**

14. At present, a firm is manufacturing 2000 items. It is estimated that the change of production P w.r.t. additional number of workers is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is

(a) 3000 (b) 3500

(c) 4500 (d) 2500 **[2013]**

15. Consider the differential equation

$$\frac{dy}{dx} = \frac{y^3}{2(xy^2 - x^2)}$$

Statement 1: The substitution $z = y^2$ transforms the above equation into first order homogeneous differential equation

Statement 2: The solution of this differential equation is $y^2 e^{-y^2/x} = C$. **[2013, online]**

16. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{d\,p(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ equals

(a) $400 - 300\, e^{t/2}$ (b) $300 - 200\, e^{-t/2}$

(c) $600 - 500\, e^{t/2}$ (d) $400 - 300\, e^{-t/2}$ **[2014]**

17. If the differential equation representing the family of all circles touching x-axis at the origin is $(x^2 - y^2)\frac{dy}{dx} = g(x)y$, then $g(x)$ equals

(a) $\frac{1}{2}x$ (b) $2x^2$

(c) $2x$ (d) $\frac{1}{2}x^2$ **[2014, online]**

18. If the general solution of the differential equation $y' = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$, for some function ϕ is given by $y \log |cx| = x$, where C is an arbitrary constant, then $\phi(2)$ is equal to

(a) 4 (b) $\frac{1}{4}$

(c) – 4 (d) $-\frac{1}{4}$ **[2014, online]**

19. The general solution of the differential equation $\sin 2x\left(\frac{dy}{dx} - \sqrt{\tan x}\right) - y = 0$ is

(a) $y\sqrt{\tan x} = x + C$

(b) $y\sqrt{\cot x} = \tan x + C$

(c) $y\sqrt{\tan x} = \cot x + C$

(d) $y\sqrt{\cot x} = x + C$ **[2014, online]**

20. If $\frac{dy}{dx} + y \tan x = \sin 2x$ and $y(0) = 1$, then $y(\pi)$ is equal to

(a) 1 (b) – 1

(c) – 5 (d) 5 **[2014, online]**

21. Let $y(x)$ be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x$ $(x \geq 1)$.

Then $y(e)$ is equal to

(a) e (b) 0

(c) 2 (d) $2e$ **[2015]**

22. The solution of the differential equation $y\, dx - (x + 2y^2)dy = 0$ is $x = f(y)$. If $f(-1) = 1$, then $f(1)$ is equal to

(a) 4 (b) 3

(c) 2 (d) 1 **[2015, online]**

23. If $y(x)$ is the solution of the differential equation $(x + 2)\frac{dy}{dx} = x^2 + 4x - 9$, $x \neq -2$ and $y(0) = 0$, then $y(-4)$ is equal to

(a) 0 (b) 1

(c) –1 (d) 2 **[2015, online]**

24. For $x \in \mathbf{R}$, $x \neq 0$ if $y(x)$ is a differentiable function such that

$$x\int_1^x y(t)\,dt = (x+1)\int_1^x t\,y(t)\,dt, \text{ then } y(x)$$

equals: (where C is a constant)

(a) $C x^3 e^{1/x}$ (b) $\dfrac{C}{x^2} e^{-1/x}$

(c) $\dfrac{C}{x} e^{-1/x}$ (d) $\dfrac{C}{x^3} e^{-1/x}$ **[2016, online]**

25. The solution of the differential equation

$$\frac{dy}{dx} + \frac{y}{2}\sec x = \frac{\tan x}{2y}, \text{ where } 0 \le x \le \frac{\pi}{2}, \text{ and } y(0) =$$

0 is given by

(a) $y^2 = 1 + \dfrac{x}{\sec x + \tan x}$

(b) $y = 1 + \dfrac{x}{\sec x + \tan x}$

(c) $y = 1 - \dfrac{x}{\sec x + \tan x}$

(d) $y^2 = 1 - \dfrac{x}{\sec x + \tan x}$

[2016, online]

26. If a curve $y = f(x)$ passes through the point $(1, -1)$ and differential equation, $y(1 + xy)dx = x\,dy$, then $f\left(-\dfrac{1}{2}\right)$ is equal to

(a) $-\dfrac{2}{5}$ (b) $-\dfrac{4}{5}$

(c) $\dfrac{2}{5}$ (d) $\dfrac{4}{5}$ **[2016]**

Previous Years' B-Architecture Entrance Examination Questions

1. A particular solution of the initial value differential equation

$$\log \frac{dy}{dx} = 3x + 4y,\ y(0) = 0$$

(a) $16y = 3(4x - 3 + 3e^{4x})$
(b) $3e^{-4y} - 4e^{3x} = 1$
(c) $4e^{3x} + 3e^{-4y} = 7$
(d) $16y = -3(4x + 3 - 3e^{4x})$ **[2006]**

2. The equation of motion of a particle are given by $\dfrac{dx}{dt} = t(t + 1)$, $\dfrac{dy}{dt} = \dfrac{1}{t+1}$

where the particle is at $(x(t))$, $y(t)$ at time t. If the particle is at the origin at $t = 0$ then

(a) $6x = (e^y + 1)(e^y - 1)^2$
(b) $6x = (2e^y - 1)(e^y + 1)^2$
(c) $6x = (e^y - 1)(e^y + 1)^2$
(d) $6x = (2e^y + 1)(e^y - 1)^2$ **[2007]**

3. The degree of the differential equation which has a solution $y^2 = 4a(x + a^2)$ where a is arbitrary constant

(a) 2 (b) 3
(c) 4 (d) 1 **[2009]**

4. Let $u(t)$ and $v(t)$ be two solutions of the differential equation $\dfrac{dy}{dt} = e^{t^2} y(t) + \sin t$ with $u(2) < v(2)$

Statement 1: $u(t) < v(t)$ for all t

Statement 2: $u - v$ is proportional to a positive function of t **[2010]**

5. If $y(x)$ is a solution of the differential equation $\dfrac{dy}{dx} + 3y = 2$, then $\lim\limits_{x\to\infty} y(x)$ is equal to

(a) 1 (b) 0

(c) $\dfrac{3}{2}$ (d) $\dfrac{2}{3}$ **[2011]**

6. The differential equation $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{y}$ determines a family of circles with

(a) variable radius and fixed centre
(b) variable radius and variable centre
(c) fixed radius and variable centre on x-axis
(d) fixed radius and variable centre on y-axis

[2012]

7. The general solution of the differential equation $\dfrac{dy}{dx} + \sin\dfrac{x+y}{2} = \sin\dfrac{x-y}{2}$ is:

(a) $\log\tan\dfrac{y}{2} + 2\sin x = C$

(b) $\log\tan\dfrac{y}{4} + 2\sin\dfrac{x}{2} = C$

(c) $\log \cot \frac{y}{2} + 2 \sin x = C$

(d) $\log \cot \frac{y}{4} + 2 \sin \frac{x}{2} = C$ **[2013]**

8. Consider the differential equation, $ydx - (x + y^2)\, dy = 0$. If for $y = 1$, x takes value 1, then the value of x when $y = 4$ is
(a) 9 (b) 16
(c) 36 (d) 64 **[2014]**

9. The general solution of the differential equation $y\, dy + \sqrt{1+y^2}\; dx = 0$ represent a family of
(a) circles
(b) ellipses other than circles
(c) hyperbolas
(d) parabolas **[2015]**

10. The solution of differential equation

$$\frac{ydx + xdy}{ydx - xdy} = \frac{x^2 e^{xy}}{y^4},$$ satisfying $y(0) = 1$, is

(a) $x^3 = 3y^3\,(1 - e^{-xy})$ (b) $x^3 = 3y^3(-1 + e^{xy})$
(c) $x^3 = 3y^3(1 - e^{xy})$ (d) $x^3 = 3y^3(-1 + e^{-xy})$
[2016]

Answers

Concept-based

1. (c)	**2.** (a)	**3.** (d)	**4.** (c)
5. (d)	**6.** (b)	**7.** (a)	**8.** (c)
9. (d)	**10.** (a)		

Level 1

11. (a)	**12.** (b)	**13.** (b)	**14.** (d)
15. (c)	**16.** (b)	**17.** (c)	**18.** (c)
19. (b)	**20.** (c)	**21.** (a)	**22.** (a)
23. (c)	**24.** (c)	**25.** (a)	**26.** (d)
27. (b)	**28.** (d)	**29.** (b)	**30.** (c)
31. (b)	**32.** (d)	**33.** (c)	**34.** (b)
35. (c)	**36.** (c)	**37.** (d)	**38.** (a)
39. (d)	**40.** (a)		

Level 2

41. (b)	**42.** (b)	**43.** (c)	**44.** (a)
45. (b)	**46.** (d)	**47.** (a)	**48.** (a)
49. (d)	**50.** (c)	**51.** (a)	**52.** (d)
53. (b)	**54.** (c)	**55.** (b)	**56.** (a)
57. (c)	**58.** (c)	**59.** (a)	**60.** (b)
61. (d)			

Previous Years' AIEEE/JEE Main Questions

1. (a)	**2.** (a)	**3.** (b)	**4.** (a)
5. (a)	**6.** (c)	**7.** (c)	**8.** (b)
9. (c)	**10.** (b)	**11.** (c)	**12.** (c)
13. (d)	**14.** (b)	**15.** (b)	**16.** (a)
17. (c)	**18.** (d)	**19.** (d)	**20.** (c)
21. (c)	**22.** (b)	**23.** (a)	**24.** (d)
25. (d)	**26.** (d)		

Previous Years' B-Architecture Entrance Examination Questions

1. (c)	**2.** (d)	**3.** (b)	**4.** (a)
5. (d)	**6.** (c)	**7.** (a)	**8.** (b)
9. (c)	**10.** (a)		

Hints and Solutions

Concept-based

1. The given equation is a linear equation with I.F. = $e^{-\int x dx} = e^{-\frac{1}{2}x^2}$. so

$$\frac{d}{dx}\left(y\, e^{-\frac{1}{2}x^2}\right) = 4x\, e^{-\frac{1}{2}x^2}$$

$$\Rightarrow y\; e^{-\frac{1}{2}x^2} = 4\int x\, e^{-\frac{1}{2}x^2}\, dx + C = -4\, e^{-\frac{1}{2}x^2} + C$$

so $y = -\,4 + Ce^{-\frac{1}{2}x^2}$

2. I.F. = $e^{\int \tan\, t\, dt} = e^{\log\ \sec\ t} = \sec\, t$. Multiplying with I. F. we have $\frac{d}{dt}\,(x \sec t) = \sec^2 t$

$\Rightarrow x \sec t = \tan t + C$

Since $x(0) = 1$ so $C = 1$. Thus $x \sec t \neq \tan^{-1} t + 1$

$\Rightarrow x = \sin t + \cos t$

3. $\frac{dy}{dx} = e^{7x + 8y} = e^{7x}\, e^{8y} \Rightarrow e^{-8y}\, dy = e^{7x}\, dx$

so $\frac{e^{-8y}}{-8} = \frac{e^{7x}}{7} + C$. Since $y(0) = 0$ so $-\frac{1}{8} = \frac{1}{7} + C$

$\Rightarrow C = -\frac{1}{8} - \frac{1}{7} = -\frac{15}{56}$. Thus $\frac{e^{-8y}}{-8} = \frac{e^{7x}}{7} - \frac{15}{56}$

$\Rightarrow -7e^{-8y} = 8e^{7x} - 15 \Rightarrow 7e^{-8y} + 8e^{7x} = 15.$

4. $\frac{dx}{dt} - \frac{2}{t+1}x = (t+1)^3$. The I. F. is $e^{-2\int \frac{dt}{t+1}} = \frac{1}{(t+1)^2}$

The equation redues to $\frac{d}{dt}\left(x\,\frac{1}{(t+1)^2}\right) = (t + 1)$

⇒ $\frac{x}{(t+1)^2} = \frac{(t+1)^2}{2} + C$. Since $x(0) = h\,2$,

So, $C = \frac{3}{2}$. Thus $\frac{1}{(t+1)^2} = \frac{(t+1)^2}{2} + \frac{3}{2}$

⇒ $x = \frac{(t+1)^4}{2} + \frac{3}{2}(t+1)^2$. $\lim_{t \to 1} x(t) = 8 + 6 = 14$.

5. $\left(y + x\frac{dy}{dx}\right)^4 \left(\frac{dy}{dx} + x\right) = 1$

⇒ the degree is 5.

6. Put $u = y + 2x \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 2$, so the given equation reduces to

$\frac{du}{dx} - 2 = \frac{1-2u}{1+u} \Rightarrow \frac{du}{dx} = \frac{1-2u+2+2u}{1+u}$

⇒ $\frac{1+u}{3}\,du = dx$

⇒ $\left(u + \frac{u^2}{2}\right) = 3x + C$

i. e. $(y + 2x + \frac{(y+2x)^2}{2}) = 3x + C$

$y + \frac{1}{2}(y + 2x)^2 = x + C$

7. Dividing by x, we get

$\left(1 - \frac{y}{x}\cos\frac{y}{x}\right)dx + \cos\frac{y}{x}\,dy = 0$

Put $y = Vx \Rightarrow \frac{dy}{dx} = V + x\frac{dV}{dx}$

$\cos V\left(V + x\frac{dV}{dx}\right) = (V\cos V - 1)$

⇒ $x\cos V\,\frac{dv}{dx} = -1$

⇒ $\cos V\,dV + \frac{dx}{x} = 0$

⇒ $\sin V + \log|x| = C$

i. e. $\sin\frac{y}{x} + \log|x| = C$. Since $y(1) = 0$ so $C = 0$

Thus $\sin\frac{y}{x} + \log|x| = 0$

8. $a\,y^{n-1}\,y' + y^n = x$, Put $y^n = u$

⇒ $n\,y^{n-1}\frac{dy}{dx} = \frac{du}{dx}$. The given equation reduces to

$\frac{a}{n}\frac{du}{dx} + u = x \Rightarrow \frac{du}{dx} + \frac{n}{a}u = \frac{n}{a}x$

I. F. = $e^{\frac{n}{a}x}$. Multiplying with $e^{\frac{n}{a}x}$, we have

$\frac{d}{dx}\left(ue^{\frac{n}{a}x}\right) = \frac{n}{a}\left[xe^{\frac{n}{a}x}\right]$

$u\,e^{\frac{n}{a}x} = \frac{n}{a}\left[\frac{x\,e^{\frac{n}{a}x}}{n/a} - \frac{1}{(n/a)^2}e^{\frac{n}{a}x}\right] + C$

So $u = x - \frac{a}{n} + Ce^{-\frac{n}{a}x}$

⇒ $n\,y^n = nx - a + Ce^{-\frac{n}{a}x}$

9. Let the curve be $y = f(x)$ so $f(0) = 0$.
Equation of normal at any point (x, y) is

$Y - y = -\frac{1}{f'(x)}(X - x)$

This passes through (1, 2) if

$2 - y = -\frac{1}{f'(x)}(1 - x)$

$2 - y = -\frac{dx}{dy}(1 - x)$

⇒ $(2 - y)dy = -dx\,(1 - x)$

$2y - \frac{y^2}{2} = -x + \frac{x^2}{2} + C$

Since $y(0) = 0$, so $C = 0$. Hence

$\frac{x^2}{2} + \frac{y^2}{2} - (x + y) = 0$

$x^2 + y^2 - 2x - 4y = 0$

which is a circle with centre (1, 2).

10. Put $x + y = u \Rightarrow 1 + y' = \frac{du}{dx}$. The equation reduces to $\frac{du}{dx} - 1 = u^2 \Rightarrow \frac{du}{1+u^2} = x$

⇒ $\tan^{-1}(x + y) = x + C$

But $y(0) = 0$ so $C = 0$

⇒ $x + y = \tan x$.

Level 1

11. $\frac{dy}{dx} = \frac{1}{x^2}ae^{-1/x} = \frac{1}{x^2}(y - b)$, so $b = 0$ and $a = 1$

12. $\frac{dy}{dx} = \frac{y - 2x}{2y + x}$. Putting $y = vx$, we get

$v + x\frac{dv}{dx} = \frac{v-2}{2v+1} \Rightarrow \frac{2v+1}{1+v^2}\,dv = -2\frac{dx}{x}$

⇒ $\log(1 + v^2) + \tan^{-1}v = -\log x^2 + C$

$$\Rightarrow \log(x^2 + y^2) + \tan^{-1}\frac{y}{x} = C$$

Putting $y(1) = 1$, we obtain $C = \dfrac{\pi}{4} + \log 2$.

13. The given equation can be written as

$$x^2\frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = -(x\,dy - y\,dx)$$

$$\Rightarrow \frac{1}{2}\frac{d(x^2 + y^2)}{\sqrt{x^2 + y^2}} = -\frac{(x\,dy - y\,dx)}{x^2} = -d\left(\frac{y}{x}\right)$$

Integrating we have, $\sqrt{x^2 + y^2} = -\dfrac{y}{x} + C$

14. $f''(x) - g''(x) = 0 \Rightarrow f'(x) - g'(x) = C$. Putting $x = 1$, we obtain $c = 2$. Thus $f(x) - g(x) = 2x + C_1$. Putting $x = 2$, we get $C_1 = -10$. Hence $f(4) - g(4) = 8 - 10 = -2$.

15. The degree of the equation in (*a*), (*b*), (*c*) is clearly 1. The equation in (a) can be written as $(y - 2y_1)^2 = 9(1 - y_1)^2$ which is of degree 2.

16. Putting $\dfrac{y}{x} = u$, we have $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$. The given differential equation can be written as $u + x\dfrac{du}{dx} = u + \dfrac{1}{u}\dfrac{\phi(u^2)}{\phi(u^2)}$

$$\Rightarrow \frac{u\,\phi'(u^2)}{\phi(u^2)}du = \frac{dx}{x}$$

$$\Rightarrow \log\phi(u^2) = \log x^2 + \text{const} \Rightarrow x^2 = k\,\phi\left(\frac{y^2}{x^2}\right).$$

17. $\dfrac{dy}{dx} = e^{3x + 4y} = e^{3x}\,.\,e^{4y} \Rightarrow e^{-4y}\,dy = e^{3x}\,dx$

Thus $\dfrac{e^{-4y}}{-4} - \dfrac{e^{3x}}{3} = \text{Const}$. But $y(0) = 0$, so

$-\dfrac{1}{4} - \dfrac{1}{3} = \text{C}$. Hence $3e^{-4y} + 4e^{3x} = 7$.

18. The given equation can be written as

$$x\,dy + y\,dx + y\,dy - x\,dx = 0$$
$$\Rightarrow d(xy + y^2/2 - x^2/2) = 0$$
$$\Rightarrow 2xy + y^2 - x^2 = \text{const.}$$

19. Write the given equation as

$$dx + (1 + x^2)(y\,dx + x\,dy) = 0$$
$$\Rightarrow \frac{dx}{1 + x^2} + d(xy) = 0 \Rightarrow \tan^{-1}x + xy = c.$$

20. $y + x\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{y}{x}$. Replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$, we have $\dfrac{dx}{dy} = \dfrac{y}{x} \Rightarrow y^2 - x^2 = \text{Const.}$

21. Equation of tangent at (x, y) to curve $y = f(x)$ is $Y - y = f'(x)(X - x)$. Putting $X = 0$, the initial ordinate of the tangent is $y - x\,f'(x)$. The subnormal at this point is given by $y\dfrac{dy}{dx}$, so we have

$$y\frac{dy}{dx} = y - x\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{y}{x + y}$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 1 \text{ which is a linear equation.}$$

22. Multiplying the given equation by $1/x^2$, we get

$$\frac{y\,dx}{x^2} - \frac{x\,dy}{x^2} + \frac{\log x}{x^2}dx = 0$$

$$-y\,d\left(\frac{1}{x}\right) - \left(\frac{1}{x}\right)dy - \log x\,d\left(\frac{1}{x}\right) = 0$$

$$\Rightarrow d\left(\frac{y}{x}\right) + d\left(\frac{\log x}{x}\right) - \frac{dx}{x^2} = 0$$

$$\Rightarrow \frac{y}{x} + \frac{\log x}{x} + \frac{1}{x} + C = 0.$$

The curve passes through $(1, -1)$, so $C = 0$

Thus the required curve is $y + \log x + 1 = 0$.

23. $y_1 = 5\,b\,e^{5x} - 7\text{c}\,e^{-7x}$ (i)

$y_2 = 25\,b\,e^{5x} + 49\,c^{\,e-7x}$ (ii)

and $y_3 = 125\,b\,e^{5x} - 343\,c\,e^{-7x}$ (iii)

Multiplying (i) by 7 and the result to (ii) gives

$y_2 + 7y_1 = 60\text{ b }e^{5x}$ (iv)

Multiplying (i) by 5 and substracting the result from (ii), we get

$y_2 - 5y_1 = 84\,C\,e^{-7x}$ (v)

use (iv) and (v) to replace to e^{5x} and $C\,e^{-7x}$ in (iii)

$$y_3 = 125\left(\frac{y_2 + 7y_1}{60}\right) - 343\left(\frac{y_2 - 5y_1}{84}\right)$$

$$\Rightarrow y_3 + 2y_2 - 35y_1 = 0.$$

24. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + C = 0$, it passes through $(a, 0)$ and $(-a, 0)$ of $a^2 + 2ga + C = 0$ and $a^2 - 2ga + C = 0$. Substracting we obtain $g = 0$ and so $C = -a^2$. Hence the equation of circle reduces to $x^2 + y^2 + 2fy - a^2 = 0$. Differentiating, we have $x + yy_1 + fy_1 = 0$

$\Rightarrow f = -\dfrac{x + yy_1}{y_1}$. Thus, we have

$$x^2 + y^2 - a^2 - \frac{2x(x + yy_1)}{y_1} = 0$$

$$\Rightarrow x^2 - y^2 - a^2 - \frac{2xy}{y_1} = 0$$

$$\Rightarrow y_1(y_2 - x_2 + a_2) + 2xy = 0.$$

25. Put $x - y = u$. Then $\dfrac{du}{dx} = 1 - \dfrac{du}{dx}$, and the given equation becomes $u^2\left(1 - \dfrac{du}{dx}\right) = a^2$

$$\Rightarrow \quad dx = \left(\frac{u^2}{u^2-a^2}\right)du = \left(1+\frac{a^2}{u^2-a^2}\right)du$$

Integrating, we obtain

$$x = u + \frac{a^2}{2a}\log\left|\frac{u-a}{u+a}\right| + C$$

$$= x - y + \frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right| + C$$

$$\Rightarrow \quad y = \frac{a}{2}\log\left|\frac{x-y-a}{x-y+a}\right| + C.$$

26. Putting $x + y = u$, the given equation becomes

$$\frac{du}{1+\sin u+\cos u} = dx$$

$$\Rightarrow \quad \int\frac{du}{1+\sin u+\cos u} = x + C$$

Substituting $\tan(u/2) = t$, to calculate the integral on the L.H.S, we obtain its value as $\log|1+t|$ so that the required solution is

$$\log\left|1+\tan\frac{x+y}{2}\right| = x + C.$$

27. The given equation can be written as

$$\frac{dy}{dx} - y\tan x = \sec x$$

which is linear equation with integrating factor

$$e^{-\int \tan x\,dx} = e^{-\log\sec x} = \cos x.$$

Multiplying with the integrating factor and integrating we obtain $y\cos x = x + C$. Since it passes through $(0, 0)$ so we have $C = 0$.

28. The given equation can be written as

$$d\left(\frac{1}{2}(x^2+y^2)\right) + \frac{x\,dy - y\,dx}{x^2+y^2} = 0$$

$$d\left(\frac{1}{2}(x^2+y^2)\right) + \frac{d(y/x)}{1+y^2/x^2} = 0$$

Integrating, we obtain

$$\frac{1}{2}(x^2+y^2) + \tan^{-1} y/x = C$$

29. Put $y/x = u$, so that $\dfrac{dy}{dx} = x\dfrac{du}{dx} + u$,

From the given equation, we have

$$x\frac{du}{dx} + u = e^u + u \Rightarrow e^{-u}\,du = \frac{dx}{x}$$

$$\Rightarrow -e^{-u} = \log x + \text{Const}$$

$$\Rightarrow \log|Cx| = -e^{-y/x}$$

30. $y' = \sin\dfrac{x-y}{2} - \sin\dfrac{x+y}{2} = -2\cos\dfrac{x}{2}\sin\dfrac{y}{2}$

$$\Rightarrow \quad \frac{dy}{\sin y/2} = -2\cos x/2\,dx.$$

Integrating, we have

$$2\log\left|\tan\frac{y}{4}\right| = \text{const} - 4\sin x/2$$

$$\Rightarrow \log\left|\tan\frac{y}{4}\right| = C - 2\sin x/2.$$

31. Differentiating the given equation $y^2 = 2c\,(x + \sqrt{c}\,)$ (1)

$$2y\frac{dy}{dx} = 2c \quad\Rightarrow\quad c = y\frac{dy}{dx}$$

Putting this value in (1)

$$y^2 = 2y\frac{dy}{dx}\left(x + \sqrt{y\frac{dy}{dx}}\right)$$

$$\Rightarrow y - 2x\frac{dy}{dx} = 2\left(y\frac{dy}{dx}\right)^{3/2}$$

$$\Rightarrow \quad \left(y - 2x\frac{dy}{dx}\right)^2 = 4y^3\left(\frac{dy}{dx}\right)^3$$

Thus the order of the differential equation is 1 and the degree is 3.

32. Equation of tangent a $P(x, y)$ to the curve $y = f(x)$ is

$$Y - y = f'(x)\,(X - x)$$

Intercept on Y-axis is $y - xf'(x)$

We are given, $y - xf'(x) = x^3$

$$\Rightarrow \frac{dy}{dx} - \frac{1}{x}y = -x^2$$

This is a linear equation whose I.F. is $\dfrac{1}{x}$.

Multiplying with I.F., the above equation reduces to

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = -x \quad\Rightarrow\quad f(x) = y = -\frac{1}{2}x^3 + Cx$$

As $f(1) = 1$ so $C = 3/2$. Thus $f(x) = -\dfrac{1}{2}x^3 + \dfrac{3}{2}x$, so $f(3) = 9$.

33. Putting $y = Vx$, the given equation reduces

$$\frac{dx}{dx} = -\frac{V^2+1}{2V} \quad\Rightarrow\quad \frac{2V\,dV}{1+V^2} = -\frac{dx}{x}$$

$\Rightarrow (1 + V^2)x = \text{Const}$

$\Rightarrow x^2 + y^2 = Cx \quad\Rightarrow\quad (x - C/2)^2 + y^2 = C^2/4$

which represents circle with centre $(C/2, 0)$ i.e., on x-axis.

34. $9y\,dy = -4x\,dx \quad\Rightarrow\quad \dfrac{9}{2}y^2 = -2x^2 + \text{const}$

$\Rightarrow \quad \dfrac{x^2}{9} + \dfrac{y^2}{4} = \text{Const}$, so represents a family of ellipses.

35. $\frac{dy}{y} = -2x\,dx \quad \Rightarrow \quad \log y = -x^2 + C$

$\Rightarrow \quad y = Ce^{-x^2}$

For $\quad x = 0, y = 1 \quad$ so $C = 1$

Thus $\quad y = e^{-x^2}$ which is bell shaped.

36. Put $x - y = u \quad \Rightarrow \quad 1 - \frac{dy}{dx} = \frac{du}{dx}$. The given equation reduces to $1 - \frac{du}{dx} = \cos u$

$\Rightarrow \quad \frac{du}{1 - \cos u} = dx \Rightarrow \frac{1}{2}\int \operatorname{cosec}^2 (u/2)\,du = x + C$

$\Rightarrow -\cot\frac{u}{2} = x + C$ i.e. $-\cot\frac{x-y}{2} = x + C$

Putting $y(0) = -\pi$, we have $C = 0$

$y = x - 2\cot^{-1}(-x)$.

37. Let (x, y) be the point of tangency. Equation of tangent is $Y - y = f'(x)\,(X - x)$. This intersects x-axis at $\left(x - \frac{1}{f'(x)}y, 0\right)$. The mid point is $\left(x - \frac{1}{2f'(x)}y, \frac{y}{2}\right)$ is on Y-axis so

$$x - \frac{1}{2f'(x)}y = 0$$

$$\Rightarrow \quad 2\frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad y^2 = Cx$$

which represents parabola with latus ractum parallel to Y-axis.

38. $\frac{dy}{dx} + 2y = 3x + 5 \quad \Rightarrow ye^{2x} = \int (3x + 5)\,e^{2x} + C$

$$= (3x + 5)\,\frac{e^{2x}}{2} - \frac{3}{4}e^{2x} + C$$

$\Rightarrow \quad y = \left(\frac{3x+5}{2} - \frac{3}{4}\right) + Ce^{-2x}$

$\Rightarrow \quad 4y - 6x - 7 = Ce^{-2x}$

39. It is linear equation with I.F. $e^{\int \tan x\,dx} = \sec x$, Required solution is $y = \sin x + \cos x$.

40. I.F. $= e^{\int \frac{1}{x}dx} = x$ so $\frac{d}{dx}(xy) = e^x$

$\Rightarrow \quad xy = e^x + C$ Put $y(a) = b$

$ab - e^a = C$, so $xy = e^x + ab - e^a$.

Level 2

41. The tangent at any point (x, y) meets the Y-axis at $\left(0, y - x\frac{dy}{dx}\right)$. According to the given condition

$\frac{1}{2}x^2$ = Area of the trapezoid

$$= \left(y + y - x\frac{dy}{dx}\right)x$$

$\Rightarrow \frac{dy}{dx} - 2\frac{y}{x} = -\frac{1}{2}$ (which is a linear equation)

I. F. $= e^{-2\int \frac{1}{x}dx} = x^{-2}$

Multiplying with I.F. and integrating, we have

$$yx^{-2} = -\frac{1}{2}\int x^{-2}\,dx + C$$

$$= \frac{1}{2}x^{-1} + C$$

$\Rightarrow y = \frac{1}{2}x + Cx^2$.

42. Equation of normal at any point (x, y)

$$Y - y = -\frac{1}{f'(x)}\,(X - x)$$

This passes through (x_0, y_0) if

$$y_0 - y = \frac{1}{f'(x)}\,(x_0 - x)$$

$(y_0 - y)\,dy + (x_0 - x)\,dx = 0$

$$\frac{(y_0 - y)^2}{2} + \frac{(x_0 - x)^2}{2} = C$$

$(y - y_0)^2 + (x - x_0)^2 = \text{Const}$

Since $y(0) = 0$ so Const $= y_0^2 + x_0^2$

Thus $y^2 - 2yy_0 + y_0^2 + x^2 + x_0^2 - 2xx_0 = x_0^2 + y_0^2$

$x^2 + y^2 = 2(xx_0 + yy_0)$.

43. $(2a - x)\,2y\frac{dy}{dx} - y^2 = 3x^2$ (Differentiating given equation)

$$2\,(2a - x)y\frac{dy}{dx} = 3x^2 + y^2$$

Eliminating a between the last equation and given equation i.e. $(2a - x)\,y^2 = x^2$, we have

$$y\,(3x^2 + y^2)\,\frac{dx}{dy} = 2x^3$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ we have

$$2x^3 + y(3x^2 + y^2)\frac{dy}{dx} = 0$$

$\frac{dy}{dx} = \frac{-2x^3}{(3x^2 + y^2)\,y}$, which is a homogeneous equation.

Putting $y = Vx$

$$V + x\frac{dV}{dx} = -\frac{2}{(3+V^2)V}$$

$$\Rightarrow x\frac{dV}{dx} = -\left[\frac{2+3V^2+V^4}{(3+V^2)V}\right]$$

$$\frac{dx}{x} = -\frac{(3+V^2)V}{(1+V^2)(2+V^2)} = -\left[\frac{2V}{1+V^2} - \frac{V}{2+V^2}\right]$$

$$\Rightarrow \log x^2 = -\log\frac{(1+V^2)^2}{2+V^2} + \text{Const}$$

$$\Rightarrow \frac{x^2\,(1+y^2/x^2)^2}{2+y^2/x^2} = \text{Const}$$

$$\Rightarrow (x^2 + y^2)^2 = C(y^2 + 2x^2).$$

44. $xy_5 = y_4 \Rightarrow x\frac{d}{dx}y_4 = y_4$

$$\Rightarrow \frac{dy_4}{y_4} = \frac{dx}{x} \Rightarrow y_4 = (\text{Const})\,x$$

So $y_3 = \text{const}\ x^2 + \text{const}$

$$\Rightarrow y_2 = C_1'\,x^3 + C_2'\,x + C_3'$$

$$\Rightarrow y_1 = C_1''\,x^4 + C_2''\,x^2 + C_3''\,x + C_4''$$

$$\Rightarrow y = C_1\,x^5 + C_2\,x^3 + C_3\,x^2 + C_4\,x + C_5.$$

45. $yy'' = y'^2 \Rightarrow \frac{y''}{y'} = \frac{y'}{y}$ Integrating, we get

$\log y' = \log y + \text{const} \Rightarrow y' = Cy$

$$\Rightarrow \frac{dy}{y} = C\,dx \Rightarrow \log y = Cx + C'$$

$$y = C_2\,e^{C_1 x}.$$

46. According to the given condition

$$\frac{y\frac{dy}{dx}}{x+y} = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{x+y}{x}$$

Putting $y = Vx$, $V + \frac{dV}{dx} = 1 + V$

$$\Rightarrow dV = \frac{dx}{x}$$

$$\Rightarrow V = \log x + \text{const}$$

$$\Rightarrow y = x\log Cx.$$

47. $\int_0^x f(t)\,dt = K\,y^{n+1}$

Differentiating, we have

$$y = f(x) = K\,(n+1)\,y^n\frac{dy}{dx}$$

$$\Rightarrow dx = K\,(n+1)\,y^{n-1}\,dy$$

$$\Rightarrow x + C = K\frac{(n+1)}{n}y^n$$

Since $f(0) = 0$, $C = 0$

$f(1) = 1$, $1 = \frac{K\,(n+1)}{n} \Rightarrow K = \frac{n}{n+1}$

Thus $x = y^n$.

48. Putting $y = Vx$

$$\text{V} + x\frac{dV}{dx} = \frac{V^2-2V-1}{V^2+2V-1}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{V^2-2V-1-V^3-2V^2+V}{V^2+2V-1}$$

$$\frac{dx}{x} = -\frac{V^2+2V-1}{V^3+V^2+V+1}dV$$

$$= -\left(-\frac{1}{V+1} + \frac{2V}{V^2+1}\right)dV$$

Integrating both sides, we get

$\log x - \log(V+1) + \log(V^2+1) + \log C = \text{O}$

$$\Rightarrow C\left(\frac{x(V^2+1)}{V+1}\right) = 1 \Rightarrow C\,(y^2 + x^2) = y + x$$

If this curve passes through $(-1, 1)$, we get $2C = 0$ i.e.

$C = 0$. Hence the required curve is $y + x = 0$ which is a straight line.

49. According to the given condition

$$\frac{y}{f'(x)} = nx$$

$$y\frac{dx}{dy} = nx \Rightarrow \frac{dy}{y} = \frac{1}{n}\frac{dx}{x}$$

Integrating $y^n = Cx$. This passes through

$(2, 3)$ if $3^n/2 = C$

So $y^n = \frac{3^n x}{2}$

$\Rightarrow \left(\frac{y}{3}\right)^n = \frac{x}{2}$. For $n = 2$. This represents a

parabola whose axes coincide with x-axis.

50. $x = \left(\frac{1}{2p} - \frac{p}{2}\right)y$. Differentiating both sides

w·r·t y $\left(p=\frac{dy}{dx}\right)$

$$\frac{1}{p} = \frac{1}{2p}-\frac{p}{2} + y\left(-\frac{1}{2p^2}-\frac{1}{2}\right)\frac{dp}{dy}$$

$$\frac{1}{2p}+\frac{p}{2} = -y\left(\frac{1+p^2}{2p^2}\right)\frac{dp}{dy}$$

$$\Rightarrow \frac{1+p^2}{2p} = -y\left(\frac{1+p^2}{2p^2}\right)\frac{dp}{dy}$$

$$-\frac{dy}{y} = \frac{dp}{p} \Rightarrow py = C$$

i.e. $y\,dy = c\,dx \Rightarrow y^2 = 2\,Cx + C^2$.

51. According to the given condition

$$\int_0^x f(t)\,dt = \frac{y^3}{x}$$

Differentiating, we get

$$y = \frac{3y^2}{x}\frac{dy}{dx}-\frac{y^3}{x^2}$$

$$\frac{x^2y+y^3}{x^2} = \frac{3y^2}{x}\frac{dy}{dx}$$

$$\Rightarrow \frac{x^2y+y^3}{3y^2x} = \frac{dy}{dx}$$

Putting $y = Vx$

$$\frac{V+V^3}{3V^2} = V + x\frac{dV}{dx}$$

$$\Rightarrow \frac{1+V^2}{3V}-V = x\frac{dV}{dx}$$

$$\Rightarrow \frac{1-2V^2}{3V} = x\frac{dV}{dx}$$

$$\Rightarrow \frac{3V}{1-2V^2}\,dV = \frac{dx}{x}$$

Integrating, we have

$\Rightarrow \frac{-3}{4}\log(1-2V^2) = \log x + \text{Const}$

$\Rightarrow x^4(1-2V^2)^3 = \text{Const}$

$\Rightarrow (2y^2-x^2)^3 = Cx^2$.

52. The initial ordinate of tangent is equal to

$y - x\,\frac{dy}{dx}$, so

$$y - x\frac{dy}{dx} = x-2$$

$$\Rightarrow \frac{dy}{dx}-\frac{1}{x}y = -1+\frac{2}{x}$$

This is a linear equation with I.F. $\frac{1}{x}$

$$\frac{d}{dx}\left(y\cdot\frac{1}{x}\right) = -\frac{1}{x}+\frac{2}{x^2}$$

$$y\cdot\frac{1}{x} = -\log|x| - 2^{x-1} + C$$

$y = Cx - x\log|x| - 2$.

53. $y' = \frac{x+y-2}{y-x-4}$. Put $x = X + h$, $y = Y + K$

$$\frac{dY}{dX} = \frac{X+Y+(h+K-2)}{Y-X+(-h+K-4)}$$

Select h, K such that $h + K - 2 = 0$, $-h + K - 4 = 0$

$\Rightarrow K = 3$, $h = -1$

$$\frac{dY}{dX} = \frac{X+Y}{Y-X}$$

Putting $Y = v\,X$, we have

$$v+X\frac{dV}{dX} = \frac{1+V}{V-1}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{1+V}{V-1}-V = \frac{1+2V-V^2}{V-1}$$

$$\frac{dX}{X} = \frac{-1}{2}\,\frac{-2V+2}{1+2V-V^2}\,dV$$

Integrating, we have $X^2(1 + 2V - V^2) = \text{Const}$

$X^2 - Y^2 + 2XY = \text{Const}$

$(x + 1)^2 - (y - 3)^2 + 2(x + 1)(y - 3) = \text{Const}$

$x^2 + 2xy - y^2 - 4x + 8y = \text{Const}$.

54. $y = xp - 3p^3$, $p = \frac{dy}{dx}$. Differentiating w·r·t·x, we have $p = p + x\frac{dp}{dx} - 9p^2\frac{dp}{dx}$

$\Rightarrow (x - 9p^2)\frac{dp}{dx} = 0$

$\frac{dp}{dx} = 0 \Rightarrow p = \text{Const}$ so $y = Cx - 3C^3$

$y = x - 3$ is solution with $C = 1$.

55. $2x + 2y\frac{dy}{dx} = 2a \Rightarrow a = x + y\frac{dy}{dx}$

Eliminating a, we have

$$x^2 + y^2 = 2\left(x+y\frac{dy}{dx}\right)x$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$x^2 + y^2 = 2\left(x^2 - xy\frac{dx}{dy}\right)$$

$$\Rightarrow \frac{y^2 - x^2}{-2xy} = \frac{dx}{dy}$$

$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$. Putting $y = Vx$, we have

$$V + x\frac{dV}{dx} = \frac{2V}{1-V^2}$$

$$\frac{1-V^2}{V\left(1+V^2\right)}dV = \frac{dx}{x}$$

$$\left[\frac{1}{V} - \frac{2V}{1+V^2}\right]dV = \frac{dx}{x}$$

$$\Rightarrow \frac{V}{x(1+V^2)} = \text{const}$$

$$\Rightarrow y = C(x^2 + y^2).$$

56. $m^2 - 4m + 3 = 0 \Rightarrow (m - 1)(m - 3) = 0$

so $m = 1, 3$

Hence $y = C_1 e^x + C_2 e^{3x}$

$6 = y(0) = C_1 + C_2$, $10 = y'(0) = C_1 + 3C_2$

$C_2 = 2$, $C_1 = 4$. Thus $y = 4e^x + 2e^{3x}$.

57. Since $C_1 + C_2$ is one constant $C_4 e^{x + C_5}$
$= C_1 e^{C_5} e^x$

Thus there are three arbitrary constants. So the order of the differential equation is 3.

58. Put $x + y = u \Rightarrow 1 + \frac{dy}{dx} = \frac{du}{dx}$. The given equation reduces to

$$\frac{du}{dx} - 1 = \sec u$$

$$\Rightarrow \frac{du}{1 + \sec u} = dx$$

$$\Rightarrow \left(1 - \frac{1}{1 + \cos u}\right)du = dx$$

$$\Rightarrow u - \tan u/2 = x + C$$

$$\Rightarrow x + y - \tan\frac{x+y}{2} = x + C$$

$$\Rightarrow y - \tan\frac{x+y}{2} = C.$$

59. Clearly $y_3 - 7y_1 + 6y = 0$ is the required equation

60. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = x + e^{3x}$. Integrating, we have

$$\frac{dy}{dx} = \frac{x^2}{2} + \frac{e^{3x}}{3} + C_1$$

$$\Rightarrow y = \frac{x^3}{6} + \frac{e^{3x}}{9} + C_1 x + C_2$$

Since $y(0) = 0$ so $C_2 = -\frac{1}{9}$ and $\frac{1}{3} = y_1(0)$

$$= \frac{1}{3} + C_1 \Rightarrow C_1 = 0$$

Hence $y = \frac{x^3}{6} + \frac{e^{3x}}{9} - \frac{1}{9} = \frac{1}{18}(3x^3 + 2e^{3x} - 2)$.

61. $\frac{dy}{dx} = \int e^x \sin x\, dx = \frac{e^x \sin(x - \pi/4)}{\sqrt{2}} + C_1$

$$\Rightarrow y = \frac{1}{\sqrt{2}}\frac{e^x \sin(x - \pi/2)}{\sqrt{2}} + C_1 x + C_2$$

$$0 = y\left(\frac{\pi}{2}\right) = 0 + C_1\frac{\pi}{2} + C_2$$

$$0 = y_1\left(\frac{\pi}{4}\right) = \frac{1}{2}e^{\pi/4}\left[-\sin\frac{\pi}{4} + \cos\pi/4\right] + C_1$$

$\Rightarrow C_1 = 0$ So $C_2 = 0$

Hence $y = \frac{1}{2}e^x \sin\left(x - \frac{\pi}{2}\right)$.

Previous Years' AIEEE/JEE Main Questions

1. $(1 + y^2) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$

$$\Rightarrow \left(1+y^2\right)\frac{dx}{dy} + x = e^{\tan^{-1} y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{1}{1+y^2}e^{\tan^{-1} y}$$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

$$\frac{d}{dy}\left(xe^{\tan^{-1} y}\right) = \frac{e^{2\tan^{-1} y}}{1+y^2}$$

$$\Rightarrow xe^{\tan^{-1} y} = \int\frac{e^{\tan^{-1} y}}{1+y^2}dy + \text{const}$$

$$= \frac{1}{2}e^{2\tan^{-1} y} + \text{const}$$

$$\Rightarrow 2xe^{\tan^{-1} y} = e^{\tan^{-1} y} + k$$

2. $x^2 + y^2 - 2ay = 0 \Rightarrow 2a = \dfrac{x^2+y^2}{y}$

Differentiating, we get

$$\frac{(2x+2yy')y-(x^2+y^2)y'}{y^2} = 0$$

$\Rightarrow (x^2 - y^2)y' = 2xy$

3. We can write the given differential equation as

$$(ydx + xdy) + \frac{1}{y}(xy)^2\, dy = 0$$

$$\Rightarrow \frac{d(xy)}{(xy)^2}+\frac{1}{y}dy = 0$$

$$\Rightarrow -\frac{1}{xy}+\log y = C.$$

4. The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y}{x}\left[\log\frac{y}{x}+1\right] \qquad (1)$$

Put $y/x = v$, so that

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting this in (1), we get

$$v+x\frac{dv}{dx} = v(\log v + 1)$$

$$\Rightarrow \frac{dv}{v\log v} = \frac{dx}{x}$$

$$\Rightarrow \int\frac{dv}{v\log v} = \int\frac{dx}{x}$$

$\Rightarrow \log\log v = \log x + \log c$

$$\Rightarrow \log v = cx \text{ or } \log\left(\frac{y}{x}\right) = cx$$

5. Differentiating w.r.t. x, we get

$$2Ax + 2By\frac{dy}{dx} = 0$$

$$\Rightarrow -Ax = By\frac{dy}{dx} \qquad (1)$$

Again differentiating w.r.t. x, we get

$$-A = B\left(\frac{dy}{dx}\right)^2+By\left(\frac{d^2y}{dx^2}\right)$$

$$\Rightarrow -Ax = Bx\left(\frac{dy}{dx}\right)^2+Bxy\left(\frac{d^2y}{dx^2}\right)$$

$$\Rightarrow By\frac{dy}{dx} = Bx\left(\frac{dy}{dx}\right)+Bxy\left(\frac{d^2y}{dx^2}\right) \text{ [from (1)]}$$

$$\Rightarrow xy\frac{d^2y}{dx^2}+(x-y)\frac{dy}{dx} = 0$$

which is of second order and first degree.

6. Equation of any circle through origin and having centre on the x-axis is

$$(x - a)^2 + y^2 = a^2 \qquad (1)$$

Differentiating w.r.t. x, we get

$2(x - a) + 2yy' = 0$

$\Rightarrow a = x + yy'$

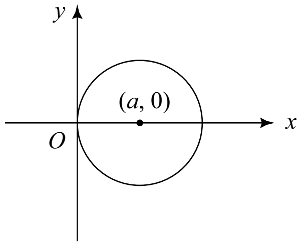


Fig. 15.2

Putting this in (1), we get

$(-yy')2 + y^2 = (x + yy')^2$

$$\Rightarrow y^2 = x^2 + 2xy\frac{dy}{dx}$$

7. Let the centre be at $(a, 2)$. Equation of circle of radius 5 is

$$(x - a)^2 + (y - 2)^2 = 25 \qquad (1)$$

Differentiating, we have

$$2(x - a) + 2(y - 2)\frac{dy}{dx} = 0$$

$$\Rightarrow x - a = -(y - 2)\frac{dy}{dx}$$

Putting this value in (1), we have

$$(y - 2)^2\left(\frac{dy}{dx}\right)^2 + (y - 2)^2 = 25$$

$(y - 2)^2 y'^2 = 25 - (y - 2)^2.$

8. $y = c_1 e^{c_2 x}$

$\Rightarrow y' = c_1 c_2 e^{c_2 x}$

$\Rightarrow y'' = c_1 c_2^2 e^{c_2 x}$

$\therefore\ yy'' = c_1^2 c_2^2 e^{c_2 x} e^{c_2 x}$

$= \left(c_1 c_2 e^{c_2 x}\right)^2 = (y')^2$

9. We can write the given differential equation as

$\dfrac{dy}{dx} = \dfrac{y(\sin x - y)}{\cos x} \Rightarrow \dfrac{1}{y^2}\dfrac{dy}{dx} - \dfrac{1}{y} = -\sec x$

Put $-\dfrac{1}{y} = z$, so that the above equation becomes

$$\frac{dz}{dx} + (\tan x)z = -\sec x \qquad (1)$$

I.F. $= e^{\int \tan x dx} = e^{\log \sec x} = \sec x$

Multiplying (1) by sec x, we get

$\Rightarrow \dfrac{d}{dx}[(\sec x)z] = -\sec^2 x \Rightarrow (\sec x)z = c_1 - \tan x$

$\Rightarrow \dfrac{-\sec x}{y} = c_1 - \tan x \Rightarrow \sec x = (c + \tan x)\, y$

Where $c = -c_1$.

10. $\dfrac{dy}{dx} = y + 3$

$\Rightarrow \int \dfrac{dy}{y+3} = \int dx \Rightarrow \ln (y + 3) = x + c$

When $x = 0$, $y = 2$, therefore, $\ln 5 = c$

Thus, $\ln (y + 3) = x + \ln 5$

When $x = \ln 2$, we get

In $(y + 3) = \ln 2 + \ln 5 \Rightarrow y + 3 = 10$ or $y = 7$

11. $\dfrac{dV}{dt} = -k(T - t),\ k > 0$

$\Rightarrow dV = -k(T - t)dt$

$\Rightarrow \int dV = -k\int(T - t)dt$

$\Rightarrow V(t) = \dfrac{k}{2}(T - t)^2 + C$

We have $V(0) = I$, therefore

$I = \dfrac{k}{2} T^2 + C \Rightarrow C = I - \dfrac{k}{2}T^2$

Also, scrap value $= V(T) = C$

$= I - \dfrac{k}{2} T^2$

12. $y^2 dx + \left(x - \dfrac{1}{y}\right)dy = 0$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2}x = \frac{1}{y^3} \qquad (1)$$

I.F. $= e^{\int (1/y^2)dy} = e^{-1/y}$

Multiplying (1) by $e^{-1/y}$, we get

$\dfrac{d}{dx}\left[xe^{-1/y}\right] = \dfrac{1}{y^3}e^{-1/y}$

$\Rightarrow xe^{-1/y} = \int \dfrac{1}{y^3}e^{-1/y}dy = I$ (say)

To evaluate I, put $-1/y = t$, so that

$I = \int(-t)e^t\, dt = -te^t + \int e^t\, dt$

$= e^t(1 - t) = e^{-1/y}(1 + 1/y)$.

Thus,

$xe^{-1/y} = e^{-1/y}(1 + 1/y) + c$

$\Rightarrow x = 1 + \dfrac{1}{y} + ce^{1/y}$

When $x = 1$, $y = 1$.

Hence,

$1 = 1 + 1 + ce \Rightarrow c = -1/e$

$\therefore\ x = 1 + \dfrac{1}{y} - \dfrac{e^{1/y}}{e}$

13. $\dfrac{d}{dt} p(t) - 0.5p(t) = -450$. I.F. $= e^{-0.5t}$

$\Rightarrow \dfrac{d}{dt}(p(t)\, e^{-0.5t}) = -450\, e^{-0.5t}$

$\Rightarrow p(t)e^{-0.5t} = 450 \times 2.\ e^{-0.5t} + C$

Putting $t = 0$, $850 = 900 + C$

$\Rightarrow C = -50$

$\Rightarrow p(t) = 900\, e^{-0.5t} - 50$

$p(t) = 0 \Rightarrow e^{-0.5t} = \frac{1}{18} \Rightarrow -0.5t = -\log 18$

$\Rightarrow t = 2 \log 18.$

14. $\frac{dP}{dx} = 100 - 12\sqrt{x}$

$\Rightarrow dP = \left(100 - 12\sqrt{x}\right)dx$

$\Rightarrow P = \int\left(100 - 12\sqrt{x}\right)dx$

$P(x) = 100\,x - 8x^{3/2} + c$

We have $P(0) = 2000$,

Therefore, $c = 2000$.

Thus, $P(25) = 100\,(25) - 8(25)^{3/2} + 2000 = 3500$

15. $z = y^2 \Rightarrow \frac{dz}{dx} = 2y\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{y^3}{2(xy^2 - x^2)} \Rightarrow 2y\frac{dy}{dx} = \frac{2y^4}{2(xy^2 - x^2)}$

$\Rightarrow \frac{dz}{dx} = \frac{z^2}{\left(xz - x^2\right)}$

which a first order homogeneous differential equation. Put $z = ux \Rightarrow \frac{dz}{dx} = u + x\frac{du}{dx}$

$u + x\frac{du}{dx} = \frac{u^2x^2}{\left(x^2u - x^2\right)} = \frac{u^2}{u-1}$

$\Rightarrow x\frac{du}{dx} = \frac{u^2 - u^2 + u}{u-1} = \frac{u}{u-1}$

$\Rightarrow \frac{u-1}{u}du = \frac{dx}{x}$

$\Rightarrow u - \log x = \log x + C$

$\Rightarrow u = \log ux + C$

$\Rightarrow ux = ce^u$

$\Rightarrow ze^{-z/x} = c$

$\Rightarrow y^2e^{-y^2/x} = c$

16. $\frac{d}{dt}p(t) - \frac{1}{2}p(t) = -200.$

I.F. is $e^{-\frac{1}{2}t}$, so

$\frac{d}{dt}\left(e^{-\frac{1}{2}t}p(t)\right) = -200\,e^{-\frac{1}{2}t}$

$\Rightarrow e^{-\frac{1}{2}t}p(t) = 400\,e^{-\frac{1}{2}t} + C$

Putting $t = 0$

$100 = p(0) = 400 + C$

$\Rightarrow C = -300$

Thus $p(t) = 400 - 300\; e^{\frac{1}{2}t}$

17. An equation of circle touching the x-axis at the origin is

$$x^2 + (y - a)^2 = a^2 \qquad (1)$$

where a is parameter.

Differentiating (1) w.r.t. x,

we get

$2x + 2(y - a)y' = 0$

$\Rightarrow a = y + \frac{x}{y'}$

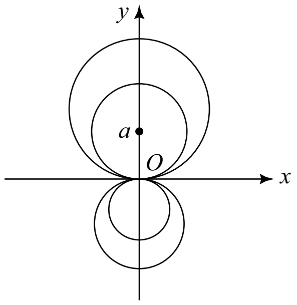


Fig. 15.3

Putting this in (1), we get

$x^2 + \left(\frac{x}{y'}\right)^2 = \left(y + \frac{x}{y'}\right)^2$

$\Rightarrow x^2 = y^2 + (2xy)\;\frac{1}{y'}$

$\Rightarrow (x^2 - y^2)\;\frac{dy}{dx} = (2x)y$

$\therefore\; g(x) = 2x$

18. $y' \ln |cx| + y\left(\frac{1}{x}\right) = 1$

$$\Rightarrow\ y'\left(\frac{x}{y}\right)+\frac{y}{x} = 1$$

$$\Rightarrow y' = \frac{y}{x}-\left(\frac{y}{x}\right)^2 = \frac{y}{x}+\phi\left(\frac{y}{x}\right)$$

$$\Rightarrow \phi(u) = -\frac{1}{u^2}$$

$$\therefore\ \phi(2) = -\frac{1}{4}$$

19. Given differential equation is

$$\frac{dy}{dx} - \operatorname{cosec}(2x)y = \sqrt{\tan x} \qquad (1)$$

$$\text{I.F.} = e^{-\int \operatorname{cosec}(2x)dx}$$

$$= e^{-(1/2)\log(\tan x)} = \frac{1}{\sqrt{\tan x}}$$

Multiplying (1) $\dfrac{1}{\sqrt{\tan x}}$, we get

$$\frac{d}{dx}\left(\frac{y}{\sqrt{\tan x}}\right) = 1$$

$$\Rightarrow\ y\sqrt{\cot x} = x + c$$

20. I.F. $= e^{\int(\tan x)dx} = e^{\log \sec x} = \sec x$

Multiplying the given differential equation by sec x, we get

$$\frac{d}{dx}((\sec x)y) = 2\sin x$$

$\Rightarrow (\sec x)\, y = -2\cos x + C$

Putting $x = 0$, $y = 1$, we get

$1 = -2 + C \Rightarrow C = 3$

When $x = \pi$, we get

$(-1)y = -2\cos \pi + 3 \Rightarrow y = -5$

21. We can write the given differential equation as

$$\frac{dy}{dx}+\frac{1}{x\log x}y = 2 \qquad (1)$$

I.F. $= e^{\int dx/x \log x} = e^{\log(\log x)} = \log x$

Multiplying (1) by log x, we get

$$(\log x)\ \frac{dy}{dx}+\frac{1}{x}y = 2\log x$$

$$\Rightarrow\ \frac{d}{dx}\ [(\log x)y] = 2\log x$$

$\Rightarrow (\log x)y = 2(x\log x - x) + C$

When $x = 1$, we get

$0 = 2(0 - 1) + C \Rightarrow C = 2$

When $x = e$, we get

$(\log e)\ y(e) = 2(e\log e - e) + 2$

$\Rightarrow y(e) = 2$

22. Write the differential equation as

$$\frac{dx}{dy}+\left(-\frac{1}{y}\right)x = 2y$$

$$\text{I.F.} = e^{\int(-1/y)dy} = e^{-\log y} = \frac{1}{y}$$

Multiplying (1) by $1/y$, we get

$$\frac{1}{y}\frac{dx}{dy}+\left(-\frac{1}{y^2}\right)x = 2$$

$$\Rightarrow\ \frac{dx}{dy}\left(\frac{x}{y}\right) = 2$$

$$\Rightarrow\ \frac{x}{y} = 2y + c$$

$\Rightarrow x = 2y^2 + cy$

When $y = -1$, $x = 1$,

Therefore, $1 = 2 - c \Rightarrow c = 1$

Thus, $f(y) = 2y^2 + y \Rightarrow f(1) = 3$.

23. We can write differential equation as

$$\frac{dy}{dx} = \frac{x^2+4x-9}{x+2} = x + 2 - \frac{13}{x+2}$$

$$\Rightarrow y = \frac{1}{2}\ (x+2)^2 - 13\log|x+2| + C$$

As $y(0) = 0$, we get

$0 = 2 - 13\log|2| + C$ or $C = 13\log 2 - 2$

Also, $y(-4) = 2 - 13\log 2 + 13\log 2 - 2$

$= 0.$

24. $x\int_1^x y(t)\,dt = (x+1)\int_1^x ty(t)\,dt$

Differentiating both the sides, w.r.t. x, we get

$$\int_1^x y(t)dt + xy(x) = \int_1^x ty(t) + (x+1)xy(x)$$

$$\Rightarrow \int_1^x y(t)dt = \int_1^x ty(t)dt + x^2 y(x)$$

Differentiating both the sides w.r.t. x, we get

$y(x) = xy(x) + 2xy(x) + x^2 y'(x)$

$\Rightarrow (1 - 3x)\, y(x) = x^2 y'(x)$

$$\Rightarrow \frac{y'(x)}{y(x)} = \frac{1}{x^2} - \frac{3}{x}$$

Integrating we get

$$\Rightarrow \ln|y(x)| = \frac{1}{x} - 3\ln|x| + A$$

$$\Rightarrow \ln|x^3 y(x)| = -\frac{1}{x} + A$$

$x^3 y(x) = \pm\, e^A e^{-1/x} = Ce^{-1/x}$

where C is a constant

$$\Rightarrow y(x) = \frac{c}{x^3}\, e^{-1/x}$$

25. $$\frac{dy}{dx} + \frac{1}{2}y\sec x = \frac{\tan x}{2y}$$

$$\Rightarrow \frac{d}{dx}(y^2) + (\sec x)y^2 = \tan x \qquad (1)$$

$\text{I.F.} = e^{\int \sec x\, dx} = e^{\ln(\sec x + \tan x)}$

$= \sec x + \tan x$

Multiplying (1) by $\sec x + \tan x$, we get

$$\frac{d}{dx}\,[y^2(\sec x + \tan x)] = \tan x\,(\sec x + \tan x)$$

$\Rightarrow y^2(\sec x + \tan x) = \sec x + \tan x - x + C$

When $x = 0$, $y = 1$, so that

$1 = 1 + C \Rightarrow C = 0$

Thus, $$y^2 = 1 - \frac{x}{\sec x + \tan x}$$

26. We can write the differential equation as

$$y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = 1 \qquad (1)$$

Put $-y^{-1} = t \Rightarrow y^{-2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

We can write (1) as

$$\frac{dt}{dx} + \frac{1}{x}t = 1$$

$$\Rightarrow x\frac{dt}{dx} + t = x$$

$$\Rightarrow \frac{dt}{dx}(tx) = x$$

$$\Rightarrow tx = \frac{1}{2}x^2 + C$$

$$\Rightarrow -\frac{1}{y}x = \frac{1}{2}x^2 + C$$

As it passes through $(1, -1)$, $1 = \dfrac{1}{2} + C \Rightarrow C = \dfrac{1}{2}$

$\therefore\ -\dfrac{1}{y}x = \dfrac{1}{2}x^2 + \dfrac{1}{2}$. When $x = -\dfrac{1}{2}$,

$$-\frac{1}{y}\left(-\frac{1}{2}\right) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

$$\Rightarrow y = \frac{4}{5}.$$

Previous Years' B-Architecture Entrance Examination Questions

1. $\log\dfrac{dy}{dx} = 3x + 4y$

$$\Rightarrow \frac{dy}{dx} = e^{3x+4y} = e^{3x}\, e^{4y}$$

$\Rightarrow e^{-4y}\, dy = e^{3x}\, dx$

$$\Rightarrow -\frac{e^{-4y}}{4} = \frac{e^{3x}}{3} + \text{const}$$

$y(0) = 0 \Rightarrow -\dfrac{1}{4} = \dfrac{1}{3} + \text{const} \Rightarrow \text{const} = -\dfrac{7}{12}$

so $-3e^{-4y} = 4e^{3x} - 7$

$\Rightarrow 4e^{3x} + 3e^{-4y} = 7$

2. $\dfrac{dx}{dt} = t(t+1) \Rightarrow x = \dfrac{t^3}{3} + \dfrac{t^2}{2} + C_1$

But at $t = 0$, $x = 0$ so $C_1 = 0$

Thus $$x = \frac{t^3}{3} + \frac{t^2}{2} \qquad (1)$$

$\dfrac{dy}{dx} = \dfrac{1}{t+1} \Rightarrow y = \log(t+1) + C_2$

But at $t = 0$, $y = 0$ so $C_2 = 0$

Thus $y = \log(t+1) \Rightarrow t = e^y - 1$.

Putting in (1), we have

$6x = t^2\ (2t + 3)$

$= (e^y - 1)^2\ (Ze^y + 1).$

3. Differentiating $y^2 = 4a(x + a^2)$ (1)

We have, $2y\dfrac{dy}{dx} = 4a \Rightarrow a = \dfrac{yy'}{2}$

Putting in (1)

$$y^2 = 4\frac{yy'}{2}\left(x+\frac{1}{4}y^2y'^2\right)$$

$= 2xy\ y' + y^3\ y'^3$

So the degree is 3.

4. $u'(t) = e^{t^2}\ u(t) + \sin t,\ v'(t) = e^{t^2}\ v(t) + \sin t$

Subtracting, we have

$u'(t) - v'(t) = e^{t^2}(u(t)-v(t))$

$\Rightarrow \log\ (u - v) = \int e^{t^2}dt + c$

$u(t) - v(t) = C\ e^{\int et^2 dt}$

so $u - v$ is proportional to a positive function of t. Since $u(2) < v(2)$ so $C < 0$.

Hence $u(t) < v(t)$ for all t.

5. $\dfrac{dy}{dx} + 3y = 2 \Rightarrow$ I.F. $= e^{3x}$. So

$\dfrac{d}{dx}\left(ye^{3x}\right) = 2e^{3x}$

$\Rightarrow ye^{3x} = \dfrac{2}{3}e^{3x} + C \Rightarrow y = \dfrac{2}{3}+Ce^{-3x}$

$\lim\limits_{x\to\infty} y(x) = \dfrac{2}{3}.$

6. $\dfrac{dy}{dx} = \dfrac{\sqrt{1-y^2}}{y} \Rightarrow -\dfrac{1}{2}\dfrac{-2y}{\sqrt{1-y^2}}dy = x$

$\Rightarrow -\sqrt{1-y^2} = x + c$

$\Rightarrow 1 - y^2 = (x + c)^2$

$\Rightarrow (x + c)^2 + y^2 = 1,$

which represent a circle with radius 1 and centre on x-axis so fixed radius and variable centre on x-axis.

7. $\dfrac{dy}{dx} = \sin\dfrac{x-y}{2} - \sin\dfrac{x+y}{2}$

$= -2\cos\dfrac{x}{2}\sin\dfrac{y}{2}$

$\Rightarrow \dfrac{dy}{\sin y/2} = -2\cos\dfrac{x}{2}dx$

$\Rightarrow 2\log\ \tan\ y/4 = -4\sin\dfrac{x}{2}+c$

$\Rightarrow \log\ \tan\dfrac{y}{4}+2\sin\dfrac{x}{2} = C$

8. $y\ dx - (x + y^2)dy = 0$

$\Rightarrow \dfrac{ydx - xdy}{y^2} = dy$

$\Rightarrow d\left(\dfrac{x}{y}\right) = dy$

$\Rightarrow \dfrac{x}{y} = y + c$

Since $y(1) = 1$ so $c = 0$

$\Rightarrow x = y^2$

$x = 16$, when $y = 4$.

9. $y\ dy + \sqrt{1+y^2}dx = 0$

$\Rightarrow dx + \dfrac{y}{\sqrt{1+y^2}}dy = 0$

$\Rightarrow x + \sqrt{1+y^2} = c$

$\Rightarrow (c - x)^2 = 1 + y^2$

$\Rightarrow (x - c)^2 - y^2 = 1,$

which represents a hyperbola.

10. Write the given differential equation as

$e^{-xy}\ d(xy) = \left(\dfrac{x}{y}\right)^2 d\left(\dfrac{x}{y}\right)$

Integrating, we get

$-\ e^{-xy} = \dfrac{1}{3}\left(\dfrac{x}{y}\right)^3 + C$

When $y(0) = 1$, we get

$-\ e^0 = \dfrac{1}{3}(0) + C \Rightarrow C = -1$

$\therefore\ 1 - e^{-xy} = \dfrac{1}{3}\left(\dfrac{x}{y}\right)^3$

$\Rightarrow 3y^3(1 - e^{-xy}) = x^3$

CHAPTER SIXTEEN

Cartesian System of Rectangular Coordinates and Straight Lines

RESULTS REGARDING POINTS IN A PLANE

Distance formula

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The distance from the origin O (0, 0) to the point $P(x_1, y_1)$ is $OP = \sqrt{x_1^2 + y_1^2}$.

Illustration 1

Show that the triangle with vertices (3, 0), (–1, –1) and (2, 4) is isosceles and right angled.h

Solution: Let the vertices of the triangle be A (3, 0), B (–1, –1) and C (2, 4) then using distance formula, we have

$$AB = \sqrt{(3+1)^2 + (0+1)^2} = \sqrt{17}$$

$$BC = \sqrt{(-1-2)^2 + (-1-4)^2} = \sqrt{9+25} = \sqrt{34}$$

$$\text{and } CA = \sqrt{(2-3)^2 + (4-0)^2} = \sqrt{17}.$$

Since $AB = AC = \sqrt{17}$, the triangle ABC is isosceles. Also $(AB)^2 + (AC)^2 = (BC)^2$ shows the triangle is right angled.

Section formula

If $R(x, y)$ divides the join of $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $m : n$ ($m, n > 0, m \neq n$) then

$$x = \frac{mx_2 \pm nx_1}{m \pm n} \text{ and } y = \frac{my_2 \pm ny_1}{m \pm n}$$

The positive sign is taken for internal division and the negative sign for external division. The *mid-point* of $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ which corresponds to internal division when $m = n$. Note that for external division $m \neq n$.

Illustration 2

The line $x - y + k = 0$ passes through the point which divides the segment joining the points (2, 3) and (4, 5) in the ratio 2 : 3. Find the value of k.

Solution: Using the section formula, we get the coordinates of the point of division as

$$\left(\frac{2 \times 4 + 3 \times 2}{5}, \frac{2 \times 5 + 3 \times 3}{5}\right) = \left(\frac{14}{5}, \frac{19}{5}\right)$$

Since it lies on the line $x - y + k = 0$

$$\frac{14}{5} - \frac{19}{5} + k = 0 \Rightarrow k = 1$$

Centroid of a triangle

If $G(x, y)$ is the centroid of the triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

$$x = \frac{x_1 + x_2 + x_3}{3} \quad \text{and} \quad y = \frac{y_1 + y_2 + y_3}{3}$$

Illustration 3

Show that the centroid of the triangle with vertices ($5 \cos \theta$, $4 \sin \theta$), ($4 \cos \theta$, $5 \sin \theta$) and (0, 0) lies on the circle $x^2 + y^2 = 9$.

Solution: Coordinates of the centroid of the given triangle are,

$$\left(\frac{5\cos\theta + 4\cos\theta + 0}{3}, \frac{4\sin\theta + 5\sin\theta + 0}{3}\right)$$

$= (3\cos\theta, 3\sin\theta)$

which lies on the circle $x^2 + y^2 = 9$

as $(3\cos\theta)^2 + (3\sin\theta)^2 = 9$.

Note

- When A, B, C are taken as vertices of a triangle, it is assumed that they are not collinear.

Incentre of a triangle

If $I(x, y)$ is the incentre of the triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then

$$x = \frac{ax_1 + bx_2 + cx_3}{a + b + c} \quad \text{and}$$

$$y = \frac{ay_1 + by_2 + cy_3}{a + b + c}$$

a, b and c being the lengths of the sides BC, CA and AB, respectively of the triangle ABC.

Illustration 4

Find the coordinates of the in center of the triangle with vertices A (–1, 12), B (–1, 0) and C (4, 0).

Solution: We have

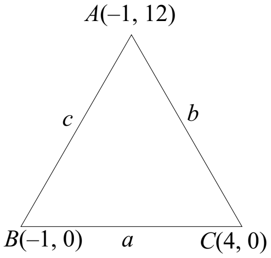


Fig. 16.1

$$a = BC = \sqrt{(-1-4)^2 + 0} = 5$$

$$b = CA = \sqrt{(4+1)^2 + (0-12)^2} = 13$$

$$c = AB = \sqrt{(-1+1)^2 + (12-0)^2} = 12$$

So using the formula for the coordinates of the in-center we get the required coordinates as

$$\left(\frac{5(-1) + 13(-1) + 12(4)}{5 + 13 + 12}, \frac{5(12) + 13(0) + 12(0)}{5 + 13 + 12}\right)$$

= (1, 2)

Area of triangle

ABC with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is

$$\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

and is generally denoted by Δ. Note that if one of the vertex (x_3, y_3) is at $O(0, 0)$, then $\Delta = \frac{1}{2}\left|x_1 y_2 - x_2 y_1\right|$.

Illustration 5

If the area of the triangle with vertices at the points whose coordinates are (2, 5), (0, 3) and (4, k) is 4 units, then find the value of k.

Solution: We have

$$\frac{1}{2}\left|\begin{vmatrix} 2 & 5 & 1 \\ 0 & 3 & 1 \\ 4 & k & 1 \end{vmatrix}\right| = 4$$

$\Rightarrow \quad 2(3 - k) + 4(5 - 3) = \pm 8$

$\Rightarrow 7 - k = \pm 4 \Rightarrow k = 3$ or 11

Condition of collinearity

Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Slope of a line

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ $(x_1 \neq x_2)$ be any two points. Then the slope of the line joining A and B is defined as

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

where θ is the angle which the line makes with the positive direction of the x-axis, $0° \leq \theta \leq 180°$, except at $\theta = 90°$. Which is possible only if $x_1 = x_2$ and the line is parallel to the y-axis.

Condition for the points $Z_k = x_k + iy_k$ $(k = 1, 2, 3)$ to form an equilateral triangle is

$Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$ (See Complex numbers)

STANDARD FORMS OF THE EQUATION OF A LINE

1. An equation of a line parallel to the x-axis is $y = k$ and that of the x-axis itself is $y = 0$.

Illustration 6

Find the equation of the line parallel to x-axis passing through the intersection of the lines $3ax + 2by + 7b = 0$ and $3bx - 2ay - 7a = 0$, where $(a, b) \neq (0, 0)$

Solution: Equation of the line parallel to x-axis is $y = k$. So we need to find the y-coordinates of the point of intersection of the given lines. Multiplying the first equation by b and the second by a and subtracting we get.

$2(b^2 + a^2)y + 7(b^2 + a^2) = 0$

$\Rightarrow y = -\frac{7}{2}$ which is the required equation.

2. An equation of a line parallel to the y-axis is $x = h$ and that of the y-axis itself is $x = 0$.

3. An equation of a line passing through the origin and (a) making an angle θ with the positive direction of the x-axis is $y = x \tan \theta$; (b) having a slope m is $y = mx$; and (c) passing through the point (x_1, y_1) is $x_1 y = y_1 x$.

4. *Slope-Intercept form* An equation of a line with slope m and making an intercept c on the y-axis is $y = mx + c$.

5. *Point-Slope form* An equation of a line with slope m and passing through (x_1, y_1) is $y - y_1 = m(x - x_1)$.

6. *Two-Point form* An equation of a line passing through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

7. *Intercept form* An equation of a line making intercepts a and b on the x-axis and y-axis respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Illustration 7

Find the equation of the straight line passing through the point (10, –7) and making intercepts on the coordinate axes whose sum is 12.

Solution: Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ where $a + b = 12$.

Since it passes through (10, – 7), $\frac{10}{a} - \frac{7}{b} = 1$

$\Rightarrow 10b - 7a = ab$
$\Rightarrow 10(12 - a) - 7a = a(12 - a)$
$\Rightarrow a^2 - 29a + 120 = 0$
$\Rightarrow a = 5, 24.$

When $a = 5$, $b = 7$; the required equation is $\frac{x}{5} + \frac{y}{7} = 1$

When a = 24, $b = -12$, the required equation is $\frac{x}{24} - \frac{y}{12} = 1$

8. *Parametric form* An equation of a line passing through a fixed point $A(x_1, y_1)$ and making an angle θ, $0 \le \theta \le \pi$, $\theta \ne \pi/2$ with the positive direction of the x-axis is

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$$

where r is the distance of *any* point $P(x, y)$ on the line from the point $A(x_1, y_1)$. Note that $x = x_1 + r \cos \theta$ and $y = y_1 + r \sin \theta$.

9. *Normal form* An equation of a line such that the length of the perpendicular from the origin on it is p and the angle which this perpendicular makes with the positive direction of the x-axis is α, is $x \cos \alpha + y \sin \alpha = p$.

10. *General form* In general, an equation of a straight line is of the form $ax + by + c = 0$, where a, b and c are real numbers and a and b cannot both be zero simultaneously. From this general form of the equation of the line, we can calculate the following:

(i) The slope is $-a/b$ $(b \ne 0)$.

(ii) The intercept on the x-axis is $-c/a$ $(a \ne 0)$, and the intercept on the y-axis is $-c/b$ $(b \ne 0)$.

(iii) $p = \frac{|c|}{\sqrt{a^2 + b^2}}$, $\cos \alpha = \pm \frac{|a|}{\sqrt{a^2 + b^2}}$ and

$\sin \alpha = \pm \frac{|b|}{\sqrt{a^2 + b^2}}$, the positive sign being taken if c is negative and vice versa.

(iv) If p_1 denotes the length of the perpendicular from (x_1, y_1) on this line, then

$$p_1 = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

(v) The points (x_1, y_1) and (x_2, y_2) lie on the same side of the line if the expressions $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same sign, and on the opposite side if they have the opposite signs.

SOME RESULTS FOR TWO OR MORE LINES

1. Two lines given by the equations $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are

(i) *parallel* (i.e., the slopes are equal), if $ab' = a'b$

(ii) *perpendicular* (i.e., the product of their slopes is –1), if $aa' + bb' = 0$

(iii) identical if $ab'c' = a'b'c = a'c'b$.

(iv) not parallel, then

(a) angle θ between them at their point of intersection is given by

$$\tan \theta = \pm \frac{m - m'}{1 + mm'} = \pm \frac{a'b - ab'}{aa' + bb'}$$

m, m' being the slopes of the two lines.

(b) the coordinates of their point of intersection are

$$\left(\frac{bc' - b'c}{ab' - a'b}, \frac{ca' - c'a}{ab' - a'b} \right)$$

(c) An equation of any line through their point of intersection is

$$(ax + by + c) + \lambda(a'x + b'y + c') = 0$$

where λ is a real number.

2. An equation of a line parallel to the line $ax + by + c = 0$ is $ax + by + c' = 0$, and the distance between these lines is

$$\frac{|c - c'|}{\sqrt{a^2 + b^2}}$$

Illustration 8

The line L is given by $2x + by = 7$, passes through (8, 3). The line K is parallel to L and has the equation $cx + 2y = c$. Find the distance between L and K.

Solution: Equation of L is $2x - 3y = 7$ as it passes through (8, 3) ⇒ slope of L is (2/3).

$$\Rightarrow \text{slope of } K = -\frac{c}{2} = \frac{2}{3} \Rightarrow c = -\frac{4}{3}.$$

and thus the equation of K is $2x - 3y = 2$.

So the required distance is $\dfrac{7-2}{\sqrt{2^2 + (-3)^2}} = \dfrac{5}{\sqrt{13}}$

3. Three lines $a_1 x + b_1 y + c_1 = 0$, $a_2 x + b_2 y + c_2 = 0$ and $a_3 x + b_3 y + c_3 = 0$ are *concurrent* (intersect at a point) if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

4. Equations of the *bisectors of the angles* between two intersecting lines $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

Any point on the bisectors is equidistant from the given lines. If ϕ is the angle between one of the bisectors and one of the lines $ax + by + c = 0$, such that $|\tan \phi| < 1$, i.e., $-\pi/4 < \phi < \pi/4$, then that bisector bisects the acute angle between the two lines, i.e., it is the *acute angle bisector* of the two lines. The other equation then represents the *obtuse angle bisector* between the two lines.

5. Equations of the lines through (x_1, y_1) and making an angle ϕ with the line $ax + by + c = 0$, $b \neq 0$ are

$$y - y_1 = m_1 (x - x_1) \text{ where } m_1 = \frac{\tan\theta - \tan\phi}{1 + \tan\theta \tan\phi}$$

and $$y - y_1 = m_2 (x - x_1) \text{ where } m_2 = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \tan\phi}$$

where $\tan\theta = -a/b$ is the slope of the given line. Note that $m_1 = \tan(\theta - \phi)$, $m_2 = \tan(\theta + \phi)$ and when $b = 0$, $\theta = \pi/2$.

SOME USEFUL POINTS

To show that A, B, C, D are the vertices of a

Parallelogram

Show that the diagonals AC and BD bisect each other.

Illustration 9

If $P(4, 7)$, $Q(7, 2)$ $R(a, b)$ and $S(3, 8)$ are the vertices of a parallelogram then find the value of a and b.

Solution: Since the diagonals of a parallelogram bisect each other. Mid points of PR and QS are same.

$$\text{so } \left(\frac{4+a}{2}, \frac{7+b}{2}\right) = \left(\frac{7+3}{2}, \frac{8+2}{2}\right)$$

$\Rightarrow a = 6, b = 3$

Rhombus

Show that the diagonals AC and BD bisect each other and a pair of adjacent sides, say, AB and BC are equal.

Square

Show that the diagonals AC and BD are equal and bisect each other, a pair of adjacent sides say AB and BC are equal.

Rectangle

Show that the diagonals AC and BD are equal and bisect each other.

LOCUS OF A POINT

To obtain the equation of a set of points satisfying some given condition(s) called locus, proceed as follows.

1. Let P (h, k) be any point on the locus.
2. Write the given condition involving h and k and simplify. If possible draw a figure.
3. Eliminate the unknowns, if any.
4. Replace h by x and k by y and obtain an equation in terms of (x, y) and the known quantities. This is the required locus.

Illustration 10

Find the locus of the mid point of the portion between the axes of the line $x \cos \alpha + y \sin \alpha = \sin \alpha \tan \alpha$.

Solution: The given line meets x-axis at the point A $(\tan^2 \alpha, 0)$ and y-axis at the point B $(0, \tan \alpha)$

Let $P(h, k)$ be the mid point of AB, then

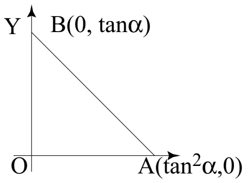


Fig. 16.2

$$h = \frac{\tan^2\alpha}{2},\ k = \frac{\tan\alpha}{2}.$$

Eliminating α, we get
$2k^2 = h$
So the required locus is $2y^2 - x = 0$.

CHANGE OF AXES

Rotation of Axes

If the axes are rotated through an angle θ in the anticlockwise direction keeping the origin fixed, then the coordinates (X, Y) of a point $P(x, y)$ with respect to the new system of coordinate are given *by* $X = x \cos\theta + y \sin\theta$ and $Y = y \cos\theta - x \sin\theta$.

Translation of Axes

The shifting of origin of axes without rotation of axes is called *Translation of axes*.

If the origin (0, 0) is shifted to the point (h, k) without rotation of the axes then the coordinates (X, Y) of a point $P(x, y)$ with respect to the new system of coordinates are given by $X = x - h$, $Y = y - k$.

Illustration 11

If origin is shifted to the point (2, 3) and the axes are rotated through an angle $\pi/4$ in the anticlockwise direction, then find the coordinates of the point (7, 11) in the new system of coordinates.

Solution: When origin is shifted to the point (2, 3), the coordinates of $P(7, 11)$ are $(7 - 2, 11 - 3) = (5, 8)$.

Now when the axes are rotated through an angle $\pi/4$ in the anticlockwise direction, the coordinates of P are $(5 \cos \pi/4 + 8 \sin \pi/4, 8 \cos \pi/4 - 5 \sin \pi/4.) = \left(\frac{13}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

which are the required coordinates.

EQUATION OF FAMILY OF LINES

A first degree equation $ax + by + c = 0$ represents a straight line involving three constants a, b and c which can be reduced to two by dividing both the sides of the equation by a non zero constant. For instance if $a \neq 0$, we can write the equation as

$$x + \frac{b}{a}y + \frac{c}{a} = 0$$

or $$x + By + C = 0$$

So in order to determine an *equation of a line* we need two conditions on the line to determine these constants. For instance, if we know two points on the line or a point on the line and its slope etc., we know the line. But if we know just one condition, we have infinite number of lines satisfying the given condition. In this case the equation of the line contains an arbitrary constant and for different values of the constant we have different lines satisfying the given condition and the constant is called a parameter.

Some Equations of Family of Lines

1. *Family of lines with given slope* (Family of parallel lines)
   $y = mx + k$, where k is *a* parameter represents a family of lines in which each line has slope m.
2. *Family of lines through a point*
   $y - y_0 = k(x - x_0)$, where k is a parameter represents a family of lines in which each line passes through the point (x_0, y_0).
3. *Family of lines parallel to a given line*
   $ax + by + k = 0$, where k is a parameter represents a family of lines which are parallel to the line $ax + by + c = 0$.
4. *Family of lines through intersection of two given lines*
   $a_1 x + b_1 y + c_1 + k(a_2 x + b_2 y + c_2) = 0$; where k is a parameter represents a family of lines, in which each line passes through the point of intersection of two intersecting lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

Illustration 12

Two sides of a triangle given by $2x + 3y - 5 = 0$ and $5x - 4y + 10 = 0$ intersect at A. Centroid of the triangle is at the origin. find the equation of the median through A.

Solution: Equation of any line through A is $(2x + 3y - 5) + \lambda(5x - 4y + 10) = 0$. This will represent the median through A if it passes through the centroid (0, 0)

$\Rightarrow -5 + 10\lambda = 0 \Rightarrow \lambda = 1/2$ and the required equation is $2(2x + 3y - 5) + (5x - 4y + 10) = 0$

$\Rightarrow 9x + 2y = 0$

5. *Family of lines perpendicular to a given line*
   $bx - ay + k = 0$, where k is a parameter represents a family of lines, in which each line is perpendicular to the line $ax + by + c = 0$.
6. *Family of lines making a given intercept on axes*
   (a) $\frac{x}{a} + \frac{y}{k} = 1$, where k is a parameter, represents a family of lines, in which each line makes an intercept a on the axis of x.
   (b) $\frac{x}{k} + \frac{y}{b} = 1$, where k is a parameter, represents a family of lines, in which each line makes an intercept b on the axis of y.
7. *Family of lines at a constant distance from the origin*
   $x \cos\alpha + y \sin\alpha = p$, where α is a parameter, represents a family of lines, in which each line is at a distance p from the origin.

Circumcentre of a triangle

The circumcentre of a triangle is the centre of the circle passing through the vertices of the triangle, so it is equi distant from the vertices of the triangle.

For example–The circumcentre (x, y) of the triangle with vertices (0, 0), (3, 4) and (4, 3) is given by

$$x^2 + y^2 = (x - 3)^2 + (y - 4)^2 = (x - 4)^2 + (y - 3)^2$$

Solving, we get $(x, y) = (25/14, 25/14)$ as the required circumcentre of the triangle.

Orthocentre of a triangle

The orthocentre of a triangle is the point of intersection of any two altitudes of the triangle.

For example–The orthocentre (p, q) of the triangle with vertices (0, 0), (3, 4) and (4, 3) is the point of intersection of the altitudes $y - x = 0$ through (0, 0) and $4x + 3y = 24$ through (3, 4) which is given by $(p, q) = \left(\frac{24}{7}, \frac{24}{7}\right)$

Note

- ☐ We know from geometry that circumcentre, centroid and orthocentre of a triangle lie on a line.

We observe that the centroid $\left(\frac{7}{3}, \frac{7}{3}\right)$ circumcentre $\left(\frac{25}{14}, \frac{25}{14}\right)$ and orthocentre $\left(\frac{24}{7}, \frac{24}{7}\right)$ of the triangle in the above example lie on the line $y = x$

SOLVED EXAMPLES

Concept-based Straight Objective Type Questions

Example 1: The base of an equilateral triangle with side $2a$ lies along the y-axis such that the midpoint of the base is at the origin if the centroid of the triangle lies on the + ve x-axis, coordinates of the vertex are.

(a) $(a, 0)$ (b) $(-a, 0)$

(c) $(a\sqrt{3}, 0)$ (d) $(-a\sqrt{3}, 0)$

Ans. (c)

Solution: Coordinates of the base BC of the triangle are $B(0, a)$ and $C(0, -a)$ as origin is the mid-point.

Since the triangle is equilateral vertex A lies on the perpendicular bisector of the base which is x-axis. Let the coordinates of A be $(x, 0)$.

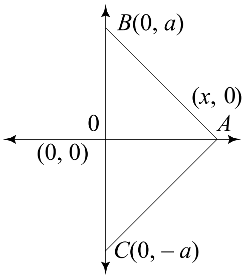


Fig. 16.3

Since the triangle is equilateral $AB = BC$

$\Rightarrow \quad x^2 + a^2 = (2a)^2$

$\Rightarrow \quad x^2 = 3a^2 \Rightarrow x = \pm\sqrt{3}a$

But since the centroid of the triangle lies on positive x-axis, coordinates of the vertex A are $(a\sqrt{3}, 0)$

Example 2: If three points $(h, 0)$, (a, b) and (o, k) lie on a line then

(a) $ah + bk = 1$ (b) $\frac{a}{h} + \frac{b}{k} = 1$

(c) $ak + bh = 1$ (d) $\frac{a}{k} + \frac{b}{h} = 1$

Ans. (b)

Solution: If given points lie on a line, then

$$\begin{vmatrix} h & 0 & 1 \\ a & b & 1 \\ 0 & k & 1 \end{vmatrix} = 0$$

Expanding along first row

$$h(b - k) + 1.\,(ak - b \times 0) = 0$$

$\Rightarrow \quad bh + ak = hk$

$\Rightarrow \quad \frac{a}{h} + \frac{b}{k} = 1$

Alternatively: Equation of the line joining $(h, 0)$ and $(0, k)$ is $\frac{x}{h} + \frac{y}{k} = 1$ as it makes intercepts h on x-axis and k on y-axis

If the point (a, b) also lies on it. then $\frac{a}{h} + \frac{b}{k} = 1$

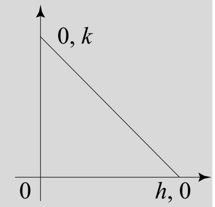


Fig. 16.4

● **Example 3:** The line joining the points (2, x) and (3, 1) is perpendicular to the line joining the points (x, 4) and (7, 5). The value of x is

(a) 2 (b) 3
(c) 4 (d) 7

Ans. (c)

◎ **Solution:** Slope of the line joining (2, x) and (3, 1) is $\frac{1-x}{3-2}$ and that of the line joining (x, 4) and (7, 5) is $\frac{5-4}{7-x}$.

Since the lines are perpendicular the product of the slopes is – 1

$$\Rightarrow \quad \frac{1-x}{1} \times \frac{1}{7-x} = -1$$

$$\Rightarrow \quad x = 4.$$

● **Example 4:** If the point (a, b) divides a line between the axes in the ratio 2 : 3, the equation of the line is

(a) $ax + by = 5$ (b) $bx + ay = 5$
(c) $\frac{2x}{a} + \frac{3y}{b} = 5$ (d) $2\,ax + 3by = 5$

Ans. (c)

◎ **Solution:** Let the line meet x-axis at (h, 0) and y-axis at (o, k), then the equation of the line is $\frac{x}{h} + \frac{y}{k} = 1$

(Intercept form)

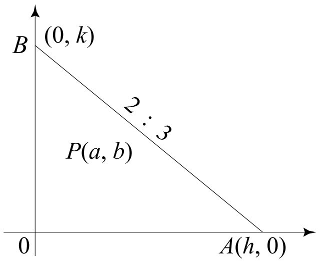


Fig. 16.5

Since $P(a, b)$ divides BA in the ratio 2 : 3 we get

$$\frac{2h + 0 \times 3}{5} = a, \quad \frac{2 \times 0 + 3k}{5} = b$$

$\Rightarrow h = \frac{5a}{2}$, $k = \frac{5b}{3}$ and the required equation is $\frac{2x}{a} + \frac{3y}{b} = 5$

● **Example 5:** P is a point on x-axis, Q is a point on y-axis. Both are equidistant from the points (7, 6) and (3, 4). Distance between P and Q is

(a) $2\sqrt{5}$ (b) $15\sqrt{5}$
(c) $15\sqrt{5}/2$ (d) $5\sqrt{15}/2$

Ans. (c)

◎ **Solution:** Let the coordinates of P be (x, 0) and of Q be (0, y), then

$(x - 7)^2 + 6^2 = (x - 3)^2 + 4^2 \Rightarrow x = 15/2$

and $7^2 + (y - 6)^2 = 3^2 + (y - 4)^2 \Rightarrow y = 15$

So coordinates of P are (15/2, 0) and of Q are (0, 15)

$$\Rightarrow \quad PQ = \sqrt{\left(\frac{15}{2}\right)^2 + (15)^2} = \frac{15\sqrt{5}}{2}$$

● **Example 6:** The slope of a line is 3 times the slope of the other line and the tangent of the angle between them is 4/13, sum of their slopes is equal to

(a) 2 (b) 4
(c) 6 (d) 8

Ans. (d)

◎ **Solution:** Let the slopes of the given lines be m and $3\,m$. if θ is the angle between them then

$$\tan\theta = \frac{|3m - m|}{1 + (3m)(m)} = \frac{4}{13}$$

$$\Rightarrow \quad \pm\, 2m \times 13 = 4\,(1 + 3m^2)$$

$$\Rightarrow \quad 6m^2 \pm 13m + 2 = 0$$

$$\Rightarrow \quad m = \frac{1}{6} \text{ or } 2 \text{ and } m = -\frac{1}{6} \text{ or } -2.$$

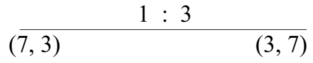
For $m = 2$, the sum of the slopes is $4m = 8$

● **Example 7:** A line perpendicular to the line segment joining the points (7, 3) and (3, 7) divides it in the ratio 1 : 3, the equation of the line is

(a) $x + y - 10 = 0$ (b) $x - y + 4 = 0$
(c) $x - y + 2 = 0$ (d) $x - y - 2 = 0$

Ans. (d)

◎ **Solution:** Slope of the given line is $\frac{7-3}{3-7} = -1$ and the coordinates of the point which divides the given line segment in the ratio 1 : 3



is $\left(\frac{1 \times 3 + 3 \times 7}{4}, \frac{1 \times 7 + 3 \times 3}{4}\right) = (6, 4)$

So equation of the line passing through the point (6, 4) and perpendicular to the given line is

$$y - 4 = x - 6 \Rightarrow x - y - 2 = 0.$$

● **Example 8:** If the distance between the parallel lines $3\,x + 4y + 7 = 0$ and $ax + y + b = 0$ is 1, the integral value of b is

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (c)

◎ **Solution:** Since the given lines are parallel. Slopes of the lines are equal so $a = \frac{3}{4}$ and the two lines are $3x + 4y + 7 = 0$ and $3x + 4y + 4b = 0$.

Distance between them is $\frac{|7-4b|}{\sqrt{3^2+4^2}} = 1$

$\Rightarrow 7 - 4b = \pm 5 \Rightarrow b = 1/2$ or 3

Example 9: If P and Q are the lengths of the perpendiculars from the origin to the lines $x \cos\theta + y \sin\theta = k \cos 2\theta$ and $x \sec\theta - y \operatorname{cosec}\theta = k$ respectively then

(a) $p^2 - 4q^2 = k^2$ (b) $p^2 + 4q^2 = k^2$
(c) $4p^2 + q^2 = k^2$ (d) $p^2 + q^2 = k^2$

Ans. (b)

Solution: $p = \frac{|k \cos 2\theta|}{\sqrt{\cos^2\theta + \sin^2\theta}} = |k \cos 2\theta|$

$$q = \frac{|k|}{\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}} = |k \sin\theta \cos\theta|$$

$$= |k\,(1/2)\sin 2\theta|$$

So $p^2 + 4q^2 = k^2(\cos^2 2\theta + \sin^2 2\theta) = k^2.$

Example: 10: If the lines $3x - y + 1 = 0$ and $x - 2y + 3 = 0$ are equally inclined to the line $y = mx$, then the value of m is given by

(a) $2m^2 - 7m - 7 = 0$ (b) $7m^2 - 7m - 2 = 0$
(c) $7m^2 - 2m - 7 = 0$ (d) $2m^2 - 7m - 2 = 0$

Ans. (c)

Solution: If θ is the angle between the lines then

$\tan\theta = \left|\frac{m-3}{1+3m}\right| = \left|\frac{m-(1/2)}{1+m(1/2)}\right|$

$\Rightarrow \quad (m-3)(2+m) = \pm(2m-1)(3m+1)$
$\Rightarrow \quad m^2 - m - 6 = \pm(6m^2 - m - 1)$
$\Rightarrow \quad 7m^2 - 2m - 7 = 0$ or $m^2 + 1 = 0$

Example 11: Equation of the line which makes an intercept of length 2 on positive x-axis and an intercept of length 3 on the negative y-axis is

(a) $2x - 3y + 6 = 0$ (b) $3x - 2y - 6 = 0$
(c) $2x - 3y - 6 = 0$ (d) $3x - 2y + 6 = 0$

Ans. (b)

Solution: Equation of the required line is $\frac{x}{2} + \frac{y}{-3} = 1$

$\Rightarrow \quad 3x - 2y = 6$

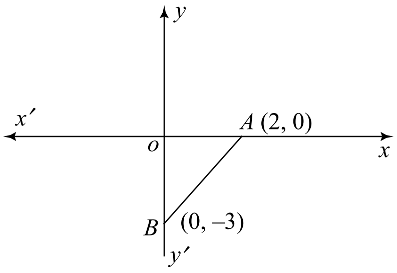


Fig. 16.6

Example 12: The perpendicular from the origin to a line L meets it at the point $(3, -9)$, equation of the line L is

(a) $x - 3y = 30$ (b) $3x - y = 18$
(c) $x + 3y + 24 = 0$ (d) $3x + y = 18$

Ans. (a)

Solution: Slope of the perpendicular line is $\frac{-9}{3} = -3$

$\Rightarrow$ slope of L is (1/3) and as it passes through $(3, -9)$

equation of L is

$$y + 9 = (1/3)(x - 3)$$

$\Rightarrow \quad x - 3y = 30$

Example 13: The distance of the point (2, 3) from the line $4x - 3y + 26 = 0$ is same as its distance from the line $3x - 4y + p = 0$. The value of p can be

(a) 5 (b) 25
(c) 31 (d) – 31

Ans. (c)

Solution: We have

$$\left|\frac{4\times2-3\times3+26}{\sqrt{4^2+3^2}}\right| = \left|\frac{3\times2-4\times3+p}{\sqrt{4^2+3^2}}\right|$$

$\Rightarrow \quad \pm 25 = 6 - 12 + p$
$\Rightarrow \quad p = \pm 25 + 6$
$\Rightarrow \quad p = 31$ or -19.

Example 14: If the segment of the line between the lines $x - y + 2 = 0$ and $x + y + 4 = 0$ is bisected at the origin, equation of the line is

(a) $y + 3x = 0$ (b) $x + 3y = 0$
(c) $y - 3x = 0$ (d) $x - 3y = 0$

Ans. (c)

Solution: Let the required line meet the given lines at (α_1, β_1) and (α_2, β_2) respectively.
then $\alpha_1 - \beta_1, + 2 = 0$, $\alpha_2 + \beta_2 + 4 = 0$

Also $\frac{\alpha_1 + \alpha_2}{2} = 0$, $\frac{\beta_1 + \beta_2}{2} = 0$

$\Rightarrow \quad \alpha_2 = -1, \beta_2 = -3$ and $\alpha_1 = 1, \beta_1 = 3$
Equation of the line passing through (1, 3) and (0, 0) is

$$y = 3x \quad \Rightarrow \quad y - 3x = 0$$

Example 15: A ray of light passing through the point (3, 7) reflects on the x-axis at a point A and the reflected ray passes through the point (2, 5), the coordinates of A are

(a) $\left(\frac{29}{12}, 0\right)$ (b) $\left(\frac{1}{2}, 0\right)$
(c) $\left(-\frac{1}{2}, 0\right)$ (d) $\left(-\frac{29}{12}, 0\right)$

Ans. (a)

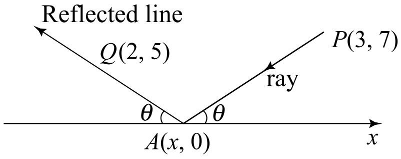


Fig. 16.7

◎ **Solution:** Let P be the given point (3, 7) and the coordinates of A be $(x, 0)$ so, if the ray PA makes an angle θ with the positive direction of x-axis, the reflected ray AQ , where Q is (2, 5), will make the same angle θ with the negative direction of x-axis. So the slope of PA = – (the slope of QA)

$$\Rightarrow \quad \frac{7}{3-x} = -\frac{5}{2-x}$$

$$\Rightarrow \quad 7(2-x) + 5(3-x) = 0$$

$$\Rightarrow \quad x = \frac{29}{12}$$

$\Rightarrow$ coordinates of A are $\left(\frac{29}{12}, 0\right)$.

LEVEL 1

Straight Objective Type Questions

Example 16: If a vertex of a triangle is (1, 1) and the mid points of two sides through this vertex are (– 1, 2) and (3, 2), then centroid of the triangle is

(a) (1, 7/3)	(b) (1/3, 7/3)
(c) (– 1, 7/3)	(d) (– 1/3, 7/3)

Ans. (a)

◎ **Solution:** Let (x_1, y_1) and (x_2, y_2) be the other two vertices of the triangle, then $\frac{x_1+1}{2} = -1$, $\frac{y_1+1}{2} = 2$

$\Rightarrow (x_1, y_1) = (-3, 3)$; similarly $(x_2, y_2) = (5, 3)$

centroid of the triangle is $\left(\frac{1-3+5}{3}, \frac{1+3+3}{3}\right) = (1, 7/3)$

Example 17: The points $(a, b + c)$, $(b, c + a)$ and $(c, a + b)$ are

(a) vertices of an equilateral triangle
(b) concyclic
(c) vertices of a right angled triangle
(d) none of these

Ans. (d)

◎ **Solution:** As the given points lie on the line $x + y = a + b + c$, they are collinear.

Example 18: If the lines $x + 2ay + a = 0$, $x + 3by + b = 0$ and $x + 4cy + c = 0$ are concurrent, then a, b, c are in

(a) A.P.	(b) G.P.
(c) H.P.	(d) none of these

Ans. (c)

◎ **Solution:** Since the given lines are concurrent

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0 \Rightarrow \quad -bc + 2ac - ab = 0$$

$$\Rightarrow \qquad b = \frac{2ac}{a+c} \quad \Rightarrow \quad a, b, c \text{ are in H.P.}$$

Example 19: If p is the length of the perpendicular from the origin on the line $\frac{x}{a} + \frac{y}{b} = 1$ and a^2, p^2, b^2 are in A.P., then $a^4 - 2p^2a^2 + 2p^4 =$

(a) – 1	(b) 0
(c) 1	(d) none of these

Ans (b)

◎ **Solution:** We have $p = \frac{|a\,b|}{\sqrt{a^2+b^2}} \quad \Rightarrow \quad p^2 = \frac{a^2b^2}{a^2+b^2}$

Also, since a^2, p^2, b^2 are in A.P.,

$$a^2 + b^2 = 2p^2 \text{ thus } a^2b^2 = p^2(a^2 + b^2) = 2p^4$$

$$\Rightarrow \qquad 2p^4 = a^2(2p^2 - a^2) \Rightarrow a^4 - 2p^2a^2 + 2p^4 = 0$$

Example 20: If a, x_1, x_2 are in G.P. with common ratio r, and b, y_1, y_2 are in G.P. with common ratio s where $s - r = 2$, then the area of the triangle with vertices (a, b), (x_1, y_1) and (x_2, y_2) is

(a) $\lvert ab(r^2 - 1)\rvert$	(b) $ab(r^2 - s^2)$
(c) $ab(s^2 - 1)$	(d) $abrs$

Ans. (a)

◎ **Solution:** Area of the triangle

$$= \frac{1}{2}\left|\begin{vmatrix} a & b & 1 \\ ar & bs & 1 \\ ar^2 & bs^2 & 1 \end{vmatrix}\right| = \frac{1}{2}\lvert ab(r-1)(s-1)(s-r)\rvert$$

$$= \lvert ab(r-1)(r+1)\rvert = \lvert ab(r^2-1)\rvert$$

Example 21: The line joining A $(b \cos \alpha, b \sin \alpha)$ and B $(a \cos \beta, a \sin \beta)$ is produced to the point $M(x, y)$ so that $AM : MB = b : a$, then $x \cos \frac{\alpha+\beta}{2} + y \sin \frac{\alpha+\beta}{2} =$

(a) – 1 (b) 0
(c) 1 (d) $a^2 + b^2$

Ans. (b)

Solution: As M divides AB externally in the ratio $b : a$

$$x = \frac{b(a\cos\beta) - a(b\cos\alpha)}{b-a} \text{ and}$$

$$y = \frac{b(a\sin\beta) - a(b\sin\alpha)}{b-a}$$

$$\Rightarrow \quad \frac{x}{y} = \frac{\cos\beta - \cos\alpha}{\sin\beta - \sin\alpha} = \frac{2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}$$

$$\Rightarrow \quad x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = 0.$$

Example 22: If the circumcentre of a triangle lies at the origin and the centroid is the middle point of the line joining the points $(a^2 + 1, a^2 + 1)$ and $(2a, -2a)$; then the orthocentre lies on the line

(a) $y = (a^2 + 1)x$ (b) $y = 2ax$
(c) $x + y = 0$ (d) $(a - 1)^2 x - (a + 1)^2 y = 0$

Ans. (d)

Solution: We know from geometry that the circumcentre, centroid and orthocentre of a triangle lie on a line. So the orthocentre of the triangle lies on the line joining the circumcentre (0, 0) and the centroid $\left(\frac{(a+1)^2}{2}, \frac{(a-1)^2}{2}\right)$

i.e. $\frac{(a+1)^2}{2}y = \frac{(a-1)^2}{2}x$

or$(a - 1)^2 x - (a + 1)^2 y = 0$.

Example 23: If p_1, p_2 denote the lengths of the perpendiculars from the origin on the lines $x \sec\alpha + y \operatorname{cosec}\alpha = 2a$ and $x\cos\alpha + y\sin\alpha = a\cos 2\alpha$ respectively, then $\left(\frac{p_1}{p_2} + \frac{p_2}{p_1}\right)^2$ is equal to

(a) $4\sin^2 4\alpha$ (b) $4\cos^2 4\alpha$
(c) $4\operatorname{cosec}^2 4\alpha$ (d) $4\sec^2 4\alpha$

Ans. (c)

Solution: $p_1^2 + p_2^2 = \frac{4a^2}{\sec^2\alpha + \operatorname{cosec}^2\alpha} + \frac{a^2\cos^2 2\alpha}{\cos^2\alpha + \sin^2\alpha}$

$$= a^2(\sin^2 2\alpha + \cos^2 2\alpha) = a^2$$

and $p_1^2 p_2^2 = a^4\sin^2 2\alpha\cos^2 2\alpha = (1/4)a^4\sin^2 4\alpha$

$$\therefore \left(\frac{p_1}{p_2} + \frac{p_2}{p_1}\right)^2 = \frac{\left(p_1^2 + p_2^2\right)^2}{p_1^2 p_2^2} = \frac{4}{\sin^2 4\alpha} = 4\operatorname{cosec}^2 4\alpha.$$

Example 24: *OPQR* is a square and M, N are the middle points of the sides PQ and QR respectively then the ratio of the areas of the square and the triangle OMN is

(a) 4 : 1 (b) 2 : 1
(c) 8 : 3 (d) 4 : 3

Ans. (c)

Solution: Taking the coordinates of vertices O, P, Q, R as (0, 0), (a, 0), (a, a), (0, a) respectively we get the coordinates of M as (a, a/2) and of N as (a/2, a)

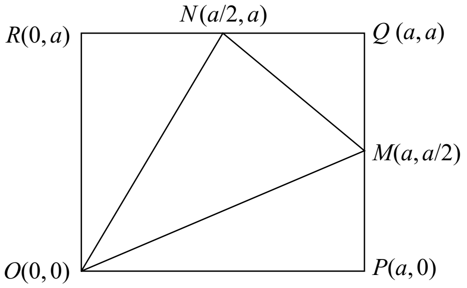


Fig. 16.8

$$\therefore \quad \text{Area of the } \Delta OMN = \frac{1}{2}\left|\begin{vmatrix} 0 & 0 & 1 \\ a & a/2 & 1 \\ a/2 & a & 1 \end{vmatrix}\right| = \frac{3a^2}{8}$$

Area of the square = a^2

$\therefore$ the required ratio is 8 : 3.

Example 25: The locus of the point of intersection of the lines $x\sin\theta + (1 - \cos\theta)y = a\sin\theta$ and $x\sin\theta - (1 + \cos\theta)y + a\sin\theta = 0$ is

(a) $x^2 - y^2 = a^2$ (b) $x^2 + y^2 = a^2$
(c) $y^2 = ax$ (d) none of these

Ans. (b)

Solution: From the given equations we have

$$\frac{1-\cos\theta}{\sin\theta} = \frac{a-x}{y} \text{ and } \frac{1+\cos\theta}{\sin\theta} = \frac{a+x}{y}$$

Multiplying we get $\frac{1-\cos^2\theta}{\sin^2\theta} = \frac{a^2-x^2}{y^2} \Rightarrow x^2 + y^2 = a^2$

Example 26: Two points (a, 3) and (5, b) are the opposite vertices of a rectangle. If the other two vertices lie on the line $y = 2x + c$ which passes through the point (a, b) then the value of c is

(a) – 7 (b) – 4
(c) 0 (d) 7

Ans. (a)

Solution: Mid point of the line joining the given points lie on the given line

$$\frac{3+b}{2} = 2\left(\frac{a+5}{2}\right) + c$$

$\Rightarrow$ $2a + 2c - b + 7 = 0$ (i)

Also since the given line passes through (a, b)

$b = 2a + c$ (ii)

Solving (i) and (ii) we get $c = -7$

Example 27: If every point on the line $(a_1 - a_2)x + (b_1 - b_2)y = c$ is equidistant from the points (a_1, b_1) and (a_2, b_2) then $2c =$

(a) $a_1^2 - b_1^2 + a_2^2 - b_2^2$ (b) $a_1^2 + b_1^2 + a_2^2 + b_2^2$

(c) $a_1^2 + b_1^2 - a_2^2 - b_2^2$ (d) none of these

Ans. (c)

Solution: Let (h, k) be any point on the given line then

$$(h - a_1)^2 + (k - b_1)^2 = (h - a_2)^2 + (k - b_2)^2$$

$$\Rightarrow \quad 2(a_1 - a_2)h + 2(b_1 - b_2)k = a_1^2 + b_1^2 - a_2^2 - b_2^2$$

$$\Rightarrow \quad (a_1 - a_2)h + (b_1 - b_2)k = (1/2)(a_1^2 + b_1^2 - a_2^2 - b_2^2) \qquad \text{(i)}$$

Since (h, k) lies on the given line

$$(a_1 - a_2)h + (b_1 - b_2)k = c \qquad \text{(ii)}$$

Comparing (i) and (ii) we get $c = (1/2)(a_1^2 + b_1^2 - a_2^2 - b_2^2)$.

Example 28: Equations of the straight lines passing through the point (4, 3) and making intercepts on the coordinate axes whose sum is – 1 are

(a) $x/2 + y/3 = 1$ and $x/2 + y/1 = 1$

(b) $x/2 - y/3 = -1$ and $x/(-2) + y/1 = -1$

(c) $x/2 + y/3 = -1$ and $x/(-2) + y/1 = -1$

(d) $x/2 - y/3 = 1$ and $x/(-2) + y/1 = 1$

Ans. (d)

Solution: Let the equation of the line be $\dfrac{x}{a} + \dfrac{y}{-1-a} = 1$.

Since it passes through (4, 3), $\dfrac{4}{a} + \dfrac{3}{-1-a} = 1$

$\Rightarrow \quad a^2 = 4 \quad \Rightarrow \quad a = \pm 2$

and the required equations are

$$\frac{x}{2} + \frac{y}{-3} = 1 \quad \text{and} \quad \frac{x}{-2} + \frac{y}{1} = 1.$$

Example 29: If non zero numbers a, b, c are in H.P, then the straightline $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{1}{c} = 0$ always passes through a fixed point. That point is

(a) (1, – 2) (b) (1, – 1/2)

(c) (– 1, 2) (d) (– 1, – 2)

Ans. (a)

Solution: a, b, c are in H.P.

$$\Rightarrow \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\Rightarrow \frac{1}{a} - \frac{2}{b} + \frac{1}{c} = 0$$

which shows that the given line passes through the point (1, – 2).

Example 30: The line parallel to x-axis passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$ where $(a, b) \neq (0, 0)$ is

(a) above x-axis at a distance 3/2 from it.

(b) above x-axis at a distance 2/3 from it.

(c) below x-axis at a distance 3/2 from it.

(d) below x-axis at a distance 2/3 from it.

Ans. (c)

Solution: Eliminating x, we get

$(2b^2 + 2a^2)y + 3b^2 + 3a^2 = 0$

$\Rightarrow \quad y = -3/2$ which is the required line and hence below x-axis at a distance 3/2 from it.

Example 31: A straight line through the point $A(3, 4)$ is such that its intercept between the axes is bisected at A. Its equation is

(a) $3x + 4y = 25$ (b) $x + y = 7$

(c) $3x - 4y + 7 = 0$ (d) $4x + 3y = 24$

Ans. (d)

Solution: Let the equation be $\dfrac{x}{a} + \dfrac{y}{b} = 1$ where $\dfrac{a}{2} = 3$, $\dfrac{b}{2} = 4$

$\Rightarrow \quad a = 6$, $b = 8$ and the required equation is $8x + 6y = 48$ or $4x + 3y = 24$

Example 32: Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the set of values which k can take is given by

(a) {1, 3} (b) {0, 2}

(c) {– 1, 3} (d) {– 3, – 2}

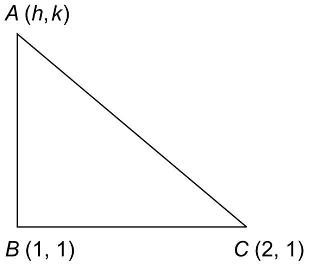


Fig 15.9

Ans. (c)

Solution: Equation of BC is $y = 1$. AB is perpendicular to BC through $B(1, 1)$ so in equation is $x = 1 \Rightarrow h = 1$.

Area of the $\Delta ABC = \dfrac{1}{2}\ AB \times BC = 1$

$$\Rightarrow \quad AB = 2 \quad \Rightarrow \quad |k - 1| = 2$$

$$\Rightarrow \quad k = -1 \text{ or } 3$$

◉ **Example 33:** Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a triangle ABC. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line

(a) $3x + 2y = 5$ (b) $2x - 3y = 7$
(c) $2x + 3y = 9$ (d) $3x - 2y = 3$

Ans. (c)

◎ **Solution:** Let $C(h, k)$ be the vertex, then the centroid is $\left(\frac{h+2-2}{3}, \frac{k-3+1}{3}\right)$ i.e. $(h/3, (k-2)/3)$ which lies on $2x + 3y = 1$

$$\Rightarrow \quad 2\,\frac{h}{3}+\frac{3(k-2)}{3} = 1$$

$\Rightarrow$ $2h + 3k = 9$ and the locus of (h, k) is $2x + 3y = 9$.

◉ **Example 34:** If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product, then the value of c is

(a) 2 (b) – 1
(c) 1 (d) – 2

Ans. (a)

◎ **Solution:** If m_1, m_2 are the slopes, then $m_1 + m_2 = -2c/7$, $m_1 m_2 = -1/7$

$$m_1 + m_2 = 4\, m_1 m_2$$

$$\Rightarrow \quad c = 2$$

◉ **Example 35:** Locus of mid point of the portion between the axes of $x \cos \alpha + y \sin \alpha = p$, where p is constant is

(a) $x^2 + y^2 = 4/p^2$ (b) $x^2 + y^2 = 4p^2$
(c) $1/x^2 + 1/y^2 = 2/p^2$ (d) $1/x^2 + 1/y^2 = 4/p^2$

Ans. (d)

◎ **Solution:** If (h, k) is the mid-point, then

$$h = p/2 \cos\alpha,\; k = p/2 \sin\alpha$$

so $(p/2h)^2 + (p/2k)^2 = \cos^2 \alpha + \sin^2 \alpha = 1$

$$\Rightarrow \quad 1/h^2 + 1/k^2 = 4/p^2$$

Locus of (h, k) is $1/x^2 + 1/y^2 = 4/p^2$

◉ **Example 36:** The lines $x + y = |a|$ and $ax - y = 1$ intersect each other in the first quadrant. Then the set of all possible values of a is in the interval.

(a) $(0, \infty)$ (b) $(1, \infty)$
(c) $(-1, \infty)$ (d) $(-1, 1)$

Ans.(b)

◎ **Solution:** Adding $x + y = |a|$ and $ax - y = 1$

We obtain $(1 + a)\, x = |a| + 1 \Rightarrow x = \frac{|a|+1}{a+1}$, $a \neq -1$

Also $y = a\left(\frac{|a|+1}{a+1}\right) - 1 = \frac{a|a|-1}{a+1}$

As $x > 0$, $y > 0$, $|a| + 1 > 0$, we get $a + 1 > 0$

and $a\,|a| - 1 > 0 \Rightarrow a > -1$ and $a\,|a| > 1$

As $|a| \geq 0$, and $a\,|a| > 1$, we get $a > 0$ and thus $a^2 > 1$ or $a > 1$

◉ **Example 37:** A square of side a lies above the x-axis and has one vertex at the origin. The side passing through the origin makes an angle α $(0 < \alpha < \pi/4)$ with the positive direction of x-axis. The equation of its diagonal not passing through the origin is

(a) $y (\cos \alpha + \sin \alpha) + x (\sin \alpha - \cos \alpha) = a$
(b) $y (\cos \alpha + \sin \alpha) + x (\sin \alpha + \cos \alpha) = a$
(c) $y (\cos \alpha + \sin \alpha) + x (\cos \alpha - \sin \alpha) = a$
(d) $y (\cos \alpha - \sin \alpha) - x (\sin \alpha - \cos \alpha) = a$

Ans. (c)

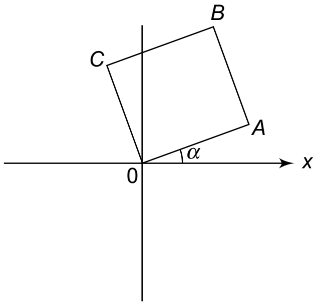


Fig. 16.10

◎ **Solution:** Coordinates of A are $(a \cos \alpha, a \sin \alpha)$ and of C are

$(a \cos (\alpha + \pi/2), a \sin (\alpha + \pi/2))$
i.e. $(-a \sin \alpha, a \cos \alpha)$
So the equation of the diagonal AC is

$$y - a \sin \alpha = \frac{a(\sin\alpha - \cos\alpha)}{a(\cos\alpha + \sin\alpha)}\,(x - a \cos \alpha)$$

or $y (\cos \alpha + \sin \alpha) + x (\cos \alpha - \sin \alpha)$

$$= a (\sin^2 \alpha + \sin \alpha \cos \alpha - \sin \alpha \cos \alpha + \cos^2 \alpha)$$
$$= a$$

◉ **Example 38:** Locus of the centroid of the triangle whose vertices are $(a \cos t, a \sin t)$, $(b \sin t, -b \cos t)$ and $(1, 0)$; where t is *a* parameter is

(a) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
(b) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$
(c) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$
(d) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$

Ans. (a)

◎ **Solution:** If (h, k) is the centroid, then

$$h = \frac{a\cos t + b \sin t + 1}{3},$$

$$k = \frac{a \sin t - b\cos t + 0}{3}$$

$\Rightarrow (3h - 1)^2 + (3k)^2 = (a \cos t + b \sin t)^2 + (a \sin t - b \cos t)^2$
$= a^2 + b^2$

Locus of (h, k) is $(3x - 1)^2 + (3y)^2 = a^2 + b^2$

Example 39: If x_1, x_2, x_3 and y_1, y_2, y_3 are both in G.P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3)

(a) lie on an ellipse
(b) lie on a circle
(c) are vertices of a triangle
(d) lie on a straight line

Ans. (d)

Solution: Let $x_2 = x_1 r$, $x_3 = x_1 r^2$

$$y_2 = y_1 r, \; y_3 = y_1 r^2$$

The given points lie on the line

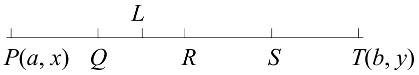
$x_1 y - y_1 x = 0$ for all values of r.

Example 40: Q, R and S are the points on the line joining the points $P(a, x)$ and $T(b, y)$ such that $PQ = QR = RS = ST$, then $\left(\dfrac{5a+3b}{8}, \dfrac{5x+3y}{8}\right)$ is the mid point of the segment

(a) PQ (b) QR
(c) RS (d) ST

Ans. (b)

Solution: The point $L\left(\dfrac{5a+3b}{8}, \dfrac{5x+3y}{8}\right)$ divides PT in the ratio 3 : 5 and hence is the middle point of QR.



Example 41: If a, b, c form a G.P. with common ratio r, the sum of the ordinates of the points of intersection of the line $ax + by + c = 0$ and the curve $x + 2y^2 = 0$ is

(a) $-r^2/2$ (b) $-r/2$
(c) $r/2$ (d) $r^2/2$

Ans. (c)

Solution: The equation of the given line is

$ax + by + c = 0$

$$\Rightarrow \quad ax + ary + ar^2 = 0 \quad \Rightarrow \quad x + ry + r^2 = 0 \qquad \text{(i)}$$

(i) intersects the curves $x + 2y^2 = 0$ at the points whose ordinates are given by

$$-2y^2 + ry + r^2 = 0 \quad \text{or} \quad 2y^2 - ry - r^2 = 0$$

Therefore required sum of the ordinates = $r/2$.

Example 42: If α, β, γ are the real roots of the equation $x^3 - 3px^2 + 3qx - 1 = 0$, then the centroid of the triangle with vertices $\left(\alpha, \dfrac{1}{\alpha}\right)$, $\left(\beta, \dfrac{1}{\beta}\right)$ and $\left(\gamma, \dfrac{1}{\gamma}\right)$ is at the point

(a) (p, q) (b) $(p/3, q/3)$
(c) $(p + q, p - q)$ (d) $(3p, 3q)$

Ans. (a)

Solution: The centroid of the given triangle is the point

$$\left(\frac{\alpha+\beta+\gamma}{3}, \frac{\dfrac{1}{\alpha}+\dfrac{1}{\beta}+\dfrac{1}{\gamma}}{3}\right) = \left(\frac{3p}{3}, \frac{\alpha\beta+\beta\gamma+\gamma\alpha}{3\alpha\beta\gamma}\right)$$

$= (p, q)$ [$\because$ $\alpha + \beta + \gamma = 3p$, $\alpha\beta + \beta\gamma + \gamma\alpha = 3q$, $\alpha\beta\gamma = 1$]

Example 43: The line L has intercepts a and b on the coordinate axes. The coordinate axes are rotated through a fixed angle, keeping the origin fixed. If p and q are the intercepts of the line L on the new axes, then $\dfrac{1}{a^2} - \dfrac{1}{p^2} + \dfrac{1}{b^2} - \dfrac{1}{q^2}$ is equal to

(a) -1 (b) 0
(c) 1 (d) none of these

Ans. (b)

Solution: Equation of the line L in the two coordinate systems is $\dfrac{x}{a} + \dfrac{y}{b} = 1$, $\dfrac{X}{p} + \dfrac{Y}{q} = 1$

where (X, Y) are the new coordinates of a point (x, y) when the axes are rotated through a fixed angle, keeping the origin fixed. As the length of the perpendicular from the origin has not changed.

$$\frac{1}{\sqrt{\dfrac{1}{a^2}+\dfrac{1}{b^2}}} = \frac{1}{\sqrt{\dfrac{1}{p^2}+\dfrac{1}{q^2}}} \quad \Rightarrow \quad \frac{1}{a^2}+\frac{1}{b^2} = \frac{1}{p^2}+\frac{1}{q^2}$$

$$\text{or} \qquad \frac{1}{a^2}-\frac{1}{p^2}+\frac{1}{b^2}-\frac{1}{q^2} = 0.$$

Example 44: Lines $ax + by + c = 0$ where $3a + 2b + 4c = 0$ $a, b, c \in$ R are concurrent at the point.

(a) (3, 2) (b) (2, 4)
(c) (3, 4) (d) (3/4, 1/2)

Ans. (d)

Solution: $3a + 2b + 4c = 0$

$$\Rightarrow \qquad \frac{3}{4}a + \frac{1}{2}b + c = 0$$

$\Rightarrow$ $ax + by + c$ passes through (3/4, 1/2) for all values of a, b, c.

Example 45: If $P(1, 0)$, $Q(-1, 0)$ and $R(2, 0)$ are three given points. The point S satisfies the relation $SQ^2 + SR^2 = 2SP^2$. The locus of S meets PQ at the point

(a) (0, 0) (b) (2/3, 0)
(c) (– 3/2, 0) (d) (0, – 2/3)

Ans. (c)

Solution: Let S be the point (x, y)

then $\quad (x + 1)^2 + y^2 + (x - 2)^2 + y^2 = 2[(x - 1)^2 + y^2]$

$\Rightarrow 2x + 3 = 0$, the locus of S and equation of PQ is $y = 0$.

So the required points is (– 3/2, 0).

Example 46: If algebraic sum of distances of a variable line from points (2, 0), (0, 2) and (– 2, – 2) is zero, then the line passes through the fixed point

(a) (– 1, – 1) (b) (0, 0)
(c) (1, 1) (d) (2, 2)

Ans. (b)

Solution: Let the equation of the variable line be

$$ax + by + c = 0$$

then according to the given condition

$$\frac{2a+c}{\sqrt{a^2+b^2}} + \frac{2b+c}{\sqrt{a^2+b^2}} + \frac{-2a-2b+c}{\sqrt{a^2+b^2}} = 0$$

$$\Rightarrow \quad c = 0$$

which shows that the line passes through (0, 0) for all values of a and b.

Example 47: An equation of a straight line passing through the inter-section of the straight lines $3x – 4y + 1 = 0$ and $5x + y – 1 = 0$ and making non-zero, equal intercepts on the axes is

(a) $22x + 22y = 13$ (b) $23x + 23y = 11$
(c) $11x + 11y = 23$ (d) $8x – 3y = 0$

Ans. (b)

Solution: Equation of any line through the point of intersection of the given lines is

$$(3x – 4y + 1) + k (5x + y – 1) = 0 \qquad (1)$$

or $$(3 + 5k)x + (k – 4)y + 1 – k = 0$$

or $$\frac{x}{(k-1)/(3+5k)} + \frac{y}{(k-1)/(k-4)} = 1$$

Since x-intercept = y-intercept

$$\Rightarrow \frac{k-1}{3+5k} = \frac{k-1}{k-4} \quad \Rightarrow \quad (k – 1)(3 + 5k – k + 4) = 0$$

$$\Rightarrow \quad k = 1 \text{ or } k = – 7/4$$

For $k = 1$, (1) becomes $8x – 3y = 0$ which makes zero intercepts on the axes.

$\therefore \quad k = – 7/4 \quad \Rightarrow$ The required equation is

$$4(3x – 4y + 1) – 7(5x + y – 1) = 0$$

$$\Rightarrow \quad 23x + 23y = 11.$$

Example 48: If the point (3, 4) lies on the locus of the point of intersection of the lines $x \cos\alpha + y \sin\alpha = a$ and $x \sin\alpha - y \cos\alpha = b$ (α is a variable), the point (a, b) lies on the line $3x – 4y = 0$ then $|a + b|$ is equal to

(a) 1 (b) 7
(c) 12 (d) 5

Ans. (b)

Solution: Squaring and adding the given equations of the lines we get $x^2 + y^2 = a^2 + b^2$ as the locus of the point of intersection of these lines.

Since (3, 4) lies on the locus, we get

$$9 + 16 = a^2 + b^2 \quad \text{i.e.} \quad a^2 + b^2 = 25 \qquad \text{(i)}$$

Also, (a, b) lies on $3x – 4y = 0$

so $$3a – 4b = 0 \quad \Rightarrow \quad b = (3/4)\, a \qquad \text{(ii)}$$

From (i), $a^2 + (9/16)a^2 = 25 \Rightarrow a^2 = 16$

So that $|a + b|^2 = (7/4)^2 a^2 = 49 \qquad |a + b| = 7$

Example 49: Equations to the sides of a triangle are $x – 3y = 0$, $4x + 3y = 5$ and $3x + y = 0$. The line $3x – 4y = 0$ passes through the

(a) incentre
(b) centroid
(c) circumcentre
(d) orthocentre of the triangle

Ans. (d)

Solution: Sides $x – 3y = 0$ and $3x + y = 0$ of the triangle being perpendicular to each other, the triangle is right angled at the origin, the point of intersection of these sides. So that origin is the orthocentre of the triangle and the line $3x – 4y = 0$ passes through this orthocentre.

Example 50: Locus of the mid-points of the intercepts between the coordinate axes by the lines passing through $(a, 0)$ does not intersect

(a) x-axis (b) y-axis
(c) $y = x$ (d) $y = a$

Ans. (b)

Solution: Equation of any line through $(a, 0)$ be $\frac{x}{a} + \frac{y}{b} = 1$, where b is a parameter. This line meets y-axis at $(0, b)$ and if (h, k) denotes the mid-point of the intercept of the line between the coordinate axes, then $h = a/2$, $k = b/2$ and then the locus of (h, k) is $x = a/2$. This clearly does not intersect y-axis.

Example 51: If h denotes the arithmetic mean and k denotes the geometric mean of the intercepts made on the coordinate axes by the lines passing through the point (1, 1), then the point (h, k) lies on

(a) a circle (b) a parabola
(c) a straight line (d) a hyperbola

Ans. (b)

Solution: Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ If it passes through (1, 1), then $\frac{1}{a} + \frac{1}{b} = 1$

$$\Rightarrow \quad a + b = ab \qquad \text{(i)}$$

Since the line makes intercepts of lengths a and b on the coordinate axes, we have

$$h = \frac{a+b}{2} \quad \text{and} \quad k = \sqrt{ab}$$

so that from (i) we get $2h = k^2$ and thus the point (h, k) lies on $2x = y^2$ which is a parabola.

Example 52: If a, b, c are in A.P., a, x, b are in G.P. and b, y, c are in G.P., the point (x, y) lies on

(a) a straight line (b) a circle
(c) an ellipse (d) a hyperbola

Ans. (b)

Solution: We have $2b = a + c$, $x^2 = ab$, $y^2 = bc$ so that $x^2 + y^2 = b(a + c) = 2b^2$ which is a circle.

Example 53: If one of the diagonals of a square is along the line $x = 2y$ and one of its vertices is $(3, 0)$, then its side through this vertex nearer to the origin is given by the equation.

(a) $y - 3x + 9 = 0$ (b) $3y + x - 3 = 0$
(c) $x = 3y - 3 = 0$ (d) $3x + y - 9 = 0$

Ans. (b)

Solution: The point $(3, 0)$ does not lie on the diagonal $x = 2y$. Let the equation of a side through the vertex $(3, 0)$ be

$$y - 0 = m\,(x - 3)$$

Since the angle between a side and a diagonal of a square is $\pi/4$, we have

$$\pm\tan\frac{\pi}{4} = \frac{m - 1/2}{1 + m(1/2)} = \frac{2m - 1}{2 + m} \quad \text{(i)}$$

$$\Rightarrow \quad m = 3, -1/3$$

Thus the equation of a side through $(3, 0)$ is $y = 3(x - 3)$ or $y = \left(-\frac{1}{3}\right)(x - 3)$ and the one nearer to the origin is $3y + x - 3 = 0$

Example 54: A line which is parallel to x-axis and crosses the curve $y = \sqrt{x}$ at angle of 45° is

(a) $x = 1/4$ (b) $y = 1/4$
(c) $y = 1/2$ (d) $y = 1$

Ans. (c)

Solution: Equation of a line parallel to x-axis is $y = k$. Which meets the curve $y = \sqrt{x}$ at the point (k^2, k). Since the line $y = k$ crosses the curve at an angle of 45°, the angle between the tangent to the curve $y = \sqrt{x}$ at (k^2, k) and the line $y = k$ is 45°. The line $y = k$ being parallel to x-axis. We have

$$\left[\frac{dy}{dx}\right]_{(k^2,\,k)} = \pm 1 \text{ on the curve } y = \sqrt{x}\,.$$

$$\Rightarrow \quad \left[\frac{1}{2\sqrt{x}}\right]_{(k^2,\,k)} = \pm 1 \quad \Rightarrow \quad k = \frac{1}{2}$$

$k > 0$, as $y = \sqrt{x}$ lies in the first quadrant

Example 55: A rectangle has two opposite vertices at the points $(1, 2)$ and $(5, 5)$. If the other vertices lie on the line $x = 3$, the coordinates of the vertex nearer the axis of x are

(a) $(3, 1)$ (b) $(3, 2)$
(c) $(3, 4)$ (d) $(3, 6)$

Ans. (a)

Solution: Let the coordinates of the other vertices lying on $x = 3$ be $(3, y_1)$ and $(3, y_2)$.

Since the diagonals of a rectangle are equal and bisect each other, we have

$$\frac{y_1 + y_2}{2} = \frac{5 + 2}{2} \Rightarrow y_1 + y_2 = 7$$

and $$|y_2 - y_1| = \sqrt{(5-1)^2 + (5-2)^2} = \sqrt{16 + 9} = 5$$

If $y_2 > y_1$ then $y_2 = 6$, $y_1 = 1$

So that the coordinates of the vertex nearer the axis of x are $(3, 1)$.

Example 56: A straight line through the origin O meets the parallel lines $4x + 2y = 9$ and $2x + y + 6 = 0$ at points P and Q respectively. The point O divides the segment PQ in the ratio

(a) 1 : 2 (b) 3 : 4
(c) 2 : 1 (d) 4 : 3

Ans. (b)

Solution: It is clear that the lines lie on opposite side of the origin O. Let the equation of any line through O be $\frac{x}{\cos\theta} = \frac{y}{\sin\theta}$. If $OP = r_1$ and $OQ = r_2$ then the coordinates of P are $(r_1\cos\theta, r_1\sin\theta)$ and that of Q are $(-r_2\cos\theta, -r_2\sin\theta)$

Since P lies on $4x + 2y = 9$, $2r_1(2\cos\theta + \sin\theta) = 9$
and Q lies on $2x + y + 6 = 0$, $-r_2(2\cos\theta + \sin\theta) = -6$

so that $\frac{r_1}{r_2} = \frac{9}{12} = \frac{3}{4}$

and the required ratio is thus 3 : 4.

Alternately Let the equation of the line through O be $y = mx$, then coordinates of P and Q are respectively $\left(\frac{9}{4 + 2m}, \frac{9m}{4 + 2m}\right)$ and $\left(\frac{-6}{2 + m}, \frac{-6m}{2 + m}\right)$ so that

$$\frac{OP}{OQ} = \frac{9}{|4 + 2m|} \times \frac{|2 + m|}{6} = \frac{3}{4}$$

Example 57: Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (3, 3\sqrt{3})$ be three points. Then the equation of the bisector of the angle PQR is

(a) $\frac{\sqrt{3}}{2} + y = 0$ (b) $x + \sqrt{3}y = 0$
(c) $\sqrt{3}x + y = 0$ (d) $x + \frac{\sqrt{3}}{2}y = 0$

Ans. (c)

◎ **Solution** Let the equation of QS, the bisector of angle PQR be $y = mx$.

Slope of $QR = \sqrt{3} = \tan 60°$

$$\Rightarrow \quad \angle PQR = 120° \Rightarrow \angle PQS = 60°$$

$\Rightarrow$ $m = -\tan 60° = -\sqrt{3}$ and thus the required equation of the bisector is $\sqrt{3}x + y = 0$.

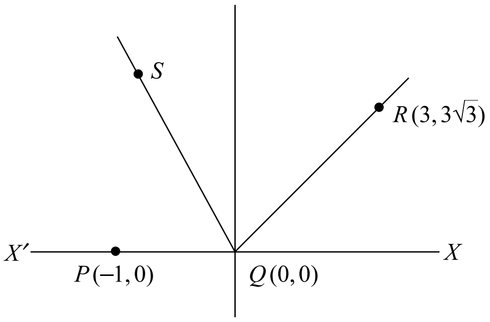


Fig. 16.11

◉ **Example 58:** The number of integer value of m, for which the x-coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer is

(a) 2 (b) 0

(c) 4 (d) 1

Ans. (a)

◎ **Solution:** x-coordinate of the point of intersection is given by

$$3x + 4(mx + 1) = 9$$

$$\Rightarrow \quad x = \frac{5}{3 + 4m}$$

As x is an integer, $3 + 4m$ must be a divisor of 5

so $3 + 4m = \pm 1$ or ± 5

$\Rightarrow$ $m = -1$ or -2

(considering the integer value only)

◉ **Example 59:** If the line $2x + y = k$ passes through the point which divides the line segment joining the points (1,1) and (2,4) in the ratio 3:2, then k equals

(a) 6 (b) 11/5

(c) 29/5 (d) 5

Ans. (a)

◎ **Solution:** Let A (1,1) and B(2,4)

If P (x,y) divides the line segment AB in the ratio 3:2, then

$$x = \frac{3(2)+2(1)}{5}, \; y = \frac{3(2)+2(1)}{5}$$

$$P\,(8/5,\ 14/5)$$

As the line $2x + y = k$ passes through P

$2 \times (8/5) + (14/5) = k \Rightarrow k = 6$

◉ **Example 60:** A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a triangle OPQ, where O in the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is

(a) – 2 (b) – 1/2

(c) – 1/4 (d) – 4

Ans. (a)

◎ **Solution:** Equation of the line be $y = mx + c$.

Since it passes through (1, 2), $2 = m + c$.

$P\,(0, c)$, $Q\,(-c/m, 0)$. If A is the area of the triangle, then

$$A = \frac{1}{2}\,|c|\left|\frac{c}{m}\right| = \frac{c^2}{2m} = \frac{(2-m)^2}{2m} = \frac{m^2 - 4m + 4}{2m}$$

$$= \frac{m}{2} - 2 + \frac{2}{m}$$

$$\frac{dA}{dm} = 0 \Rightarrow \frac{1}{2} - \frac{2}{m^2} = \Rightarrow m^2 = 4 \Rightarrow m = \pm 2$$

◉ **Example 61:** Let PQR be a right angled isosceles triangle right angled at P (2, 1). If the equation of the line QR is $2x + y = 3$, then the equations representing the pair of lines PQ and PR is

(a) $x + 3y = 5$, $3x - y = 5$

(b) $x + y = 3$, $x - y = 1$

(c) $2x + 3y = 7$, $3x - 2y = 4$

(d) $3x + y = 7$, $x - 3y = -1$

Ans. (a)

◎ **Solution:** Let the slopes of PQ and PR be m and $-1/m$ respectively. Since PQR is an isosceles triangle $\angle PQR = \angle PRQ$

$$\Rightarrow \quad \left|\frac{m+2}{1-2m}\right| = \left|\frac{-\frac{1}{m}+2}{1+\frac{2}{m}}\right|$$

[∵ slope of $QR = -2$]

$\Rightarrow m + 2 = \pm(1 - 2m) \Rightarrow m = 3$ or $-1/3$

So the equations of PQ and PR are

$(y - 1) = 3(x - 2)$ and $y - 1 = (-1/3)(x - 2)$

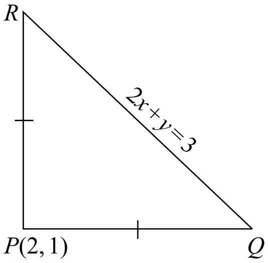


Fig. 16.12

$$\Rightarrow \quad 3x - y = 5,\ x + 3y = 5$$

Example 62: The orthocentre of the triangle formed by the lines xy = 0 and 2x + 3y – 5 = 0 is

(a) (2, 3) (b) (3, 2)
(c) (0, 0) (d) (5, – 5)

Ans. (c)

Solution: The given triangle is right angled at (0, 0) which is therefore the orthocentre of the triangle.

Example 63: The equations of the sides of a parallelogram are $x = 2$, $x = 3$ and $y = 1$, $y = 5$. Equations to the pair of diagonals are

(a) $4x + y = 7, 4x - y = 13$
(b) $x + 4y = 13, x - 4y = 7$
(c) $x + 4y = 7, x - 4y = 13$
(d) $4x + y = 13, 4x - y = 7$

Ans. (d)

Solution: The vertices of the parallelogram are

$$A(2, 1), B(2, 5), C(3, 5) \text{ and } D(3, 1)$$

Equation of the diagonal AC is

$$y - 1 = \frac{5-1}{3-2}(x - 2) \Rightarrow 4x - y = 7$$

Equation of the diagonal BD is

$$y - 5 = \frac{1-5}{3-2}(x - 2) \Rightarrow 4x + y = 13$$

Example 64: The lines joining the origin to the points of intersection of $3x^2 + \lambda xy - 4x + 1 = 0$ and $2x + y - 1 = 0$ are at right angles for

(a) $\lambda = -4$ (b) $\lambda = 4$
(c) $\lambda = 7$ (d) all values of λ

Ans. (d)

Solution: Equation of the lines joining the origin to the points of intersection of the given lines is

$$3x^2 + \lambda xy - 4x(2x + y) + 1\cdot(2x + y)^2 = 0$$

(Making the equation of the pair of lines homogeneous with the help of the equation of the line)

$$\Rightarrow \quad x^2 - \lambda xy - y^2 = 0$$

which are perpendicular for all values of λ as the product of the slopes is – 1.

Example 65: The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2x + (p^2 + 1)y + 2q = 0$ are perpendicular to a common line for

(a) exactly two values of p
(b) more than two values of p
(c) no value of p
(d) exactly one value of p

Ans. (d)

Solution: Two lines will be perpendicular to a common line if these two are parallel.

$$\therefore \quad p(p^2 + 1) = -\frac{(p^2+1)^2}{p^2+1}$$

$$\Rightarrow \quad (p + 1)(p^2 + 1) = 0 \Rightarrow p = -1$$

Example 66: The line L is given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then the distance between L and K is

(a) $17/\sqrt{15}$ (b) $23/\sqrt{17}$
(c) $23/\sqrt{15}$ (d) $\sqrt{15}$

Ans. (b)

Solution: As $L : \frac{x}{5} + \frac{y}{b} = 1$ passes through (13, 32), we get

$$\frac{13}{5} + \frac{32}{b} = 1 \Rightarrow b = -20$$

As $K : \frac{x}{c} + \frac{y}{3} = 1$ is parallel to L

$$-\frac{b}{5} = -\frac{3}{c} \Rightarrow c = -\frac{3}{4}.$$

Now equation of L and K are respectively $4x - y - 20 = 0$ and $4x - y + 3 = 0$ and the distance between them is

$$\frac{|-20-3|}{\sqrt{16+1}} = \frac{23}{\sqrt{17}}.$$

Example 67: Consider three points $P(\cos\alpha, \sin\beta)$, $Q(\sin\alpha, \cos\beta)$ and $R(0, 0)$. Where $0 < \alpha, \beta < \pi/4$, Then

(a) P lies on the line segment RQ
(b) Q lies on the line segment PR
(c) R lies on the line segment QP
(d) P, Q, R are non-collinear.

Ans. (d)

Solution: Consider $\Delta = \begin{vmatrix} \cos\alpha & \sin\beta & 1 \\ \sin\alpha & \cos\beta & 1 \\ 0 & 0 & 1 \end{vmatrix}$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$= \cos(\alpha + \beta) \neq 0 \text{ as } 0 < \alpha + \beta < \pi/2$$

Hence the given points are non-collinear.

Example 68: Consider the lines given by $L_1 : x + 3y - 5 = 0$, $L_2 : 3x - ky - 1 = 0$ $L_3 : 5x + 2y - 12 = 0$. If one of L_1, L_2, L_3 is parallel to at least one of the other two, then k satisfies

(a) $k^2 + 4k - 45 = 0$
(b) $5k^2 + 51k + 45 = 0$
(c) $5k^2 - 19k - 6 = 0$
(d) $5k^2 + 51k + 54 = 0$

Ans. (*d*)

Solution: L_1 and L_3 are not parallel.

L_2 is parallel to L_1 if $\frac{-1}{3} = \frac{3}{k} \Rightarrow k = -9$

L_2 is parallel to L_3 if $\frac{-5}{2} = \frac{3}{k} \Rightarrow k = \frac{-6}{5}$.

So the required values of k are given by $(k + 9)(k + 6/5) = 0$

or $\quad 5k^2 + 51k + 54 = 0$

Example 69: Consider the lines

$L_1 : x + y = 10, \quad L_2 : x + y = 60$

$L_3 : x = 40, \quad L_4 : y = 40.$

L_1 meets x-axis and y-axis at A and B respectively.

L_4 meets y-axis at C and L_2 at D

L_3 meets L_2 at E and x-axis at F.

Perimeter of the hexagon $ABCDEF$ is

(a) $100 + 30\sqrt{2}$ (b) $50 + 40\sqrt{2}$

(c) 100 (d) none of these

Ans. (*a*)

Solution:

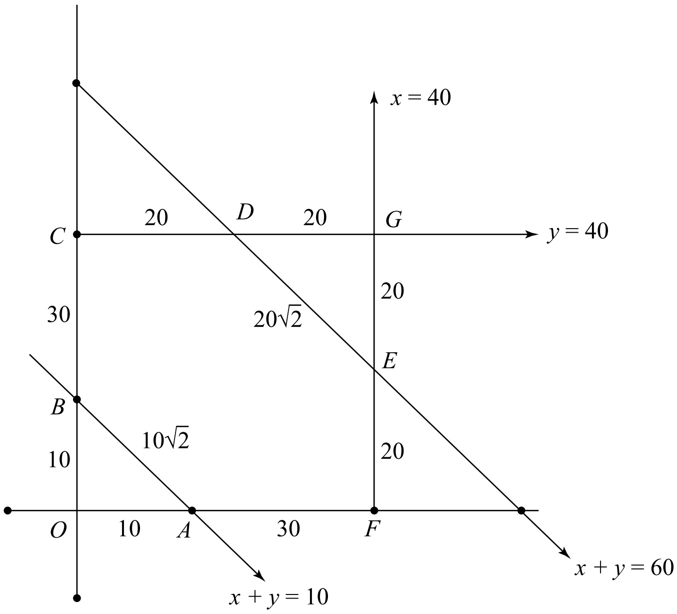


Fig. 16.13

From the figure it is clear that $OF = 40$, $OA = 10 \Rightarrow AF = 30$

Similarly $BC = 30$.

Coordinates of D are $(20, 40)$

so $\quad CD = DG = 20$

Similarly $\quad GE = EF = 20$

$$AB = \sqrt{10^2 + 10^2} = 10\sqrt{2}$$

$$ED = \sqrt{(20)^2 + (20)^2} = 20\sqrt{2}$$

Hence the required perimeter is $AB + BC + CD + DE + EF + FA$

$$= 10\sqrt{2} + 30 + 20 + 20\sqrt{2} + 20 + 30$$

$$= 100 + 30\sqrt{2}$$

Example 70: If a line through $P(a, 2)$ making an angle $\pi/4$ with the positive direction of x-axis, meets the curve $4x^2 + 9y^2 = 36$ at A and D and meets the axes at B and C, so that PA, PB, PC, PD are in G.P, then a is equal to

(a) $\frac{2}{13}$ (b) $\frac{13}{2}$

(c) $\frac{13}{5}$ (d) $\frac{5}{13}$

Ans. (*b*)

Solution: Equation of any line through $P(a, 2)$ making an angle $\theta = \pi/4$ with the +ve direction of x-axis is $\frac{x-a}{1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}}$. Any point on this line at a distance r from P is $(a + r/\sqrt{2}, 2 + r/\sqrt{2})$. If it lies on the curve, then

$$4\left(a + r/\sqrt{2}\right)^2 + 9\left(2 + r/\sqrt{2}\right)^2 = 36$$

This gives two values of r representing PA and PD

So that $\quad PA \cdot PD = \frac{4a^2 \times 2}{4+9} = \frac{8a^2}{13}$.

Since it meets the axes at B and C

$$PB = -a\sqrt{2}, \ PC = -2\sqrt{2}$$

So $\quad PB \cdot PC = 4a$

As PA, PB, PC, PD are in G.P

$$PA \cdot PD = PB \cdot BC$$

$\Rightarrow \quad \frac{8a^2}{13} = 4a \Rightarrow a = \frac{13}{2}$.

Example 71: The quadrilateral formed by the lines $y = ax + c$, $y = ax + d$ and $y = bx + c$ and $y = bx + d$ has area 18. The quadrilateral formed by the lines $y = ax + c$, $y = ax - d$, $y = bx + c$ and $y = bx - d$ has area 72. If a, b, c, d are positive integers then the minimum value of the sum $a + b + c + d$ is

(a) 13 (b) 14

(c) 15 (d) 16.

Ans. (d)

Solution: Coordinates of A are $(0, c)$ and of C are $(0, d)$

So Length of $AC = |c - d|$

Length of the perpendicular from B on AC $(x = 0)$ is equal to the x-coordinate $\frac{c-d}{a-b}$ of B. Thus area of this quadrilateral is equal to $\frac{(c-d)^2}{|a-b|} = 18$

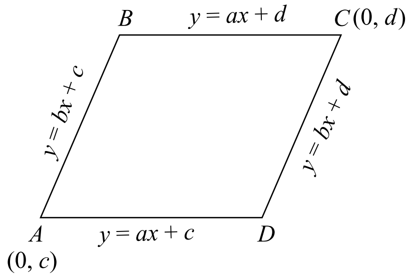


Fig. 16.14

Similarly area of the other quadrilateral is $\frac{(c+d)^2}{|a-b|} = 72$

As a, b, c, d, are positive integers minimum value of $|a - b| = 2$

$$\frac{c+d}{c-d} = \pm 2$$

$\Rightarrow$ $c = 3d$ or $d = 3c$.

We can take $a = 3$ or $b = 1$ or $a = 1$, $b = 3$

and $c = 3, d = 9$ or $d = 3, c = 9$.

In all the cases

$$a + b + c + d = 3 + 1 + 3 + 9 = 16.$$

Example 72: A straight line L through the point $(3, -2)$ is inclined at an angle 60° to the line $\sqrt{3}\,x + y = 1$. If L also intersects the x-axis, then the equation of L is

(a) $y + \sqrt{3}\,x + 2 - 3\sqrt{3} = 0$
(b) $y - \sqrt{3}\,x + 2 + 3\sqrt{3} = 0$
(c) $\sqrt{3}\,y - x + 3 + 2\sqrt{3} = 0$
(d) $\sqrt{3}\,y + x - 3 + 2\sqrt{3} = 0$

Ans. (b)

Solution: Let the slope of L be m

then $$\left|\frac{m+\sqrt{3}}{1-\sqrt{3}m}\right| = \tan 60° = \sqrt{3}$$

$\Rightarrow$ $m + \sqrt{3} = \sqrt{3}\,(1 - \sqrt{3}\,m)$

or $m + \sqrt{3} = -\sqrt{3}\,(1 - \sqrt{3}\,m)$

$\Rightarrow$ $m = 0$ *or* $m = \sqrt{3}$

but $m \neq 0$ as L intersects x- axis

Hence equation of L is

$$y + 2 = \sqrt{3}\,(x - 3)$$

or $$y - \sqrt{3}\,x + 2 + 3\sqrt{3} = 0$$

Example 73: Given four lines with equations $x + 2y - 3 = 0$, $3x + 4y - 7 = 0$, $2x + 3y - 4 = 0$ $4x + 5y - 6 = 0$

(a) they are all concurrent
(b) they are sides of a qudrilateral
(c) only three lines are concurrent
(d) none of these

Ans. (c)

Solution: $x + 2y - 3 = 0$, $2x + 3y - 4 = 0$ intersect at $(-1, 2)$. $4x + 5y - 6 = 0$ passes through this point but $3x + 4y - 7 = 0$ does not pass through it, so only three of them are concurrent

Example 74: The point (4, 1) undergoes the following three transformations successively.

(i) reflection about the line $y = x$
(ii) transformation through a distance 2 units along the positive direction of x- axis
(iii) rotation through an angle $\pi/4$ about the origin in the counter clockwise direction. Then the final position of the point is given by the coordinates

(a) $\left(\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (b) $\left(-\sqrt{2}, 7\sqrt{2}\right)$

(c) $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$ (d) $\left(\sqrt{2}, 7\sqrt{2}\right)$

Ans. (c)

Solution: Reflection of the point (4, 1) about the line $y = x$ is (1,4) which is transfered to the point (3,4) at a distance 2 units along the positive direction of x- axis.

Finally a rotation through an angle $\pi/4$ takes it to

$$(3 \cos (\pi/4) - 4 \sin (\pi/4)), (3\sin (\pi/4) + 4 \cos (\pi/4))$$

$$= \left(\frac{-1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$$

Example 75: If the sum of the distances of a point from two perpendicular lines is 1 then its locus is

(a) square
(b) circle
(c) straight line
(d) two intersecting lines.

Ans. (a)

Solution: Let us take the two perpendicular line as the axes of coordinates. If (h, k) is any point of the locus, then $|h| + |k| = 1$

Locus is $|x| + |y| = 1$

This consists of four line segment

$$x + y = \pm 1, x - y = \pm 1$$

which form a square

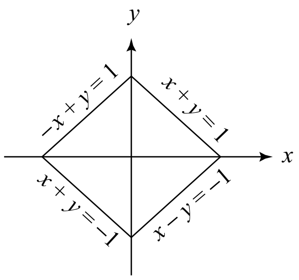


Fig. 16.15

Example 76: The diagonals of a parallelgram *PQRS* are along the lines $x + 3y = 4$ and $6x - 2y = 7$. Then *PQRS* must be a

(a) rectangle (b) square
(c) cyclic quadrilateral (d) rhombus

Ans.(d)

Solution: Since the diagonals of the parallelogram are at right angles, it is a rhombus.

Example 77: Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius then the product of the lengths of the line segments A_0A_1, A_0A_2 and A_0A_4 is

(a) 3/4 (b) $3\sqrt{3}$
(c) 3 (d) $3\sqrt{3}/2$

Ans. (c)

Solution: The vertices of the hexagon be as shown in the figure

$$(A_0A_1)^2 = \frac{1}{4}+\frac{3}{4}=1$$

$$(A_0A_2)^2 = \frac{9}{4}+\frac{3}{4}=3$$

$$(A_0A_4)^2 = (A_0A_2)^2 = 3$$

So $(A_0A_1)\,(A_0A_2)\,(A_0A_4) = \sqrt{1\times3\times3} = 3$

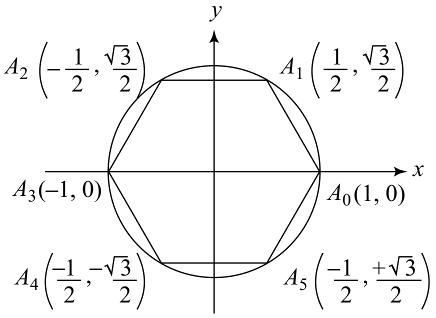

y
$A_2\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$A_1\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
x
$A_3(-1, 0)$
$A_0(1, 0)$
$A_4\left(\frac{-1}{2},\frac{-\sqrt{3}}{2}\right)$
$A_5\left(\frac{-1}{2},\frac{+\sqrt{3}}{2}\right)$

Fig. 16.16

Example 78: The incentre of the triangle with vertices $(1, \sqrt{3})$ (0,0) and (2,0) is

(a) $(1,\sqrt{3}/2)$ (b) $(2/3, 1/\sqrt{3})$
(c) $(2/3,\sqrt{3}/2)$ (d) $(1, 1/\sqrt{3})$

Ans. (d)

Solution: Length of each side of the triangle is 2 so it is an equilaltral triangle and hence it incentre is the centroid $(1, 1/\sqrt{3})$ of the triangle.

Example 79: A ray of light travels along the line $2x - 3y + 5 = 0$ and strikes a plane mirror lying along the line $x + y = 2$. The equation of the straight line containing the reflected ray is

(a) $2x - 3y + 3 = 0$ (b) $3x - 2y + 3 = 0$
(c) $21x - 7y + 1 = 0$ (d) $21x + 7y - 1= 0$

Ans. (b)

Solution: Let θ be the angle which the ray $2x - 3y + 5 = 0$ makes with the line $x + y = 2$ then $\tan\theta = \left|\frac{2/3+1}{1-2/3\,(1)}\right|$ $= |5| = 5$

Reffected ray makes angle $-\theta$ with $x + y = 2$, so if the slope of the reflected ray is m then $\frac{m+1}{1-m} = \tan(-\theta) = -5$

$\Rightarrow \quad m = 3/2.$

Also, the reflected ray passes through (1/5, 9/5), the point of intersection of the ray and the line

Hence equation of the reflected ray is

$$(y - 9/5) = 3/2\,(x - 1/5)$$

$\Rightarrow 3x - 2y + 3 = 0$

Example 80: Let $0 < \alpha < \pi/2$ be a fixed angle. If $P = (\cos\theta, \sin\theta)$ and $Q = (\cos(\alpha - \theta), \sin(\alpha - \theta))$

Then Q is obtained from P by

(a) clockwise rotation around origin through an angle α
(b) anti clockwise rotation around origin through an angle α
(c) reflection in the line through the origin with slope $\tan\alpha$
(d) reflection in the line through the origin with slope $\tan\alpha/2$

Ans. (d)

Solution: Clockwise rotation of P through an angle α takes it to the point $(\cos(\theta - \alpha), \sin(\theta - \alpha))$ and anticlockwise takes it to $(\cos(\alpha + \theta), \sin(\alpha + \theta))$

Now slope of PQ

$$= \frac{\sin\theta-\sin(\alpha-\theta)}{\cos\theta-\cos(\alpha-\theta)}$$

$$= \frac{2\cos(\alpha/2)\sin(\theta-\alpha/2)}{-2\sin(\alpha/2)\sin(\theta-\alpha/2)}$$

$$= -\cot(\alpha/2)$$

$\Rightarrow PQ$ is perpendicular to the line with slope $\tan(\alpha/2)$

Hence Q is the reflection of P in the line through the origin with slope $\tan(\alpha/2)$.

Assertion-Reason Type Questions

Example 81: Consider the lines

$$L_1 : p^2x + py - 1 = 0$$
$$L_2 : q^2x + qy + 6 = 0$$

L_1 passes through the point (3, 2) and L_2 passes through the point (2, 7)

Statement-1: If the product of the slopes of L_1 and L_2 is 2, then they intersect at the point (– 4, – 5)

Statement-2: L_1 and L_2 are neither parallel nor perpendicular.

Ans. (b)

Solution: we have $3p^2 + 2p - 1 = 0$ and

$$2q^2 + 7q + 6 = 0$$

$$\Rightarrow \quad p = -1 \text{ or } \frac{1}{3} \text{ and } q = -2 \text{ or } \frac{-3}{2}$$

Slope of $L_1 = -p$, slope of $L_2 = -q$, product of the slopes = $pq \neq -1$ for any values of p and q. So L_1 and L_2 are not perpendicular. Also as $p \neq q$ for any value, they are not parallel.

Thus the statement-2 is true. Next, if the product of the slope is 2 then $p = -1$, $q = -2$ and the equations of L_1 and L_2 are respectively

$$x - y - 1 = 0 \text{ and } 4x - 2y + 6 = 0$$

which intersect at (– 4, – 5) so the statement-1 is also true but does not follow from statement-1.

Example 82: Statement-1: Consider the point $A(0, 1)$ and $B(2, 0)$ and P be a point on the line $4x + 3y + 9 = 0$, then the coordinates of P such that $|PA - PB|$ is maximum is (– 24/5, 17/5)

Statement-2: If A and B are two fixed points and P is any point in a plane, then $|PA - PB| \leq AB$.

Ans. (a)

Solution: From geometry $|PA - PB| < AB$ and $|PA - PB| = AB$ if P lies on extended line segment AB so statement-2 is true. Using in statement-1, $|PA - PB|$ is maximum if P lies on AB whose equation is $x + 2y = 2$

The given line is $4x + 3y + 9 = 0$, on solving we get the required coordinates of P as (– 24/5, 17/5) and hence statement-1 is also true.

Example 83: Statement-1: The points $A(3, 4)$, $B(2, 7)$, $C(4, 4)$ and $D(3, 5)$ are such that one of them lies inside the triangle formed by the other points.

Statement-2: Centroid of a triangle lies inside the triangle.

Ans. (a)

Solution: $\frac{3+2+4}{3} = 3, \ \frac{4+7+4}{3} = 5$

$\Rightarrow$ $D(3, 5)$ is the centroid of the triangle ABC.

Using statement-2 which is true, statement-1 is also true.

Example 84: Statement-1: If the circumcentre of a triangle lies at the origin and centroid is the mid point of the line joining the points (2, 3) and (4, 7), then its orthocentre lies on the line $5x - 3y = 0$

Statement-2: Circumcentre, centroid and orthocentre of a triangle lie on the same line

Ans. (a)

Solution: From geometry, statement-2 is True. Using it in statement-1, orthocentre lies on the line joining (0, 0) and $\left(\frac{4+2}{2}, \frac{7+3}{2}\right)$ i.e. (3, 5) which is $5x - 3y = 0$ and so the statement-1 is also true.

Example 85: Statement-1: The points $A(-2, 2)$, $B(2, -2)$ and $C(1, 1)$ are the vertices of an obtuse angled isosceles triangle.

Statement-2: Every obtuse angled triangle is isosceles

Ans. (c)

Solution: Equation of AB is $y + x = 0$ in statement-1 and $y = x$ is the perpendicular bisector of AB. $C(1, 1)$ lies on it so the triangle ABC is isosceles, $AC = BC$ slope of $AC = -1/3$, slope of $AB = -1$

$$\tan A = \frac{-1/3 + 1}{1 + 1/3} = \frac{1}{2} \quad \Rightarrow \quad A < \pi/4.$$

(A cannot be obtuse as $A = B$)

$\Rightarrow A + B < \pi/2 \quad \Rightarrow \quad C > \pi/2$ and the triangle is obtuse angled. So statement-1 is True.

Statement-2 is false as a triangle with angles 120°, 40°, 20° is obtuse angled but not isosceles.

Example 86: Statement-1: The lines

$(a + b)x + 2(a - b)y = 2a$ are concurrent at the point (1, 1/2)

Statement-2: The lines $x + 2y - 2 = 0$ and $x - 2y = 0$ intersect at the point(1, 1/2)

Ans. (a)

Solution: Statement-2 is true. Lines in statement-1 can be written as $a(x + 2y - 2) + b(x - 2y) = 0$ which passes through the point of intersection(1, 1/2) of the lines $x + 2y - 2 = 0$, $x - 2y = 0$, follows from statement-1

Example 87: Statement-1: Lines $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line $L_3 : y + 2 = 0$ at P and Q respectively. The bisector of the acute angle between L_1 and L_2 intersect L_3 at R.

Statement-1: The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.

Statement-2: The bisector of an angle of a triangle divides the triangle into two similar triangles.

Ans. (c)

◎ **Solution:** Vertices of the triangle in statement-1 are $O(0, 0)$, $P(-2, -2)$, $Q(1, -2)$. Since the bisector of an angle of a triangle divides the opposite side. In the ratio of the side containing the angle.

$PR : RQ = OP : OQ = 2\sqrt{2} : \sqrt{5}$. So statement-1 is true

Statement-2 is false, because for example if $A = 50°$, $B = 60°$, $C = 70°$ then bisector of angle B does not divide the triangle into two equiangular and hence similar triangles.

● **Example 88: Statement-1:** If the perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept equal to -4, then $k^2 - 16 = 0$

Statement-2: Centroid of an isoceles triangle ABC lies on the perpendicular bisector of the base BC where $B = C$.

Ans. (b)

◎ **Solution:** Any point $L(x, y)$ on the perpendicular bisector in statement-1 is equidistant from P and Q. Locus of L is $(x-1)^2 + (y-4)^2 = (x-k)^2 + (y-3)^2$

$$\Rightarrow \quad 2(k+1)x - 2y = k^2 - 8$$

$$y\text{-intercept} = -\frac{k^2-8}{2} = -4 \quad \Rightarrow \quad k^2 - 16 = 0$$

So statement-1 is true but does not follow from statement-2 which is also true, as the perpendicular bisector of BC is also the median through A.

● **Example 89: Statement-1:** Circumcentre of the triangle formed by the lines $x + y = 0$, $x - y = 0$ and $x - 7 = 0$ is $(7, 0)$

Statement-2: Circumcentre of a triangle lies inside the triangle.

Ans. (c)

◎ **Solution:** In statement-1, the triangle is right angled with hypotenuse $x - 7 = 0$ and the vertices of the hypotenuse are $(7, 7)$ and $(7, -7)$, circumcentre is the mid-point $(7, 0)$ of the hypotenuse. So statement-1 is True. Statement-2 is false as the circumcentre of an obtuse angled triangle lies outside the triangle and that of the right angled is on the hypotenuse.

● **Example 90: Statement-1:** Equation of the pair of lines through the origin perpendicular to the pair of lines $7x^2 - 55xy - 8y^2 = 0$ is $8x^2 - 55xy - 7y^2 = 0$

Statement-2: Equation of the pair of lines through the origin perpendicular to the pair to lines $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$

Ans. (a)

◎ **Solution:** Statement-2 is True because if m_1 and m_2 are the slopes of $ax^2 + 2hxy + by^2 = 0$, then $m_1 + m_2 = -\frac{2h}{b}$, $m_1 m_2 = \frac{a}{b}$.

$\Rightarrow \quad \frac{1}{m_1} + \frac{1}{m_2} = -\frac{2h}{a}$ and $\frac{1}{m_1 m_2} = \frac{b}{a}$. Equation of the lines with slope $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$ is $y^2 + \left(\frac{1}{m_1} + \frac{1}{m_2}\right)xy + \frac{1}{m_1 m_2} x^2 = 0$

$$\Rightarrow y^2 - \frac{2h}{a}xy + \frac{b}{a}x^2 = 0 \quad \Rightarrow \quad bx^2 - 2hxy + ay^2 = 0$$

Using it, statement-1 is also true.

● **Example 91: Statement-1:** The area enclosed with in the curve $|x| + |y| = 1$ is 2 square units.

Statement-2: The curve $|x| + |y| = 1$ is a square of each side $\sqrt{2}$ units

Ans. (a)

◎ **Solution:** The statement-2 is true as it represents the sides $x + y = \pm 1$ and $x - y = \pm 1$, of a square of each side $\sqrt{2}$ units which shows that the statement-1 is also true

● **Example 92:**

Statement-1 If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

then the two triangle with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3) and (a_1,b_1), (a_2,b_2), (a_3,b_3) must be congruent

Statement-2 Two congruent triangles have the same area

Ans. (d)

◎ **Solution:** Statement-2 is true, but two triangles having the same area may not be congruent for example, the triangles with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$ and $(0, 0)$, $(2, 0)$ and $(1, 4)$ have same area but are not congruent and thus statement-1 is false.

● **Example 93:** Let θ_1 be the angle between two lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_2 = 0$ and θ_2 be the angle between two lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_3 = 0$ where c_1, c_2, c_3 are any real number.

Statement 1: If c_1 and c_2 are proportional then $\theta_1 = \theta_2$

Statement 2: $\theta_1 = \theta_2$ for all c_2 and c_3.

Ans. (a)

◎ **Solution:** Lines $-x + 5y + c_2 = 0$ and $-x + 5y + c_3 = 0$ are parallel for all values of c_2 and c_3 so both will make the same angle with the line $2x + 3y + c_1 = 0$ and $\theta_1 = \theta_2$ for all c_2 and c_3 showing that statement-2 is true and hence statement-1 is also true.

Example 94: $L_1 : x - y + 1 + \lambda_1 (2x - y - 2) = 0$ and L_2 : $5x + 3y - 2 + \lambda_2 (3x - y - 4) = 0$ where λ_1, λ_2 are arbitrary numbers.

Statement 1: $5x - 2y - 7 = 0$ is the equation of a line belonging to both the family of lines L_1 and L_2.

Statement 2: There is only one line belonging to both the family of lines L_1 and L_2

Ans. (a)

Solution: L_1 represent the lines passing through the point (3, 4), the point of intersection of $x - y + 1 = 0$ and $2x - y - 2 = 0$ and L_2 represents the lines passing through the point (1, – 1), the point of intersection of the lines $5x + 3y - 2 = 0$ and $3x - y - 4 = 0$. So the line belonging to both the families will be the only one line joining (3, 4) and (1, – 1) which is

$$y + 1 = \frac{4+1}{3-1}(x - 1)$$

or $$5x - 2y - 7 = 0.$$

So statement-2 is true and is a correct explanation for statement-1 to be true.

Example 95: Statement-1: The line $2x + y + 6 = 0$ is perpendicular to the line $x - 2y + 5 = 0$ and second line passes through (1, 3)

Statement-2: Product of the slopes of any two parallel lines is equal to – 1

Ans. (c)

Solution: Statement-2 is false as the product of the slopes of two perpendicular line is – 1. Product of the slopes of the lines in statement-1 is $-2 \times \frac{1}{2} = -1$ so they are perpendicular and the second line passes through (1, 3) so the statement-1 is true.

LEVEL 2

Straight Objective Type Questions

Example 96: If a, b, c are unequal and different from 1 such that the points $\left(\frac{a^3}{a-1}, \frac{a^2-3}{a-1}\right), \left(\frac{b^3}{b-1}, \frac{b^2-3}{b-1}\right)$ and $\left(\frac{c^3}{c-1}, \frac{c^2-3}{c-1}\right)$, are collinear, then

(a) $bc + ca + ab + abc = 0$
(b) $a + b + c = abc$
(c) $bc + ca + ab = abc$
(d) $bc + ca + ab - abc = 3(a + b + c)$

Ans. (d)

Solution: Suppose the given points lie on the line

$$lx + my + n = 0$$

then a, b, c are the roots of the equation.

$$lt^3 + m(t^2 - 3) + n(t - 1) = 0$$

or $$lt^3 + mt^2 + nt - (3m + n) = 0$$

$$\Rightarrow \quad a + b + c = -m/l$$

$$bc + ca + ab = n/l$$

$$abc = (3m + n)/l$$

Eliminating l, m, n, we get

$$abc = -3(a + b + c) + bc + ca + ab$$

$$\Rightarrow \quad bc + ca + ab - abc = 3(a + b + c)$$

Example 97: If two vertices of a triangle are (–2, 3) and (5, –1), orthocentre lies at the origin and centroid on the line $x + y = 7$, then the third vertex lies at

(a) (7, 4)	(b) (8, 14)
(c) (12, 21)	(d) none of these

Ans. (d)

Solution: Let $O(0, 0)$ be the orthocentre; $A(h, k)$ the third vertex; and $B(-2, 3)$ and $C(5, -1)$ the other two vertices. Then the slope of the line through A and O is k/h, while the line through B and C has the slope – 4/7. By the property of the orthocentre, these two lines must be perpendicular, so we have

$$\left(\frac{k}{h}\right)\left(-\frac{4}{7}\right) = -1 \Rightarrow \frac{k}{h} = \frac{7}{4} \qquad \text{(i)}$$

Also $$\frac{5-2+h}{3} + \frac{-1+3+k}{3} = 7$$

$$\Rightarrow \quad h + k = 16 \qquad \text{(ii)}$$

Which is not satisfied by the points given in (*a*), (*b*) or (*c*).

Example 98: The points A (2, 3); B (3, 5), C (7, 7) and D (4, 7) are such that

(a) $ABCD$ is a parallelogram
(b) A,B,C and D are collinear
(c) D lies inside the triangle ABC
(d) D lies on the boundary of the triangle ABC

Ans. (d)

◎ **Solution:** A, B, C are not collinear

as slope of $BA \neq$ slope of BC slope of AD = slope of BD
$\Rightarrow D$ lies on the side AB of the triangle ABC

◉ **Example 99:** If $p, x_1, x_2 \ldots x_i, \ldots$ and $q, y_1, y_2 \ldots, y_i, \ldots$ are in A.P., with common difference a and b respectively, then the centre of mean position of the points $A_i\ (x_i, y_i)$, $i = 1, 2 \ldots n$ lies on the line

(a) $ax - by = aq - bp$
(b) $bx - ay = ap - bq$
(c) $bx - ay = bp - aq$
(d) $ax - by = bq - ap$

$$\left[\textbf{Note.}\text{ Centre of Mean Position }\left(\frac{\Sigma xi}{n}, \frac{\Sigma yi}{n}\right)\right]$$

Ans. (c)

◎ **Solution:** Let the coordinates of the centre of mean position of the points

$$A_i,\ i = 1, 2, \ldots n \text{ be } (x, y), \text{ then}$$

$$x = \frac{x_1 + x_2 + \cdots + x_n}{n},\ y = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

$$\Rightarrow \quad x = \frac{n\,p + a\,(1 + 2 + \cdots + n)}{n},$$

$$y = \frac{n\,q + b\,(1 + 2 + \cdots + n)}{n}$$

$$\Rightarrow \quad x = p + \frac{n(n+1)}{2n}\ a,\ y = q + \frac{n(n+1)}{2n}\ b$$

$$\Rightarrow \quad x = p + \frac{n+1}{2}\,a,\ y = q + \frac{n+1}{2}\,b$$

$$\Rightarrow \quad 2\,\frac{(x-p)}{a} = 2\,\frac{(y-q)}{b} \quad \Rightarrow \quad bx - ay = bp - aq.$$

◉ **Example 100:** If P is a point (x, y) on the line. $y = -3x$ such that P and the point (3, 4) are on the opposite sides of the line $3x - 4y = 8$, then

(a) $x > 8/15, y < -8/5$ (b) $x > 8/5, y < -8/15$
(c) $x = 8/15, y = -8/5$ (d) none of these

Ans. (a)

◎ **Solution:** Let $k = 3x - 4y - 8$

then the value of k at $(3, 4) = 3 \times 3 - 4 \times 4 - 8 = -15 < 0$

$\therefore$ For the point $P\ (x, y)$ we should have $k > 0$

$$\Rightarrow \quad 3x - 4y - 8 > 0$$

$$\Rightarrow \quad 3x - 4(-3x) - 8 > 0$$

$$[\because P\ (x, y) \text{ lies on } y = -3x]$$

$$\Rightarrow \quad x > 8/15$$

$$\Rightarrow \quad y < -8/5.$$

◉ **Example 101:** The area enclosed by $2|x| + 3|y| \leq 6$ is

(a) 3 sq. units (b) 4 sq. units
(c) 12 sq. units (d) 24 sq. units

Ans. (c)

◎ **Solution:** The given inequality is equivalent to the following system of inequalities.

$$2x + 3y \leq 6, \text{ when } x \geq 0, y \geq 0$$
$$2x - 3y \leq 6, \text{ when } x \geq 0, y \leq 0$$
$$-2x + 3y \leq 6, \text{ when } x \leq 0, y \geq 0$$
$$-2x - 3y \leq 6, \text{ when } x \leq 0, y \leq 0$$

which represents a rhombus with sides

$$2x \pm 3y = 6 \text{ and } 2x \pm 3y = -6$$

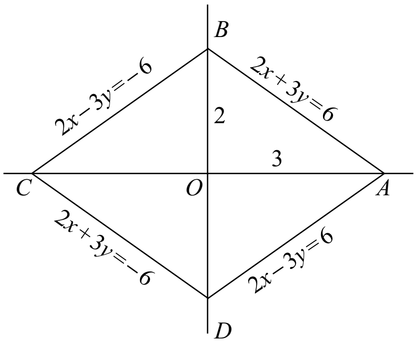


Fig. 16.17

Length of the diagonals is 6 and 4 units along x-axis and y-axis.

$\therefore$ The required area

$$= \frac{1}{2} \times 6 \times 4 = 12 \text{ sq. units.}$$

◉ **Example 102:** Let O be the origin, A (1, 0) and B (0, 1) and P (x, y) are points such that $xy > 0$ and $x + y < 1$, then

(a) P lies either inside the triangle OAB or in the third quadrant
(b) P cannot lie inside the triangle OAB
(c) P lies inside the triangle OAB
(d) P lies in the first quadrant only

Ans. (a)

◎ **Solution:** Since $xy > 0$, P either lies in the first quadrant or in the third quadrant. The inequality $x + y < 1$ represents all points below the line $x + y = 1$. So that $xy > 0$ and $x + y < 1$ imply that either P lies inside the triangle OAB or in the third quadrant.

◉ **Example 103:** A line has intercepts a and b on the coordinate axes. When the axes are rotated through an angle α, in the anti clockwise direction keeping the origin fixed, the line makes equal intercepts on the coordinate axes, then $\tan\alpha =$

(a) $\dfrac{a+b}{a-b}$ (b) $\dfrac{a-b}{a+b}$

(c) $a^2 - b^2$ (d) none of these

Ans. (b)

Solution: Let the equation of the line in the original coordinate system be $\dfrac{x}{a}+\dfrac{y}{b}=1$. If (X, Y) denote the coordinates of any point $P(x, y)$ in the new coordinate system obtained by rotation of the axes through an angle α, then

$$x = X\cos\alpha - Y\sin\alpha,\ y = X\sin\alpha + Y\cos\alpha$$

So that the equation of the line with reference to new system of coordinates is

$$\frac{X\cos\alpha - Y\sin\alpha}{a}+\frac{X\sin\alpha + Y\cos\alpha}{b}=1$$

or $$X\left(\frac{\cos\alpha}{a}+\frac{\sin\alpha}{b}\right)+Y\left(\frac{\cos\alpha}{b}-\frac{\sin\alpha}{a}\right)=1$$

Since it makes equal intercepts on the coordinates axes.

$b\cos\alpha + a\sin\alpha = a\cos\alpha - b\sin\alpha$

$\Rightarrow (a-b)\cos\alpha = (a+b)\sin\alpha$

$\Rightarrow \quad \tan\alpha = \dfrac{a-b}{a+b}$

Example 104: The equations to the sides of a triangle are $x - 3y = 0$, $4x + 3y = 5$ and $3x + y = 0$. The line $3x - 4y = 0$ passes through the

(a) incentre
(b) centroid
(c) circumcentre
(d) orthocentre of the triangle

Ans. (d)

Solution: Two sides $x - 3y = 0$ and $3x + y = 0$ of the triangle being perpendicular to each other, the triangle is right angled at the origin, the point of intersection of these sides. So that origin is the orthocentre of the triangle and the line $3x - 4y = 0$ passes through this orthocentre.

Example 105: A line through $P(1, 2)$ is such that it makes unequal intercepts on the axes, and the intercept between the axes is trisected at P, an equation of the line is

(a) $x + y = 3$ (b) $4x + y = 6$

(c) $4x - y = 6$ (d) $3x + 2y = 1$

Ans. (b)

Solution: Let the equation of the line be $\dfrac{x}{a}+\dfrac{y}{b}=1$, which meets x-axis at $A\,(a, 0)$ and y-axis at $B\,(0, b)$. Then the coordinates of the points which divide AB in the ratio 1 : 2 and 2 : 1 are respectively $\left(\dfrac{2a}{3},\dfrac{b}{3}\right)$ and $\left(\dfrac{a}{3},\dfrac{2b}{3}\right)$

Now if $\left(\dfrac{2a}{3},\dfrac{b}{3}\right)$ represents $P\,(1, 2)$ then $a = 3/2$, $b = 6$ and the required equation of the line is $\dfrac{2x}{3}+\dfrac{y}{6}=1 \Rightarrow 4x + y = 6$ which is given in (b)

If $\left(\dfrac{a}{3},\dfrac{2b}{3}\right)$ represents $P\,(1, 2)$ then $a = b = 3$ and the equation of the line is $x + y = 3$ which makes equal intercepts on the axes and so we do not consider this.

Example 106: The point (4, 1) undergoes the following successive transformations:

(i) reflection about the line $y = x$
(ii) translation through a distance 2 units along the positive x-axis.

then, the final coordinates of the point are

(a) (4, 3) (b) (3, 4)

(c) (1, 4) (d) (4, 4)

Ans. (b)

Solution: Let $Q\,(x, y)$ be the reflection of $P(4, 1)$ about the line $y = x$, then mid-point of PQ lies on this line and PQ is perpendicular to it. So we have $\dfrac{y+1}{2}=\dfrac{x+4}{2}$ and $\dfrac{y-1}{x-4} = -1$.

$\Rightarrow \quad x - y = -3$ and $x + y = 5$

$\Rightarrow \quad x = 1, y = 4$

Therefore reflection of (4, 1) about $y = x$ is (1, 4). Next, this point is shifted 2 units along the positive x-axis, the new coordinates are $(1 + 2, 4 + 0) = (3, 4)$

Example 107: Two adjacent sides of a parallelogram are $4x + 5y = 0$ and $7x + 2y = 0$. If an equation to one of the diagonals is $11x + 7y - 9 = 0$, then an equation of the other diagonal is

(a) $x + y = 0$ (b) $7x - 11y = 0$

(c) $x - y = 0$ (d) none of these

Ans. (c)

Solution: Since the given lines intersect at the origin O, one of the vertex is $O\,(0, 0)$. Let A and B be the points of intersection of the sides $4x + 5y = 0$ and $7x + 2y = 0$ respectively with the diagonal. $11x + 7y - 9 = 0$, then the coordinates of A and B are respectively $\left(\dfrac{5}{3},\dfrac{-4}{3}\right)$ and $\left(-\dfrac{2}{3},\dfrac{7}{3}\right)$ the coordinates of the mid point of AB are $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$. Since the other diagonal passes through the vertex (0, 0) and the mid-point $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$ of AB, its equation is $y = x$.

Example 108: If a line joining two points A (2, 0) and B (3, 1) is rotated about A in anticlockwise direction through an angle 15°, then equation of the line in the new position is

(a) $\sqrt{3}\,x + y = 2\sqrt{3}$ (b) $\sqrt{3}\,x - y = 2\sqrt{3}$

(c) $x + \sqrt{3}y = 2\sqrt{3}$ (d) $x - \sqrt{3}y = 2\sqrt{3}$

Ans. (b)

Solution: Slope of $AB = \dfrac{1-0}{3-2} = 1$

$\Rightarrow$ $\angle BAX = 45°$ (Ref. Fig. 16.18)

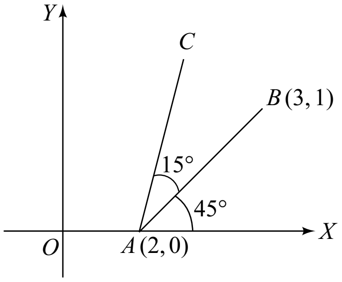


Fig. 16.18

If AC is the new position of the line AB then $\angle CAX = 45° + 15° = 60°$.
and thus its equation is

$$y = \tan 60° (x - 2)$$

$$\Rightarrow \quad y = \sqrt{3}\,(x-2) \quad \Rightarrow \quad \sqrt{3}x - y = 2\sqrt{3}$$

Example 109: If the medians AD and BE of the triangle with vertices $A(0, b)$, B (0, 0) and C $(a, 0)$ are perpendicular, then an equation of the line through (a, b) perpendicular to AC is

(a) $y = \sqrt{2}x + b$ (b) $y = -\sqrt{2}x + b$

(c) $y = \sqrt{2}x - b$ (d) $x = \sqrt{2}y - a$

Ans. (c)

Solution: Slope of $AD = \dfrac{b-0}{0-a/2} = -\dfrac{2b}{a}$

Slope of $BE = \dfrac{b/2}{a/2} = \dfrac{b}{a}$

Since AD and BE are perpendicular

$$\frac{-2b}{a}.\frac{b}{a} = -1$$

$$\Rightarrow \quad a^2 = 2b^2 \qquad \text{(i)}$$

Slope of AC is $-b/a$

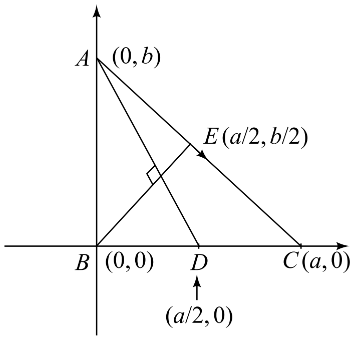


Fig. 16.19

$\therefore$ Equation of the line through (a, b) perpendicular to AC is

$$y - b = \frac{a}{b}\,(x - a)$$

$$\Rightarrow \quad ax - by = a^2 - b^2$$

$$\Rightarrow \pm\sqrt{2}bx - by = b^2 \text{ from (i)}$$

$$\pm\sqrt{2}x - y = b \quad \Rightarrow \quad y = \pm\sqrt{2}x - b$$

So an equation of required line is $y = \sqrt{2}x - b$

Example 110: $A_1, A_2 \ldots A_n$ are points on the line $y = x$ lying in the positive quadrant such that $OA_n = nOA_{n-1}$, O being the origin. If $OA_1 = 1$ and the coordinates of A_n are $(2520\sqrt{2}, 2520\sqrt{2})$, then $n =$

(a) 5 (b) 6
(c) 7 (d) 8

Ans. (c)

Solution: We have

$$OA_n = n.\ OA_{n-1} = n\,(n-1)\,OA_{n-2} \ldots$$
$$= n\,(n-1) \ldots 2(OA_1) = n!\ OA_1 = n!$$
$$\Rightarrow \quad \sqrt{2}\,(2520\sqrt{2}) = n!$$
$$\Rightarrow \quad 2520 \times 2 = n!$$
$$\Rightarrow \quad 7 \times 6 \times 5 \times 4 \times 3 \times 2 = n!$$
$$\Rightarrow \quad n = 7$$

Example 111: If x_1, x_2, x_3 are the abcissa of the points A_1, A_2, A_3 respectively where the lines $y = m_1 x$, $y = m_2 x$, $y = m_3 x$ meet the line $2x - y + 3 = 0$ such that m_1, m_2, m_3 are in A.P., then x_1, x_2, x_3 are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Ans. (c)

Solution: We have $x_1 = \dfrac{3}{m_1 - 2}, x_2 = \dfrac{3}{m_2 - 2}, x_3 = \dfrac{3}{m_3 - 2}$

$$\Rightarrow \quad \frac{1}{x_1} + \frac{1}{x_3} = \frac{m_1 - 2 + m_3 - 2}{3} = \frac{m_1 + m_3 - 4}{3}$$

$$= \frac{2m_2 - 4}{3} = \frac{2}{x_2}$$

(as m_1, m_2, m_3 are in A.P.)

⇒ x_1, x_2, x_3 are in H.P.

Example 112: If the lines $x = k$; $k = 1, 2 \ldots, n$ meet the line $y = 3x + 4$ at the points $A_k(x_k, y_k)$, $k = 1, 2, \ldots, n$ then the ordinate of the centre of Mean position of the points A_k, $k = 1, 2, \ldots, n$ is

(a) $\frac{n+1}{2}$ (b) $\frac{3n+11}{2}$

(c) $\frac{3(n+1)}{2}$ (d) none of these

Ans. (b)

Solution: We have $y_k = 3k + 4$, the ordinate of A_k, the point of intersection of $x = k$ and $y = 3x + 4$. So the ordinate of the centre of Mean position of the points A_k, $k = 1, 2, \ldots n$ is

$$\frac{1}{n}\sum_{k=1}^{n} y_k = \frac{1}{n}\sum_{k=1}^{n} (3k + 4) = \frac{3}{n}\sum_{k=1}^{n} k + 4$$

$$= \frac{3n(n+1)}{n \cdot 2} + 4 = \frac{3n+11}{2}.$$

Example 113: If $y = ax$ is one of the lines belonging to the family of lines representing the sides of an equilateral triangle with one vertex at the origin, then the product of the slopes of all the lines of this family is

(a) a^3 (b) $a(a^2 - 3)$

(c) $a(1 - 3a^2)$ (d) $\frac{a(a^2-3)}{1-3a^2}$

Ans. (d)

Solution: Let $a = \tan\theta$, then the slopes of the lines making an equilateral triangle with one vertex at the origin are:

$\tan\theta$, $\tan(\theta + 60°)$, $\tan(\theta + 120°)$, $\tan(\theta + 180°) = \tan\theta$

So the product of the slopes is

$$\tan\theta \tan(\theta + 60°) \tan(\theta + 120°)$$

$$= \tan\theta \tan(\theta + 60°) \tan(\theta - 60°)$$

$$= a \cdot \frac{a+\sqrt{3}}{1-a\sqrt{3}} \cdot \frac{a-\sqrt{3}}{1+a\sqrt{3}} = \frac{a(a^2-3)}{1-3a^2}.$$

Example 114: The sum of the intercepts cut off by the axes on the lines $x + y = a$, $x + y = ar$, $x + y = ar^2$, … where $a \neq 0$ and $r = 1/2$ is

(a) $2a$ (b) $a\sqrt{2}$

(c) $2\sqrt{2}a$ (d) $a/\sqrt{2}$

Ans. (c)

Solution: The intercepts between the axes made by the given lines are $a\sqrt{2}$, $ar\sqrt{2}$, $ar^2\sqrt{2}$, …

So the required sum $= \sqrt{2}\,[a + ar + ar^2 + \ldots]$

$$= \sqrt{2}\frac{a}{1-r} = \frac{\sqrt{2}a}{1-(1/2)} = 2\sqrt{2}a.$$

Example 115: The distance between the parallel lines given by $(x + 7y)^2 + 4\sqrt{2}\,(x + 7y) - 42 = 0$ is

(a) 4/5 (b) $4\sqrt{2}$

(c) 2 (d) $10\sqrt{2}$

Ans. (c)

Solution: The lines given by the equation are

$$(x + 7y - 3\sqrt{2})(x + 7y + 7\sqrt{2}) = 0$$

$$\Rightarrow \quad x + 7y - 3\sqrt{2} = 0 \quad \text{and} \quad x + 7y + 7\sqrt{2} = 0$$

$$\text{distance between these lines} = \left|\frac{7\sqrt{2} - (-3\sqrt{2})}{\sqrt{1^2 + 7^2}}\right|$$

$$= \frac{10\sqrt{2}}{5\sqrt{2}} = 2.$$

Example 116: If the lines joining the origin to the intersection of the line $y = mx + 2$ and the curve $x^2 + y^2 = 1$ are at right angles, then

(a) $m^2 = 1$ (b) $m^2 = 3$

(c) $m^2 = 7$ (d) $2m^2 = 1$

Ans. (c)

Solution: Joint equation of the lines joining the origin and the point of intersection of the line $y = mx + 2$ and the curve $x^2 + y^2 = 1$ is

$$x^2 + y^2 = \left(\frac{y - mx}{2}\right)^2$$

$$\Rightarrow \quad x^2(4 - m^2) + 2mxy + 3y^2 = 0$$

Since these lines are at right angles

$4 - m^2 + 3 = 0 \quad \Rightarrow m^2 = 7.$

Example 117: If one of the lines given by the equation $2x^2 + axy + 3y^2 = 0$ coincide with one of those given by $2x^2 + bxy - 3y^2 = 0$ and the other lines represented by them be perpendicular, then

(a) $a = -5$, $b = 1$ (b) $a = 5$, $b = -1$

(c) $a = 5$, $b = 1$ (d) none of these

Ans. (c)

Solution: Let $\frac{2}{3}x^2 + \frac{a}{3}xy + y^2 = (y - mx)(y - m'x)$

and $$\frac{2}{-3}x^2 + \frac{b}{-3}xy + y^2 = \left(y + \frac{1}{m}x\right)(y - m'x)$$

then $$m + m' = -\frac{a}{3},\ mm' = \frac{2}{3} \quad \text{(i)}$$

$$\frac{1}{m} - m' = \frac{-b}{3},\ -\frac{m'}{m} = -\frac{2}{3} \quad \text{(ii)}$$

$$\Rightarrow \quad m^2 = 1 \Rightarrow m = \pm 1$$

$$\text{If } m = 1,\ m' = \frac{2}{3} \Rightarrow a = -5,\ b = -1$$

$$\text{If } m = -1,\ m' = -\frac{2}{3} \Rightarrow a = 5,\ b = 1.$$

Example 118: If the equation $ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$ ($a, b, c, d, \neq 0$) represents three coincident lines then

(a) $a = d$ (b) $b = c$

(c) $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$ (d) $ac = bd$

Ans. (c)

Solution: Let the given equation represent three coincident lines $y - mx = 0$, then

$$y^3 + \frac{3c}{d}y^2x + \frac{3b}{d}yx^2 + \frac{a}{d}x^3 = (y - mx)^3$$

$$= y^3 - 3y^2(mx) + 3y(mx)^2 - m^3x^3$$

$$\Rightarrow \quad m^3 = -\frac{a}{d},\ m^2 = \frac{b}{d},\ m = -\frac{c}{d}$$

$$\Rightarrow \quad m = -\frac{a}{b} = -\frac{b}{c} = -\frac{c}{d}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{b}{c} = \frac{c}{d}.$$

Example 119: Two of the lines represented by $x^3 - 6x^2y + 3xy^2 + dy^3 = 0$ are perpendicular for

(a) all real values of d

(b) two real values of d

(c) three real values of d

(d) no real value of d

Ans. (b)

Solution: Let m_1, m_2, m_3 be the slopes of the three lines represented by the given equation such that

$$m_1 m_2 = -1$$

We have $m_1 m_2 m_3 = -\frac{1}{d}$ so that $m_3 = \frac{1}{d}$

Since $y = m_3 x$ i.e. $x = dy$ satisfies the given equation, we get

$$d^3 - 6d^2 + 3d + d = 0$$

$$\Rightarrow \quad d(d^2 - 6d + 4) = 0$$

If $d = 0$, the given equation represents the lines $x = 0$ and $x^2 - 6xy + 3y^2 = 0$ which are not perpendicular.

$\therefore$ $d \neq 0$ and $d^2 - 6d + 4 = 0$

$$\Rightarrow \quad d = \frac{6 \pm \sqrt{36 - 16}}{2} = 3 \pm \sqrt{5}$$

which gives two real values of d.

Example 120: The line $x + y = 1$ meets the lines represented by the equation $y^3 - xy^2 - 14x^2y + 24x^3 = 0$ at the points A, B, C. If O is the point of intersection of the lines represented by the given equation then $OA^2 + OB^2 + OC^2$ is equal to

(a) 22/9 (b) 85/72

(c) 181/72 (d) 221/72

Ans. (d)

Solution: The given cubic can be written as

$$(y - 2x)(y - 3x)(y + 4x) = 0$$

$\therefore$ The three lines given by this equation are

$y = 2x$, $y = 3x$ and $y = -4x$, they intersect at O (0,0) and meet the line $x + y = 1$ at the points

A (1/3, 2/3), B (1/4, 3/4) and C (–1/3, 4/3)

$$\therefore \quad OA^2 + OB^2 + OC^2 = \frac{5}{9} + \frac{10}{16} + \frac{17}{9} = \frac{221}{72}$$

Example 121: A line passing through the point $P(2, 3)$ meets the lines represented by $x^2 - 2xy - y^2 = 0$ at the points A and B such that $PA \cdot PB = 17$, the equation of the line is

(a) $x = 2$ (b) $y = 3$

(c) $3x - 2y = 0$ (d) none of these

Ans. (b)

Solution: Let the equation of the line through $P(2, 3)$ making an angle θ with the positive direction of x-axis be

$$\frac{x - 2}{\cos\theta} = \frac{y - 3}{\sin\theta}.$$

Then the coordinates of any point on this line at a distance r from P are $(2 + r\cos\theta, 3 + r\sin\theta)$. If $PA = r_1$ and $PB = r_2$, then r_1, r_2 are the roots of the equation.

$$(2 + r\cos\theta)^2 - 2(2 + r\cos\theta)(3 + r\sin\theta) - (3 + r\sin\theta)^2 = 0$$

$$\Rightarrow r^2(\cos 2\theta - \sin 2\theta) - 2r(\cos\theta + 5\sin\theta) - 17 = 0$$

$$\Rightarrow 17 = \mathbf{PA} \cdot \mathbf{PB} = r_1 r_2 = \frac{17}{\cos 2\theta - \sin 2\theta}$$

$\Rightarrow \cos 2\theta - \sin 2\theta = 1$ which is satisfied by $\theta = 0$ and thus the equation of the line is $y = 3$.

Example 122: If $(\lambda, 2)$ is an interior point of the triangle formed by the lines $x + y = 4$, $3x - 7y = 8$ and $4x - y = 31$, then

(a) $2 < \lambda < 22/3$ (b) $2 < \lambda < 33/4$

(c) $22/3 < \lambda < 33/4$ (d) $\lambda > 9$

Ans. (c)

Solution: The point $(\lambda, 2)$ lies on the line $y = 2$ which meets the line $x + y = 4$ at a point outside the triangle and the line $3x - 7y = 8$ at $D = (22/3, 2)$ and the line $4x - 3y = 31$ at $E = (33/4, 2)$.

So x-coordinate of the point on the line $y = 2$, lying inside the triangle lies between 22/3 and 33/4.

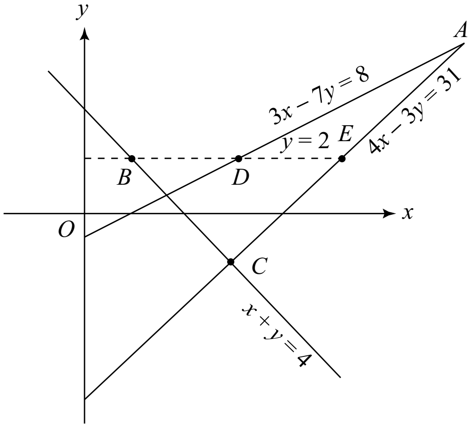


Fig. 16.20

Example 123: The lines given by L_1: $x + 3y - 5 = 0$, L_2: $3x - k y - 1 = 0$ L_3: $5x + 2y - 12 = 0$
form a triangle if k is

(a) 5 (b) 5/6
(c) – 6/5 (d) – 9

Ans. (b)

Solution: L_1 and L_3 intersect at (2,1)
The lines will form a triangle if no two of them are parallel and all of them are not concurrent.
Lines are concurrent at (2,1) if k = 5
L_2 is parallel to L_1 if $k = -9$
L_3 is parallel to L_1 if $k = -6/5$
So the lines form a triangle for any value of k other than 5, – 9, – 6/5

Example 124: Let the algebraic sum of the perpendicular distances from the points (2, 0) (0, 2) and (1, 1) to a variable straight line be zero, then the line passes through a fixed point whose coordinates are

(a) (1, – 1) (b) (– 1, 1)
(c) (1, 1) (d) (– 1, – 1)

Ans. (c)

Solution: Let the equation of the line be $ax + by + c = 0$

then $\dfrac{2a+c}{\sqrt{a^2+b^2}} + \dfrac{2b+c}{\sqrt{a^2+b^2}} + \dfrac{a+b+c}{\sqrt{a^2+b^2}} = 0$

$\Rightarrow \quad 3a + 3b + 3c = 0$

$\Rightarrow \quad a + b + c = 0$

So the equation of the line can be $ax + by - (a + b) = 0$

$\Rightarrow \quad a(x - 1) + b(y - 1) = 0$

which passes through (1,1)

Example 125: The locus of the point P(h, k), when the area of the triangle formed by the lines $y = x$, $x + y = 2$ and the line through P (h,k) and parallel to the x-axis is $4h^2$ is

(a) $x + 2y - 1 = 0$ (b) $2x + y - 1 = 0$
(c) $2x - y - 1 = 0$ (d) $x - 2y + 1 = 0$

Ans. (b)

Solution: Let the line through P (h, k) be $y = k$
Vertices of the triangle are (1,1), (k,k), (2 – k, k)

$$4h^2 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ k & k & 1 \\ 2 & -k & 1 \end{vmatrix} \Rightarrow 4h^2 = (k-1)^2$$

$= 2h = \pm (k - 1)$

Hence locus of P is $2x \pm (y - 1) = 0$

i.e $\quad 2x + y - 1 = 0$ or $2x - y + 1 = 0$

EXERCISE

Concept-based Straight Objective Type Questions

1. A is a point on the positive x-axis at a distance 3 units from the origin and B is a point on the positive y-axis at a distance 4 units from the origin. If P divides AB in the ratio 1 : 2, the coordinates of P are
(a) (1, 8/3) (b) (2, 4/3)
(c) 8/3, 1) (d) (4/3, 2)

2. The slopes of the line which passes through the origin, and the mid-point of the line segment joining the points. P(3, – 4) and Q (– 5, – 2) is
(a) 3 (b) – 3
(c) – 1 (d) 1

3. Distance between P (x_1, y_1) and Q (x_2, y_2) when PQ is parallel to y-axis is
(a) $x_1 - x_2$ (b) $|x_1 - x_2|$
(c) $y_1 - y_2$ (d) $|y_1 - y_2|$

4. The lines parallel to the axes and passing through the point (4, – 5) are
(a) $x = -5, y = 4$ (b) $x = 5, y = -4$
(c) $x = 4, y = -5$ (d) $x = -4, y = 5$

5. The equation of the line whose perpendicular distance from the origin is 3 units and the angle which the normal makes with the positive direction of x-axis is 30° is

(a) $x + \sqrt{3}\,y = 3$ (b) $\sqrt{3}\,x + y = 6$
(c) $\sqrt{3}\,x + y = 1$ (d) $x + \sqrt{3}\,y = 6$

6. Points (8, 2), (– 2, – 2) and (3, 0) are the vertices of
(a) an equilateral triangle (b) an isosceles triangle
(c) right angled triangle (d) none of these

7. If the angle between the lines $\sqrt{3}\,y - x + 4 = 0$ and $x + y - 6 = 0$ is θ, then $\tan\theta$ is equal to.
(a) $\sqrt{3} + 1$ (b) $\sqrt{3} - 1$
(c) $2 + \sqrt{3}$ (d) $3 + \sqrt{2}$

8. Equation of the line passing through the point $(a - 1, a + 1)$ and making zero intercept on both axes is
(a) $ax + ay - 1 = 0$ (b) $(a + 1)x + (a - 1)y = 0$
(c) $(a - 1)x - (a + 1)y = 0$ (d) $(a + 1)x - (a - 1)y = 0$

9. The angle which the normal to the line $x - \sqrt{3}\,y + 8 = 0$ passing through the origin, makes with the positive x-axis is
(a) 30° (b) 60°
(c) 120° (d) 150°

10. If the line through the points $(h, 7)$ and (2, 3) intersects the line $3x - 4y - 5 = 0$ at right angles, then the value of h is
(a) – 1 (b) 1
(c) 5 (d) – 5

11. Equation of a line passing through the intersection of the lines $7x - y + 2 = 0$ and $x - 3y + 6 = 0$ parallel to x-axis is
(a) $y = 2$ (b) $y = -2$
(c) $y = 3$ (d) $y = -3$

12. The value of p for which the lines $2x + y - 3 = 0$, $3x - y - 2 = 0$ and $x - py + 5 = 0$ may intersect at a point is
(a) 2 (b) 3
(c) 5 (d) 6

13. Equation of the line equidistant from the parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$ is
(a) $6x + 4y + 5 = 0$ (b) $18x + 12y + 11 = 0$
(c) $18x + 12y - 11 = 0$ (d) $12x + 8y + 7 = 0$

14. Area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $y = k$ in square units is
(a) $2k$ (b) k^2
(c) $2k^2$ (d) $k^2/2$.

15. The distance of the line $2x + 3y - 5 = 0$ from the point (3, 5) along the line $5x - 3y = 0$ in units is
(a) $\dfrac{2\sqrt{34}}{21}$ (b) $\dfrac{16}{21}$
(c) $\dfrac{4\sqrt{34}}{21}$ (d) $\dfrac{3\sqrt{43}}{21}$

LEVEL 1

Straight Objective Type Questions

16. The points $(k - 1, k + 2)$, $(k, k + 1)$, $(k + 1, k)$ are collinear for
(a) any value of k (b) $k = -1/2$ only
(c) no value of k (d) integral values of k only

17. The area of the triangle formed by the points $(k, 2 - 2k)$, $(-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ is 70 units. For
(a) four real values of k
(b) no integral value of k
(c) two integral values of k
(d) only one integral value of k

18. The quadrilateral $ABCD$ formed by the points A (0, 0); B (3, 4), C (7, 7) and D (4, 3) is a
(a) rectangle (b) square
(c) rhombus (d) parallelogram

19. The triangle with vertices A (2, 7), B $(4, y)$ and C (–2, 6) is right angled at A if
(a) $y = -1$ (b) $y = 0$
(c) $y = 1$ (d) none of these

20. The join of the points (–3, –4) and (1, –2) is divided by y-axis in the ratio.
(a) 1 : 3 (b) 2 : 3
(c) 3 : 1 (d) 3 : 2

21. The straight lines $x + y - 4 = 0$, $3x + y - 4 = 0$ and $x + 3y - 4 = 0$ form a triangle which is
(a) isosceles (b) right angled
(c) equilateral (d) none of these

22. The points $P\,(a, b + c)$, $Q\,(b, c + a)$ and $R\,(c, a + b)$ are such that $PQ = QR$ if
(a) a, b, c are in A.P. (b) a, b, c are in G.P.
(c) a, b, c are in H.P. (d) none of these

23. If a, b, c are in A.P. then the points (a, x), (b, y) and (c, z) are collinear if
(a) $x^2 = y$ (b) $x = z^2$
(c) $y^2 = z$ (d) x, y, z are in A.P.

24. The centroid of a triangle lies at the origin and the coordinates of its two vertices are $(-8, 7)$ and $(9, 4)$. The area of the triangle is

(a) 95/6 (b) 285/2
(c) 190/3 (d) 285

25. The mid points of the sides AB and AC of a triangle ABC are $(2, -1)$ and $(-4, 7)$ respectively, then the length of BC is

(a) 10 (b) 20
(c) 25 (d) 30

26. If the vertices of a triangle ABC are $A\ (-4, -1)$, $B\ (1, 2)$ and $C\ (4, -3)$, then the coordinates of the circumcentre of the triangle are,

(a) $(1/3, -2/3)$ (b) $(0, -4)$
(c) $(0, -2)$ (d) $(-3/2, 1/2)$

27. The extremities of a diagonal of a parallelogram are the points $(3, -4)$ and $(-6, 5)$. If third vertex is $(-2, 1)$ then the coordinates of the fourth vertex are

(a) $(1, 0)$ (b) $(0, 0)$
(c) $(1, 1)$ (d) none of these

28. If the vertices A and B of a triangle ABC are given by $(2, 5)$ and $(4, -11)$ and C moves along the line $L_1 : 9x + 7y + 4 = 0$, the locus of the centroid of the triangle ABC is a straight line parallel to

(a) AB (b) BC
(c) CA (d) L_1

29. The number of lines that can be drawn through the point $(4, -5)$ at a distance 12 from the point $(-2, 3)$ is

(a) 0 (b) 1
(c) 2 (d) infinite

30. If O is the origin and the coordinates of A and B are (x_1, y_1) and (x_2, y_2) respectively then $OA \times OB \cos \angle AOB$ is equal to

(a) $x_1 y_1 + x_2 y_2$ (b) $x_1 x_2 + y_1 y_2$
(c) $x_1 y_2 + x_2 y_1$ (d) $x_1 x_2 - y_1 y_2$

31. If the lines $x + ay + a = 0$, $bx + y + b = 0$ and $cx + cy + 1 = 0$ (a, b, c being distinct and $\neq 1$) are concurrent, then the value of $\frac{a}{a-1} + \frac{b}{b-1} + \frac{c}{c-1}$ is

(a) -1 (b) 0
(c) 1 (d) none of these

32. The line $\frac{x}{3} + \frac{y}{4} = 1$ meets the axis of y and axis of x at A and B respectively. A square $ABCD$ is constructed on the line segment AB away from the origin, the coordinates of the vertex of the square farthest from the origin are

(a) $(7, 3)$ (b) $(4, 7)$
(c) $(6, 4)$ (d) $(3, 8)$

33. A line $2x + 3y - 1 = 0$ intersects the three sides BC, CA and AB of a triangle ABC in P, Q, R respectively then $\frac{BP}{PC} \times \frac{CQ}{QA} \times \frac{AR}{RB} =$

(a) -1 (b) 1
(c) 2 (d) 3

34. If the vertices P, Q, R of a triangle are rational points, which of the following points of the triangle PQR is (are) always rational point (s)?

(a) centroid (b) incentre
(c) circumcentre (d) orthocentre

(A rational point is a point both of whose coordinates are rational number)

35. Let $A_0\ A_1\ A_2\ A_3\ A_4\ A_5$ be a regular hexagon with vertex A_0 and A_3 at $(1, 0)$ and $(-1, 0)$ respectively. The equations representing $A_1\ A_4$ and $A_2\ A_5$ are

(a) $y = \pm\sqrt{3}x$ (b) $x = \pm\sqrt{3}y$
(c) $y = \pm x$ (d) none of these

36. Let PS be the median of the triangle with vertices $P\ (2, 2)$, $Q\ (6, -1)$ and $R\ (7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is

(a) $2x - 9y - 7 = 0$ (b) $2x - 9y - 11 = 0$
(c) $2x + 9y - 11 = 0$ (d) $2x + 9y + 7 = 0$

37. Two equal sides of an isosceles triangle are given by $7x - y + 3 = 0$ and $x + y - 3 = 0$, the slope m of the third side is given by

(a) $m^2 - 1 = 0$ (b) $m^2 - 3 = 0$
(c) $3m^2 - 1 = 0$ (d) $3m^2 + 8m - 3 = 0$

38. Two sides of a rhombus $ABCD$ are parallel to the line $y = x + 2$ and $y = 7x + 3$. If the diagonals of the rhombus intersect at the point $(1, 2)$ and the vertex A is on Y-axis at a distance a from the origin, then a is equal to

(a) 2 (b) 5
(c) 2/5 (d) 5/2

39. A line which makes an acute angle θ with the positive direction of x-axis is drawn through the point $P\ (3, 4)$ to meet the lines $x = 6$ and $y = 8$ at R and S respectively then $RS = 8 - 2\sqrt{3}$, if

(a) $\theta = \pi/3$ (b) $\theta = \pi/4$
(c) $\pi/6$ (d) $\pi/12$

40. In a right angled triangle ABC right angled at C: $CA = a$, $CB = b$. If the angular points A and B slide along x-axis and y-axis respectively then C lies on

(a) $bx \pm ay = 0$ (b) $ax \pm by = 0$
(c) $\frac{x}{a} \pm \frac{y}{b} = 1$ (d) $\frac{x}{b} + \frac{y}{a} = 1$

41. *ABCD* is a quadrilateral. *P* (3, 7) and *Q* (7, 3) are the middle points of the diagonals *AC* and *BD* respectively. The coordinates of the mean point (or the centre of mean position) of the vertices of the quadrilateral are
    (a) (0, 0)
    (b) (3, 3)
    (c) (5, 5)
    (d) (7, 7)

42. A line meets *x*-axis at *A* and *y*-axis at *B* such that in-centre of the triangle *OAB* is (1, 1). Equation of *AB* is
    (a) $2x + y = 2$
    (b) $3x + 4y = 12$
    (c) $2x + y = 6$
    (d) $2x + y = 4$

43. *ABCD* is a rectangle in the clockwise direction. The coordinates of *A* are (1, 3) and of *C* are (5, 1), vertices *B* and *D* lie on the line $y = 2x + c$, then the coordinates of *D* are
    (a) (2, 0)
    (b) (4, 4)
    (c) (0, 2)
    (d) (2, 4)

44. The area of a triangle, two of whose vertices are (2, 1) and (3, – 2) is 5. The coordinates of the third vertex can not be
    (a) (6, – 1)
    (b) (4, 5)
    (c) (– 1, 20)
    (d) (2, 9)

45. The diagonals of a parallelogram *PQRS* are along the lines $x + 3y = 4$ and $6x - 2y = 7$, the *PQRS* must be *a*
    (a) rectangle
    (b) square
    (c) cyclic quadrilateral
    (d) rhombus

46. If *a, b, c* are in A.P., then $ax + by + c = 0$ represents
    (a) a single line
    (b) a family of concurrent lines
    (c) a family of parallel lines
    (d) none of these

47. If area of the triangle formed by the line *L* perpendicular to $5x - y = 1$ and the coordinate axes is 5, then the distance of *L* from the origin is
    (a) $5\sqrt{2}$
    (b) $5/\sqrt{13}$
    (c) $5\sqrt{13}$
    (d) none of these

48. The area enclosed by the lines $|x| + |y| = 2$ is
    (a) 1 sq unit
    (b) 2 sq units
    (c) 4 sq units
    (d) none of these

49. If the circumcentre of a triangle lies at the point (*a*, *a*) and the centroid is the mid-point of the line joining the points $(2a + 3, a + 4)$ and$(a - 4, 2a - 3)$; then the orthocentre of the triangle lies on the line
    (a) $y = x$
    (b) $(a - 1)x + (a + 1)y = 0$
    (c) $(a - 1)x - (a + 1)y = 0$
    (d) $(a + 1)x - (a - 1)y = 2a$

50. If $A\ (at^2, 2at)$, $B(a/t^2, -2a/t)$ and $S\ (a, 0)$ are three points, then $\dfrac{1}{SA} + \dfrac{1}{SB}$ is independent of
    (a) *a*
    (b) *t*
    (c) both *a* and *t*
    (d) none of these

51. $qx + py + (p + q - r) = 0$ is the reflection of the line $px + qy + r = 0$ in the line
    (a) $x - y = p + q$
    (b) $x - y = p - q$
    (c) $x + y + 1 = 0$
    (d) $x + y - 1 = 0$

52. Coordinates of the vertices *B* and *C* of the base of a triangle *ABC* are (– *a*, 0) and (*a*, 0) respectively. If $C - B = \pi/3$, the vertex *A* lies on the curve
    (a) $x^2 - y^2 + 2\sqrt{3}\ xy - a^2 = 0$
    (b) $x^2 + y^2 + 2\sqrt{3}\ xy - a^2 = 0$
    (c) $\sqrt{3}\ (x^2 - y^2) + 2xy - \sqrt{3}\,a^2 = 0$
    (d) $\sqrt{3}\ (x^2 + y^2) - 2xy + \sqrt{3}\,a^2 = 0$

53. If the line $\sqrt{5}x = y$ meets the lines $x = 1$, $x = 2$, … $x = n$ at points $A_1, A_2, \ldots, A_n$ respectively, then $(0\ A_1)^2 + (0\ A_2)^2 + \ldots + (0\ A_n)^2$ is equal to
    (a) $3n^2 + 3n$
    (b) $2n^3 + 3n^2 + n$
    (c) $3n^3 + 3n^2 + 2$
    (d) $(3/2)\ (n^4 + 2n^3 + n^3)$

54. One diagonal of a square is the portion of the line $3x + 2y = 12$ intercepted between the axes. The coordinates of the extremity of the other diagonals not lying in the first quadrant are
    (a) (1, – 1)
    (b) (– 1, – 1)
    (c) (– 1, 1)
    (d) none of these

55. If $y = m_i x + \dfrac{1}{m_i}$ $(i = 1, 2, 3)$ represents three straight lines whose slopes are the roots of the equation. $2m^3 - 3m^2 - 3m + 2 = 0$, *A* and *B* are the algebraic sum of the intercepts made by the lines on *x*-axis and *y*-axis respectively, then $\alpha A + \beta B = 0$ if (α, β) is
    (a) (4, 7)
    (b) (2, 7)
    (c) (7, 2)
    (d) (– 1,– 7)

56. If $u = a_1x + b_1y + c_1$, $v = a_2x + b_2y + c_2 = 0$ and $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ then $u + kv = 0$ represents
    (a) a family of concurrent lines
    (b) a family of parallel lines
    (c) $u = 0$ or $v = 0$
    (d) none of these

57. Reflection of the line $x + y + 1 = 0$ in the line $lx + my + n = 0$ is
    (a) $(x + y + 1)(l + m) - 2(l^2 + m^2)(lx + my + n) = 0$
    (b) $(x + y + 1)(l^2 + m^2) - 2(l + m)(lx + my + n) = 0$
    (c) $(l + m + 1)(x + y) - 2(lx + my)(l^2 + m^2) = 0$
    (d) none of these

58. *ABCD* is a square in which *A* lies on the positive *y*-axis and *B* lies on the positive *x*-axis. If *D* is the point (12, 17), the coordinates of *C* are

(a) (17, 12)
(b) (17, 5)
(c) (14, 16)
(d) (15, 3)

59. *ABCD* is a rhombus. Its diagonals *AC* and *BD* intersect at the point *M* and satisfy $BD = 2AC$. If the coordinates of *D* and *M* are (1, 1) and (2, – 1) respectively, the coordinates of *A* are

(a) (3, 1/2)
(b) (1, – 3/2)
(c) (3/2, 1)
(d) (1/2, 3)

60. A straight line passes through the point (1, 1) and the portion of the line intercepted between *x* and *y* axes is divided at the point in the ratio 3 : 4. An equation of the line is

(a) $3x + 4y = 7$
(b) $4x + 3y = 7$
(c) $4x - 3y = 1$
(d) $3x - 4y + 1 = 0$

61. The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of perpendicular lines if

(a) $a = 3$, $b = 4$
(b) $a = 4$, $b = -3$
(c) $h = -1$
(d) $a = 11$, $b = -11$

62. The equation $ax^2 + 2hxy + ay^2 = 0$ represent a pair of coincident lines through the origin if

(a) $h = 2a$
(b) $2h = a$
(c) $h^2 = a$
(d) $h^2 = a^2$

63. The equation $ax^2 + by^2 + cx + cy = 0$ represents a pair of straight lines if

(a) $a = 0$
(b) $b = 0$
(c) $a + b = 0$
(d) none of these

64. The equation $x^3 - 6x^2y + 11xy^2 - 6y^3 = 0$ represent three straight lines passing through the origin, the slops of which form

(a) an A.P.
(b) a G.P.
(c) an H.P.
(d) none of these

65. If the slopes of the lines given by $8x^3 + ax^2y + bxy^2 + y^3 = 0$ are in G.P., then

(a) $a = b$
(b) $2a = b$
(c) $a = 2b$
(d) $a + b = 0$

66. If the slope of one of the lines given by $6x^2 + axy + y^2 = 0$ exceeds the slope of the other by one, then *a* is equal to

(a) ± 2
(b) 5
(c) – 5
(d) ± 5

67. If the slope of one of the lines represented by $ax^2 + (3a + 1)\,xy + 3y^2 = 0$ be reciprocal of the slope of the other, then the slopes of the lines are

(a) 3/2, 2/3
(b) 1/2, 2/1
(c) 1/3, 3
(d) – 1/3, – 3

68. The equations of the pairs of opposite sides of a rectangle are $x^2 - 7x + 6 = 0$ and $y^2 - 14y + 40 = 0$, the equation of the diagonal nearer the origin is

(a) $5x - 6y + 14 = 0$
(b) $6x - 5y + 14 = 0$
(c) $6x + 5y - 56 = 0$
(d) $6x + 5y - 14 = 0$

69. The equation of a line bisecting the join of (2010, 1600) and (– 1340, 1080) and having intercept on the axes in the ratio 1 : 2 is

(a) $2x + y = 1680$
(b) $x + 2y = 1680$
(c) $2x + y = 2010$
(d) none of these

70. Let the coordinates of *P* be (*x*, *y*) and of *Q* be (α, β) where α is the geometric and β is the arithmetic mean of the coordinates of *P*. If the mid point of *PQ* is (42, 31) the coordinates of *P* are

(a) (61, 21)
(b) (49, 25)
(c) (31, 31)
(d) none of these

71. Image of the point (–1, 3) with respect to the line $y = 2x$ is

(a) (7/5, 14/5)
(b) (1, 2)
(c) (3, 1)
(d) (5, 1)

72. The point (a^2, a) lies between the straight lines $x + y = 6$ and $x + y = 2$ for

(a) all values of *a*
(b) no value of *a*
(c) $|2a - 3| < 1$
(d) $|2a + 5| > 1$

73. Perimeter of the quadrilateral bounded by the coordinate axis and the lines $x + y = 50$ and $3x + y = 90$ is

(a) $80 + 20\sqrt{2}$
(b) $80 + 10\sqrt{10}$
(c) $80 + 20\sqrt{2} + 10\sqrt{10}$
(d) 110.

74. If the sum of the slopes of the lines given by $3x^2 - 2cxy - 5y^2 = 0$ is twice their product, then the value of *c* is

(a) 2
(b) 3
(c) 6
(d) none of these

75. If one of the lines given by $2cx^2 + 2xy - (c^2 - 1)y^2 = 0$ is $2x + 3y = 0$, then the integral value of *c* is

(a) 2
(b) 3
(c) 4
(d) 8

Assertion-Reason Type Questions

76. Let $A(2, -3)$ and $B(-2, 1)$ be the vertices of a triangle ABC.
    Statement-1: If the centroid of the triangle moves on the line $x + y = 5$, the vertex moves on the line $x + y = 17$.
    Statement-2: If the centroid of the triangle moves the line $x - y + 1 = 0$ $(x \neq 0)$, the triangle is either isosceles or equilateral.
77. **Statement-1:** $x^2y - 3xy - 2x^2 + 6x - 4y + 8 = 0$ represents three straight lines two of which are parallel and the third is perpendicular to the other two
    Statement-2: $xy - 2x + y - 2 = 0$ represents a pair of straight lines one of which is common to the pair of straight lines $xy + 2x - y - 2 = 0$
78. **Statement-1:** The locus represented by the equation $(2x + 3y - 1)^2 + (x + y + 1)^2 = 0$ is a pair of straight lines $2x + 3y - 1 = 0$ and $x + y + 1 = 0$.
    Statement-2: The equation $(x + 4y - 2)^2 + (4x + y - 8)^2 + (x \pm y - k)^2 = 0$ represents a point if $k = 2$.
79. **Statement-1:** Points $P = (-\sin(\beta - \alpha), -\cos\beta)$, $Q = (\cos(\beta - \alpha), \sin\beta)$ and $R = (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$ where $0 < \alpha, \beta, \theta < \pi/4$ are non collinear
    Statement-2: If the slopes of the lines PQ and QR are not equal the points P, Q and R are non collinear.
80. **Statement-1:** If $x + ky = 1$ and $x = a$ are the equations of the hypotenuse and a side of a right angled isosceles triangle then $k = \pm a$.
    Statement-2: Each side of a right angled isosceles triangle makes an angle $\pi/4$ with the hypotenuse.
81. **Statement-1:** Locus of the centroid of the triangle whose vertices are $(a\cos t, a\sin t)$, $(b\sin t, -b\cos t)$ and $(1, 0)$ where t is a parameter is $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
    Statement-2: The centroid of a triangle is equidistance from the vertices of the triangle.
82. **Statement-1:** If non zero numbers a, b, c are in Harmonic progression, then the straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ always passes through the fixed point $(1, 1)$
    Statement-2: If a, b, c are in Harmonic progression. then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression.
83. **Statement-1:** The straight line. $(\sin\theta + \sqrt{3}\cos\theta)x + (\sqrt{3}\sin\theta - \cos\theta)y + (5\sin\theta - 7\cos\theta) = 0$ for all values of θ except $\theta = n\pi/2$, n is an integer; passes through the point of intersection of the lines $x + \sqrt{3}y + 5 = 0$ and $\sqrt{3}x - y - 7 = 0$.
    Statement-2: $L_1 + \lambda L_2 = 0$ represents a line through the point of intersection of the lines $L_1 = 0$, $L_2 = 0$ for all non-zero finite values of λ.
84. **Statement-1:** If the area of the triangle formed by the lines $y = x$, $x + y = 2$ and the lines through $P(h, k)$ parallel to x-axis is $4h^2$, then P lies on the lines $4x^2 + 2y - y^2 - 1 = 0$.
    Statement-2: Area of the triangle formed by the lines $y = x$, $x + y = 2$ and the axis of x is equal to half the area of the triangle formed by the line $x + y = 2$ and the coordinate axes.
85. **Statement-1:** Reflexion of the point $(6, 1)$ in the line $x + y = 0$ is $(-1, -6)$.
    Statement-2: Reflexion of a point $P(\alpha, \beta)$ in the line $ax + by + c = 0$ is $Q(\alpha', \beta')$ if $R\left(\frac{\alpha + \alpha'}{2}, \frac{\beta + \beta'}{2}\right)$ lies on the line.
86. **Statement-1:** $4x^2 + 12xy + 9y^2 = 0$ represents a pair of perpendicular lines through the origin.
    Statement-2: $ax^2 + 2hxy + by^2 = 0$ represents a pair of coincident lines if $h^2 = ab$.
87. **Statement-1:** $x^2y^2 - x^2 - y^2 + 1 = 0$ represents the sides of a square of area 4 square units.
    Statement-2: $3x^2 + \lambda xy - 3y^2 = 0$ represents a pair of perpendicular lines for all values of λ

LEVEL 2

Straight Objective Type Questions

88. If $(0, 1)$, $(1, 1)$ and $(1, 0)$ are the mid points of the sides of a triangle, the coordinates of its incentre are
    (a) $(2 + \sqrt{2}, 2 + \sqrt{2})$
    (b) $((2 + \sqrt{2}), -(2 + \sqrt{2}))$
    (c) $((2 - \sqrt{2}), (2 - \sqrt{2}))$
    (d) $((2 - \sqrt{2}), -(2 - \sqrt{2}))$

89. The vertices of the triangle ABC are $A(1, 2)$, $B(0, 0)$ and $C(2, 3)$, then the greatest angle of the triangle is

(a) 75°
(b) 105°
(c) 120°
(d) None of these

90. The points $(0, 8/3)$, $(1, 3)$ and $(82, 30)$ are the vertices of

(a) obtuse angled triangle
(b) acute angled triangle
(c) right angled triangle
(d) none of these

91. Area of the rhombus enclosed by the lines $ax \pm by \pm c = 0$ is

(a) $2a^2/bc$
(b) $2b^2/ca$
(c) $2c^2/ab$
(d) none of these

92. If $x \cos \alpha + y \sin \alpha = -\sin \alpha \tan \alpha$ be the equation of a line, then the length of the perpendiculars on the line from the points $(a^2, 2a)$, $(ab, a + b)$ and $(b^2, 2b)$ are in

(a) A.P.
(b) G.P.
(c) H.P.
(d) none of these

93. The coordinates of the points A and B are respectively $(-3, 2)$ and $(2, 3)$. P and Q are points on the line joining A and B such that $AP = PQ = QB$. A square $PQRS$ is constructed on PQ as one side, the coordinates of R are

(a) $(-4/3, 7/3)$
(b) $(0, 13/3)$
(c) $(1/3, 8/3)$
(d) $(2/3, 1)$

94. If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of a triangle, then the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

represents

(a) the median through A
(b) the altitude through A
(c) the perpendicular bisector of BC
(d) the line joining the centroid with a vertex

95. Given four lines whose equations are $x + 2y - 3 = 0$, $2x + 3y - 4 = 0$, $3x + 4y - 7 = 0$ and $4x + 5y - 6 = 0$, then the lines are

(a) concurrent
(b) the sides of a quadrilateral with one vertex at $(3, 0)$
(c) the sides of a cyclic quadrilateral
(d) none of these

96. A ray of light coming from the point $(1, 2)$ is reflected at a point A on the axis of x and then passes through the point $(5, 3)$. The coordinates of the point A are

(a) $(5/13, 0)$
(b) $(-7, 0)$
(c) $(13/5, 0)$
(d) $(15, 0)$

97. If Δ_1, Δ_2, Δ_3 are the areas of the triangles with vertices $(0, 0)$, $(a \tan \alpha, b \cot \alpha)$, $(a \sin \alpha, b \cos \alpha)$; (a, b), $(a \sec^2 \alpha, b \operatorname{cosec}^2 \alpha)$, $(a + a \sin^2 \alpha, b + b \cos^2 \alpha)$ and $(0, 0)$, $(a \tan \alpha, -b \cot \alpha)$, $(a \sin \alpha, b \cos \alpha)$, then Δ_1, Δ_2, Δ_3 are in G.P. for

(a) all values of α
(b) only one value of α
(c) finite number of values of α
(d) no value of α

98. The orthocentre of the triangle formed by the lines $y = 0$, $(1 + t)x - ty + t(1 + t) = 0$ and $(1 + u)x - uy + u(1 + u) = 0$ $(t \neq u)$ for all values of t and u lies on the line

(a) $x - y = 0$
(b) $x + y = 0$
(c) $x - y + 1 = 0$
(d) $x + y + 1 = 0$

99. If $A(3, 0)$ and $B(6, 0)$ are two fixed points and $U(x_1, y_1)$ is a variable point in the plane. AU and BU meet y-axis at C and D respectively and AD meets OU at V, then for all positions of U in the plane, CV passes through the point

(a) $(0, 0)$
(b) $(0, 2)$
(c) $(2, 0)$
(d) none of these

100. The incentre of the triangle with vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$ is

(a) $(1, \sqrt{3}/2)$
(b) $(2/3, 1/\sqrt{3})$
(c) $(2/3, \sqrt{3}/2)$
(d) $(1, 1/\sqrt{3})$

101. If $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ form three sides of a square, the equation of the fourth side nearer the point $(1, -1)$ is

(a) $2x - y - 6 = 0$
(b) $2x - y + 6 = 0$
(c) $2x - y - 14 = 0$
(d) $2x - y + 14 = 0$

102. The distance between the orthocentre and the circumcentre of the triangle with vertices $(0, 0)$, $(0, a)$ and $(b, 0)$ is

(a) $\sqrt{a^2 - b^2}/2$
(b) $a + b$
(c) $a - b$
(d) $\sqrt{a^2 + b^2}/2$

103. The centroid of a triangle lies at the origin and the coordinates of its two vertices are $(-8, 0)$ and $(9, 11)$, the area of the triangle in sq. units is

(a) 11/8
(b) 8/11
(c) 88
(d) none of these

104. The line $3x + 2y = 24$ meets the y-axis at A and the x-axis at B; C is a point on the perpendicular bisector of AB such that the area of the triangle ABC is 91 sq. units. The coordinates of C can be

(a) $(29/2, -1)$ (b) $(29/2, 13)$
(c) $(-13/2, 1)$ (d) $(-13/2, 13)$

105. The straight lines $4x - 3y - 5 = 0$, $x - 2y = 0$, $7x + y - 40 = 0$ and $x + 3y + 10 = 0$ form
(a) a rectangle
(b) a parallelogram
(c) a cyclic quadrilateral
(d) none of these

106. If the straight lines $x + 2x - 9 = 0$, $3x + 5y - 5 = 0$ and $ax + by + 1 = 0$ are concurrent, then the straight line $35x - 22y - 1 = 0$ passes through
(a) (a, b) (b) (b, a)
(c) $(a, -b)$ (d) $(-a, b)$

107. If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ be the square of the other, then $\frac{a+b}{h} + \frac{8h^2}{ab}$ is equal to
(a) 0 (b) 1
(c) 6 (d) 8

108. The locus represented by the equation
$(x - y + c)^2 + (x + y - c)^2 = 0$ is
(a) a line parallel to x-axis
(b) a point
(c) a pair of straight lines
(d) a line parallel of y-axis

Previous Years' AIEEE/JEE Main Questions

1. If the pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on y-axis then
(a) $2fgh = bg^2 + ch^2$ (b) $bg^2 \neq ch^2$
(c) $abc = 2fgh$ (d) none of these **[2002]**

2. Locus of mid point of the portion between the axes of $x \cos \alpha + y \sin \alpha = p$ where p is constant is
(a) $x^2 + y^2 = 4/p^2$
(b) $x^2 + y^2 = 4p^2$
(c) $1/x^2 + 1/y^2 = 2/p^2$
(d) $1/x^2 + 1/y^2 = 4/p^2$ **[2002]**

3. A triangle with vertices $(4, 0)$, $(-1, -1)$, $(3, 5)$ is
(a) isosceles and right angled
(b) isosceles but not right angled
(c) right angled but not isosceles
(d) neither right angled nor isosceles. **[2002]**

4. The sides of a triangle are $3x + 4y = 0$, $4x - 3y = 0$ and $5x + 5y = 1$ then the triangle is
(a) right angled (b) obtuse angled
(c) equilateral (d) none of these **[2002]**

5. A square of side a lies above the x-axis and has one vertex at the origin. The side passing through the origin makes an angle α $(0 < \alpha < \pi/4)$ with the positive direction of x-axis. The equation of the diagonal not passing through the origin is
(a) $y (\cos \alpha + \sin \alpha) + x (\sin \alpha - \cos \alpha) = a$
(b) $y (\cos \alpha + \sin \alpha) + x (\sin \alpha + \cos \alpha) = a$
(c) $y (\cos \alpha + \sin \alpha) + x (\cos \alpha - \sin \alpha) = a$
(d) $y (\cos \alpha - \sin \alpha) - x (\sin \alpha - \cos \alpha) = a$ **[2003]**

6. If the pair of straight lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, then
(a) $p = -q$ (b) $pq = 1$
(c) $pq = -1$ (d) $p = q$ **[2003]**

7. Locus of centroid of the triangle whose vertices are $(a \cos t, a \sin t)$,$(b \sin t, -b \cos t)$ and $(1, 0)$, where t is a parameter is
(a) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
(b) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$
(c) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$
(d) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$ **[2003]**

8. If x_1, x_2, x_3 and y_1, y_2, y_3 are both in G.P. with the same common ratio, then the points (x_1, y_1), (x_2, y_2) and (x_3, y_3)
(a) lie on an ellipse
(b) lie on a circle
(c) are vertices of a triangle
(d) lie on a straight line **[2003]**

9. If the equation of the locus of point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2) x + (b_1 - b_2) y + c = 0$, then $c =$
(a) $a_1^2 - a_2^2 + b_1^2 - b_2^2$
(b) $(1/2) (a_1^2 + a_2^2 + b_1^2 + b_2^2)$
(c) $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$
(d) $(1/2) (a_2^2 + b_2^2 - a_1^2 - b_1^2)$ **[2003]**

10. Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a triangle ABC. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line
(a) $3x + 2y = 5$ (b) $2x - 3y = 7$
(c) $2x + 3y = 9$ (d) $3x - 2y = 3$ **[2004]**

11. The equation of the straight line passing through the point (4, 3) and making intercepts on the coordinate axes whose sum is – 1 is

(a) $\frac{x}{2}+\frac{y}{3}=1$ and $\frac{x}{2}+\frac{y}{1}=1$

(b) $\frac{x}{2}-\frac{y}{3}=-1$ and $\frac{x}{-2}-\frac{y}{1}=-1$

(c) $\frac{x}{2}+\frac{y}{3}=-1$ and $\frac{x}{-2}-\frac{y}{1}=-1$

(d) $\frac{x}{2}-\frac{y}{3}=1$ and $\frac{x}{-2}+\frac{y}{1}=1$ **[2004]**

12. If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product, then c has the value

(a) 2
(b) – 1
(c) 1
(d) – 2 **[2004]**

13. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$, then c equals

(a) 3
(b) – 1
(c) 1
(d) – 3 **[2004]**

14. The line parallel to x-axis passing through the intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$ where $(a, b) \neq (0, 0)$ is

(a) above x-axis at a distance 3/2 from it
(b) above x-axis at a distance 2/3 from it
(c) below x-axis at a distance 3/2 from it
(d) below x-axis at a distance 2/3 from it. **[2005]**

15. If a vertex of a triangle is (1, 1) and the mid points of two sides through this vertex are (– 1, 2) and (3, 2), then the centroid of the triangle is

(a) (1, 7/3)
(b) (1/3, 7/3)
(c) (– 1, 7/3)
(d) (– 1/3, 7/3) **[2005]**

16. If non zero numbers a, b, c are in H.P., then the straight line $\frac{x}{a}+\frac{y}{b}+\frac{1}{c}=0$ always passes through a fixed point. That point is

(a) (1, – 2)
(b) (1, – 1/2)
(c) (– 1, 2)
(d) (– 1, – 2) **[2005]**

17. If the pair of lines $ax^2 + 2(a + b)\, xy + by^2 = 0$ lie along diameters of a circle and divide the circle into four sectors such that the area of one of the sector is thrice the area of the another sector, then

(a) $3a^2 + 10ab + 3b^2 = 0$
(b) $3a^2 + 2ab + 3b^2 = 0$
(c) $3a^2 - 10ab + 3b^2 = 0$
(d) $3a^2 - 2ab + 3b^2 = 0$ **[2005]**

18. A straight line through the point $A(3, 4)$ is such that its intercept between the axes is bisected at A. Its equation is

(a) $3x + 4y = 25$
(b) $x + y = 7$
(c) $3x - 4y + 7 = 0$
(d) $4x + 3y = 24$ **[2006]**

19. Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right angled with AC as its hypotenuse. If the area of the triangle is 1, then the set of values which 'k' can take is given by

(a) {1, 3}
(b) {0, 2}
(c) {– 1, 3}
(d) {– 3, – 2} **[2007]**

20. Let $P = (-1, 0)$, $Q = (0, 0)$ and $R = (\sqrt{3}, 3)$, be three points. The equation of the bisector of angle PQR is

(a) $\sqrt{3}\ x + y = 0$
(b) $x + (\sqrt{3}/2)\ y = 0$
(c) $(\sqrt{3}/2)\ x + y = 0$
(d) $x = \sqrt{3}\ y = 0$ **[2007]**

21. If one of the lines of $my^2 + (1 - m^2)\, xy - mx^2 = 0$ is a bisector of the angle between the line $xy = 0$ then m is

(a) – 1/2
(b) – 2
(c) 1
(d) 2 **[2007]**

22. The perpendicular bisector of the lines segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept-4. Then a possible value of k is

(a) – 4
(b) 1
(c) 2
(d) – 2 **[2008]**

23. The lines $p(p^2 + 1)\, x - y + q = 0$ and $(p^2 + 1)^2 x + (p^2 + 1)\, y + 2q = 0$ are perpendicular to a common line for

(a) exactly two values of p
(b) more than two values of p
(c) no value of p
(d) exactly one value of p **[2009]**

24. The line L is given by $\frac{x}{5}+\frac{y}{b}=1$ passes through the point (13, 32). The line K is parallel to L and has the equation $\frac{x}{c}+\frac{y}{3}=1$. Then the distance between L and K is

(a) $\frac{17}{\sqrt{15}}$
(b) $\frac{23}{\sqrt{17}}$
(c) $\frac{23}{\sqrt{15}}$
(d) $\sqrt{15}$ **[2010]**

25. The line $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ intersect the line L_3: $y + 2 = 0$ at P and Q respectively The bisector of the acute angle between L_1 and L_2 intersect L_3 at **R**.

Statement-1 : The ratio $PR:RQ$ equals $2\sqrt{2} : \sqrt{5}$

Statement-2 : In any triangle, bisector of an angle divides the triangle into two similar triangles **[2011]**

26. The lines $x + y = |a|$ and $ax - y = 1$ intersect each other in the first quadrant then the set of all possible values of a is in the interval.

(a) $(0, \infty)$ (b) $(1, \infty)$
(c) $(-1, \infty)$ (d) $(-1, 1)$ **[2011]**

27. If A (2, – 3), B (–2, 1) are two vertices of a triangle and the third vertex moves on the line $2x + 3y = 9$, then the locus of the centroid of the triangle is

(a) $x - y = 1$ (b) $2x + 3y = 1$
(c) $2x + 3y = 3$ (d) $2x - 3y = 1$ **[2011]**

28. If the line $2x + y = k$ passes through the point which divides the segment joining the points (1, 1) and (2, 4) in the ratio 3:2, then k equals

(a) 6 (b) 11/5
(c) 29/5 (d) 5 **[2012]**

29. A line is drawn through the point (1, 2) to meet the coordinate axes at P and Q such that it forms a triangle OPQ, where O is the origin. If the area of the triangle OPQ is least, then the slope of the line PQ is

(a) – 2 (b) –1/2
(c) –1/4 (d) – 4 **[2012]**

30. A ray of light along $x + \sqrt{3}\,y = \sqrt{3}$ gets reflected upon reaching the x-axis, an equation of the reflected ray is

(a) $\sqrt{3}y = x - \sqrt{3}$ (b) $y = \sqrt{3}x - \sqrt{3}$
(c) $\sqrt{3}y = x - 1$ (d) $y = x + \sqrt{3}$ **[2013]**

31. Equation of the line passing through the points of intersection of the parabola $x^2 = 8y$ and the ellipse $\frac{x^2}{3} + y^2 = 1$ is:

(a) $y - 3 = 0$ (b) $y + 3 = 0$
(c) $3y + 1 = 0$ (d) $3y - 1 = 0$ **[2013, online]**

32. A light ray emerging from the point source placed at $P(1, 3)$ is reflected at a point Q in the axis of x. If the reflected ray passes through the point $R(6, 7)$, then the abscissa of Q is:

(a) 1 (b) 3
(c) 7/2 (d) 5/2 **[2013, online]**

33. If three lines $x - 3y = p$, $ax + 2y = q$ and $ax + y = r$ form a right-angled triangle, then:

(a) $a^2 - 9a + 18 = 0$ (b) $a^2 - 6a - 12 = 0$
(c) $a^2 - 6a - 18 = 0$ (d) $a^2 - 9a + 12 = 0$

[2013, online]

34. The x-coordinates of the incentre of the triangle that has the coordinates of mid points of its sides as (0, 1), (1, 1) and (1, 0) is

(a) $2 - \sqrt{2}$ (b) $1 + \sqrt{2}$
(c) $1 - \sqrt{2}$ (d) $2 + \sqrt{2}$ **[2013]**

35. If x-intercept of some line L is double as that of the line $3x + 4y = 12$ and the y-intercept of L is half as that of the same line, then the slope of L is:

(a) – 3 (b) – 3/8
(c) – 3/2 (d) – 3/16 **[2013, online]**

36. If the extremities of the base of an isosceles triangle are the points $(2a, 0)$ and $(0, a)$ and the equation of one of the sides is $x = 2a$, then the area of the triangle, in square units, is:

(a) $\frac{5}{4}a^2$ (b) $\frac{5}{2}a^2$
(c) $\frac{25}{4}a^2$ (d) $5a^2$ **[2013, online]**

37. Let θ_1, be the angle between two lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_2 = 0$ and θ_2 be the angle between two lines $2x + 3y + c_1 = 0$ and $-x + 5y + c_3 = 0$, where c_1, c_2, c_3 are any real numbers:

Statement 1: If c_2 and c_3 are proportional, then $\theta_1 = \theta_2$

Statement 2: $\theta_1 = \theta_2$ for all c_2 and c_3.

(a) Statement-1 is true, statement-2 is true and statement-2 is a correct explanation for statement-1.
(b) Statement-1 is true, statement-2 is true but statement-2 is Not a correct explanation for statement-1
(c) Statement-1 is false, Statement-2 is true.
(d) Statement-1 is true, Statement-2 is false.

[2013, online]

38. If the image of point P (2, 3) in a line L is Q (4, 5) then the image of point R(0, 0) in the same line is:

(a) (2, 2) (b) (4, 5)
(c) (3, 4) (d) (7, 7) **[2013, online]**

39. Let A(– 3, 2) and B(– 2, 1) be the vertices of a triangle ABC. If the centroid of this triangle lies on the line $3x + 4y + 2 = 0$, then vertex C lies on the line.

(a) $4x + 3y + 5 = 0$ (b) $3x + 4y + 3 = 0$
(c) $4x + 3y + 3 = 0$ (d) $3x + 4y + 5 = 0$

[2013, online]

40. Let a, b, c and d be non zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5\,bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes then:

(a) $2bc - 3ad = 0$ (b) $2bc + 3ad = 0$
(c) $3bc - 2ad = 0$ (d) $3bc + 2ad = 0$ **[2014]**

41. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing $(1, -1)$ and parallel to PS is:
(a) $4x - 7y - 11 = 0$ (b) $2x + 9y + 7 = 0$
(c) $4x + 7y + 3 = 0$ (d) $2x - 9y - 11 = 0$
[2014]

42. Let a and b be any two numbers satisfying $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{4}$. Then, the foot of perpendicular from the origin on the variable line $\frac{x}{a} + \frac{y}{b} = 1$, lies on:
(a) a hyperbola with each semi-axis = $\sqrt{2}$
(b) a hyperbola with each semi-axis = 2
(c) a circle of radius = 2
(d) a circle of radius = $\sqrt{2}$ **[2014, online]**

43. Given three points P, Q, R with $P(5, 3)$ and R lies on the x-axis. If equation of RQ is $x - 2y = 2$ and PQ is parallel to x-axis, then centroid of $\Delta\ PQR$ lies on the line
(a) $2x + y - 9 = 0$ (b) $x - 2y + 1 = 0$
(c) $5x - 2y = 0$ (d) $2x - 5y = 0$
[2014, online]

44. The base of an equilateral triangle is along the line given by $3x + 4y = 9$. If a vertex of the triangle is $(1, 2)$, then length of a side of the triangle is:
(a) $\frac{2\sqrt{3}}{15}$ (b) $\frac{4\sqrt{3}}{15}$
(c) $\frac{4\sqrt{3}}{5}$ (d) $\frac{2\sqrt{3}}{5}$ **[2014, online]**

45. If a line intercepted between the coordinates axes is trisected at a point A (4, 3), which is nearer to the x-axis, then its equation is
(a) $4x - 3y = 7$ (b) $3x + 2y = 18$
(c) $3x + 8y = 36$ (d) $x + 3y = 13$
[2014, online]

46. If three distinct lines $x + 2ay + a = 0$, $x + 3by + b = 0$ and $x + 4ay + a = 0$ are concurrent, then the point (a, b) lies on a:
(a) circle (b) hyperbola
(c) straight-line (d) parabola
[2014, online]

47. The circumcentre of a triangle lies at the origin and its centroid is the mid point of the line segment joining the points $(a^2 + 1, a + 1)$ and $(2a, -2a)$, $a \neq 0$. Then for any a, the orthocentre of this triangle lies on the line:
(a) $y - 2ax = 0$
(b) $y - (a^2 + 1)x = 0$
(c) $y + x = 0$
(d) $(a - 1)^2 x - (a + 1)^2 y = 0$ **[2014, online]**

48. If a line L is perpendicular to the line $5x - y = 1$, and the area of the triangle formed by the line L and the coordinate axes, is 5, then the distance of L from the line $x + 5y = 0$ is
(a) $7/\sqrt{5}$ (b) $5/\sqrt{13}$
(c) $7/\sqrt{13}$ (d) $5/\sqrt{7}$ **[2014, online]**

49. The number of points having both coordinates as integer, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is
(a) 901 (b) 861
(c) 820 (d) 780 **[2015]**

50. Locus of the image of the point (2, 3) in the line $(2x - 3y + 4) + k\,(x - 2y + 3) = 0$, $k \in \mathbf{R}$, is *a*:
(a) straight line parallel to x-axis
(b) straight line parallel to y-axis
(c) circle of radius $\sqrt{2}$
(d) circle of radius $\sqrt{3}$ **[2015]**

51. A straight line L through the point $(3, -2)$ is inclined at an angle of 60° to the line $\sqrt{3}x + y = 1$. If L also intersects x-axis, then equation of L is:
(a) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$
(b) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$
(c) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$
(d) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$ **[2015, online]**

52. The points $\left(0, \frac{8}{3}\right)$, (1, 3) and (82, 30)
(a) form an obtuse angled triangle
(b) form an acute angled triangle
(c) form a right angled triangle
(d) lie on a straight line. **[2015, online]**

53. Let L be the line passing through the point $P(1, 2)$ such that its intercepted segment between the coordinate axes is bisected at P. If L_1 is the line perpendicular to L and passing through the point $(-2, 1)$, then the point of intersection of L and L_1 is:
(a) $\left(\frac{4}{5}, \frac{12}{5}\right)$ (b) $\left(\frac{11}{20}, \frac{29}{10}\right)$
(c) $\left(\frac{3}{10}, \frac{17}{5}\right)$ (d) $\left(\frac{3}{5}, \frac{23}{10}\right)$ **[2015,online]**

54. Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombus?
(a) $(-3, -9)$ (b) $(-3, -8)$
(c) $\left(\frac{1}{3} - \frac{8}{3}\right)$ (d) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$ **[2016]**

55. If a variable line drawn through the intersection on the lines $\frac{x}{3}+\frac{y}{4}=1$ and $\frac{x}{4}+\frac{y}{3}=1$, meets the coordinate axes at A and B, $(A \neq B)$, then the locus of the midpoint of AB is:
(a) $7xy = 6(x + y)$
(b) $4(x + y)^2 - 28(x + y) + 49 = 0$
(c) $6xy = 7(x + y)$
(d) $14(x + y)^2 - 97(x + y) + 168 = 0$ **[2016, online]**

56. The point (2, 1) is translated parallel to the line $L : x - y - 4 = 0$ by $2\sqrt{3}$ units. If the new point Q lies in the third quadrant, then the equation of the line passing through Q and perpendicular to L is:
(a) $x + y = 2 - \sqrt{6}$ (b) $2x + 2y = 1 - \sqrt{6}$
(c) $x + y = 3 - 3\sqrt{6}$ (d) $x + y = 3 - \sqrt{6}$
[2016, online]

57. A straight line through origin O meets the lines $3y = 10 - 4x$ and $8x + 6y + 5 = 0$ at points A and B respectively. Then O divides the segment AB in the ratio:
(a) 2 : 3 (b) 1 : 2
(c) 4 : 1 (d) 3 : 4 **[2016, online]**

58. A ray of light is incident along a line which meets another line $7x - y + 1 = 0$, at the point (0, 1). The ray is then reflected from this point along the line, $y + 2x = 1$. Then the equation of the line of incidence of the ray of light is:
(a) $41x - 25y + 25 = 0$ (b) $41x + 25y - 25 = 0$
(c) $41x - 38y + 38 = 0$ (d) $41x + 38y - 38 = 0$
[2016, online]

Previous Years' B-Architecture Entrance Examination Questions

1. The y-axis and the lines $(a^5 - 2a^3)x + (a + 2)y + 3a = 0$ and $(a^5 - 3a^2)x + 4y + a - 2 = 0$ are concurrent for
(a) Two values of a (b) Three values of a
(c) Five values of a (d) no value of a **[2007]**

2. If the point of intersection of the lines $2px + 3qy + r = 0$ and $px - 2qy - 2r = 0$ lies strictly in the fourth quadrant and is equidistant from the two axes, then
(a) $5p + 4q = 0$ (b) $4p - 5q = 0$
(c) $4p + 5q = 0$ (d) $5p - 4q = 0$ **[2009]**

3. Consider a triangle ABC with vertices at $(0, -3)$, $(-2\sqrt{3}, 3)$ and $(2\sqrt{3}, 3)$ respectively. The incentre of the triangle with vertices at the mid points of the sides of triangle ABC is
(a) (0, 0) (b) (0, 1)
(c) $\left(-\sqrt{3}, -\sqrt{3}\right)$ (d) $\left(\sqrt{3}, \sqrt{3}\right)$ **[2009]**

4. The equation of a straight line belonging to both the families of lines $x - y + 1 + \lambda_1 (2x - y - 2) = 0$ and $5x + 3y - 2 + \lambda_2 (3x - y - 4) = 0$, where λ_1, λ_2 are arbitrary numbers is:
(a) $5x - 2y - 7 = 0$ (b) $2x + 5y - 7 = 0$
(c) $5x + 2y - 7 = 0$ (d) $2x - 5y - 7 = 0$ **[2010]**

5. **Statement-1:** The equation $|x| + |y| = 2$ represents a parallelogram.
Statement-2: Lines $x + y = 2$ and $x + y = -2$ are parallel. Also lines $x - y = 2$ and $-x + y = 2$ are parallel.
(a) Statement-1 is false, Statement-2 is true;
(b) Statement-1 is true, Statement-2 is true; statement-2 is not correct explanation for statement-1.
(c) Statement-1 is true, Statement-2 is true; statement-2 is **not** *a* correct explanation for statement-1.
(d) Statement-1 is true, Statement-2 is false. **[2011]**

6. If the line joining the points $A(2, 0)$ and $B(3, 1)$ is rotated about A in anti-clockwise direction through an angle of 15°, then the equation of the line in new position is
(a) $\sqrt{3}\,x - y = 2\sqrt{3}$ (b) $\sqrt{3}\,x + y = 2\sqrt{3}$
(c) $x + \sqrt{3}\,y = 2$ (d) $x - \sqrt{3}\,y = 2$ **[2012]**

7. If m_1 and m_2 are the roots of the equation $x^2 + (\sqrt{3} + 2)\,x + (\sqrt{3} - 1) = 0$, then the area of the triangle formed by the lines $y = m_1 x$, $y = -m_2 x$ and $y = 1$ is
(a) $\frac{1}{2}\left(\frac{\sqrt{3}+2}{\sqrt{3}-1}\right)$ (b) $\frac{1}{2}\left(\frac{\sqrt{3}+2}{\sqrt{3}+1}\right)$
(c) $\frac{1}{2}\left(\frac{-\sqrt{3}+2}{\sqrt{3}-1}\right)$ (d) $\frac{1}{2}\left(\frac{-\sqrt{3}+2}{\sqrt{3}+1}\right)$ **[2012]**

8. **Statement-1:** The line $2x + y + 6 = 0$ is perpendicular to the line $x - 2y + 5 = 0$ and second line passes through (1, 3).

   Statement-2: Product of the slopes of any two parallel lines is equal to – 1.

   (a) Statement-1 is true, statement-2 is true; statement-2 is a *correct* explanation for statement-1
   (b) Statement-1 is true, statement-2 is true; statement-2 is *not* correct explanation for statement-1.
   (c) Statement-1 is true, statement-2 is false.
   (d) Statement-1 is false, statsement-2 is true **[2013]**

9. If a variable line, passing through the point of intersection of the lines $x + 2y - 1 = 0$ and $2x - y - 1 = 0$, meet the coordinate axes in A and B, then the locus of the mid-point of AB is

   (a) $x + 3y = 0$ (b) $x + 3y = 10$
   (c) $x + 3y = 10xy$ (d) $x + 3y + 10xy = 0$
   [2013]

10. If the point $(p, 5)$ lies on the line parallel to the y-axis and passing through the intersection of the lines $2(a^2 + 1)x + by + 4(a^3 + a) = 0$ and $(a^2 + 1)x - 3by + 2(a^3 + a) = 0$, then p is equal to

    (a) $-2a$ (b) $-3a$
    (c) $2a$ (d) $3a$ **[2014]**

11. If the points $(x, -3x)$ and (3, 4) lie on the opposite side of the line $3x - 4y = 8$, then

    (a) $x > \frac{8}{15}, y < \frac{-8}{5}$ (b) $x > \frac{8}{5}, y > \frac{-8}{15}$
    (c) $x < \frac{8}{15}, y > \frac{-8}{5}$ (d) $x = \frac{8}{15}, y = \frac{-8}{5}$ **[2015]**

12. Two vertices of a triangle are (3, – 2) and (–2, 3) and its orthocentre is (–6, 1). Then the third vertex of this triangle can NOT lie on the line:

    (a) $4x + y = 2$ (b) $5x + y = 2$
    (c) $3x + y = 3$ (d) $6x + y = 0$ **[2016]**

13. A line passing through the point $P(1, 2)$ meets the line $x + y = 7$ at the distance of 3 units from P. Then the slope of this line satisfies the equation

    (a) $7x^2 - 18x + 7 = 0$ (b) $16x^2 - 39x + 16 = 0$
    (c) $7x^2 - 6x - 7 = 0$ (d) $8x^2 - 9x + 1 = 0$ **[2016]**

Answers

Concept-based

1. (b)	**2.** (a)	**3.** (d)	**4.** (c)
5. (b)	**6.** (d)	**7.** (c)	**8.** (d)
9. (c)	**10.** (a)	**11.** (a)	**12.** (d)
13. (b)	**14.** (b)	**15.** (c)	

Level 1

16. (a)	**17.** (d)	**18.** (c)	**19.** (a)
20. (c)	**21.** (a)	**22.** (a)	**23.** (d)
24. (b)	**25.** (b)	**26.** (c)	**27.** (d)
28. (d)	**29.** (a)	**30.** (b)	**31.** (c)
32. (b)	**33.** (a)	**34.** (a)	**35.** (a)
36. (d)	**37.** (d)	**38.** (d)	**39.** (c)
40. (b)	**41.** (c)	**42.** (b)	**43.** (a)
44. (d)	**45.** (d)	**46.** (b)	**47.** (b)
48. (d)	**49.** (d)	**50.** (b)	**51.** (c)
52. (c)	**53.** (b)	**54.** (c)	**55.** (b)
56. (b)	**57.** (b)	**58.** (b)	**59.** (b)
60. (b)	**61.** (d)	**62.** (d)	**63.** (c)
64. (c)	**65.** (c)	**66.** (d)	**67.** (d)
68. (b)	**69.** (c)	**70.** (b)	**71.** (c)
72. (c)	**73.** (c)	**74.** (b)	**75.** (c)
76. (b)	**77.** (c)	**78.** (d)	**79.** (a)
80. (d)	**81.** (c)	**82.** (d)	**83.** (a)
84. (b)	**85.** (c)	**86.** (d)	**87.** (b)

Level 2

88. (c)	**89.** (d)	**90.** (d)	**91.** (c)
92. (b)	**93.** (d)	**94.** (a)	**95.** (d)
96. (c)	**97.** (d)	**98.** (b)	**99.** (c)
100. (d)	**101.** (b)	**102.** (d)	**103.** (d)
104. (b)	**105.** (c)	**106.** (a)	**107.** (c)
108. (b)			

Previous Years' AIEEE/JEE Main Questions

1. (a)	**2.** (d)	**3.** (a)	**4.** (a)
5. (c)	**6.** (c)	**7.** (a)	**8.** (d)
9. (d)	**10.** (c)	**11.** (d)	**12.** (a)
13. (d)	**14.** (c)	**15.** (a)	**16.** (a)
17. (b)	**18.** (d)	**19.** (c)	**20.** (a)
21. (c)	**22.** (a)	**23.** (d)	**24.** (b)
25. (c)	**26.** (b)	**27.** (b)	**28.** (a)
29. (a)	**30.** (a)	**31.** (d)	**32.** (d)
33. (a)	**34.** (a)	**35.** (d)	**36.** (b)
37. (a)	**38.** (d)	**39.** (b)	**40.** (c)
41. (b)	**42.** (c)	**43.** (d)	**44.** (b)
45. (b)	**46.** (c)	**47.** (d)	**48.** (b)

49. (d)	**50.** (c)	**51.** (a)	**52.** (d)
53. (a)	**54.** (c)	**55.** (a)	**56.** (d)
57. (c)	**58.** (c)		

Previous Years' B-Architecture Entrance Examination Questions

1. (a)	**2.** (d)	**3.** (b)	**4.** (a)
5. (b)	**6.** (a)	**7.** (a)	**8.** (c)
9. (c)	**10.** (a)	**11.** (a)	**12.** (b)
13. (a)			

Hints and Solutions

Concept-based

1. Coordinate of A are (3, 0) and of B are (0, 4) so coordinate of P are $\left(\frac{1\times0+2\times3}{3}, \frac{1\times4+2\times0}{3}\right)=\left(2, \frac{4}{3}\right)$.

2. Coordinate of the mid-point of the line segment joining P and Q are $\left(\frac{3-5}{2}, \frac{-4-2}{2}\right) = (-1, -3)$

   so slope of the required line is $\frac{0-(-3)}{0-(-1)} = 3$

3. When PQ is parallel to y-axis $x_1 = x_2$ and the required distance is $|y_1 - y_2|$

4. Line parallel to x-axis is $y = -5$ and parallel to y-axis is $x = 4$.

5. Equation of the line is $x \cos \alpha + y \sin \alpha = p$ where $p = 3$, $\alpha = 30°$. So the required equation is $\left(\frac{\sqrt{3}}{2}\right)x + \left(\frac{1}{2}\right)y = 3$ or $\sqrt{3}\,x + y = 6$

6. Slope of the line joining (8, 2), (– 2, – 2) is $\frac{2}{5}$ and the slope of the line joining (8, 2), (3, 0) is also $\frac{2}{5}$, so the points are collinear.

7. Slopes of the given lines are $\frac{1}{\sqrt{3}}$ and – 1, so

$$\tan\theta = \left[\frac{\frac{1}{\sqrt{3}}+1}{1-\frac{1}{\sqrt{3}}}\right] = \left[\frac{\sqrt{3}+1}{\sqrt{3}-1}\right] = 2+\sqrt{3}.$$

8. Since the line makes zero intercept on both the axes, it must pass through the origin. So equation of the line is $y = \frac{a+1}{a-1}x$ as the slope of the line is $\frac{a+1-0}{a-1-0}$

   or $(a + 1)\, x - (a - 1)\, y = 0$

9. We write the equation of the given line in the normal form as $-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = \frac{8}{2}$

   or $x \cos \alpha + y \sin \alpha = p$ where $\cos \alpha = \frac{-1}{2}$, $\sin \alpha = \frac{\sqrt{3}}{2}$ and $p = 4$, α being the required angle.

   $\Rightarrow \alpha = 120°$

10. Product of the slopes of the given lines is – 1.

    $\Rightarrow \frac{3}{4} \times \frac{7-3}{h-2} = -1$

    $h = -1$

11. Equation of any line through the intersection of given line is

    $7x - y + 2 + \lambda\,(x - 3y + 6) = 0$

    It will be parallel to x-axis if

    $7 + \lambda = 0 \Rightarrow \lambda = -7$ and the required equation is $y = 2$.

12. First two lines intersect at (1, 1) and the third line passes through (1, 1) if $p = 6$

    Alternatively $\begin{bmatrix} 1 & -p & 5 \\ 2 & 1 & -3 \\ 3 & -1 & -2 \end{bmatrix} = 0.$

13. The given lines are $3x + 2y + 6 = 0$ and $3x + 2y - \frac{7}{3} = 0.$

    Let the required line be $3x + 2y + k = 0$. Since it is equidistant from the given lines

    $$\frac{k+\frac{7}{3}}{\sqrt{9+4}} = \frac{6-k}{\sqrt{9+4}} \quad \Rightarrow \quad k = \frac{11}{6}.$$

14. Vertices of the triangle are (0, 0), (k, k), $(-k, k)$

    $$\text{Area} = \left(\frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ k & k & 1 \\ -k & k & 1 \end{bmatrix}\right) = k^2$$

15. Two lines intersect at the point $\left(\frac{5}{7}, \frac{25}{21}\right)$. Distance of this point from the given point (3, 5) is

    $$\sqrt{\left(3-\frac{5}{7}\right)^2+\left(5-\frac{25}{21}\right)^2} = \frac{4\sqrt{34}}{21}$$

Level 1

16. Slope of the line joining any two points is independent of k

17. $\begin{vmatrix} k & 2-2k & 1 \\ -k+1 & 2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} = 140$
which is true for only one integral value of k.

18. Opposite sides are parallel, all sides are equal. AB is not perpendicular to AD.

19. Product of the slopes of AB and AC is -1

20. $\dfrac{\lambda-3}{\lambda+1} = 0 \Rightarrow \lambda = 3$, ratio is $3:1$

21. Vertices are (0, 4), (4, 0), (1, 1) and the sides are $\sqrt{10}, \sqrt{10}$ and $\sqrt{32}$.

22. $PQ = \sqrt{2}\ |a-b|$, $QR = \sqrt{2}\ |b-c|$ so $PQ = QR$ if $a - b = b - c$ *i.e.* a, b, c are in A.P.

23. $\dfrac{y-x}{b-a} = \dfrac{z-y}{c-b} \Rightarrow y - x = z - y$
$(\because\ b - a = c - b)$
$\Rightarrow$ x, y, z are in A.P.

24. Area is thrice the area of the triangle with third vertex at the centroid.

25. Length of BC is twice the length of the line joining the mid-point of AB and AC.

26. Triangle is right angled at B so mid-point of AC is the circumcentre.

27. Diagonals of a parallelogram bisect each other.

28. Let C (h, k), then Centroid is $\left(x = \dfrac{h+2+4}{3},\ y = \dfrac{k+5-1}{3}\right)$ and $9h + 7k + 4 = 0 \Rightarrow 9(3x - 6) + 7(3y - 4) + 4 = 0$ is the locus of the centroid parallel to L_1.

29. The distance between the given points is less than 12.

30. $2OA \times OB \cos \underline{|AOB} = (OA)^2 + (OB)^2 - (AB)^2$

31. $\dfrac{x}{a} + y + 1 = 0, \quad x + \dfrac{y}{b} + 1 = 0,$

$x + y + \dfrac{1}{c} = 0$

$x = \dfrac{c-1}{a-1}\cdot\dfrac{a}{c}, \quad y = \dfrac{c-1}{b-1}\cdot\dfrac{b}{c}$

$x + y + \dfrac{1}{c} = 0 \Rightarrow \dfrac{a}{a-1} + \dfrac{b}{b-1} + \dfrac{1}{c-1} = 0$

$\Rightarrow \dfrac{a}{a-1} + \dfrac{b}{b-1} + \dfrac{c}{c-1} = 1$

32. Coordinates of D are $(5 \cos \alpha, 4 + 5 \sin \alpha)$
$\cos \alpha = 4/5$, $\sin \alpha = 3/5$
check the coordinates of C.

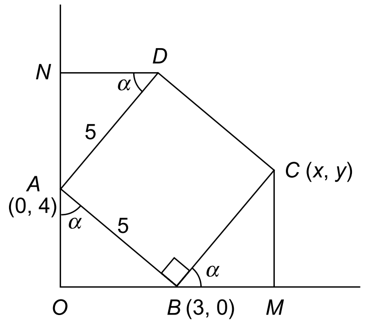


Fig. 16.21

33. Let P divide BC in the ratio $\lambda : 1$. As P lies on the line

$$2\left(\frac{\lambda x_3 + x_2}{\lambda+1}\right) + 3\left(\frac{\lambda y_3 + y_2}{\lambda+1}\right) - 1 = 0$$

$$\Rightarrow \lambda = -\frac{2x_2 + 3y_2 - 1}{2x_3 + 3y_3 - 1} = \frac{BP}{PC}$$

Similarly $\dfrac{CQ}{QA} = -\dfrac{2x_3 + 3y_3 - 1}{2x_1 + 3y_1 - 1}$

and $\dfrac{AR}{RB} = -\dfrac{2x_1 + 3y_1 - 1}{2x_2 + 3y_2 - 1}$

34. Sum of rational numbers is a rational number and a rational number divided by 3 is again a rational number.

35. $A_1 A_4$ and $A_2 A_5$ pass through the mid point of $A_0 A_3$ *i.e.* (0, 0) and make an angle of 60° with $A_0 A_3$ which is x-axis.

36. Slope of the line joining $P(2, 2)$ and $S(13/2, 1)$ is $-2/9$.

37. $\dfrac{m-7}{1+7m} = -\dfrac{m-(-1)}{1-m}$

38. The diagonals of the rhombus are parallel to the angle bisectors of the given lines. Slope of the diagonal through $A(0, a)$ is $2 - a$.

39. $\dfrac{x-3}{\cos\theta} = \dfrac{y-4}{\sin\theta} = r$
$PR = 3/\cos \theta$, $PS = 4/\sin \theta$
$RS = PS - PR$

40. Let $A(p, 0)$, B$(0, q)$ and $C(h, k)$, then
$(h - p)^2 + k^2 = a^2$, $h^2 + (k - q)^2 = b^2$
and $p^2 + q^2 = a^2 + b^2$
Eliminate p, q

41. Centre of mean position is the mid point of PQ.

42. Length of the perpendicular from (1, 1) to the line is equal to 1.

43. Mid point (3, 2) of AC lies on BD, $c = -4$. Let the coordinates of D be $(x, 2x - 4)$. $AD \perp CD \Rightarrow x = 4, 2$. The required coordinates are (4, 4) or (2, 0). As D lies below A, its coordinates are (2, 0)

44. $\begin{vmatrix} x & y & 1 \\ 2 & 1 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 10 \Rightarrow 3x + y - 7 = 10$

45. Diagonals of a rhombus are at right angles.

46. $ax + \dfrac{(a+c)}{2} y + c = 0$

$a(2x + y) + c(y + 2) = 0$

which represents a family of lines passing through the intersection of $2x + y = 0$ and $y + 2 = 0$

47. L: $y = -(1/5)x + c$, meets axes at $(0, c)$ and $(5c, 0)$, $(1/2) \times 5c \times c = 5 \Rightarrow c^2 = 2$ Distance of

L from the origin $= \dfrac{c}{\sqrt{1+1/25}} = \dfrac{5\sqrt{2}}{\sqrt{26}}$

48. The lines enclose a square, the length of each side being $2\sqrt{2}$.

49. Circumcentre, centroid and the orthocentre lie on the same line.

50. $SA = a(1 + t^2)$, $SB = a(1 + t^2)/t^2$

51. Equation of the bisectors of the angle between the given lines are

$$\frac{px + qy + r}{\sqrt{p^2 + q^2}} = \pm \frac{qx + py + (p + q - r)}{\sqrt{p^2 + q^2}}$$

Taking the negative sign we get $x + y + 1 = 0$ so one is the reflection of the other line, in the line $x + y + 1 = 0$.

52. Let the coordinates of A be (x, y), then

$\tan C = \dfrac{y}{a - x}$, $\tan B = \dfrac{y}{a + x}$

$\tan(C - B) = \tan(\pi/3) = \sqrt{3}$

53. A_n $(n, n\sqrt{5})$, $(0\ A_n)^2 = 6n^2$, required

$\text{Sum} = 6\Sigma n^2 = \dfrac{6n(n+1)(2n+1)}{6}$.

54. Line meets the coordinate axes at $(4, 0)$ and $(0, 6)$. If (x, y) are the coordinates of an extremity of the other diagonal, then $(x - 4)^2 + y^2 = x^2 + (y - 6)^2$

$\Rightarrow$ $2x - 3y + 5 = 0$ which is satisfied by the coordinates in (c).

55. $(m + 1)(2m - 1)(m - 2) = 0$

$\Rightarrow m_1 = -1, m_2 = 1/2, m_3 = 2$

$$A = -\sum \frac{1}{m_i^2} = -\frac{21}{4}$$

$$B = \sum \frac{1}{m_i} = \frac{3}{2}$$

$$-\frac{21}{4}\alpha + \frac{3}{2}\beta = 0$$

$$\Rightarrow \qquad -21\alpha + 6\beta = 0$$

which is satisfied by (b)

56. $u = 0$, $v = 0$ are parallel lines but not identical. $u + kv = 0$ represents a family of lines parallel to these lines.

57. Required line is

$(x + y + 1) + \lambda(lx + my + n) = 0$ such that $lx + my + n = 0$ is equally inclined to $x + y + 1 = 0$ and this line.

58. Triangles NDA, MBC and OAB are congruent.

$\Rightarrow$ $MB = OA = ND = 12$

and $CM = OB = NA = ON - OA = 5$

$OM = OB + BM = 5 + 12 = 17$

coordinates of $C = (OM, CM) = (17, 5)$

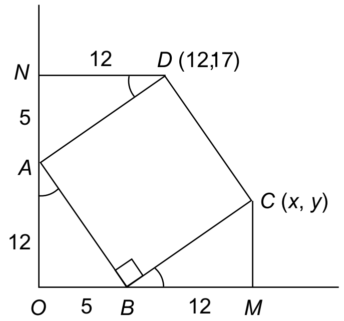


Fig. 16.22

59. Let the coordinates of A be (x, y)

AM is perpendicular to MD and $AM = (1/2)\ MD$.

then $\dfrac{y+1}{x-2} \times \dfrac{1+1}{1-2} = -1 \Rightarrow x - 2y - 4 = 0$,

$MD = \sqrt{5}$.

which is satisfied by (b) only.

60. Let the equation of the line be $\dfrac{x}{a} + \dfrac{y}{b} = 1$. Meets the coordinate axes at $A(a, 0)$ and $B(0, b)$ $(1, 1)$ divides AB in the ratio $3 : 4$.

61. Lines are perpendicular if $a + b = 0$

62. Lines are coincident if $(h)^2 = a\ .\ a$

63. If $a + b = 0$, the lines are

$(x + y)(ax - ay + c) = 0$

64. We can write $(x - y)(x - 2y)(x - 3y) = 0$

65. Let the slopes be m/r, m, mr, then product

$= m^3 = -8 \Rightarrow m = -2$

and m satisfies $8 + am + bm^2 + m^3 = 0$

66. Let the slopes be m and $m + 1$, then

$2m + 1 = -a$, $m(m + 1) = 6$.

67. Product of the slopes is $1 = a/3 \Rightarrow a = 3$

68. Vertices of the rectangle are $A(1, 4)$ $B(6, 4)$, $C(6, 10)$ and $D(1, 10)$

Equation of AC is $6x - 5y + 14 = 0$

Equation of BD is $6x + 5y - 56 = 0$

AC is nearer the origin.

69. Mid point is $P(335, 1340)$. Let the required equation be $\frac{x}{k}+\frac{y}{2k} = 1$, it passes through P if $k = 1005$.

70. $Q = (\alpha, \beta) = \left(\sqrt{xy}, \frac{x+y}{2}\right)$,

$$\text{mid-point of } PQ = \left(\frac{x+\sqrt{xy}}{2}, \frac{\frac{x+y}{2}+y}{2}\right)$$

$\Rightarrow x + \sqrt{xy} = 84$, $x + 3y = 124$. both are satisfied by (b) $(49, 25)$

71. Equation of the line through $(-1, 3)$ perpendicular to $y = 2x$ is $x + 2y = 5$ which meets $y = 2x$ at $(1, 2)$. Required point is (α, β) such that $\frac{\alpha-1}{2} = 1$, $\frac{\beta+3}{2} = 2$.

72. (a^2, a) lies between the given lines
if $a^2 + a - 6 > 0$ and $a^2 + a - 2 < 0$ ... (1)
or $a^2 + a - 6 < 0$ and $a^2 + a - 2 > 0$ ... (2)
(1) does not hold for any value of a
From (2) $-3 < a < 2$ and $[a < -2$ or $a > 1]$
$\Rightarrow -3 < a < -2$ or $1 < a < 2$.
$\Rightarrow |2a + 5| < 1$ or $|2a - 3| < 1$

73. Coordinates of the vertices are $(0, 0)$, $(30, 0)$, $(20, 30)$, $(0, 50)$. Lengths of the sides are 30, $10\sqrt{10}$, $20\sqrt{2}$, 50.

74. $\frac{m_1+m_2}{m_1m_2} = 2 \Rightarrow \frac{2c}{3} = 2 \Rightarrow c = 3$

75. Let the slope of the other line be m

then $-\frac{2}{3} \times m = \frac{2c}{-(c^2-1)}$

and $-\frac{2}{3} + m = \frac{-2}{-(c^2-1)}$

$\Rightarrow -\frac{2}{3} + \frac{3c}{c^2-1} = \frac{2}{c^2-1}$

$\Rightarrow 2c^2 - 9c + 4 = 0$

76. In statement-1, let the vertex C be (α, β), then centroid is $\left(\frac{\alpha+2-2}{3}, \frac{\beta-3+1}{3}\right)$ which lies on $x + y = 5 \Rightarrow \alpha + \beta = 17$
In statement-2, $x - y + 1 = 0$ is the equation of the perpendicular bisector of AB which is also the median through C.

77. Statement -1 is true, as the lines are $x + 1 = 0$, $y - 2 = 0$, $x - 4 = 0$
Statement -2 is false as the two sets of lines are $x + 1 = 0$, $y - 2 = 0$ and $x - 1 = 0$, $y + 2 = 0$

78. Statement -2 is true, the given equation represents points which satisfy $x + 4y - 2 = 0$, $4x + y - 8 = 0$ and $x \pm y - k = 0$. Point of intersection of first two is $(2, 0)$ which also satisfies the third equation if $k = 2$.
Statement -1 is false, as the locus represents a point, the point of intersection of the lines given.

79. Statement-2 is true and using it statement-1 is also true as the slope of $PQ = \frac{\sin\beta+\cos\beta}{\cos(\beta-\alpha)+\sin(\beta-\alpha)}$,

$$\text{Slope of } QR = \frac{\sin\beta-\sin(\beta-\theta)}{\cos(\beta-\alpha)-\cos(\beta-\alpha+\theta)}$$

$$= \frac{\cos(\beta-(\theta/2))}{\sin(\beta-\alpha+(\theta/2))}$$

80. Statement-2 is true, using it in statement-1 hypotenuse makes an angle $\pi/4$ with $x = a$ which is parallel to y-axis, so it makes an angle $\pi/4$ with x-axis also and then the slope of the hypotenuse is ±1
$\Rightarrow$ $k = \pm 1$, statement-1 is false

81. Statement-2 is false, In statement-1

$$x = \frac{a\cos t+b\sin t-1}{3},\ y = \frac{a\sin t-b\cos t}{3}$$

$\Rightarrow$ $(3x - 1)^2 + (3y)^2 = a^2 + b^2$, statement-1 is true.

82. Statement-2 is true, using it $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}$ $\Rightarrow$ the line $\frac{x}{a}+\frac{y}{b}+\frac{1}{c} = 0$ passes through the point $(1, -2)$, so the statement-1 is false.

83. Statement-2 is true, equation in statement-1 is

$$\left(x+\sqrt{3}.y+5\right)\ \sin\theta + \left(\sqrt{3}x-y-7\right)\ \cos\theta = 0$$

$$\text{or } \left(x+\sqrt{3}y+5\right) + \tan\theta\left(\sqrt{3}x-y-7\right) = 0$$

$(\cos\theta \ne 0)$
Using statement-2, statement-1 is true.

84. Statement-1, equation of the third line is $y = k$, vertices are $(1, 1)$,(k, k) and $(2 - k, k)$

$$\Rightarrow \quad 4h^2 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ k & k & 1 \\ 2-k & k & 1 \end{vmatrix} \Rightarrow (k-1)^2$$

So statement-1 is true. Area of the first triangle in statement-2 is $\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 1$ and the area of the second is $\left(\frac{1}{2}\right)(2)(2) = 2$ and statement-2 is also true but does not lead to statement-1

85. Statement-2 is false, PQ should be perpendicular to the line and mid point of PQ lies on the line. Using this correct statement, statement-1 is true.
86. Statement-1 is false, the lines are coincident as the statement-2 is true and statement-1 satisfies the condition in statement-2.
87. In statement-1, $(x^2 - 1)(y^2 - 1) = 0 \Rightarrow x = \pm 1$, $y = \pm 1$ which enclose a square of each side equal 2, so statement-1 is true. Statement-2 is also true as the sum of the coefficients of x^2 and y^2 is zero but does not imply statement-1

Level 2

88. The given triangle is right-angled with vertices $A(0, 0)$, $B(2, 0)$ and $C(0, 2)$.

Coordinates of the incentre are

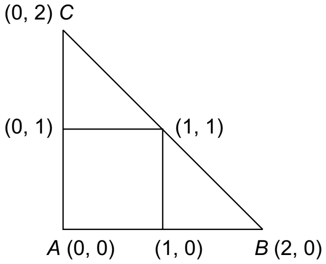


Fig. 16.23

$$\left(\frac{a\,x_1 + b\,x_2 + c\,x_3}{a+b+c}, \frac{a\,y_1 + b\,y_2 + c\,y_3}{a+b+c}\right)$$

$$= \left(\frac{2\sqrt{2}\times 0 + 2\times 2 + 2\times 0}{2\sqrt{2}+2+2}, \frac{2\sqrt{2}\times 0 + 2\times 0 + 2\times 2}{2\sqrt{2}+2+2}\right)$$

$$= \left(2-\sqrt{2}, 2-\sqrt{2}\right)$$

89. Sides of the triangle are

$a = BC = \sqrt{4+9} = \sqrt{13}$, $b = \sqrt{2}$, $c = \sqrt{5}$

The greatest angle A is given by

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc} = \frac{2+5-13}{2\times\sqrt{2}\times\sqrt{5}} = \frac{-3}{\sqrt{10}}$$

90. The given points are collinear.
91. Area of the rhombus

$= \frac{1}{2} \times$ product of the diagonals

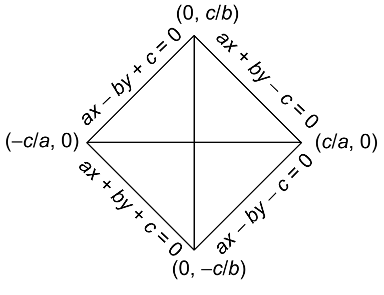


Fig. 16.24

$$= \frac{1}{2}\times\frac{2\,c}{a}\times\frac{2\,c}{b} = \frac{2\,c^2}{ab}$$

92. $l_1 = a^2\cos\alpha + 2a\sin\alpha + \sin\alpha\tan\alpha$

$l_2 = ab\cos\alpha + (a + b)\sin\alpha + \sin\alpha\tan\alpha$

$l_3 = b^2\cos\alpha + 2b\sin\alpha + \sin\alpha\tan\alpha$

$\Rightarrow l_1 l_3 = l_2^2 \Rightarrow l_1, l_2, l_3$ are in G.P.

93. Length of $AB = \sqrt{26}$

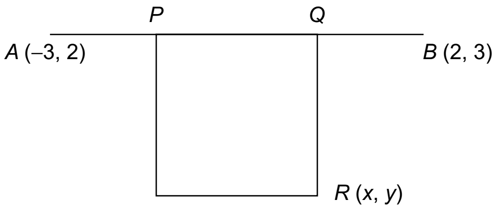


Fig. 16.24

Length of $PQ = \sqrt{26}/3$

Coordinates of Q are $\left(\frac{2\times 2-3}{3}, \frac{2\times 3+2}{3}\right)$

$= \left(\frac{1}{3}, \frac{8}{3}\right)$

$R(x, y)$ lies on the line through Q perpendicular to $AB \Rightarrow y - (8/3) = -5\,(x - (1/3))$

$\Rightarrow 5x + y = 13/3$ (1)

Equation of AB is $x - 5y + 13 = 0$

Distance of R from AB is equal to PQ.

$$\Rightarrow \left|\frac{x-5y+13}{\sqrt{1+25}}\right| = \frac{\sqrt{26}}{3}$$

$\Rightarrow x - 5y + 13 = \pm\, 26/3$ (2)

Solving (1) & (2) we get $R(x, y) = (0, 13/3)$ or $(2/3, 1)$

94. The given equation can be written as

$$x(y_1 - y_2 + y_1 - y_3) - y(x_1 - x_2 + x_1 - x_3) + x_1y_2 - x_2y_1 + x_1y_3 - y_1x_3 = 0$$

$$\Rightarrow x\,(2y_1 - y_2 - y_3) - y\,(2x_1 - x_2 - x_3) + y_1\,(2x_1 - x_2 - x_3) - x_1(2y_1 - y_2 - y_3) = 0$$

$$\Rightarrow y - y_1 = \frac{2y_1 - y_2 - y_3}{2x_1 - x_2 - x_3}(x - x_1)$$

which represents the median through A

95. Three lines $x + 2y - 3 = 0$, $2x + 3y - 4 = 0$ and $4x + 5y - 6 = 0$ are concurrent at $(-1, 2)$. But the fourth line $3x + 4y - 7 = 0$ does not pass through this point.

96. Slope of the incident ray PA is $2/(1 - x)$ and that of the reflected ray QA is $3/(5 - x)$

so $\frac{2}{1-x} = \frac{-3}{5-x} \Rightarrow x = \frac{13}{5} \Rightarrow A\left(\frac{13}{5}, 0\right)$

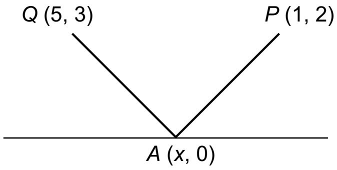


Fig. 16.26

97. $\Delta_1 = \frac{1}{2}\left|\begin{vmatrix} 0 & 0 & 1 \\ a\tan\alpha & b\cot\alpha & 1 \\ a\sin\alpha & b\cos\alpha & 1 \end{vmatrix}\right|$

$= \frac{1}{2} ab\,|(\sin\alpha - \cos\alpha)|$

$\Delta_2 = \frac{1}{2}\left|\begin{vmatrix} a & b & 1 \\ a\sec^2\alpha & b\,\cos es^2\alpha & 1 \\ a + a\sin^2\alpha & b + b\cos^2\alpha & 1 \end{vmatrix}\right|$

$= \frac{1}{2} ab\ [\text{cosee}^2\ \alpha - 1 - \cos^2\ \alpha + 1 + \sin^2\ \alpha -$

$\sec^2\ \alpha + \sec^2\alpha + 1 - \text{cosec}^2\alpha - 1]$

$= \frac{1}{2}\ ab\ |(\sin^2\ \alpha - \cos^2\ \alpha)|$

and $\Delta_3 = \frac{1}{2}\left|\begin{vmatrix} 0 & 0 & 1 \\ a\tan\alpha & -b\cot\alpha & 1 \\ a\sin\alpha & b\cos\alpha & 1 \end{vmatrix}\right|$

$= \frac{1}{2} ab\,|\,(\sin\alpha + \cos\alpha)\,|$

Now $\Delta_1\,\Delta_3 = (1/4)\ a^2\ b^2\ |\sin^2\alpha - \cos^2\alpha| = \Delta_2^2$

$\Rightarrow |\sin^2\ \alpha - \cos^2\ \alpha| = 0$ or 1

$\Rightarrow \alpha = \pm\ \pi/4$ or $\alpha = \pm\ \pi/2$

But for these values of α, the triangles are not defined.

Hence Δ_1, Δ_2, Δ_3 are in G.P. for no value of α.

98. The orthocentre (x, y) lies on the lines.

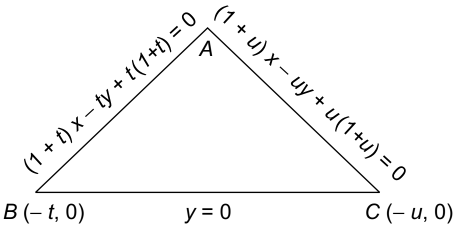


Fig. 16.27

$y = \frac{-u}{1+u}(x+t)$

and $y = \frac{-t}{1+t}(x+u)$

$\Rightarrow [(1 + u) - (1 + t)]\ y + (u - t)\ x = 0$

$\Rightarrow (u - t)\ (x + y) = 0$

$\Rightarrow x + y = 0$ as $u \neq t$

99. Equation of AU is

$y - y_1 = \frac{0 - y_1}{3 - x_1}(x - x_1)$

so coordinates of C are $(0,\ 3y_1/(3 - x_1))$

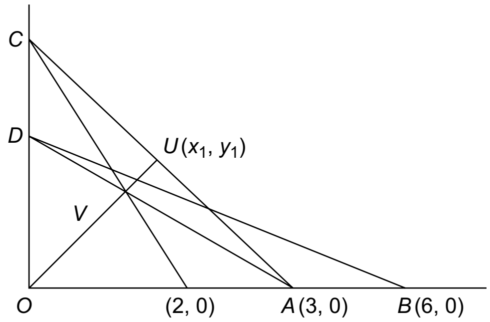


Fig. 16.28

Similarly coordinates of D are $(0,\ 6y_1/(6 - x_1))$

Equation of AD is $\frac{x}{3} + \frac{y(6 - x_1)}{6y_1} = 1$

Equation of OU is $y_1x = x_1y$

Solving, we get the coordinates of

$V = \left(\frac{6x_1}{6 + x_1}, \frac{6y_1}{6 + x_1}\right)$

Equation of CV is

$y = \frac{3y_1}{3 - x_1} - \frac{3y_1}{2(3 - x_1)}\,x = \frac{3y_1}{3 - x_1}\left(1 - \frac{x}{2}\right)$

which passes through (2, 0) for all values of (x_1, y_1)

100. The triangle is equilateral, so the incentre is the centroid $(1,\ 1/\sqrt{3}\,)$ of the triangle

101. Equation of the fourth side is $2x - y - k = 0$

such that $\frac{4 - k}{\sqrt{5}} = \pm\frac{3 + 7}{\sqrt{5}} \Rightarrow k = -6$ or 14.

So the required equation is $2x - y + 6 = 0$, as it is nearer to $(1, -1)$

102. The triangle is right angled at (0, 0) so the orthocentre is (0, 0) and circumcentre is $(a/2,\ b/2)$, the mid point of the hypotenuse.

103. Third vertex of the triangle is (–1, –11)

104. $A(0, 12)$, $B(8, 0)$, $C(x, y)$ lies on the line

$$2x - 3y + 10 = 0 \text{ and } \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 0 & 12 & 1 \\ 8 & 0 & 1 \end{vmatrix} = \pm 91$$

Solving we get $(x, y) = (-13/2, -1)$ or $(29/2, 13)$

105. $$\tan A = \frac{\frac{1}{2}+7}{1-\frac{7}{2}} = -3$$

$$\tan C = \frac{\frac{4}{3}+\frac{1}{3}}{1-\frac{4}{3}\times\frac{1}{3}} = 3$$

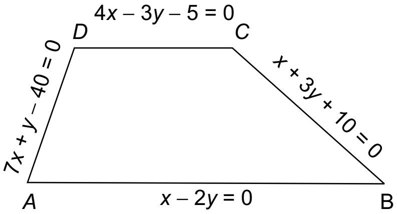


Fig. 16.29

$\Rightarrow \tan(A + C) = 0$

$\Rightarrow A + C = 180°$

$\Rightarrow$ *ABCD* is a cyclic quadrilateral

Note: *ABCD* cannot be a rectangle or a parallelogram, so verify for cyclic quadrilateral.

106. If given lines are concurrent at (–35, 22), then

$35a - 22b - 1 = 0$

$\Rightarrow$ The line $35x - 22y - 1 = 0$ passes through (a, b).

107. We can write $\left(\frac{y}{x}\right)^2 + \frac{2h}{b}\left(\frac{y}{x}\right) + \frac{a}{b} = 0$

If the slopes of the lines are m and m^2 then

$$m + m^2 = -\frac{2h}{b} \text{ and } mm^2 = \frac{a}{b}$$

$$\Rightarrow \left(\frac{a}{b}\right)^{1/3} + \left(\frac{a}{b}\right)^{2/3} = -\frac{2h}{b}$$

$$\Rightarrow \frac{a}{b} + \left(\frac{a}{b}\right)^2 + 3\left(\frac{a}{b}\right)\left(-\frac{2h}{b}\right) = -\frac{8h^3}{b^3}$$

$$\Rightarrow \frac{a}{b^2}[b + a - 6h] + \frac{8h^3}{b^3} = 0$$

$$\Rightarrow a + b - 6h + \frac{8h^3}{ab} = 0$$

$$\Rightarrow \frac{a+b}{h} + \frac{8h^2}{ab} = 6.$$

108. From the given equation we have

$x - y + c = 0$ and $x + y - c = 0$ which represents the point $(0, c.)$

Previous Years' AIEEE/JEE Main Questions

1. Since the given equation represents a point lines we have

$$abc + 2fgha - af^2 - bg^2 - ch^2 = 0 \quad (1)$$

as they intersect on y-axis,

$$by^2 + 2fy + c = 0 \quad (2)$$

(2) gives a unique value of y if $4f^2 = 4bc$

$\Rightarrow af^2 = abc$ and from (1) we have

$$2fgh = bg^2 + ch^2.$$

2. Let (h, k) be mid-point of the portion, then

$$h = \frac{p}{2\cos\alpha}, k = \frac{p}{2\sin\alpha}$$

$$\Rightarrow 1 = \cos^2\alpha + \sin^2\alpha = \frac{p^2}{4h^2} + \frac{p^2}{4k^2}$$

$$\Rightarrow \frac{1}{h^2} + \frac{1}{k^2} = \frac{4}{p^2}$$

Locus of (h, k) is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$

3. Let A (4, 0), B (–1, – 1), C (3, 5)

$(AB)^2 = 26$, $(BC)^2 = 52$, $(AC)^2 = 26$

so the triangle is right angled.

4. $3x + 4y = 0$ and $4x - 3y = 0$ are at right angle so the triangle is right angled.

5. Coordinates of A are $(a \cos \alpha, a \sin \alpha)$ and of C are $(a \cos(\alpha + \pi/2)a \sin(\alpha + \pi/2))$

i.e. $(-a \sin\alpha, a \cos\alpha)$

So the equation of the diagonal AC is

$$y - a\sin\alpha = \frac{a(\sin\alpha - \cos\alpha)}{a(\cos\alpha - \sin\alpha)}(x - a\cos\alpha)$$

$\Rightarrow y(\cos\alpha + \sin\alpha + x(\sin\alpha - \cos\alpha) = a$

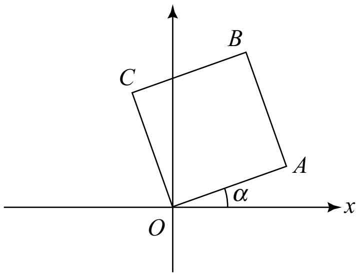


Fig. 16.30

6. Equation of the bisectors of $x^2 - 2pxy - y^2 = 0$ is

$$\frac{x^2-y^2}{2} = \frac{xy}{-p}$$

or $x^2 + \frac{2}{p}xy - y^2 = 0$

Comparing with $x^2 - 2qxy - y^2 = 0$ we get $pq = -1$

7. Let (h, k) be the centroid of the triangle, then

$$h = \frac{a\cos t + b\sin t + 1}{3},\ k = \frac{a\sin t - b\cos t}{3}$$

$\Rightarrow (3h - 1)^2 + (3k)^2 = a^2 + b^2$

Locus of (h, k) is

$(3x - 1)^2 + (3y)^2 = a^2 + b^2$

8. Let $x_2 = x_1r$, $x_3 = x_1r^2$

$y_2 = y_1r$, $y_3 = y_1r^2$

then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_1r & y_1r & 1 \\ x_1r^2 & y_1r^2 & 1 \end{vmatrix} = 0$$

So the given points lie on a straight line.

9. Mid-point $\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}\right)$ of the given points lies on the locus, so

$$(a_1 - a_2)\frac{(a_1+a_2)}{2} + (b_1 - b_2)\frac{(b_1+b_2)}{2} + c = 0$$

$$\Rightarrow \quad c = \frac{1}{2}\left(a_2^2 + b_2^2 - a_1^2 - b_1^2\right)$$

10. Let the third vertex be $C(h, k)$. Centroid of ΔABC is

$$\frac{1}{2}\left(a_2^2 + b_2^2 - a_1^2 - b_1^2\right) = G\left(\frac{h}{3}, \frac{k-2}{3}\right)$$

which lies on $2x + 3y = 1$.

$$\Rightarrow \quad \frac{2h}{3} + 3\left(\frac{k-2}{3}\right) = 1 \Rightarrow 2h + 3k = 9$$

11. Let the intercepts on the axes be a and $-a-1$. Equation of such a straight line

$$\frac{x}{a} - \frac{y}{a+1} = 1$$

As it pass through (4, 3) we get

$$\frac{4}{a} - \frac{3}{a+1} = 1 \quad \Rightarrow 4a + 4 - 3a = a(a+1)$$

$\Rightarrow \quad a^2 = 4 \Rightarrow a = \pm 2$

Thus, required lines are

$$\frac{x}{2} - \frac{y}{3} = 1 \text{ and } \frac{x}{-2} + \frac{y}{1} = 1$$

12. Slopes m_1, m_2 of two lines are given by

$$1 - 2cm - 7m^2 = 0$$

We are given $m_1 + m_2 = 4m_1m_2$

$$\Rightarrow \quad \frac{-2c}{7} = 4\left(\frac{-1}{7}\right) \Rightarrow c = 2$$

13. Slopes of the two lines are connected by

$$6 - m + 4cm^2 = 2$$

As $y = -\frac{3}{4}x$ is one of the lines

$$6 + \frac{3}{4} + 4c\left(-\frac{3}{4}\right)^2 = 0$$

$$\Rightarrow \quad \frac{9}{4}c = \frac{-27}{4} \Rightarrow c = -3$$

14. A straight line passing through the point of intersection of given lines is

$ax + 2by + 3b + k(bx - 2ay - 3a) = 0$

or $(a + kb)x + 2(b - ka)y + 3(b - ka) = 0$

This line will be parallel to the x-axis if $a + kb = 0$ or $k = -a/b$

Thus, the line parallel to the x-axis is

$\frac{2}{b}(a^2 + a^2)y + \frac{3}{b}(b^2 + a^2) = 0$ or $2y + 3 = 0$

$\Rightarrow \quad y = -3/2$

This line is below the x-axis at a distance of (3/2) from it.

15. Let coordinates of B be (x, y). Then

$$\frac{x+1}{2} = -1, \frac{y+1}{2} = 2$$

$\Rightarrow \quad x = -3, y = 3$

$\therefore$ Coordinates of $B(-3, 3)$

Similarly coordinates of C are (5, 3)

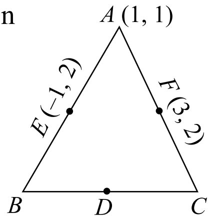


Fig. 16.31

Thus, centroid of ΔABC is

$$\left(\frac{1-3+5}{3}, \frac{1+3+3}{3}\right) = \left(1, \frac{7}{3}\right)$$

16. a, b, c are in H.P.

$$\Rightarrow \quad \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are A.P.}$$

$$\Rightarrow \quad \frac{1}{a} - \frac{2}{b} + \frac{1}{c} = 0$$

Thus, $\frac{x}{a}+\frac{y}{b}+\frac{1}{c}=0$ passes through (1, –2).

17. As area of one of the sectors thrice that of the other, $\pi - \theta = 3\theta$ or $\theta = \pi/4$.

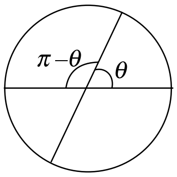


Fig. 16.32

Thus, angle between two lines is $\pi/4$ or $3\pi/4$

$$\therefore \quad \pm 1=\frac{2\sqrt{(a+b)^2-ab}}{a+b}$$

$\Rightarrow \quad (a + b)^2 = 4(a^2 + b^2 + ab)$

$\Rightarrow \quad 3a^2 + 2ab + 3b^2 = 0$

18. Let equation of straight line be

$\frac{x}{a}+\frac{y}{b}=1$

It meets the axes in $P(a, 0)$ and $Q(0, b)$. As the mid-point of PQ is $A(3, 4)$, we get

$$\left(\frac{a}{2},\frac{b}{2}\right)=(3,4)$$

$\Rightarrow \quad a = 6,\ b = 8$

Thus, required line is

$$\frac{x}{6}+\frac{y}{8}=1$$

or $\quad 4x + 3y = 24$

19. $AB^2 + BC^2 = AC^2$

$\Rightarrow (h - 1)^2 + (k - 1)^2 + 1$
$= (h - 2)^2 + (k - 1)^2$

$\Rightarrow \quad h = 1$

Area of $\quad \Delta ABC = 1$

$\Rightarrow \quad \frac{1}{2}(1)\,|k-1|=1$

$\Rightarrow \quad |k - 1| = 2 \quad \Rightarrow \quad k - 1 = \pm 2$

$\Rightarrow \quad k = 3, -1$

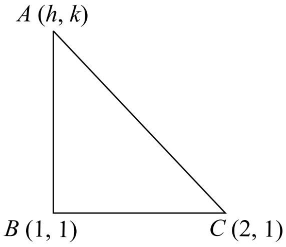


Fig. 16.33

20. $\angle RQX = 60°$

Let QS be the bisector of $\angle PQR$, then

$\angle RQS = 60°$

$\Rightarrow \quad \angle SQX = 120°$

$\Rightarrow$ Slope of $QS = -\sqrt{3}$

$\therefore$ Equation of QS is

$y = -\sqrt{3}\,x$ or $\quad \sqrt{3}\,x + y = 0$

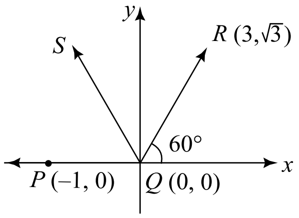


Fig. 16.34

21. Bisectors of lines $xy = 0$ i.e. $x = 0$, $y = 0$ are $y = \pm x$.

For $m = 1$, the given equation reduces to $y^2 - x^2 = 0$

22. Mid-point of PQ is $\left(\frac{k+1}{2},\frac{7}{2}\right)$

Equation of the perpendicular bisector of PQ in

$$y-\frac{7}{2}=-\frac{k-1}{3-4}\left(x-\frac{k-1}{2}\right)$$

$$y\text{-intercept} = \frac{(k-1)(k+1)}{-2}+\frac{7}{2}=4$$

$\Rightarrow \quad k^2 = 16 \quad \Rightarrow k = \pm 4$

23. Slope of two lines are

$$m_1 = \frac{1}{p(p^2+1)},\ m_2 = -\frac{1}{p\left(p^2+1\right)}=-\frac{1}{p^2+1}$$

Two lines will be perpendicular to a common line if these two lines are parallel.

$$\therefore \quad -\frac{1}{p\left(p^2+1\right)}=-\frac{1}{p^2+1}$$

$\Rightarrow \quad p = -1$

For $p = -1$, two lines become

$-2x - y + q = 0$

and $4x + 2y + 2q = 0$ which are parallel.

24. As L: $\frac{x}{5}+\frac{y}{b}=1$ passes through (13, 32), we get

$$\frac{13}{5}+\frac{32}{b}=1 \ \Rightarrow\ \frac{32}{b}=1-\frac{13}{5}=-\frac{8}{5}$$

$\Rightarrow \quad b = -20$

$\therefore$ equation of line L is $\frac{x}{5}-\frac{y}{20}=1$

As K: $\frac{x}{c}+\frac{y}{3}=1$ is parallel to L.

$$\frac{1/5}{1/c}=\frac{-1/20}{1/3}\Rightarrow\frac{c}{5}=\frac{-3}{20}\Rightarrow c=\frac{-3}{4}$$

Thus, equation of K is $\frac{-4x}{3}+\frac{y}{3}=1$.

$\therefore$ Equation of L and K are

$4x-y-20=0$ and $4x-y+3=0$

Distance between L and K is

$$\frac{|-20-3|}{\sqrt{16+1}}=\frac{23}{\sqrt{17}}$$

25. We have $OP=2\sqrt{2}$ and $OQ=\sqrt{5}$.

R divides PQ is the ratio $2\sqrt{2}:\sqrt{5}$, that is,

$PR: RQ=2\sqrt{2}:\sqrt{5}$

However, statement-2 is false.

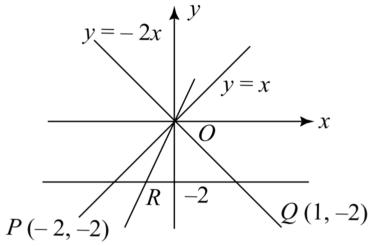


Fig. 16.35

26. If $a>0$, the given lines intersect at the point $(1, a-1)$. Since this point lies in the first quadrant, $a-1>0\Rightarrow a>1$. If $a<0$, then the equation $ax-y=1$ gives $y=ax-1$ which is negative for all positive values of x. Hence $a<0$ is not possible and the required values of a are in the interval $(1, \infty)$.

27. Let $G(h, k)$ be centroid of ΔABC, then coordinates of C are $(3h-2+2, 3k+3-1)=(3h, 3k+2)$

As C lies on $2x+3y=9$, we get

$$2(3h)+3(3k+2)=9$$

$$\Rightarrow \quad 2h+3k=1$$

Thus, locus of centroid G is $2x+3y=1$

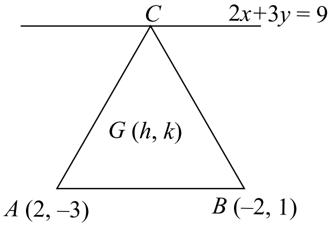


Fig. 16.36

28. Coordinates of the point dividing the join of the points (1, 1), (2, 4) in the ratio 3 : 2 are $\left(\frac{3\times2+2\times1}{3+2},\frac{3\times4+2\times1}{3+2}\right)=\left(\frac{8}{5},\frac{14}{5}\right)$

$2x+y=k$ passes through this points.

if $2\times(8/5)+(14/5)=k \Rightarrow k=6$

29. Let equation of line through (1, 2) be

$$y-2=m(x-1).$$

It meets the coordinate axes in $P\left(\frac{m-2}{m},0\right)$ and $Q(0, 2-m)$.

Let A = Area of ΔOPQ = $\frac{1}{2}\left|\frac{m-2}{m}\right||m-2|$

$$=\frac{1}{2}\frac{(m-2)^2}{|m|}$$

$$=\frac{1}{2}\left||m|+\frac{4}{|m|}-\frac{4m}{|m|}\right|$$

$$=\frac{1}{2}\left[\left(\sqrt{|m|}-\frac{2}{\sqrt{|m|}}\right)^2+4\left(1-\frac{m}{|m|}\right)\right]$$

Note A is least if $|m|=2$ or $m=\pm 2$.

For $m=2$, the line passes through the origin, and P and Q coincide.

$\therefore$ required value of m is -2.

30. Ray of light $x+\sqrt{3}y=\sqrt{3}$ meets the x-axis at $(\sqrt{3}, 0)$. The reflect ray makes an angle of 30° with the x-axis. Its equation is

$$y-0=\frac{1}{\sqrt{3}}(x-\sqrt{3})\Rightarrow\sqrt{3}y=x-\sqrt{3}$$

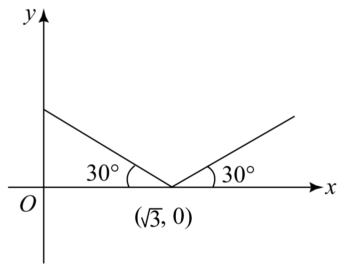


Fig. 16.37

31. Solving the given equation we get

$\frac{8y}{3}+y^2=1 \Rightarrow 3y^2+8y-3=0$

$\Rightarrow y=\frac{1}{3}$ or $y=-3$ but $y\neq-3$.

so $y=\frac{1}{3}$ and the line $3y-1=0$ passes through the point of intersection of the the two curves.

32. Slope of PQ = – (Slope of RQ)

$$\Rightarrow \quad \frac{3}{1-x} = \frac{-7}{6-x}$$

$$\Rightarrow \quad x = \frac{5}{2}$$

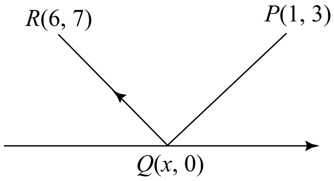


Fig. 16.38

33. Slopes of the given lines are

$$\frac{1}{3}, -\frac{a}{2}, -a$$

Two of the lines are perpendicular.

$$\text{if} \quad \frac{1}{3}\left(-\frac{a}{2}\right) = -1 \quad \text{or} \quad \frac{1}{3}(-a) = -1$$

$\Rightarrow$ $a = 6$ or $a = 3$

$\Rightarrow$ a satisfies the equation

$a^2 - 9a + 18 = 0$

34. Vertices of the triangle are

$A(0, 0)$, $B(2, 0)$, $C(0, 2)$.

$AB = AC = 2$ and $BC = 2\sqrt{2}$

So x-coordinates of the incentre is

$$\frac{2 \times 0 + 2 \times 2 + 2\sqrt{2} \times 0}{2 + 2 + 2\sqrt{2}}$$

$$= \frac{4}{4 + 2\sqrt{2}} = 2 - \sqrt{2}$$

35. $3x + 4y = 12$

$$\Rightarrow \quad \frac{x}{4} + \frac{y}{3} = 1$$

Equation of L is $\dfrac{x}{8} + \dfrac{4}{3/2} = 1$

Slope of L = $-\dfrac{3}{2} \times \dfrac{1}{8} = -\dfrac{3}{16}$

36. Let $A(2a, 0)$, $B(0, a)$ and $C(2a, y)$

$BC = AC \Rightarrow (2a)^2 + (y - a)^2 = y^2$

$$\Rightarrow \quad y = \frac{5a}{2}$$

Area of the triangle

$$= \frac{1}{2} AC \times 2a = \frac{1}{2} y \times 2a = ay$$

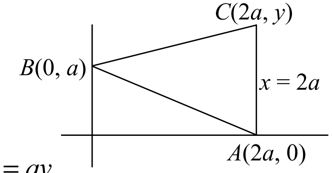


Fig. 16.39

$$= \frac{5a^2}{2} \text{ sq. units}$$

37. $2x + 3y + c_1 = 0$ makes the same angle with the lines $-x + 5y + c_2 = 0$ and $-x + 5y + c_1 = 0$ as they are parallel. Hence $\theta_1 = \theta_2$ for all c_1 and c_2. So statement-2 is true and thus it follows that statement-1 in also true.

38. Slope of PQ = 1.

Mid point of PQ is (3, 4). So equation of L is

$$y - 4 = -(x - 3)$$

$\Rightarrow$ $x + y - 7 = 0$

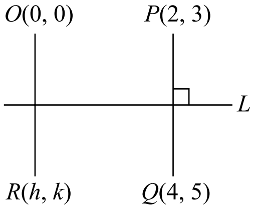


Fig. 16.40

Let the image of θ (0, 0) be R (h, k) the slope of OR = slope of PQ

$$\Rightarrow \quad \frac{k}{h} = 1 \quad \Rightarrow \quad k = h$$

Also mid-point $\left(\dfrac{h}{2}, \dfrac{k}{2}\right)$ of OR lies on L

$$\therefore \quad \frac{h}{2} + \frac{k}{2} - 7 = 0 \quad \Rightarrow \quad h = k = 7$$

and the required point is (7, 7)

39. See Question No. 10 (2004)

40. Let the point of intersection be (h, – h) then

$4ah - 2ah + c = 0$ and $5bh - 2bh + d = 0$

$$\Rightarrow \quad h = \frac{-c}{2a} = \frac{-d}{3b} \quad \Rightarrow \quad 3bc - 2ad = 0$$

41. Coordinates of S are $\left(\dfrac{13}{2}, 1\right)$.

Slope of PS is $\dfrac{2-1}{2-\dfrac{13}{2}} = -\dfrac{2}{9}$

Required equation of the live is

$$y + 1 = -\frac{2}{9}(x - 1)$$

$\Rightarrow$ $2x + 9y + 7 = 0$

42. Let (h, k) be the foot of perpendicular from (0, 0) to the line $\dfrac{x}{a} + \dfrac{4}{b} = 1$. If this length of perpendicular is p, then

$$p = \frac{\left|\dfrac{0}{a} + \dfrac{0}{b} - 1\right|}{\sqrt{(1+a)^2 + (1/b)^2}} = 2$$

Now,

$$\sqrt{h^2 + k^2} = p = 2$$

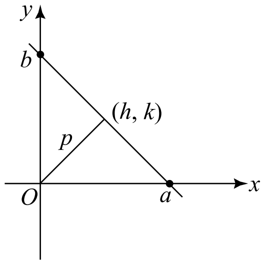


Fig. 16.41

$\Rightarrow h^2 + k^2 = 4$

Thus (h, k) likes on a circle of radius 2.

43. Coordinates of R are (2, 0). As PQ is parallel to the x-axis, and meets $x - 2y = 2$ in Q, coordinates of Q are (8, 3). Centroid of ΔPQR is (5, 2) and it lies on $2x - 5y = 0$

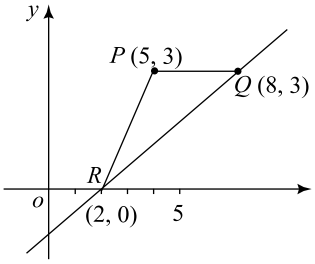


Fig. 16.42

44. If a is the side of the equilateral triangle, then

$$\frac{\sqrt{3}}{2}a = \text{AM}$$

$$= \frac{|3(1)+4(2)-9|}{\sqrt{3^2+4^2}}$$

$$\Rightarrow \quad a = \frac{2}{5}\cdot\frac{2}{\sqrt{3}} = \frac{4}{15}\sqrt{3}$$

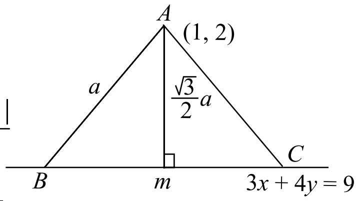


Fig. 16.43

45. We have

$$\left(\frac{2h}{3}, \frac{k}{3}\right) = (4, 3)$$

$\Rightarrow h = 6, k = 9$

Equation of line is

$\frac{x}{6} + \frac{y}{9} = 1$ or $3x + 2x = 18$

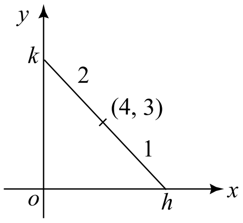


Fig. 16.44

46. As the lines are distinct $a \neq 0$. As the lines are concurrent.

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4a & a \end{vmatrix} = 0$$

$R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 0 & 2a & 0 \end{vmatrix} = 0$$

$\Rightarrow \quad -2a(b - a) = 0 \Rightarrow a = b$

Thus, (a, b) lies on a straight line.

47. We know that circumcentre. centroid and othocentre of a triangle lie on a straight line, say L.
centroid of triangle is

$$\left(\frac{a^2+1+2a}{2}, \frac{a^2+1-2a}{2}\right) = \left(\frac{1}{2}(a+1)^2, \frac{1}{2}(a-1)^2\right)$$

Equation of line L is

$$\frac{x-0}{(a+1)^2/2-0} = \frac{y-0}{(a-1)^2/2-0}$$

$\Rightarrow \quad (a - 1)^2 x - (a + 1)^2 y = 0$

48. An equation of line perpendicular to $5x - y = 1$ is $x + 5y = c$

Now, $5 = $ Area of ΔOAB

$$= \frac{1}{2}|c|\left|\frac{c}{5}\right|$$

$\Rightarrow \quad c^2 = 50$

$\Rightarrow \quad |c| = 5\sqrt{2}$

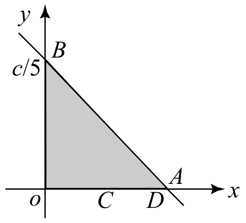


Fig. 16.45

Distance between $x + 5y = c$ and $x + 5y = 0$ is

$$\frac{|c|}{\sqrt{1+5^2}} = \frac{5\sqrt{2}}{\sqrt{26}} = \frac{5}{\sqrt{13}}$$

49. For $1 \le k \le 40$, the number of integral points on the line $x = k$ and lying in the interior of triangles is $k - 1$.

Thus, required number of points $= \sum_{k=1}^{40}(k-1)$

$$\frac{1}{2}(39)(40) = 780$$

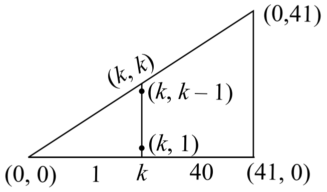


Fig. 16.46

50. $(2x - 3y + 4) + k(x - 2y + 3) = 0, k \in R$**,** represents a family of lines passing through the intersection of the lines $2x - 3y + 4 = 0$ and $x - 2y + 3 = 0$, that is, through (1, 2).

If (x, y) is image of (2, 3) in (1), then both (x, y) and (2, 3) are equidistant from (1, 2), thus,

$$(x - 1)^2 + (y - 2)^2 = (2 - 1)^2 + (3 - 2)^2 = 2.$$

Thus, (x, y) lies on a circle of radius $\sqrt{2}$

51. Let slope of required line L be m, then

$$\pm \tan 60^\circ = \frac{m-\left(-\sqrt{3}\right)}{1+m\left(-\sqrt{3}\right)}$$

$$\Rightarrow \pm\sqrt{3} = m+\sqrt{3}$$

$$\Rightarrow m = 0 \text{ or } m = -\sqrt{3}$$

As the desired line intersects the x-axis, $m = -\sqrt{3}$.

Thus, equation of required line is

$$y + 2 = -\sqrt{3}(x-3)$$

or $y+\sqrt{3}x+2-3\sqrt{3}=0$

52. Slope of line joining $A(0, 8/3)$ and $B(1, 3)$ is

$$m_1 = \frac{3-8/3}{1-0} = \frac{1}{3}$$

and slope of line joining $B(1, 3)$ and $C(82, 30)$ is

$$m_2 = \frac{30-3}{82-1} = \frac{1}{3}$$

As $m_1 = m_2$, $AB \parallel AC$. But B is common, therefore A, B, C are collinear.

53. Suppose L meets the axes at $A(a, 0)$ and $B(0, b)$. As $P(1, 2)$ is the mid-point of AB,

$$\frac{a}{2}=1 \text{ and } \frac{b}{2}=2$$

$$\Rightarrow \quad a = 2,\ b = 4.$$

Thus, equation of line L is

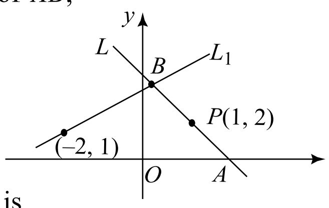


Fig. 16.47

$$\frac{x}{2}+\frac{y}{4}=1 \quad \text{or } 2x+y=4 \qquad (1)$$

and the equation of L_1 is $(x + 2) - 2(y - 1) = 0$

or $\quad x - 2y + 4 = 0 \qquad (2)$

solving (1) and (2) we get the required point as $\left(\frac{4}{5}, \frac{12}{5}\right)$

54. A vertex of the rhombus is (1, 2), the point of intersection of $x - y + 1 = 0$ and $7x - y - 5 = 0$

Another vertex of the rhombus is

$(2(-1) - 1, 2(-2) - 2) = (-3, -6)$

Other two sides to the rhombus are

$(x + 3) - (y + 6) = 0$ and $7(x + 3) - (y + 6) = 0$

or $x - y - 3 = 0$ and $7x - y + 15 = 0$.

A vertex can be obtained by solving $7x - y - 5 = 0$ and $x - y - 3 = 0$

Thus a vertex of rhombus is $\left(\frac{1}{3}, -\frac{8}{3}\right)$

55. An equation of line through intersection of $\frac{1}{3}x+\frac{1}{4}y-1=0$ and $\frac{1}{4}x+\frac{1}{3}y-1=0$ is

$$\left(\frac{1}{3}x+\frac{1}{4}y-1\right)+k\left(\frac{1}{4}x+\frac{1}{3}y-1\right)=0$$

or $\left(\frac{1}{3}+\frac{k}{4}\right)x+\left(\frac{1}{4}+\frac{k}{3}\right)y-(1+k)=0$

It meets the x-axis is in $A\left(\frac{12(k+1)}{4+3k}, 0\right)$ and the y-axis in $B\left(0, \frac{12(k+1)}{3+4k}\right)$

Let (α, β) be the mid-point of AB, therefore

$$2\alpha = \frac{12(k+1)}{4+3k},\ 2\beta = \frac{12(k+1)}{3+4k}$$

$$\Rightarrow \frac{1}{2\alpha}+\frac{1}{2\beta} = \frac{7(k+1)}{12(k+1)} = \frac{7}{12}$$

$$\Rightarrow 6(\alpha + \beta) = 7\alpha\beta$$

Thus, locus of mid-point AB is $6(x + y) = 7xy$

56. An equation of line through (2, 1) and parallel to $x - y = 4$ is $(x - 2) - (y - 1) = 0$ or $x - y - 1 = 0$.

A point on this line is $(t, t - 1)$.

Its distance from (2, 1) is

$$d = \sqrt{(t-2)^2+(t-1-1)^2} = \sqrt{2}|t-2|$$

Now $\sqrt{2}|t-2| = 2\sqrt{3}$

$$\Rightarrow t - 2 = \pm\sqrt{6} \Rightarrow t = 2 \pm \sqrt{6}$$

As the point lies in the third quadrant, we take $t = 2 - \sqrt{6}$

$\therefore$ point Q is $(2 - \sqrt{6}, 1 - \sqrt{6})$

An equation of required line through Q and perpendicular to L is

$$(x - (2 - \sqrt{6})) + (y - (1 - \sqrt{6})) = 0$$

or $x + y = 3 - 2\sqrt{6}$

57. Two given lines are

$4x + 3y - 10 = 0$

and $8x + 6y + 5 = 0$

These lines are parallel.

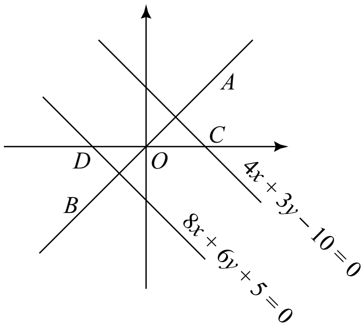


Fig. 16.48

Thus $OA : OB = OC : OD$

$= \frac{10}{4} : \frac{5}{8} = 4 : 1$

58. An equation of normal AN is $(x - 0) + 7(y - 1) = 0$

or $x + 7y - 7 = 0$

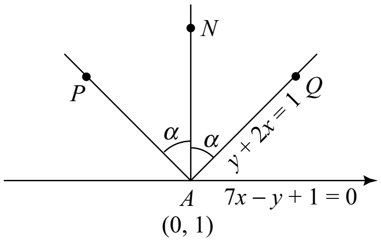


Fig. 16.49

Let an equation of PA be $y = mx + 1$,

then $\angle PAN = \angle NAQ$

$$\frac{-1/7 - m}{1 - m/7} = \frac{-2 - (-1/7)}{1 + (2/7)}$$

$$\Rightarrow \left(\frac{1}{7} + m\right)(9) = 13\left(1 - \frac{m}{7}\right)$$

$$\Rightarrow \left(9 + \frac{13}{7}\right)m = 13 - \frac{9}{7}$$

$\Rightarrow 76m = 82 \Rightarrow m = 41/38$

Thus, required equation is

$y = \frac{41}{38}x + 1$

or $41x - 38y + 38 = 0$

Previous Years' B-Architecture Entrance Examination Questions

1. Let the lines intersect at the point k.

   then $(a + 2)k + 3a = 0$ and $4k + a - 2 = 0$

   $$\Rightarrow \frac{3a}{a+2} = \frac{a-2}{4} \Rightarrow a^2 - 12a + 4 = 0$$

   which gives us two values of a.

2. Let the point of intersection be $(h, -h)$; $h > 0$ then $2ph - 3qh + r = 0$ and $ph + 2qh - 2r = 0$,

   $$\Rightarrow \frac{r}{2p - 3q} = \frac{-2r}{p + 2q}$$

   $\Rightarrow p + 2q + 2(2p - 3q) = 0$

   $\Rightarrow 5p - 4q = 0$

3. Mid-points of the sides of the triangle ABC are $P(0, 3)$, $Q\left(-\sqrt{3}, 0\right)$, $R\left(\sqrt{3}, 0\right)$

   PQR is an equilateral triangle with each side of length $2\sqrt{3}$. So its in-centre is same as its centroid and hence its coordinates are (0, 1).

4. $(1 + 2\lambda_1)x - (1 + \lambda_1)y + (1 - 2\lambda_1) = 0$

   $$(5 + 3\lambda_2)x + (3 - \lambda_2)y - (2 + 4\lambda_2) = 0$$

   They represent the same line.

   if $\frac{5 + 3\lambda_2}{1 + 2\lambda_1} = \frac{3 - \lambda_2}{-1 - \lambda_1} = \frac{-2 - 4\lambda_2}{1 - 2\lambda_1}$

   Solving, we get $\lambda_1 = -3$, $\lambda_2 = -25$ and the required equation of the straight line is

   $$5x - 2y - 7 = 0$$

5. Statement-2 is true and the statement-1 represents the lines given in statement so statement-1 is also true.

6. See Example 108. Page 15.26.

7. $m_1 + m_2 = -\left(\sqrt{3} + 2\right)$, $m_1 m_2 = \sqrt{3} - 1$

   vertices the triangle are (0, 0), $\left(\frac{1}{m_1}, 1\right)$, $\left(\frac{-1}{m_2}, 1\right)$

   so the area of the triangle is

   $$\frac{1}{2}\left|\begin{vmatrix} 0 & 0 & 1 \\ \frac{1}{m_1} & 1 & 1 \\ \frac{-1}{m_2} & 1 & 1 \end{vmatrix}\right| = \frac{1}{2}\left[\frac{1}{m_1} + \frac{1}{m_2}\right]$$

   $$= \frac{1}{2}\left|\frac{m_1 + m_2}{m_1 m_2}\right| = \frac{1}{2}\left|\frac{-(\sqrt{3} + 2)}{\sqrt{3} - 1}\right| = \frac{1}{2}\left[\frac{\sqrt{3} + 2}{\sqrt{3} - 1}\right]$$

8. Statement-1 is true but the statement-2 is false as the slopes of parallel lines are same.

9. Equation of a line through the point of intersection of the given lines is

   $$(1 + 2\lambda)x + (2 - \lambda)y - (1 + \lambda) = 0$$

   coordinates of A are $\left(\frac{1 + \lambda}{1 + 2\lambda}, 0\right)$ and of B are

$\left(0, \frac{1+\lambda}{2-\lambda}\right)$

If (h, k) is the mid-point of AB, then

$$h = \frac{1+\lambda}{2(1+2\lambda)}, \quad \frac{1+\lambda}{2(2-\lambda)}$$

$$\Rightarrow \quad \lambda = \frac{1+2h}{4h-1} = \frac{4k-1}{2k+1}$$

$$\Rightarrow \quad (1-2h)(2k+1) = (4h-1)(4k-1)$$

$$\Rightarrow \quad 2h + 6k - 20\,hk = 0$$

Locus of (h, k) is $x + 3y = 10xy$.

10. The point of intersection of the given lines is $(-2a, 0)$ and the equation of the line parallel to y-axis passing through this point is $x = -2a$, $(p, 5)$ lies on it so $p = -2a$.

11. $[3x - 4(-3x) - 8]\,[3(3) - 4(4) - 8] < 0$

$\Rightarrow (15x - 8)(-17) < 0 \Rightarrow 15x - 8 > 0$

$\Rightarrow \quad x > \frac{8}{15}$.

Also, $y = -3x < -8/5$.

12. Let m_1 = slope of

$BH = \frac{1-(-2)}{-6-3} = \frac{-1}{3}$

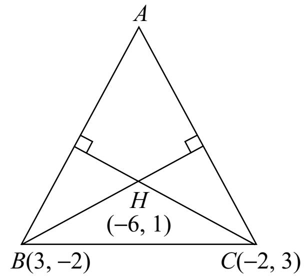


Fig. 16.50

$\Rightarrow$ Slope of $AC = 3$

$\therefore$ Equation of AC is

$y - 3 = 3(x + 2)$ or $3x - y + 9 = 0$ (1)

Let m_2 = slope of $CH = \frac{1-3}{-6+2} = \frac{1}{2}$

$\Rightarrow$ Slope of $AB = -2$

$\therefore$ Equation of AB is $y + 2 = -2(x - 3)$

or $2x + y - 4 = 0$ (2)

As A lies on (1) and (2), coordinates of A are $(-1, 6)$ and it does not lie on $5x + y = 2$.

13. An equation of line through $(1, 2)$ is $y - 2 = m(x - 1)$

A point on this line is of the form $Q(x, mx - m + 2)$.

This point will lie on the line $x + y = 7$ if

$x + mx - m + 2 = 7 \Rightarrow x = \frac{m+5}{m+1}$

As $PQ = 3$, $PQ^2 = 9$

$\Rightarrow (x - 1)^2 + (mx - m)^2 = 9$

$\Rightarrow (x - 1)^2\ 1 + m^2 = 9$

$\Rightarrow 16m^2 + 16 = 9m^2 + 18m + 9$

$\Rightarrow 7m^2 - 18m + 7 = 0$

$\Rightarrow m$ satisfies the equation $7x^2 - 18x + 7 = 0$

CHAPTER SEVENTEEN

Circles and Systems of Circles

DEFINITION OF A CIRCLE

A circle is the locus of a point which moves in a plane, so that its distance from a *fixed point* in the plane is always *constant*.

The fixed point is called the *centre* of the circle and the constant distance is called its *radius*.

EQUATIONS OF A CIRCLE

1. An equation of a circle with centre (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

Illustration 1

A circle of area 9π square units has two of its diameters along the lines $x + y = 5$ and $x - y = 1$. Find the equation of the circle.

Solution: The centre of the circle is the point of intersection of the diameters $x + y = 5$ and $x - y = 1$, which is $(3, 2)$. If r is the radius of the circle, then $\pi r^2 = 9\pi \Rightarrow r = 3$ and the equation of the circle is $(x - 3)^2 + (y - 2)^2 = 3^2$

$$\Rightarrow \quad x^2 + y^2 - 6x - 4y + 4 = 0$$

2. An equation of a circle with centre $(0, 0)$ and radius r is

$$x^2 + y^2 = r^2$$

3. An equation of the circle on the line segment joining (x_1, y_1) and (x_2, y_2) as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Illustration 2

The line $3x + 4y - 12 = 0$ meets the coordinate axes at A and B. Find the equation of the circle drawn on AB as diameter.

Solution: $3x + 4y - 12 = 0$ meets x-axis at $A(4, 0)$ and y-axis at $B(0, 3)$. Equation of the circle on the line joining A and B as diameter is

$$(x - 0)(x - 4) + (y - 0)(y - 3) = 0$$

$$\Rightarrow \quad x^2 + y^2 - 4x - 3y = 0$$

4. General equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where g, f and c are constants.

(i) Centre of this circle is $(-g, -f)$.

(ii) Its radius is $\sqrt{g^2 + f^2 - c}$, $(g^2 + f^2 \ge c)$.

(iii) Length of the intercept made by this circle on the x-axis is $2\sqrt{g^2 - c}$ if $g^2 - c \ge 0$, and that on the y-axis is $2\sqrt{f^2 - c}$ if $f^2 - c \ge 0$.

Illustration 3

Show that $x + y - 1 = 0$ is the equation of a diameter of the circle $x^2 + y^2 - 6x + 4y + c = 0$ for all values of c.

Solution: Centre of the circle is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

i.e $(3, -2)$ which lies on the line $x + y - 1 = 0$ and any line passing through the centre of the circle is a diameter of the circle. Hence $x + y - 1 = 0$ is a diameter of the circle.

5. General equation of second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

in x and y, represents a circle if and only if

(i) coefficient of x^2 equals coefficient of y^2, i.e., $a = b \ne 0$.

(ii) coefficient of xy is zero, i.e., $h = 0$.

(iii) $g^2 + f^2 - ac \ge 0$

Illustration 4

Find the centre and radius of the circle

$3x^2 + (a + 1)y^2 + 6x - 9y + a + 4 = 0$

Solution: Since the given equation represents a circle, co-efficient of x^2 = co-efficient of y^2

$\Rightarrow$ $3 = a + 1 \Rightarrow a = 2$ and the equation of the circle becomes

$$3x^2 + 3y^2 + 6x - 9y + 6 = 0 \text{ or } x^2 + y^2 + 2x - 3y + 2 = 0$$

whose centre is $(-1, 3/2)$ and

$$\text{radius} = \sqrt{1 + (3/2)^2 - 2} = \frac{\sqrt{5}}{2}.$$

SOME RESULTS REGARDING CIRCLES

1. *Position of a point with respect to a circle.* Point $P(x_1, y_1)$ lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, according as $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c >, =$ or < 0.

Illustration 5

Find the values of a for which the point (a, a), $a > 0$, lies outside the circle $x^2 + y^2 - 2x + 6y - 6 = 0$

Solution: The point (a, a) lies outside the given circle if $a^2 + a^2 - 2a + 6a - 6 > 0$

$\Rightarrow 2\,(\text{a}^2 + 2a - 3) > 0 \quad \Rightarrow \ (a + 3)\,(a - 1) > 0$

$\Rightarrow a > 1$ as $a > 0$

2. *Parametric coordinates* of any point on the circle $(x - h)^2 + (y - k)^2 = r^2$ are given by $(h + r\cos\theta, k + r\sin\theta)$, with $0 \le \theta < 2\pi$. In particular, parametric coordinates of any point on the circle $x^2 + y^2 = r^2$ are $(r\cos\theta, r\sin\theta)$ with $0 \le \theta < 2\pi$.

3. An equation of the *tangent* to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x_1, y_1) on the circle is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

4. An equation of the *normal* to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at point (x_1, y_1) on the circle is

$$\frac{y - y_1}{y_1 + f} = \frac{x - x_1}{x_1 + g}$$

5. Equations of the *tangent* and *normal* to the circle $x^2 + y^2 = r^2$ at the point (x_1, y_1) on the circle are, respectively,

$$xx_1 + yy_1 = r^2 \quad \text{and} \quad \frac{x}{x_1} = \frac{y}{y_1}$$

Illustration 6

Find the equation of the tangent to the circle $x^2 + y^2 = 25$ at the point in the first quadrant where the diameter $4x - 3y = 0$ meets the circle.

Solution: The diameter meets the given circle at the point (3, 4) in the first quadrant.

Equation of the tangent to the circle at this point is $x(3) + y(4) = 25 \Rightarrow 3x + 4y = 25$

6. The line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = r^2$ if and only if $c^2 = r^2\,(1 + m^2)$.

7. The lines $y = mx \pm r\sqrt{1 + m^2}$ are tangents to the circle $x^2 + y^2 = r^2$, for all finite values of m. If m is infinite, the tangents are $x \pm r = 0$.

Illustration 7

The line $y = \sqrt{3x + 4}$ touches a circle with centre at the origin. Find the radius of the circle.

Solution: Let the radius of the circle be r, so its equation is $x^2 + y^2 = r^2$. As the line $y = \sqrt{3x + 4}$ touches this circle,

$$(4)^2 = r^2\,(1 + (\sqrt{3}\,)^2) \quad \Rightarrow \ r^2 = 4 \Rightarrow r = 2$$

so the required radius is 2.

8. An equation of the *chord* of the circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$, whose mid-point is (x_1, y_1), is $T = S_1$, where

$$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

and $$S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

In particular, an equation of the chord of the circle $x^2 + y^2 = r^2$, whose mid-point is (x_1, y_1), is $xx_1 + yy_1 = x_1^2 + y_1{}^2$.

9. An equation of the *chord of contact* of the tangents drawn from a point (x_1, y_1) outside the circle $S = 0$, is $T = 0$. (S and T are as defined in (8) above.)

10. *Length of the tangent* drawn from a point (x_1, y_1) outside the circle $S = 0$, to the circle, is $\sqrt{S_1}$. (S and S_1 are as defined in (8) above.)

Illustration 8

Find the length of a tangent drawn from the point (3, 4) to the circle $x^2 + y^2 - 4x + 6y - 3 = 0$.

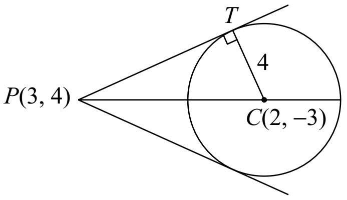


Fig. 17.1

Solution: Centre of the circle is (2, –3) and radius $= \sqrt{2^2 + (-3)^2 + 3} = 4$. So if PT is the tangent from $P(3, 4)$ to the circle, then

$$(PT)^2 = (PC)^2 - (CT)^2$$
$$= (3 - 2)^2 + (4 + 3)^2 - 4^2 = 34.$$

The required length of the tangent is $\sqrt{34}$.

Note

If $S \equiv x^2 + y^2 - 4x + 6y - 3$ then length of the tangent from $P(3,4)$ to this circle is $S = \sqrt{(3)^2 + (4)^2 - 4(3) + 6(4) - 3} = \sqrt{34}$.

11. Two circles with centres $C_1(x_1, y_1)$ and $C_2(x_2, y_2)$, and radii r_1, r_2 respectively,

(i) *touch each other externally* if $|C_1 C_2| = r_1 + r_2$. The point of contact is

$$\left(\frac{r_1 x_2 + r_2 x_1}{r_1 + r_2}, \frac{r_1 y_2 + r_2 y_1}{r_1 + r_2}\right)$$

Illustration 9

Show that the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 - 8x + 12 = 0$ touch each other. Find the point of contact.

Solution: Centre of the first circle is (1, 0) and its radius is 1. Centre of the second circle is (4, 0) and its radius in 2. Since the distance between the centres = 3, which is equal to the sum of the radii. The two circles touch each other externally and the point of contact is the point (2, 0) which divides the join of (1, 0) and (4, 0) in the ratio 1 : 2.

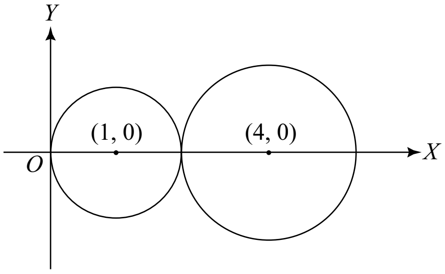


Fig. 17.2

(ii) *touch each other internally* if $|C_1 C_2| = |r_1 - r_2|$, $r_1 \neq r_2$. The point of contact is

$$\left(\frac{r_1 x_2 - r_2 x_1}{r_1 - r_2}, \frac{r_1 y_2 - r_2 y_1}{r_1 - r_2}\right)$$

12. An equation of the *family of circles* passing through the points (x_1, y_1) and (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

13. An equation of the family of circles which touch the line $y - y_1 = m(x - x_1)$ at (x_1, y_1) for any finite value of m, is

$$(x - x_1)^2 + (y - y_1)^2 + \lambda[(y - y_1) - m(x - x_1)] = 0$$

If m is infinite, equation becomes $(x - x_1)^2 + (y - y_1)^2 + \lambda(x - x_1) = 0$.

14. Let QR be a chord of a circle passing through the point $P(x_1, y_1)$, and let the tangents at the extremities Q and R of this chord intersect at the point $L(h, k)$ (Fig. 17.3). Then locus of L is called as *polar* of P with respect to the circle, and P is called the *pole* of its polar.

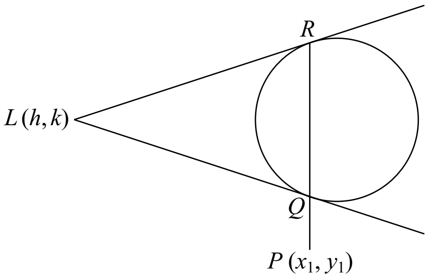


Fig. 17.3

(i) *Equation of the polar* of $P(x_1, y_1)$ with respect to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, is $T = 0$, where T is as defined in (8) above.

(ii) If the polar of P with respect to a circle passes through Q, then the polar of Q with respect to the same circle passes through P. Two such points P and Q are called *conjugate points* of the circle.

15. If lengths of the tangents drawn from a point P to the two circles

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and

$$S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

are equal, then the locus of P is called the *radical axis* of the two circles $S_1 = 0$ and $S_2 = 0$, and its equation is $S_1 - S_2 = 0$, i.e.,

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

(i) Radical axes of two circles is perpendicular to the line joining their centres.

(ii) Radical axes of three circles, taken in pairs, pass through a fixed point called the *radical centre* of the three circles, if the centres of these circles are non-collinear.

SPECIAL FORMS OF EQUATION OF A CIRCLE

1. An equation of a circle with centre (r, r), radius $|r|$ is $(x - r)^2 + (y - r)^2 = r^2$. This touches the coordinate axes at the points $(r, 0)$ and $(0, r)$.

Illustration 10

A circle of radius 3 units touches both the axes. Find its equation.

Solution: As the circle touches both the axes, the distance of the centre from both the axes in 3 units. So the centre of the circle can be (± 3, ± 3) and hence there are four circles with radius 3, touching both the axes and their equations are

$$x^2 + y^2 \pm 6x \pm 6y + 9 = 0.$$

2. An equation of a circle with centre (x_1, r), radius $|r|$ is $(x - x_1)^2 + (y - r)^2 = r^2$. This touches the x-axis at $(x_1, 0)$.

3. An equation of a circle with centre (r, y_1), radius $|r|$ is $(x - r)^2 + (y - y_1)^2 = r^2$. This touches the y-axis at $(0, y_1)$.

4. An equation of a circle with centre $(a/2, b/2)$ and radius $\sqrt{(a^2 + b^2)/4}$ is

$$x^2 + y^2 - ax - by = 0$$

This circle passes through the origin (0, 0), and has intercepts a and b on the x and y axes, respectively.

SYSTEMS OF CIRCLES

Let $S \equiv x^2 + y^2 + 2gx + 2fy + c$, $S' \equiv x^2 + y^2 + 2g'x + 2f'y + c'$ and $L \equiv ax + by + k'$.

1. If two circles $S = 0$ and $S' = 0$ intersect at real and distinct points, then $S + \lambda S' = 0$ $(\lambda \neq -1)$ represents a *family of circles* passing through these points (λ being a parameter),

and $S - S' = 0$ (for $\lambda = -1$) represents the *common chord* of the circles.

2. If two circles $S = 0$ and $S' = 0$ touch each other, then $S - S' = 0$ represents equation of the *common tangent* to the two circles at their point of contact.

3. If two circles $S = 0$ and $S' = 0$ intersect each other *orthogonally* (the tangents at the point of intersection of the two circles are at right angles), then

$$2gg' + 2ff' = c + c'.$$

4. If the circle $S = 0$ intersects the line $L = 0$ at two real and distinct points, then $S + \lambda L = 0$ represents a family of circles passing through these points.

Illustration 11

Find the equation of the circle described on the chord $3x + y + 5 = 0$ of the circle $x^2 + y^2 = 16$ as diameter.

Solution: Equation of any circle passing through the points of intersection of the chord and the circle is

$$x^2 + y^2 - 16 + \lambda(3x + y + 5) = 0$$

The chord $3x + y + 5 = 0$ is a diameter of this circle if the centre $\left(\frac{-3\lambda}{2}, \frac{-\lambda}{2}\right)$ of the circle lies on the chord.

$$\Rightarrow \quad 3\left(\frac{-3\lambda}{2}\right) - \frac{\lambda}{2} + 5 = 0$$

$\Rightarrow$ $\lambda = 1$ and the required equation of the circle is $x^2 + y^2 + 3x + y - 11 = 0$.

5. If $L = 0$ is a tangent to the circle $S = 0$ at P, then $S + \lambda L = 0$ represents a family of circles touching $S = 0$ at P, and having $L = 0$ as the common tangent at P.

6. *Coaxial Circles* A system of circle is said to be coaxial if every pair of circles of the system have the same radical axis.

The simplest form of the equation of a coaxial system of circles is $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is constant, the common radical axis of the system being y-axis and the line of centres being x-axis.

The Limiting points of the coaxial system of circles are the members of the system which are of zero radius. Thus the limiting points of the coaxial system of circles

$$x^2 + y^2 + 2gx + c = 0 \text{ are } \left(\pm\sqrt{c}, 0\right) \text{ if } c \geq 0.$$

The equation $S + \lambda S' = 0$ ($\lambda \neq -1$) represents a family of coaxial circles, two of whose members are given to be $S = 0$ and $S' = 0$.

Conjugate systems (or orthogonal systems) of circles Two system of circles such that every circle of one system cuts every circle of the other system orthogonally are said to be conjugate system of circles. For instance,

$$x^2 + y^2 + 2gx + c = 0 \qquad \text{and} \qquad x^2 + y^2 + 2fy - c = 0,$$

where g and f are variables and c is constant, represent two systems of coaxial circles which are conjugate.

COMMON TANGENTS TO TWO CIRCLES

If $(x - g_1)^2 + (y - f_1)^2 = a_1^2$ and $(x - g_2)^2 + (y - f_2)^2 = a_2^2$ are two circles with centres C_1 (g_1, f_1) and C_2 (g_2, f_2) and radii a_1 and a_2 respectively, then we have the following results regarding their common tangents.

1. When $C_1 C_2 > a_1 + a_2$ i.e., distance between the centres is greater than the sum of their radii, the two circles do not intersect with each other and *four* common tangents can be drawn to two circles. Two of them are *direct common tangents* and the other two are *transverse common tangents*. The points T_1, T_2 of intersection of direct common tangents and transverse common tangents respectively, always lie on the line joining the centres of the two circles and divide it externally and internally respectively in the ratio of their radii.

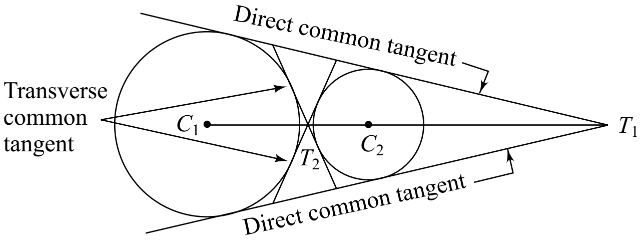


Fig. 17.4

$$\frac{C_1T_1}{T_1C_2} = \frac{a_1}{a_2} \text{ (externally)} \quad \frac{C_1T_2}{T_2C_2} = \frac{a_1}{a_2} \text{ internally}$$

2. When $C_1 C_2 = a_1 + a_2$ i.e., the distance between the centres is equal to the sum of the radii, the two circles touch each other externally, two direct common tangents are real and distinct and the transverse common tangents coincide.

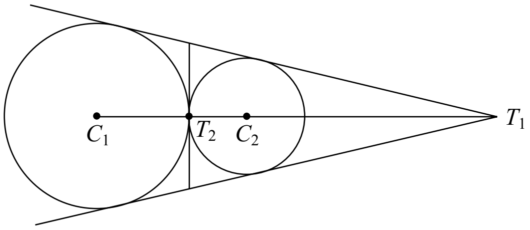


Fig. 17.5

3. When $C_1 C_2 < a_1 + a_2$ i.e., the distance between the centres is less than the sum of the radii, the circles intersect at two real and distinct points, the two direct common tangents are real and distinct while the transverse common tangents are imaginary.

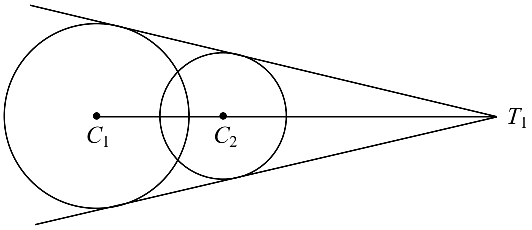


Fig. 17.6

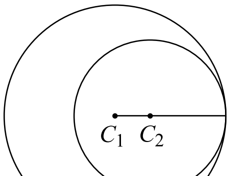


Fig. 17.7

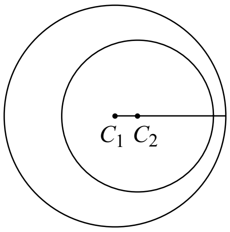


Fig. 17.8

4. When $C_1\ C_2 = |a_1 - a_2|$ $(a_1 \neq a_2)$ i.e., the distance between the centres is equal to the difference of their radii, the circles touch each other internally, two direct common tangents are real and coincident while the transverse common tangents are imaginary. (Fig 16.7)
5. When $C_1\ C_2 < |a_1 - a_2|$ $a_1 \neq a_2$ i.e., the distance between the centres is less than the difference of the radii, one circle with smaller radius lies inside the other and the four common tangents are all imaginary. (Fig 16.8)

Illustration 12

Find the number of common tangents to the circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2y = 0$.

Solution: Centre of the circle $x^2 + y^2 = 16$ is (0, 0) and its radius is 4. Centre of the circle $x^2 + y^2 - 2y = 0$ is (0, 1) and its radius is 1.

Distance between the centre = 1 which is less than the difference between the radii.

Second circle lies inside the first circle so there is no real common tangent to the given circles.

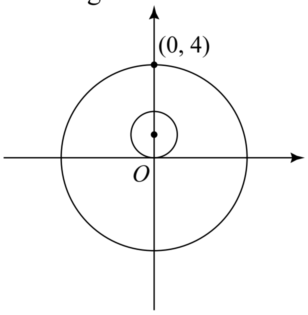


Fig. 17.9

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: If each of the lines $5x + 8y = 13$ and $4x - y = 3$ contains a diameter of the circle $x^2 + y^2 - 2(a^2 - 7a + 11)x - 2(a^2 - 6a + 6)y + b^3 + 1 = 0$, then

(a) $a = 5$ and $b \notin (-1, 1)$
(b) $a = 1$ and $b \notin (-1, 1)$
(c) $a = 2$ and $b \in (-\infty, 1)$
(d) $a = 5$ and $b \in (-\infty, 1)$

Ans. (d)

Solution: The point of intersection (1, 1) of the given lines is the centre of the circle.

$\Rightarrow \quad a^2 - 7a + 11 = 1,\ a^2 - 6a + 6 = 1$

$\Rightarrow \quad a = 5$ and the equation of the circle is $x^2 + y^2 - 2x - 2y + b^3 + 1 = 0$

so radius of the circle = $\sqrt{1 + 1 - (b^3 + 1)} = \sqrt{1 - b^3}$

For radius to be real $b^3 < 1 \Rightarrow b \in (-\infty, 1)$

Example 2: If a circle touches the axis of x at (5, 0) and passes through the point (4, – 1), it also passes through the point

(a) (–1, 6)	(b) (6, –1)
(c) (1, 6)	(d) (6, 1)

Ans. (b)

Solution: Centre of the circle is on the line $x = 5$ at a distance equal to the radius of the circle from the axis of x.

Let the coordinates of the centre be $(5, k)$, so the equation of the circle is

$$(x - 5)^2 + (y - k)^2 = k^2$$

As it passes through the point (4, – 1)

$$(4 - 5)^2 + (-1 - k)^2 = k^2 \quad \Rightarrow \quad k = -1$$

and the equation of the circle is

$$(x-5)^2+(y+1)^2=1$$

which passes through the point (6, – 1)

Note

(6, – 1) is the other end of the diameters through (4, –1) of the circle.

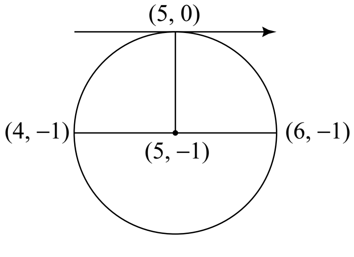


Fig. 17.10

◉ **Example 3:** For all values of a, equation of a diameter of the circle $x^2+y^2-2ax+2ay+a^2=0$ is

(a) $x+y=a$ (b) $x-y=a$
(c) $x+y=2a$ (d) $x-y=2a$.

Ans. (d)

◎ **Solution:** Centre of the circle is $(a, -a)$ and the line $x-y=2a$ passes through it. So it is a diameter of the given circle.

◉ **Example 4:** S: $x^2+y^2-6x+4y-3=0$ is a circle and L : $4x+3y+19=0$ is a straight line.

(a) L is a chord of S. (b) L is a diameter of S
(c) L is a tangent to S (d) none of these

Ans. (d)

◎ **Solution:** Centre of S is (3, –2) and its radius is $\sqrt{3^2+(-2)^2+3}=4$

Length of the perpendicular of (3, – 2) from the line L is

$$\left|\frac{4(3)+3(-2)+19}{\sqrt{(4)^2+(3)^2}}\right|=5$$ which is greater than the radius of S. So the line L lies outside the circle S.

◉ **Example 5:** Circles $x^2+y^2+2x-8y-8=0$ and $x^2+y^2+2x-6y-6=0$

(a) touch each other internally at (–1, – 1)
(b) touch each other externally at (–1, –1)
(c) intersect each other at (–1, –1)
(d) none of these.

Ans. (a)

◎ **Solution:** Centre of the first circle is (–1, 4) and its radius is $\sqrt{1+16+8}=5$. Centre of the second circle is (– 1, 3) and its radius is $\sqrt{1+9+6}=4$.

Distance between the centres = 1 = difference between the radii, so the two circles touch each other internally at a common point (– 1, –1)

◉ **Example 6:** Vertices of an isosceles triangle of area a^2 are $(-a, 0)$ and $(a, 0)$. Equation of the circumcircle of the triangle is

(a) $x^2+y^2+2ax-2ay+a^2=0$
(b) $x^2+y^2-2ax+2ay+a^2=0$
(c) $x^2+y^2=a^2$
(d) none of these

Ans. (c)

◎ **Solution:** Since the triangle is isosceles, third vertex lies on the line $x=0$, perpendicular to the base and passing through the mid-point (0, 0) of the base. As the area is a^2, distance of the vertex from the base is a as the length of the base is $2a$. So vertex of the triangle is $(0, \pm a)$ and let the equation of the circle passing through the vertices of the triangle $x^2+y^2+2gx+2fy+c=0$, then $a^2+2ga+c=0$, $a^2-2ga+c=0$

and $a^2\pm 2fa+c=0$.

$\Rightarrow$ $c=-a^2$, $g=f=0$ and the equation of the required circle is $x^2+y^2=a^2$.

◉ **Example 7:** If the circle $x^2+y^2-6x-8y+(25-a^2)=0$ touches the axis of y, then a equals.

(a) 0 (b) ±4
(c) ±2 (d) ±3

Ans. (d)

◎ **Solution:** If the circle touches the axis of y, distance of its centre (3, 4) from the axis of y is equal to the radius $\sqrt{(3)^2+(4)^2-(25-a^2)}$ of the circle.

$$\Rightarrow\quad 3=\sqrt{a^2}\quad\Rightarrow\quad a=\pm 3$$

◉ **Example 8:** A circle is described on the line joining the points (2, – 3) and (– 4, 7) as a diameter. The circle

(a) passes through the origin
(b) touches the axis of x.
(c) touches the axis of y
(d) origin lies inside the circle

Ans. (d)

◎ **Solution:** Equation of the circle is

$$(x-2)(x+4)+(y+3)(y-7)=0$$

$$\Rightarrow\quad x^2+y^2+2x-4y-29=0.$$

Centre is (1, – 2), radius = $\sqrt{34}$

Distance of the centre from the origin is $\sqrt{5}<\sqrt{34}$.

So the origin lies inside the circle.

◉ **Example 9:** A circle with centre at the centroid of the triangle with vertices (2, 3), (6, 7) and (7, 5) passes through the origin, radius of the circle in units is

(a) 5 (b) $5\sqrt{2}$
(c) 4 (d) none of these

Ans. (b)

◎ **Solution:** Centroid of the triangle is $\left(\frac{2+6+7}{3}, \frac{3+7+5}{3}\right)$ = (5, 5). Let the equation of the circle be $(x-5)^2 + (y-5)^2 = r^2$, r being the radius.

$\Rightarrow\ x^2 + y^2 - 10x - 10y + 50 - r^2 = 0$ which passes through the origin if $r^2 = 50 \Rightarrow r = 5\sqrt{2}$.

◉ **Example 10:** $S: x^2 + y^2 - 8x + 10y = 0$ and $L: x - y - 9 = 0$ are the equations of a circle and a line.

(a) L is a normal to the circle S.
(b) S is the only circle having radius $\sqrt{41}$ and a diameter along L.
(c) L is a tangent to the circle S.
(d) L does not intersect the circle S.

Ans. (a)

◎ **Solution:** L passes through the centre (4, –5) of the circle S and hence is a diameter and every diameter is a normal to the circle.

◉ **Example 11:** A circle has its centre on the y-axis and passes through the origin, touches another circle with centre (2, 2) and radius 2, then the radius of the circle is

(a) 1 (b) 1/2
(c) 1/3 (d) 1/4.

Ans. (b)

◎ **Solution:** Let the centre of the circle be (0, k). As it passes through the origin, its radius is k.
Since it touches the second circle

$$\sqrt{(0-2)^2 + (k-2)^2} = k + 2$$

$$\Rightarrow (k+2)^2 - (k-2)^2 = 4 \Rightarrow k = \frac{1}{2}$$

◉ **Example 12:** If the point (3, 4) lies inside and the point (– 3, – 4) lies outside the circle $x^2 + y^2 - 7x + 5y - p = 0$, then the set of all possible values of p is

(a) (24, 25) (b) (25, 26)
(c) (24, 26) (d) (0, 24)

Ans. (c)

◎ **Solution:** (3, 4) lies inside the circle
$\Rightarrow\ (3)^2 + (4)^2 - 7(3) + 5(4) - p < 0 \ \Rightarrow\ 24 < p$
(–3, – 4) lies outside the circle
$\Rightarrow\ (-3)^2 + (-4)^2 - 7(-3) + 5(-4) - p > 0.$
$\Rightarrow\ p < 26.$
So $p \in (24, 26)$

◉ **Example 13:** The mid point of chord by the circle $x^2 + y^2 = 16$ on the line $x + y + 1 = 0$ is

(a) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (b) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$
(c) $\left(\frac{1}{2}, \frac{-3}{2}\right)$ (d) $\left(\frac{3}{4}, -\frac{7}{4}\right)$

Ans. (b)

◎ **Solution:** The line meets the circle at points whose x-coordinates are the roots of the equation

$$x^2 + (x+1)^2 = 16 \ \Rightarrow\ 2x^2 + 2x - 15 = 0$$

$$\Rightarrow\ x_1 + x_2 = -1$$

So if (h, k) is the required point.

$$h = \frac{x_1 + x_2}{2} = -\frac{1}{2} \text{ and } h + k + 1 = 0 \ \Rightarrow\ k = -\frac{1}{2}$$

and the required point is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

◉ **Example 14:** If a chord of a circle $x^2 + y^2 = 25$ with one extremity at (4, 3) subtends a right angle at the centre of this circle, then the coordinates of the other extremity of this chord can be.

(a) (– 3, –4) (b) (4, –3)
(c) (3, 4) (d) (–3, 4)

Ans. (d)

◎ **Solution:** Let P be the point (4, 3) and Q (h, k) be the other extremity of the chord through P. Centre of the circle is the origin 0. Then OP is perpendicular to OQ.

$$\Rightarrow\ \frac{k}{h} \times \frac{3}{4} = -1 \ \Rightarrow\ \frac{k}{h} = -\frac{4}{3}$$

which is satisfied by (h, k) = (–3, 4)

◉ **Example 15:** Equation of a common chord of the circles $x^2 + y^2 + 6x - 10y + 9 = 0$ and $x^2 + y^2 - 10x + 6y + 25 = 0$ is

(a) $x + y + 4 = 0$ (b) $x - y + 4 = 0$
(c) $x + y + 1 = 0$ (d) $x - y - 1 = 0$

Ans. (d)

◎ **Solution:** Equation of the common chord is
$x^2 + y^2 + 6x - 10y + 9 - (x^2 + y^2 - 10x + 6y + 25) = 0$
$\Rightarrow\ x - y - 1 = 0$

◉ **Example 16:** $x + ay = a^2 + 1$ is a tangent to the circle $x^2 + y^2 = 10$ for

(a) any values of a (b) only one value of a
(c) two values of a (d) no value of a.

Ans. (c)

◎ **Solution:** If $x + ay = a^2 + 1$ touches the circle, length of the perpendicular of the centre (0, 0) of the circle from the line is $\sqrt{10}$, the radius of the circle.

$$\Rightarrow\ \left|\frac{0 + 0.9 - (a^2+1)}{\sqrt{1+a^2}}\right| = \sqrt{10}$$

$$\Rightarrow\ \sqrt{1+a^2} = \sqrt{10} \ \Rightarrow\ a^2 = 9 \ \Rightarrow a = \pm 3$$

◉ **Example 17:** If the circles $x^2 + y^2 + 5x - 6y - 1 = 0$ and $x^2 + y^2 + ax - y + 1 = 0$ intersect orthogonally (the tangents at the point of intersection of the circles are at right angles), the value of a is

(a) 6/5 (b) 5/6
(c) – 6/5 (d) – 5/6

Ans. (c)

Solution: Let $C_1\left(-\frac{5}{2}, 3\right)$ be the centre and $r_1 = \sqrt{\left(-\frac{5}{2}\right)^2 + (3)^2 + 1}$ be the radius of the first circle and $C_2\left(-\frac{a}{2}, \frac{1}{2}\right)$ be the centre and $r_2 = \sqrt{\left(-\frac{a}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 1}$ be the radius of the second circle. If P is a point of intersection of the two circles then $(C_1 P)^2 + (C_2 P)^2 = (C_1 C_2)^2$

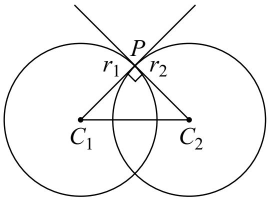


Fig. 17.11

$$\Rightarrow \left(-\frac{5}{2}\right)^2 + (3)^2 + 1 + \left(-\frac{a}{2}\right)^2 + \left(\frac{1}{2}\right)^2 - 1 = \left(\frac{-5}{2} + \frac{a}{2}\right)^2 + \left(3 - \frac{1}{2}\right)^2$$

$$\Rightarrow \quad a = -\frac{6}{5}$$

Note

We could also use $2gg' + 2ff' = c + c'$

Example 18: Length of a tangent drawn from the origin to the circle $x^2 + y^2 - 6x + 4y + 8 = 0$ in units is

(a) 4 (b) 6
(c) 8 (d) $2\sqrt{2}$

Ans. (d)

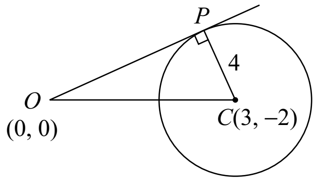


Fig. 17.12

Solution: If P is the point of contact of the tangent from O and C is the centre of the circle, then

$$(OP)^2 = (OC)^2 - (CP)^2$$
$$= 3^2 + (-2)^2 - (3^2 + (-2)^2 - 8) = 8$$

Example 19: A circle passing through the intersection of the circles $x^2 + y^2 + 5x + 4 = 0$ and $x^2 + y^2 + 5y - 4 = 0$ also passes through the origin. The centre of the circle is

(a) $\left(\frac{5}{2}, \frac{5}{2}\right)$ (b) $\left(\frac{5}{4}, \frac{5}{4}\right)$
(c) $\left(-\frac{5}{4}, -\frac{5}{4}\right)$ (d) $\left(-\frac{5}{2}, -\frac{5}{2}\right)$

Ans. (c)

Solution: Let the equation of the circle be
$x^2 + y^2 + 5x + 4 + \lambda (x^2 + y^2 + 5y - 4) = 0$
It passes through the origin if $4 - 4\lambda = 0$
$\Rightarrow \quad \lambda = 1$ and the equation of the circle is
$2x^2 + 2y^2 + 5x + 5y = 0$
Centre of the circle is $\left(-\frac{5}{4}, -\frac{5}{4}\right)$

Example 20: The locus of the point from which mutually perpendicular tangents can be drawn to the circle $x^2 + y^2 = 36$ is

(a) $x^2 + y^2 = 42$ (b) $x^2 + y^2 = 48$
(c) $x^2 + y^2 = 60$ (d) $x^2 + y^2 = 72$.

Ans. (d)

Solution: $y = mx \pm 6\sqrt{1 + m^2}$ is the equation of any tangent to the circle. If it passes through (h, k), then $k = mh \pm 6\sqrt{1 + m^2}$

$\Rightarrow \quad (k - mh)^2 = 36\,(1 + m^2)$
$\Rightarrow \quad (36 - h^2)\, m^2 + 2mhk + (36 - k^2) = 0$

which gives two values of m say m_1 and m_2, slopes of two tangents passing thought (h, k). These tangents are perpendicular

$$\text{If} \quad m_1 m_2 = -1 \;\Rightarrow\; \frac{36 - k^2}{36 - h^2} = -1$$

$\Rightarrow \quad h^2 + k^2 = 72 \quad \Rightarrow$ Locus of (h, k) is $x^2 + y^2 = 72$.

LEVEL 1

Straight Objective Type Questions

Example 21: Four distinct points (2, 3), (0, 2), (4, 5) and (0, t) are concyclic if the value of t is

(a) – 2 (b) 2
(c) 17 (d) – 17

Ans. (c)

Solution: An equation of a circle through (2, 3) and (4, 5) is

$$(x - 2)(x - 4) + (y - 3)(y - 5) + \lambda \begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 0$$

It will pass through (0, 2) if

$$(-2)(-4)+(-1)(-3)+\lambda\begin{vmatrix}0 & 2 & 1\\2 & 3 & 1\\4 & 5 & 1\end{vmatrix}=0$$

$\Rightarrow \quad 11 + 2\lambda = 0$

$\Rightarrow \quad \lambda = -11/2$ and it will pass through $(0, t)$ if

$$(-2)(-4)+(t-3)(t-5)-\frac{11}{2}\begin{vmatrix}0 & t & 1\\2 & 3 & 1\\4 & 5 & 1\end{vmatrix}=0$$

$\Rightarrow \quad t^2 - 19t + 34 = 0$

$\Rightarrow \quad t = 2, 17$

◉ **Example 22:** The locus of the middle points of the chords of the circle $x^2 + y^2 = 4a^2$ which subtend a right angle at the centre of the circle is

(a) $x + y = 2a$ (b) $x^2 + y^2 = a^2$
(c) $x^2 + y^2 = 2a^2$ (d) $x^2 + y^2 = x + y$

Ans. (c)

◎ **Solution:** Let $M(h, k)$ be the middle point of the chord AB of the given circle $x^2 + y^2 = 4a^2$, with centre at O (0, 0) and radius equal to $2a$.

Then OM is perpendicular to AB

Since AOB is a right angled triangle

$$4(AM)^2 = (AB)^2 = (OA)^2 + (OB)^2$$
$$= (2a)^2 + (2a)^2 = 8a^2$$

$\Rightarrow \quad AM = \sqrt{2}\ a$

Also $\quad (OA)^2 = (OM)^2 + (AM)^2$

$\Rightarrow \quad (2a)^2 = h^2 + k^2 + 2a^2$

$\Rightarrow \quad h^2 + k^2 = 2a^2$

$\Rightarrow \quad$ Locus of (h, k) is $x^2 + y^2 = 2a^2$.

◉ **Example 23:** Two tangents are drawn from the origin to a circle with centre at (2, – 1). If the equation of one of the tangents is $3x + y = 0$, the equation of the other tangent is

(a) $3x - y = 0$ (b) $x + 3y = 0$
(c) $x - 3y = 0$ (d) $x + 2y = 0$

Ans. (c)

◎ **Solution:** Let the equation of the other tangent from the origin be $y = mx$, then length of the perpendiculars from the centre (2, –1) on the two tangents is same.

$$\Rightarrow \quad \left|\frac{2m+1}{\sqrt{1+m^2}}\right| = \left|\frac{6-1}{\sqrt{9+1}}\right| = \frac{5}{\sqrt{10}}$$

$\Rightarrow \quad 10(2m + 1)^2 = 25(1 + m^2)$

$\Rightarrow \quad 3m^2 + 8m - 3 = 0$

$\Rightarrow \quad (3m - 1)(m + 3) = 0$

$\Rightarrow \quad m = -3$ or $1/3$.

$m = -3$ represents the given tangent hence the slope of the required tangent is 1/3 and its equation is $y = (1/3)x$

$\Rightarrow \quad x - 3y = 0.$

◉ **Example 24:** A line meets the coordinate axes in A and B. A circle is circumscribed about the triangle OAB. If the distances from A and B of the tangent to the circle at the origin O be m and n, then the diameter of the circle is

(a) $m(m + n)$ (b) $m + n$
(c) $n(m + n)$ (d) $m^2 + n^2$

Ans. (b)

◎ **Solution:** Let the coordinates of A be $(a, 0)$ and of B be $(0, b)$, then AOB being a right angled triangle AB is a diameter of the circle, so equation of the circle is $(x - a)(x - 0) + (y - b)(y - 0) = 0$

$$\Rightarrow x^2 + y^2 - ax - by = 0$$

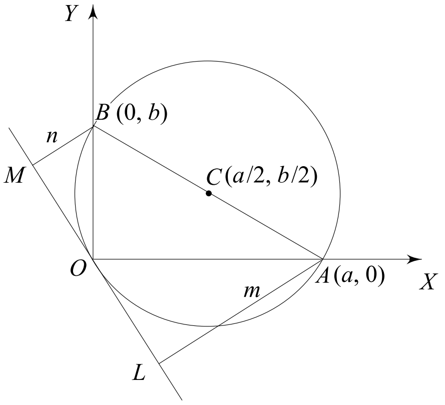


Fig. 17.13

Equation of the tangent at the origin is $ax + by = 0$ (i)

Let AL and BM be the perpendicular from A and B on (i)

$$\text{then} \quad AL = \left|\frac{a^2}{\sqrt{a^2+b^2}}\right| = m \text{ and } BM = \left|\frac{b^2}{\sqrt{a^2+b^2}}\right| = n$$

$\Rightarrow \quad m + n = \sqrt{a^2 + b^2}$ = diameter of the circle.

◉ **Example 25:** A variable chord of the circle $x^2 + y^2 - 2ax = 0$ is drawn through the origin. Locus of the centre of the circle drawn on this chord as diameter is

(a) $x^2 + y^2 + ax = 0$ (b) $x^2 + y^2 + ay = 0$
(c) $x^2 + y^2 - ax = 0$ (d) $x^2 + y^2 - ay = 0$

Ans. (c)

◎ **Solution:** Let (h, k) be the centre of the required circle then (h, k) is the middle point of the chord say OA of the given circle $x^2 + y^2 - 2ax = 0$, O being the origin.

$\Rightarrow \quad A(2h, 2k)$ lies on the circle

$\Rightarrow \quad h^2 + k^2 - ah = 0$

$\Rightarrow \quad$ Locus of (h, k) is $x^2 + y^2 - ax = 0$.

◉ **Example 26:** If a circle passes through the point (3, 4) and cuts the circle $x^2 + y^2 = a^2$ orthogonally, the equation of the locus of its centre is

(a) $3x + 4y - a^2 = 0$
(b) $6x + 8y = a^2 + 25$
(c) $6x + 8y + a^2 + 25 = 0$
(d) $3x + 4y = a^2 + 25$

Ans. (b)

◎ **Solution:** Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{(i)}$$

Since it passes through (3, 4)

$$9 + 16 + 6g + 8f + c = 0$$

$$\Rightarrow \quad 6g + 8f + c = -25 \qquad \text{(ii)}$$

As (i) cuts the circle $x^2 + y^2 = a^2$ orthogonally

$$2g \times 0 + 2f \times 0 = c - a^2$$

$$\Rightarrow \quad c = a^2.$$

So from (ii) we get

$$6g + 8f + a^2 + 25 = 0$$

Hence locus of the centre $(-g, -f)$ is

$$6x + 8y - (a^2 + 25) = 0.$$

◉ **Example 27:** An equation of the circle passing through the origin, having its centre on the line $x + y = 4$ and cutting the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally is

(a) $x^2 + y^2 - 2x - 6y = 0$
(b) $x^2 + y^2 - 6x - 2y = 0$
(c) $x^2 + y^2 - 4x - 4y = 0$
(d) $x^2 + y^2 - 8x = 0$

Ans. (c)

◎ **Solution:** Let the centre of the circle lying on the line $x + y = 4$ be $(g, 4 - g)$. Since the required circle passes through the origin, equation of the circle is

$$x^2 + y^2 - 2gx - 2(4 - g)y = 0$$

Since it cuts the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally,

$$-2g(-2) - 2(4 - g)(1) = 4$$

$$\Rightarrow \quad 6g = 12 \quad \Rightarrow \quad g = 2$$

Hence an equation of the required circle is

$$x^2 + y^2 - 4x - 4y = 0.$$

◉ **Example 28:** If O is the origin and OP, OQ are the tangents from the origin to the circle $x^2 + y^2 - 6x + 4y + 8 = 0$, the circumcentre of the triangle OPQ is

(a) (3, –2) (b) (3/2, – 1)
(c) (3/4, – 1/2) (d) (– 3/2, 1)

Ans. (b)

◎ **Solution:** We note that PQ is the chord of contact of the tangents from the origin to the circle

$$x^2 + y^2 - 6x + 4y + 8 = 0 \qquad \text{(i)}$$

Equation of PQ is $3x - 2y - 8 = 0$ (ii)

Equation of a circle passing through the intersection of (i) and (ii) is

$$x^2 + y^2 - 6x + 4y + 8 + \lambda(3x - 2y - 8) = 0 \qquad \text{(iii)}$$

If this represents the circumcircle of the triangle OAB, it passess through O (0, 0), so $\lambda = 1$ and the equation (iii) becomes $x^2 + y^2 - 3x + 2y = 0$.

So that the required coordinates of the centre are $(3/2, -1)$

◉ **Example 29:** The circle passing through three distinct points $(1, t)$, $(t, 1)$ and (t, t) passes through the point

(a) (1, 1) (b) (–1, –1)
(c) (–1, 1) (d) (1, –1)

for all values of t

Ans. (a)

◎ **Solution:** Equation of a circle passing through $(1, t)$ and $(t, 1)$ is

$$(x - 1)(x - t) + (y - t)(y - 1) + \lambda \begin{vmatrix} x & y & 1 \\ 1 & t & 1 \\ t & 1 & 1 \end{vmatrix} = 0$$

It passes through (t, t) if $0 + 0 + \lambda \begin{vmatrix} t & t & 1 \\ 1 & t & 1 \\ t & 1 & 1 \end{vmatrix} = 0$

$$\Rightarrow \quad \lambda = 0$$

$\Rightarrow$ The circle through the given points is $(x - 1)(x - t) + (y - t)(y - 1) = 0$, which clearly passes through (1,1)

◉ **Example 30:** If OA and OB are tangents from the origin O, to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, $c > 0$ and C is the centre of the circle, then area of the quadrilateral $OACB$ is

(a) $\frac{1}{2}\sqrt{c(g^2 + f^2 - c)}$ (b) $\sqrt{c(g^2 + f^2 - c)}$

(c) $c\sqrt{g^2 + f^2 - c}$ (d) $\sqrt{\frac{g^2 + f^2 - c}{c}}$

Ans. (b)

◎ **Solution:** Since $OA = OB = \sqrt{c}$ (length of the tangent from the origin to the circle) and $CA = CB = \sqrt{g^2 + f^2 - c}$

Area of the quadrilateal $OACB$

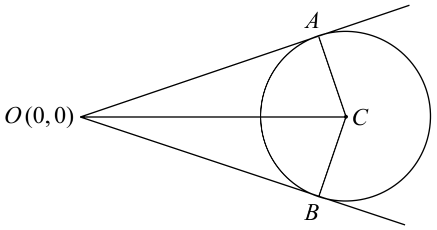


Fig. 17.14

$$= 2 \text{ Area of the triangle } OAC$$

$$= 2 \times (1/2)\, OA \times CA$$

$$= \sqrt{c}\,\sqrt{g^2 + f^2 - c}$$

◉ **Example 31:** If the line $y = x + 3$ meets the circle $x^2 + y^2 = a^2$ at A and B, then equation of the circle on AB as diameter is

(a) $x^2 + y^2 + 3x - 3y - a^2 + 9 = 0$
(b) $x^2 + y^2 - 3x + 3y - a^2 - 9 = 0$
(c) $x^2 + y^2 + 3x + 3y + a^2 - 9 = 0$
(d) $x^2 + y^2 - 3x + 3y - a^2 + 9 = 0$

Ans. (a)

◎ **Solution:** Equation of any circle passing through the intersection of the given circle and the given line is $x^2 + y^2 - a^2 + \lambda(y - x - 3) = 0$ (i)

If AB is a diameter of the circle, the centre $\left(\frac{\lambda}{2}, -\frac{\lambda}{2}\right)$ of (i) lies on the given line $y - x - 3 = 0$

$$\Rightarrow \quad -\frac{\lambda}{2} - \frac{\lambda}{2} - 3 = 0 \quad \Rightarrow \quad \lambda = -3.$$

So that equation of the required circle is

$$x^2 + y^2 + 3x - 3y - a^2 + 9 = 0.$$

◉ **Example 32:** The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$, and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. The radius of the circle with AB as diameter is

(a) $\sqrt{a^2 + b^2 + p^2 + q^2}$

(b) $\sqrt{a^2 + p^2}$

(c) $\sqrt{b^2 + q^2}$

(d) none of these

Ans. (a)

◎ **Solution:** Let the coordinates of A and B be (x_1, y_1) and (x_2, y_2), respectively. From the given conditions, we then have $x_1 + x_2 = -2a$, $x_1x_2 = -b^2$, $y_1 + y_2 = -2p$ and $y_1y_2 = -q^2$. Therefore the required radius is

$$\begin{aligned}(1/2)\,AB &= (1/2)\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\ &= (1/2)\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2 - 4(x_1\,x_2 + y_1\,y_2)} \\ &= (1/2)\sqrt{4a^2 + 4p^2 + 4(b^2 + q^2)} \\ &= \sqrt{a^2 + b^2 + p^2 + q^2}\;.\end{aligned}$$

◉ **Example 33:** If two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g_1x + 2f_1y = 0$ touch each other, then

(a) $f_1g = fg_1$ (b) $ff_1 = gg_1$

(c) $f^2 + g^2 = f_1^2 + g_1^2$ (d) none of these

Ans. (a)

◎ **Solution:** Both the circles clearly pass through the origin (0, 0). They will therefore touch each other if they have a common tangent at the origin. Now the tangent to the first circle at (0, 0) is $gx + fy = 0$, and that of the second circle is $g_1x + f_1y = 0$. If these two equations are to represent the same line, we must have $g/g_1 = f/f_1$, i.e., $f_1g = fg_1$.

◉ **Example 34:** If two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ cut the coordinate axes at concyclic points, then

(a) $a_1a_2 + b_1b_2 = 0$ (b) $a_1a_2 - b_1b_2 = 0$

(c) $a_1b_1 + a_2b_2 = 0$ (d) $a_1b_1 - a_2b_2 = 0$

Ans. (b)

◎ **Solution:** The given lines meet x-axis at $(-c_1/a_1, 0)$ and $(-c_2/a_2, 0)$, y-axis at $(0, -c_1/b_1)$ and $(0, -c_2/b_2)$. If an equation of the circle passing through these points is $x^2 + y^2 + 2gx + 2fy + c = 0$, then $-c_1/a_1$, $-c_2/a_2$ are the roots of $x^2 + 2gx + c = 0$ $\Rightarrow$ $c = c_1c_2/a_1a_2$. Similarly $c = c_1c_2/b_1b_2$ $\Rightarrow a_1a_2 - b_1b_2 = 0$.

◉ **Example 35:** The length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + \alpha = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + \beta = 0$ is

(a) $\sqrt{\beta - \alpha}$ (b) $\sqrt{\alpha - \beta}$

(c) $\sqrt{\alpha\beta}$ (d) $\sqrt{\alpha/\beta}$

Ans. (a)

◎ **Solution:** Let (h, k) be any point on the first circle, then

$$h^2 + k^2 + 2gh + 2fk + \alpha = 0 \qquad (1)$$

length of the tangent from (h, k) to the second circle is

$$\sqrt{h^2 + k^2 + 2gh + 2fk + \beta} = \sqrt{\beta - \alpha}\,.$$

◉ **Example 36:** The coordinates of the point on the circle $x^2 + y^2 - 2x - 4y - 11 = 0$ farthest from the origin are

(a) $\left(2 + 8/\sqrt{5}, 1 + 4/\sqrt{5}\right)$

(b) $\left(1 + 4/\sqrt{5}, 2 + 8/\sqrt{5}\right)$

(c) $\left(1 + 8/\sqrt{5}, 2 + 4/\sqrt{5}\right)$

(d) none of these

Ans. (b)

◎ **Solution:** The required point lies on the normal to the circle through the origin, i.e on the line $2x = y$ which gives

$$\begin{aligned} & x^2 + 4x^2 - 2x - 8x - 11 = 0 \\ \Rightarrow\quad & 5x^2 - 10x - 11 = 0 \\ \Rightarrow\quad & x = 1 \pm 4/\sqrt{5} \text{ and } y = 2\,(1 \pm 4/\sqrt{5}\,)\end{aligned}$$

and the coordinates of the required point farthest from the origin are $(1 + 4/\sqrt{5}, 2 + 8/\sqrt{5})$.

◉ **Example 37:** C_1 is a circle with centre at the origin and radius equal to r and C_2 is a circle with centre at $(3r, 0)$ and radius equal to $2r$. The number of common tangents that can be drawn to the two circles are

(a) 1 (b) 2

(c) 3 (d) 4

Ans. (c)

◎ **Solution:** The given circles C_1 and C_2 touch each other externally as the distance between the centres of the two circles is equal to the sum of their radii, hence the number of common tangents to these circles is 3.

◉ **Example 38:** An equilateral triangle is inscribed in the circle $x^2 + y^2 = a^2$ with the vertex at $(a, 0)$. The equation of the side opposite to this vertex is

(a) $2x - a = 0$ (b) $x + a = 0$
(c) $2x + a = 0$ (d) $3x - 2a = 0$

Ans. (c)

◎ **Solution:** $A\ (a, 0)$ be the vertex of the equilateral triangles ABC inscribed in the circle $x^2 + y^2 = a^2$

Let M be the middle point of the side BC, then MOA is perpendicular to BC and O being the centroid of the triangle $OA = 2\ (OM)$

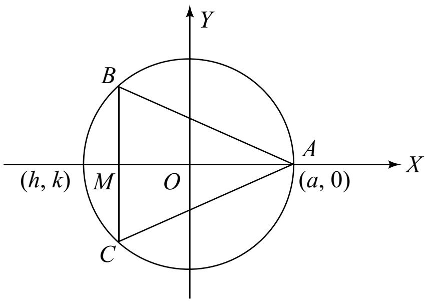


Fig. 17.15

(Circumcentre and Centroid of an equilateral triangle are same.)

So if (h, k) be the coordinates of M,

then $\dfrac{2h + a}{3} = 0$ and $\dfrac{2k + 0}{3} = 0$

$\Rightarrow\quad h = -(a/2)$ and $k = 0$

and hence the equation of BC is $x = -a/2$ or $2x + a = 0$.

◉ **Example 39:** Length of the common chord of the circles

$(x - 1)^2 + (y + 1)^2 = c^2$ and $(x + 1)^2 + (y - 1)^2 = c^2$ is

(a) $\frac{1}{2}\sqrt{c^2 - 2}$ (b) $\sqrt{c^2 - 2}$

(c) $2\sqrt{c^2 - 2}$ (d) $c + 2$

Ans. (c)

◎ **Solution:** Equation of the common chord AB of the given circles is

$$(x - 1)^2 + (y + 1)^2 - (x + 1)^2 - (y - 1)^2 = 0$$

$\Rightarrow\quad y = x$

Let $C_1\ (1, -1)$ be the centre of the first circle and M be the mid-point of AB, then

$$C_1A = c,\ C_1M = \left|\frac{1+1}{\sqrt{2}}\right| = \sqrt{2}$$

and $AB = 2AM = 2\sqrt{(C_1A)^2 - (C_1M)^2} = 2\sqrt{c^2 - 2}$.

◉ **Example 40:** The lines $2x - 3y = 5$ and $3x - 4y = 7$ are the diameters of a circle of area 154 square units. An equation of this circle is $(\pi = 22/7)$

(a) $x^2 + y^2 + 2x - 2y = 62$
(b) $x^2 + y^2 + 2x - 2y = 47$
(c) $x^2 + y^2 - 2x + 2y = 47$
(d) $x^2 + y^2 - 2x + 2y = 62$

Ans. (c)

◎ **Solution:** The centre of the circle is the point of intersection of the given diameters $2x - 3y = 5$ and $3x - 4y = 7$, which is $(1, -1)$ and if r is the radius of the circle, then

$\pi r^2 = 154 \quad \Rightarrow \quad r^2 = 154 \times \dfrac{7}{22}$

$\Rightarrow\quad r = 7.$

Hence an equation of the required circle is

$$(x - 1)^2 + (y + 1)^2 = 7^2 \quad \Rightarrow \quad x^2 + y^2 - 2x + 2y = 47.$$

◉ **Example 41:** The equation of a circle with origin as centre and passing through the vertices of an equilateral triangle whose median is of length $3a$ is

(a) $x^2 + y^2 = 9a^2$ (b) $x^2 + y^2 = 16a^2$
(c) $x^2 + y^2 = 4a^2$ (d) $x^2 + y^2 = a^2$

Ans. (c)

◎ **Solution:** As the centre of the circumcircle of an equilateral triangle is its centroid and the distance of the centroid from the vertex of the triangle is 2/3 of its median through that vertex. So the distance of the centre from a vertex of the triangle is $(2/3) \times 3a = 2a$ and hence the equation of the circumcircle is

$$x^2 + y^2 = 4a^2.$$

◉ **Example 42:** The circles $x^2 + y^2 - 10x + 16 = 0$ and $x^2 + y^2 = r^2$ intersect each other in two distinct points if

(a) $r < 2$ (b) $r > 8$
(c) $2 < r < 8$ (d) $2 \le r \le 8$

Ans. (c)

◎ **Solution:** Centres of the given circles are (5, 0) and (0, 0) and their radii are 3 and r respectively. The two circles will intersect in two distinct points if the distance between their centres is greater than the difference and less than the sum of their radii

$\Rightarrow\quad |3 - r| < 5 < 3 + r \quad \Rightarrow \quad 2 < r < 8.$

◉ **Example 43:** A line is drawn through the point $P(3, 11)$ to cut the circle $x^2 + y^2 = 9$ at A and B. Then $PA \cdot PB$ is equal to

(a) 9 (b) 121
(c) 205 (d) 139

Ans. (b)

◎ **Solution:** From geometry we know $PA\ .\ PB = (PT)^2$ where PT is the length of the tangent from P to the circle.

Hence $\ PA\ .\ PB = (3)^2 + (11)^2 - 9 = 11^2 = 121.$

Alternate Solution

Equation of any line through (3, 11) is

$$\frac{x-3}{\cos\theta} = \frac{y-11}{\sin\theta} = r \text{ (say)}$$

Then the coordinates of a point on this line at a distance r from (3, 11) are $(3 + r\cos\theta, 11 + r\sin\theta)$ and if this lies on the given circle $x^2 + y^2 = 9$

then $(3 + r\cos\theta)^2 + (11 + r\sin\theta)^2 = 9$

$\Rightarrow \quad 9 + 121 + 2r(3\cos\theta + 11\sin\theta) + r^2 = 9$

$\Rightarrow \quad r^2 + 2r(3\cos\theta + 11\sin\theta) + 121 = 0$

which is quadratic in r, gives two values of r say r_1 and r_2 and hence the distances of the points A and B from P.

Thus, $PA \cdot PB = r_1 r_2 = 121$.

◉ **Example 44:** If the line $x\cos\alpha + y\sin\alpha = p$ represents the common chord $APQB$ of the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ $(a > b)$ as shown in the Fig. 17.16, then AP is equal to

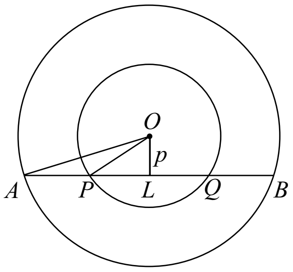


Fig. 17.16

(a) $\sqrt{a^2 + p^2} + \sqrt{b^2 + p^2}$

(b) $\sqrt{a^2 - p^2} + \sqrt{b^2 - p^2}$

(c) $\sqrt{a^2 - p^2} - \sqrt{b^2 - p^2}$

(d) $\sqrt{a^2 + p^2} - \sqrt{b^2 + p^2}$

Ans. (c)

◎ **Solution:** The given circles are concentric with centre at (0, 0) and the length of the perpendicular from (0, 0) on the given line is p. Let $OL = p$

then $\quad AL = \sqrt{(OA)^2 - (OL)^2} = \sqrt{a^2 - p^2}$

and $\quad PL = \sqrt{(OP)^2 - (OL)^2} = \sqrt{b^2 - p^2}$

$\Rightarrow \quad AP = \sqrt{a^2 - p^2} - \sqrt{b^2 - p^2}$

◉ **Example 45:** A circle touches both the coordinate axes and the line $x - y = a\sqrt{2}$ $(a > 0)$, the coordinates of the centre of the circle can be

(a) (a, a)	(b) $(a, -a)$
(c) $(-a, a)$	(d) none of these

Ans. (a)

◎ **Solution:** Let r be the radius of the circle. Since it touches the coordinate axes and the line $x - y = a\sqrt{2}$, the coordinates of the centre of the circle can be (r, r), $(-r, -r)$ or $(r, -r)$ (As $r > 0$ and the line $x - y = a\sqrt{2}$ meets the coordinates axes at $(a\sqrt{2}, 0)$ and $(0, -a\sqrt{2})$).

If the centre is (r, r) or $(-r, -r)$, then

$$\left|\frac{-a\sqrt{2}}{\sqrt{1+1}}\right| = r \qquad \Rightarrow \qquad r = a$$

So (a, a) can be the coordinates of the centre of the circle,

If the centre is $(r, -r)$ we have

$$\left|\frac{r + r - a\sqrt{2}}{\sqrt{2}}\right| = r \qquad \Rightarrow \qquad r = \left(\sqrt{2} \pm 1\right)a.$$

◉ **Example 46:** If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on the y-axis, then the length of PQ is

(a) 4	(b) $2\sqrt{5}$
(c) 5	(d) $3\sqrt{5}$

Ans. (c)

◎ **Solution:** The line $5x - 2y + 6 = 0$ meets y-axis at (0, 3). So the coordinates of Q are (0, 3) and PQ is the length of the tangent from Q (0, 3) to the circle $x^2 + y^2 + 6x + 6y - 2 = 0$. Hence

$$PQ = \sqrt{0 + 3^2 + 6\times 0 + 6\times 3 - 2} = \sqrt{25} = 5.$$

◉ **Example 47:** A triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If Q and R have coordinates (3, 4) and (– 4, 3) respectively, then $\angle QPR$ is equal to

(a) $\pi/2$	(b) $\pi/3$
(c) $\pi/4$	(d) $\pi/6$

Ans. (c)

◎ **Solution:** We know that $\angle QPR = \frac{1}{2}\angle QOR$; O being the centre (0, 0) of the given circle $x^2 + y^2 = 25$. Let m_1 = slope of $OQ = 4/3$ and m_2 = slope of $OR = -3/4$

As $\quad m_1 m_2 = -1$, $\angle QOR = \pi/2$

$\Rightarrow \quad \angle QPR = \pi/4$

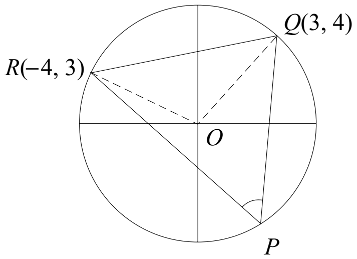


Fig. 17.17

Example 48: Two vertices of an equilateral triangle are $(-1, 0)$ and $(1, 0)$ and its third vertex lies above the x-axis, an equation of the circumcircle is

(a) $3x^2 + 3y^2 - 2\sqrt{3}y = 3$

(b) $2x^2 + 2y^2 - 3\sqrt{2}y = 2$

(c) $x^2 + y^2 - 2y = 1$

(d) none of these

Ans. (a)

Solution: Let A $(-1, 0)$, B $(1, 0)$ and C $(0, b)$ be the vertices of the triangle, as C lies on the locus of points equidistance from $A(-1, 0)$ and $B(1, 0)$ i.e., y-axis.

Then $\quad AB = AC \Rightarrow \sqrt{1+b^2} = 2$

$$\Rightarrow \quad b^2 = 3 \quad \Rightarrow \quad b = \sqrt{3} \qquad [\because b > 0]$$

Since the triangle is equilateral, the centre of the circumcircle is at the centroid of the triangle which is $(0, \sqrt{3}/3)$. Thus the equation of the circumcircle is

$$(x - 0)^2 + (y - 1/\sqrt{3})^2 = (1 - 0)^2 + (0 - 1/\sqrt{3})^2$$

$$\Rightarrow \quad x^2 + y^2 - (2/\sqrt{3})\,y + 1/3 = 1 + 1/3$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 2\sqrt{3}\,y = 3.$$

Example 49: An isosceles triangle is inscribed in the circle $x^2 + y^2 - 6x - 8y = 0$ with vertex at the origin and one of the equal sides along the axis of x. Equation of the other side through the origin is

(a) $7x - 24y = 0$ (b) $24x - 7y = 0$

(c) $7x + 24y = 0$ (d) $24x + 7y = 0$

Ans. (d)

Solution: Centre of the circle is $(3, 4)$ and it passes through the origin. If $y = mx$ is the equation of the required line, then length of the perpendicular from the centre on this line is equal to the length of the perpendicular from the centre on the axis of x.

$$\Rightarrow \quad \frac{3m-4}{\sqrt{1+m^2}} = \pm 4$$

$$\Rightarrow \quad 9m^2 - 24m + 16 = 16(1 + m^2)$$

$$\Rightarrow \quad m = -24/7$$

$(\because m = 0$ corresponds to x-axis$)$

and hence the required equation is $24x + 7y = 0$.

Example 50: Equations of the common tangents of the circles $x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$ are

(a) $x = 0$ (b) $y = 4$

(c) $3x + 4y = 10$ (d) all of these

Ans. (d)

Solution: Equations of the given circles can be written as C_1: $(x - 1)^2 + (y - 3)^2 = 1$ having centre at $(1, 3)$ and radius 1 and C_2: $(x + 3)^2 + (y - 1)^2 = 9$ having centre at $(-3, 1)$ and radius 3.

Both the circles clearly touch $x = 0$ and $y = 4$

Now $\quad \left|\dfrac{3\times1+4\times3-10}{\sqrt{3^2+4^2}}\right| = 1$

and $\quad \left|\dfrac{3(-3)+4(1)-10}{\sqrt{3^2+4^2}}\right| = 3$

Shows that both the circles touch $3x + 4y = 10$ also.

Example 51: A circle whose radius is 5 and which touches externally the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point $(5, 5)$ intersects in real distinct points the line

(a) $x = 0$ (b) $y = 0$

(c) $y = x$ (d) none of these

Ans. (c)

Solution: Centre of the given circle is A $(1, 2)$ and its radius is $\sqrt{1+(2)^2+20} = 5$. Point of contact P is $(5, 5)$. Let B (h, k) be the centre of the required circle of radius 5, then P is the mid-point of AB, so that

$$\frac{h+1}{2} = 5 \text{ and } \frac{k+2}{2} = 5 \quad \Rightarrow \quad h = 9, k = 8$$

and an equation of the required circle is

$$x^2 + y^2 - 18x - 16y + 120 = 0$$

If $x = 0$, $y^2 - 16y + 120 = 0$ does not give real values of y

If $y = 0$, $x^2 - 18x + 120 = 0$ does not give real values of x

If $\quad y = x,\ 2x^2 - 34x + 120 = 0$

or $\quad x^2 - 17x + 60 = 0 \quad \Rightarrow \quad x = 5, 12$

This shows that the circle intersects the line $y = x$ at two real distinct points.

Example 52: A point moves such that the sum of the squares of its distances from the sides of a square of side unity is equal to 9. The locus of such a point is a circle

(a) inscribed in the square

(b) circumscribing the square

(c) inside the square

(d) containing the square

Ans. (d)

Solution: Let P (x, y) be any point on the locus and $OABC$ be the square of side unity.

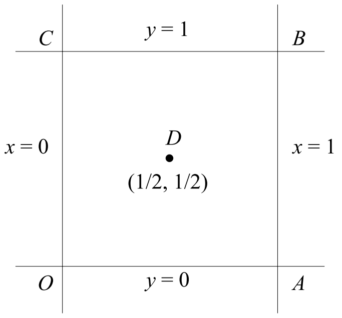


Fig. 17.18

Then, according to the given condition

$$x^2 + y^2 + (x-1)^2 + (y-1)^2 = 9$$

$$\Rightarrow \quad 2(x^2 + y^2) - 2x - 2y - 7 = 0$$

$$\Rightarrow \quad x^2 + y^2 - x - y - 7/2 = 0$$

which is a circle with centre (1/2, 1/2) coinciding with the centre D of the square and the radius of the circle is $\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{7}{2}} = 2$, showing that the circle contains the square.

◉ **Example 53:** Equation of a circle with centre (–4, 3) touching internally and containing the circle $x^2 + y^2 = 1$ is

(a) $x^2 + y^2 + 8x - 6y + 9 = 0$
(b) $x^2 + y^2 - 8x + 6y + 9 = 0$
(c) $x^2 + y^2 + 8x - 6y - 11 = 0$
(d) $x^2 + y^2 - 8x + 6y - 11 = 0$

Ans. (c)

◎ **Solution:** Let the equation of the required circle be

$$(x+4)^2 + (y-3)^2 = r^2 \qquad \text{(i)}$$

If (i) touches the circle $x^2 + y^2 = 1$ internally (ii) the distance between the centres (– 4, 3) and (0, 0) of these circles is equal to the difference of their radii

$$\Rightarrow \quad \sqrt{4^2+3^2} = r - 1 \Rightarrow r = 5 + 1 \Rightarrow r = 6$$

So that an equation of the required circle is

$$x^2 + y^2 + 8x - 6y - 11 = 0.$$

◉ **Example 54:** The tangents drawn from the origin to the circle $x^2 + y^2 - 2px - 2qy + q^2 = 0$ are perpendicular if

(a) $p^2 + q^2 = 1$ (b) $p^2 - q^2 = 1$
(c) $p^2 - q^2 = 0$ (d) none of these

Ans. (c)

◎ **Solution:** Equation of the given circle can be written as

$$(x-p)^2 + (y-q)^2 = p^2 \qquad \text{(i)}$$

This has (p, q) as the centre and p as the radius showing that it touches y-axis. So one of the tangents from the origin to the circle is y-axis.

⇒ Other tangent from the origin to the circle must be x-axis. which is possible if $q = \pm\, p$.

$$\Rightarrow \quad p^2 - q^2 = 0.$$

◉ **Example 55:** A square is inscribed in the circle $x^2 + y^2 - 2x + 4y - 93 = 0$ with its sides parallel to the axes of coordinates. The coordinates of the vertex farthest from the origin are

(a) (9, – 8) (b) (8, – 9)
(c) (8, 5) (d) (– 6, 9)

Ans. (b)

◎ **Solution:** The centre of the given circle is (1, –2) and its radius is $\sqrt{98}$, so the coordinates of the vertices of the square inscribed in the given circle are $(1 + \sqrt{98}\cos\theta, -2 + \sqrt{98}\sin\theta)$ where $\theta = \pm\,\pi/4, \pm\,3\pi/4$.

S^2 = square of the distance of a vertex from the origin.
$= 5 + 98 + 2\sqrt{98}\,(\cos\theta - 2\sin\theta)$ which is maximum when
$\theta = -\,\pi/4$, so the required point is $(1 + 7, -2 - 7)$
$= (8, -9)$

◉ **Example 56:** An equation of a circle touching the axes of coordinates and the line $x\cos\alpha + y\sin\alpha = 2$ is $x^2 + y^2 - 2gx + 2gy + g^2 = 0$ where $g =$

(a) $2(\cos\alpha + \sin\alpha + 1)^{-1}$
(b) $2(\cos\alpha - \sin\alpha + 1)^{-1}$
(c) $2(\cos\alpha + \sin\alpha - 1)^{-1}$
(d) $-2(\cos\alpha - \sin\alpha - 1)^{-1}$

Ans. (b)

◎ **Solution:** Centre of the circle is $(g, -g)$ and radius is $|g|$. If it touches the line $x\cos\alpha + y\sin\alpha = 2$, the

$$g\cos\alpha - g\sin\alpha - 2 = \pm\, g$$

$$\Rightarrow \quad g(\cos\alpha - \sin\alpha \pm 1) = 2$$

$$\Rightarrow \quad g = \frac{2}{\cos\alpha - \sin\alpha \pm 1}$$

which is satisfied by (b).

◉ **Example 57:** If common chord of the circle C with centre at (2, 1) and radius r and the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a diameter of the second circle, then the value of r is

(a) 3 (b) 2
(c) 3/2 (d) 1

Ans. (a)

◎ **Solution:** Equation of C is $(x-2)^2 + (y-1)^2 = r^2$

or $\quad x^2 + y^2 - 4x - 2y = r^2 - 5$

Equation of the common chord is

$$x^2 + y^2 - 4x - 2y - (r^2 - 5) - [x^2 + y^2 - 2x - 6y + 6] = 0$$

$$\Rightarrow \quad 2x - 4y + r^2 + 1 = 0$$

Since it is a diameters of the second circle, the centre (1, 3) lies on it.

$$\Rightarrow \quad 2 - 12 + r^2 + 1 = 0 \quad \Rightarrow \quad r = 3.$$

◉ **Example 58:** The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point in the set is

(a) $4 \le x^2 + y^2 \le 64$ (b) $x^2 + y^2 \le 25$
(c) $x^2 + y^2 \ge 25$ (d) $3 \le x^2 + y^2 \le 9$.

Ans. (a)

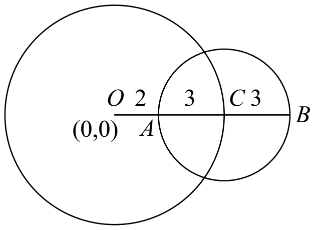


Fig. 17.19

◎ **Solution:** Distance of any point (x, y) on a circle with centre C lying on the circle $x^2 + y^2 = 25$ and radius 3 from the origin lies between OA (= 2) and OB (= 8)
Hence $4 \le x^2 + y^2 \le 64$.

● **Example 59:** If a circle passes through (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is

(a) $2ax - 2by + (a^2 + b^2 + 4) = 0$
(b) $2ax + 2by - (a^2 + b^2 + 4) = 0$
(c) $2ax + 2by + (a^2 + b^2 + 4) = 0$
(d) $2ax - 2by - (a^2 + b^2 + 4) = 0$

Ans. (b)

◎ **Solution:** Let the equation of the circle be
$x^2 + y^2 - 2gx - 2fy + c = 0$. As it passes through (a, b)
$$a^2 + b^2 - 2ga - 2fb + c = 0$$
Since it intersects the circle $x^2 + y^2 = 4$
orthogonally
$$2g \times 0 + 2f \times 0 = c - 4 \Rightarrow c = 4$$
and the locus of (g, f), the centre is
$$2ax + 2by - (a^2 + b^2 + 4) = 0$$

● **Example 60:** Consider a family of circles which are passing through the point $(-1, 1)$ and are tangent to x-axis. If (h, k) are the coordinates of the centre of the circles, then the set of values of k is given by the interval

(a) $0 < k < 1/2$ (b) $k \ge 1/2$
(c) $-1/2 \le k \le 1/2$ (d) $k \le 1/2$.

Ans. (b)

◎ **Solution:** Equation of the circle with centre (h, k) and touching x-axis is
$$(x - h)^2 + (y - k)^2 = k^2$$
which passes through $(-1, 1)$
if $h^2 + 2h - 2k + 2 = 0$
or if $(h + 1)^2 = 2k - 1$
For real values of h, $2k - 1 \ge 0$.
$\Rightarrow$ $k \ge 1/2$

● **Example 61:** If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q then the line $5x + by - a = 0$ passes through P and Q for

(a) Infinitely many values of a
(b) Exactly two values of a
(c) Exactly one value of a
(d) No value of a

Ans. (d)

◎ **Solution:** Equation of the line passing through the points P and Q is
$$x^2 + y^2 + 2ax + cy + a - (x^2 + y^2 - 3ax + dy - 1) = 0$$
$$\Rightarrow \quad 5ax + (c - d)y + a + 1 = 0$$
which coincides with $5x + by - a = 0$
if $$\frac{5a}{5} = \frac{c-d}{b} = \frac{a+1}{-a}.$$
or if $a^2 + a + 1 = 0$ which does not give any real value of a.

● **Example 62:** A circle touches the x-axis and also touches the circle with centre at (0, 3) and radius 2. The locus of the centre of the circle is

(a) a hyperbola (b) a parabola
(c) an ellipse (d) a circle

Ans. (b)

◎ **Solution:** Let (h, k) be the centre of the circle. Since it touches x-axis, its radius is k. As it also touches the circle with centre (0, 3) and radius 2 (externally as x-axis lies outside the given circle) distance between the centres is equals to the sum of their radii.
So $h^2 + (k - 3)^2 = (k + 2)^2$
$\Rightarrow$ $h^2 = 5(2k - 1)$
Locus of the centre (h, k) is
$$x^2 = 5(2y - 1)$$
which is a parabola.

● **Example 63:** Three distinct points A, B, C, are given in the two dimensional coordinate plane such that the ratio of the distance of anyone of them from the point (1, 0) to the distance from the point (–1, 0) is equal to 1/3. Then the circumcentre of the triangle ABC is at the point

(a) (5/2, 0) (b) (5/3, 0)
(c) (0, 0) (d) (5/4,0)

Ans. (d)

◎ **Solution:** Let $Q = (1, 0)$ and $R = (-1, 0)$
Let $P(x, y)$ be any point such that
$$\frac{PQ}{PR} = \frac{1}{3} \Rightarrow 9(PQ)^2 = (PR)^2$$
$$\Rightarrow \quad 9[(x - 1)^2 + y^2] = (x + 1)^2 + y^2$$
$$\Rightarrow \quad 8x^2 + 8y^2 - 20x + 8 = 0$$
A, B, C lie on this locus of P which is a circle circumscribing the triangle ABC.
So the circumcentre of the triangle ABC is the centre (5/4, 0) of this circle.

● **Example 64:** Let $A(1,2)$ $B(3, 4)$ be two points and $C(x, y)$ be a point such that area of the triangle ABC is 3 sq. units and $(x - 1)(x - 3) + (y - 2)(y - 4) = 0$. Then maximum number of positions of C, in the xy plane is

(a) 2 (b) 4
(c) 8 (d) none of these

Ans. (d)

◎ **Solution:** C lies on the circle on AB as diameter of length $2\sqrt{2}$.

So radius of the circle is $\sqrt{2}$.

Area of the triangle $ABC = 3 = (1/2)\, AB \times$ altitude from C on AB

$$\Rightarrow \quad \text{altitude} = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}} > \sqrt{2}.$$

Maximum altitude from C on AB is equal to the radius of the circle.

So no such C exists.

◉ **Example 65:** A circle touches y-axis at (0, 3) and makes an intercept of 2 units on the positive x-axis. Intercept made by the circle on the line $\sqrt{10}\ x - 3y = 1$ in units is

(a) 3 (b) 6
(c) $2\sqrt{10}$ (d) 10.

Ans. (c)

◎ **Solution:**

$$AB = 2, AD = 1$$
$$CD = 3 = OE$$
$$AC = \sqrt{(AD)^2 + (CD)^2}$$
$$= \sqrt{1+3^2} = \sqrt{10} = CE$$

So the coordinates of C are $\left(\sqrt{10}, 3\right)$

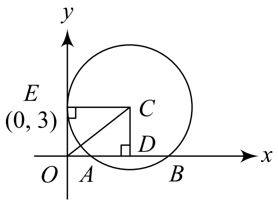


Fig. 17.20

The line $\sqrt{10}\,x - 3y = 1$ passes through the centre $C\left(\sqrt{10}, 3\right)$ of the circle and hence is a diameter and the required intercept is twice the radius of the circle.

◉ **Example 66:** Equation of the circle which circumscribes the square formed by the lines $xy - 9x - 2y + 18 = 0$ and $xy - 5x - 6y + 30 = 0$ is

(a) $x^2 + y^2 - 4x - 10y + 21 = 0$
(b) $x^2 + y^2 - 8x - 14y + 57 = 0$
(c) $x^2 + y^2 - 12x - 10y + 53 = 0$
(d) none of these.

Ans. (b)

◎ **Solution:** Given equations represent the lines $x = 2$, $y = 9$, $x = 6$, $y = 5$ which forms a square of each side equal to 4, vertex $A(2, 5)$, $C(6, 9)$. Centre of the circle is the mid point of AC and the radius is $(1/2)\,AC$. Hence the equation of the circle is

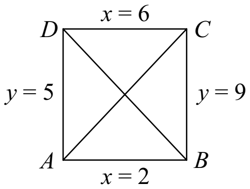


Fig. 17.21

$$(x-4)^2 + (y-7)^2 = \left(2\sqrt{2}\right)^2$$

$$\Rightarrow \quad x^2 + y^2 - 8x - 14y + 16 + 49 - 8 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 8x - 14y + 57 = 0.$$

◉ **Example 67:** If the centre of the circle passing through the origin and the points of intersection of the pair of straight lines $xy - 7x + 3y - 21 = 0$ with the coordinate axes lies on the $x + y = k$, then k is equal to.

(a) 0 (b) 1
(c) 2 (d) 4

Ans. (c)

◎ **Solution:** Equation the circle passing through the points (– 3, 0) and (0, 7) where the lines meet the axes, and the origin is $x^2 + y^2 + 3x - 7y = 0$. Whose centre (– 3/2, 7/2) lies on the line $x + y = k$

$$\Rightarrow \quad k = 2$$

◉ **Example 68:** If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and (1, 1) for

(a) all except two values of p
(b) exactly one value of p
(c) all values of p
(d) all except one value of p.

Ans. (d)

◎ **Solution:** Equation of a circle passing through P and Q is $x^2 + y^2 + 3x + 7y + 2p - 5 + \lambda\,(x^2 + y^2 + 2x + 2y - p^2) = 0$

which passes through (1, 1)

$$\text{if} \quad (7 + 2p) - \lambda\,(p^2 - 6) = 0$$

$$\Rightarrow \quad \lambda = \frac{7+2p}{p^2-6} \neq -1$$

$$\Rightarrow \quad p^2 - 6 + 7 + 2p \neq 0 \quad \Rightarrow \quad (p+1)^2 \neq 0$$

$$\Rightarrow \quad p \neq -1$$

Example 69: Equation of circle passing through (1, 5) and (4, 1) and touching y-axis is $x^2 + y^2 - 5x - 6y + 9 + \lambda(4x + 3y - 19) = 0$ where λ is equal to

(a) 0, – 40/9	(b) 0
(c) 40/9	(d) – 40/9

Ans. (a)

Solution: Equation of any circle through the given points is $(x - 1)(x - 4) + (y - 5)(y - 1) + \lambda(4x + 3y - 19) = 0$ where $4x + 3y - 19 = 0$ is the equation of the line joining (1, 5) and (4, 1)

$$\Rightarrow \qquad x^2 + y^2 - 5x - 6y + 9 + \lambda(4x + 3y - 19) = 0$$

Which touches y-axis if the roots of the equation

$y^2 - 6y + 9 + \lambda(3y - 19) = 0$ are equal. (Taking the intersection of the circle with $x = 0$)

$$\Rightarrow \qquad (3\lambda - 6)^2 = 4(9 - 19\lambda)$$

$$\Rightarrow \qquad 9\lambda^2 + 40\lambda = 0 \Rightarrow \lambda = 0, -40/9.$$

Example 70: The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if

(a) $15 < m < 65$

(b) $35 < m < 85$

(c) $-85 < m < -35$

(d) $-35 < m < 15$

Ans. (d)

Solution: The line intersects the circle at two distinct points if length of the perpendicular from (2, 4), the centre of the circle, to the line is less than the radius.

$$\Rightarrow \qquad \frac{|(3)(2) - 4(4) - m|}{\sqrt{(3)^2 + (4)^2}} < \sqrt{4 + 16 + 5}$$

$$\Rightarrow \qquad |m+10| < 25$$

$$\Rightarrow \qquad -25 < m + 10 < 25$$

$$\Rightarrow \qquad -35 < m < 15.$$

Example 71: The equation of the circle passing through the points (1, 0) and (0, 1) and having the smallest radius is

(a) $x^2 + y^2 - 2x - 2y + 1 = 0$

(b) $x^2 + y^2 - x - y = 0$

(c) $x^2 + y^2 + 2x + 2y - 7 = 0$

(d) $x^2 + y^2 + x + y - 2 = 0$

Ans. (b)

Solution: Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

As it passes through (1, 0) and (0, 1), we get

$1 + 2g + c = 0,\ 1 + 2f + c = 0$

If r is the radius of the circle, then

$$r^2 = g^2 + f^2 - c = \frac{1}{4}(1 + c)^2 + \frac{1}{4}(1 + c)^2 - c$$

$$= \frac{1}{2}(1 + c^2)$$

Now r^2 will be least if $c = 0$ and the required equation is $x^2 + y^2 - x - y = 0$

Example 72: The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ ($c > 0$) touch each other

(a) $\|a\| = 2c$	(b) $2\|a\| = c$
(c) $\|a\| = c$	(d) $a = 2c$

Ans. (c)

Solution: Given circles can touch each other internally see the figures. This is possible if

$$c = a \qquad \text{or} \qquad c = -a$$

$$\Rightarrow \qquad c = |a|.$$

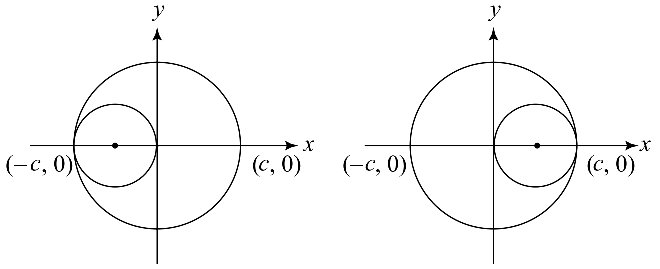


Fig. 17.22

Example 73: The circle passing through the point (–1, 0) and touching y-axis at (0, 2) also passes through the point

(a) (–3/2, 0)	(b) (–5/2, 2)
(c) (–3/2, 5/2)	(d) (–4, 0)

Ans. (d)

Solution: Let he equation of the circle be

$$(x - k)^2 + (y - 2)^2 = k^2$$

which passes through the point (–1, 0)

if $\quad (-1 - k)^2 + 4 = k^2 \Rightarrow k = -5/2$

and the equation of the circle is

$$x^2 + y^2 + 5x - 4y + 4 = 0$$

which passes through (–4, 0)

Example 74: The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is

(a) $20(x^2 + y^2) - 36x + 45y = 0$

(b) $20(x^2 + y^2) + 36x - 45y = 0$

(c) $36(x^2 + y^2) - 20x + 45y = 0$

(d) $36(x^2 + y^2) + 20x - 45y = 0$

Ans. (a)

Solution: Equation of the chord contact in terms of the mid-point (α, β) is $x\alpha + y\beta = \alpha^2 + \beta^2$

Let $P(5t, 4t - 4)$ be a point on $4x - 5y = 20$

Equation of the chord of contact of P is

$x(5t) + y(4t - 4) = 9$

comparing the equations we get

$$\frac{5t}{\alpha} = \frac{4t - 4}{\beta} = \frac{9}{\alpha^2 + \beta^2} = \frac{4(5t) - 5(4t - 4)}{4\alpha - 5\beta}$$

$\Rightarrow \quad 20(\alpha^2+\beta^2) - 36\alpha + 45\beta = 0$

Locus of (α, β) is $20(x^2+y^2) - 36x + 45y = 0$

Example 75: The length of the diameter of the circle which touches the x-axis at the point (1, 0) and passes through the point (2, 3) is

(a) 6/5 (b) 5/3

(c) 10/3 (d) 3/5

Ans. (c)

Solution: Let the equation of the circle be $(x-1)^2 + (y-k)^2 = k^2$. Since it passes through (2, 3)

$1 + (3-k)^2 = k^2 \Rightarrow k = 5/3$

$\Rightarrow$ diameter = $2k = 10/3$.

Assertion-Reason Type Questions

Example 76: Statement-1: Limiting points of a family of coaxial circles are (1, 2) and (2, 1). No circle of this family passes through the origin.

Statement-2 : Equation of a circle passing through (1, 2) and (2, 1), the centre of which does not lie on the join of these points is $x^2 + y^2 - 3x - 3y + 4 = 0$

Ans. (c)

Solution: In statement 1, two members of the family are $(x-1)^2 + (y-2)^2 = 0$ and $(x-2)^2 + (y-1)^2 = 0$. Equation of any member of this family is $x^2 + y^2 - 2x - 4y + 5 + \lambda(x^2 + y^2 - 4x - 2y + 5) = 0$ which passes through the origin if $\lambda = -1$, in which case the equation does not represent a circle. So the statement is true.

In statement-2, Equation represents the circle on the join of the given points as diameter. So the statement is false.

Example 77: Statement-1: Equation of a tangent to the circle $x^2 + y^2 = 50$ at a point which has positive integral coordinates (α, β) $(\alpha \neq \beta)$ is $x + 7y = 50$ or $7x + y = 50$.

Statement-2 : There are 12 points on the circle $x^2 + y^2 = 50$ with integral coordinates.

Ans. (a)

Solution: Statement-2 is true as the required points are, $(\pm 1, \pm 7)$, $(\pm 7, \pm 1)$ $(\pm 5, \pm 5)$ out of which the points satisfying the conditions in statement are (1, 7) and (7, 1) only and the tangents at these points are respectively $x + 7y = 50$ and $7x + y = 50$ so statement–1 is also true.

Example 78: Statement-1: The centre of the circle passing through the points (3, 8) and (5, 4) and having smallest radius is (4, 6)

Statement-2 : The centre of a circle passing through two given points lies at the mid-point of the line joining the points.

Ans. (c)

Solution: In statement-1, the radius is smallest when the line joining the given points is a diameter of the circle and hence its centre is the mid-point $\left(\frac{3+5}{2}, \frac{4+8}{2}\right)$ i.e (4, 6) of the line joining the given points. So statement-1 is True. Statement-2 is false as the line joining the given points may not be a diameter for each circle passing through these points.

Example 79: Statement-1: The circle $x^2 + y^2 - 8x - 4y + 16 = 0$ touches x-axis at the point (4, 0)

Statement-2 : The circle $(x-a)^2 + (y-r)^2 = r^2$ touches x-axis at the point $(a, 0)$

Ans. (a)

Solution: Statement-2 is true as the centre of the circle is (a, r) and its radius is r, centre is at a distance r from x-axis. In statement-2, the circle is $(x-4)^2 + (y-2)^2 = 4 = 2^2$, using statement-2, statement-1 is also true.

Example 80: Statement-1 : Point (3, – 1) lies outside the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$

Statement-2 : A point (α, β) lies outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ if $\alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c > 0$

Ans. (d)

Solution: Statement-2 is true because the distance between the point (α, β) and the centre $(-g, -f)$ of the circle is greater than its radius.

$$\Rightarrow \quad (\alpha + g)^2 + (\beta + f)^2 > g^2 + f^2 - c$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + 2g\alpha + 2f\beta + c > 0.$$

Using it in statement-1, $2 \times 9 + 2 \times 1 - 3 \times 3 + 5(-1) - 7 = -1 < 0$ so statement-1 is false.

Example 81: Tangents are drawn from the point (17, 7) to the circle $x^2 + y^2 = 169$

Statement-1: Tangents are mutually perpendicular.

Statement-2: The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$

Ans. (a)

Solution: Statement-2 is true, because equation of any tangent to the circle is $y = mx \pm 13\sqrt{1+m^2}$. If it passes through (h, k), then $k = mh \pm 13\sqrt{1+m^2}$

$$\Rightarrow \quad (169 - h^2)m^2 + 2mhk + (169 - k^2) = 0$$

which gives two values of m say m_1, m_2, the slopes of the tangents drawn from (h, k) to the circle

$$m_1m_2 = -1 \quad \Rightarrow \quad \frac{169 - k^2}{169 - h^2} = -1 \quad \Rightarrow \quad h^2 + k^2 = 338$$

$\Rightarrow$ Locus of (h, k) is $x^2 + y^2 = 338$
Since the point (17, 7) in statement-1 satisfies the above equation, using statement-2, statement-1 is also true.

Example 82: Statement-1: Limiting points of a family of co-axial system of circles are (1, 1) and (3, 3). The member of this family passing through the origin is $2x^2 + 2y^2 - 3x - 3y = 0$

Statement-2: Equation of the tangent to the circle $2x^2 + 2y^2 - 3x - 3y = 0$ at the origin is $x + y = 0$
Ans. (b)

Solution: In statement-1, two members of the co-axial system are circles with centres at (1, 1), (3, 3) and radius zero. So the equation of the system of circles is

$$(x - 1)^2 + (y - 1)^2 + \lambda[(x - 3)^2 + (y - 3)^2] = 0$$

If it passes through the origin.
$1 + 1 + \lambda[9 + 9] = 0 \quad \Rightarrow \quad \lambda = -1/9$ and the required circle is $2x^2 + 2y^2 - 3x - 3y = 0$
So statement-1 is true.
Statement-2 is also true but does not lead the statement-1.

Example 83: Statement 1: $x^2 + y^2 - 6x - 10y - 2 = 0$ is the only circle of radius 6 units having a diameter along the line $5x - 2y - 5 = 0$.

Statement-2: $5x - 2y - 5 = 0$ is a normal to the circle $x^2 + y^2 - 6x - 10y - 2 = 0$.
Ans. (d)

Solution: Given circle is $(x - 3)^2 + (y - 5)^2 = 36$
$\Rightarrow$ centre of the circle is (3, 5) and radius is 6. Since the line $5x - 2y - 5 = 0$ passes through (3, 5) it is a diameter of the circle. So this is a circle of radius 6 units having $5x - 2y - 5 = 0$ as a diameter. But any other point on this can also be taken as the centre, giving us another circle of radius 6 with the same diameter. So statement-1 is false. Since every diameter is normal to the circle, statement-2 is true.

Example 84: Statement-1: The circle $x^2 + y^2 - 8x + 10y + 32 = 0$ does not touch or intersect any axis.

Statement-2: A circle with centre (a, b) and radius r does not touch or intersect any axis if $r < \min[|a|, |b|]$
Ans. (a)

Solution: We find by fig. $r < |a|$ and $r < |b|$
$\Rightarrow$ $r < \min(|a|, |b|)$
So statement-2 is true, using it statement-1 is also true, as the centre of the circle is (4, – 5) and $r = 3$.

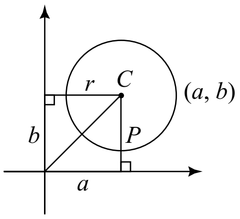


Fig. 17.23

Example 85: C_1: $x^2 + y^2 + 2x - 2y + 2 - a^2 = 0$
C_2: $x^2 + y^2 - 10x - 14y + 74 - a^2 = 0$
L: $x + y - 6 = 0$

Statement 1: L is a common tangent to both the circles C_1 and C_2 if $a = 3\sqrt{2}$.

Statement 2: L is a common chord of both the circles C_1 and C_2 for all values of a.
Ans. (c)

Solution: $C_1 : (x + 1)^2 + (y - 1)^2 = a^2$
$C_2 : (x - 5)^2 + (y - 7)^2 = a^2$

Centre of C_1 is (– 1, 1) and of C_2 is (5, 7) and both have same radius equal to a. Distance between the centre $= 6\sqrt{2} = 2a$ (when $a = 3\sqrt{2}$) = sum of the radii, so the circle touch each other externally and the equation of a common tangent is $(x^2 + y^2 + 2x - 2y + 2 - a^2) - (x^2 + y^2 - 10x - 14y + 74 - a^2) = 0$
$\Rightarrow \quad 12x + 12y - 72 = 0$ or $x + y - 6 = 0$
So statement-1 is true and statement-2 is false as for $a < 3\sqrt{2}$, circles C_1 and C_2 will not touch or intersect each other, L will lie outside both the circles.

LEVEL 2

Straight Objective Type Questions

Example 86: Let $ABCD$ be a quadrilateral with area 18, side AB parallel to the side CD and $AB = 2CD$. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

(a) 3	(b) 2
(c) 3/2	(d) 1

Ans. (b)

Solution: Let $CD = a$, then $AB = 2a$ and r be the radius of the circle, then $AD = 2r$. Let A be the origin and AB and AD as x-axis and y-axis respectively.
The coordinates of A, B, C, D are respectively

(0, 0), (2a, 0), (a, 2r), (0, 2r)

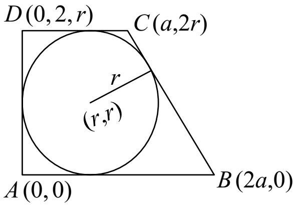


Fig. 17.24

Area $(ABCD) = (1/2)(a + 2a)(2r) = 18$

$\Rightarrow \quad ar = 6.$

Equation of BC is $2rx + ay - 4ar = 0$ and the coordinates of the centre of the circle are (r, r)

Since the circle touches BC,

$$\frac{2r^2 + ar - 4ar}{\sqrt{4r^2 + a^2}} = r$$

$\Rightarrow \quad 4r^4 - 72r^2 + 324 = 4r^4 + 36$

$\Rightarrow \quad r = 2.$

◉ **Example 87:** An equation of the chord of the circle $x^2 + y^2 = a^2$ passing through the point (2, 3) farthest from the centre is

(a) $2x + 3y = 13$ (b) $3x - y = 3$
(c) $x - 2y + 4 = 0$ (d) $x - y + 1 = 0$

Ans. (a)

◎ **Solution:** Let P (2, 3) be the given point, M be the middle point of a chord of the circle $x^2 + y^2 = a^2$ through P. Then the distance of the centre O of the circle from the chord is OM.

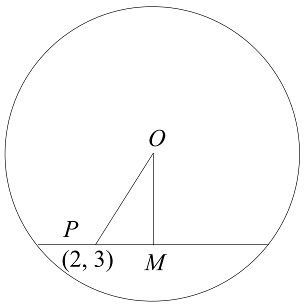


Fig. 17.25

and $(OM)^2 = (OP)^2 - (PM)^2$ which is maximum when PM is minimum, i.e., M coincides with P i.e., P is the middle point of the chord. Hence the equation of the chord is

$2 \cdot x + 3 \cdot y - a^2 = (2)^2 + (3)^2 - a^2$

$\Rightarrow \quad 2x + 3y = 13.$

◉ **Example 88:** An isosceles right angled triangle is inscribed in the circle $x^2 + y^2 = r^2$. If the coordinates of an end of the hypotenuse are (a b), the coordinates of the vertex are

(a) $(-a, -b)$ (b) $(b, -a)$
(c) (b, a) (d) $(-b, -a)$

Ans. (b)

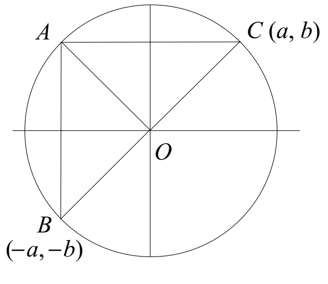


Fig. 17.26

◎ **Solution:** Since the hypotenuse of a right angled triangle inscribed in a circle is a diameter of the circle, if the coordinates of the end C of the hypotenuse BC are (a, b), the coordinates of B are ($-a$, $-b$).

Equation of BC is $\dfrac{y}{x} = \dfrac{b}{a}$. If A is the vertex of the isosceles triangle then OA is perpendicular to BC and the equation of AO is $\dfrac{y}{x} = \dfrac{-a}{b}$ which meets the circle $x^2 + y^2 = r^2$ at points for which

$$\left(\frac{a^2}{b^2} + 1\right) x^2 = r^2 = a^2 + b^2$$

[$\because$ (a, b) lies on $x^2 + y^2 = r^2$]

$\Rightarrow \quad x^2 = b^2 \;\Rightarrow\; x = \pm b \;\Rightarrow\; y = \mp a.$

coordinates of A are ($-b$, a) or (b, $-a$).

◉ **Example 89:** Two rods of lengths a and b slide along the x-axis and y-axis respectively in such a manner that their ends are concyclic. The locus of the centre of the circle passing through the end points is

(a) $4(x^2 + y^2) = a^2 + b^2$ (b) $x^2 + y^2 = a^2 + b^2$
(c) $4(x^2 - y^2) = a^2 - b^2$ (d) $x^2 - y^2 = a^2 - b^2$

Ans. (c)

◎ **Solution:** Let C (h, k) be the centre of the circle passing through the end points of the rod AB and PQ of lengths a and b respectively, CL and CM be perpendiculars from C on AB and PQ respectively. (Fig. 17.27)

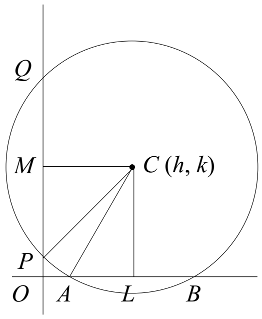


Fig. 17.27

then $AL = (1/2)\,AB = a/2$

$PM = (1/2)\,PQ = b/2$

and $CA = CP$ (radii of the same circle)

$$\Rightarrow \quad k^2 + \frac{a^2}{4} = h^2 + \frac{b^2}{4}$$

$\Rightarrow \quad 4(h^2 - k^2) = a^2 - b^2$

so that locus of (h, k) is $4(x^2 - y^2) = a^2 - b^2$

◉ **Example 90:** If the point (1, 4) lies inside the circle $x^2 + y^2 - 6x - 10y + p = 0$ and the circle does not touch or intersect the coordinate axes, then

(a) $0 < p < 34$ (b) $25 < p < 29$
(c) $9 < p < 25$ (d) $9 < p < 29$

Ans. (b)

◎ **Solution:** Since the circle does not touch or intersect the coordinates axes, the absolute values of x and y coordinates of the centre are greater than the radius of the circle. Coordinates of the centre of the circle are (3, 5) and the radius is $\sqrt{9+25-p}$

so that $$3 > \sqrt{9+25-p} \Rightarrow p > 25 \quad (1)$$

$$5 > \sqrt{9+25-p} \Rightarrow p > 9 \quad (2)$$

and the point (1, 4) lies inside the circle

$\Rightarrow \quad 1 + 16 - 6 - 10 \times 4 + p < 0$

$\Rightarrow \quad p < 29 \quad (3)$

From (1), (2), (3) we get

$$25 < p < 29.$$

◉ **Example 91:** The lengths of the intercepts made by any circle on the coordinates axes are equal if the centre lies on the line (s) represented by

(a) $x + y + 1 = 0$ (b) $x + y = 1$
(c) $x^2 - y^2 = 0$ (d) $x - y = 1$

Ans. (c)

◎ **Solution:** Let the equation of any circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{(i)}$$

For intercept made by the circle on x-axis, put $y = 0$ in (i)

$$\Rightarrow \quad x^2 + 2gx + c = 0$$

If x_1, x_2 are roots of (*ii*), then length of the intercept on x-axis is

$$|x_1 - x_2| = \sqrt{(x_1+x_2)^2 - 4x_1x_2} = 2\sqrt{g^2 - c}$$

Similarly length of the intercept of the y-axis is $2\sqrt{f^2 - c}$

Since the lengths of these intercepts are equal

$\sqrt{g^2 - c} = \sqrt{f^2 - c}$

$\Rightarrow \quad g^2 = f^2 = (-g)^2 = (-f)^2$

Therefore, centre lies on $x^2 - y^2 = 0$

◉ **Example 92:** If $a > 2b > 0$ then the positive value of m for which $y = mx - b\sqrt{1+m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is

(a) $\dfrac{2b}{\sqrt{a^2 - 4b^2}}$ (b) $\dfrac{\sqrt{a^2 - 4b^2}}{2b}$

(c) $\dfrac{2b}{a - 2b}$ (d) $\dfrac{b}{a - 2b}$

Ans. (a)

◎ **Solution:** $y = mx - b\sqrt{1+m^2}$ is a tangent to the circle $x^2 + y^2 = b^2$ for all values of m. If it also touches the circle $(x - a)^2 + y^2 = b^2$, then the length of the perpendicular from its centre $(a, 0)$ on this line is equal to the radius b of the circle, which gives

$$\frac{ma - b\sqrt{1+m^2}}{\sqrt{1+m^2}} = \pm b$$

Taking negative value on R.H.S. we get $m = 0$, so we neglect it. Taking the positive value on R.H.S. we get

$$ma = 2b\sqrt{1+m^2}$$

$\Rightarrow \quad m^2(a^2 - 4b^2) = 4b^2$

$\Rightarrow \quad m = \dfrac{2b}{\sqrt{a^2 - 4b^2}}.$

◉ **Example 93:** Let PQ and RS be tangents at the extremities of a diameter PR of a circle of radius r. Such that PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals.

(a) $\sqrt{PQ.RS}$ (b) $\dfrac{PQ + RS}{2}$

(c) $\dfrac{2PQ.RS}{PQ + RS}$ (d) $\sqrt{\dfrac{PQ^2 + RS^2}{2}}$

Ans. (a)

◎ **Solution:** From Fig. 17.28, we have

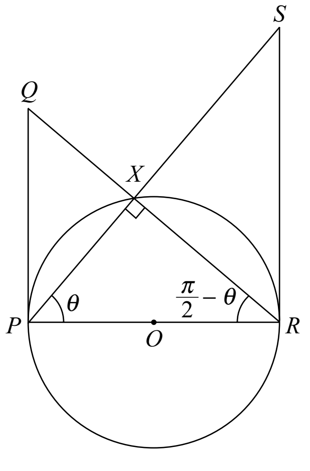


Fig. 17.28

$$\frac{PQ}{PR} = \tan(\pi/2 - \theta) = \cot\theta.$$

and $$\frac{RS}{PR} = \tan\theta$$

$\Rightarrow \quad \dfrac{PQ}{PR} \cdot \dfrac{RS}{PR} = 1$

$\Rightarrow \quad (PR)^2 = PQ \,.\, PS$

$\Rightarrow \quad (2r)^2 = PQ \,.\, PS$

$\Rightarrow \quad 2r = \sqrt{PQ.PS}$

◉ **Example 94:** If two distinct chords, drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $pq \neq 0$) are bisected by the x-axis, then

(a) $p^2 = q^2$ (b) $p^2 = 8q^2$

(c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

Ans. (d)

◎ **Solution:** Let PQ be a chord of the given circle passing through P (p, q) and the coordinates of Q be (x, y). Since PQ is bisected by the x-axis, the mid-point of PQ lies on the x-axis which gives $y = -q$.

Now Q lies on the circle $x^2 + y^2 - px - qy = 0$

so $\quad x^2 + q^2 - px + q^2 = 0$

$$\Rightarrow \quad x^2 - px + 2q^2 = 0 \qquad \text{(i)}$$

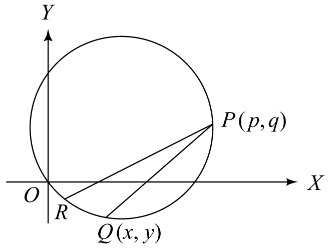


Fig. 17.29

which gives two values of x and hence the coordinates of two points Q and R (say), so that the chords PQ and PR are bisected by x-axis. If the chords PQ and PR are distinct, the roots of (i) are real distinct.

$\Rightarrow$ the discriminant $p^2 - 8q^2 > 0$

$\Rightarrow \quad p^2 > 8q^2$.

◉ **Example 95:** Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then L_1 can be represented by

(a) $x + y = 0$ (b) $x - y = 0$

(c) $7x + y = 0$ (d) $x - 7y = 0$

Ans. (b)

◎ **Solution:** Let the equation of L_1 be $y = mx$. Since the intercepts made by the circle on L_1 and L_2 are equal, their distances from the centre of the circle are also equal. Centre of the given circle is $(1/2, -3/2)$, so that we have

$$\left|\frac{\frac{1}{2} - \frac{3}{2} - 1}{\sqrt{1+1}}\right| = \left|\frac{m \times \frac{1}{2} + \frac{3}{2}}{\sqrt{m^2+1}}\right|$$

$$\Rightarrow \quad \frac{2}{\sqrt{2}} = \frac{|m+3|}{2\sqrt{m^2+1}}$$

$$\Rightarrow \quad 8(m^2 + 1) = (m + 3)^2$$

$$\Rightarrow \quad 7m^2 - 6m - 1 = 0$$

$$\Rightarrow \quad (m - 1)(7m + 1) = 0$$

$$\Rightarrow \quad m = 1 \quad \text{or} \quad m = -1/7$$

So the equations representing L_1 are

$$y = x$$

or $$y = (-1/7)\,x$$

$$\Rightarrow \quad x - y = 0 \quad \text{or} \quad x + 7y = 0$$

So (b) is the correct answer

◉ **Example 96:** Let $A_0 A_1 A_2 A_3 A_4 A_5$ be a regular hexagon inscribed in a unit circle with centre at the origin. Then the product of the lengths of the line segments $A_0 A_1$, $A_0 A_2$ and $A_0 A_4$ is

(a) 3/4 (b) $3\sqrt{3}$

(c) 3 (d) $3\sqrt{3}/2$

Ans. (c)

◎ **Solution:**

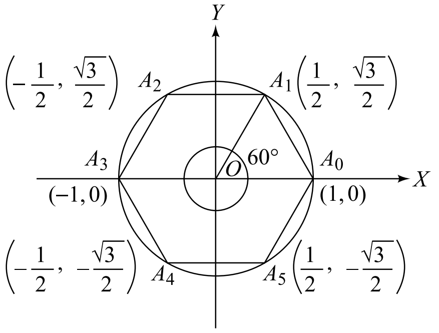

Y
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
A_2
$A_1\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
A_3
O
60°
A_0
X
(−1,0)
(1,0)
$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
A_4
$A_5\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Fig. 17.30

Let O be the centre of the circle of unit radius and the coordinates of A_0 be $(1, 0)$.

Since each side of the regular hexagon makes an angle of 60° at the centre O.

Coordinates of A_1 are $(\cos 60°, \sin 60°) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

A_2 are $(\cos 120°, \sin 120°) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

A_3 are $(-1, 0)$

A_4 are $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and A_5 are $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$$\text{Now} \quad A_0 A_1 = \sqrt{\left(1 - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$A_0 A_2 = \sqrt{\left(1 + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{3}{4}}$$

$$= \sqrt{3} = A_0 A_4$$

$$\text{So that} \quad (A_0 A_1)(A_0 A_2)(A_0 A_4) = 3$$

◉ **Example 97:** For each natural number k, let C_k denote the circle with radius k centimeters and centre at the origin O. On the circle C_k a particle moves k centimetres in the counter-clockwise direction. After completing its motion on C_k, the particle moves to C_{k+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at $(1, 0)$. If the particle crosses the positive direction of x-axis for the first time on the circle C_n, then $n =$

(a) 4 (b) 5

(c) 6 (d) 7

Ans. (d)

◎ **Solution:** The motion of the particle on the first four circles is shown with bold line in the figure. Note that on every circle the particle travels just one radian. The particle crosses the positive direction of x-axis first time on C_n, where n is the least positive integer such that $n \geq 2\pi \Rightarrow n = 7$.

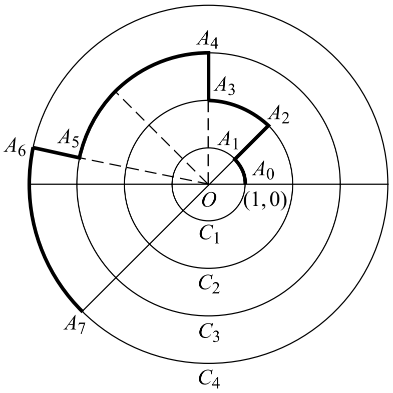


Fig. 17.31

◉ **Example 98:** C_1 and C_2 are circles of unit radius with centres at (0, 0) and (1, 0) respectively. C_3 is a circle of unit radius, passes through the centres of the circles C_1 and C_2 and have its centre above x-axis. Equation of the common tangent to C_1 and C_3 which does not pass through C_2 is

(a) $x - \sqrt{3}\,y + 2 = 0$ (b) $\sqrt{3}\,x - y + 2 = 0$

(c) $\sqrt{3}\,x - y - 2 = 0$ (d) $x + \sqrt{3}\,y + 2 = 0$

Ans. (b)

◎ **Solution:** Equation of any circle through (0, 0) and (1, 0) is

$$(x - 0)(x - 1) + (y - 0)(y - 0) + \lambda \begin{vmatrix} x & y & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x^2 + y^2 - x + \lambda y = 0$$

If it represents C_3, its radius = 1

$$\Rightarrow \quad 1 = (1/4) + (\lambda^2/4) \quad \Rightarrow \quad \lambda = \pm\sqrt{3}$$

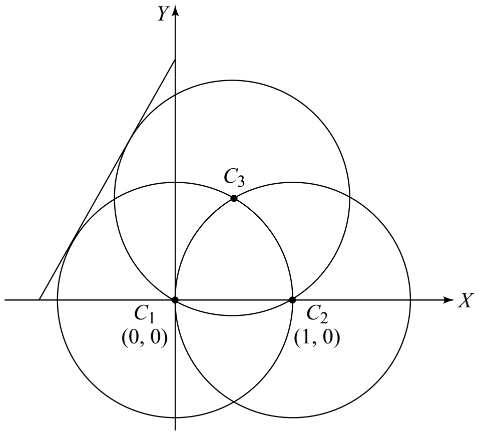


Fig. 17.32

As the centre of C_3, lies above the x-axis, we take $\lambda = -\sqrt{3}$ and thus an equation of C_3 is $x^2 + y^2 - x - \sqrt{3}y = 0$

Since C_1 and C_3 intersect and are of unit radius, their common tangents are parallel to the line joining their centres (0, 0) and (1/2, $\sqrt{3}$ /2).

So, let the equation of a common tangent be

$$\sqrt{3}\,x - y + k = 0$$

It will touch C_1, if

$$\left|\frac{k}{\sqrt{3+1}}\right| = 1 \Rightarrow \quad k = \pm 2$$

From the figure (16.32), we observe that the required tangent makes positive intercept on the y-axis and negative on the x-axis and hence its equation is $\sqrt{3}\,x - y + 2 = 0$.

◉ **Example 99:** A chord of the circle $x^2 + y^2 - 4x - 6y = 0$ passing through the origin subtends an angle $\tan^{-1}$ (7/4) at the point where the circle meets positive y-axis. Equation of the chord is

(a) $2x + 3y = 0$ (b) $x + 2y = 0$

(c) $x - 2y = 0$ (d) $2x - 3y = 0$

Ans. (c)

◎ **Solution:** The given circle passes through the origin O and meets the positive Y-axis at B (0, 6). Let OP be the chord of the circle passing through the origin subtending an angle θ at B, where $\tan\theta = 7/4$

$\Rightarrow \quad \angle OBP = \theta$

Equation of the tangent OT at O to the given circle is

$$2x + 3y = 0$$

$\Rightarrow$ slope of the tangent = – 2/3

So that, if $\angle XOT = \alpha$, $\tan\alpha = 2/3$

From geometry, $\angle POT = \angle OBP = \theta$

$\Rightarrow \quad \angle POX = \theta - \alpha$

and
$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$$

$$= \frac{\frac{7}{4} - \frac{2}{3}}{1 + \frac{7}{4} \times \frac{2}{3}} = \frac{13}{26} = \frac{1}{2}$$

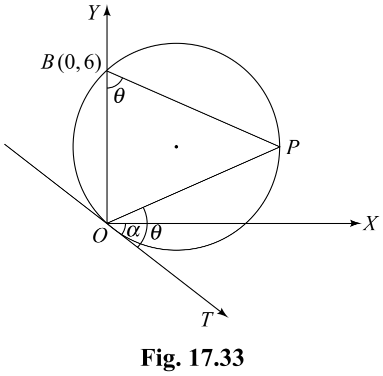


Fig. 17.33

Hence the equation of OP is $y = x \tan(\theta - \alpha)$

$\Rightarrow \quad x - 2y = 0$

Example 100: On the line joining the points A (0, 4) and B (3, 0), a square $ABCD$ is constructed on the side of the line away from the origin. Equation of the circle having centre at C and touching the axis of x is

(a) $x^2 + y^2 - 14x - 6y + 49 = 0$
(b) $x^2 + y^2 - 14x - 6y + 9 = 0$
(c) $x^2 + y^2 - 6x - 14y + 49 = 0$
(d) $x^2 + y^2 - 6x - 14y + 9 = 0$

Ans. (a)

Solution: Let $\angle ABO = \theta$ then $\angle CBL = 90^\circ - \theta$, CL being perpendicular to x-axis. The coordinates of C are (OL, LC)

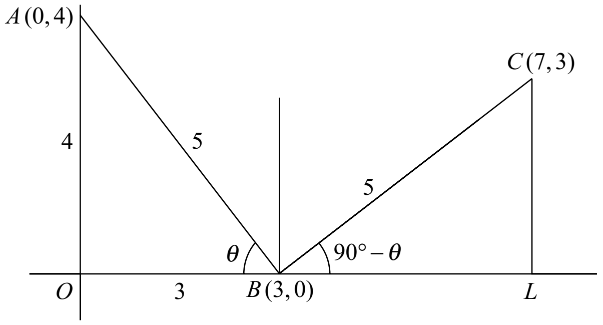


Fig. 17.34

$$OL = OB + BL = 3 + 5 \sin\theta$$

$$= 3 + 5 \times (4/5) = 7$$

$$CL = 5 \cos\theta = 5 \times (3/5) = 3$$

So the coordinate of C are (7, 3) and the equation of the circle having C as centre and touching x-axis is

$(x - 7)^2 + (y - 3)^2 = (CL)^2 = 9$

$\Rightarrow \quad x^2 + y^2 - 14x - 6y + 49 = 0.$

Example 101: A circle with centre at the origin and radius equal to a meets the axis of x at A and B. $P(\alpha)$ and $Q(\beta)$ are two points on this circle so that $\alpha - \beta = 2\gamma$, where γ is a constant. The locus of the point of intersection of AP and BQ is

(a) $x^2 - y^2 - 2ay \tan\gamma = a^2$
(b) $x^2 + y^2 - 2ay \tan\gamma = a^2$
(c) $x^2 + y^2 + 2ay \tan\gamma = a^2$
(d) $x^2 - y^2 + 2ay \tan\gamma = a^2$

Ans. (b)

Solution: Coordinates of A are $(-a, 0)$ and of P are $(a\cos\alpha, a\sin\alpha)$

$\therefore$ Equation of AP is

$$y = \frac{a\sin\alpha}{a(\cos\alpha + 1)}(x + a)$$

or $\qquad y = \tan(\alpha/2)(x + a) \qquad$ (i)

Similarly equation of BQ is

$$y = \frac{a\sin\beta}{a(\cos\beta - 1)}(x - a)$$

or $\qquad y = -\cot(\beta/2)(x - a) \qquad$ (ii)

We now eliminate α, β from (i) and (ii)

From (i) and (ii) $\tan(\alpha/2) = \dfrac{y}{a + x}$, $\tan(\beta/2) = \dfrac{a - x}{y}$

Now $\quad \alpha - \beta = 2\gamma$

$$\Rightarrow \qquad \tan\gamma = \frac{\tan(\alpha/2) - \tan(\beta/2)}{1 + \tan(\alpha/2)\tan(\beta/2)} = \frac{\dfrac{y}{a + x} - \dfrac{a - x}{y}}{1 + \dfrac{y}{a + x}\cdot\dfrac{a - x}{y}}$$

$$\Rightarrow \qquad \tan\gamma = \frac{y^2 - (a^2 - x^2)}{(a + x)y + (a - x)y} = \frac{x^2 + y^2 - a^2}{2ay}$$

$$\Rightarrow \qquad x^2 + y^2 - 2ay\tan\gamma = a^2$$

which is the required locus.

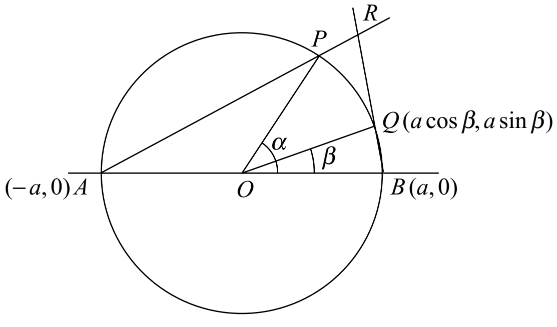


Fig. 17.35

Example 102: Equation of a circle having radius equal to twice the radius of the circle $x^2 + y^2 + (2p + 3)x + (3 - 2p)y + p - 3 = 0$ and touching it at the origin is

(a) $x^2 + y^2 + 9x - 3y = 0$
(b) $x^2 + y^2 - 9x + 3y = 0$
(c) $x^2 + y^2 + 18x + 6y = 0$
(d) $x^2 + y^2 + 18x - 6y = 0$

Ans. (d)

Solution: Since the given circle passes through the origin $p - 3 = 0 \Rightarrow p = 3$ and the equation of the given circle is

$$x^2 + y^2 + 9x - 3y = 0$$

Equation of the tangent at the origin to this circle is

$$9x - 3y = 0 \qquad \text{(i)}$$

Let the equation of the required circle which also passes through the origin be

$$x^2 + y^2 + 2gx + 2fy = 0.$$

Equation of the tangent at the origin to this circle is

$$gx + fy = 0 \qquad \text{(ii)}$$

If (i) and (ii) represent the same line, then

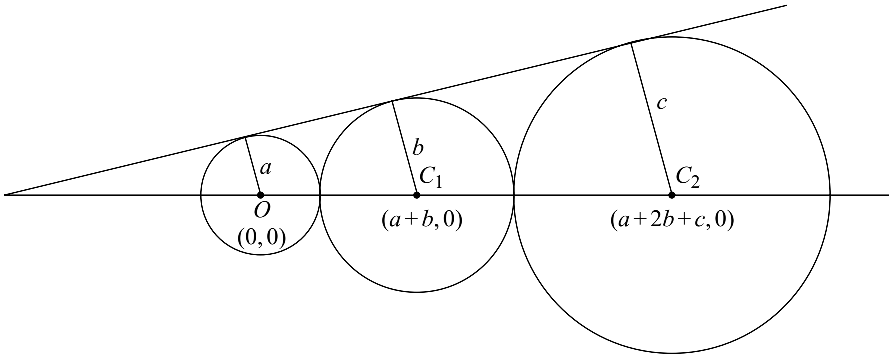


Fig. 17.36

$$\frac{g}{9} = \frac{f}{-3} = k \text{ (say)} \qquad \text{(iii)}$$

We are given that $\sqrt{g^2 + f^2} = 2\sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{-3}{2}\right)^2} = \sqrt{81+9}$

From (iii) we get $|k|\sqrt{9^2 + 3^2} = \sqrt{90} \quad \Rightarrow \quad k = \pm 1$

For $k = 1$,

$g = 9, f = -3$ and the equation of the required circle is $x^2 + y^2 + 18x - 6y = 0$.

Example 103: A circle C_1 of radius b touches the circle $x^2 + y^2 = a^2$ externally and has its centre on the positive x-axis; another circle C_2 of radius c touches the circle C_1 externally and has its centre on the positive x-axis. Given $a < b < c$, then the three circles have a common tangent if a, b, c are in

(a) A.P.
(b) G.P.
(c) H.P.
(d) none of these

Ans. (b)

Solution: Refer Fig. 17.36. The centre of C_1 is $(a + b, 0)$ and the centre of C_2 is $(a + 2b + c, 0)$

Let $y = mx + k$ be a tangent common to the three circles. Since it touches $x^2 + y^2 = a^2$, C_1 and C_2

$$\frac{k}{\sqrt{1+m^2}} = \pm a, \quad \frac{m(a+b)+k}{\sqrt{1+m^2}} = \pm b$$

$$\text{and } \frac{m(a+2b+c)+k}{\sqrt{1+m^2}} = \pm c$$

As the centre of the three circles lie on the same side of the line $y = mx + k$, taking the same sign, say positive, in the three relations we get,

$$\frac{k}{\sqrt{1+m^2}} = a = b - \frac{m+1}{1-m} = c - m\left(\frac{1}{5}, \frac{9}{5}\right)$$

$\Rightarrow \quad \dfrac{a+b}{b-a} = \dfrac{a+2b+c}{c-a}$ (eliminating m)

$\Rightarrow \quad (a + b)(c - a) = (b - a)(a + 2b + c)$

$\Rightarrow \quad ac - a^2 + bc - ba = ba - a^2 + 2b^2 - 2ab + bc - ac$

$\Rightarrow \quad 2ac = 2b^2 \quad \Rightarrow \quad ac = b^2$

$\Rightarrow \quad a, b, c$ are in G.P.

Example 104: If two circles, each of radius 5 units, touch each other at (1, 2) and the equation of their common tangent is $4x + 3y = 10$, then equation of the circle, a portion of which lies in all the quadrants is

(a) $x^2 + y^2 - 10x - 10y + 25 = 0$
(b) $x^2 + y^2 + 6x + 2y - 15 = 0$
(c) $x^2 + y^2 + 2x + 6y - 15 = 0$
(d) $x^2 + y^2 + 10x + 10y + 25 = 0$

Ans. (b)

Solution: The centres of the two circles will lie on the line through P (1, 2) perpendicular to the common tangent $4x + 3y = 10$. If C_1 and C_2 are the centres of these circles then $PC_1 = 5 = r_1$, $PC_2 = -5 = r_2$. Also C_1, C_2 lie on the line $\dfrac{x-1}{\cos\theta} = \dfrac{y-2}{\sin\theta} = r$ where $\tan\theta = 3/4$. When $r = r_1$ the coordinates of C_1 are $(5\cos\theta + 1, 5\sin\theta + 2)$ or (5, 5) as $\cos\theta = 4/5$, $\sin\theta = 3/5$.

When $r = r_2$, the coordinates of C_2 are $(-3, -1)$

The circle with centre C_1 (5, 5) and radius 5 touches both the coordinate axes and hence lies completely in the first quadrant.

Therefore the required circle is with centre $(-3, -1)$ and radius 5, so its equation is

$$(x + 3)^2 + (y + 1)^2 = 5^2$$

$$\text{or} \quad x^2 + y^2 + 6x + 2y - 15 = 0$$

Since the origin lies inside the circle, a portion of the circle lies in all the quadrants.

Example 105: The locus of the point of intersection of the tangents at the extremities of a chord of the circle $x^2 + y^2 = a^2$ which touches the circle $x^2 + y^2 - 2ax = 0$ is

(a) $y^2 = a(a - 2x)$
(b) $x^2 = a(a - 2y)$
(c) $x^2 + y^2 = (x + a)^2$
(d) $x^2 + y^2 = (y + a)^2$

Ans. (a)

◎ **Solution:** Let $P(h, k)$ be the point of intersection of the tangents at the extrimities of the chord AB of the circle $x^2 + y^2 = a^2$. Then AB is the chord of contact of the tangents from P to the circle and so its equation is

$$hx + ky = a^2 \qquad \text{(i)}$$

If (i) touches the circle $x^2 + y^2 - 2ax = 0$

then $$\frac{(h)(a) + (k)(0) - a^2}{\sqrt{h^2 + k^2}} = \pm a$$

⇒ $(h - a)^2 = h^2 + k^2$

⇒ locus of (h, k) is $(x - a)^2 = x^2 + y^2$

⇒ $y^2 = a(a - 2x)$.

◉ **Example 106:** If θ is the angle subtended by the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at a point $P(x_1, y_1)$ outside the circle and $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$, then $\cos\theta$ is equal to

(a) $\dfrac{S_1 + c - g^2 - f^2}{S_1 - c + g^2 + f^2}$ (b) $\dfrac{S_1 - c + g^2 + f^2}{S_1 + c - g^2 + f^2}$

(c) $\dfrac{S_1 + c + g^2 - f^2}{S_1 - c + g^2 - f^2}$ (d) $\dfrac{S_1 - c + g^2 + f^2}{S_1 + c + g^2 - f^2}$

Ans. (a)

◎ **Solution:** Let PA and PB be the tangents from $P(x_1, y_1)$ to the given circle with centre $C(-g, -f)$ such that $\angle APB = \theta$ then $\angle APC = \angle CPB = \theta/2$

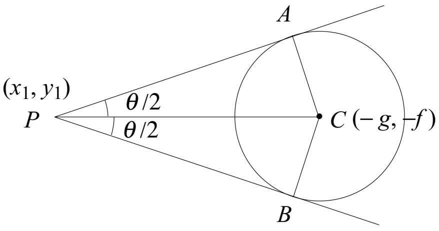


Fig. 17.37

From right angled triangle PAC

$$\tan\frac{\theta}{2} = \frac{CA}{PA} = \sqrt{\frac{g^2 + f^2 - c}{S_1}}$$

Now $$\cos\theta = \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} = \frac{S_1 - \left(g^2 + f^2 - c\right)}{S_1 + \left(g^2 + f^2 - c\right)}$$

$$= \frac{S_1 + c - g^2 - f^2}{S_1 - c + g^2 + f^2}$$

◉ **Example 107:** If the area of the quadrilateral formed by the tangent from the origin to the circle $x^2 + y^2 + 6x - 10y + c = 0$ and the pair of radii at the points of contact of these tangents to the circle is 8 square units, then c is a root of the equation

(a) $c^2 - 32c + 64 = 0$ (b) $c^2 - 34c + 64 = 0$
(c) $c^2 + 2c - 64 = 0$ (d) $c^2 + 34c - 64 = 0$

Ans. (b)

◎ **Solution:** Let OA, OB be the tangents from the origin to the given circle with centre $C(-3, 5)$ and radius $\sqrt{9 + 25 - c}$ $= \sqrt{34 - c}$.

Then area of the quadrilateral $OACB$

$$= 2 \times \text{area of the triangle } OAC$$
$$= 2 \times (1/2) \times OA \times AC$$

Now OA = length of the tangent from the origin to the given circle = $\sqrt{c}$

and AC = radius of the circle = $\sqrt{34 - c}$

so that $\sqrt{c}\,\sqrt{34 - c} = 8$ (given) ⇒ $c(34 - c) = 64$

⇒ $c^2 - 34c + 64 = 0$

◉ **Example 108:** An equation of the circle in which the chord joining the points (1, 2) and (2, – 1) subtends an angle of $\pi/4$ at any point on the circumference is

(a) $x^2 + y^2 - 15 = 0$
(b) $x^2 + y^2 - 6x - 2y + 5 = 0$
(c) $x^2 + y^2 + 6x + 2y - 15 = 0$
(d) $x^2 + y^2 - 2x - 4y + 4 = 0$

Ans. (b)

◎ **Solution:** Let $P(x, y)$ be any point on the circle passing through $A(1, 2)$ and $B(2, -1)$ the slope of $AP = \dfrac{y - 2}{x - 1}$ and slope of $BP = \dfrac{y + 1}{x - 2}$

Since they include an angle of $\pi/4$ so

$$\frac{\dfrac{y + 1}{x - 2} - \dfrac{y - 2}{x - 1}}{1 + \dfrac{y - 2}{x - 1} \times \dfrac{y + 1}{x - 2}} = \pm \tan\frac{\pi}{4} = \pm 1$$

⇒ $(y + 1)(x - 1) - (x - 2)(y - 2) = \pm[(x - 1)(x - 2) + (y - 2)(y + 1)]$

⇒ $3x + y - 5 = \pm[x^2 + y^2 - 3x - y]$

⇒ $x^2 + y^2 - 5 = 0$ or $x^2 + y^2 - 6x - 2y + 5 = 0$.

◉ **Example 109:** An equation of a common tangent to the circles $x^2 + y^2 + 14x - 4y + 28 = 0$ and

$x^2 + y^2 - 14x + 4y - 28 = 0$ is

(a) $x - 7 = 0$ (b) $y + 7 = 0$
(c) $28x + 45y + 371 = 0$ (d) None of these

Ans. (c)

◎ **Solution:** By subtracting the equation of one of the circles from that of the other, we get the equation of their common chord as $7x - 2y + 14 = 0$. This chord intersects the circles at two real, distinct points, so the given circles have two common tangents. Let PA_1A_2 be a common tangent to these circle, whose centres are $O_1(-7, 2)$ and $O_2(7, -2)$, and radii 5 and 9, respectively. Then triangles PO_1A_1 and PO_2A_2 are similar, so

$$\frac{PO_1}{O_1A_1} = \frac{PO_2}{O_2A_2} \quad \Rightarrow \quad \frac{PO_1}{5} = \frac{PO_2}{9}.$$

⇒ P divides O_1O_2 externally in the ratio 5 : 9, so its coordinates are $(-49/2, 7)$. Thus any line through P is given by

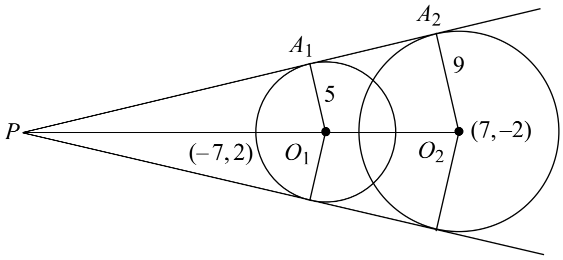


Fig. 17.38

$$y - 7 = m\left(x + \frac{49}{2}\right). \qquad (1)$$

This line will touch the circle of radius 5 centred at O_1 if

$$\frac{m\left(-7 + \frac{49}{2}\right) - 2 + 7}{\sqrt{1+m^2}} = \pm 5$$

$$\Rightarrow \quad \frac{35m}{2} + 5 = \pm 5\sqrt{1+m^2}$$

$$\Rightarrow \quad \left(\frac{7m}{2} + 1\right)^2 = 1 + m^2$$

$$\Rightarrow \quad \frac{45m^2}{4} + 7m = 0,$$

⇒ $m = 0$ or $m = -28/45$.

Substituting $m = -28/45$ in (1) we get the equation given in (c).

Example 110: The locus of the point, the sum of the squares of whose distances from n fixed points $A_i\ (x_i, y_i)$, $i = 1, 2, \ldots, n$ is equal to k^2 is a circle

(a) passing through the origin
(b) with centre at the origin
(c) with centre at the point of mean position of the given points
(d) none of these

Ans. (c)

Solution: Let (x, y) be any point on the locus, then

$$\sum_{i=1}^{n} [(x - x_i)^2 + (y - y_i)^2] = k^2$$

$$\Rightarrow \quad n(x^2 + y^2) - 2x\sum_{i=1}^{n} x_i - 2y\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - k^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)x - 2\left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)y + \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - k^2\right) = 0$$

which is a circle with centre $\left(\frac{1}{n}\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i\right)$ the point of mean position of the given points.

Example 111: An isosceles triangle ABC is inscribed in a circle $x^2 + y^2 = a^2$ with the vertex A at $(a, 0)$ and the base angles B and C each equal to 75°, then length of the base BC is

(a) $a/2$	(b) a
(c) $2a/\sqrt{3}$	(d) $\sqrt{3}\,a/2$

Ans. (b)

Solution: $\angle B = \angle C = 75°$

⇒ $\angle BAC = 30°$

⇒ $\angle BOC = 60°$

⇒ BOC is an equilateral triangle

⇒ $BC = OB$ = the radius of the circle

⇒ $BC = a$.

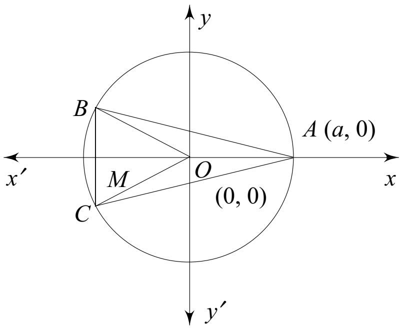


Fig. 17.39

Example 112: If the locus of a point which moves so that the line joining the points of contact of the tangents drawn from it to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = a^2$, is the circle $x^2 + y^2 = c^2$, then a, b, c are in

(a) A.P.	(b) G.P.
(c) H.P.	(d) none of these

Ans. (b)

Solution: Let $P\ (h, k)$ be any point on the locus. Equation of the chord of contact of P with respect to the circle $x^2 + y^2 = b^2$ is $hx + ky = b^2$.

If it touches the circle $x^2 + y^2 = a^2$, then

$$\left|\frac{-b^2}{\sqrt{h^2+k^2}}\right| = a \quad \Rightarrow \quad a^2(h^2+k^2) = b^4$$

So that the locus of $P(h, k)$ is $x^2 + y^2 = (b^2/a)^2$

$$\therefore \qquad c^2 = \left(\frac{b^2}{a}\right)^2 \quad \Rightarrow \quad ac = b^2$$

$\Rightarrow a, b, c$ are in G.P.

Example 113: If the line $3x - 4y - k = 0$, $(k > 0)$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b), then $k + a + b$ is equal to

(a) 20 (b) 22

(c) −30 (d) −28

Ans. (a)

Solution: Since the given line touches the given circle, the length of the perpendicular from the centre (2, 4) of the circle to the line $3x - 4y - k = 0$ is equal to the radius $\sqrt{4+16+5} = 5$ of the circle.

$$\Rightarrow \qquad \frac{3\times2-4\times4-k}{\sqrt{9+16}} = \pm 5$$

$$\Rightarrow \qquad k = 15 \qquad [\because k > 0]$$

Now equation of the tangent at (a, b) to the given circle is

$$xa + yb - 2(x + a) - 4(y + b) - 5 = 0$$

$$\Rightarrow \qquad (a - 2)x + (b - 4)y - (2a + 4b + 5) = 0.$$

If it represents the given line $3x - 4y - k = 0$

$$\text{then} \quad \frac{a-2}{3} = \frac{b-4}{-4} = \frac{2a+4b+5}{k} = l \text{ (say)}$$

then $a = 3l + 2$, $b = 4 - 4l$ and $2a + 4b + 5 = kl$ (1)

$$\Rightarrow \qquad 2(3l + 2) + 4(4 - 4l) + 5 = 15l \qquad (\because k = 15)$$

$$\Rightarrow \qquad l = 1 \quad \Rightarrow \quad a = 5, b = 0 \text{ and } k + a + b = 20.$$

Example 114: The circles $x^2 + y^2 + 2g_1x - a^2 = 0$ and $x^2 + y^2 + 2g_2x - a^2 = 0$ cut each other orthogonally. If p_1, p_2 are perpendiculars from $(0, a)$ and $(0, -a)$ on a common tangent of these circles then $p_1 p_2 =$

(a) $a^2/2$ (b) a^2

(c) $2a^2$ (d) $a^2 + 2$

Ans. (b)

Solution: Since the given circles cut each other orthogonally

$$g_1 g_2 + a^2 = 0 \qquad (1)$$

If $lx + my = 1$ is a common tangent of these circles, then

$$\frac{-lg_1-1}{\sqrt{l^2+m^2}} = \pm\sqrt{g_1^2+a^2}$$

$$\Rightarrow \qquad (lg_1 + 1)^2 = (l^2 + m^2)(g_1^2 + a^2)$$

$$\Rightarrow \qquad m^2 g_1^2 - 2lg_1 + a^2(l^2 + m^2) - 1 = 0$$

Similarly $m^2 g_2^2 - 2lg_2 + a^2(l^2 + m^2) - 1 = 0$

So that g_1, g_2 are the roots of the equation

$$m^2g^2 - 2lg + a^2(l^2 + m^2) - 1 = 0$$

$$\Rightarrow \qquad g_1g_2 = \frac{a^2(l^2+m^2)-1}{m^2} = -a^2 \text{ by (1)}$$

$$\Rightarrow \qquad a^2(l^2 + m^2) = 1 - a^2m^2 \qquad (2)$$

$$\text{Now} \qquad p_1p_2 = \frac{|ma-1|}{\sqrt{l^2+m^2}}\cdot\frac{|-ma-1|}{\sqrt{l^2+m^2}}$$

$$= \frac{|1-m^2a^2|}{l^2+m^2} = a^2 \text{ by (2)}$$

Example 115: *A* circle *C* touches the *x*-axis and the circle $x^2 + (y - 1)^2 = 1$ externally, then locus of the centre of the circle *C* is given by

(a) $\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \le 0\}$.

(b) $\{(x, y) : y = x^2\} \cup \{(0, y) : y \le 0\}$

(c) $\{(x, y) : x^2 + (y - 1)^2 = 0\} \cup \{(0, y) : y \le 0\}$

(d) $\{(x, y) : x^2 + 4y = 0\} \cup \{(0, y) : y \le 0\}$

Ans. (a)

Solution: Let the centre of *C* be $(h, \pm r)$ and radius r because it touches *x*-axis.

Since it touches the given circle with radius 1 and centre (0, 1), externally

$$h^2 + (1 \pm r)^2 = (r + 1)^2.$$

$$\Rightarrow \qquad h^2 = (r + 1)^2 - (r \pm 1)^2$$

$$\Rightarrow \qquad h^2 = 0 \text{ or } h^2 = 4r.$$

$\therefore$ Locus of (h, r) is

$\{(x, y) : x^2 = 4y\} \cup \{(0, y) : y \le 0\}$.

EXERCISE

Concept-based Straight Objective Type Questions

1. Equation of the circle passing through the origin and having its centre on the line $y = 3x$ at a distance $\sqrt{10}$ from the origin is

(a) $x^2 + y^2 - 2x + 6y = 0$ (b) $x^2 + y^2 + 2x - 6y = 0$

(c) $x^2 + y^2 - 2x - 6y = 0$ (d) none of these

2. The radius of the circle

   $3x^2 + by^2 + 4bx - 6by + b^2 = 0$ is

   (a) 1 (b) 3

   (c) $\sqrt{10}$ (d) $\sqrt{11}$

3. Equation of the circle described on the line segment of $3x + 4y = 12$ intercepted between the axes as a diameter is

   (a) $x^2 + y^2 - 4x - 3y = 0$
   (b) $x^2 + y^2 + 4x - 3y = 0$
   (c) $x^2 + y^2 - 4x + 3y = 0$
   (d) $x^2 + y^2 + 4x + 3y = 0$

4. The point (1, 2) lies inside and (3, 4) outside the circle $x^2 + y^2 - 7x + 15y - c = 0$, if

   (a) $c = 25$ (b) $c = 35$

   (c) $c = 65$ (d) c takes any real value

5. $S : x^2 + y^2 + 6x - 14y - 6 = 0$ is a circle and $L : 7x + 3y + 58 = 0$ is a straight line

   (a) L is a diameter of S (b) L is a chord of S

   (c) L is a tangent to S (d) none of these

6. The angle between the two tangents from the origin to the circle $(x - 7)^2 + (y + 1)^2 = 25$ is

   (a) $\pi/6$ (b) $\pi/4$

   (c) $\pi/3$ (d) $\pi/2$

7. A line passes through the point $P(5, 6)$ outside the circle $x^2 + y^2 = 12$ and meets the circle at A and B. The value of $PA \cdot PB$ is equal to

   (a) 25 (b) 36

   (c) 49 (d) 61

8. The tangent to the circle $x^2 + y^2 = 5$ at $(1, -2)$ touches the circle $x^2 + y^2 - 8x + 6y + 20 = 0$ at the point

   (a) (3, 1) (b) (3, – 1)

   (c) (– 3, 1) (d) (1, – 3)

9. Two circles of equal radius of 5 units have their centres at the origin and the point (2, – 3). Equation of the common chord of these circles is

   (a) $4x - 6y - 13 = 0$ (b) $2x - 3y + 13 = 0$

   (c) $4x - 6y + 12 = 0$ (d) $2x - 3y - 12 = 0$

10. Two circles touch each other externally at the point $(0, k)$ and y-axis is the common tangent to these circles. Centres of these circle lie on the line

    (a) $x = k$ (b) $y = k$

    (c) $x + y = k$ (d) $x - y = k$

11. A circle has radius 3 units and its centre lies on the line $y = x - 1$. If the circle passes through the point (7, 3), then an equation of the circle is

    (a) $x^2 + y^2 - 8x - 6y + 16 = 0$
    (b) $x^2 + y^2 + 6x + 8y + 16 = 0$
    (c) $x^2 + y^2 - 14x - 12y - 76 = 0$
    (d) $x^2 + y^2 + 12x + 14y - 76 = 0$

12. The line $3x - y - 17 = 0$ meets the circle $x^2 + y^2 - 8x + 10y + 5 = 0$ at the point A and B. P is any point on the circle other than A or B, then the triangle APB is

    (a) equilateral (b) isosceles

    (c) right angled (d) obtuse angled

13. A circle passes through the origin and its centre is on the line $y = x$. If it cuts the circle $x^2 + y^2 - 4x - 6y + 10 = 0$ orthogonally, then the equation of the circle is

    (a) $x^2 + y^2 + 2x + 2y = 0$ (b) $x^2 + y^2 - 2x - 2y = 0$

    (c) $2x^2 + 2y^2 - x - y = 0$ (d) $2x^2 + 2y^2 + x + y = 0$

14. Equation of the circle on the common chord of the circles $x^2 + y^2 - ax = 0$ and $x^2 + y^2 - ay = 0$ as a diameter is

    (a) $2x^2 + 2y^2 - ax - ay = 0$
    (b) $x^2 + y^2 - ax - ay = 0$
    (c) $2x^2 + 2y^2 + ax + ay = 0$
    (d) $x^2 + y^2 - x - y + a = 0$

15. A circle touches the lines $x - y - 1 = 0$ and $x - y + 1 = 0$, the centre of the circle lies on the line

    (a) $x + y - 1 = 0$ (b) $x + y + 1 = 0$

    (c) $x - y = 0$ (d) none of these

16. The number of common tangents that can be drawn to the circles $x^2 + y^2 - 4x - 6y - 3 = 0$ and $x^2 + y^2 + 2x + 2y + 1 = 0$ is

    (a) 0 (b) 1

    (c) 2 (d) 3

17. If the circle $(x - 2)^2 + (y - 3)^2 = a^2$ lies entirely with in the circle $x^2 + y^2 = b^2$, then

    (a) $a = b$ (b) $a - b > \sqrt{13}$

    (c) $b - a > \sqrt{13}$ (d) $b - a = \sqrt{13}$

18. There are four circles each of radius 1 unit touching both the axis. The equation of the smaller circle touching all these circles is

    (a) $x^2 + y^2 = 2$ (b) $x^2 + y^2 = \left(\sqrt{2} - 1\right)^2$

    (c) $x^2 + y^2 = 4$ (d) $x^2 + y^2 = 1$

19. The locus of the point which moves so that the ratio of the lengths of the tangents to the circle $x^2 + y^2 - 4y + 3 = 0$ and $x^2 + y^2 + 6y + 5 = 0$ is

    (a) $5x^2 + 5y^2 + 70x - 33 = 0$
    (b) $5x^2 + 5y^2 + 60y + 7 = 0$
    (c) $5x^2 + 5y^2 + 70y + 33 = 0$
    (d) $5x^2 + 5y^2 + 60x + 7 = 0$

20. A circle has two of its diameters along the lines $2x + 3y - 18 = 0$ and $3x - y - 5 = 0$ and touches the line $x + 2y + 4 = 0$. Equation of the circle is

    (a) $x^2 + y^2 + 6x - 8y + 16 = 0$
    (b) $x^2 + y^2 - 6x + 8y + 16 = 0$
    (c) $x^2 + y^2 - 6x - 8y + 9 = 0$
    (d) $x^2 + y^2 - 6x - 8y + 16 = 0$

LEVEL 1

Straight Objective Type Questions

21. Equation of the circle with centre $\left(\frac{a}{2}, \frac{b}{2}\right)$ and radius $\sqrt{\frac{a^2+b^2}{4}}$ is

(a) $x^2 + y^2 - ax - by = (a + b)^2$
(b) $x^2 + y^2 + ax + by = (a - b)^2$
(c) $x^2 + y^2 - ax - by = \frac{(a^2+b^2)}{4}$
(d) $x^2 + y^2 - ax - by = 0$

22. The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle if

(a) $a = h = 2, b = 0$
(b) $b = h = 2, a = 0$
(c) $a = b = 2, h = 0$
(d) none of these

23. The radius of the circle passing through the point (6, 2), two of whose diameter, are $x + y = 6$ and $x + 2y = 4$ is

(a) 10
(b) $2\sqrt{5}$
(c) 6
(d) 4

24. An equation of the circle through (1, 1) and the points of intersection of $x^2 + y^2 + 13x - 3y = 0$ and $2x^2 + 2y^2 + 4x - 7y - 25 = 0$ is

(a) $4x^2 + 4y^2 - 30x - 10y - 32 = 0$
(b) $4x^2 + 4y^2 + 30x - 13y - 25 = 0$
(c) $4x^2 + 4y^2 - 43x + 10y + 25 = 0$
(d) none of these

25. An equation of the normal at the point (2, 3) to the circle $x^2 + y^2 - 2x - 2y - 3 = 0$ is

(a) $2x + y - 7 = 0$
(b) $x + 2y - 3 = 0$
(c) $2x - y - 1 = 0$
(d) $x - 2y + 1 = 0$

26. The line $x + y \tan \theta = \cos \theta$ touches the circle $x^2 + y^2 = 4$ for

(a) $\theta = \pi/6$
(b) $\theta = \pi/3$
(c) $\theta = \pi/2$
(d) no value of θ

27. If $(x, 3)$ and (3, 5) are the extremities of a diameter of a circle with centre at $(2x, y)$, then the values of x and y are

(a) $x = 1, y = 4$
(b) $x = 4, y = 1$
(c) $x = 8, y = 2$
(d) none of these

28. The circle $(x - r)^2 + (y - r)^2 = r^2$ touches

(a) x-axis at the origin
(b) y-axis at the origin
(c) both the coordinate axes
(d) none of these

29. The number of circles touching the coordinate axes and the line $x + y = 1$ is

(a) exactly one
(b) two
(c) three
(d) four

30. The centre of circle passing through the points (0 0), (1, 0) and touching the circle $x^2 + y^2 = 9$ is the point whose coordinates are

(a) $(1/\sqrt{2}, 2)$
(b) $(1/2, \sqrt{2})$
(c) $(\sqrt{2}, 1/2)$
(d) $(2, 1/\sqrt{2})$

31. Equation of a circle which touches y-axis at (0, 2) and cuts off an intercept of 3 units from the x-axis is $x^2 + y^2 - 2\alpha x - 4y + 4 = 0$ where $\alpha^2 =$

(a) 5/2
(b) 25/4
(c) 4/25
(d) 29

32. The circles $x^2 + y^2 + 2x - 2y + 1 = 0$ and $x^2 + y^2 - 2x - 2y + 1 = 0$

(a) touch each other externally
(b) touch each other internally
(c) intersect on the y-axis
(d) intersect on x-axis

33. The tangent at any point to the circle $x^2 + y^2 = r^2$ meets the coordinate axes at A and B. If lines drawn parallel to the coordinate axes through A and B intersect at P, the locus of P is

(a) $x^2 + y^2 = r^{-2}$
(b) $x^{-2} + y^{-2} = r^2$
(c) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{r^2}$
(d) $\frac{1}{x^2} - \frac{1}{y^2} = r^2$

34. Equation of a circle with centre $C(h, k)$ and radius 5 such that $h^2 - 3h + 2 = 0$ and $k^2 + 5k - 6 = 0$ is

(a) $x^2 + y^2 - 2x - 4y = 0$
(b) $x^2 + y^2 - 2x - 2y - 6 = 0$
(c) $x^2 + y^2 - 4x - 2y - 20 = 0$
(d) $x^2 + y^2 - 4x - 6y - 12 = 0$.

35. If $S \equiv x^2 + y^2 - 2x - 4y - 4 = 0$, $L \equiv 2x + 2y + 15 = 0$ and $P(3, 4)$ represent a circle, a line and a point respectively then

(a) L is a tangent to S at P
(b) L is polar of P with respect to S
(c) L is the chord of contact of P with respect to S
(d) P is inside and L is outside S

36. If PQR is the triangle formed by the common tangents to the circles $x^2 + y^2 + 6x = 0$ and $x^2 + y^2 - 2x = 0$, then the centroid of the triangle is at the point

(a) $(1, 0)$ (b) $(0, 0)$
(c) $(-1, 0)$ (d) none of these

37. A rectangle $ABCD$ is inscribed in the circle $x^2 + y^2 + 3x + 12y + 2 = 0$. If the coordinates of A and B are respectively $(3, -2)$ and $(-2, 0)$, then the coordinates of the mid-point of CD are

(a) $(-3/2, -6)$ (b) $(-7/2, 11)$
(c) $(5/2, -1)$ (d) $(-5/2, -1)$

38. A line makes equal intercepts of length a on the coordinate axes. A circle is circumscribed about the triangle which the line makes with the coordinate axes. The sum of the distance of the vertices of the triangle from the tangent to this circle at the origin is

(a) $a/\sqrt{2}$ (b) a
(c) $a\sqrt{2}$ (d) $a + \sqrt{2}$

39. α, β and γ are the parametric angles of three points P, Q and R, respectively, on the circle $x^2 + y^2 = 1$, and A is the point $(-1, 0)$. If the lengths of the chords AP, AQ and AR are in G.P., then $\cos(\alpha/2)$, $\cos(\beta/2)$ and $\cos(\gamma/2)$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

40. If P is a point with integral coordinates on the circle $x^2 + y^2 = 9$, Q is a point on the line $7x + y + 3 = 0$, and the line $x - y + 1 = 0$ is the perpendicular bisector of PQ, the coordinates of P are

(a) $(3, 0)$ (b) $(0, 3)$
(c) $(-72, 21)$ (d) $(-1, 1)$

41. If the circle C_1: $x^2 + y^2 = 16$ intersects another circle C_2 with centre at (a, b) and radius 5 in such a manner that the common chord is of maximum length, then $a^2 + b^2 =$

(a) 9 (b) 12
(c) 15 (d) 25

42. If OA and OB are equal chords of the circle $x^2 + y^2 - 2x + 4y = 0$ perpendicular to each other passing through the origin, then the slopes of OA and OB are the roots of the equation

(a) $2m^2 - 5m - 2 = 0$ (b) $3m^2 - 8m - 3 = 0$
(c) $6m^2 + 5m - 6 = 0$ (d) $m^2 + 1 = 0$

43. Equation of the circle passing through $(1, 0)$ and $(0, 1)$ and having the smallest possible radius is

(a) $x^2 + y^2 - x - y = 0$
(b) $x^2 + y^2 - 2x - 2y + 1 = 0$
(c) $x^2 + y^2 = 1$
(d) none of these

44. From the point A $(0, 3)$ on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point M such that $AM = 2AB$. Equation of the locus of M is

(a) $x^2 + 2x + (y - 3)^2 = 0$
(b) $(x - 3)^2 + 4y + y^2 = 0$
(c) $x^2 + 8x + (y - 3)^2 = 0$
(d) $x^2 + 4x + (y + 3)^2 = 0$

45. If the lines $3x - 4y + 4 = 0$ and $6x - 8y - 7 = 0$ are tangents to a circle, radius of the circle is

(a) 3/4 (b) 1
(c) 7/8 (d) 11/10

46. If the distances from the origin of the centres of three circles $x^2 + y^2 - 2\lambda_i x = c^2$ $(i = 1, 2, 3)$ are in G.P., then the lengths of the tangents drawn to them from any point on the circle $x^2 + y^2 = c^2$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

47. The distance between the chords of contact of the tangents to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the origin and the point (g, f) is

(a) $2\sqrt{g^2 + f^2 - c}$
(b) $(g^2 + f^2 - c)/2\sqrt{g^2 + f^2}$
(c) $(1/2)(g^2 + f^2 + c)$
(d) $g^2 + f^2 - c$

48. A variable circle passes through the fixed point A (p, q) and touches the axis of x, the locus of the other end of the diameter through A is

(a) $(x - p)^2 = 4qy$ (b) $(x - q)^2 = 4py$
(c) $(y - p)^2 = 4qx$ (d) $(y - q)^2 = 4px$

49. The locus of a point, which moves such that the lengths of the tangents from it two concentric circles of radii a and b are inversely proportional to their radii is a circle with radius

(a) $a + b$ (b) $\sqrt{|a^2 - b^2|}$
(c) $\sqrt{a^2 + b^2}$ (d) none of these

50. The circle that can be drawn to touch the coordinate axes and the line $4x + 3y = 12$ cannot lie in

(a) first quadrant (b) second quadrant
(c) third quadrant (d) fourth quadrant

51. Area of an equilateral triangle inscribed in a circle of radius a is

(a) $\sqrt{3}\,a^2/4$ (b) $\pi a^2/3$
(c) $3\sqrt{3}\,a^2/4$ (d) $\sqrt{3}\,\pi a^2/4$

52. A point P moves such that $PA/PB = p$ where A and B are two fixed points with $AB = a$, the locus of P is a circle with radius

(a) $a/(1-p^2)$ (b) $p/(1-a^2)$
(c) $ap/(1-p^2)$ (d) $ap/(1-a^2)$

53. If the lengths of the tangents from two points A and B to a circle are l and l' respectively and points are conjugate with respect to the circle (i.e. each passes through the polar of the other with respect to the circle), then $(AB)^2 =$

(a) $\left|l^2 - l'^2\right|$ (b) $(l + l')^2$
(c) $l^2 + l'^2$ (d) $(l - l')^2$

54. If two circles which pass through the points $(a, 0)$ and $(-a, 0)$ touch the line $y = mx + c$ and cut orthogonally then

(a) $c^2 = a^2(1 + m^2)$ (b) $c^2 = a^2(1 + 2m^2)$
(c) $c^2 = a^2(1 + m)^2$ (d) $c^2 = a^2 + m^2$

55. No portion of the circle $x^2 + y^2 - 16x + 18y + 1 = 0$ lies in the

(a) first quadrant (b) second quadrant
(c) third quadrant (d) fourth quadrant

56. The greatest distance of the point $P(10, 7)$ from the points on the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is

(a) 5 (b) $5\sqrt{3}$
(c) 10 (d) 15

57. The length of the longest ray drawn from the point $(4, 3)$ to the circle $x^2 + y^2 + 16x + 18y + 1 = 0$ is equal to

(a) the radius of the circle
(b) the diameter of the circle
(c) circumference of the circle
(d) the distance of the centre of the circle from the origin

58. If $y = \pm a$ is a pair of tangents to the circle $x^2 + y^2 = a^2$ meeting the tangent at any point C on the circle at P and Q, then $CP.CQ =$

(a) 1 (b) a^2
(c) $1/a^2$ (d) none of these

59. The circle $x^2 + y^2 = 9$ is contained in the circle $x^2 + y^2 - 6x - 8y + 25 = c^2$ if

(a) $c = 2$ (b) $c = 3$
(c) $c = 5$ (d) $c = 10$

60. The line $(x - 1)\cos\theta + (y - 1)\sin\theta = 1$, for all values of θ touches the circle

(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 - 2x = 0$
(c) $x^2 + y^2 - 2y = 0$ (d) $x^2 + y^2 - 2x - 2y + 1 = 0$

61. Equation of the circle which cuts each of the circles $x^2 + y^2 + 2gx + c = 0$, $x^2 + y^2 + 2g_1x + c = 0$ and $x^2 + y^2 + 2hx + 2ky + a = 0$ orthogonally is

(a) $h(x^2 + y^2) + (a - c)x - ch = 0$
(b) $k(x^2 + y^2) + (a - c)y - ck = 0$
(c) $x^2 + y^2 + (a - c)x + c = 0$
(d) $x^2 + y^2 + (a - c)y - c = 0$

62. The point at which the circles $x^2 + y^2 + 2x + 6y + 4 = 0$ and $x^2 + y^2 + 6x + 2y + 7 = 0$ subtend equal angles lies on the circle

(a) $x^2 + y^2 + 10x - 2y + 10 = 0$
(b) $x^2 + y^2 - 2x + 10y + 10 = 0$
(c) $x^2 + y^2 + 10x - 2y - 10 = 0$
(d) $x^2 + y^2 - 10x + 10y - 10 = 0$

63. Equation of a circle which passes through the point $(2, 0)$ and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + \alpha)x + 5\alpha^2 y = 1$ as α tends to 1 is

(a) $25(x^2 + y^2) + 20x - 2y - 140 = 0$
(b) $25(x^2 + y^2) - 20x + 2y - 60 = 0$
(c) $9(x^2 + y^2) - 20x + 2y + 4 = 0$
(d) $9(x^2 + y^2) - 2x - 20y + 4 = 0$

64. If the limiting points of the system of circles $x^2 + y^2 + 2gx + \lambda(x^2 + y^2 + 2fy + k) = 0$, where λ is a parameter, subtend a right angle at the origin, then $k/f^2 =$

(a) -1 (b) 1
(c) 2 (d) none of these

65. If the chord of the circle $x^2 + y^2 - 4y = 0$ along the line $x + y = 1$ subtends an angle θ at a point on the circumference of the larger segment then $\cos\theta =$

(a) $1/2$ (b) $1/\sqrt{2}$
(c) $\sqrt{3}/2$ (d) $1/2\sqrt{2}$

66. Locus of the centre of the circle touching the line $3x + 4y + 1 = 0$ and having radius equal to 5 units is

(a) a pair of perpendicular lines
(b) a pair of parallel lines
(c) a pair of straight lines
(d) none of these

67. Chord of the circle $x^2 + y^2 = 81$ bisected at the point $(-2, 3)$ meets the diameter $x + 5y = 0$ at a point

(a) on the circle (b) inside the circle
(c) outside the circle (d) none of these

68. An equilateral triangle is inscribed in the circle $x^2 + y^2 = 1$ with one vertex at the point $(1, 0)$. Length of each side of the triangle is

(a) 1 (b) $\sqrt{2}$
(c) $\sqrt{3}/2$ (d) $\sqrt{3}$

69. The curve parametrically described by the equations $x = 2 + 3\cos\theta$, $y = 3 + 3\sin\theta$ represents a circle which touches

(a) x-axis at $(2, 0)$ (b) y-axis at $(0, 3)$
(c) $x + y = 2$ at $(1, 1)$ (d) $x + y = 3$ at $(1, 2)$

70. Consider two circles $x^2 + y^2 = a^2 - \lambda$ and $x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta + \lambda = 0$. Each circle passes through the centre of the other for
(a) all values of λ (b) no value of λ
(c) only one value of λ (d) more than one value of λ

71. Tangents at two points A and B on the circle $x^2 + y^2 - 6x - 4y - 11 = 0$ intersect at the point (1, 8). Equation of the chord AB is
(a) $x + 3y - 15 = 0$ (c) $3x - y + 15 = 0$
(c) $x - 3y + 15 = 0$ (d) $3x + 2y + 11 = 0$

72. Equation of the circle on AB as diameter, where A and B are the points of intersection of the circle $x^2 + y^2 - 8x = 0$ and the curve $4x^2 - 9y^2 = 36$ is
(a) $x^2 + y^2 - 12x + 24 = 0$ (b) $x^2 + y^2 + 12x + 24 = 0$
(c) $x^2 + y^2 + 24x - 12 = 0$ (d) $x^2 + y^2 - 24x - 12 = 0$

73. A polygon of nine sides, each of length 2, is inscribed in a circle with centre at the origin. Equation of the circle is $x^2 + y^2 = r^2$, where $1/r$ is equal to
(a) cos 20° (b) sin 20°
(c) cos 40° (d) sin 40°

Assertion-Reason Type Questions

74. **Statement-1:** The radius of the circle
$2x^2 + 2y^2 - 4x\cos\theta + 4y\sin\theta - 1 - 4\cos\theta - \cos2\theta = 0$ is $1 + \cos\theta$

Statement-2: The radius of the circle
$(x - \cos\theta)(x - \sin\theta) + (y - \cos\theta)(y - \sin\theta) = 0$ $(\theta \neq \pi/4)$ is $|\cos\theta - \sin\theta|$

75. **Statement-1:** The points of intersection of the two curves whose equations are
S_1: $x^2 + 2y^2 - 6x - 12y + 23 = 0$ and
S_2: $4x^2 + 2y^2 - 20x - 12y + 35 = 0$ lie on a circle with centre at (8/3, 3) and radius equal to $\sqrt{47}/3\sqrt{2}$.

Statement-2: Equation of a curve passing through the intersection of $S_1 = 0$ and $S_2 = 0$ in Statement-1 is $S_1 + \lambda S_2 = 0$ which represents a circle if there exists a real value of λ for which the coefficients of x^2 and y^2 are equal.

76. **Statement-1:** The locus of the point of intersection of the tangents to the circle $x = a\cos\theta$, $y = a\sin\theta$ at points whose parametric angles differ by $\pi/2$ is $x^2 + y^2 = 2a^2$

Statement-2: Tangents at the extremities of a diameter of a circle are parallel.

77. **Statement-1:** The centre of a circle passing through the points (0, 0), (1, 0) and touching the circles $C : x^2 + y^2 = 9$ lies inside the circle C.

Statement-2: If a circle C_1 passes through the centre of the circle C_2 and also touches the circle, the radius of the circle C_2 is twice the radius of the circle C_1.

78. $C_1 : (x - 3)^2 + (y - 4)^2 = a^2$

Statement-1: C_1 touches the axis of x if $a = 4$

Statement-2: C_1 touches the line $y = x$ if $a = 3$.

79. **Statement-1:** The common chord of the circles $x^2 + y^2 - 10x + 16 = 0$ and $x^2 + y^2 = r^2$ is of maximum length if $r^2 = 34$.

Statement-2: The common chord of two circles is of maximum length if it passes through the centre of the circle with smaller radius.

80. **Statement-1:** The equation $x^2 - y^2 - 4x - 4y = 0$ represents a circle with centre (2, 2) passing through the origin.

Statement-2: The equation $x^2 + y^2 + 4x + 6y + 13 = 0$ represents a point.

LEVEL 2

Straight Objective Type Questions

81. The circles $x^2 + y^2 + ax = 0$ and $x^2 + y^2 = c^2$ $(c > 0)$ touch each other if
(a) $a^2 + c^2 = 1$ (b) $a^2 - c^2 = 0$
(c) $a^2 - c^2 = 1$ (d) none of these

82. If $(a\cos\theta_i, a\sin\theta_i)$ $i = 1, 2, 3$ represent the vertices of an equilateral triangle inscribed in a circle, then
(a) $\cos\theta_1 + \cos\theta_2 + \cos\theta_3 = 0$
(b) $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 \neq 0$
(c) $\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = 0$
(d) $\cot\theta_1 + \cot\theta_2 + \cot\theta_3 = 0$

83. A circle C is drawn on the line joining the centres of the circles C_1: $x^2 + y^2 - 4 = 0$ and C_2: $x^2 + y^2 - 8x + 7 = 0$ as a diameter. The length of the intercept made

on the common chord of the circles C_1 and C_2 by the circle C is

(a) $\sqrt{231}/8$ (b) $\sqrt{231}/4$

(c) $\sqrt{231}/2$ (d) $11/8$

84. If the equation $ax^2 + 2(a^2 + ab - 16)xy + by^2 + 2ax + 2by - 4\sqrt{2} = 0$ represents a circle, the radius of the circle is

(a) 2 (b) $2\sqrt{2}$

(c) $\sqrt{2}$ (d) $4\sqrt{2}$

85. If a circle passes through the points of intersection of the coordinate axes with the line $x - \lambda y + 1 = 0$ $(\lambda \neq 0)$ and $x - 2y + 3 = 0$, then the value of λ is a root of the equation

(a) $6\lambda^2 - 7\lambda + 2 = 0$ (b) $7\lambda^2 - 6\lambda - 2 = 0$

(c) $2\lambda^2 - 6\lambda + 1 = 0$ (d) $2\lambda^2 - 7\lambda + 6 = 0$

86. If a square with length of each side equal to a is inscribed in the circle $x^2 + y^2 + 4x + 10y + 21 = 0$, then a is equal to

(a) $\sqrt{2}$ (b) 2

(c) $2\sqrt{2}$ (d) 4

87. The locus of a point which divides the join of $A\ (-1, 1)$ and a variable point P on the circle $x^2 + y^2 = 4$ in the ratio $3 : 2$ is a circle whose centre is at the point

(a) $(2/5, 2/5)$ (b) $(-2/5, 2/5)$

(c) $(2/5, -2/5)$ (d) $(-5/14, 5/14)$

88. If (α, β) are the roots of the equation $15x^2 - 22x + 8 = 0$ and (α', β') are the roots of the equation $8x^2 - 22x + 15 = 0$, then the equation of the circle on $A\ (\alpha, \alpha')$ and $B\ (\beta, \beta')$ as diameter is

(a) $120\ (x^2 + y^2) - 22\ (8x + 15y) + 289 = 0$

(b) $120\ (x^2 + y^2) - 22\ (15y + 8x) + 289 = 0$

(c) $23\ (x^2 + y^2) + 22\ (8x + 15y) - 120 = 0$

(d) none of these

89. If two circles cut a third circle orthogonally, the radical axis of the two circles

(a) touches the third circle

(b) passes through the centre of the third circle

(c) lies out side the third circle

(d) none of these

90. If a circle of constant radius $3k$ passes through the origin and meets the axes at A and B, the locus of the centroid of the triangle OAB is the circle

(a) $x^2 + y^2 + kx + ky = 0$

(b) $x^2 + y^2 = 4k^2$

(c) $x^2 + y^2 = k^2$

(d) $x^2 + y^2 + 2kx - 2ky + k^2 = 0$

91. The locus of a point which moves such that the sum of the squares of its distances from the three vertices of a triangle is a constant, is a circle whose centre is the

(a) incentre of the triangle

(b) circumcentre of the triangle

(c) orthocentre of the triangle

(d) centroid of the triangle

92. The tangents to the circle $x^2 + y^2 = 48$, which are inclined at angle of 60° with the axis of x form a rhombus, the length of whose sides is

(a) $2\sqrt{3}$ (b) 3

(c) $3\sqrt{3}$ (d) 4

93. An equation of the circle described on the chord $x \cos \alpha + y \sin \alpha - p = 0$ of the circle $x^2 + y^2 = a^2$ as diameter is

(a) $x^2 + y^2 + 2px \cos \alpha + 2py \sin \alpha - (2p^2 + a^2) = 0$

(b) $x^2 + y^2 - 2px \cos \alpha - 2py \sin \alpha + 2p^2 - a^2 = 0$

(c) $x^2 + y^2 - 2px \cos \alpha + 2py \sin \alpha - 2p^2 + a^2 = 0$

(d) $x^2 + y^2 + 2px \cos \alpha - 2py \sin \alpha + 2p^2 - a^2 = 0$

94. The locus of a point which moves in a plane so that the sum of the squares of its distances from the line $ax + by + c = 0$ and $bx - ay + d = 0$ is r^2, is a circle of radius

(a) r (b) $r\sqrt{a^2 + b^2}$

(c) $r\sqrt{ab}$ (d) none of these

95. Two circles are such that one is inscribed in and the other is circumscribed about a square $A_1 A_2 A_3 A_4$. If the length of each side of the square is a and P, Q are two points respectively on these circles, then

$$\left| \sum_{i=1}^{4} (PAi)^2 - \sum_{i=1}^{4} (QAi)^2 \right| =$$

(a) $a^2/4$ (b) $a^2/2$

(c) a^2 (d) $2a^2$

96. Two tangents T_1 and T_2 are drawn from $(-2, 0)$ to the circle $C : x^2 + y^2 = 1$. Equation of a circle touching C, having T_1, T_2 as a pair of tangents from $(-2, 0)$ and radius greater than the radius of C is

(a) $x^2 + y^2 - 6x + 5 = 0$

(b) $x^2 + y^2 - 8x + 7 = 0$

(c) $9x^2 + 9y^2 + 24x + 15 = 0$

(d) none of these

97. The locus of the point of intersection of tangents to the circle $x = a \cos \theta$, $y = a \sin \theta$ at points whose parametric angles differ by $\pi/4$ is

(a) $x^2 + y^2 = 2\left(\sqrt{2} - 1\right)^2 a^2$

(b) $x^2 + y^2 = 2(2 - \sqrt{2})a^2$

(c) $x^2 + y^2 = \left(\sqrt{2}+1\right)^2\ a^2$

(d) none of these

98. The locus of the centre of the circle which cuts the circles $x^2 + y^2 + 4x - 6y + 9 = 0$ and $x^2 + y^2 - 4x + 6y + 4 = 0$ orthogonally is

(a) $12x + 8y + 5 = 0$ (b) $8x - 12y + 5 = 0$
(c) $5x - 12y + 8 = 0$ (d) none of these

99. An equation of the tangent at the point (5, 2) of a circle is given by $3x - 2y - 11 = 0$. If the circle passes through the origin, an equation of the circle is

(a) $x^2 + y^2 - 23x - 102y = 0$
(b) $11x^2 + 11y^2 - 102x - 23y = 0$
(c) $11x^4 + 11y^2 - 23x - 102y = 0$
(d) $x^2 + y^2 - 102x - 23y = 0$

100. A circle passes through the origin O and cuts the axes at A $(a, 0)$ and B $(0, b)$. The reflection of origin O in the line AB is the point

(a) $\left(\dfrac{2ab^2}{a^2+b^2}, \dfrac{2a^2b}{a^2+b^2}\right)$ (b) $\left(\dfrac{2a^2b}{a^2+b^2}, \dfrac{2ab^2}{a^2+b^2}\right)$

(c) $\left(\dfrac{2ab}{a^2+b^2}, \dfrac{2ab}{a^2+b^2}\right)$ (d) (a, b)

101. If a circle cuts $x^2 + y^2 = a^2$ orthogonally and passes through the point $\left(\dfrac{a^2p}{p^2+q^2}, \dfrac{a^2q}{p^2+q^2}\right)$, then it will also pass through

(a) $(p, 0)$ (b) $(0, q)$
(c) (p, q) (d) (ap, aq)

102. Two circles intersect at the point P (2, 3) and the line joining the other extremity of the two diameter through P makes an angle $\pi/6$ with x-axis, then the equation of the common chord of the two circles is

(a) $x + \sqrt{3}y - (2 + 3\sqrt{3}) = 0$

(b) $x + \sqrt{3}y - (2\sqrt{3} + 2) = 0$

(c) $\sqrt{3}x + y - (2\sqrt{3} + 3) = 0$

(d) $\sqrt{3}x + y - (2 + 3\sqrt{3}) = 0$

103. A system of circles is drawn through two fixed points (–1, 0) and (1, 0); tangents are drawn to these circles parallel to the line $y = x$, the locus of the points of contact is

(a) $x^2 - y^2 - 2xy = 1$
(b) $x^2 + y^2 = 1$
(c) $x^2 + y^2 + 2xy = 0$
(d) $x^2 - y^2 + 2xy + 1 = 0$

104. The radius upto one place of decimal of the smallest circle which touches the line $3x - y = 6$ at $(1, -3)$ and also touches the line $y = x$ is

(a) 1.2 (b) 1.6
(c) 1.8 (d) 2.1

105. Of the two concentric circles the smaller one has the equation $x^4 + y^2 = 4$. If each of the two intercepts on the line $x + y = 2$ made between the two circles is 1, the equation of the larger circle is

(a) $x^2 + y^2 = 5$ (b) $x^2 + y^2 = 5 + 2\sqrt{2}$
(c) $x^2 + y^2 = 7 + 2\sqrt{2}$ (d) $x^2 + y^2 = 11$

106. An equation of the circle situated symmetrically opposite to the circle $x^2 + y^2 - 2x = 0$ with respect to the line $x + y = 2$ is

(a) $x^2 + y^2 + 2y - 4 = 0$
(b) $x^2 + y^2 - 4x - 2y + 4 = 0$
(c) $x^2 + y^2 - x + y - 2 = 0$
(d) $x^2 + y^2 - 3x - y + 2 = 0$

107. If the chord along the line $y - x = 3$ of the circle $x^2 + y^2 = k^2$ subtends an angle of 30° in the major segment of the circle cut off by the chord then $k^2 =$

(a) 3 (b) 6
(c) 9 (d) 36

108. Three concentric circles of which the biggest is $x^2 + y^2 = 1$ have their radii in A.P. with common difference d (> 0). If the line $y = x + 1$ cuts all the circles in real distinct points, then

(a) $d < \dfrac{2-\sqrt{2}}{4}$ (b) $d > \dfrac{2+\sqrt{2}}{4}$

(c) $d > 1 + \dfrac{1}{\sqrt{2}}$ (d) d is any real number

109. Radius of the circle centred at (3, – 2), of maximum area contained in the circle $x^2 + y^2 - 4x + y = 0$ is

(a) (1/2) $\left(\sqrt{17} - \sqrt{13}\right)$ (b) $1/\sqrt{2}$

(c) $\sqrt{2}$ (d) none of these

110. Equation of a family of circles, such that each member of the family passes through the origin and makes an intercept on the line $y = 2x$ which is twice the intercept made on the line $x = 2y$ is (λ being a parameter)

(a) $x^2 + y^2 - 2\lambda y = 0$
(b) $x^2 + y^2 - 2\lambda x = 0$
(c) $x^2 + y^2 - 2\lambda x - 2\lambda y = 0$
(d) none of these

Previous Years' AIEEE/JEE Main Questions

1. Two common tangents to the circle $x^2 + y^2 = 2a^2$ and parabola $y^2 = 8ax$ are
   (a) $x = \pm (y + 2a)$ (b) $y = \pm (x + 2a)$
   (c) $x = \pm (y + a)$ (d) $y = \pm (x + a)$ **[2002]**
2. If the chord $y = mx + 1$ of the circle $x^2 + y^2 = 1$ subtends an angle of measure 45° at the major segment of the circle then the value of m is
   (a) 2 (b) $t - 2$
   (c) -1 (d) none of these **[2002]**
3. The centres of a set of circles, each of radius 3, lie on the circle $x^2 + y^2 = 25$. The locus of any point in the set is
   (a) $4 \le x^2 + y^2 \le 64$ (b) $x^2 + y^2 \le 25$
   (c) $x^2 + y^2 \le 25$ (d) $3 \le x^2 + y^2 \le 9$ **[2002]**
4. The centre of the circle passing through (0, 0) and (1, 0) and touching the circle $x^2 + y^2 = 9$ is
   (a) (1/2, 1/2) (b) $(1/2, -\sqrt{2})$
   (c) (3/2, 1/2) (d) (1/2, 3/2) **[2002]**
5. The equation of a circle with origin as centre and passing through equilateral triangle whose medium is of length $3a$ is
   (a) $x^2 + y^2 = 9a^2$ (b) $x^2 + y^2 = 16a^2$
   (c) $x^2 + y^2 = 4a^2$ (d) $x^2 + y^2 = a^2$ **[2002]**
6. If two circle $(x - 1)^2 + (y - 3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points then
   (a) $r < 2$ (b) $r = 2$
   (c) $r > 2$ (d) $2 < r < 8$
7. The lines $2x - 3y = 5$ and $3x - 4y = 7$ are diameters of a circle having area as 154 sq. units. Then the equation of the circle is
   (a) $x^2 + y^2 + 2x - 2y = 47$
   (b) $x^2 + y^2 - 2x + 2y = 47$
   (c) $x^2 + y^2 - 2x + 2y = 62$
   (d) $x^2 + y^2 + 2x - 2y = 62$ **[2003]**
8. If a circle passes through (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is
   (a) $2ax - 2by + (a^2 + b^2 + 4) = 0$
   (b) $2ax + 2by - (a^2 + b^2 + 4) = 0$
   (c) $2ax + 2by + (a^2 + b^2 + 4) = 0$
   (d) $2ax - 2by - (a^2 + b^2 + 4) = 0$ **[2004]**
9. A variable circle passes through the fixed point A(p, q) and touches x-axis. The locus of the other end of the diameter through A is
   (a) $(y - p)^2 = 4qx$ (b) $(x - q)^2 = 4py$
   (c) $(x - p)^2 = 4qy$ (d) $(y - q)^2 = 4px$ **[2004]**
10. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π, then the equation of the circle is
    (a) $x^2 + y^2 + 2x + 2y - 23 = 0$
    (b) $x^2 + y^2 - 2x - 2y - 23 = 0$
    (c) $x^2 + y^2 - 2x + 2y - 23 = 0$
    (d) $x^2 + y^2 + 2x - 2y - 23 = 0$ **[2004]**
11. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB. Equation of the circle on AB as diameter is
    (a) $x^2 + y^2 + x + y = 0$
    (b) $x^2 + y^2 - x + y = 0$
    (c) $x^2 + y^2 - x - y = 0$
    (d) $x^2 + y^2 + x - y = 0$ **[2004]**
12. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect in two distinct points P and Q then the line $5x + by - a = 0$ passes through P and Q for
    (a) infinitely many values of a
    (b) exactly two values of a
    (c) exactly one value of a
    (d) no value of a **[2005]**
13. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = p^2$ orthogonally, then the equation of the locus of its centre is
    (a) $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - p^2) = 0$
    (b) $2ax + 2by - (a^2 + b^2 + p^2) = 0$
    (c) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - p^2) = 0$
    (d) $2ax + 2by - (a^2 - b^2 + p^2) = 0$ **[2005]**
14. A circle touches the x-axis and also touches the circle with centre at (0, 3) and radius 2. The locus of the centre of the circle is
    (a) a hyperbola (b) a parabola
    (c) an ellipse (d) a circle **[2005]**
15. Let C be the circle with centre (0, 0) and radius 3 units. The equation of the locus of the mid points of the chords of the circle C that subtends an angle $2\pi/3$ at its centre is
    (a) $x^2 + y^2 = 9/4$ (b) $x^2 + y^2 = 3/2$
    (c) $x^2 + y^2 = 1$ (d) $x^2 + y^2 = 27/4$ **[2006]**
16. If the lines $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ are two diameters of a circle of area 49π square units, the equation of the circle is

(a) $x^2 + y^2 - 2x + 2y - 47 = 0$
(b) $x^2 + y^2 + 2x - 2y - 47 = 0$
(c) $x^2 + y^2 + 2x - 2y - 62 = 0$
(d) $x^2 + y^2 - 2x + 2y - 62 = 0$ **[2006]**

17. Consider a family of circles with are passing through the point $(-1, 1)$ and are tangent to x-axis. If (h, k) are the coordinates of the centre of the circles, then the set of values of k is given by the interval

(a) $0 < k < 1/2$ (b) $k \geq 1/2$
(c) $-1/2 \leq k \leq 1/2$ (d) $k \leq 1/2$ **[2007]**

18. The point diametrically opposite to the point $P(1, 0)$ on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is

(a) $(3, 4)$ (b) $(3, -4)$
(c) $(-3, 4)$ (d) $(-3, -4)$ **[2008]**

19. Three distinct points A, B and C are given in the 2-dimensional coordinate plane such that the ratio of the distance of any one of them from the point $(1, 0)$ to the distance from the point $(-1, 0)$ is equal to $1/3$. Then the circumcentre of the triangle ABC is at the point

(a) $(5/2, 0)$ (b) $(5/3, 0)$
(c) $(0, 0)$ (d) $(5/4, 0)$ **[2009]**

20. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and $(1, 1)$ for

(a) all except two values of p.
(b) exactly one value of p
(c) all values of p
(d) all except one value of p **[2009]**

21. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if

(a) $15 < m < 65$
(b) $35 < m < 85$
(c) $-85 < m < -35$
(d) $-35 < m < 15$ **[2010]**

22. The equation of the circle passing through the points $(1, 0)$ and $(0, 1)$ and having the smallest radius is

(a) $x^2 + y^2 - 2x - 2y + 1 = 0$
(b) $x^2 + y^2 - x - y = 0$
(c) $x^2 + y^2 + 2x + 2y - 7 = 0$
(d) $x^2 + y^2 + x + y - 2 = 0$ **[2011]**

23. The two circles $x^2 + y^2 = ax$, $x^2 + y^2 = c^2$ $(c > 0)$ touch each other

(a) $|a| = 2c$ (b) $2|a| = c$
(c) $|a| = c$ (d) $a = 2c$ **[2011]**

24. The length of the diameter of the circle which touches the x-axis at the point $(1, 0)$ and passes through the point $(2, 3)$ is

(a) $6/5$ (b) $5/3$
(c) $10/3$ (d) $3/5$ **[2012]**

25. A circle passing through $(1, -2)$ and touching the axis of x at $(3, 0)$ also passes through the point

(a) $(2, -5)$ (b) $(5, -2)$
(c) $(-2, 5)$ (d) $(-5, 2)$ **[2013]**

26. If each of the lines $5x + 8y = 13$ and $4x - y = 3$ contains a diameter of the circle $x^2 + y^2 - 2(a^2 - 7a + 11)x - 2(a^2 - 6a + 6)y + b^3 + 1 = 0$, then

(a) $a = 5$ and $b \notin (-1, 1)$
(b) $a = 1$ and $b \notin (-1, 1)$
(c) $a = 2$ and $b \in (-\infty, 1)$
(d) $a = 5$ and $b \in (-\infty, 1)$ **[2013, online]**

27. If a circle C passing through $(4, 0)$ touches the circle $x^2 + y^2 + 4x - 6y - 12 = 0$ externally at a point $(1, -1)$, then the radius of the circle C is:

(a) 5 (b) $2\sqrt{5}$
(c) 4 (d) $\sqrt{57}$ **[2013, online]**

28. If two vertices of an equilateral triangle are $A(-a, 0)$ and $B(a, 0)$, $a > 0$, and the third vertex C lies above x-axis, then the equation of the circumcircle of ΔABC is:

(a) $3x^2 + 3y^2 - 2\sqrt{3}\,ay = 3a^2$
(b) $3x^2 + 3y^2 - 2ay = 3a^2$
(c) $x^2 + y^2 - 2ay = a^2$
(d) $x^2 + y^2 - \sqrt{3}\,ay = a^2$ **[2013, online]**

29. If the circle $x^2 + y^2 - 6x - 8y + (25 - a^2) = 0$ touches the axis of x, then a equals

(a) 0 (b) ± 4
(c) ± 2 (d) ± 3 **[2013, online]**

30. **Statement-1:** The only circle having radius $\sqrt{10}$ and a diameter along line $2x + y = 5$ is $x^2 + y^2 - 6x + 2y = 0$.

Statement-2: $2x + y = 5$ is a normal to the circle $x^2 + y^2 - 6x + 2y = 0$

(a) Statement-1 is false, Statement 2 is true.
(b) Statement-1 is true, Statement 2 is true, Statement-2 is a correct explanation for Statement-1.
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is true, Statement-2 is true. Statement-2. is **not** a correct explanation for Statement-1. **[2013, online]**

31. If a circle of unit radius is divided into two parts by an arc of another circle subtending an angle 60° on the circumference of the first circle, then the radius of the arc is:

(a) $\sqrt{3}$ (b) $1/2$
(c) 1 (d) $\sqrt{2}$ **[2013, online]**

32. Let C be the circle with circle with centre at (1, 1) and radius = 1. If T is the circle centre at (0, y), passing through origin and touching the circle C externally, then radius of T is equal to

(a) $\sqrt{3}/\sqrt{2}$ (b) $\sqrt{3}/2$

(c) 1/2 (d) 1/4 **[2014]**

33. If a point (1, 4) lies inside the circle $x^2 + y^2 - 6x - 10y + p = 0$ and the circle does not touch or intersect the coordinate axes, then the set of all possible values of p is the interval:

(a) (0, 25) (b) (25, 39)

(c) (9, 25) (d) (25, 29) **[2014, online]**

34. The set of all real values of λ for which exactly two common tangents can be drawn to the circle $x^2 + y^2 - 4x - 4y + 6 = 0$ and $x^2 + y^2 - 10x - 10y + \lambda = 0$ is the interval

(a) (12, 32) (b) (18,42)

(c) (12,24) (d) (18, 48) **[2014, online]**

35. For the two circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2y = 0$ there is/are

(a) one pair of common tangents

(b) two pairs of common tangents

(c) three common tangents

(d) no common tangent **[2014, online]**

36. The equation of the circle described on the chord $3x + y + 5 = 0$ of the circle $x^2 + y^2 = 16$ as diameter is

(a) $x^2 + y^2 + 3x + y - 11 = 0$

(b) $x^2 + y^2 + 3x + y + 1 = 0$

(c) $x^2 + y^2 + 3x + y - 2 = 0$

(d) $x^2 + y^2 + 3x + y - 22 = 0$ **[2014, online]**

37. The number of common tangents to the circles $x^2 + y^2 -4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ is:

(a) 1 (b) 2

(c) 3 (d) 4

38. If incentre of an equilateral triangle is (1, 1) and the equation of one side is $3x + 4y + 3 = 0$ than the equation of the circumcircle of this triangle is:

(a) $x^2 + y^2 - 2x - 2y - 2 = 0$

(b) $x^2 + y^2 - 2x - 2y - 14 = 0$

(c) $x^2 + y^2 - 2x - 2y + 2 = 0$

(d) $x^2 + y^2 - 2x - 2y - 7 = 0$ **[2015, online]**

39. If a circle passing through the point (– 1, 0) touches y-axis at (0, 2), then length of the chord of the circle along the x-axis is:

(a) $\frac{3}{2}$ (b) $\frac{5}{2}$

(c) 3 (d) 5 **[2015, online]**

40. If $y + 3x = 0$ is the equation of a chord of the circle, $x^2 + y^2 - 30x = 0$, then the equation of the circle with their chord as diameter is:

(a) $x^2 + y^2 + 3x + 9y = 0$ (b) $x^2 + y^2 - 3x + 9y = 0$

(c) $x^2 + y^2 - 3x - 9y = 0$ (d) $x^2 + y^2 + 3x + 9y = 0$

[2015, online]

41. The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on

(a) a circle

(b) an ellipse which is not a circle

(c) a hyperbola (d) a parabola **[2016]**

42. If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at (–3, 2), then the radius of S is:

(a) $5\sqrt{2}$ (b) $5\sqrt{3}$

(c) 5 (d) 10 **[2016]**

43. A circle passes through (–2, 4) and touches the y-axis at (0, 2). Which one of the following equations can represent a diameter of this circle?

(a) $2x - 3y + 10 = 0$ (b) $3x + 4y - 3 = 0$

(c) $4x + 5y - 6 = 0$ (d) $5x + 2y + 4 = 0$

[2016, online]

44. Equation of the tangent to the circle, at the point (1, –1), where centre is the point of intersection of the straight lines $x - y = 1$ and $2x + y = 3$ is:

(a) $x + 4y + 3 = 0$ (b) $3x - y - 4 = 0$

(c) $x - 3y - 4 = 0$ (d) $4x + y - 3 = 0$

[2016, online]

Previous Years' B-Architecture Entrance Examination Questions

1. The line $x \sin \alpha - y \cos \alpha = a$ touches the circle $x^2 + y^2 = a^2$, then.

(a) $\alpha \in [0, \pi]$ (b) $\alpha \in [-\pi, \pi]$

(c) α can have any value (d) $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **[2006]**

2. If a circle of area 16π has two of its diameters along the line $2x - 3y + 5 = 0$ and $x + 3y - 11 = 0$, then the equation of the circle is

(a) $x^2 + y^2 - 4x + 6y - 13 = 0$

(b) $x^2 + y^2 - 4x - 6y - 3 = 0$

(c) $x^2 + y^2 - 4x - 6y - 13 = 0$
(d) $x^2 + y^2 - 4x + 6y - 3 = 0$ **[2006]**

3. The circle passing through the distinct points $(1, t)$, $(t, 1)$ and (t, t) for all values if t passes through the point
(a) $(1, -1)$ (b) $(-1, 1)$
(c) $(-1, -1)$ (d) $(1, 1)$ **[2006]**

4. The value of k for which the circle $x^2 + y^2 - 4x + 6y + 3 = 0$ will bisect the circumference of the circle $x^2 + y^2 + 6x - 4y + k = 0$ is
(a) 53 (b) – 53
(c) 47 (d) – 47 **[2007]**

5. If the point $(2, k)$ lies outside the circles $x^2 + y^2 = 13$ and $x^2 + y^2 + x - 2y - 14 = 0$, then
(a) $K \in\ ]-\infty, -2\ [U]\ 3, \infty\ [$
(b) $K \in\ ]-3, -2\ [U]\ 3, 4[$
(c) $K \in\ ]-3, -4[$
(d) $K \in\ ]-\infty, -3\ [U]\ 4, \infty\ [$ **[2007]**

6. The shortest distance between the circles $x^2 + y^2 = 1$ and $(x - 9)^2 + (y - 12)^2 = 4$ is
(a) 9 (b) 11
(c) 12 (d) 14 **[2008]**

7. The midpoint of the chord intercepted by the circle $x^2 + y^2 = 16$ on the line through the point $(1, -2)$ and $(0, -1)$ is
(a) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, -\frac{3}{2}\right)$
(c) $\left(-\frac{1}{4}, -\frac{1}{4}\right)$ (d) $\left(\frac{3}{4}, \frac{1}{4}\right)$ **[2009]**

8. The circle $x^2 + y^2 - 6x - 10y + p = 0$ does not touch or intersect the axes and the point $(1, 4)$ lies inside the circle for all p in the interval
(a) $(25, 35)$ (b) $(25, 29)$
(c) $(0, 25)$ (d) $(0, 29)$ **[2010]**

9. The equation of a circle of area 22π square units for which each of the two lines $2x + y = 2$ and $x - y = -5$ is diameter, is:
(a) $x^2 + y^2 - 2x + 8y - 5 = 0$
(b) $x^2 + y^2 - 2x - 8y - 5 = 0$
(c) $x^2 + y^2 + 2x - 8y - 5 = 0$
(d) $x^2 + y^2 + 2x + 8y - 5 = 0$ **[2011]**

10. If a chord of a circle $x^2 + y^2 = 4$ with one extremity at $(1, \sqrt{3})$ subtends a right angle at the centre of this circle, then the coordinates of the other extremity of this chord can be:
(a) $(-1\ \sqrt{3})$ (b) $(1, -\sqrt{3})$
(c) $(-\sqrt{3}, -1)$ (d) $(\sqrt{3}, -1)$ **[2012]**

11. Consider $L_1 : 3x + y + \alpha - 2 = 0$ and $L_2 : 3x + y - \alpha + 3 = 0$, where α is a positive real number, and $C : x^2 + y^2 - 2x + 4y - 4 = 0$.

Statement-1: If L_1 is a chord of the circle C, then the line L_2 is not always a diameter of the circle C.

Statement-2: If L_1 is a diameter of the circle C, then the line L_2 is not a chord of the circle C. then
(a) both the statements are true
(b) both the statements are false
(c) Statement-1 is true and statement-2 is false
(d) statement-2 is true and statement-1 is false
[2013]

12. A circle has two of its diameters along the lines $x + y = 5$ and $x - y = 1$ and has area 9π, then equation of the circle is
(a) $x^2 + y^2 - 6x - 4y - 3 = 0$
(b) $x^2 + y^2 - 6x - 4y - 4 = 0$
(c) $x^2 + y^2 - 6x - 4y + 3 = 0$
(d) $x^2 + y^2 - 6x - 4y + 4 = 0$ **[2014]**

13. The number of interger values of k for which the equation $x^2 + y^2 + (k - 1)x - ky + 5 = 0$ represents a circle whose radius can not excced 3, is:
(a) 10 (b) 11
(c) 4 (d) 5 **[2015]**

14. If the line $ax + by = 2$ $(a \neq 0)$ touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, thus $a + b$ is equal to:
(a) $\frac{-4}{3}$ (b) $\frac{-5}{3}$
(c) $\frac{-1}{3}$ (d) $\frac{1}{4}$ **[2015]**

Answers

Concept-based

1. (c)	**2.** (c)	**3.** (a)	**4.** (b)
5. (b)	**6.** (d)	**7.** (c)	**8.** (b)
9. (a)	**10.** (b)	**11.** (a)	**12.** (c)
13. (b)	**14.** (a)	**15.** (c)	**16.** (d)
17. (c)	**18.** (b)	**19.** (c)	**20.** (d)

Level 1

21. (d)	**22.** (c)	**23.** (b)	**24.** (b)
25. (c)	**26.** (d)	**27.** (a)	**28.** (c)
29. (d)	**30.** (b)	**31.** (b)	**32.** (a)
33. (c)	**34.** (c)	**35.** (d)	**36.** (a)
37. (b)	**38.** (c)	**39.** (b)	**40.** (a)
41. (a)	**42.** (b)	**43.** (a)	**44.** (c)
45. (a)	**46.** (b)	**47.** (b)	**48.** (a)
49. (c)	**50.** (c)	**51.** (c)	**52.** (c)

53. (c)	**54.** (b)	**55.** (b)	**56.** (d)
57. (a)	**58.** (b)	**59.** (d)	**60.** (d)
61. (b)	**62.** (a)	**63.** (a)	**64.** (c)
65. (d)	**66.** (b)	**67.** (b)	**68.** (d)
69. (a)	**70.** (c)	**71.** (c)	**72.** (a)
73. (b)	**74.** (b)	**75.** (a)	**76.** (b)
77. (a)	**78.** (c)	**79.** (a)	**80.** (d)

Level 2

81. (b)	**82.** (a)	**83.** (b)	**84.** (a)
85. (a)	**86.** (d)	**87.** (b)	**88.** (a)
89. (b)	**90.** (b)	**91.** (d)	**92.** (d)
93. (b)	**94.** (a)	**95.** (c)	**96.** (b)
97. (b)	**98.** (b)	**99.** (c)	**100.** (c)
101. (c)	**102.** (c)	**103.** (a)	**104.** (b)
105. (b)	**106.** (b)	**107.** (b)	**108.** (a)
109. (a)	**110.** (a)		

Previous Years' AIEEE/JEE Main Questions

1. (b)	**2.** (c)	**3.** (a)	**4.** (b)
5. (c)	**6.** (d)	**7.** (b)	**8.** (b)
9. (c)	**10.** (c)	**11.** (c)	**12.** (d)
13. (b)	**14.** (b)	**15.** (a)	**16.** (a)
17. (b)	**18.** (d)	**19.** (d)	**20.** (d)
21. (d)	**22.** (b)	**23.** (c)	**24.** (c)
25. (b)	**26.** (d)	**27.** (a)	**28.** (a)
29. (b)	**30.** (a)	**31.** (c)	**32.** (d)
33. (c)	**34.** (b)	**35.** (d)	**36.** (a)
37. (c)	**38.** (b)	**39.** (c)	**40.** (b)
41. (d)	**42.** (b)	**43.** (a)	**44.** (a)

Previous Years' B-Architecture Entrance Examination Questions

1. (c)	**2.** (b)	**3.** (d)	**4.** (a)
5. (d)	**6.** (c)	**7.** (a)	**8.** (b)
9. (c)	**10.** (d)	**11.** (c)	**12.** (d)
13. (c)	**14.** (c)		

Hints and Solutions

Concept-based

1. Centre of the circle is (1, 3) or (– 1, – 3) and radius is $\sqrt{10}$.

2. Given equation represents a circle if $b = 3$ and the equation is
   $x^2 + y^2 + 4x - 6y + 3 = 0$ whose radius $= \sqrt{10}$.

3. Line meets the axes at (4, 0) and (0, 3). Equation of the circle on the line joining this points as a diameter is
   $(x - 0)(x - 4) + (y - 0)(y - 3) = 0.$
   $\Rightarrow x^2 + y^2 - 4x - 3y = 0.$

4. (1, 2) lies inside the circle
   $\Rightarrow 1 + (2)^2 - 7 + 15(2) - c < 0 \Rightarrow c < 28$
   (3, 4) lies outside the circle
   $\Rightarrow (3)^2 + (4)^2 - 7(3) + 15(4) - c > 0 \Rightarrow c < 64.$

5. Centre of the circle S is (– 3, 7) and its radius is 8
   distance of (– 3, 7) from the line is $\left|\dfrac{-21+21+58}{\sqrt{49+9}}\right|$
   $= \sqrt{58} < 8$
   So the line L is a chord of S.

6. Let $y = mx$ be a tangent through the origin
   $\Rightarrow \dfrac{m(7)-(-1)}{\sqrt{1+m^2}} = \sqrt{25} \Rightarrow m = \dfrac{3}{4}$ or $\dfrac{-4}{3}$
   Product of the slopes = – 1 ⇒ The required angle is $\dfrac{\pi}{2}$.

7. From Geometry $PA \cdot PB = (PC)^2$
   $= (5)^2 + (6)^2 - 12 = 49.$

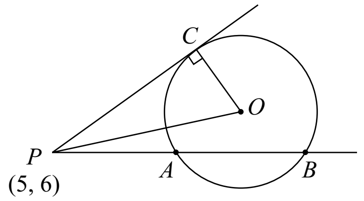


Fig. 17.40

8. Equation of the tangent is $x - 2y = 5$, which meets the second circle at points for which
   $(2y + 5)^2 + y^2 - 8(2y + 5) + 6y + 20 = 0$
   $\Rightarrow 5y^2 + 10y + 5 = 0 \Rightarrow y = -1$ and $x = 3$ so the required point of contact is (3, – 1)

9. Two circles are $x^2 + y^2 = 25$ and $(x - 2)^2 + (y + 3)^2 = 25$. So the equation of the required chord is
   $(x - 2)^2 + (y + 3)^2 - (x^2 + y^2) = 0.$
   $\Rightarrow 4x - 6y - 13 = 0$

10. Centres will lie on the line through (0, k) perpendicular to y-axis ($x = 0$) required line $y = k$.

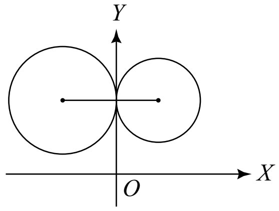


Fig. 17.41

11. Let the centre be $(h, h - 1)$ then the equation of the circle is $(x - h)^2 + (y - h + 1)^2 = 9$. As it passes through (7, 3), $(7 - h)^2 + (3 - h + 1)^2 = 9$.

    $\Rightarrow h = 4$ or 7 and the equation of the circle is
    $x^2 + y^2 - 8x - 6y + 16 = 0$
    or $x^2 + y^2 - 14x - 12x + 76 = 0$.

12. The line passes through the centre of the circle, so AB is a diameter of the circle and $\underline{|APB}$ being an angle in a semi circle is a right angle.

13. Let the equation of the circle be $x^2 + y^2 + 2gx + 2gy = 0$. As it cuts the given circle orthogonally.
    $2g(-2) + 2g(-3) = 10 \Rightarrow g = -1$.

14. Equation of the common chord is $y = x$ and the end points of which are (0, 0) and $(a/2, a/2)$. Equation of the circle on this chord as diameter is

    $x(x - a/2) + y(y - a/2) = 0$.

    $2x^2 + 2y^2 - ax - ay = 0$.

15. Centre will lie on the line parallel to both the given lines and equidistant from both.

16. Centre of the first circle is (2, 3) and radius 4-centre of the second circle is (– 1, – 1) and radius, distance between the centre = 5 = 4 + 1. So they touch externally and hence 3 common tangents.

17. $r_1 - r_2 > C_1 C_2$ where $r_1 = b$, $r_2 = a$.
    C_1 (0, 0) , C_2 (2 , 3)
    $\Rightarrow b - a > \sqrt{(0-2)^2 + (0-3)^2} = \sqrt{13}$

18. Equation of the given circle can be $(x \pm 1)^2 + (y \pm 1)^2 = 1$

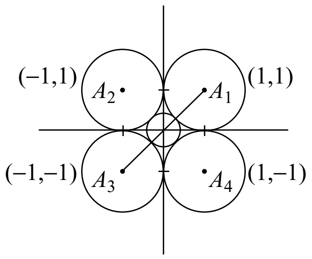


Fig. 17.42

Centre of the required circle is the origin and radius

$= \frac{1}{2}(A_1 A_3 - (1 + 1))$

$= \frac{1}{2}(2\sqrt{2} - 2) = \sqrt{2} - 1$

So its equation is $x^2 + y^2 = (\sqrt{2} - 1)^2$

19. If (h, k) is any point on the locus,

    $$\frac{\sqrt{h^2 + k^2 - 4k + 3}}{\sqrt{h^2 + k^2 + 6k + 5}} = \frac{3}{2}$$

    $\Rightarrow 5h^2 + 5k^2 + 70k + 33 = 0$

    Required locus is $5x^2 + 5y^2 + 70y + 33 = 0$

20. Diameters intersect at the point (3, 4) so the centre of the circle is (3, 4) and the distance of the centre from the line $x + 2y + 4 = 0$ is $\left|\frac{3+2(4)+4}{\sqrt{3^2+4^2}}\right| = 3$.

    So the equation of the circle is

    $(x - 3)^2 + (y - 4)^2 = 9$ or $x^2 + y^2 - 6x - 8y + 16 = 0$

Level 1

21. $(x - a/2)^2 + (y - b/2)^2 = (a^2 + b^2)/4$
22. Coefficient of x^2 = coefficient of y^2 and coefficient of $xy = 0$
23. radius = the distance between the given point and the point of intersection of the diameters.
24. $(2x^2 + 2y^2 + 4x - 7y - 25) + \lambda(x^2 + y^2 + 13x - 3y) = 0$ passes through (1, 1)
25. Normal is the line joining the point and the centre of the circle.
26. $\frac{\cos\theta}{\sqrt{1 + \tan^2\theta}} = \pm 2$. (Length of the perpendicular from the centre on the tangent is equal to the radius of the circle). $\Rightarrow \cos\theta = \pm\sqrt{2}$
27. $\frac{x+3}{2} = 2x, \frac{3+5}{2} = y$
28. Distance of the centre from both the axes is equal to the radius.
29. Let the radius of the circle be r (> 0), then the coordinates of the centre are $(\pm r, \pm r)$ as it touches both the axes, since it touches the line $x + y = 1$

    $\frac{\pm r \pm r - 1}{\sqrt{2}} = \pm r \Rightarrow \quad (2r \pm 1)^2 = 2r^2$ or $r = 1/\sqrt{2}$.

    $\Rightarrow 2r^2 \pm 4r + 1 = 0 \quad \Rightarrow \quad r = \frac{\pm 4 \pm \sqrt{16-8}}{4}$, which gives only two positive values of r and hence two circles in Iquadrant. $r = 1/\sqrt{2}$ also gives two circles one in II and the other in IV quadrant.

    (Draw the figure and find the answer)

30. As the circle passes through the centre of the given circle, it will touch the given circle internally and so its radius is half the radius of the given circle.
31. $| x_1 - x_2 | = 3$ where $x_1 + x_2 = 2\alpha$, $x_1x_2 = 4$.
32. Distance between the centers is equal to the sum of the radii.
33. Equation of a tangent is $x \cos \theta + y \sin \theta = r$,

$$A\left(\frac{r}{\cos\theta}, 0\right) B\left(0, \frac{r}{\sin\theta}\right), P\left(\frac{r}{\cos\theta}, \frac{r}{\sin\theta}\right) = P(x, y)$$

$$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{r^2}$$

34. $h = 1, 2$; $k = 1, -6$; $(h, k) = (1, 1), (1, -6), (2, 1), (2, -6)$ circle is $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - 25 = 0$
35. $P(3, 4)$ does not lie on the line L, $S(3, 4) < 0$ so P is inside the circle length of the perpendicular from the centre $(1, 2)$ on L is greater than the radius of S.
36. $x = 0$ is one of the common tangents and the others two tangents intersect at P on x-axis, the line joining the centres. The triangle being isosceles, the centroid lies on x-axis. Let $y = mx + c$ be a common tangent then $P(-c/m, 0)$

Also $\dfrac{-3m + c}{\sqrt{1 + m^2}} = 3$, $\dfrac{m + c}{\sqrt{1 + m^2}} = 1 \quad \Rightarrow \quad c/m = -3$ and

$P(3, 0)$. Distance of the centroid from the base $x = 0$ is equal to (1/3) the distance of P from the base.

37. C and D are the other ends of the diameters through. A and B respectively.
38. Equation of the circle is $x^2 + y^2 - ax - ay = 0$, the tangent at the origin is $x + y = 0$. Vertices of the triangle are $(a, 0)$, $(0, a)$ and $(0, 0)$.
39. $(AP)^2 = (\cos \alpha + 1)^2 + \sin^2 \alpha = 2(1 + \cos \alpha)$ $= 4 \cos^2 \alpha/2$ etc.
40. The points in (*c*) and (*d*) do not lie on the circle. Verify the result for the points in (*a*) and (*b*).
41. Common chord is of maximum length when it passes through the centre of the circle with smaller radius.

$C_2 : x^2 + y^2 - 2ax - 2by + a^2 + b^2 - 25 = 0$. Equation of the common chord is $2ax + 2by - a^2 - b^2 + 9 = 0$ which passes through the origin if $a^2 + b^2 = 9$.

42. OA, OB make equal angles $\pi/4$ with the line joining the origin 0 and the centre $(1, -2)$ of the circle.

$$\frac{m + 2}{1 - 2m} = \pm 1 \quad \Rightarrow \quad 3m^2 - 8m - 3 = 0.$$

43. The radius is smallest when the chord is a diameter.
44. B is the mid-point of AM and lies on the circle.
45. Radius is half the distance between the given parallel tangents.
46. $\lambda_1, \lambda_2, \lambda_3$ are in G.P. Length of the tangents are

$l_i = \sqrt{2\lambda_i\, c \cos\theta}$ $(i = 1, 2, 3)$

47. Chords of contacts from the origin and from (g, f) are respectively $gx + fy + c = 0$ and $2gx + 2fy + g^2 + f^2 + c = 0$
48. Let the other end of the diameter be (h, k), then the centre of the circle is $\left(\dfrac{p + h}{2}, \dfrac{k + q}{2}\right)$ and as it touches x-axis, its radius is $\dfrac{k + q}{2}$

$$\text{so } \left(p - \frac{p + h}{2}\right)^2 + \left(q - \frac{k + q}{2}\right)^2 = \left(\frac{k + q}{2}\right)^2.$$

49. Let the centre be at the origin, then

$$\frac{\sqrt{x^2 + y^2 - a^2}}{\sqrt{x^2 + y^2 - b^2}} = \frac{b}{a}.$$

$\Rightarrow x^2 + y^2 = a^2 + b^2$.

50. No part of the line lies in the third quadrant.
51. Let each side of the triangle be x, each side subtends an angle of 120° at the centre of the circle.

$$\text{so } \cos 120^\circ = \frac{a^2 + a^2 - x^2}{2a^2}$$

$\Rightarrow x^2 = 3a^2$ and area of the triangle is $\dfrac{\sqrt{3}}{4} x^2$.

52. Let $A(0, 0)$, $B(a, 0)$ and $P(h, k)$, then

$$\frac{h^2 + k^2}{(h - a)^2 + k^2} = p^2$$

Locus is $x^2 + y^2 + \dfrac{2\,ap^2}{1 - p^2}\, x - \dfrac{a^2\, p^2}{1 - p^2} = 0$.

53. Let $A\ (x_1, y_1)$, $B(x_2, y_2)$ and the circle be $x^2 + y^2 = a^2$ $l_1^2 = x_1^2 + y_1^2 - a^2$, $l_2^2 = x_2^2 + y_2^2 - a^2$, and $x_1x_2 + y_1y_2 = a^2$ as A and B are conjugate w.r.t. the circle.
54. Centres of the circles lie on y-axis as x-axis is the common chord. Let the centres be $(0, f_1)$ and $(0, f_2)$. The circles intersect orthogonally, so $\dfrac{f_1}{a} \times \dfrac{f_2}{a} = -1$

$\Rightarrow \quad f_1f_2 = -a^2$.

As they touch $y = mx + c$.

$$\frac{f_1 - c}{\sqrt{1 + m^2}} = a^2 + f_1^2$$

$\Rightarrow \quad m^2f_1^2 + 2cf_1 + (1 + m^2)a^2 - c^2 = 0$ which gives two values of f_1 say, f_1 and f_2 such that

$$f_1f_2 = \frac{(1+m^2)\,a^2 - c^2}{m^2} = -\,a^2$$

$\Rightarrow \quad c^2 = (2m^2 + 1)a^2.$

55. Centre is (8, – 9), radius is 12, origin lies outside the circle. Centre being in the fourth quadrant no part of the circle lies in second quadrant.

56. Centre is C (2, 1), radius is 5, the greatest distance is $PC + 5$.

57. Longest ray is the tangent from (4, 3) to the circle whose length = 12 = radius of the circle.

58. $y = a$ is a tangent at $A(0, a)$ and $y = -\,a$ is a tangent at $B(0, -\,a)$, tangent at C $(a \cos \theta, a \sin \theta)$ meet the line $y = a$ at P such that $PA = \dfrac{a\,(1-\sin\theta)}{\cos\theta}$, similarly

$$QB = \frac{a\,(1+\sin\theta)}{\cos\theta}.$$

$PA = PC$ and $QB = QC \quad \Rightarrow \quad PC.QC = a^2.$

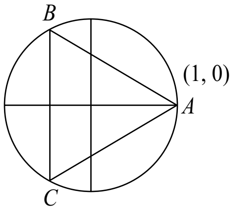


Fig. 17.43

59. Centre of the bigger circle is $A(3, 4)$ and radius is c, centre of the smaller circle is $O(0, 0)$ and radius is 3. $c > OA + 3$ i.e., $c > 8$.

60. $(x-1)\cos\theta + (y-1)\sin\theta = 1$ is a tangent to the circle $(x-1)^2 + (y-1)^2 = 1$.

61. Let the equation of the circle be $x^2 + y^2 + 2Gx + 2Fy + \lambda = 0$. Then $2Gg = \lambda + c = 2Gg_1$

$\Rightarrow \quad G = 0, \lambda = -\,c$ and

$2Gh + 2Fk = \lambda + a \quad \Rightarrow \quad 2F = \dfrac{a-c}{k}.$

62. Let $P(h, k)$ be the point where the common tangents to these circles intersect. Then the circles subtend equal angles at P. Lengths of the tangents from P to two circles are proportional to their radii.

$$\frac{h^2+k^2+6h+2k+7}{h^2+k^2+2h+6k+4} = \frac{3}{6}$$

$\Rightarrow \quad h^2 + k^2 + 10h - 2k + 10 = 0.$

63. The point of intersection is given by

$(3\alpha^2 - \alpha - 2)x = \alpha^2 - 1 \Rightarrow (3\alpha + 2)x = \alpha + 1\,[\alpha \neq 1]$

$\lim\limits_{\alpha\to 1} \Rightarrow x = \dfrac{2}{5}, y = \dfrac{1}{25}.$

64. Limiting points are $A\left(-\dfrac{g}{1+\lambda_1}, \dfrac{-\lambda_1 f}{1+\lambda_1}\right)$ and $B\left(-\dfrac{g}{1+\lambda_2}, \dfrac{-\lambda_2 f}{1+\lambda_2}\right)$ where λ_1, λ_2 are the roots of

$\lambda^2(f^2 - k) - k\lambda + g^2 = 0$

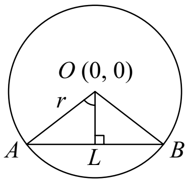


Fig. 17.44

OA is perpendicular to OB

$$\Rightarrow \frac{\lambda_1 f}{g} \times \frac{\lambda_2 f}{g} = -\,1$$

$$\Rightarrow \lambda_1\lambda_2 = -\,\frac{g^2}{f^2}.$$

$$\Rightarrow \frac{g^2}{f^2-k} = -\,\frac{g^2}{f^2} \Rightarrow \frac{k}{f^2} = 2.$$

65. Centre is (0, 2), radius is 2, length of the perpendicular from the centre on the line is $1/\sqrt{2}$

so $\cos\theta = \dfrac{1/\sqrt{2}}{2} = \dfrac{1}{2\sqrt{2}}$

66. Required locus is $\dfrac{3x+4y+1}{\sqrt{9+16}} = \pm\,5$

67. Equation of the chord is

$y - 3 = (2/3)(x + 2) \Rightarrow 2x - 3y = -\,13$

which intersects the diameter $x + 5y = 0$ at (– 5, 1) whose distance from the centre is $\sqrt{25+1} < 9$, the radius.

68. Equations of AB and AC are

$$y = \pm\,\frac{1}{\sqrt{3}}(x-1)$$

which intersect the circle $x^2+y^2=1$ at points for which

$\Rightarrow 3(1 - x^2) = (x - 1)^2$

$\Rightarrow x = 1$ or $x = -\,1/2$

Taking $x = -\,1/2$, $B\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$, $C\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$

Length of each side = $\sqrt{3}$.

69. Equation of the circle is

$(x - 2)^2 + (y - 3)^2 = 3^2$, centre is (2,3) and the radius 3 so it touches x-axis at (2, 0)

70. For $\lambda = 0$, each passes through the centre of the other.

71. AB is the chord of contact of the tangents from (1, 8) to the given circle so its equation is

$x(1) + y(8) - 3(x + 1) - 2(y + 8) - 11 = 0$

$\Rightarrow x - 3y + 15 = 0$

72. Two curves intersect at the points for which

$4x^2 - 9(8x - x^2) = 36$

$\Rightarrow 13x^2 - 72x - 36 = 0$

$\Rightarrow (x - 6)(13x + 6) = 0$

As the circle lies on the positive side of x-axis, $x = 6$

$\Rightarrow y = \pm 2\sqrt{3}$ and the required circle is

$(x - 6)(x - 6) + (y - 2\sqrt{3})(y + 2\sqrt{3}) = 0$

73. Each side subtends an angle $\frac{2\pi}{9}$ at the origin.

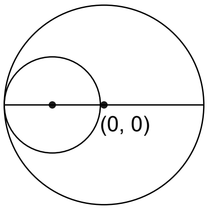


Fig. 17.45

$AB = 2 \Rightarrow AL = 1$

$\underline{|AOL} = 20^\circ$

$1 = AL = r \sin 20^\circ \Rightarrow 1/r = \sin 20^\circ$

74. Radius in statement-1 is

$$\sqrt{\cos^2\theta + \sin^2\theta + \frac{1 + 4\cos\theta + \cos 2\theta}{2}}$$

$$= \sqrt{(1/2)(3 + 4\cos\theta + 2\cos^2\theta - 1)}$$

$$= \sqrt{(1 + \cos\theta)^2} \Rightarrow \text{Statement–1 is true.}$$

In statement-2 the circle is on the line joining $(\cos\theta, \sin\theta)$ and $(\sin\theta, \cos\theta)$ as diameter so the radius is $\frac{1}{2}\sqrt{(\cos\theta - \sin\theta)^2 + (\sin\theta - \cos\theta)^2}$ and the statement is true.

75. Statement-2 is true and using in statement-1, we have $x^2 + 2y^2 - 6x - 12y + 23 + \lambda(4x^2 + 2y^2 - 20x - 12y + 35) = 0$ which represents a circle if $1 + 4\lambda = 2 + 2\lambda$ $\Rightarrow \lambda = 1/2$ and the equation of the circle through the points of intersection of $S_1 = 0$ and $S_2 = 0$ is
$6(x^2 + y^2) - 32x - 36y + 81 = 0$
whose centre is $(8/3, 3)$ and radius is $\sqrt{47}/3\sqrt{2}$. So the statement-1 is also true.

76. $x\cos\theta + y\sin\theta = a$, $x\cos(\pi/2 + \theta) + y\sin(\pi/2 + \theta) = a$
$\Rightarrow (x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 = 2a^2$
$\Rightarrow x^2 + y^2 = 2a^2$
$\Rightarrow$ statement-1 is true, statement-2 is also true but does not lead to statement-1.

77. Statement-2 is true as circle C_1 touches the circle C_2 internally and the diameter of C_1 is equal to the radius of the circle C_2. Using it in statement-1, the centre of the required circle lies inside C.

78. Statement-1 is true as the distance of the centre (3, 4) from x-axis is equal to the radius. Statement-2 is false, because $\frac{4-3}{\sqrt{2}} = a \Rightarrow a = 1/\sqrt{2}$.

79. Statement-2 is True, using it in statement-1, common chord is $10x = r^2 + 16$, which passes through (5, 0) if $r^2 = 34$ (if it passes through (0, 0), $r^2 + 16 = 0$ which is not possible for any real value of r)

80. Statement-1 is not true as the equation represents a pair of straight lines. In statement-2, $(x + 2)^2 + (y + 3)^2 = 0$ which gives a circle of zero radius and hence the point $(-2, -3)$. So statement-2 is true.

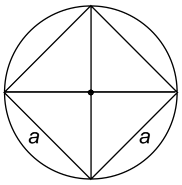


Fig. 17.46

Level 2

81. $x^2 + y^2 + ax = 0$ passes through the centre (0, 0) of the circle $x^2 + y^2 = c^2$. Circles will touch each other if the radius $a/2$ of the first circle is equal to half the radius of the 2nd circle

$\Rightarrow c = \pm a \Rightarrow c^2 - a^2 = 0$

82. Since each side subtends an angle of 120° at the centre (0, 0) of the circle.

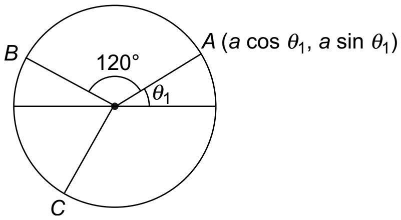


Fig. 17.47

$\theta_2 = 120^\circ + \theta_1$, $\theta_3 = 240^\circ + \theta_1$

So $\cos\theta_1 + \cos\theta_2 + \cos\theta_3$

$= \cos\theta_1 + \cos(120^\circ + \theta_1) + \cos(240^\circ + \theta_1)$

$= \cos\theta_1 + 2\cos(180^\circ + \theta_1)\cos(-120^\circ)$

$= \cos\theta_1 + 2 \times \cos\theta_1(-1/2) = 0$

Similarly $\sin\theta_1 + \sin\theta_2 + \sin\theta_3 = 0$

83. Equation of the circle $C: x(x - 4) + y^2 = 0$

Equation of the common chord of C_1 and C_2 is $8x - 11 = 0$ which meets the circle C at points for which $x = 11/8$ and $y^2 = \frac{4 \times 11}{8} - \left(\frac{11}{8}\right)^2 = \frac{231}{64}$

Hence the required intercept

$= 2 \times \sqrt{\frac{231}{64}} = \frac{\sqrt{231}}{4}$

84. We have $a = b$ and $a^2 + ab - 16 = 0$

$\Rightarrow a = b = \pm 2\sqrt{2}$

Equation of the circle is $x^2 + y^2 + 2x + 2y \pm 2 = 0$

Radius of the circle is $\sqrt{(-1)^2 + (-1)^2 \pm 2} = 0$ or 2.

85. Circle passing through the points (–3, 0), (–1, 0) and (0, 3/2) is $x^2 + y^2 + 4x - (7/2)y + 3 = 0$

which passes through the point (0, $1/\lambda$) if

$$\frac{1}{\lambda^2} - \frac{7}{2\lambda} + 3 = 0 \Rightarrow 6\lambda^2 - 7\lambda + 2 = 0$$

86. $2a^2 = (2r)^2$

where $r^2 = (-2)^2 + (-5)^2 - 21$

$\Rightarrow a^2 = 2 \times 8 = 16$

$\Rightarrow a = 4$

87. A (–1, 1), $P(2\cos\theta, 2\sin\theta)$

coordinates (x, y) of R which divides the join of A and P in the ratio 3:2 are

$$x = \frac{6\cos\theta - 2}{5} \quad y = \frac{6\sin\theta + 2}{5}$$

Locus of R is $(5x + 2)^2 + (5y - 2)^2 = 36$

which is a circle with centre (–2/5, 2/5)

88. Equation of the circle is

$(x - \alpha)(x - \beta) + (y - \alpha')(y - \beta') = 0$

$\Rightarrow x^2 + y^2 - (\alpha + \beta)x - (\alpha' + \beta')y + \alpha\beta + \alpha'\beta' = 0$

$\Rightarrow x^2 + y^2 - -\frac{22}{15}x - \frac{22}{8}y + \frac{8}{15} + \frac{15}{8} = 0$

$\Rightarrow 120(x^2 + y^2) - 22(8x + 15y) + 289 = 0$

89. Let $C_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$

$C_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

$C : x^2 + y^2 + 2gx + 2fy + c = 0$

If C_1 and C_2 cut C orthogonally.

then $2gg_1 + 2ff_1 = c + c_1$

$2gg_2 + 2ff_2 = c + c_2$

$\Rightarrow 2(g_1 - g_2)g + 2(f_1 - f_2)f = c_1 - c_2 \cdots I$

Radical axis of C_1 and C_2 is

$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$

which passes through $(-g, -f)$, the centre of C (using I)

90. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy = 0$ s.t $\sqrt{g^2 + f^2} = 3k$

centroid of the triangle OAB is

$$(x, y) = \left(\frac{-2g}{3}, \frac{-2f}{3}\right)$$

$g^2 + f^2 = 9k^2 \Rightarrow 9(x^2 + y^2) = 4 \times 9k^2$

$\Rightarrow x^2 + y^2 = 4k^2$

91. Let the vertices of the triangle be $A_i\ (x_i, y_i)$, $i = 1, 2, 3$. If (x, y) is any point on the locus then $\sum_{i=1}^{3}\left[(x - x_i)^2 + (y - y_i)^2\right] = c^2$, c being constant.

$$\Rightarrow 3(x^2 + y^2) - 2x\sum_{i=1}^{3} x_i - 2y\sum_{i=1}^{3} y_i + \sum_{i=1}^{3} x_i^2 + \sum_{i=1}^{3} y_i^2 - c^2 = 0$$

Which represents a circle with centre at $\left(\frac{\sum_{i=1}^{3} x_i}{3}, \frac{\sum_{i=1}^{3} y_i}{3}\right)$, the centroid of the triangle $A_1 A_2 A_3$

92. Equation of a tangent is

$y = \pm\sqrt{3}x \pm 4\sqrt{3}\sqrt{1 + 3}$

which meets the axis of x at $x = \pm 8$

Length of the tangent from (± 8, 0) of the circle $x^2 + y^2 = 48$ is $\sqrt{(\pm 8)^2 - 48} = \sqrt{16} = 4$

which is the required length of the side of the rhombus.

93. Equation of any circle on this chord is

$x^2 + y^2 - a^2 + \lambda(x\cos\alpha + y\sin\alpha - p) = 0$

Chord is a diameter of the circle if the centre $(-\lambda\cos\alpha/2, -\lambda\sin\alpha/2)$ lies on the chord.

$\Rightarrow \lambda = -2p$

94. $(ax + by + c)^2 + (bx - ay + d)^2 = r^2(a^2 + b^2)$

$\Rightarrow (a^2 + b^2)(x^2 + y^2) + 2(ac + bd)x$
$+ 2(bc - ad)y + c^2 + d^2 = r^2(a^2 + b^2)$

which is a circle with radius r

95. Vertices of the square be (± $a/2$, ± $a/2$)

Equation of the inscribed circle is

$$x^2 + y^2 = a^2/4 \quad (1)$$

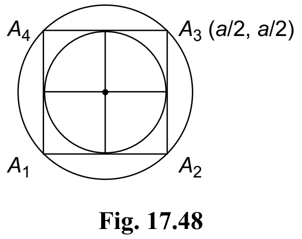


Fig. 17.48

Equation of the circumscribed circle is

$x^2 + y^2 = a^2/2$ (2)

If $P(x, y)$ is a point on (1), then

$\sum_{i=1}^{4} (PA_i)^2 = [x \pm (a/2)]^2 + [y \pm (a/2)]^2$

$= 4(x^2 + y^2) + 4\left(\frac{a^2 + a^2}{4}\right) = a^2 + 2a^2 = 3a^2$

Similarly $\sum_{i=1}^{4} (QA_i)^2 = 2a^2 + 2a^2 = 4a^2$

96. Equation of a tangent from (–2,0) to the circle $x^2 + y^2 = 1$ is $\sqrt{3}y = x + 2$

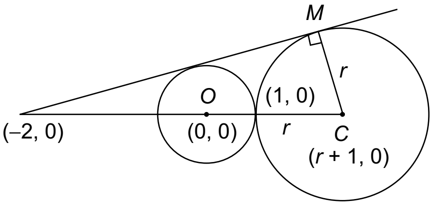


Fig. 17.49

Let the centre of the required circle be $C\ (r + 1, 0)$ and radius r.

Then $CM = \frac{r + 1 + 2 - 0}{\sqrt{1 + 3}} = r$

$\Rightarrow r = 3$ and the equation of the circle is $(x - 4)^2 + y^2 = (3)^2 \Rightarrow x^2 + y^2 - 8x + 7 = 0$

97. Let the equations of the two tangents be

$x \cos\theta + y \sin\theta = a$ and
$x \cos(\pi/4 + \theta) + y \sin(\pi/4 + \theta) = a$

$\Rightarrow x\ [\cos\theta - (1/\sqrt{2})\ (\cos\theta - \sin\theta)]$

$+\ y\ [\sin\theta - (1/\sqrt{2})\ (\cos\theta + \sin\theta)] = 0$

$\Rightarrow x[(\sqrt{2} - 1) \cos\theta + \sin\theta] - [\cos\theta -$

$(\sqrt{2} - 1) \sin\theta]\ y = 0$

$\Rightarrow \frac{x}{\cos\theta - (\sqrt{2} - 1)\sin\theta} = \frac{y}{(\sqrt{2} - 1)\cos\theta + \sin\theta}$

$= k = \frac{x^2 + y^2}{(\sqrt{2} - 1)^2 + 1}$

then from $x \cos\theta + y \sin\theta = a$, we get

$\Rightarrow k \cos\theta\ [\cos\theta - (\sqrt{2} - 1) \sin\theta]$

$+\ k \sin\theta\ [(\sqrt{2} - 1) \cos\theta + \sin\theta] = a$

$\Rightarrow k = a$

$\Rightarrow x^2 + y^2 = [(\sqrt{2} - 1)^2 + 1]\ a^2 = 2(2 - \sqrt{2})a^2$

98. Let the equation of the circle cutting the given circle orthogonally be

$x^2 + y^2 + 2gx + 2fy + c = 0$

then $2g(2) + 2f(-3) = c + 9$

and $2g(-2) + 2f(3) = 4 + c$

$\Rightarrow 8g - 12f = 5$

So locus of the centre $(-g, -f)$ is $8x - 12y + 5 = 0$

99. Let the equation of the circle be

$x^2 + y^2 - 2gx - 2fy = 0$

then $(g - 5)^2 + (f - 2)^2 = g^2 + f^2$

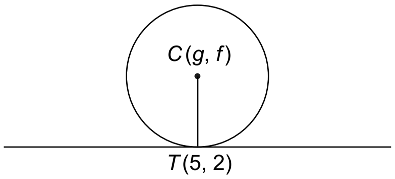


Fig. 17.50

$\Rightarrow 10g + 4f - 29 = 0$ (1)

$C\ (g, f)$ lies on the line through $T(5, 2)$ perpendicular to the

tangent $3x - 2y - 11 = 0$

so $2g + 3f - 16 = 0$ (2)

Solving (1) and (2) we get $g = \frac{23}{22}, f = \frac{51}{11}$

100. Equation of AB is $x/a + y/b = 1$ or $bx + ay = ab$

If $P(h, k)$ is the required point of reflection

then OP is perpendicular to AB and AB is equidistant from O and P.

So $\frac{k}{h} = \frac{a}{b}$ and $\frac{-ab}{\sqrt{a^2 + b^2}} = \pm\frac{hb + ak - ab}{\sqrt{a^2 + b^2}}$

Taking the –ve sign, we get $hb + ak = 2ab$

Solving, $h = \frac{2ab^2}{a^2 + b^2}$, $k = \frac{2a^2b}{a^2 + b^2}$

Note + ve sign gives $(h, k) = (0, 0)$

101. Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

As it cuts the circle $x^2 + y^2 = a^2$ orthogonally

$c = a^2$ and since it passes through the given point

$\left(\frac{a^2 p}{p^2 + q^2}\right)^2 + \left(\frac{a^2 q}{p^2 + q^2}\right)^2 + \frac{2ga^2 p}{p^2 + q^2} + \frac{2fa^2 q}{p^2 + q^2} + a^2$

$= 0$

$\Rightarrow a^2 + 2gp + 2fq + p^2 + q^2 = 0$

$\Rightarrow p^2 + q^2 + 2\,pg + 2qf + a^2 = 0$

Showing that the circle passes through (p, q).

102. Let the other extremities of the diameter through $P\ (2, 3)$ be $A\ (a_1, b_1)$, $B\ (a_2, b_2)$ then equations of the two circles are

$(x - a_1)\ (x - 2) + (y - b_1)\ (y - 3) = 0$

and $(x - a_2)\ (x - 2) + (y - b_2)\ (y - 3) = 0$

So equation of the common chord is

$(x - 2)\ (a_2 - a_1) + (y - 3)\ (b_2 - b_1) = 0$

Since AB makes an angle $\pi/6$ with x–axis

$$\frac{b_2 - b_1}{a_2 - a_1} = \frac{1}{\sqrt{3}}$$

103. Equation of the circle passing through $(-1, 0)$ and $(1, 0)$ is $x^2 + y^2 + 2fy - 1 = 0$ (f being a parameter)

Let (h, k) be the point of contact of the tangent parallel to $y = x$

Equation of the tangent at (h, k) is

$hx + (k + f)\ y + fk - 1 = 0$

so that $\dfrac{-h}{k+f} = 1 \Rightarrow f = -(h + k)$

since (h, k) lies on the circle

$h^2 + k^2 + 2fk - 1 = 0$

$\Rightarrow h^2 + k^2 - 2k\ (h + k) -1 = 0$

$\Rightarrow h^2 - k^2 - 2hk - 1 = 0$

Locus of (h, k) is $x^2 - y^2 - 2xy = 1$

104. Line $y = x$ meets the

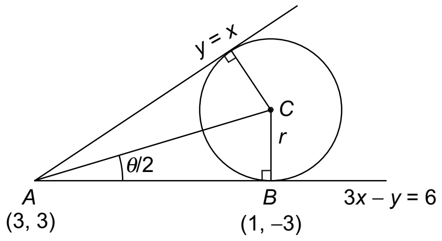


Fig. 17.51

line $3x - y = 6$ at $A(3, 3)$, B be $(1, -3)$, C the centre of the circle, and $\angle CAB = \theta/2$

Then the radius $r = A\,B \tan \theta/2 = \sqrt{40}\ \tan(\theta/2)$

Also $\tan\theta = \dfrac{3-1}{1+3} = \dfrac{1}{2} \Rightarrow \tan(\theta/2) = -\,2+\sqrt{5}$

$\Rightarrow r = \sqrt{40}\,(-2+\sqrt{5}) = 1.6.$

105. Let the equation of the larger circle be $x^2 + y^2 = a^2$ and the line $x + y = 2$ meet the smaller circle at A and B, the larger circle at C and D.

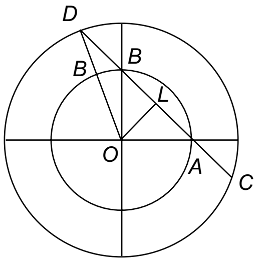


Fig. 17.52

$\Rightarrow OA = OB = 2$ and $OC = OD = a$

Let OL be perpendicular to AB

then $OL = \dfrac{2}{\sqrt{1+1}} = \sqrt{2}$, $LD = LB + BD$

$= (1/2)\sqrt{4+4} + 1$

$\Rightarrow a^2 = (OD)^2 = (OL)^2 + (LD)^2 = 2 + \left(\sqrt{2}+1\right)^2$

$= 5 + 2\sqrt{2}$.

106. Centre of the required circle is the point of reflection of the centre $(1, 0)$ in the line $x + y = 2$ and its radius is equal to the radius 1 of the given circle. So its equation is $(x - 2)^2 + (y - 1)^2 = 1$

107. If the chord AB makes an angle of 30° in the major segment, it makes an angle of 60° at the centre so that the length of the chord is equal to the radius of the circle. Let OL be perpendicular to AB.

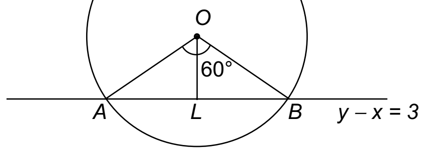


Fig. 17.53

then $(OL)^2 = \left(\dfrac{3}{\sqrt{2}}\right)^2 = \dfrac{9}{2}$

$(AL)^2 = (OA)^2 - (OL)^2 = k^2 - (9/2)$

$k^2 = (AB)^2 = 4\ (AL)^2 = 4(k^2 - (9/2))$

$\Rightarrow k^2 = 6$

108. Let the radii of the three circles be

$1,\ 1 - d,\ 1 - 2d$

Line $y = x + 1$ will cut all the three circles in real distinct points if the distance of the centre $(0, 0)$

from the line is less than the radius of the circle with smallest radius.

$$\frac{1}{\sqrt{2}} < 1 - 2d \Rightarrow d < \frac{2-\sqrt{2}}{4}$$

109. Circle with centre at (3, – 2) must touch the given circle internally. If r is the required radius then.

Distance between the centre = difference of the radii

$$\sqrt{(3-2)^2 + (-2 + (1/2))^2} = \sqrt{2^2 + (1/2))^2} - r$$

$$\Rightarrow r = (1/2)\left(\sqrt{17} - \sqrt{13}\right)$$

110. Let the equation of the required circles be

$x^2 + y^2 + 2gx + 2fy = 0$

Intercept on the line $y = 2x$ is $\frac{2g + 4f}{5}$

Intercept on the line $x = 2y$ is $\frac{4g + 2f}{5}$

$$\frac{2g+4f}{5} = 2 \times \frac{4g + 2f}{5} \Rightarrow g = 0$$

and the required equation is

$x^2 + y^2 + 2fy = 0$ or $x^2 + y^2 - 2\lambda y = 0$

where λ is a parameter.

Previous Years' AIEEE/JEE Main Questions

1. Equation of any tangent to the parabola $y^2 = 8ax$ is

$$y = mx + \frac{2a}{m}$$

Since it touches the circle $x^2 + y^2 = 2a^2$

$$\left|\frac{\frac{2a}{m}}{\sqrt{1+m^2}}\right| = \sqrt{2}a$$

$\Rightarrow (m^2 + 2)(m^2 - 1) = 0 \quad \Rightarrow \quad m = \pm 1$

and the required equations of the tangents are

$$y = \pm (x + 2a)$$

2. Chord will subtenes a right angle at the centre (0, 0) of the circle.

Equation of the pair of lines through (0, 0) and the points of intersection of the circle and the chord is

$x^2 + y^2 = (y - mx)^2$

$\Rightarrow \quad (1 - m)^2x^2 + 2mxy = 0$

These are at rightangles if $1 - m^2 = 0$ $\Rightarrow \quad m = \pm 1$

3. Required points will lie on or outside the circle $x^2 + y^2 = 4$ and on or inside the circle $x^2 + y^2 = 64$ and hence $4 \le x^2 + y^2 \le 64$

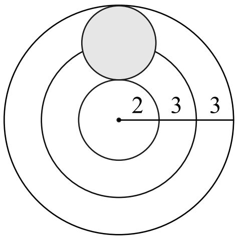


Fig. 17.54

4. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2f = 0$$

Since it passes through (1, 0), $g = -\frac{1}{2}$.

As the circle passes through the centre of (0, 0) of the circle $x^2 + y^2 = 9$, it will touch the circle internally. So the radius of the required circle is 3/2.

$\Rightarrow \quad g^2 + f^2 = \frac{9}{4} \quad \Rightarrow \quad f^2 = 2 \quad \Rightarrow \quad f = \pm\sqrt{2}$

and the centre of the required circle in

$(-g, -f) = \left(\frac{1}{2}, \pm\sqrt{2}\right)$

5. Radius of the circle = $\frac{2}{3}\times$ length of the median. So the required equation of the circle is

$x^2 + y^2 = \left(\frac{2}{3}\times 3a\right)^2 \quad \Rightarrow \quad x^2 + y^2 = 4a^2$

6. Distance between the centre of the given circles

$= \sqrt{(1-4)^2 + (3+1)^2} = 5$

Radius of the given circles are r and $\sqrt{16+1-8} = 3$ circle will intersect in two distinct points if $r - 3 < 5 < r + 3$ or $2 < r < 8$.

7. $\pi r^2 = 154 \quad \Rightarrow \quad r = 7$

Point of intersection of the diameters is (1, –1) which is the centre of the circle and hence its equation is

$(x - 1)^2 + (y + 1)^2 = 49$

or $x^2 + y^2 - 2x + 2y = 47$

8. Let (h, k) be centre of the required circle. As this circle passes through (a, b) its equation is

$(x - h)^2 + (y - k)^2 = (a - h)^2 + (b - k)^2$

or $x^2 + y^2 - 2hx - 2ky - (a^2 + b^2 - 2ha - 2kb) = 0$

This circle will cut

$x^2 + y^2 - 4 = 0$

orthogonally if

$(-2h)(0) + (-2k)(0) = -4 - (a^2 + b^2 - 2ha - 2kb)$

$\Rightarrow \quad 2ha + 2kb - (a^2 + b^2 + 4) = 0$

Thus, required is $2ax + 2by - (a^2 + b^2 + 4) = 0$

9. Let the other end of the diameter be $B(h, k)$. Its equation is

$(x - p)(x - h) + (y - q)(y - k) = 0$

As this circle touches the x-axis, i.e. $y = 0$, the equation

$(x - p)(x - h) + kq = 0$ or $x^2 - (p + h)x + ph + kq = 0$ must have equal roots, i.e.

$(p + h)^2 - 4(ph + kq) = 0$

$\Rightarrow \quad (p - h)^2 = 4kq$

Thus required locus is $(x - p)^2 = 4qy$

10. As centre lies on both $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$, centre of the circle must be $(1, -1)$. If r is the radius of the circle, we must have

$2\pi r = 10\,\pi \Rightarrow r = 5$

$\therefore$ Required equation is

$(x - 1)^2 + (y + 1)^2 = 25$

or $\quad x^2 + y^2 - 2x + 2y - 23 = 0$

11. Equation of any circle through A and B is

$$x^2 + y^2 - 2x + \lambda(x - y) = 0$$

Its centre is $\left(1 - \frac{\lambda}{1}, \frac{\lambda}{2}\right)$.

AB will be a diameter if centre lies on AB, that is, if

$$1 - \frac{\lambda}{2} = \frac{\lambda}{2} \text{ or } \lambda = 1.$$

Thus, equation of desired circle is

$$x^2 + y^2 - 2x + x - y = 0$$

or $\qquad x^2 + y^2 - x - y = 0$

12. P and Q lie on the common chord of the two given circles. An equation of the common chord is $S_1 - S_2 = 0$

or $(x^2 + y^2 + 2ax + cy + a) - (x^2 + y^2 - 3ax + dy - 1) = 0$

or $5ax + (c - d)y + a + 1 = 0 \qquad (1)$

Since $5x + by - a = 0$ passes through P and Q. This line must be identical to (1). Therefore

$$\frac{5a}{5} = \frac{c-d}{b} = \frac{a+1}{-a}$$

$\Rightarrow \quad -a^2 = a + 1 \quad$ or $\quad a^2 + a + 1 = 0$

There is no real value of a satisfying this equation.

13. Let (h, k) be centre of circle passing through (a, b). Then its equation is

$(x - h)^2 + (y - k)^2 = (a - h)^2 + (b - k)^2$

or $x^2 + y^2 - 2hx - 2ky - (a^2 + b^2 - 2ah - 2bk) = 0$

This will meet the circle $x^2 + y^2 - p^2 = 0$ orthogonally if

$(0)(-2h) + (0)(-2k) = -p^2 - (a^2 + b^2 - 2ah - 2bk)$

$\Rightarrow 2ah + 2bk - (a^2 + b^2 + p^2) = 0$

This, required locus is

$2ax + 2by - (a^2 + b^2 + p^2) = 0$

14. Let centre of circle be (α, β). Radius of circle $= \beta$. As the circle touches the circle with centre at $(0, 3)$ and radius 2, distance between centres = sum of the radii

$\Rightarrow \quad \sqrt{(\alpha - 0)^2 + (\beta - 3)^2} = \beta + 2$

$\Rightarrow \alpha^2 + \beta^2 - 6\beta + 9 = \beta^2 + 4\beta + 4$

$\Rightarrow \alpha^2 = 10\beta - 5$

$\therefore$ locus of centre $x^2 = 10y - 5$, which is a parabola.

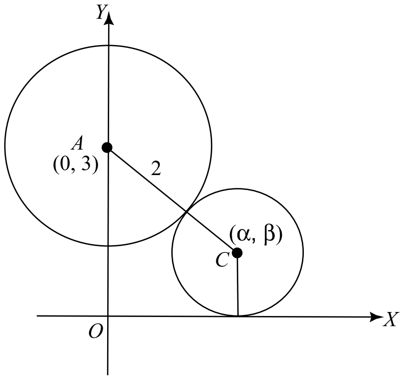


Fig. 17.55

15. Let $M(p, q)$ be the mid-point of a chord AB of the circle subtending an angle of $2\pi/3$ at the centre.

As ΔOAB is an isosceles triangle $OM \perp AB$.

$AM^2 = OA^2 - OM^2$

$= 9 - (p^2 + q^2)$

$\Rightarrow \quad AM = \sqrt{9 - (p^2 + q^2)}$

$\Rightarrow \quad AB = 2(AM) = 2\sqrt{9 - (p^2 + q^2)}$

By the law of cosines

$$\cos\frac{2\pi}{3} = \frac{OA^2 + OB^2 - AB^2}{2(OA)(OB)}$$

$$\Rightarrow\ -\frac{1}{2} = \frac{9+9-4(9-p^2-q^2)}{2(3)(3)}$$

$\Rightarrow$ $-9 = -18 + 4(p^2 + q^2)$

$\Rightarrow$ $p^2 + q^2 = 9/4$

Thus, required locus is

$x^2 + y^2 = 9/4$

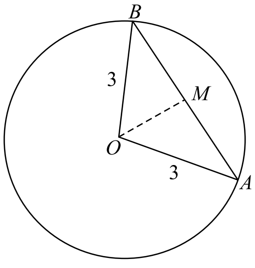


Fig. 17.56

16. Centre is the point of intersection of the diameters

$3x - 4y - 7 = 0$

and $2x - 3y - 5 = 0$

that is, the centre is (1, – 1). Also, radius of circle is 7.

$\therefore$ Equation of circle is

$(x - 1)^2 + (y + 1)^2 = 7^2$

or $x^2 + y^2 - 2x + 2y - 47 = 0$

17. Radius of circle = $|k|$

Equation of circle

$(x - h)^2 + (y - k)^2 = |k|^2$

As it passes through (– 1, 1), we get

$(-1 - h)^2 + (1 - k)^2 = k^2$

$\Rightarrow$ $h^2 + 2h + 2(1 - k) = 0$

As h is real,

$\Rightarrow$ $4 - 8(1 - k) \geq 0$

$\Rightarrow$ $k > \frac{1}{2}$

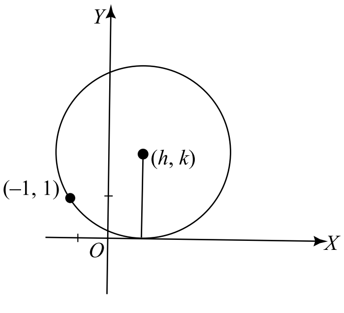


Fig. 17.57

18. Let the coordinates of the required point Q be (α, β) then mid-point of PQ is the centre of the circle so

$$\frac{\alpha+1}{2} = -1,\ \frac{\beta}{2} = -2$$

$\Rightarrow$ $\alpha = -3, \beta = -4$

So the required point is (– 3, – 4)

19. Let S be (1, 0) and R be (– 1, 0)

Let $P(x, y)$ be a point such that

$$\frac{PS}{PR} = \frac{1}{3} \qquad (1)$$

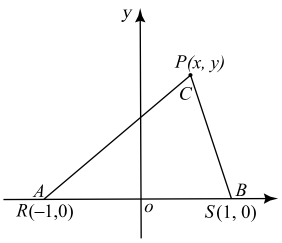


Fig. 17.58

Note that A, B, C lie on the locus of P. Equation (1) gives

$9\ PS^2 = PR^2$

$\Rightarrow 9[(x - 1)^2 + y^2] = (x + 1)^2 + y^2$

$\Rightarrow 8x^2 + 8y^2 - 20x + 8 = 0$

$$\Rightarrow x^2 + y^2 - \frac{5}{2}x + 1 = 0 \qquad (2)$$

Centre of this circle is (5/4, 0)

As A, B, C lie on (2), circumcentre of ΔABC is (5/4, 0)

20. Equation of any circle through P and Q is

$S_1 + \lambda(S_2 - S_1) = 0$

$\Rightarrow x^2 + y^2 + 3x + 7y + 2p - 5$

$+ \lambda(-x - 5y - p^2 - 2p + 5) = 0$

It will pass through (1, 1) is $(1 + 1 + 3 + 7 + 2p - 5)$ $+ l(-1 - 5 - p^2 - 2p + 5) = 0$

$\Rightarrow (7 + 2p =) - \lambda(p + 1)^2 = 0$

$$\Rightarrow\ \lambda = \frac{7+2p}{(p+1)^2}$$

Thus, there exist a circle through P, Q and (1, 1) for all values of p except – 1.

21. The line $3x - 4y = m$ will intersect the circle in two distinct points if length of perpendicular from the centre (2, 4) to the line < radius

$$\Rightarrow\ \frac{|(3)(2) - (4)(4) - m|}{\sqrt{9+16}} < \sqrt{4+16+5}$$

$\Rightarrow$ $|m + 10| < 25 \Rightarrow -25 < m + 10 < 25$

$\Rightarrow 35 < m < 15$

22. Let equation of circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$

As it passes through (1, 0) and (0, 1), we get

$1 + 2g + c = 0, \quad 1 + 2f + c = 0$

If r is radius of the circle, then

$$r^2 = g^2 + f^2 - c$$

$$= \frac{1}{4}(1+c)^2 + \frac{1}{4}(1+c)^2 - c$$

$$= \frac{1}{2}(1+c^2)$$

Note that r^2 will be least if $c = 0$.

$\therefore \quad g = f = -1/2, \; c = 0$

Thus, equation of required circle is $x^2 + y^2 - x - y = 0$

23. $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ can touch each other only internally. See figures. This is possible if $a = c$ or $c = -a$, that is, if $c = | a |$.

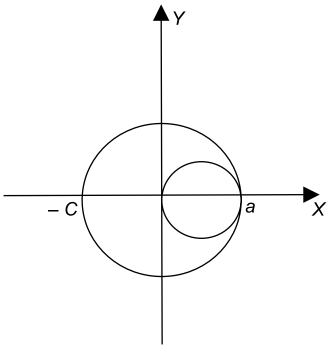


Fig. 17.59

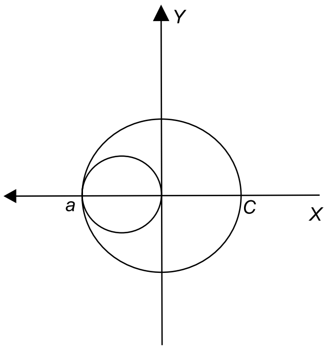


Fig. 17.60

24. As the circle touches the x-axis at (1, 0) its centre is $(1, k)$ where $|k|$ is radius of the circle. Equation of the circle is

$$(x - 1)^2 + (y - k)^2 = k^2$$

As it passe through (2, 3), we get $1^2 + (3 - k)^2 = k^2$ $\Rightarrow 10 - 6k = 0$

$\Rightarrow 2k = 10/3$

Thus, diameter of the circle is 10/3.

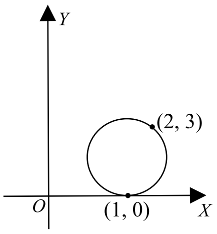


Fig. 17.61

25. Centre of circle lies on the line $x = 3$. Let centre be $(3, k)$, then radius of circle is $|k|$. We have

$(3 - 1)^2 + (k + 2)^2 = |k|^2$

$\Rightarrow k^2 + 4k + 8 = k^2 \Rightarrow k = -2.$

This circle passes through (5, –2) as $(5 - 3)^2 + (-2 + 2)^2 = 4 = (\text{radius})^2$.

Alternatively, note that (5, – 2) is the other end point of the diameter with one end point at (1, – 2).

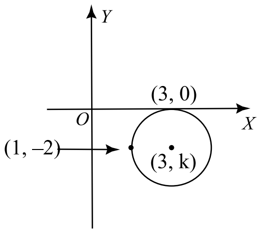


Fig. 17.62

26. Lines intersect at (1, 1)

$\Rightarrow \quad a^2 - 7a + 11 = 1 \quad \text{and} \quad a^2 - 6a + 6 = 1$

$\Rightarrow \quad a = 5$

So the equation of the circle is

$(x - 1)^2 + (y - 1)^2 = 1 - b^3.$

Radius of the circle is $1 - b^3 > 0 \Rightarrow \quad b^3 < 1$

$\Rightarrow \quad b \in (-\infty, 1)$

so $\quad a = 5, \; b \in (-\infty, 1)$

27. Let the centre of the circle c be (h, k) then $(h - 1)^2 + (k + 1)^2 = (h - 4)^2 + k^2$

$\Rightarrow \quad 3h + k = 7 \qquad (1)$

Equation of the tangent at (1, – 1) to the given circle is

$x - y + 2(x + 1) - 3(y - 1) - 12 = 0$

$\Rightarrow \quad 3x - 4y - 7 = 0$

(h, k) lies on the line $\perp$ to this tangent.

$\Rightarrow \quad 4h + 3k - 1 = 0 \qquad (2)$

Solving (1) and (2) we get $h = 4$, $k = -5$ so that the radius of the circle is 5.

28. Length of each side is $2a$.

Length of the median

$= \sqrt{(4a)^2 - a^2} = \sqrt{3}a$

centre of the circumcircle is

$\left(0, \frac{1}{3}(\sqrt{3}a)\right) = \left(0, \frac{a}{\sqrt{3}}\right)$

Hence the equation of the circumcirele is

$$x^2 + \left(y - \frac{a}{\sqrt{3}}\right)^2 = a^2 + \left(\frac{a}{\sqrt{3}}\right)^2$$

$\Rightarrow 3x^2 + 3y^2 - 2\sqrt{3}ay = 3a^2$

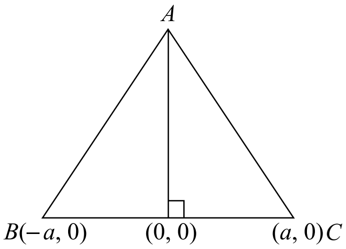


Fig. 17.63

29. y-coordinate of the centre of the circle is equal to its radius

$\Rightarrow \quad 4 = (3)^2 + (4)^2 - (25 - a^2)$

$\Rightarrow \quad a^2 = 16 \quad \Rightarrow \quad a = \pm 4$

30. Statement-2 is true, as every diameter of the circle is a normal to the circle and $2x + y = 5$ is a diameter. Statement-1 is false as any point on the diameter can be taken as the centre of the circle.

31. Let the equation of the circle be $x^2 + y^2 = 1$ and the common chord QR of the two circles subtend an angle of 60° at $P(-1, 0)$

Equation of PQ is

$$y = \frac{1}{\sqrt{3}}(x+1)$$

which meets the circle $x^2 + y^2 = 1$ at $Q\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Line through Q perpendicular to PQ is

$$y - \frac{\sqrt{3}}{2} = -\sqrt{3}\left(x - \frac{1}{2}\right)$$

which meets x-axis at (1, 0), the centre of the arc.

Hence the radius of the arc $= \sqrt{\left(1-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$

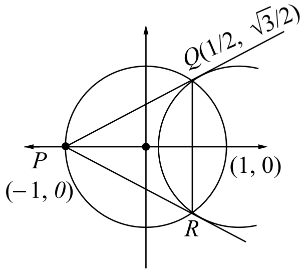


Fig. 17.64

32. $A\ (0, y)$ and $B\ (1, 1)$ be the given centres of the circles

AB = sum of the radii

$= y + 1$

Also $\quad AB = \sqrt{(y-1)^2 + 1}$

$\Rightarrow \quad (y + 1)^2 = (y - 1)^2 + 1 \quad \Rightarrow \quad y = \frac{1}{4}$

So the required radius of T is $\frac{1}{4}$

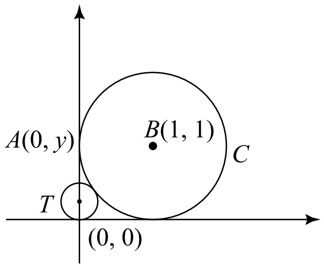


Fig. 17.65

33. The given circle is

$(x - 3)^2 + (y - 5)^2 = 34 - p \quad (1)$

Its centre is (3, 5) and radius is $\sqrt{34 - p}$ As (1) does not touch or intersect the axes, $\sqrt{34 - p} <$ min {3, 5}

$\Rightarrow 25 < p$.

Also, (1, 4) lies inside (1),

$(1 - 3)^2 + (4 - 5)^2 < 34 - p$

$\Rightarrow p < 29$

$\therefore p \in (25, 29)$

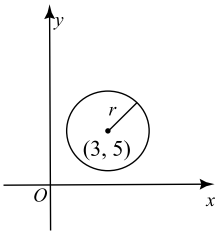


Fig. 17.66

34. There will be exactly two tangents if the two circles intersect, that is, if

$$|r_1 - r_2| < C_1 C_2 < r_1 + r_2 \qquad (1)$$

where $r_1 = \sqrt{2}$

$r_2 = \sqrt{50-\lambda}$

$C_1 = (2, 2)$

$C_2 = (5, 5)$

Thus, (1) gives

$$\left|\sqrt{50-\lambda} - \sqrt{2}\right| < \sqrt{9+9} < \sqrt{50-\lambda} + \sqrt{2}$$

$$\Rightarrow \quad -3\sqrt{2} < \sqrt{50-\lambda} - \sqrt{2} < 3\sqrt{2} < \sqrt{50-\lambda} + \sqrt{2}$$

$$\Rightarrow \quad 0 \le \sqrt{50-\lambda} < 4\sqrt{2}, \ \sqrt{50-\lambda} > 2\sqrt{2}$$

$$\Rightarrow \quad 50 - \lambda < 32, \ 50 - \lambda > 8$$

$$\Rightarrow \quad 18 < \lambda < 42 \Rightarrow \quad \lambda \in (18, 42)$$

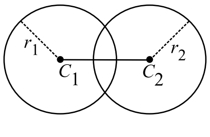


Fig. 17.67

35. We can write $x^2 + y^2 - 2y = 0$ as $x^2 + (y-1)^2 = 1$

This circle lies inside the $x^2 + y^2 = 4$

$\therefore$ There is no common tangent to the two circles.

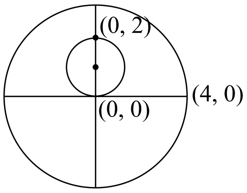


Fig. 17.68

36. An equation of circle passing through intersection of

$$x^2 + y^2 = 16 \qquad (1)$$

and $\quad 3x + y + 5 = 0 \qquad (2)$

is $\quad x^2 + y^2 - 16 + \lambda(3x + y + 5) = 0$

Its centre is $\left(-\frac{3}{2}\lambda, -\frac{1}{2}\lambda\right)$

Line (2) will be a diameter if centre lies on L, that is, if

$$3\left(-\frac{3}{2}\lambda\right) + \left(-\frac{1}{2}\lambda\right) + 5 = 0 \Rightarrow \quad \lambda = 1$$

$\therefore$ An equation of required circle is

$$x^2 + y^2 + 3x + y - 11 = 0$$

37. Centres and radii of two circles are:

$$C_1(2, 3); \ r_1 = 5$$

and $C_2\ (-3, -9)$; $r_2 = 8$

As $C_1C_2 = 13 = r_1 + r_2$

Two circles touch externally, therefore they have three tangents in common.

38. In an equilateral triangle

$$\frac{r}{R} = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$\Rightarrow \quad R = 2r$

Here $r = \dfrac{|3+4+3|}{\sqrt{9+16}} = 2$

Thus, $R = 4$

Also, in an equilateral triangle incentre and circumcentre coincide.

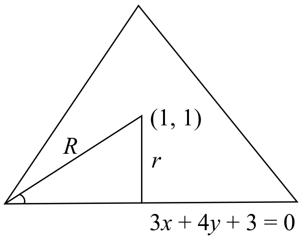


Fig. 17.69

Thus, equation of circumcircle is

$$(x-1)^2 + (y-1)^2 = 4^2$$

or $\qquad x^2 + y^2 - 2x - 2y - 14 = 0.$

39. Centre of the circle lies on the line $y = 2$. Let centre of circle be $(a, 2)$, then its radius is $|a|$.

Thus, $(a+1)^2 + 2^2 = |a|^2$

$\Rightarrow a = -5/2$

An equation of circle

$$(x + 5/2)^2 + (y-2)^2 = 25/4$$

For point of intersection with the x-axis, put $y = 0$, so that

$$(x+\frac{5}{2})^2 + 4 = \frac{25}{4} \Rightarrow x = -\frac{5}{2} \pm \frac{3}{2} = -4, -1$$

Thus, length of chord of the along the x-axis is $-1 - (-4) = 3$.

40. An equation of circle through intersection of $x^2 + y^2 - 30x = 0$ and $3x + y = 0$ is

$$x^2 + y^2 - 30 + \lambda\,(3x + y) = 0 \qquad (1)$$

where λ is some real number. Centre of (1) is $\left(15 - \frac{3}{2}\lambda, -\frac{1}{2}\lambda\right)$.

The line $3x + y = 0$ will be a diameter of (1) if

$$3\left(15-\frac{3}{2}\lambda\right)+\left(-\frac{1}{2}\lambda\right)=0$$

$\Rightarrow 45 - 5\lambda = 0$ or $\lambda = 9$.

Thus, (1) can be written as

$x^2 + y^2 - 3x + 9y = 0$

41. Centre and radius of circle

$x^2 + y^2 - 8x - 8y - 4 = 0$ (1)

is (4, 4) and 6.

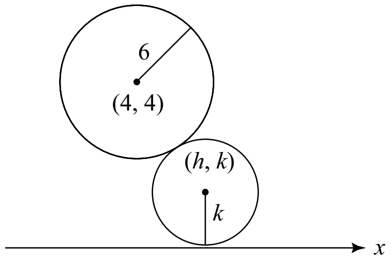


Fig. 17.70

Let (h, k) be centre of the circle touching (1) and the x-axis. Then its radius is k. We have

$$\sqrt{(h-4)^2+(k-4)^2}=k+6$$

$(h - 4)^2 = (k + 6)^2 - (k - 4)^2 = 10(2k + 2)$

Thus, equation of required locus is

$(x - 4)^2 = 20(y + 1)$ which is a parabola.

42. An equation of circle with centre at (–3, 2) and radius r is

$(x + 3)^2 + (y - 2)^2 = r^2$

or $x^2 + y^2 + 6x - 4y + 13 - r^2 = 0$ (1)

Common chord of (1) and

$x^2 + y^2 - 4x + 6y - 12 = 0$ (2)

is $10x - 10y + 25 - r^2 = 0$

It will be diameter of (2) if it passes through centre of (2), that is, if it passes through (2, –3). Therefore $10(2) - 10(-3) + 25 - r^2$

$\Rightarrow r^2 = 75 \quad \Rightarrow r = 5\sqrt{3}$

43. Note that centre of circle lies on $y = 2$. Let centre of circle be $(h, 2)$, then radius of circle is $|h|$.

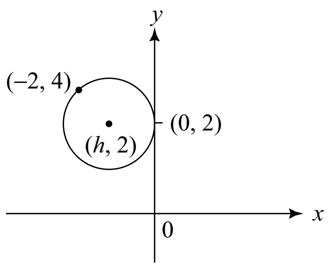


Fig. 17.71

$\therefore$ Equation of circle is $(x - h)^2 + (y - 2)^2 = |h|^2$

As it passes through (–2, 4)

$(-2 - h)^2 + (4 - 2)^2 = h^2$

or $(2 + h)^2 + 2^2 = h^2$

$\Rightarrow 4 + 4h + h^2 + 4 = h^2$

$\Rightarrow h = -2$

Thus centre of circle is (–2, 2)

A diameter of circle is $2x - 3y + 10 = 0$

44. Centre of circle is (4/3, 1/3)

Thus equation of circle is

$$\left(x-\frac{4}{3}\right)^2+\left(y-\frac{1}{3}\right)^2=\left(1-\frac{4}{3}\right)^2+\left(-1-\frac{1}{3}\right)^2$$

or $x^2+y^2-\frac{8}{3}x-\frac{2}{3}y=0$

An equation of tangent at (1, –1) is

$$x(1)+(-1)y-\frac{4}{3}(x+1)-\frac{1}{3}(y-1)=0$$

or $3x - 3y - 4x - 4 - y + 1 = 0$

or $-x - 4y - 3 = 0$ or $x + 4y + 3 = 0$

Previous Years' B-Architecture Entrance Examination Questions

1. Length of the perpendicular from the centre (0, 0) of the circle to the line is equal to a, the radius of the circle

$$\Rightarrow \frac{|a|}{\sqrt{\sin^2\alpha+\cos^2\alpha}} = |a| \quad \text{which is true values of}$$

α.

2. $\pi r^2 = 16\pi \quad \Rightarrow \quad r = 4$

centre of the circle is (2, 3), the point of intersection of the diameters. Hence the required equation is.:

$(x + 2)^2 + (y - 3)^2 = 16$

$\Rightarrow x^2 + y^2 - 4x - 6y - 3 = 0$

3. Equation of the circle passing through $(1, t)$, $(t, 1)$ and (t, t) is $x^2 + y^2 - (t + 1)(x + y) + 2t = 0$

which passes through (1, 1) for all values of t.

4. The common chord of the two circles is a diameter of the circle $x^2 + y^2 + 6x - 4y + x = 0$

equation of the common chord is

$10x - 10y + (k - 3) = 0$ which passes through (3, 2) if

$$10(-3) - 10(2) + (k - 3) = 0 \qquad \Rightarrow k = 53$$

5. $2^2 + k^2 > 13 \quad \Rightarrow \quad k^2 > 9$ (1)

$$\Rightarrow \quad \left(2+\frac{1}{2}\right)^2 + (k-1)^2 > \left(\frac{1}{2}\right)^2 + (1)^2 + 14 = \frac{61}{4}$$

$\Rightarrow \quad (k-1)^2 > 9$ (2)

From (1) and (2) we get

$k \in \,]-\infty, -3[\, \cup \,]\, 4, \infty[$

6. Required shortest distance between the circles is AB $= C_1C_2 - (C_1A) - (BC_2)$

$= \sqrt{9^2 + 12^2} - 1 - 2$
$= 15 - 1 - 2 = 12.$

Note C_1 (0, 0), $C_2(a, 12)$ are the 1 and 2 are the radii of the two circles respectively.

7. Equation of the line through (1, –2) and (0, –1) is $x + y + 1 = 0$ which meets the circle $x^2 + y^2 = 16$ at points for which $x^2 + (x + 1)^2 = 16$

$\Rightarrow \quad \frac{x_1 + x_2}{2} = -\frac{1}{2}$, so x-coordinate of the mid-point of the chord is $-\frac{1}{2}$. Substituting in $x + y + 1 = 0$, y-coordinates of the mid-point is also $-\frac{1}{2}$ and hence the required point is $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

8. radius of the circle < min (3, 5)

$\Rightarrow \quad \sqrt{9 + 25 - p} < 3 \quad \Rightarrow 25 < p$

distance of (1, 4) from the centre < radius

$\Rightarrow \quad (3 - 1)^2 + (4 - 5)^2 < 34 - p$
$\Rightarrow \quad p < 29.$

Hence $p \in (25, 29)$

9. $\pi r^2 = 22\pi \quad \Rightarrow r = \sqrt{22}$

centre of the circle is (– 1, 4), the point of intersection of the given lines.

Hence the equation of the circle is

$(x + 1)^2 + (y - 4)^2 = 22$

$\Rightarrow \quad x^2 + y^2 + 2x - 8y - 5 = 0$

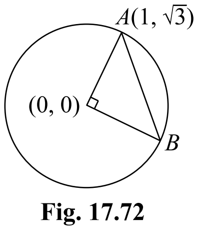


Fig. 17.72

10. Other extremity lies on the line

$$y = \frac{1}{\sqrt{3}}x$$

$\Rightarrow \quad x = -\sqrt{3}y$

Required point is $\left(\sqrt{3}, -1\right)$

11. Centre of the circle C is (1, – 2) and radius = 3

L_1, is a chord of C

if $\frac{3 - 2 - \alpha + 3}{\sqrt{1+9}} < 3 \quad \Rightarrow \quad (\alpha - 1)^2 < 90$

$L2$, is a diameter of c

if $3 - 2 - \alpha + 3 = 0 \quad \Rightarrow \quad \alpha = 4$

Showing that statement-1 is true

Next, L_1 is a diameter of C

if $3 - 2 + \alpha - 2 = 0 \quad \Rightarrow \quad \alpha = 1$

L_2, is a chord of C

if $\frac{3 - 2 - \alpha + 3}{\sqrt{1+9}} < 3$ which is true for $\alpha = 1$

showing that statement-2 is false.

12. $\pi r^2 = 9 \quad \Rightarrow \quad r = 3$

Centre of the circle is (3, 2)

Equation of the circle is $(x - 3)^2 + (y - 2)^2 = 9$

$\Rightarrow \quad x^2 + y^2 - 6x - 4y + 4 = 0$

13. $r \le 3 \quad \Rightarrow \quad r^2 \le 9$

$\Rightarrow \quad 0 \le \frac{1}{4}(k-1)^2 + \frac{1}{4}k^2 - 5 \le 9$

$\Rightarrow \quad 0 \le k^2 - 2k + 1 + k^2 - 20 \le 36$

$\Rightarrow \quad 19 \le 2\,k^2 - 2k \le 55$

$\Rightarrow \quad 39 \le (2k - 1)^2 \le 111$

As k is an integer,

$2k - 1 = \pm\, 7, \pm\, 9$

$\Rightarrow k$ can take four integral values.

14. As $ax + by = 2$ passes through centre (0, 2) of the second circle,

$$a(0) + 2b = 2 \quad \Rightarrow \quad b = 1.$$

Also, as $ax + by - 2 = 0$ touches

$$x^2 + y^2 - 2x - 3 = 0$$

$\frac{|a + b(0) - 2|}{\sqrt{a^2 + b^2}} = 2 \quad \Rightarrow \quad |a - 2| = 2\sqrt{a^2 + 1}$

$\Rightarrow a^2 - 4a + 4 = 4a^2 + 4$

$\Rightarrow 3a^2 + 4a = 0 \quad \Rightarrow a = -\,4/3$ as $a \ne 0$.

Now, $a + b = -\frac{4}{3} + 1 = -\frac{1}{3}$

CHAPTER EIGHTEEN

Parabola

CONIC SECTION

A *conic* (or a *conic section*) is the locus of a point which moves in a plane such that its distance from a fixed point bears a constant ratio to its distance from a fixed line. The fixed point is called the *focus* and the fixed line is called the *directrix* of the conic. The constant ratio is known as the *eccentricity* of the conic and is denoted by the letter *e*. The conic is an *ellipse*, a *parabola* or a *hyperbola*, according as $e < = \text{or} > 1$.

Geometrically, these curves can be obtained as sections of a right circular cone by a plane in various positions and that is why they are called conic sections. A conic (or a conic section) is given by a second degree equation. Thus a conic means a pair of straight lines, a circle, a parabola, an ellipse or a hyperbola. A circle is a limiting case of an ellipse.

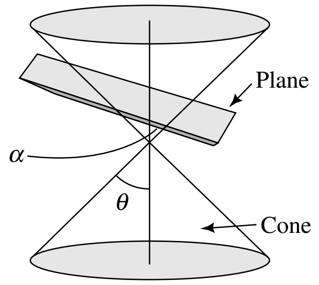


Fig. 18.1

Circle, Ellipse, Parabola and Hyperbola

When the plane cuts the cone with semivertical angle θ at an angle α with the vertical axis of the cone (other than at the vertex) we have the following:

1. If $\alpha = 90°$, the section is *a circle*
2. If $\theta < \alpha < 90°$, the section is *an ellipse*
3. If $\alpha = \theta$, the section is *a parabola*

(In each of the above situations, the plane cuts entirely across one side of the cone above or below the vertex.)

4. If $0 \le \alpha < \theta$, the plane cuts through both the sides of the cone and the curve of intersection is *a hyperbola*.

Equation of a Conic Section

Let the focus of the conic be (α, β) and the directrix be the line $ax + by + c = 0$, e being the eccentricity, then equation of the conic is the locus of the point $P(h, k)$, such that

$$\sqrt{(h-\alpha)^2 + (k-\beta)^2} = e\left|\frac{ah+bk+c}{\sqrt{a^2+b^2}}\right|$$

$$(x-\alpha)^2 + (y-\beta)^2 = \frac{e^2(ax+by+c)^2}{a^2+b^2}$$

PARABOLA

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (called the focus) is equal to its distance from a fixed line (called the directrix).

Standard Forms of the Equation of a Parabola.

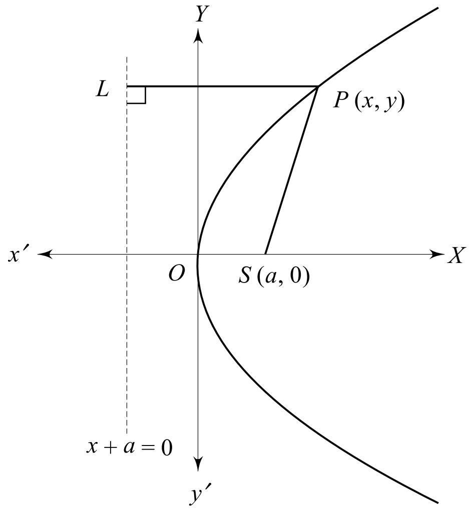


Fig. 18.2

Let the focus S be $(a, 0)$ and directrix be the line $x = -a$, with $a > 0$. If $P(x, y)$ is any point on the locus, then

$$(x-a)^2 + y^2 = (x+a)^2 \Rightarrow y^2 = (x+a)^2 - (x-a)^2 = 4ax$$

$y^2 = 4ax$ is a *standard form* of the equation of a parabola. Four standard forms of a parabola are

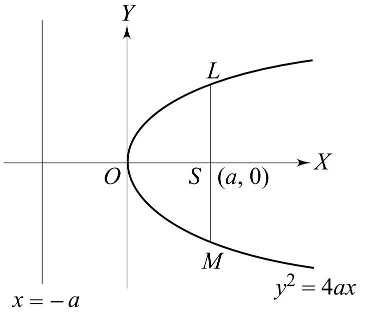


Fig. 18.3 (a)

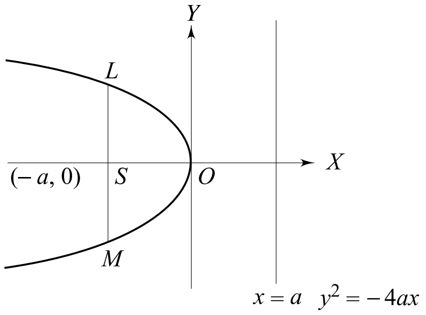


Fig. 18.3 (b)

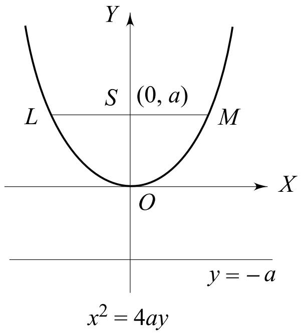


Fig. 18.3 (c)

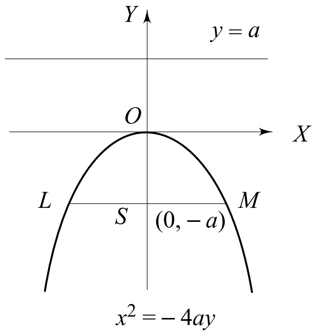


Fig. 18.3 (d)

Illustration 1

Find the equation of the parabola whose focus is (0, –4) and directrix is $y = 4$.

◎ **Solution:** Let P (x, y) be any point on the parabola. S (0, – 4) be the focus, then by definition of the parabola.

$$\sqrt{(x-0)^2+(y+4)^2} = \sqrt{(y-4)^2}$$

$$\Rightarrow \quad x^2 = (y-4)^2 - (y+4)^2$$

$$\Rightarrow \quad x^2 = -16y$$

which is the required equation of the parabola.

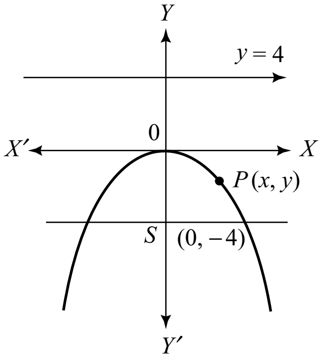


Fig. 18.4

The following terms are used in context of the parabola $y^2 = 4ax$:

1. The point $O(0, 0)$ is the *vertex* of the parabola, and the tangent to the parabola at the vertex is $x = 0$.
2. The line joining the vertex O and the focus $S(a, 0)$ is the *axis* of *the parabola* and its equation is therefore $y = 0$.
3. Any chord of the parabola perpendicular to its axis is called a *double ordinate*.
4. Any chord of the parabola passing through its focus is called a *focal chord.*
5. The focal chord of the parabola perpendicular to its axis is called *its latus rectum*; the length of this latus rectum is therefore $4a$.

Illustration 2

Consider the parabola $y^2 = 12x$. The equation can be written as $y^2 = 4ax$ where $a = 3$. Then

(a) Coordinates of the focus are $(a, 0) = (3, 0)$:

(b) The coordinates of vertex are (0, 0)

(c) Equation of the directrix is $x = -a$ i.e. $x = -3$

(d) Equation of the tangent at the vertex is $x = 0$.

(e) Equation of the latus rectum is $x = a$ i.e. $x = 3$

(f) Length of the latus rectum is $4a = 12$.

(g) Let $P(x,\ 2\sqrt{3x})$ and $P'(x,\ -2\sqrt{3x})$, $x > 0$ be two points, then for all values of x, PP' is a double ordinate of the parabola.

6. The points on a parabola, the normals at which are concurrent, are called *co-normal points* of the parabola. If (x_1, y_1), (x_2, y_2) and (x_3, y_3) are

conormal points of the parabola $y^2 = 4ax$, then $y_1 + y_2 + y_3 = 0$.

7. A line which bisects a system of parallel chords of a parabola is called a *diameter* of the parabola.

The following are some *standard results for the parabola* $y^2 = 4ax$:

1. The *parametric equations* of the parabola or the coordinates of any point on it are $x = at^2$, $y = 2at$.
2. The *tangent* to the parabola at (x', y') is $yy' = 2a\,(x + x')$ and that at $(at^2, 2at)$ is $ty = x + at^2$.
3. The condition that the line $y = mx + c$ is a tangent to the parabola is $c = a/m$ and the equation of any tangent to it (not parallel to the y-axis) is therefore $y = mx + (a/m)$.

Illustration 3

Find the equation of the tangent with slopes 5 to the parabola $y^2 = 20x$, also find the coordinates of the point of contact of the tangent to the parabola.

◎ **Solution:** Equation of the parabola can be written as $y^2 = 4ax$, $a = 5$.

Equation of the tangent with slope 5 is

$$y = 5x + \frac{5}{5} \quad \Rightarrow y = 5x + 1.$$

Parametric coordinates of any point on the parabola are $(5t^2, 10t)$ and the equation of the tangent at this point is

$$ty = x + 5t^2$$

Comparing with $y = 5x + 1$, we get $t = \dfrac{1}{5}$.

So the coordinates of the point of contact are

$$\left(5\left(\frac{1}{5}\right)^2, 10\left(\frac{1}{5}\right)\right) = \left(\frac{1}{5}, 2\right)$$

4. The *chord of contact* (defined as in circles) of (x', y') w.r.t. the parabola is $yy' = 2a\,(x + x')$.
5. The *polar* (defined as in circle) of (x', y') w.r.t. the parabola is $yy' = 2a(x + x')$.
6. The *chord with mid-Point* (x', y') of the parabola is $T = S'$, where $T = yy' - 2a\,(x + x')$ and $S' = y'^2 - 4ax'$.
7. The equation of the *pair of tangents* from (x', y') to the parabola is $T^2 = SS'$. Where $S = y^2 - 4ax$.
8. The *normal* at $(at^2, 2at)$ to the parabola is $y = -tx + 2at + at^3$. If m is the slope of this normal, then its equation is $y = mx - 2am - am^3$, which is the normal to the parabola at $(am^2, -2am)$.

Illustration 4

A normal of slope 4 at a point P on the parabola $y^2 = 28x$, meets the axis of the parabola at Q. Find the length PQ.

◎ **Solution:** $y^2 = 28x = 4ax$, $a = 7$ also $m = 4$, equation of the normal at $P(am^2 - 2am)$ is $y = mx - 2am - am^3$

So coordinates of P are $(112, -56)$ and the equation of the normal is

$$y = 4x - 56 - 448 \quad \text{or} \quad y = 4x - 504.$$

which meets the axis $y = 0$ at $Q\,(126, 0)$

and $\quad (PQ)^2 = (126 - 112)^2 + (0 + 56)^2 = 14^2 \times 17$

$\Rightarrow \quad PQ = 14\sqrt{17}$.

9. A *diameter* of the parabola is the locus of the middle points of a system of parallel chords of the parabola and the equation of a diameter is $y = 2a/m$ where m is the slope of the parallel chords which are bisected by it.
10. The equation of a chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$.
11. If the chord joining the points having parameters t_1 and t_2 passes through the focus, then $t_1\,t_2 = -1$.
12. If the coordinates of one end of a focal chord are $(at^2, 2at)$, then the coordinates of the other end are $(a/t^2, -2a/t)$.

Illustration 5

PP' is a focal chord of the parabola $y^2 = 8x$. If the coordinates of P are $(18, 12)$, find the coordinates of P'.

◎ **Solution:** $y^2 = 8x = 4ax$, $a = 2$.

So, coordinates of the focus are $S(2, 0)$.

Equation of the line joining P and S is $3x - 4y - 6 = 0$ which meets the parabola at points for which $3y^2 - 32y - 48 = 0$

$\Rightarrow \quad y = 12, -4/3$

$y = 12$, represents P. So the coordinates of P' are $\left(\dfrac{2}{9}, -\dfrac{4}{3}\right)$

Alternately Coordinates of P are $(at^2, 2\,at) = (18, 12)$ where $a = 2$, $t = 3$, using the above result. Coordinates of P' are $\left(\dfrac{a}{t^2}, \dfrac{-2a}{t}\right) = \left(\dfrac{2}{9}, -\dfrac{4}{3}\right)$

13. For the end of the latus rectum, the values of the parameter t are ± 1.
14. The tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2\,;\, 2at_2)$ intersect at $(at_1\,t_2,\ a(t_1 + t_2))$.
15. The tangents at the extremities of any focal chord intersect at right angles on the directrix.

Illustration 6

Show that the tangents at the extremities of the focal chord PP' in the Illustration 5 are at right angles and intersect on the directrix of the parabola.

◎ **Solution:** Equations of the tangents at P and P' are

$$12y = 4(x + 18) \text{ and } y\left(-\frac{4}{3}\right) = 4\left(x + \frac{2}{9}\right)$$

$\Rightarrow x - 3y + 18 = 0$ and $3x + y + 2/3 = 0$

which are perpendiculars and intersect at $\left(-2, \dfrac{16}{3}\right)$ which lies on the directrix $x = -2$ of the parabola.

16. The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.

17. The area of the triangle formed by any three points on the parabola is twice the area of the triangle formed by the tangents at these points.

18. The circle described on any focal chord of a parabola as diameter touches the directrix.

19. If the normal at the point $(at_1^2, 2at_1)$ meets the parabola again at $(at_2^2, 2at_2)$ then $t_2 = -t_1 - 2/t_1$.

20. Three normals can be drawn from a point (x_1, y_1) to the parabola. The points where these normals meet the parabola are called *feet of the normals* or *conormal points*. The sum of the slopes of these normals is zero and the sum of the ordinates of the feet of the normals is also zero.

21. If the normals at 't_1' and 't_2' meet on the parabola then $t_1 t_2 = 2$.

22. The centroid of the triangle formed by the conormal points on the parabola lies on the axis of the parabola.

23. The pole of a focal chord of the parabola lies on its directrix.

24. A diameter of the parabola is parallel to its axis and the tangent at the point where it meets the parabola is parallel to the system of chords bisected by the diameter.

25. The semi-latus rectum of the parabola is the harmonic mean between the segments of any focal chord of the parabola.

26. If the tangent and normal at any point P on the parabola meet the axis of the parabola in T and G respectively, then

(i) $ST = SG = SP$, S being the focus

(ii) $\underline{|PSK} = \pi/2$, where K is the point where the tangent at P meets the directrix.

(iii) The tangent at P is equally inclined to the axis of the parabola and the focal distance of P.

Illustration 7

Find the locus of the intersection of the perpendicular tangents to the parabola $x^2 = 4y$.

Let $x = my + c$ be a tangent to the parabola. Then the roots of the equation $(my + c)^2 = 4y$ are equal, as the line meets the parabola at a single point.

$\Rightarrow (2\,mc - 4)^2 = 4m^2c^2 \Rightarrow c = \dfrac{1}{m}$.

So equation of any tangent to the parabola is $x = my + \dfrac{1}{m}$

and equation of the perpendicular tangent is

$$x = -\frac{1}{m}y - m$$

Solving for the point of intersection we get

$$y + 1 = 0$$

So the perpendicular tangents to the parabola intersect on the line $y + 1 = 0$ which is the directrix of the parabola.

Illustration 8

The normal at a point P (36, 36) on the parabola $y^2 = 36x$ meets the parabola again at a point Q. Find the coordinates of Q.

◎ **Solution:** Parabola is $y^2 = 4ax$, $a = 9$

and $P(36, 36) = (at^2, 2at)$, $t = 2$

Equation of the normal at $(at^2, 2at)$ is

$$y = -tx + 2at + at^3$$

So equation of the normal at P is

$$y = -2x + 36 + 72$$

or $\quad y = -2x + 108$

Which meets the parabola at points for which

$(-2x + 108)^2 = 36x$

$\Rightarrow \quad x^2 - 117x + (54)^2 = 0$

$\Rightarrow \quad x = 36, 81.$

$x = 36$, gives the point P, so other end Q of the normal at P is $(81, -54)$

Note: y-coordinate of Q cannot be +ve

Coordinates of Q are $(at'^2, 2at')$

Where $t' = -3 = -t - \dfrac{2}{t}$.

(using result 19)

Illustration 9

If the normals at $P\,(at_1^2, 2at_1)$ and $Q\,(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$ intersect on the parabola, then find the value of $t_1 t_2$.

◎ **Solution:** Equation of normal at $(at^2, 2at)$ to the parabola is $y = -tx + 2at + at^3$.

If it passes through (h, k) then

$$k = -th + 2at + at^3$$

$$\Rightarrow \quad at^3 + (2a - h)t - k = 0 \qquad (1)$$

Which gives three values of t showing that the normal at three points with $t = t_1, t_2, t_3$ pass through (h, k).

So if the normals at P and Q intersect on the parabola, then $h = at_3^2$, $k = 2at_3$.

which proves result 21.

From (7) we have $t_1 t_2 t_3 = \dfrac{k}{a} = \dfrac{2at_3}{a}$

$$\Rightarrow \quad t_1 t_2 = 2.$$

Illustration 10

If the tangent and normal at any point P on the parabola meet the axis of the parabola in T and G respectively, then show that the focus S of the parabola is the centre of the circle passing through T, P and G.

◎ **Solution:** Equations of the tangent and normal at $P(at^2, 2at)$ are respectively

$$ty = x + at^2$$
$$y = -tx + 2at + at^3$$

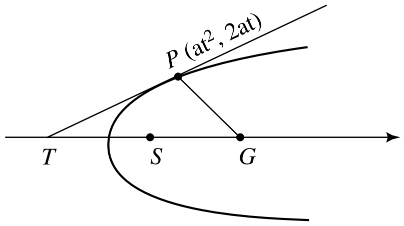


Fig. 18.5

So the coordinates of T are $(-at^2, 0)$ and of G are $(2a + at^2, 0)$

Focus S is $(a, 0)$

$SG = ST = SP = a(1 + t^2)$

Which shows that S is the centre of the circle through T, P and G.

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: The point of intersection of the normals to the parabola $y^2 = 4x$ at the ends of its latus rectum is

(a) (0, 2)	(b) (3, 0)
(c) (0, 3)	(d) (2, 0)

Ans. (b)

◎ **Solution:** Any point on the parabola is $(at^2, 2at)$ where $a = 1$. For the latus rectum $x = a \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$. So ends of latus rectum are $(1, \pm 2)$.

Equations of the normals at these points is given by $y = -tx + 2at + at^3$

$$\Rightarrow \quad x + y - 3 = 0 \text{ and } x - y - 3 = 0$$

which intersect at (3, 0).

Example 2: Equation of the line joining the foci of the parabolas $y^2 = 4x$ and $x^2 = -4y$ is

(a) $x + y - 1 = 0$	(b) $x - y - 1 = 0$
(c) $x - y + 1 = 0$	(d) $x + y + 1 = 0$

Ans. (b)

◎ **Solution:** Foci of the parabolas are (1, 0) and (0, – 1) and the equation of the line joining them is

$$y = x - 1 \quad \text{or} \quad x - y - 1 = 0$$

Example 3: The tangent at a point P on the parabola $y^2 = 8x$ meets the directrix of the parabola at Q such that distance of Q from the axis of the parabola is 3. Then the coordinates of P cannot be

(a) (2, 4)	(b) (8, 8)
(c) (1/2, 2)	(d) (8 – 8)

Ans. (a)

◎ **Solution:** Equation of the tangent at $P\ (2t^2, 4t)$ to the parabola is $ty = x + 2t^2$ which meets the directrix $x = -2$ of the parabola at Q for which $y = \dfrac{2t^2 - 2}{t}$. So that $\dfrac{2t^2 - 2}{t} = \pm 3.$

$$\Rightarrow \quad 2t^2 - 3t - 2 = 0 \quad \text{or} \quad 2t^2 + 3t - 2 = 0$$

$$\Rightarrow \quad (2t + 1)(t - 2) = 0 \quad \text{or} \quad (2t - 1)(t + 2) = 0$$

$$\Rightarrow \quad t = \pm 2, \pm\frac{1}{2}$$

So, the coordinates of P are $(8, \pm 8)$ or $\left(\dfrac{1}{2}, \pm 2\right)$.

Example 4: $y^2 = 16x$ is a parabola and $x^2 + y^2 = 16$ is a circle. Then

(a) circle passes through the vertex of the parabola
(b) circle touches the parabola at the vertex
(c) circle passes through the focus of the parabola
(d) circle lies inside the parabola.

Ans. (c)

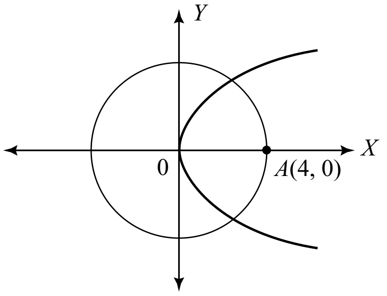


Fig. 18.6

◎ **Solution:** Centre of the circle is the origin, the vertex of the parabola and the focus of the parabola is A (4, 0). The circle passes through A.

◉ **Example 5:** The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$

(c) $\frac{1}{8}$ (d) $\frac{2}{3}$

Ans. (a)

◎ **Solution:** Equation of the tangent at $(t^2, 2t)$ to the parabola $y^2 = 4x$ is $ty = x + t^2$ (1)

and the tangent at $(-16t', -8t'^2)$ to the parabola $x^2 = -32y$ is

$$t'x = y - 8t'^2 \qquad (2)$$

Since (1) and (2) represent the same line, comparing we get

$$\frac{1}{-t'} = \frac{-t}{1} = \frac{t^2}{-8t'^2} \Rightarrow t = 2$$

Hence the required slope of the tangent is $\frac{1}{t} = \frac{1}{2}$.

Alternate Solution

$y = mx + \frac{1}{m}$ is a tangent to the parabola $y^2 = 4x$. If it touches the parabola, $x^2 = -32y$, then the roots of the equation $x^2 + 32\left(mx + \frac{1}{m}\right) = 0$ are equal.

$$\Rightarrow \quad (32m)^2 = 4 \times 32 \times \frac{1}{m} \Rightarrow m = \frac{1}{2}$$

◉ **Example 6:** Let L_1 be the length of the common chord of the curves $x^2 + y^2 = 9$ and $y^2 = 8x$, and L_2 be the length of the latus rectum of $y^2 = 8x$, then

(a) $L_1 > L_2$ (b) $L_1 = L_2$

(c) $L_1 < L_2$ (d) $\frac{L_1}{L_2} = \sqrt{2}$

Ans. (c)

◎ **Solution:** For the points of intersection of the curves $x^2 + y^2 = 9$ and $y^2 = 8x$, we have

$$x^2 + 8x - 9 = 0 \quad \Rightarrow \quad x = -9 \text{ or } x = 1$$

Since $x > 0$ for $y^2 = 8x$, the points of intersection are $(1, 2\sqrt{2})$ and $(1, -2\sqrt{2})$ so that the length of the common chord L_1 is $4\sqrt{2}$.

L_2 = length of the latus rectum of $y^2 = 8x$ is 8. So $L_1 < L_2$

◉ **Example 7:** If m is the slope of a common tangent of the parabola $y^2 = 16x$ and the circle $x^2 + y^2 = 8$, then m^2 is equal to

(a) 1 (b) 2

(c) 4 (d) 8

Ans. (a)

◎ **Solution:** Equation of a tangent to the parabola $y^2 = 16x$ is $y = mx + \frac{4}{m}$.

If this touches the circle of $x^2 + y^2 = 8$, then the distance of the centre (0, 0) of the circle from this tangent is equal to the radius $2\sqrt{2}$ of the circle.

$$\Rightarrow \quad \frac{\frac{4}{m}}{\sqrt{1+m^2}} = 2\sqrt{2}$$

$$\Rightarrow \quad 2 = m^2 + m^4$$

$$\Rightarrow \quad (m^2 + 2)(m^2 - 1) = 0 \Rightarrow m^2 = 1.$$

◉ **Example 8:** Equation of a normal to the parabola $y^2 = 32x$ passing through its focus is

(a) $x = 0$ (b) $y = 0$

(c) $x + y - 8 = 0$ (d) $x - y - 8 = 0$

Ans. (b)

◎ **Solution:** Equation of a normal to the parabola is

$y = mx - 2am - am^3$, $4a = 32 \Rightarrow a = 8$

which passes through the focus (8, 0)

if $0 = 8m - 16m - 8m^3 = 0$

$\Rightarrow m^3 + m = 0 \Rightarrow m(m^2 + 1) = 0 \Rightarrow m = 0$

and the required equation of the normal is $y = 0$.

◉ **Example 9:** P_1: $y^2 = 49x$ and P_2: $x^2 = 4ay$ are two parabolas. Equation of a tangent to the parabola P_1 at a point where it intersects the parabola P_2 is:

(a) $2x - y - 4a = 0$ (b) $y = 0$

(c) $x - 2y + 4a = 0$ (d) $x - y = 0$

Ans. (c)

◎ **Solution:** Two parabolas intersect at the points (0, 0) and $(4a, 4a)$.

Equation of the tangent to P_1 at (0, 0) is $x = 0$ and at $(4a, 4a)$ is

$$y(4a) = 2a(x + 4a)$$

$$\Rightarrow \quad x - 2y + 4a = 0$$

◉ **Example 10:** A point R divides the line segment joining the points $P(1, 3)$ and $Q(1, 1)$ in the ratio $1 : \lambda$. If R is an interior point of the parabola $y^2 = 4x$, then λ can take a value in the interval.

(a) $-1 < \lambda < \frac{3}{5}$ (b) $-\frac{3}{5} < \lambda < 5$

(c) $-2 < \lambda < 2$ (d) $-\frac{3}{5} < \lambda < 1$

Ans. (d)

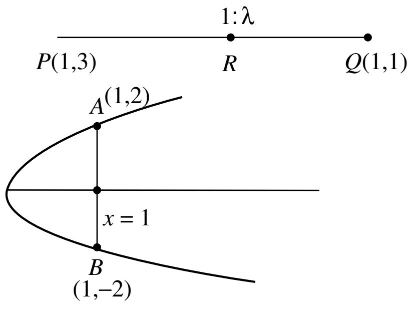


Fig. 18.7

◎ **Solution:** Coordinates of R are

$$= \left(\frac{\lambda+1}{\lambda+1}, \frac{3\lambda+1}{\lambda+1}\right)$$

$$= \left(1, \frac{3\lambda+1}{\lambda+1}\right)$$

$\Rightarrow$ For all values of λ, x-coordinate of R is 1 and y-coordinate of R varies between –2 and 2.

$$\Rightarrow \qquad -2 < \frac{3\lambda+1}{\lambda+1} < 2.$$

$$\Rightarrow \qquad -\frac{3}{5} < \lambda < 1$$

◉ **Example 11:** A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is

(a) $\left(-\frac{9}{8}, \frac{9}{2}\right)$ (b) (2, –4)

(c) (2, 4) (d) $\left(\frac{9}{8}, \frac{9}{2}\right)$

Ans. (d)

◎ **Solution:** $y^2 = 18x \Rightarrow 2y\frac{dy}{dx} = 18$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{9}{y}$$

We are given $\frac{dy}{dx} = 2$

so $\qquad \frac{9}{y} = 2 \Rightarrow y = \frac{9}{2}.$

and $\qquad x = \frac{1}{18}y^2 \Rightarrow x = \frac{9}{8}$

and the required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$

◉ **Example 12:** The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again at the point $(bt_2^2, 2bt_2)$, then

(a) $t_2 = -t_1 + \frac{2}{t_1}$ (b) $t_2 = t_1 - \frac{2}{t_1}$

(c) $t_2 = t_1 + \frac{2}{t_1}$ (d) $t_2 = -t_1 - \frac{2}{t_1}$

Ans. (d)

◎ **Solution:** Equation of the parabola is $y^2 = 4bx$ and the equation of the tangent at $(bt_1^2, 2bt_1)$ is

$$y = -t_1x + 2bt_1 + bt_1^3$$

which passes through $(bt_2^2, 2bt_2)$

$$\Rightarrow \qquad 2bt_2 = -t_1\,(bt_2^2) + 2bt_1 + bt_1^3$$

$$\Rightarrow \qquad 2(t_2 - t_1) = -t_1(t_2^2 - t_1^2)$$

$$\Rightarrow \qquad t_2 = -t_1 - \frac{2}{t_1}.$$

◉ **Example 13:** If P be the point (1, 0) and Q, a point on the locus $y^2 = 8x$. The locus of the mid point of PQ is:

(a) $x^2 + 4y + 2 = 0$ (b) $x^2 - 4y + 2 = 0$

(c) $y^2 - 4x + 2 = 0$ (d) $y^2 + 4x + 2 = 0$

Ans. (c)

◎ **Solution:** Let the coordinates of Q be $(2t^2, 4t)$

coordinates of the mid point of PQ are $\left(\frac{2t^2+1}{2}, 2t\right)$

Let $\qquad h = \frac{2t^2+1}{2}, k = 2t$

Eliminating t, we get $4h = k^2 + 2$

So the required locus is $y^2 - 4x + 2 = 0$

◉ **Example 14:** A parabola has the origin as its focus and the line $x = 2$ as the directrix. Then the vertex of the parabola is at

(a) (2, 0) (b) (0, 2)

(c) (1, 0) (d) (0, 1)

Ans. (c)

◎ **Solution:** Vertex is equidistant from the focus and the directrix and hence its coordinates are (1, 0).

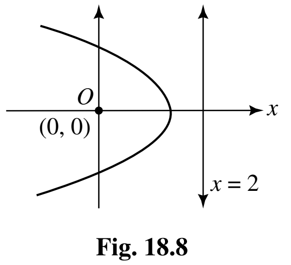


Fig. 18.8

Example 15: A chord is drawn through the focus of the parabola $y^2 = 6x$ such than its distance from the vertex of this parabola is $\frac{\sqrt{5}}{2}$, then its slope can be

(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{3}}$

Ans. (a)

Solution: Focus of the parabola is $\left(\frac{3}{2}, 0\right)$

Let the equation of the chord be

$$y = m\left(x - \frac{3}{2}\right)$$

Its distance from the vertex (0, 0) is $\left|\frac{-\frac{3m}{2}}{\sqrt{1+m^2}}\right| = \frac{\sqrt{5}}{2}$

$$\Rightarrow \quad 9m^2 = 5(1 + m^2)$$

$$\Rightarrow \quad m^2 = \frac{5}{4} \Rightarrow m = \frac{\sqrt{5}}{2}$$

LEVEL 1

Straight Objective Type Questions

Example 16: The focus of the parabola $4y^2 + 12x - 20y + 67 = 0$ is

(a) (– 7/2, 5/2) (b) (– 3/4, 5/2)

(c) (– 17/4, 5/2) (d) (5/2, – 3/4)

Ans. (c)

Solution: The given equation of the parabola can be written as

$$y^2 - 5y = -3x - 67/4 \quad \Rightarrow \quad (y - 5/2)^2 = -3\,(x + 7/2)$$

$\Rightarrow \quad Y^2 = 4aX$ where $Y = y - 5/2$, $X = x + 7/2$ and $a = -3/4$

The focus of $Y^2 = 4aX$ is $(X, Y) = (a, 0) = (-3/4, 0)$

$\Rightarrow \quad x + 7/2 = -3/4,\ y - 5/2 = 0 \quad \Rightarrow \quad x = -17/4,\ y = 5/2$

Therefore, required focus is (– 17/4, 5/2)

Example 17: A line bisecting the ordinate *PN* of a point $P\,(at^2, 2at)$, $t > 0$, on the parabola $y^2 = 4ax$ is drawn parallel to the axis to meet the curve at *Q*. If *NQ* meets the tangent at the vertex at the point *T*, then the coordinates of *T* are.

(a) (0, 4*at*/3) (b) (0, 2*at*)

(c) ($at^2/4$, *at*) (d) (0, *at*)

Ans. (a)

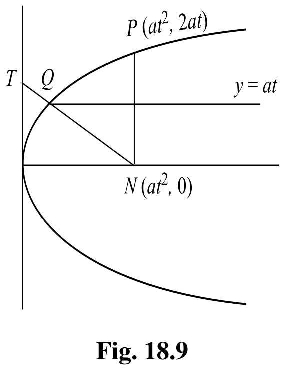


Fig. 18.9

Solution: Equation of the line parallel to the axis and bisecting the ordinate *PN* of the point $P\,(at^2, 2at)$ is $y = at$ which meets the parabola $y^2 = 4ax$ at the point $Q\left(\frac{at^2}{4}, at\right)$.

Coordinates of *N* are $(at^2, 0)$.
Equation of *NQ* is

$$y = \frac{0 - at}{at^2 - at^2/4}\left(x - at^2\right)$$

which meets the tangent at the vertex, $x = 0$, at the point $y = 4at/3$.

Example 18: If *P*, *Q*, *R* are three points on a parabola $y^2 = 4ax$ whose ordinates are in geometrical progression, then the tangents at *P* and *R* meet on

(a) the line through *Q* parallel to *x*-axis
(b) the line through *Q* parallel to *y*-axis
(c) the line joining *Q* to the vertex
(d) the line joining *Q* to the focus.

Ans. (b)

Solution: Let the coordinates of *P*, *Q*, *R* be $(at_i^2, 2at_i)$ $i = 1, 2, 3$ having ordinates in G.P. So that t_1, t_2, t_3 are also in G.P. i.e. $t_1t_3 = t_2^2$. Equations of the tangents at *P* and *R* are

$t_1\,y = x + at_1^2$ and $t_3\,y = x + at_3^2$, which intersect

at the point $\frac{x + at_1^2}{t_1} = \frac{x + at_3^2}{t_3} \quad \Rightarrow \quad x = at_1t_3 = at_2^2$

which is a line through *Q* parallel to *y*-axis.

Example 19: The locus of the mid-point of the line segment joining the focus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix

(a) $x = -a$ (b) $x = -a/2$

(c) $x = 0$ (d) $x = a/2$

Ans. (c)

Solution: The focus of the parabola $y^2 = 4ax$ is $S\,(a, 0)$, let $P\,(at^2, 2at)$ be any point on the parabola then coordinates of the mid-point of *SP* are given by

$$x = \frac{a\,(t^2+1)}{2}, \quad y = \frac{2at+0}{2}$$

Eliminating '*t*' we get the locus of the mid-point as $y^2 = 2ax - a^2$ or $y^2 = 2a\,(x - a/2)$ (1)

which is a parabola of the form $Y^2 = 4AX$ (2)

Where $Y = y$, $X = x - a/2$ and $A = a/2$

Equation of the directrix of (2) is $X = -A$

So equation the directrix of (1) is $x - a/2 = -a/2$

$\Rightarrow \quad x = 0$

Example 20: Equation of a common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

(a) $3y = 9x + 2$	(b) $y = 2x + 1$
(c) $2y = x + 8$	(d) $y = x + 2$

Ans. (d)

Solution: Equation of a tangent at $(at^2, 2at)$ to $y^2 = 8x$ is $ty = x + at^2$ where $4a = 8$ *i.e.* $a = 2$

$\Rightarrow \quad ty = x + 2t^2$ which intersects the curve $xy = -1$ at the points given by $\dfrac{x(x+2t^2)}{t} = -1$ clearly $t \neq 0$

or $x^2 + 2t^2x + t = 0$ and will be a tangent to the curve if the roots of this quadratic equation are equal, for which $4t^4 - 4t = 0 \Rightarrow t = 0$ or $t = 1$ and an equation of a common tangent is $y = x + 2$.

Example 21: The tangent at the point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4a\,(x + b)$ at Q and R, the coordinates of the mid-point of QR are

(a) $(x_1 - a, y_1 + b)$	(b) (x_1, y_1)
(c) $(x_1 + b, y_1 + a)$	(d) $(x_1 - b, y_1 - b)$

Ans. (b)

Solution: Equation of the tangent at $P(x_1, y_1)$ to the parabola $y^2= 4ax$ is $yy_1 = 2a\,(x + x_1)$

or $\quad 2ax - y_1y + 2ax_1 = 0$ (i)

If $M\,(h, k)$ is the mid-point of QR, then equation of QR a chord of the parabola $y^2= 4a(x + b)$ in terms of its mid-point is $ky - 2a\,(x + h) - 4ab = k^2 - 4a\,(h+ b)$

(using $T = S'$) or $\quad 2ax - ky + k^2 - 2ah = 0$ (ii)

Since (i) and (ii) represent the same line, we have

$$\frac{2a}{2a} = \frac{y_1}{k} = \frac{2ax_1}{k^2 - 2ah}$$

$\Rightarrow \quad k = y_1$ and $k^2 - 2ah = 2ax_1$

$\Rightarrow \quad y_1^2 - 2ah = 2ax_1 \quad \Rightarrow \quad 4ax_1 - 2ax_1 = 2ah$

(as $P(x_1, y_1)$ lies on the parabola $y^2 = 4ax$)

$\Rightarrow \quad h = x_1$ so that $h = x_1$, $k = y_1$ and the mid point of QR is (x_1, y_1)

Example 22: AB is a chord of the parabola $y^2 = 4ax$ with the end A at the vertex of the given parabola. BC is drawn perpendiculars to AB meeting the axis of the parabola at C. The projection of BC on this axis is

(a) a	(b) $2a$
(c) $4a$	(d) $8a$

Ans. (c)

Solution: Draw BD perpendicular to the axis of the parabola. Let the coordinates of B be (x, y) then slope of AB is given by $\tan\theta = y/x$

Projection of BC on the axis of the parabola is

$DC = BD \tan\theta$

$$= y\,(y/x) = y^2/x = 4ax/x = 4a$$

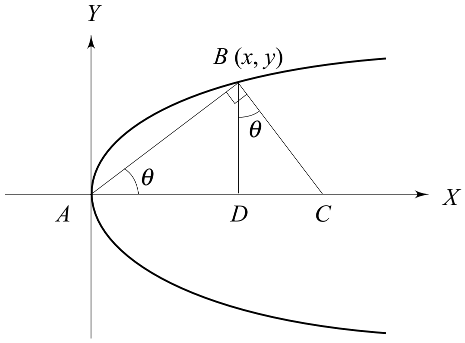


Fig. 18.10

Example 23 Equation of the common tangent touching the circle $(x - 3)^2 + y^2 = 9$ and the parabola $y^2 = 4x$ above the x-axis is

(a) $\sqrt{3}\;y = 3x + 1$	(b) $\sqrt{3}\;y = -(x + 3)$
(c) $\sqrt{3}\;y = x + 3$	(d) $\sqrt{3}\;y = -(3x + 1)$

Ans. (c)

Solution: Equation of a tangent to the parabola $y^2 = 4x$ is $y = mx + 1/m$.

It will touch the circle $(x - 3)^2 + y^2 = 9$ whose centre is (3, 0) and radius is 3 if

$$\left|\frac{0 + m\,(3) + (1/m)}{\sqrt{1+m^2}}\right| = 3$$

or if $\quad (3m + 1/m)^2 = 9(1 + m^2)$

or if $\quad 9m^2 + 6 + 1/m^2 = 9 + 9m^2$

or if $\quad m^2 = 1/3$ i.e. $m = \pm 1/\sqrt{3}$

As the tangent is above the x-axis, we take $m = 1/\sqrt{3}$ and thus the required equation is $\sqrt{3}\;y = x + 3$.

Example 24: The point of intersection of the tangents to the parabola $y^2 = 4x$ at the points where the circle $(x - 3)^2 + y^2 = 9$ meets the parabola, other than the origin, is

(a) (– 2, 0)	(b) (1, 0)
(c) (0, 0)	(d) (– 1, – 1)

Ans. (a)

◎ **Solution:** The circle meets the parabola at points given by $(x-3)^2+4x=9$

$\Rightarrow x^2-2x=0 \Rightarrow x=0, x=2$. But $x=0$ gives the origin so we take $x=2$ and $y=\pm 2\sqrt{2}$. Equation of the tangents to the parabola at $(2, 2\sqrt{2})$ and $(2, -2\sqrt{2})$ are respectively.

$$y\,(2\sqrt{2}) = 2(x+2) \text{ and } y(-2\sqrt{2}) = 2(x+2)$$

Solving these we get $y=0$ and $x=-2$.

◉ **Example 25:** Equation of the directrix of the parabola $y^2+4y+4x+2=0$ is

(a) $x=-1$ (b) $x=1$
(c) $x=-3/2$ (d) $x=3/2$

Ans. (d)

◎ **Solution:** Given equation can be written as

$$(y+2)^2 = -4x+2 = -4(x-1/2)$$

which is of the form $Y^2=4aX$

where $Y=y+2, X=x-1/2, a=-1$

The directrix of the parabola $Y^2=4aX$ is $X=-a$

$\Rightarrow \quad x-1/2=-(-1) \quad \Rightarrow \quad x=3/2$

is the equation of the directrix of the given parabola

◉ **Example 26:** If $x+y=k$ is a normal to the parabola $y^2=12x$, then it touches the parabola

(a) $y^2=-36x$ (b) $y^2=-12x$
(c) $y^2=-9x$ (d) none of these

Ans. (a)

◎ **Solution:** Since $y=mx-2am-am^3$ is a normal to the parabola $y^2=4ax$, taking $a=3$ and $m=-1$ we get

$$y=-x-2\,(3)(-1)-3\,(-1)^3$$

$\Rightarrow \quad x+y=9$ is a normal to the parabola $y^2=12x$.

Suppose it touches the parabola $y^2=4ax$.

Equation of a tangent to the parabola $y^2=4ax$ is

$$y=mx+a/m$$

If it represents the line $x+y=9$, then

$$m=-1 \text{ and } a/m=9 \quad \Rightarrow \quad a=-9$$

So an equation of the required parabola is $y^2=4(-9)x$ or $y^2=-36x$

◉ **Example 27:** If the normal chord at a point 't' on the parabola $y^2=4ax$ subtends a right angle at the vertex, then a value of t is

(a) 4 (b) $\sqrt{3}$
(c) $\sqrt{2}$ (d) 1

Ans. (c)

◎ **Solution:** Equation of the normal at 't' to the parabola $y^2=4ax$ is

$$y=-tx+2at+at^3 \qquad \text{(i)}$$

The joint equation of the lines joining the vertex (origin) to the points of intersection of the parabola and the line (i) is

$$y^2 = 4ax\left[\frac{y+tx}{2at+at^3}\right]$$

$\Rightarrow \quad (2t+t^3)y^2 = 4x\,(y+tx)$

$\Rightarrow \quad 4t\,x^2-(2t+t^3)\,y^2+4xy=0$

Since these lines are at right angles co-efficient of x^2 + coefficient of $y^2=0$

$\Rightarrow \quad 4t-2t-t^3=0 \quad \Rightarrow \quad t^2=2$

For $t=0$, the normal line is $y=0$, i.e. axis of the parabola which passes through the vertex $(0, 0)$.

◉ **Example 28:** The slopes of the normals to the parabola $y^2=4ax$ intersecting at a point on the axis of the parabola at a distance $4a$ from its vertex are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Ans. (a)

◎ **Solution:** Equation of any normal to the parabola $y^2=4ax$ is

$$y=mx-2am-am^3.$$

If it passes through $(4a, 0)$, the point on the axis $y=0$, at a distance $4a$ from the vertex $(0,0)$ then $m=0, \pm\sqrt{2}$

Therefore the slopes of the required normals are $-\sqrt{2}$, 0, $\sqrt{2}$; which are in A.P.

◉ **Example 29:** If the focus of a parabola divides a focal chord of the parabola in segments of length 3 and 2, the length of the latus rectum of the parabola is

(a) 3/2 (b) 6/5
(c) 12/5 (d) 24/5

Ans. (d)

◎ **Solution:** Let $y^2=4ax$ be the equation of the parabola, then the focus is $S\,(a, 0)$. Let $P\,(at^2{}_1, 2at_1)$ and $Q\,(at^2{}_2, 2at_2)$ be vertices of a focal chord of the parabola, then $t_1\,t_2=-1$.

Let $SP=3$, $SQ=2$

$$SP = \sqrt{a^2(1-t_1^2)^1+4a^2t_1^2} = a(1+t_1^2) = 3 \qquad \text{(i)}$$

and $$SQ = a\left(1+\frac{1}{t_1^2}\right) = 2 \qquad \text{(ii)}$$

From (i) and (ii) we get $t_1^2=3/2$ and $a=6/5$

Hence the length of the latus rectum = 24/5.

◉ **Example 30:** P is a point on the parabola whose ordinate equals its abscissa. A normal is drawn to the parabola at P to meet it again at Q. If S is the focus of the parabola then the product of the slopes of SP and SQ is

(a) -1 (b) 1/2
(c) 1 (d) 2

Ans. (a)

◎ **Solution:** Let $P(at^2, 2at)$ be a point on the parabola $y^2=4ax$, then $at^2=2at \quad \Rightarrow \quad t=2$ and thus the coordinates of P are $(4a, 4a)$.

Equation of the normal at P is $y = -tx + 2at + at^3$

$\Rightarrow \quad y = -2x + 4a + 8a \quad \Rightarrow \quad 2x + y = 12a \qquad \text{(i)}$

which meets the parabola $y^2 = 4ax$ at points given by

$$y^2 = 2a(12a - y) \quad \Rightarrow \quad y^2 + 2ay - 24a^2 = 0$$

$\Rightarrow \quad y = 4a$ or $y = -6a$

$y = 4a$ corresponds to the point P

and $y = -6a \Rightarrow \quad x = 9a$ from (i)

So that the coordinates of Q are $(9a, -6a)$. Since the coordinate of the focus S are $(a, 0)$, slope $SP = 4/3$ and slope of $SQ = -6/8$. Product of the slopes $= -1$.

◉ **Example 31:** Equation of the directrix of the parabola whose focus is (0, 0) and the tangent at the vertex is $x - y + 1 = 0$ is

(a) $x - y = 0$ (b) $x - y - 1 = 0$

(c) $x - y + 2 = 0$ (d) $x + y - 1 = 0$

Ans. (c)

◎ **Solution:** Since the directrix is parallel to the tangent at the vertex, let the equation of the directrix be.

$$x - y + \lambda = 0$$

But the distance between the focus and directrix is twice the distance between the focus and the tangent at the vertex.

Therefore $\quad \dfrac{0+0+\lambda}{\sqrt{1+1}} = 2 \times \dfrac{0-0+1}{\sqrt{1+1}}$

$\because$ focus lies on the same side of the directrix as the tangent at the vertex of the parabola.

$\Rightarrow \quad \lambda = 2$, and the required equation is $x - y + 2 = 0$

◉ **Example 32:** The common tangents to the circle $x^2 + y^2 = a^2/2$ and the parabola $y^2 = 4ax$ intersect at the focus of the parabola

(a) $x^2 = 4ay$ (b) $x^2 = -4ay$

(c) $y^2 = -4ax$ (d) $y^2 = 4a(x + a)$

Ans. (c)

◎ **Solution:** Equation of a tangent to the parabola $y^2 = 4ax$ is $y = mx + a/m$.

If it touches the circle $x^2 + y^2 = a^2/2$

$$\frac{a}{m} = \left(\frac{a}{\sqrt{2}}\right)\sqrt{1+m^2} \quad \Rightarrow \quad 2 = m^2(1+m^2)$$

$\Rightarrow \quad m^4 + m^2 - 2 = 0 \quad \Rightarrow \quad (m^2 - 1)(m^2 + 2) = 0$

$\Rightarrow \quad m^2 = 1 \quad \Rightarrow \quad m = \pm 1$

Hence the common tangents are $y = x + a$ and $y = -x - a$ which intersect at the point $(-a, 0)$ which is the focus of the parabola $y^2 = -4ax$.

◉ **Example 33:** The locus of the vertices of the family of parabolas

$$y = \frac{a^3x^2}{3} + \frac{a^2x}{2} - 2a \text{ is}$$

(a) $xy = 64/105$ (b) $xy = 105/64$

(c) $xy = 3/4$ (d) $xy = 35/16$

Ans. (b)

◎ **Solution:** Equation of the parabola can be written as

$$\frac{y}{a} = \left(\frac{ax}{\sqrt{3}} + \frac{\sqrt{3}}{4}\right)^2 - \frac{3}{16} - 2$$

or $\quad \left(x + \dfrac{3}{4a}\right)^2 = \dfrac{a^2}{3a}\left(y + \dfrac{35}{16}a\right)$

vertex is $x = -3/4a$, $y = -35a/16$

Locus of the vertex is $xy = 105/64$.

◉ **Example 34:** An equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent is

(a) $(-1, 1)$ (b) $(0, 2)$

(c) $(2, 4)$ (d) $(-2, 0)$

Ans. (d)

◎ **Solution:** $y = mx + a/m$ is a tangent for all values of m where $a = 2$.

For $m = 1$, $y = x + 2$ so the equation of the tangent perpendicular to the given tangent is $y = -x - 2$. The required point is the point of intersection of these tangents i.e., $(-2, 0)$. Alternately perpendicular tangents to a parabola intersect on the directrix. $x + 2 = 0$

◉ **Example 35:** If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is

(a) $x = -1$ (b) $2x - 1 = 0$

(c) $x = 1$ (d) $2x + 1 = 0$

Ans. (*a*)

◎ **Solution:** Equation of a tangent to the parabola is $y = mx + 1/m$. If it passes through $P(h, k)$, then $k = mh + 1/m$ $\Rightarrow m^2h - mk + 1 = 0$ which gives the slopes of two tangents passing through P. As these are at right angles, product of the slopes $= -1 \Rightarrow 1/h = -1$

Locus of P is $x = -1$ which is the directrix of the parabola.

◉ **Example 36:** The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is

(a) $\dfrac{3\sqrt{2}}{5}$ (b) $\dfrac{\sqrt{3}}{4}$

(c) $\dfrac{3\sqrt{2}}{8}$ (d) $\dfrac{2\sqrt{3}}{8}$

Ans. (c)

◎ **Solution:** Any point on the curve is (t^2, t) whose distance from the given line is

$$\left|\frac{t^2 - t + 1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}}\left|\left(t - \frac{1}{2}\right)^2 + \frac{3}{4}\right| \geq \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$$

Example 37: If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabola $y^2 = 4ax$ and $x^2 = 4ay$, then

(a) $d^2 + (2b - 3c)^2 = 0$ (b) $d^2 + (3b + 2c)^2 = 0$

(c) $d^2 + (2b + 3c)^2 = 0$ (d) $d^2 + (3b - 2c)^2 = 0$

Ans. (c)

Solution: Given parabolas intersect at (0, 0) and $(4a, 4a)$. If the given line passes through these points, then

$$d = 0 \text{ and } 2b(4a) + 3c(4a) = 0$$

$$\Rightarrow \quad 2b + 3c = 0$$

Hence $\quad d^2 + (2b + 3c)^2 = 0.$

Example 38: The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the parabola $y^2 = 4ax$ is

(a) $xy = 4a$ (b) $xy = a$

(c) $xy = a^2$ (d) $ax^3 + 2ax + y = 0$

Ans. (b)

Solution: Equation of any tangent with slope m to the parabola is $y = mx + \dfrac{a}{m}$

Comparing with $y = \alpha x + \beta$, we get

$$m = \alpha,\ \frac{a}{\alpha} = \beta \Rightarrow \alpha\beta = a.$$

and the required locus is $xy = a$.

Example 39: The point of intersection of the normals to the parabola $y^2 = 4x$ at the ends of its latus rectum is

(a) (0, 2) (b) (3, 0)

(c) (0, 3) (d) (2, 0)

Ans. (b)

Solution: Equation of the latus rectum is $x = 1$ and the coordinates of the ends of the latus rectum are (1, 2) and (1, –2). Equations of the normals at these points is $y = -tx + 2at + at^3$, where $t = \pm 1$, $a = 1 \Rightarrow y + x = 3$ and $y - x = -3$ which intersect at (3, 0).

Example 40: Two tangents are drawn from a point (–2, –1) to the curve $y^2 = 4x$. If α is the angle between them, then $|\tan\alpha|$ is equal to:

(a) $\dfrac{1}{3}$ (b) $\dfrac{1}{\sqrt{3}}$

(c) $\sqrt{3}$ (d) 3.

Ans. (d)

Solution: Equation of a tangent to the parabola $y^2 = 4x$ is $y = mx + \dfrac{1}{m}$.

If it passes through (–2, –1), then

$$-1 = -2m + \frac{1}{m}$$

$$\Rightarrow 2m^2 - m - 1 = 0$$

$$\Rightarrow (2m + 1)(m - 1) = 0 \Rightarrow m = 1, \frac{-1}{2}$$

$$\text{So } |\tan\alpha| = \left|\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right| = 3$$

Example 41: Let O be the vertex and Q be any point on the parabola $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1:3, then the locus of P is:

(a) $x^2 = y$ (b) $y^2 = x$

(c) $y^2 = 2x$ (d) $x^2 = 2y$

Ans. (d)

Solution: Let the coordinate of Q be (x', y') and that of P be (h, k), then

$$h = \frac{1}{4}x',\ k = \frac{1}{4}y'$$

As (x', y') lies on the parabola $x^2 = 8y$

we get $(4h)^2 = 8(4k)$

$$\Rightarrow \quad h^2 = 2k$$

Locus of $P(h, k)$ is $x^2 = 2y$

Example 42: Let $y^2 = 16x$ be a given parabola and L be an extremity of its latus rectum in the first quadrant. If a chord is drawn through L with slope-1, then the length of this chord is:

(a) 32 (b) $16\sqrt{2}$

(c) $16\sqrt{3}$ (d) $32\sqrt{2}$

Ans. (d).

Solution: Equation of the latus rectum is $x = 4$ and the coordinates of L are (4, 8). Equation of the chord through L with slope-1 is $y - 8 = -(x - 4)$

$\Rightarrow x + y = 12$

Solving the equation of the chord and the parabola we get $y^2 = 16(12 - y) \Rightarrow y = 8$ and –24.

So the coordinates of M the other end of the chord through L is (36, –24).

and $LM = \sqrt{(36-4)^2 + (-24-8)^2} = 32\sqrt{2}$

Example 43: The locus of the mid-points of the chords the parabola $x^2 = 4py$ having slope m is a:

(a) line parallel to x-axis at a distance $|2pm|$ from it.

(b) line parallel to y-axis at a distance $|2pm|$ from it.

(c) line parallel to $y = mx$, $m \neq 0$ at a distance $|2pm|$ from it.

(d) circle with centre at the origin and radius $|2pm|$.

Ans. (b)

◎ **Solution:** Let (h, k) be the mid point of a chord then its equation is

$hx - 2p(y - k) = h^2 - 4pk$ (using $T = S$)

slope of this chord is $\dfrac{h}{2p} = m \Rightarrow h = 2pm$

Locus of (h, k) is $x = 2pm$

which is line parallel to y-axis at a distance $|2pm|$ from it.

● **Example 44** Let (x, y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from $(0,0)$ to (x, y) in the ratio 2:3. Then locus of P is

(a) $x^2 = y$ (b) $5y^2 = 2x$

(c) $5y^2 = 8x$ (d) $5x^2 = 2y$

Ans. (c)

◎ **Solution:** Let P be (h, k)

then $h = \dfrac{2x}{5}$, $k = \dfrac{2y}{5}$

As (x, y) lies on the parabola $y^2 = 4x$.

$$\left(\frac{5k}{2}\right)^2 = 4\left(\frac{5h}{2}\right)$$

$\Rightarrow$ $5k^2 = 8h$

$\Rightarrow$ Locus of P is $5y^2 = 8x$.

● **Example 45** Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point $(9, 6)$, then L is **not** given by

(a) $y - x + 3 = 0$ (b) $y + 3x - 33 = 0$

(c) $y + x - 15 = 0$ (d) $y - 2x + 12 = 0$

Ans. (c)

◎ **Solution:** The equation of the normal is

$y = mx - 2m - m^3$.

As it passes through $(9, 6)$

$$6 = 9m - 2m - m^3 \quad \Rightarrow \quad m^3 - 7m + 6 = 0$$

$\Rightarrow$ $m = 1, 2, -3$.

Showing that L is not given by (c).

Assertion-Reason Type Questions

● **Example 46** Consider the two curves $C_1 : y^2 = 4x$, $C_2 : x^2 + y^2 - 6x + 1 = 0$.

Statement-1: C_1 and C_2 touch each other exactly at two points.

Statement-2: Equation of the tangent at $(1, 2)$ to C_1 and C_2 both is $x - y + 1 = 0$ and at $(1, -2)$ is $x + y + 1 = 0$.

Ans. (*a*)

◎ **Solution:** Solving for the points of intersection we have

$$x^2 + 4x - 6x + 1 = 0$$

$\Rightarrow$ $(x - 1)^2 = 0$

$\Rightarrow$ $x = 1 \quad \Rightarrow \quad y = \pm 2$

Thus the two curves meet at $(1, 2)$ and $(1, -2)$

Tangent at $(1, 2)$ to $y^2 = 4x$ is

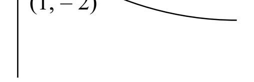


Fig. 18.11

$$y(2) = 2(x + 1) \quad \Rightarrow \quad x - y + 1 = 0$$

Tangent at $(1, 2)$ to the circle C_2 is

$$x + 2y - 3(x + 1) + 1 = 0$$

or $x - y + 1 = 0$ same as the tangent to the curve C_1. Similarly the tangent at the point $(1, -2)$ to the two curves is $x + y + 1 = 0 \Rightarrow$ statement-2 is true and hence statement-1 is also true.

● **Example 47: Statement-1:** The normal at a point P on the parabola $y^2 = 4x$ meets the x-axis at Q. If M is the mid point of the line segment PQ, then the locus of M is another parabola whose vertex is at the focus of the given parabola.

Statement-2: The tangent at a point P on the parabola $y^2 = 4x$ meets the x-axis at Q. If R is the mid point of PQ, then the locus of R is the tangent at the vertex of the parabola.

Ans. (*b*)

◎ **Solution:** Normal at the point $P(m^2, -2m)$ on the parabola $y^2 = 4x$ is $y = mx - 2m - m^3$ which meet the x-axis at $Q(2 + m^2, 0)$.

Coordinates of M are $(1 + m^2, -m) = (x, y)$

Locus of M is $y^2 = x - 1$ which is a parabola with vertex $(1, 0)$, the focus of the parabola $y^2 = 4x$. Thus statement-1 is true.

Next, tangent at a point $P(t^2, 2t)$ on the parabola $y^2 = 4x$ is $ty = x + t^2$ which meets the x-axis at $Q(-t^2, 0)$ coordinates of R are $(0, t) = (x, y)$.

Locus of R is $x = 0$, the tangent at the vertex of the parabola $y^2 = 4x$. So statement-2 is also true but does not lead to statement-1.

◉ **Example 48: Statement-1:** The curve $y = -\frac{x^2}{2} + x + 1$ is symmetrical with respect to the line $x = 1$.
Statement-2: A parabola is symmetric about its axis.
Ans. (a)

◎ **Solution:** Statement-2 is true, Equation in statement-1 is $(x - 1)^2 = -2(y - 3/2)$ which is a parabola with axis $x - 1 = 0$, using statement-2, statement-1 is also true.

◉ **Example 49: Statement-1**: A parabola has the origin as its focus and the line $y = 2$ as the directrix, then the vertex of the parabola is at the point (0, 1).
Statement-2: Vertex of a parabola is equidistance from the focus and the directrix and lies on the line through the focus perpendicular to the directrix.
Ans. (a)

◎ **Solution:** Statement-2 is true and using it in statement-1, the vertex is on the line $x = 0$ at a distance 1 from the focus (0, 0), so the vertex is at the point (0, 1) and the statement-1 is also true.

◉ **Example 50: Statement 1:** The point $\left(\frac{1}{4}, \frac{1}{2}\right)$ on the parabola $y^2 = x$ is closest to the line $y = x + 1$.

Statement-2: The tangent at $\left(\frac{1}{4}, \frac{1}{2}\right)$ to the parabola $y^2 = x$ is parallel to the line $y = x + 1$.
Ans. (b)

◎ **Solution:** Tangent at $\left(\frac{1}{4}, \frac{1}{2}\right)$ to the parabola $y^2 = x$ is $y\left(\frac{1}{2}\right) = \frac{1}{2}\left(x + \frac{1}{4}\right)$ or $y = x + \frac{1}{4}$ which is parallel to y.=.x.+.1.

So statement-2 is true. For Statement-1, any point on the parabola $y^2 = x$ is $\left(\frac{1}{4}t^2, \frac{1}{2}t\right)$ whose distance from the line $y = x + 1$ is $\left|\frac{\frac{1}{2}t - \frac{1}{4}t^2 - 1}{\sqrt{2}}\right| = \frac{(t-1)^2 + 3}{4\sqrt{2}}$

which is minimum when $t = 1$, so the coordinates on the point on the parabola closest to $y = x + 1$ is $\left(\frac{1}{4}, \frac{1}{2}\right)$ and thus statement-1 is also true but does not follow from statement-1.

◉ **Example 51: Statement-1** A circle drawn on a focal radii of the parabola $y^2 = 4ax$ as a diameter touches the tangent at the vertex of the parabola.
Statement-2 The portion of the tangent to the parabola $y^2 = 4ax$ intercepted between the point of contact, other than the vertex, and the directrix subtends a right angle at the focus of the parabola.
Ans. (b)

◎ **Solution:** For statement-2, let the coordinates of a point P on the parabola be $(at^2, 2at)$.
Equation of the tangent at P to the parabola is $ty = x + at^2$ which meets the directrix $x + a = 0$ at $Q\left(-a, a\left(t - \frac{1}{t}\right)\right)$.

Coordinates of the focus S are $(a, 0)$.

Slope of $PS = \frac{2at}{at^2 - a} = m_1$

Slope of $QS = \frac{a\left(t - \frac{1}{t}\right)}{-a - a} = \frac{t^2 - 1}{-2t} = m_2$

$m_1m_2 = -1 \Rightarrow PQ$ subtends a right angle at the focus and thus statement-2 is true.

Next, for statement-1, equation of the circle on PS as a diameter is

$(x - a)(x - at^2) + (y - 0)(y - 2at) = 0$

which meets the tangent at the vertex $x = 0$ at points for which

$a^2t^2 + y(y - 2at) = 0$

$\Rightarrow (y - at)^2 = 0$

Showing that the circle touches the tangent at the vertex and hence statement-1 is also true but does not follow from statement-2.

◉ **Example 52: Statement-1** If $x + 2y = 5$ and $2x - y = 0$ are two tangent to the parabola whose axis is $x + y = 3$, then equation of the directrix of the parabola is $x - y + 1 = 0$.
Statement-2 Perpendicular tangents to a parabola intersect at its directrix.
Ans. (a)

◎ **Solution:** For statement-2, let the equation of the parabola be $y^2 = 4ax$ and equations of two perpendicular tangents are

$$y = mx + \frac{a}{m} \text{ and } y = -\frac{1}{m}x + \frac{a}{-\frac{1}{m}}$$

Point of intersection of these tangents is given by

$mx + \frac{a}{m} = -\frac{1}{m} - am.$

$\Rightarrow x + a = 0$

which is the directrix of the parabola.

Showing that statement-2 is true.

For statement-1, $x + 2y = 5$ and $2x - y = 0$ are two perpendicular tangents which intersect at the point (1, 2).

So using statement-2, (1, 2) lies on the directrix, also directrix is perpendicular to the axis $x + y = 3$ and hence its equation is $y - 2 = (x - 1)$

$\Rightarrow x - y + 1 = 0$

Showing that the statement-1 is also true.

Example 53: Given: A circle, $2x^2 + 2y^2 = 5$ and a parabola $y^2 = 4\sqrt{5}x$

Statement-1 An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement-2 If the line $y = mx + \dfrac{\sqrt{5}}{m}$ $(m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$.
Ans. (a)

Solution: An equation of the tangent to $y^2 = 4\sqrt{5}x$ is

$y = mx + \dfrac{\sqrt{5}}{m}$ or $m^2x - my + \sqrt{5} = 0$

which will touch the circle $2x^2 + 2y^2 = 5$

$$\left|\frac{\sqrt{5}}{\sqrt{m^4 + m^2}}\right| = \frac{\sqrt{5}}{\sqrt{2}}$$

$$\Rightarrow \quad m^4 + m^2 - 2 = 0$$

$$\Rightarrow \quad m = \pm 1$$

$$m = \pm 1 \text{ satisfies } m^4 - 3m^2 + 2 = 0$$

So statement-2 is true and hence statement-1 is also true.

Example 54: Statement-1: The slope of the tangent at any point P on a parabola, whose axis is the axis of x and vertex is at the origin, is inversely proportional to the ordinate of that point.
Statement-2: The system of parabolas $y^2 = 4ax$ satisfies a differential equation of degree 1 and order 1.

Ans. (b)

Solution: $y^2 = 4ax \Rightarrow 2y\dfrac{dy}{dx} = 4a.$

$\Rightarrow \dfrac{dy}{dx} - \dfrac{2a}{y} = 0$ which is a differential equation of order 1 and degree 1 shows that the statement-2 is true.
For statement-1, slope of the tangent at any point is given by $m = \dfrac{dy}{dx} = \dfrac{2a}{y}$.

Showing that the slope m is inversely proportional to the ordinate y of the point and thus statement-1 is also true but does not follow from statement-2.

Example 55: Statement-1 Point of intersection of the tangents drawn to the parabola $x^2 = 4y$ at (4, 4) and (–4, 4) lies on the y-axis.

Statement-2 Tangents drawn at the extremities of the latus rectum of the parabola $x^2 = 4ay$ intersect on the axis of the parabola.
Ans. (b)

Solution: Coordinates of the extremities of the latus rectum of the parabola $x^2 = 4ay$ are $(\pm 2a, a)$ and the equation of the tangents at these extremities are

$x(2a) = 2a(y + a)$ and $x(-2a) = 2a(y + a)$

which intersect on $x = 0$, i.e. y-axis the axis of the parabola. Hence statement-2 is true.

In statement-1, equations of the tangents are $x(4) = 2(y + 4)$ and $x(-4) = 2(y + 4)$ which again intersect on $x = 0$ i.e. y-axis.

So statement-1 is also true but does not follow statement-2 as $(\pm 4, 4)$ are not the extremities of the latus rectum of the parabola $x^2 = 4y$.

LEVEL 2

Straight Objective Type Questions

Example 56: If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, $k \neq 0$ and the parabola intersects the circle $x^2 + y^2 = 4$ in two real distinct points, then the value of k is

(a) – 4 (b) – 8
(c) 4 (d) none of these

Ans. (b)

Solution: The equation of the parabola can be written as

$$y^2 = k(x - 8/k) \text{ which is of the form } Y^2 = 4AX$$

where $\quad Y = y$, $X = x - 8/k$ and $A = k/4$

Equation of the directrix is $X = -A \quad \Rightarrow \quad x - 8/k = -k/4$

which represents the given line $x - 1 = 0$ if $\dfrac{8}{k} - \dfrac{k}{4} = 1$

$\Rightarrow \quad k^2 + 4k - 32 = 0 \quad \Rightarrow \quad k = -8$ or 4

For $k = 4$, the parabola is $y^2 = 4(x - 2)$ whose vertex is (2, 0) and touches the circle $x^2 + y^2 = 4$ at the vertex. Therefore $k \neq 4$.

For $k = -8$, the parabola is $y^2 = -8(x + 1)$ which intersects the circle $x^2 + y^2 = 4$ at two real distinct points as the vertex $(-1, 0)$ of the parabola lies inside the circle.

◉ **Example 57:** M is the foot of the perpendicular from a point P on the parabola $y^2 = 8(x - 3)$ to its directrix and S is the focus of the parabola, if SPM is an equilateral triangle, the length of each side of the triangle is

(a) 2	(b) 3
(c) 4	(d) 8

Ans. (d)

Vertex is (3, 0) focus is (3 + 2, 0) directrix is $x = 3 - 2$.

◎ **Solution:** Since SPM is an equilateral triangle $\angle SMP = 60°$.

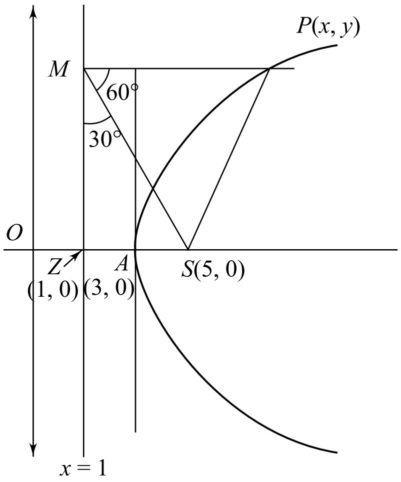


Fig. 18.12

$$\Rightarrow \quad \angle SMZ = 30°$$

From triangle SZM

$$\frac{SZ}{SM} = \sin 30° \quad \Rightarrow \quad SM = 2SZ$$

$$= 2 \times 4 = 8$$

Hence length of each side of the triangle is 8.

◉ **Example 58:** PQ is a double ordinate of a parabola $y^2 = 4ax$. The locus of its points of trisection is another parabola length of whose latus rectum is k times the length of the latus rectum of the given parabola, the value of k is

(a) 1/9	(b) 1/3
(c) 2/3	(d) none of these

Ans. (a)

◎ **Solution:** Let PQ be a double ordinate of the parabola $y^2 = 4ax$ meeting the axis at L, R and S be the points of trisection of PQ. So that $PR = RS = SQ$. (Fig. 18.13)

If the coordinates of R are (h, k), the coordinates of S are $(h, -k)$ the coordinates of P are $(h, 3k)$

Since P lies on the parabola $y^2 = 4ax$

$$(3k)^2 = 4ah \quad \Rightarrow \quad 9k^2 = 4ah$$

The locus of $R(h, k)$ is therefore

$$9y^2 = 4ax \text{ or } y^2 = (4/9)\, ax$$

which is a parabola whose length of the latus rectum = $(4a/9)$

= (1/9) (length of latus rectum of the given parabola)

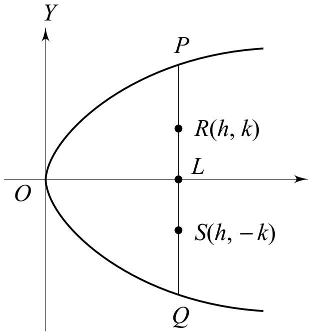


Fig. 18.13

◉ **Example 59:** An equilateral triangle is inscribed in the parabola $y^2 = 4x$ one of whose vertex is at the vertex of the parabola, the length of each side of the triangle in units is

(a) $\sqrt{3}/2$	(b) $4\sqrt{3}/2$
(c) $8\sqrt{3}/2$	(d) $8\sqrt{3}$

Ans. (d)

◎ **Solution:** Let A be the vertex and ABC be the equilateral triangle inscribed in the parabola $y^2 = 4x$, AM be perpendicular on BC.

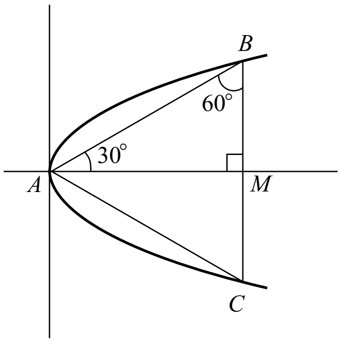


Fig. 18.14

Then if $AB = l$, $AM = l \cos 30°$

and $BM = l \sin 30°$.

Thus the coordinates of B are $(l\sqrt{3}/2, l/2)$

Since B lies on the parabola $y^2 = 4x$

$$\Rightarrow \quad (l/2)^2 = 4l\sqrt{3}/2$$

$$\Rightarrow \quad l = 8\sqrt{3}$$

Example 60: If PQ is a focal chord of the parabola $y^2 = 4ax$ with focus at S, then $\dfrac{2\,SP.SQ}{SP + SQ} =$

(a) a (b) $2a$

(c) $4a$ (d) a^2

Ans. (b)

Solution: Let the coordinates of P be $(at_1^2, 2at_1)$ and of Q be $(at_2^2, 2at_2)$. Since PQ is a focal chord, $t_1 t_2 = -1$

Focus is $S(a, 0) \Rightarrow SP = \sqrt{a^2(1-t_1^2)^2 + 4a^2t_1^2} = a(1 + t_1^2)$

and $\quad SQ = a(1 + 1/t_1^2) = \dfrac{a(1+t_1^2)}{t_1^2}$

So that $\quad \dfrac{2\,SP.SQ}{SP+SQ} = \dfrac{2a^2(1+t_1^2)^2}{t_1^2\, a\left[(1+t_1^2)+\left(1+\dfrac{1}{t_1^2}\right)\right]} = 2a$

Example 61: If the tangents at the extremities of a chord PQ of a parabola intersect at T, then the distances of the focus of the parabola from the points P, T, Q are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

Ans. (b)

Solution: Let the equation of the parabola be $y^2 = 4ax$ and $P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$ be the extremities of the chord PQ. The coordinates of T, the point of intersection of the tangents at P and Q are

$$(at_1 t_2, a(t_1 + t_2))$$

Now $\quad SP = a(1 + t_1^2)$

$\quad SQ = a(1 + t_2^2)$

and $\quad ST^2 = (at_1 t_2 - a)^2 + [a(t_1 + t_2) - 0]^2$

$= a^2(t_1^2 + t_2^2 + t_1^2 t_2^2 + 1)$

$= a^2(1 + t_1^2)(1 + t_2^2) = SP.SQ$

So that SP, ST, SQ are in G.P.

Example 62: If the normals to the parabola $y^2 = 4ax$ at the ends of the latus rectum meet the parabola again at the points P and Q then the equation of PQ is

(a) $x = 2a$ (b) $x = 3a$

(c) $x = 6a$ (d) $x = 9a$

Ans. (d)

Solution: The ends of the latus rectum are $L(a, 2a)$ and $L'(a, -2a)$. Equation of the normal to the parabola $y^2 = 4ax$ at $(am^2, -2am)$ is $y = mx - 2am - am^3$. Taking $m = -1$ and 1 respectively we get the equations of the normals at L and L' as

$$y = -x + 2a + a \text{ and } y = x - 2a - a$$

or $\quad x + y - 3a = 0$ and $x - y - 3a = 0$

Now $x + y - 3a = 0$ meets the parabola $y^2 = 4ax$ at points given by

$$(3a - x)^2 = 4ax \quad \Rightarrow \quad x^2 - 10ax + 9a^2 = 0$$

$\Rightarrow \quad x = a$ of $x = 9a$. But $x = a$ correspondence to the point L.

Therefore $x = 9a$ and $y = 3a - x = -6a$

and the coordinates of P are $(9a, -6a)$

Similarly the coordinates of Q are $(9a, 6a)$

Hence the equation of PQ is $x = 9a$

Example 63: The orthocentre of the triangle formed by three distinct tangents to the parabola $y^2 = 4ax$ lies on:

(a) $x + a = 0$ (b) $x - a = 0$

(c) $x = 0$ (d) $y = 0$

Ans. (a)

Solution: Equations of the tangents at the points $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$ and $C(at_3^2, 2at_3)$ are

$$t_1 y = x + at_1^2$$
$$t_2 y = x + at_2^2$$
$$t_3 y = x + at_3^2$$

These tangents meet at

$$P(a\,t_2t_3,\ a(t_3 + t_2))$$
$$Q(a\,t_3t_1,\ a(t_3 + t_1))$$
$$R(a\,t_1t_2,\ a(t_1 + t_2))$$

Equation of the altitude through P is $y - a(t_2 + t_3) = -t_1(x - at_2t_3)$

$\Rightarrow y - a(t_1 + t_2 + t_3 + t_1t_2t_3) = -t_1(x + a)$

which passes through the point $(-a, a(t_1 + t_2 + t_3 + t_1t_2t_3))$

From symmetry the other two altitudes also pass through this point and hence the orthocentre of the triangle lies on $x = -a$ or $x + a = 0$.

Example 64: Through the vertex of the parabola $y^2 = 4ax$, chords OA and OB are drawn at right angles to each other. For all positions of the point A, the chord AB meets the axis of the parabola at a fixed point. Coordinates of the fixed point are:

(a) $(a, 0)$ (b) $(-a, 0)$

(c) $(4a, 0)$ (d) $(-4a, 0)$

Ans. (c)

Solution: Let the coordinates of A be $(at_1^2, 2at_1)$ and of B be $(at_2^2, 2at_2)$.

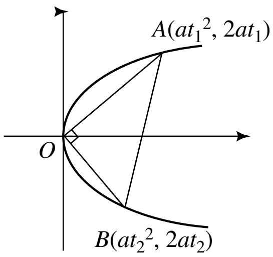


Fig. 18.15

OA is perpendicular to *OB*

$$\Rightarrow \left(\frac{2}{t_1}\right)\left(\frac{2}{t_2}\right) = -1 \Rightarrow t_1 t_2 = -4.$$

Equation of *AB* is

$$\frac{x - at_1^2}{at_2^2 - at_1^2} = \frac{y - 2at_1}{2at_2 - 2at_1}$$

$\Rightarrow 2(x - at_1^2) = (t_1 + t_2)(y - 2at_1)$

$\Rightarrow 2(x + a\,t_1 t_2) - y(t_1 + t_2) = 0$

$\Rightarrow 2(x - 4a) - y(t_1 + t_2) = 0$

which meets the axis $y = 0$ at the point $(4a, 0)$ for all values of t_1.

Example 65: *A* is a point on the parabola $y^2 = 4ax$.

The normal at *A* meets the parabola again at *B*. If *AB* subtends a right angle at the vertex of the parabola, then the slope of *AB* is

(a) ± 1 (b) $\pm\sqrt{2}$

(c) $\pm\dfrac{1}{\sqrt{2}}$ (d) $\pm 2\sqrt{2}$.

Ans. (b)

Solution: Equation of the normal at $A(at^2, 2at)$ to the parabola $y^2 = 4ax$ is

$$y = -tx + 2at + at^3$$

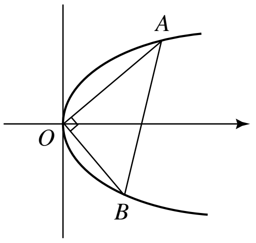


Fig. 18.16

Equation of the pair of lines *OA* and *OB* is

$$y^2 = (4ax)\frac{y + tx}{2at + at^3}$$

$$\Rightarrow \quad (t^3 + 2t)y^2 = 4tx^2 + 4xy$$

$$\Rightarrow \quad 4tx^2 + 4xy - (t^3 + 2t)y^2 = 0$$

These lines are perpendicular.

$$\Rightarrow \quad 4t - t^3 - 2t = 0$$

$$\Rightarrow \quad t^3 - 2t = 0 \Rightarrow t = 0 \text{ or } t = \pm\sqrt{2}$$

EXERCISE

Concept-based
Straight Objective Type Questions

1. A chord is drawn through the focus of the parabola $y^2 = 6x$ such that its distance from the vertex of this parabola is $\dfrac{\sqrt{5}}{2}$, then its slope can be:

   (a) $\dfrac{\sqrt{5}}{2}$ (b) $\dfrac{\sqrt{3}}{2}$

   (c) $\dfrac{2}{\sqrt{5}}$ (d) $\dfrac{2}{\sqrt{3}}$

2. An equation of the parabola whose focus is $(-3, 0)$ and the directrix is $x + 5 = 0$ is:

   (a) $y^2 = 4(x + 5)$ (b) $y^2 = 4(x + 4)$
   (c) $y^2 = 4(x + 3)$ (d) $y^2 = 4(x - 3)$

3. The centre *C* of a variable circle passing through a fixed point $(a, 0)$, $a > 0$, touches the line $y = x$. Locus of *C* is a parabola whose directrix is

   (a) $x + y = 0$ (b) $x + y = a$
   (c) $x - y = 0$ (d) $x - y = a$

4. The parabolas $y^2 = 4x$ and $x^2 = 32y$ intersect at a point *P* other than the origin. If the angle of intersection is θ, then $\tan\theta$ is equal to

   (a) $\dfrac{5}{3}$ (b) $\dfrac{4}{5}$

   (c) $\dfrac{3}{5}$ (d) 1

5. Length of the common chord of the parabola $y^2 = 8x$ and the circle $x^2 + y^2 - 2x - 4y = 0$ is:

   (a) $\dfrac{1}{2}\sqrt{5}$ (b) $\sqrt{5}$

   (c) $2\sqrt{5}$ (d) $3\sqrt{5}$

6. If *P* is the point $(1, 0)$ and *Q* lies on the parabola $y^2 = 36x$, then the locus of the mid point of *PQ* is:

   (a) $y^2 = 9(2x - 1)$ (b) $y^2 = 9(x + 2)$
   (c) $y^2 = 2(x - 9)$ (d) $x^2 = 9(y - 2)$

7. Let *Q* be the foot of the perpendicular from the origin *O* to the tangent at a point $P(\alpha, \beta)$ on the

parabola $y^2 = 4ax$, and S be the focus of the parabola, then $(OQ)^2$ (SP) is equal to

(a) α (b) $a\alpha^2$
(c) β (d) $a\beta^2$

8. An equation of the latus rectum of the parabola $x^2 + 4x + 2y = 0$ is

(a) $y = \frac{-3}{2}$ (b) $y = \frac{2}{3}$
(c) $y = \frac{3}{2}$ (d) $y = \frac{-2}{3}$

9. If a tangent to the parabola $y^2 = 4x$ makes an angle $\pi/4$ with the positive direction of the axis of x, then the coordinates of the point of contact with the parabola are:

(a) $(1, 2)$ (b) $(1, -2)$
(c) $(4, 4)$ (d) $(4, -4)$

10. Distance of a point P on the parabola $y^2 = 48x$ from the focus is l and its distance from the tangent at the vertex is d, then $l - d$ is equal to

(a) 4 (b) 8
(c) 12 (d) 16.

11. Locus of the mid points of the chords of the parabola $y^2 = 8x$ which touch the circle

$x^2 + y^2 = 4$ is

(a) $(y^2 - 4x)^2 = 16(x^2 + 4)$
(b) $(y^2 - 4x^2) = 4(x^2 + a^2)$
(c) $(y^2 - 4x)^2 = 4(16 + y^2)$
(d) $(y^2 - 4x)^2 = 16(4 + y^2)$

12. If θ is the angle between the tangents to the parabola $y^2 = 12x$ passing through the point $(-1, 2)$ then $|\tan\theta|$ is equal to

(a) 2 (b) 3
(c) $\frac{1}{2}$ (d) $\frac{2}{3}$

13. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point $(9, 6)$, then equation of L can not be

(a) $y - x + 3 = 0$ (b) $y + 3x - 33 = 0$
(c) $y + x - 15 = 0$ (d) $y - 2x + 12 = 0$

14. Normal at a point P on the parabola $y^2 = 4ax$ meets the axis at Q such that the distance of Q from the focus of the parabola is $10a$. The coordinates of P are:

(a) $(6a, 9a)$ (b) $(6a, -9a)$
(c) $(9a, 6a)$ (d) $(3a, 6a)$

15. $P_1 : y^2 = x$, $P_2 : y^2 = -x$, $P_3 : x^2 = y$, $P_4 : x^2 = -y$ are four parabola. Points of intersection of the parabola P_3 and P_4 with P_1 and P_2 (other than the origin) enclose a square of area (in sq. units).

(a) 2 (b) 4
(c) 8 (d) 16.

LEVEL 1

Straight Objective Type Questions

16. The directrix of the parabola $y^2 + 4x + 3 = 0$ is

(a) $x - 3/4 = 0$ (b) $x + 1/4 = 0$
(c) $x - 1/4 = 0$ (d) $x - 4/3 = 0$

17. The line $x + y = 6$ is a normal to the parabola. $y^2 = 8x$ at the point

(a) $(18, -12)$ (b) $(4, 2)$
(c) $(2, 4)$ (d) $(3, 3)$

18. The length of the chord of the parabola $y^2 = 4ax$ whose equation is

$y - x\sqrt{2} + 4a\sqrt{2} = 0$ is

(a) $2\sqrt{11}\,a$ (b) $4\sqrt{2}\,a$
(c) $8\sqrt{2}\,a$ (d) $6\sqrt{3}\,a$

19. If perpendiculars are drawn on any tangent to a parabola $y^2 = 4ax$ from the points $(a \pm k, 0)$ on the axis. The difference of the squares of their lengths is

(a) 4 (b) $4a$
(c) $4k$ (d) $4ak$

20. If the normal drawn from the point on the axis of the parabola $y^2 = 8ax$ whose distance from the focus is $8a$, and which is not parallel to either axes, makes an angle θ with the axis of x, then θ is equal to

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) none of these

21. Let AB be a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then the locus of the centroid of the triangle PAB as P moves on the circle is

(a) a parabola (b) a circle
(c) an ellipse (d) a pair of straight lines

22. For the parabola $y^2 + 8x - 12y + 20 = 0$, which of the following in not correct
(a) vertex (2, 6)
(b) focus (0, 6)
(c) length of the latus rectum = 4
(d) axis is $y = 6$

23. Tangents at the extremities of a focal chord of a parabola intersect
(a) on the axis of the parabola
(b) on the tangent at the vertex
(c) at the point of intersection of the directrix and the line parallel to the axis of the parabola through the mid-point of the chord.
(d) none of these

24. The coordinates of the end point of the latus rectum of the parabola $(y - 1)^2 = 2(x + 2)$, which does not lie on the line $2x + y + 3 = 0$ are
(a) (– 2, 1) (b) (– 3/2, 1)
(c) (– 3/2, 2) (d) (– 3/2, 0)

25. The point of contact of the tangent to the parabola $y^2 = 9x$ which passes through the point (4, 10) and makes an angle θ with the axis of the parabola such that $\tan \theta > 2$ is
(a) (4/9, 2) (b) (36, 18)
(c) (4, 6) (d) (1/4, 3/2)

26. Equation of the normal at a point on the parabola $y^2 = 36x$, whose ordinate is three times its abcissac is
(a) $2x + 3y + 44 = 0$ (b) $2x - 3y + 44 = 0$
(c) $2x + 3y - 44 = 0$ (d) $2x - 3y = 0$

27. Which of the following parametric equation does not represent a parabola
(a) $x = t^2 + 2t + 1, y = 2t + 2$
(b) $x = a(t^2 - 2t + 1), y = 2at - 2a$
(c) $x = 3 \sin^2 t, y = 6 \sin t$
(d) $x = a \sin t, y = 2a \cos t$

28. If the line $kx + y = 4$ touches the parabola $y = x - x^2$, then the point of contact is
(a) (– 2, 2) (b) (2, – 2)
(c) (– 2, 6) (d) (2, – 6)

29. If the normal at P to the parabola $y^2 = 4ax$ meets the parabola again at Q such that PQ subtends a right angle at the vertex of the parabola, then the coordinates of P are
(a) $(2a, 2\sqrt{2}a)$ (b) $(-2a, 2\sqrt{2}a)$
(c) $(2\sqrt{2}a, 2a)$ (d) $(2\sqrt{2}a, -2a)$

30. An equation of a common tangent to the parabola $y^2 = 4x$ and $x^2 = 4y$ is
(a) $x - y + 1 = 0$ (b) $x + y - 1 = 0$
(c) $x + y + 1 = 0$ (d) $y = 0$

31. $y = -2x + 12a$ is a normal to the parabola $y^2 = 4ax$ at the point whose distance from the directrix of the parabola is
(a) $4a$ (b) $5a$
(c) $4\sqrt{2}a$ (d) $8a$

32. PN is an ordinate of the parabola $y^2 = 9x$. A straight line is drawn through the mid-point M of PN parallel to the axis of the parabola meeting the parabola at Q. NQ meets the tangent at the vertex A, at a point T, then AT/NP =
(a) 3/2 (b) 4/3
(c) 2/3 (d) 3/4

33. If the area of the triangle inscribed in the parabola $y^2 = 4ax$ with one vertex at the vertex of the parabola and other two vertices at the extremities of a focal chord is $5a^2/2$, then the length of the focal chord is
(a) $3a$ (b) $5a$
(c) $25a/4$ (d) none of these

34. Length of the tangent drawn from an end of the latus rectum of the parabola $y^2 = 4ax$ to the circle of radius a touching externally the parabola at the vertex is
(a) $\sqrt{3}a$ (b) $2a$
(c) $\sqrt{7}a$ (d) $3a$

35. If the tangents at the extremities of a focal chord of the parabola $x^2 = 4ay$ meet the tangent at the vertex at points whose abcissac are x_1 and x_2 then $x_1 x_2$ =
(a) a^2 (b) $a^2 - 1$
(c) $a^2 + 1$ (d) $-a^2$

36. Equation of the tangent at a point P on the parabola $y^2 = 4ax$, the normal at which is at a distance $a\sqrt{5}/4$ from the focus of the parabola is
(a) $4x - 2y + a = 0$ (b) $4x - 8y + 9a = 0$
(c) $2x - y - 12a = 0$ (d) $2x + y - 12a = 0$

37. An isosceles triangle is inscribed in the parabola $y^2 = 4ax$ with its base as the line joining the vertex of the parabola and positive end of the latus rectum of the parabola. If $(at^2, 2at)$ is the vertex of the triangle then
(a) $2t^2 - 8t + 5 = 0$ (b) $2t^2 + 8t - 5 = 0$
(c) $2t^2 + 8t + 5 = 0$ (d) $2t^2 - 8t - 5 = 0$

38. Equation of a family of circles passing through the extremities of the latus rectum of the parabola $y^2 = 4ax$, g being a parametric, is

(a) $x^2 + y^2 + 2g\,(y - 2a) - 5a^2 = 0$

(b) $x^2 + y^2 + 2g\,(x + a) - 5a^2 = 0$

(c) $x^2 + y^2 + 2g(x - a) - 5a^2 = 0$

(d) $x^2 + y^2 + 2g\,(y + 2a) - 5a^2 = 0$

39. A triangle ABC is inscribed in the parabola $y^2 = 4x$ such that A lies at the vertex of the parabola and BC is a focal chord of the parabola with one extremity at (9, 6), the centroid of the triangle ABC lies at

(a) (41/27, 8/9) (b) (82/27, 16/9)
(c) (87/27, 20/9) (d) (80/27, 20/9)

40. P is a point on the parabola $y^2 = 4ax$ whose ordinate is equal to its abscissa and PQ is a focal chord, R and S are the feet of the perpendiculars from P and Q respectively on the tangent at the vertex, T is the foot of the perpendicular from Q to PR, area of the triangle PTQ is

(a) $75\,a^2/4$ (b) $85\,a^2/2$
(c) $75\,a^2/8$ (d) $45\,a^2/2$

41. P is a point on the locus of the mid-points of the chords of the parabola $y^2 = 4ax$ passing through the vertex of the parabola; S is the focus of the parabola and the line joining S and P is produced to meet the parabola at Q. If the ordinate of P is equal to its abscissa, coordinates of Q are

(a) $(4a, 4a)$ (b) $(3a, 3a)$
(c) $(2a, 2a)$ (d) none of these

42. The length of the perpendiculars from the focus and the extremities of a focal chord of a parabola on the tangent at the vertex form

(a) an A.P. (b) a G.P.
(c) an H.P. (d) none of these.

43. Locus of the point of intersection of the normals to the parabola $y^2 = 16x$ which are at right angles is

(a) $y^2 = 4(x - 4)$ (b) $y^2 = 4(x - 8)$
(c) $y^2 = -4(x - 12)$ (d) $y^2 = -8(x - 8)$

44. Equation of a common tangent to the parabola $y = x^2$ and $y = -(x - 2)^2$ is

(a) $y = 4(x - 1)$ (b) $y = 4(x + 1)$
(c) $y = 4(x - 1)$ (d) $y = -4(x + 1)$

45. Let (x_i, y_i), $i = 1, 2, 3, 4$, be the points of intersection of the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$, $c \neq 0$, then which of the following is true

(a) $y_1 - y_2 + y_3 - y_4 = 0$

(b) $x_1 + x_2 + x_3 + x_4 = 0$

(c) $y_1y_2y_3y_4 = 16a^2$

(d) $x_1x_2x_3x_4 = c^2$

Assertion-Reason Type Questions

46. **Statement-1:** The tangent to the parabola $y^2 = 4x$ at any point P and perpendicular on it from the focus S meet on the directrix of the parabola.

Statement-2: Tangents and normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square whose area is $8a^2$ sq. units.

47. **Statement-1:** If the vertex of a parabola lies at the point $(a, 0)$ and the directrix is y-axis then the focus of the parabola is at the point $(2a, 0)$.

Statement-2: Length of the common chord of the parabola $y^2 = 12x$ and the circle $x^2 + y^2 = 9$ is equal to the length of the latus rectum of the parabola.

48. **Statement-1:** The tangents at the extremities of a focal chord of a parabola intersect on its directrix.

Statement-2: The locus of the point of intersection of perpendicular tangents to the parabola is its directrix.

49. Let $P : y^2 = 2x$ be a parabola.

Statement-1 If three distinct normals of P pass through $(\alpha, 0)$, then $\alpha > 1$.

Statement-2 If $\alpha \leq 1$, exactly one normal of P passes through $(\alpha, 0)$.

50. **Statement-1** No normal of the parabola $y^2 = 4ax$ with integral slope, can touch the circle $x^2 + y^2 = a^2$.

Statement-2 The equation $x^6 + 4x^4 + 3x^2 - 1 = 0$ has no integral roots.

51. **Statement-1** If the focus of the parabola is (1, –1) and its directrix is $3x + 4y - 9 = 0$, then the length of its latus rectum is 4.

Statement-2 Length of the latus rectum of a parabola is twice the length of the perpendicular from the focus on the directrix.

52. **Statement-1** Number of common tangents to the parabola $y^2 = 2x$ and the circle $x^2 + y^2 - 4x + 3 = 0$ is 2.

    Statement-2 The circle $x^2 + y^2 - 4x + 3 = 0$ does not intersect the parabola $y^2 = 2x$.

53. **Statement-1** PQ is a focal chord of the parabola $y^2 = 8x$. If the coordinates are (32, 16) then the coordinates of Q are $\left(\frac{1}{8}, -2\right)$.

    Statement-2 If the coordinates of one end of a focal chord of a parabola are $(at^2, 2at)$ then the coordinates of the other end are $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$.

54. **Statement-1** If the tangents at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ intersect on the axis of the parabola, then $t_1 + t_2 = 0$.

    Statement-2 The tangents at the extremities of any focal chord of a parabola intersect on the axis of the parabola and include an angle of $\pi/4$.

55. **Statement-1** Length of a focal chord of the parabola $y^2 = 16x$ can not be a 9 units.

    Statement-2 $\left|t + \frac{1}{t}\right| \geq 2$ for all $t \neq 0$.

LEVEL 2

Straight Objective Type Questions

56. P is a point on the axis of the parabola $y^2 = 4ax$; Q and R are the extremities of its latus rectum, A is its vertex. If PQR is an equilateral triangle lying within the parabola and $\angle AQP = \theta$, then $\cos\theta =$

    (a) $\frac{2-\sqrt{3}}{2\sqrt{5}}$ (b) $\frac{9}{8\sqrt{5}}$

    (c) $\frac{\sqrt{5}-2}{2\sqrt{3}}$ (d) none of these

57. The points of intersection of the circle $x^2 + y^2 = a^2$ with the parabolas $y^2 = 4ax$ and $y^2 = -4ax$ form a rectangle whose area is

    (a) $8(\sqrt{5} - 2)a^2$ (b) $8(\sqrt{5} - 2)^{3/2}a^2$

    (c) $8(\sqrt{5} + 2)^{3/2}a^2$ (d) none of these

58. The circle $x^2 + y^2 = 9$ and the parabola $y^2 = 8x$ intersect at P and Q. Tangents to the circle at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S. The ratio of the areas of the triangles PQS and PQR is

    (a) $1 : \sqrt{2}$ (b) $1 : 2$

    (c) $1 : 4$ (d) $1 : 8$

59. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then the area of the quadrilateral $PQRS$ in sq. units is

    (a) 3 (b) 6

    (c) 9 (d) 20

60. Length of the chord of contact drawn from the point (–3, 2) to the parabola $y^2 = 4x$ is

    (a) 4 (b) $2\sqrt{2}$

    (c) $8\sqrt{2}$ (d) $4\sqrt{2}$

61. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose vertex is the point:

    (a) $\left(\frac{a}{3}, 0\right)$ (b) $\left(\frac{2a}{3}, 0\right)$

    (c) $(a, 0)$ (d) $\left(\frac{4a}{3}, 0\right)$

62. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following are the coordinates of P?

    (a) $(4, 2\sqrt{2})$ (b) $(9, 3\sqrt{2})$

    (c) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (d) $(2, \sqrt{2})$

63. The tangents to the parabola $y^2 = 4ax$ at the points A, B, C, taken in pair intersect at the points P, Q, R. The ratio of the areas of the ΔABC and ΔPQR is

    (a) 3 : 2 (b) 2 : 1

    (c) 3 : 1 (d) 2 : 3

64. $y = (x - 11)\cos\theta - \cos(3\theta)$ is a normal to the parabola $y^2 = 16x$ for

(a) $\theta = \pi/3$ (b) $\pi/6$

(c) $\theta = \dfrac{2\pi}{3}$ (d) all values of θ.

65. If P_1Q_1 and P_2Q_2 are two focal chords of the parabola $y^2 = 4ax$, then the lines P_1P_2 and Q_1Q_2 intersect on:

(a) the axis of the parabola.
(b) the directrix of the parabola
(c) the tangent at the vertex of the parabola
(d) latus rectum of the parabola.

Previous Years' AIEEE/JEE Main Questions

1. The normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$ then

(a) $t_2 = -t_1 + 2/t_1$ (b) $t_2 = t_1 - 2/t_1$
(c) $t_2 = t_1 + 2/t_1$ (d) $t_2 = -t_1 - 2/t_1$ **[2003]**

2. A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa is

(a) $(-9/8, 9/2)$ (b) $(2, -4)$
(c) $(2, 4)$ (d) $(9/8, 9/2)$ **[2004]**

3. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabola $y^2 = 4ax$ and $x^2 = 4ay$, then

(a) $d^2 + (2b - 3c)^2 = 0$
(b) $d^2 + (3b + 2c)^2 = 0$
(c) $d^2 + (2b + 3c)^2 = 0$
(d) $d^2 + (3b - 2c)^2 = 0$ **[2004]**

4. If P be the point (1, 0) and Q a point on the locus $y^2 = 8x$. The locus of the mid point of PQ is

(a) $x^2 + 4y + 2 = 0$ (b) $x^2 - 4y + 2 = 0$
(c) $y^2 - 4x + 2 = 0$ (d) $y^2 + 4x + 2 = 0$
[2005]

5. The locus of the vertices of the family of parabola $y = \dfrac{a^3x^2}{3} + \dfrac{a^2x}{2} - 2a$ is

(a) $xy = 64/105$ (b) $xy = 105/64$
(c) $xy = 3/4$ (d) $xy = 35/16$ **[2006]**

6. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent is

(a) $(-1, 1)$ (b) $(0, 2)$
(c) $(2, 4)$ (d) $(-2, 0)$ **[2007]**

7. A parabola has the origin as its focus and the line $x = 2$ as the directrix. Then the vertex of the parabola is at

(a) (2, 0) (b) (0, 2)
(c) (1, 0) (d) (0, 1) **[2008]**

8. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is

(a) $3\sqrt{2}/5$ (b) $\sqrt{3}/4$

(c) $3\sqrt{2}/8$ (d) $2\sqrt{3}/8$ **[2009], [2011]**

9. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is

(a) $x = -1$ (b) $2x - 1 = 0$
(c) $x = 1$ (d) $2x + 1 = 0$ **[2010]**

10. **Statement-1:** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}\,x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement-2: If the line $y = mx + 4\sqrt{3}/m, (m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}\,x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$. **[2012]**

11. Given: A circle, $2x^2 + 2y^2 = 5$ and a parabola $y^2 = 4\sqrt{5}x$.

Statement-1: An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement-2: If the line $y = mx + \sqrt{5}/m\ (m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$ **[2013]**

12. **Statement-1:** The slope of the tangent at any point P on a parabola, whose axis is the axis of x and vertex is at the origin, is inversely proportional to the ordinate of the point P.

Statement-2: The system of parabolas $y^2 = 4ax$ satisfies a differential equation of degree 1 and order 1. **[2013, online]**

13. **Statement-1:** The line $x - 2y = 2$ meets the parabola $y^2 + 2x = 0$ only at the point $(-2, -2)$.

Statement-2: The line $y = mx - 1/2\text{m}$ $(m \neq 0)$ is a tangent to the parabola, $y^2 = -2x$ at the point $\left(-\frac{1}{2m^2}, -\frac{1}{m}\right)$ **[2013, online]**

14. The point of intersection of the normals to the parabola $y^2 = 4x$ at the ends of its latus rectum is:

(a) (0, 2) (b) (3, 0)
(c) (0, 3) (d) (2, 0) **[2013, online]**

15. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is:

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$
(c) $\frac{1}{8}$ (d) $\frac{2}{3}$ **[2014]**

16. Let L_1 be the length of the common chord of the curves $x^2 + y^2 = 9$ and $y^2 = 8x$, and L_2 be the length of the latus rectum of $y^2 = 8x$, then

(a) $L_1 > L_2$ (b) $L_1 = L_2$
(c) $L_1 < L_2$ (d) $\frac{L_1}{L_2} = \sqrt{2}$

[2014, online]

17. Two tangents are drawn from a point (–2, –1) to the curve $y^2 = 4x$. If α is the angle between them, then $|.\tan \alpha.|$ is equal to:

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\sqrt{3}$ (d) 3 **[2014, online]**

18. A chord is drawn through the focus of the parabola $y^2 = 6x$ such that its distance from the vertex of this parabola is $\frac{\sqrt{5}}{2}$, then its slope can be:

(a) $\frac{\sqrt{5}}{2}$ (b) $\frac{\sqrt{3}}{2}$
(c) $\frac{2}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{3}}$ **[2014, online]**

19. Let O be the vertex and Q be any point on the parabola $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is:

(a) $x^2 = y$ (b) $y^2 = x$
(d) $y^2 = 2x$ (d) $x^2 = 2y$ **[2015]**

20. Let PQ be a double ordinate of the parabola $y^2 = -4x$, where P lies in the second quadrant. If R divides PQ in the ratio 2 : 1, then the locus of R is:

(a) $9y^2 = 4x$ (b) $9y^2 = -4x$
(d) $3y^2 = 2x$ (d) $3y^2 = -2x$

[2015, online]

21. If the tangent to the conic $y - 6 = x^2$ at (2, 10) touches the circle $x^2 + y^2 + 8x - 2y = k$ (for same fixed k) at a point (α, β); then (α, β) is:

(a) $\left(-\frac{6}{17}, \frac{10}{17}\right)$ (b) $\left(-\frac{8}{17}, \frac{2}{17}\right)$
(d) $\left(-\frac{4}{17}, \frac{1}{17}\right)$ (d) $\left(-\frac{7}{17}, \frac{6}{17}\right)$

[2015, online]

22. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre as P is

(a) $x^2 + y^2 - 4x + 8y + 12 = 0$
(b) $x^2 + y^2 - x + 4y - 12 = 0$
(d) $x^2 + y^2 - x/4 + 2y - 24 = 0$
(d) $x^2 + y^2 - 4x + 9y + 18 = 0$ **[2016]**

23. The minimum distance of a point on the curve $y = x^2 - 4$ from the origin is:

(a) $\frac{\sqrt{15}}{2}$ (b) $\sqrt{\frac{19}{2}}$
(d) $\sqrt{\frac{15}{2}}$ (d) $\sqrt{\frac{19}{2}}$ **[2016, online]**

24. P and Q two distinct points on the parabola, $y^2 = 4x$, with parameters t and t_1 respectively. If the normal at P passes through Q, then the minimum value of t_1^2 is:

(a) 8 (b) 4
(d) 6 (d) 2 **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. An equilateral triangle is inscribed in the parabola $y^2 = 8x$ with one of its vertices at the vertex of the parabola. Then the length of its side is

(a) $8\sqrt{3}$ (b) $16\sqrt{3}$
(c) 16 (d) 8 **[2006]**

2. **Statement-1:** The point $\left(\frac{1}{4}, \frac{1}{2}\right)$ on the parabola $y^2 = x$ is closest to the line $y = x + 1$.

   Statement-2: The tangent at $\left(\frac{1}{4}, \frac{1}{2}\right)$ to the parabola $y^2 = x$ is parallel to the line $y = x + 1$

   (a) Statement-1 is true, statement-2 is true, statement-2 is **not** a correct explanation for statement-1.
   (b) Statement-1 is true, statement-2 is false.
   (c) Statement-1 is false, statement-2 is true.
   (d) Statement-1 is true, statement-2 is true, statement-2 is a correct explanation for statement-1. **[2009]**

3. The acute angle between the tangents drawn from the point (1, 4) to the parabola $y^2 = 4x$ is:

   (a) $\pi/6$ (b) $\pi/2$
   (c) $\pi/3$ (d) $\pi/4$ **[2011]**

4. **Statement-1:** Point of intersection of the tangents drawn to the parabola $x^2 = 4y$ at (4, 4) and (–4, 4) lies on the y-axis.

   Statement-2: Tangents drawn at the extremities of the latus rectum of the parabola $x^2 = 4y$ intersect on the axis of the parabola.

   (a) Statement-1 is true, statement-2 is true, statement-2 is a correct explanation for statement-1.
   (b) Statement-1 is true, statement-2 is true, statement-2 is not a correct explanation for statement-1.
   (c) Statement-1 is true, statement-2 is false.
   (d) Statement-1 is false, statement-2 is true. **[2012]**

5. Let $y^2 = 16x$ be a given parabola and L be an extremity of its latus rectum in the first quadrant. If a chord is drawn through L with slope –1, then the length of this chord is

   (a) 32 (b) $16\sqrt{2}$
   (c) $16\sqrt{3}$ (d) $32\sqrt{2}$ **[2013]**

6. The locus of the mid points of the chords of the parabola $x^2 = 4py$ having slope m is a:

   (a) line parallel to x-axis at a distance $|2pm|$ from it.
   (b) line parallel to y-axis at a distance $|2pm|$ from it.
   (c) line parallel to $y = mx$, $m \neq 0$ at a distance $|2pm|$ from it.
   (d) circle with centre at the origin and radius $|2pm|$. **[2014]**

7. Let PQ be a focal chord of the parabola $y^2 = 4x$. If the centre of a circle having PQ as its diameter lies on the line $\sqrt{5}\ y + 4 = 0$, then, the length of the chord PQ is:

   (a) $\frac{26}{5}$ (b) $\frac{36\sqrt{5}}{5}$
   (c) $\frac{26\sqrt{5}}{5}$ (d) $\frac{36}{5}$ **[2016]**

Answers

Concept-based

1. (a) **2.** (b) **3.** (c) **4.** (c)
5. (c) **6.** (a) **7.** (b) **8.** (c)
9. (a) **10.** (c) **11.** (c) **12.** (a)
13. (c) **14.** (c) **15.** (b)

Level 1

16. (c) **17.** (c) **18.** (d) **19.** (d)
20. (c) **21.** (b) **22.** (c) **23.** (c)
24. (c) **25.** (a) **26.** (c) **27.** (d)
28. (b) **29.** (a) **30.** (c) **31.** (b)
32. (c) **33.** (c) **34.** (c) **35.** (d)
36. (a) **37.** (b) **38.** (c) **39.** (b)
40. (c) **41.** (d) **42.** (b) **43.** (c)
44. (a) **45.** (d) **46.** (d) **47.** (c)
48. (a) **49.** (b) **50.** (a) **51.** (a)
52. (d) **53.** (a) **54.** (c) **55.** (a)

Level 2

56. (a) **57.** (b) **58.** (c) **59.** (d)
60. (c) **61.** (b) **62.** (a) **63.** (b)
64. (d) **65.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (d) **2.** (d) **3.** (c) **4.** (c)
5. (b) **6.** (d) **7.** (c) **8.** (c)
9. (a) **10.** (a) **11.** (a) **12.** (b)
13. (a) **14.** (b) **15.** (a) **16.** (c)
17. (d) **18.** (a) **19.** (d) **20.** (b)
21. (b) **22.** (a) **23.** (c) **24.** (d)

Previous Years' B-Architecture Entrance Examination Questions

1. (b) **2.** (a) **3.** (b) **4.** (b)
5. (d) **6.** (b) **7.** (d)

Hints and Solutions

Concept-based

1. Let the equation of the chord be.

   $y - 0 = m\left(x - \frac{3}{2}\right)$

   Its distance from the vertex (0, 0) is $\left|\frac{m\left(-\frac{3}{2}\right)}{\sqrt{1+m^2}}\right| = \frac{\sqrt{5}}{2} \Rightarrow m = \pm\frac{\sqrt{5}}{2}$.

2. We have $\sqrt{(x+3)^2 + y^2} = |x + 5|$

   $\Rightarrow y^2 = (x + 5)^2 - (x + 3)^2 = 4(x + 4)$

3. C is equidistant from the point $(a, 0)$ and the line $y = x$, so locus of C is a parabola with directrix $y = x$ i.e. $x - y = 0$.

4. Coordinates of P are (16, 8)

   $m_1 = \frac{dy}{dx}$ for $y^2 = 4x$ at P

   $\Rightarrow m_1 = \left.\frac{2}{y}\right|_{(16,8)} = \frac{1}{4}$

   $m_2 = \frac{dy}{dx}$ for $x^2 = 32y$ at P

   $m_2 = \left.\frac{x}{16}\right|_{16,8} = 1$

   $\tan\theta = \frac{1-\frac{1}{4}}{1+\frac{1}{4}} = \frac{3}{5}$.

5. Any point on the parabola is $(2t^2, 4t)$ which lies on the circle if $t^4 + 3t^2 - 4t = 0 \Rightarrow t = 0, 1$

   $\Rightarrow$ end points of the common chord are (0, 0) and (2, 4) and the required length is $2\sqrt{5}$.

6. Let the coordinates of Q be $(9t^2, 18t)$ coordinates of R the mid point of PQ are $\left(\frac{9t^2+1}{2}, 9t\right) = (x, y)$.

   Locus of R is

   $2x = 9\left(\frac{y}{9}\right)^2 + 1$

   $\Rightarrow y^2 = 9(2x - 1)$

7. Let $\alpha = at^2$, $\beta = 2at$.

   Equation of the tangent at $P(\alpha, \beta)$ is $ty = x + at^2$

   $\Rightarrow OQ = \frac{at^2}{\sqrt{1+t^2}}$,

   $SP = a(t^2 + 1)$

   So $(OQ)^2\,(SP) = \frac{a^2t^4}{1+t^2} \times a(1 + t^2) = a^3t^4 = a\alpha^2$

8. We can write the equation of the parabola as $(x + 2)^2 = -2(y - 2)$

   Shifting the origin to $(-2, 2)$, the equation is $X^2 = -2Y$ where $X = x + 2$, $Y = y - 2$

   Equation of the latus rectum is $Y = -\left(\frac{2}{4}\right) = \frac{-1}{2}$

   $\Rightarrow y = \frac{3}{2}$

9. Slope of the tangent to the parabola at $(t^2, 2t)$ is $\frac{1}{t} = \tan\frac{\pi}{4} = 1 \Rightarrow t = 1$ and the required point of contact is (1, 2).

10. Any point on the parabola is $P(12t^2, 24t)$ whose distance from the focus (12, 0) is $12(t^2 + 1) = l$ and from y-axis, the tangent at the vertex is $12t^2 = d$.

    So $l - d = 12$.

11. Let (h, k) be the mid-point of the chord of the parabola $y^2 = 8x$, then its equation is

    $ky - 4(x + h) = k^2 - 8h$

    $\Rightarrow 4x - ky + k^2 - 4h = 0$

    which touches the circle $x^2 + y^2 = 4$

    $\Rightarrow \frac{k^2 - 4h}{\sqrt{16+k^2}} = 2$

    $\Rightarrow (k^2 - 4h)^2 = 4(k^2 + 16)$

    Locus of (h, k) is $(y^2 - 4x)^2 = 4(16 + y^2)$

12. Equation of any tangent is $y = mx + \frac{3}{m}$ which passes through $(-1, 2)$ if $2 = -m + \frac{3}{m}$

    $\Rightarrow m^2 + 2m - 3 = 0$

    $\Rightarrow m = 1, -3$

So $|\tan\theta| = \left|\dfrac{1+3}{1-3}\right| = 2.$

13. Equation of a normal to the parabola $y^2 = 4x$ is $y = mx - 2m - m^3$ which passes through (9, 6) if $m^3 - 7m + 6 = 0$

    $\Rightarrow m = 1, 2, -3.$

    So equation of L can not be $y + x - 15 = 0$

14. Equation of the normal at $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ is $y = -tx + 2at + at^3$ which meets the axis $y = 0$ at $x = a(2 + t^2)$. So coordinates of Q are $(a(2 + t^2), 0)$. Focus is $S(a, 0)$.

    $QS = 2a + at^2 - a = a(1 + t^2) = 10a \Rightarrow t^2 = 9$

    So the required coordinates are $(9a, \pm 6a)$

15. Points of intersection are $A(1, 1)$, $B(-1, 1)$, $C(-1, -1)$ and $D(1, -1)$

    Area of the square $ABCD = 2^2 = 4$

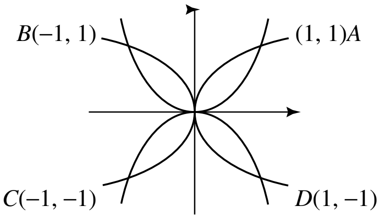


Fig. 18.17

Level 1

16. $y^2 = -4(x + 3/4)$, the directrix is $x + 3/4 - 1 = 0$
17. $y = -x + 6$ is a normal at the point $(am^2, -2am)$ where $a = 2$, $m = -1$ i.e., (2, 4)
18. Any point on the parabola is $(at^2, 2at)$;

    $2at - at^2\sqrt{2} + 4a\sqrt{2} = 0$

    $\Rightarrow \quad \sqrt{2}\,t^2 - 2t - 4\sqrt{2} = 0$

    $\Rightarrow \quad t_1 + t_2 = \sqrt{2},\ t_1t_2 = -4.$

    $\Rightarrow t_1 - t_2 = \sqrt{18}$

    Length of the chord is

    $$\sqrt{\left(at_1^2 - at_2^2\right)^2 + \left(2at_1 - 2at_2\right)^2} = a\sqrt{18}\,\sqrt{2+4} = 6\sqrt{3}a$$

19. Any tangent is $y = mx + a/m$. Required difference is

    $$\left[\frac{m(a+k) + a/m}{\sqrt{1+m^2}}\right]^2 - \left[\frac{m(a-k) + a/m}{\sqrt{1+m^2}}\right]^2$$

    $$= \frac{1}{1+m^2} \times 4(ma + a/m)mk = 4ak.$$

20. Focus is $(2a, 0)$, so point is $P(2a + 8a, 0)$. Equation of any normal is $y = mx - 2(2a)m - (2a)m^3$. If it passes through $P(10a, 0)$, then

    $6\,am - 2\,am^3 = 0 \quad \Rightarrow \quad m = 0, \pm\sqrt{3}.$

    So the required angle is $\theta = \tan^{-1}\sqrt{3} = \pi/3.$

21. Let (h, k) be the centroid, then

    $$h = \frac{r + r\cos\theta}{3},\ k = \frac{r + r\sin\theta}{3}.$$

    $\Rightarrow (3h - r)^2 + (3k - r)^2 = r^2.$

    $\Rightarrow 9(h^2 + k^2) - 6r(h + k) + r^2 = 0$

    Locus of (h, k) is a circle.

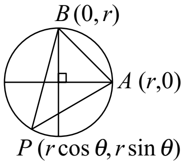


Fig. 18.18

22. $(y - 6)^2 = -8(x - 2)$ or $Y^2 = 4AX$ where $Y = y - 6$, $X = x - 2$, $A = -2$, vertex is (2, 6), focus is $(X = A, Y = 0)$ i.e., (0, 6) axis is $Y = 0$ i.e. $y = 6$ and length of the latus rectum $= |4A| = 8$.
23. Extremities of a focal chord are $(at^2, 2at)$, $(a/t^2, -2a/t)$ Tangents are $ty = x + at^2$, and $-ty = xt^2 + a$.

    They intersect at $\left(-a, \dfrac{a(t^2-1)}{t}\right)$ or $\left(-a, a\left(t - \dfrac{1}{t}\right)\right)$

    which is the point of intersection of the directrix $x = -a$ and the line $y = a\left(t - \dfrac{1}{t}\right)$ which is the line through the mid-point of the chord parallel to the axis.

24. $Y^2 = 4AX$ where $Y = y - 1$, $X = x + 2$, $A = 1/2$.

    End of the L.R. is $(X = A, Y = \pm 2A) = (-3/2, 2)$ or $(-3/2, 0)$ but $(-3/2, 0)$ lies on the line.

25. Equation of the tangent at any point $(9t^2/4, 9t/2)$ on the parabola is $ty = x + 9t^2/4$. Since it passes through (4, 10), $10t = 4 + 9t^2/4$ or $9t^2 - 40t + 16 = 0$

    $\Rightarrow t = 4/9, 4.$ $\tan\theta = 1/t > 2 \quad \Rightarrow \quad t < 1/2$. So $t = 4/9$ and the required point is (4/9, 2).

26. Equation of the normal at $(9m^2, -18m)$ on the parabola is

    $y = mx - 2 \times 9m - 9m^3$ where $-18m = 3 \times 9m^2$.

    i.e., $m = 0$ or $m = -2/3$. For $m = -2/3$, the equation of the normal is $2x + 3y - 44 = 0$.

27. $x = aT^2$, $y = 2aT$ represents a parabola.

    in(a) $a = 1$, $T = t + 1$, in (b) $a = a$, $T = (t - 1)$

    in(c) $a = 3$, $T = \sin t$. But in (d) if $2aT = 2a\cos t$

    $\Rightarrow \quad T = \cos$ t which does not satisfy $x = aT^2$.

28. $4 - kx = x - x^2$ or $x^2 - (k + 1)x + 4 = 0$ has coincident roots if $(k + 1)^2 - 16 = 0 \Rightarrow k = 3$ or -5. So $x = \pm 2$ and the points of contact are $(2, -2)$ or $(-2, -6)$.

29. Let the coordinates of P be $(am^2, -2am)$ and Q be $(am'^2, -2am')$ then $\dfrac{-2am}{am^2} \times \dfrac{-2am'}{am'^2} = -1$

$\Rightarrow mm' = -4$

Normal at P, $y = mx - 2am - am^3$. Since it passes through Q

$$-2a\left(-\frac{4}{m}\right) = am\left(\frac{16}{m^2}\right) - 2am - am^3 = 0$$

$\Rightarrow m^4 + 2m^2 - 8 = 0 \Rightarrow m = \pm\sqrt{2}$ and the coordinates of P are $(2a, \pm 2\sqrt{2}\,a)$

30. Equation of a tangent to $y^2 = 4x$ is $y = mx + 1/m$. If it touches the parabola $x^2 = 4y$ then the roots of the equation $x^2 = 4(mx + 1/m)$ are coincident. So $(4m)^2 + \dfrac{16}{m} = 0 \Rightarrow m = -1$ and the equation of the tangent is $x + y + 1 = 0$.

31. $y = -2x + 12a$ is a normal at the point $(a(-2)^2, -2a(-2))$ i.e., $(4a, 4a)$ whose distance from $x = -a$ is $5a$.

32. Let P be $(9t^2/4, 9t/2)$, then $NP = 9t/2$. $MN = 9t/4$, equation of MQ is $y = 9t/4$ which meets the parabola $y^2 = 9x$ at the point $Q(9t^2/16, 9t/4)$, N is $(9t^2/4, 0)$ equation of NQ is

$$y = -\frac{9\times 16}{4\times 27t}\left(x - \frac{9}{4}t^2\right)$$ which meets the tangent $x = 0$ (at the vertex A) at the point for which $y = 3t$ so that $AT = 3t$ and $\dfrac{AT}{NP} = \dfrac{3t\times 2}{9t} = \dfrac{2}{3}$.

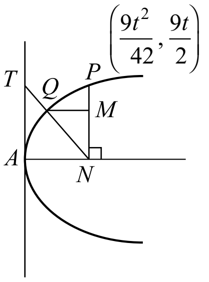


Fig. 18.19

33. Let the vertices be $O(0, 0)$, $A(at^2, 2at)$, $B\left(\dfrac{a}{t^2}, \dfrac{-2a}{t}\right)$ then

$$\frac{1}{2}\left|\begin{vmatrix} 0 & 0 & 1 \\ at^2 & 2at & 1 \\ \dfrac{a}{t^2} & \dfrac{2a}{t} & 1 \end{vmatrix}\right| = \frac{5a^2}{2} \Rightarrow 2t^2 - 5t + 2 = 0$$

$\Rightarrow t = 2$ or $1/2$ so the vertices of a focal chord are $(4a, 4a)$ and $(a/4, -a)$ (Taking $t = 2$) and length of this focal chord is $25\,a/4$.

34. Equation of the circle is $(x + a)^2 + y^2 = a^2$ and an end of the Latus rectum of the parabola is $(a, 2a)$. Required length is $\sqrt{(a+a)^2 + (2a)^2 - a^2} = a\sqrt{7}$.

35. One extremity of the focal chord be $(2at, at^2)$. Equation of the tangent is $tx = y + at^2$ which meets the tangent at the vertex, $y = 0$ at $x = at$ so $x_1 = at$ and $x_2 = a\left(-\dfrac{1}{t}\right)$ thus $x_1x_2 = -a^2$.

36. Equation of the normal with slope m is $y = mx - 2am - am^3$ whose distance from the focus $(a, 0)$ is $\dfrac{am - 2am - am^3}{\sqrt{1+m^2}} = \dfrac{\sqrt{5}}{4}a$

$\Rightarrow 16m^2(1 + m^2) = 5$

$\Rightarrow m^2 = 1/4 \Rightarrow m = \pm 1/2$

Slope of the tangent is ± 2 and the equation of the tangent is

$y = \pm 2x \pm a/2 \Rightarrow 4x \pm 2y + a = 0$.

37. $(at^2)^2 + (2at)^2 = (at^2 - a)^2 + (2at - 2a)^2$, as vertex of the parabola is $(0, 0)$ and positive end of the latus-rectum is $(a, 2a)$.

38. Taking $x = a$ in (c) we get $y = \pm 2a$ for all values of g.

39. If one extremity of the focal chord is $(9, 6)$, the other is $(1/9, -2/3)$ so the centroid of the triangle is $\left(\dfrac{0+9+1/9}{3}, \dfrac{0+6-2/3}{3}\right) = \left(\dfrac{82}{27}, \dfrac{16}{9}\right)$

40. $P(4a, 4a)$, $Q(a/4, -a)$, $R(0, 4a)$, $S(0, -a)$, $T(a/4, 4a)$. So area of the ΔPTQ is

$$\frac{1}{2} \times PT \times QT = \frac{1}{2}\left(4a - \frac{a}{4}\right) \times 5a = \frac{75}{8}a^2$$

41. $O(0, 0)$ is the vertex and $R(at^2, 2at)$ is any point on the parabola.$(at^2/2, 2at/2)$ is the mid-point of the chord OR. Whose locus is $y^2 = 2ax$, coordinates of P on this locus are $(2a, 2a)$, focus is $S(a, 0)$. Equation of SP is $y = 2(x - a)$ which meets $y^2 = 4ax$ at the point Q for which $4(x - a)^2 = 4ax$ $\Rightarrow x^2 - 3ax + a^2 = 0$ which is not satisfied by the points in (a), (b) or (c).

42. $y^2 = 4ax$ be the parabola, $P(at^2, 2at)$, $Q\left(\dfrac{a}{t^2}, \dfrac{-2a}{t}\right)$ and the focus is $S(a, 0)$. The tangent at the vertex is $x = 0$. Lengths of the perpendicular from P,

Q and S on this tangent are respectively $l_1 = at^2$, $l_2 = a/t^2$, $l_3 = a$ so that $l_1 l_2 = a^2 = l_3^2 \Rightarrow l_1, l_2, l_3$ form a G.P.

43. Let two perpendicular normals to the parabola $y^2 = 16x$ be $y = mx - 8m - 4m^3$ and $y = -\frac{1}{m}x + \frac{8}{m} + \frac{4}{m^3}$.

Solving we get $x = 4\left(m - \frac{1}{m}\right)^2 + 12$

$$y = -4\left(m - \frac{1}{m}\right)$$

Eliminating m we get the required locus as $y^2 = 4(x - 12)$.

44. Coordinates of any point on $y = x^2$ is (t, t^2)

$$\left.\frac{dy}{dx}\right|_{(t,t^2)} = 2t$$

Equation of the tangent to the parabola is $y - t^2 = 2t(x - t)$

$\Rightarrow y = 2tx - t^2$

which will touch the parabola $y = -(x - 2)^2$ if the roots of the equation $-(x - 2)^2 = 2tx - t^2$ are equal.

$\Rightarrow (t - 2)^2 - (4 - t^2) = 0 \Rightarrow t = 0$ or 2

For $t = 2$, the required tangent is $y = 4(x - 1)$

45. For the points of intersections of the given parabola and circle, we have $\frac{y^4}{16a^2} + y^2 + 2g \times \frac{y^2}{4a} + 2fy + c = 0$

$\Rightarrow y^4 + 4a(4a + 2g)y^2 + 32a^2fy + 16a^2c = 0$

$\Rightarrow y_1 + y_2 + y_3 + y_4 = 0,\ y_1y_2y_3y_4 = 16a^2c$

Also $x_1x_2x_3x_4 = \frac{(y_1y_2y_3y_4)^2}{(4a)^4} = c^2$

$$x_1 + x_2 + x_3 + x_4 = \frac{1}{4a}(y_1^2 + y_2^2 + y_3^2 + y_4^2) \neq 0.$$

46. **Statement-1** is false. Equation of any tangent to the parabola is $y = mx + \frac{1}{m}$ and equation of the perpendicular from the focus $S(1, 0)$ on it is

$y = -\frac{1}{m}x + \frac{1}{m}$ and these intersect at $x = 0$, directrix is $x = -1$.

Statement-2 is true. Tangents and normals at $(a, \pm 2a)$ are respectively $x \pm y + a = 0$ and $x \pm y - 3a = 0$ which enclose a square, length of a side $= 2\sqrt{2}a$.

47. In statement-1, focus is on the x-axis at a distance a from the vertex so statement-1 is true.

Statement-2 is false as the length of the latus-rectum of the parabola is 12 which is greater than the diameter of the circle and the common chord is of length less than the diameter.

48. **Statement-2** is true, equations of the perpendicular tangents to the parabola $y^2 = 4ax$ are $y = mx + a/m$, $y = m'x + a/m'$ where $mm' = -1$ they intersect at the point for which

$$(m - m')\,x = \frac{a}{m'} - \frac{a}{m} = \frac{a(m - m')}{mm'} \quad \Rightarrow \quad x = -a$$

which is the directrix of the parabola.
Since the extremities of a focal chord are $(at^2, 2at)$ and $(at'^2, 2at')$ where $tt' = -1$ and the slopes of the tangents at those points are $1/t$ and $1/t'$ whose product is -1, the tangents are perpendicular and hence by statement-2 they intersect on the directrix.

49. An equation of a normal to the parabola $y^2 = 2x$ is $y = mx - m - \frac{1}{2}m^3$.

It will pass through $(\alpha, 0)$ if $0 = m\alpha - m - \frac{1}{2}m^3$

$\Rightarrow m[m^2 - 2(\alpha - 1)] = 0$

If this equation has three distinct roots then $\alpha > 1$ and thus the statement-1 is true.

Also if $\alpha \leq 1$, we have only one value of m and thus exactly one normal of P passing through $(\alpha, 0)$ and statement-2 is also true but does not justify statement-1.

50. Equation of a normal to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$ which touches the circle $x^2 + y^2 = a^2$ if $(-2am - am^3)^2 = a^2(1 + m^2)$

$\Rightarrow m^6 + 4m^4 + 3m^2 - 1 = 0$

Which does not hold for any integral value of m so statement-2 is true and justify that statement-1 is also true.

51. Statement-2 is true, see theory using it in statemenet-1, length of the latus rectum

$= 2\left|\frac{3(1) + (4)(-1) - 9}{\sqrt{9 + 16}}\right| = 4$ and thus statement-1 is also true.

52. From the given equations we have $x^2 + 2x - 4x + 3 = 0 \Rightarrow (x - 1)^2 + 2 = 0$ which has no real solution and thus the statement-2 is true.

    For statement-1, equation of a tangent to the parabola $y^2 = 2x$ is $y = mx + \frac{1}{2m}$ which will touch the circle if $\left|\frac{2m - 0 + \frac{1}{2m}}{\sqrt{1+m^2}}\right| = \sqrt{2^2 - 3}$

    $$\Rightarrow \left(2m + \frac{1}{2m}\right)^2 = 1 + m^2$$

    $$\Rightarrow 3m^2 + \frac{1}{4m^2} + 1 = 0$$

    which gives no real value of m.

    So there is no common tangent to the given curves and the statement-1 is false.

53. Statement-2 is true, see theory.

    In statement-1, for the parabola $y^2 = 8x$ $a = 2$,

    So $(at^2, 2at) = (32, 16) \Rightarrow t = 4$ and using statement-2, the coordinates of Q are $\left(\frac{a}{t^2}, \frac{-2a}{t}\right) = \left(\frac{2}{16}, \frac{-2 \times 2}{2}\right) = \left(\frac{1}{8}, -2\right)$

    So statement-2 is also true.

54. Statement-2 is false as these tangents intersect on the directrix and are at right angles.

    In statement-1, the tangents intersect at the point $(at_1t_2, a(t_1 + t_2))$ which lies on the axis i.e. $y = 0 \Rightarrow t_1 + t_2 = 0$ and hence statement-1 is true.

55. If $t > 0$, $\left|t + \frac{1}{t}\right| = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2 + 2 \geq 2$

    If $t < 0$, $\left|t + \frac{1}{t}\right| = \left|\left(-t - \frac{1}{t}\right)\right| \geq 2$

    So statement-2 is true.

    Length of a focal chord of the parabola $y^2 = 16x$ is $a\left(t + \frac{1}{t}\right)^2$ where $a = 4$

    So if $4\left(t + \frac{1}{t}\right)^2 = 9$

    $$\Rightarrow \left|t + \frac{1}{t}\right| = \frac{3}{2} < 2$$

    Which is not possible by statement-2 and thus statement-1 is also true.

Level 2

56. $PQ = QR = 4a$

    $$AQ = \sqrt{a^2 + 4a^2} = \sqrt{5}\,a$$

    $$AP = AM + PM$$

    $$= a + \frac{\sqrt{3}}{2}(4a) = (1 + 2\sqrt{3})a$$

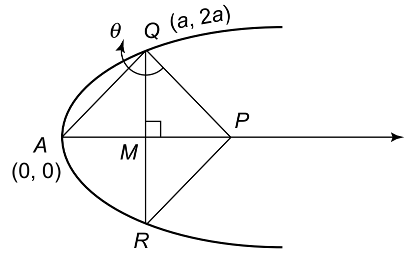


Fig. 18.20

$$\cos\theta = \frac{(AQ)^2 + (PQ)^2 - (AP)^2}{2\,(AQ)\,(PQ)}$$

$$= \frac{5 + 16 - \left(1 + 12 + 4\sqrt{3}\right)}{8\sqrt{5}} = \frac{2 - \sqrt{3}}{2\sqrt{5}}$$

57. Points of intersection of $x^2 + y^2 = a^2$ and $y^2 = 4ax$ are $\left(\left(\sqrt{5} - 2\right)a, \pm 2\left(\sqrt{5} - 2\right)^{1/2} a\right)$ and the points of intersection of $y^2 = -4ax$ and $x^2 + y^2 = a^2$ are $\left(\left(2 - \sqrt{5}\right)a, \pm 2\left(\sqrt{5} - 2\right)^{1/2} a\right)$

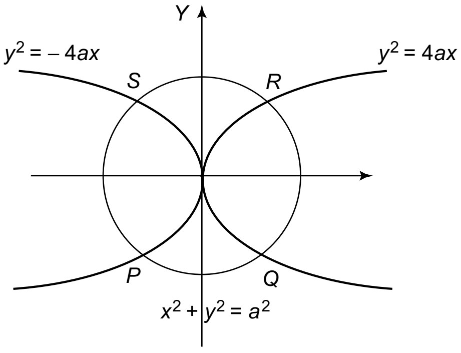


Fig. 18.21

$$\text{Required area} = 2 \times \left(\sqrt{5} - 2\right) \times 2 \times 2\left(\sqrt{5} - 2\right)^{1/2} a^2$$

$$= 8\left(\sqrt{5} - 2\right)^{3/2} a^2$$

58. Coordinates of P are $\left(1, 2\sqrt{2}\right)$ and of Q are $\left(1, -2\sqrt{2}\right)$. Tangent at $P\left(1, 2\sqrt{2}\right)$ to the circle $x^2 + y^2 = 9$ is $x + 2\sqrt{2}y = 9$ which meets x-axis at $R(9, 0)$.

Tangent at P $\left(1, 2\sqrt{2}\right)$ to the parabola $y^2 = 8x$ is $y\left(2\sqrt{2}\right) = 4(x + 1)$ which meets x-axis at $S(-1, 0)$

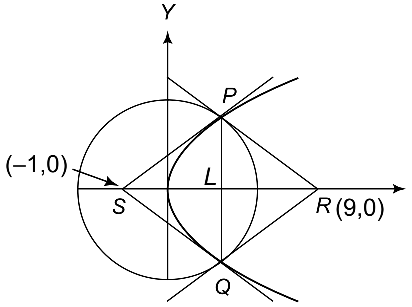


Fig. 18.22

Equation of PQ is $x = 1$

So $\dfrac{\text{Area of } \Delta\, PQS}{\text{Area of } \Delta\, PQR} = \dfrac{SL}{LR} = \dfrac{2}{8} = \dfrac{1}{4}$

59. Equation of any tangent of the parabola $y^2 = 8x$ is $y = mx + \dfrac{2}{m}$.

It will be a tangent to the circle $x^2 + y^2 = 2$ if

$$\frac{|2/m|}{\sqrt{1+m^2}} = \sqrt{2}$$

$$\Rightarrow \frac{4}{m^2} = 2(1 + m^2)$$

$$\Rightarrow m^4 + m^2 - 2 = 0$$

$$\Rightarrow (m^2 + 2)\,(m^2 - 1) = 0$$

$$\Rightarrow m = \pm\, 1.$$

Thus, two common tangents are $y = x + 2$ and $y = -x - 2$.

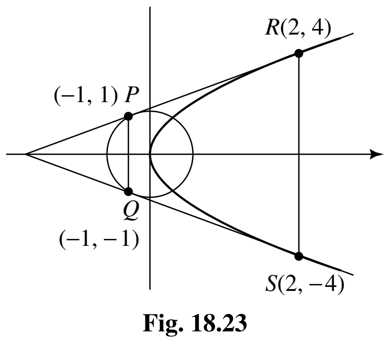


Fig. 18.23

Coordinates of P, Q, R, S are respectively $(-1, 1)$, $(-1, -1)$, $(2, 4)$, $(2, -4)$

$\therefore$ $PQ = 2$, $RS = 8$ and distance between PQ and RS is 5.

$\therefore$ area of $PQRS = \dfrac{1}{2}(2 + 8)(5) = 20(\text{unit})^2$

60. The tangents at the points $A(t_1^{\,2}, 2t_1)$ and $B(t_2^{\,2}, 2t_2)$ intersect at the point $(t_1t_2,\ t_1 + t_2)$

So $t_1t_2 = -3$ and $t_1 + t_2 = 2$.

$(AB)^2 = (t_1^{\,2} - t_2^{\,2})^2 + 4(t_1 - t_2)^2$

$= (t_1 - t_2)^2[(t_1 + t_2)^2 + 4]$

$= [(t_1 + t_2)^2 - 4t_1t_2]\ [(t_1 + t_2)^2 + 4]$

$= (4 + 12)\ (4 + 4) = 16 \times 8$

$\Rightarrow AB = 8\sqrt{2}$

61. Equations of tangent and normal at point $P(at^2, 2at)$ to the parabola are

$ty = x + at^2$

$y = -tx + 2at + at^3$ respectively.

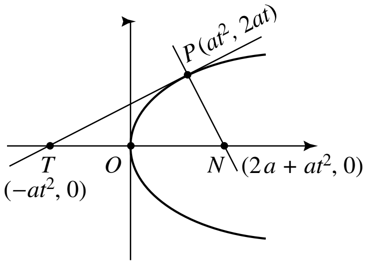


Fig. 18.24

These meet the axis of the parabola at $T(-at^2, 0)$ and $N(2a + at^2, 0)$

Let centroid of ΔPTN be (h, k), then

$h = \dfrac{1}{3}(-at^2 + 2a + at^2 + at^2) = \dfrac{1}{3}(2a + at^2)$ and

$k = \dfrac{1}{3}(0 + 0 + 2at) = \dfrac{2}{3}(at)$

Eliminating t, we get

$$\left(h - \frac{2a}{3}\right) = \frac{a}{3}\left(\frac{3k}{2a}\right)^2$$

Thus, locus of centroid is

$$y^2 = \frac{4}{3}a\left(x - \frac{2a}{3}\right)$$

Vertex of this parabola is $(2a/3, 0)$;

62. As circle with PQ as diameter passes through O, $\angle POQ = \dfrac{\pi}{2}$.

Let coordinates of P be $\left(t^2, \sqrt{2}t\right)$, where $t > 0$, and

let coordinates of Q be $\left(t_1^2, \sqrt{2}t_1\right)$.

Let m_1 = slope of $OP = \dfrac{\sqrt{2}}{t}$

m_2 = slope of $OQ = \dfrac{\sqrt{2}}{t_1}$

As $\angle POQ = \dfrac{\pi}{2}$, $m_1m_2 = -1$

$\Rightarrow tt_1 = -2.$

Area of $OPQ = \dfrac{1}{2}|x_1y_2 - x_2y_1|$

$= \dfrac{1}{2}\left|t^2\sqrt{2}t_1 - t_1^2\sqrt{2}t\right|$

$= \dfrac{\sqrt{2}}{2}(2)|t - t_1|$ $\qquad [\because tt_1 = -2]$

$= \sqrt{2}\left|t + \dfrac{2}{t}\right| = \sqrt{2}\left(t + \dfrac{2}{t}\right)$

$\Rightarrow 3\sqrt{2} = \sqrt{2}\left(t + \dfrac{2}{t}\right)$

$\Rightarrow t^2 - 3t + 2 = 0 \Rightarrow t = 1, 2$

Thus, coordinates of P are $(1, \sqrt{2})$ or $(4, 2\sqrt{2})$

63. Let $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$, $C(at_3^2, 2at_3)$ then $P(at_2t_3, a(t_2 + t_3))$ $Q(at_3t_1, a(t_3 + t_1))$ and $R(at_1t_2, a(t_1 + t_2))$

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix}$$

$= a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$

$$\text{Area of } \Delta PQR = \frac{1}{2}\begin{vmatrix} at_2t_3 & a(t_2 + t_3) & 1 \\ at_3t_1 & a(t_3 + t_1) & 1 \\ at_1t_2 & a(t_1 + t_2) & 1 \end{vmatrix}$$

$= \dfrac{1}{2}a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$

Hence the required ratio is 2 : 1.

64. Equation of a normal to the parabola $y^2 = 16x$ is $y = -tx + 8t + 4t^3$ for all values of t

Comparing with the given equation, we have

$-t = \cos\theta$, $8t + 4t^3 = -11\cos\theta - \cos3\theta$.

$\Rightarrow -8\cos\theta - 4\cos^3\theta = -11\cos\theta - \cos3\theta$.

$\Rightarrow 4\cos^3\theta - 3\cos\theta = \cos3\theta$.

Which is true for all values of θ.

65. Let the coordinates of P_1 be $(at_1^2, 2at_1)$, then the coordinates of Q_1 are $\left(\dfrac{a}{t_1^2}, \dfrac{-2a}{t_1}\right)$ and the coordinates of P_2 be $(at_2^2, 2at_2)$, then the coordinates of Q_2 are $\left(\dfrac{a}{t_2^2}, \dfrac{-2a}{t_2}\right)$

Equation of P_1P_2 is

$$\frac{x - at_1^2}{at_2^2 - at_1^2} = \frac{y - 2at_1}{2at_2 - 2at_1}$$

or $2(x + at_1t_2) = (t_1 + t_2)y$ $\qquad (1)$

Similarly equation of Q_1Q_2 is

$$2\left(x + \frac{a}{t_1t_2}\right) = -\left(\frac{1}{t_1} + \frac{1}{t_2}\right)y$$

or $2(t_1t_2x + a) = -(t_1 + t_2)y$ $\qquad (2)$

Adding (1) and (2) we get

$2(x + a)(1 + t_1t_2) = 0 \Rightarrow x + a = 0$

Showing that P_1P_2 and Q_1Q_2 intersect on $x = -a$, which is the directrix of the parabola.

Previous Years' AIEEE/JEE Main Questions

1. Equation of the normal at $(bt_1^2, 2bt_1)$ of the parabola is

$y = -t_1x + 2bt_1 + bt_1^3$

which passes through $(bt_2^2, 2bt_2)$ if $2bt_2 = -t_1(bt_2^2) + 2bt_1 + bt_1^3$.

$\Rightarrow 2(t_2 - t_1) = -t_1(t_2^2 - t_1^2)$

$\Rightarrow t_2 = -t_1 - \dfrac{2}{t_1}$

2. Given curve $y^2 = 18x \Rightarrow 2y\dfrac{dy}{dx} = 18$

$$\Rightarrow \frac{dy}{dx} = \frac{9}{y}$$

We are given $\frac{dy}{dx} = 2$

$$\therefore \ \frac{9}{y} = 2 \Rightarrow y = \frac{9}{2}$$

Thus, $x = \frac{1}{18}y^2 = \frac{1}{18}\left(\frac{9}{2}\right)^2 = \frac{9}{8}$

$\therefore$ required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$.

3. The two parabolas $y^2 = 4ax$ and $x^2 = 4ay$ meet in (0, 0) and $(4a, 4a)$. As these points lie on $2bx + 3cy + 4d = 0$, we get $d = 0$ and $2b(4a) + 3c(4a) + 4d = 0$

   $\Rightarrow d = 0$ and $2b + 3c = 0$

   Thus, $d^2 + (2b + 3c)^2 = 0$

4. Let Q be $(2t^2, 4t)$, $P(1, 0)$

   Mid point of $PQ = (h, k) = \left(\frac{2t^2+1}{2}, 2t\right)$

   Eliminating t we get

   $$h = \frac{2\left(\frac{k}{2}\right)^2 + 1}{2} \Rightarrow 4h = k^2 + 2$$

   Locus of (h, k) is $y^2 - 4x + 2 = 0$

5. $y = \frac{1}{3}a^3x^2 + \frac{1}{2}a^2x - 2a \qquad (1)$

   $$\frac{dy}{dx} = \frac{2}{3}a^3x + \frac{1}{2}a^2$$

   At the vertex of parabola $y = ax^2 + bx + c$, $\frac{dy}{dx} = 0$

   So $\frac{dy}{dx} = 0 \Rightarrow a = -\frac{3}{4x}$

   Putting $a = \frac{-3}{4x}$ in (1), we get

   $$y = \frac{1}{3}\left(\frac{-27}{64}\frac{1}{x^3}\right)x^2 + \frac{1}{2}\left(\frac{9}{16x^2}\right)x + \frac{3}{2x}$$

   $\Rightarrow xy = \frac{105}{64}$.

6. Since perpendicular tangents intersect on the directrix $x = -2$, required point is $(-2, 0)$.

7. Vertex of the parabola is equidistant from the focus and the directrix on the axis of the parabola. So its coordinates are (1, 0).

8. Any point on the parabola is (y^2, y).

   Its distance from the line $x - y + 1 = 0$ is

   $$P = \left|\frac{y^2 - y + 1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}}\left|\left(y - \frac{1}{2}\right)^2 + \frac{3}{4}\right| \geq \frac{3}{4\sqrt{2}}$$

   Hence the shortest distance is $\frac{3\sqrt{2}}{8}$ which is attained when the perpendicular is drawn from $\left(\frac{1}{4}, \frac{1}{2}\right)$ on the parabola.

9. Perpendicular tangent to the parabola $y^2 = 4x$ intersect on the directrix of the parabola i.e. $x = -1$.

10. An equation of tangent to the parabola $y^2 = 4ax$ is of the form $y = mx + a/m$.

    $\therefore$ an equation of the tangent to

    $y^2 = 16\sqrt{3}x$ is $y = mx + 4\sqrt{3}/m$.

    Now, $y = mx + 4\sqrt{3}/m$ will be tangent to $2x^2 + y^2 = 4$ if the equation

    $$2x^2 + (mx + 4\sqrt{3}/m)^2 = 4$$

    or $(2 + m^2)x^2 + 8\sqrt{3}x + 48/m^2 - 4 = 0$

    has equal roots.

    $\Leftrightarrow (8\sqrt{3})^2 - 4(2 + m^2)(48/m^2 - 4) = 0$

    $\Leftrightarrow m^4 + 2m^2 = 24$.

    $\therefore$ Statement-2 is true.

    Since $m = 2$ satisfies this equation, we get that statement-1 is also true and statement-2 is a correct reason for the statement-1.

11. An equation of tangent to $y^2 = 4\sqrt{5}x$ is

    $y = mx + \frac{\sqrt{5}}{m}$ or $m^2x - my + \sqrt{5} = 0$.

    This line will touch $x^2 + y^2 = 5/2$ if

    $$\frac{|m^2(0) - m(0) + \sqrt{5}|}{\sqrt{m^4 + m^2}} = \frac{\sqrt{5}}{\sqrt{2}}$$

$\Rightarrow m^4 + m^2 - 2 = 0$

$\Rightarrow m = \pm 1$. Note that $m = \pm 1$ satisfies $m^4 - 3m^2 + 2 = 0$.

For $m = 1$, $y = x + \sqrt{5}$ is a common tangent to the two curves.

12. $y^2 = 4ax$

$$\Rightarrow 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} - \frac{2a}{y} = 0$$

Which is a differential equation of degree 1 and order1 so statement-2 is true.

Slope of the tangent $m = \frac{dy}{dx} = \frac{2a}{y}$ at $P(x, y)$

$\Rightarrow m$ is inversely proportional to the ordinate of P.

So statement-1 is also true but does not follow from statement-1.

13. $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$, for $y^2 = -2x$, $a = -\frac{1}{2}$

$\Rightarrow y = mx - \frac{1}{2m}$ is a tangent to the parabola

at $\left(-\frac{1}{2m^2}, \frac{-1}{m}\right) \Rightarrow$ statement-2 is true.

$\Rightarrow x - 2y = 2$ is a tangent to the parabola at $(-2, -2)$ for $m = \frac{1}{2}$ and thus statement-1 is also true.

14. Ends of the latus rectum are $(1, \pm 2) = (t^2, 2t)$ $\Rightarrow t = \pm 1$

Equation of the normal at $(t^2, 2t)$ is $y = -tx + 2t + t^3$.

Equations of the normals for $t = \pm 1$ are $y = -x + 3$ and $y = x - 3$.

They intersect at $(3, 0)$.

15. $y = mx + \frac{1}{m}$ is a tangent to the parabola $y^2 = 4x$ which will touch the parabola $x^2 = -32y$ if the root of $x^2 = -32\left(mx + \frac{1}{m}\right)$ are equal.

$\Rightarrow 32m^3 - 4 = 0 \Rightarrow m = \frac{1}{2}$

16. For the points of intersection $x^2 + 8x - 9 = 0$

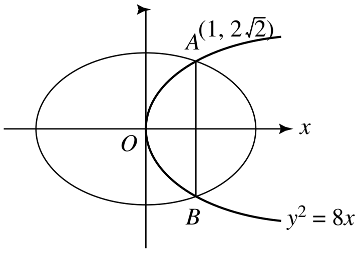


Fig. 18.25

$\Rightarrow (x - 1)(x + 9) = 0$

$\Rightarrow x = 1 \qquad [\because x > 0]$

$\therefore$ Coordinate of A are $\left(1, 2\sqrt{2}\right)$

Thus, $L_1 = 4\sqrt{2}$

Also, $L_2 = 8 = (4)\ (2)$

$\therefore L_1 < L_2$

17. Let $y = mx + \frac{1}{m}$ be a tangent to the parabola $y^2 = 4x$.

It will pass through $(-2, -1)$, we get

$$-1 = -2m + \frac{1}{m}$$

$\Rightarrow 2m^2 - m - 1 = 0 \Rightarrow m = 1, -1/2$

$$\Rightarrow |\tan\alpha| = \left|\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right| = 3$$

18. Let the slope of the line be m, then its equation is $y = m\left(x - \frac{3}{2}\right)$.

Its distance from $(0, 0)$ is

$$\frac{|m(0-3/2)-0|}{\sqrt{1+m^2}} = \frac{\sqrt{5}}{2}$$

$$\Rightarrow (3m)^2 = 5(1 + m^2) \Rightarrow m = \pm\frac{\sqrt{5}}{2}.$$

$\therefore$ Required slope is $\dfrac{\sqrt{5}}{2}$

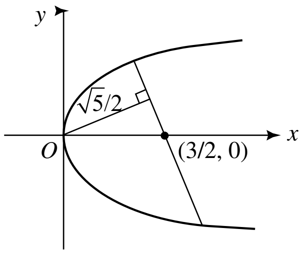


Fig. 18.26

19. Let coordinates of Q be (x', y') and that of P be (x, y), then

$$x = \frac{1}{4}x',\ y = \frac{1}{4}y'$$

As (x', y') lies on $x^2 = 8y$,

$$(4x)^2 = 8(4y) \Rightarrow x^2 = 2y$$

20. Let coordinates of P be $(-t^2, 2t)$, where $t > 0$, then coordinates of Q are $(-t^2, -2t)$.

If coordinates of R are (x, y), then

$$x = \frac{-t^2 - 2t^2}{2+1} = -t^2$$

and $y = \dfrac{1(2t) + 2(-2t)}{2+1} = -\dfrac{2}{3}t$

Eliminating t, we get

$$-x = \left(-\frac{3}{2}y\right)^2 \text{ or } 9y^2 = -4x$$

21. An equation of tangent to

$y - 6 = x^2$ at $(2, 10)$ is

$$(y + 10) - 12 = 2(2)x \text{ or } 4x - y + 2 = 0 \qquad (1)$$

An equation of normal at the point of contact (α, β) of (1) and $x^2 + y^2 + 8x - 2y = k$ is

$$(x - (-4)) + 4(y - 1) = 0$$

$$\text{or } x + 4y = 0 \qquad (2)$$

The point (α, β) is the point of intersection of (1) and (2). Solving (1) and (2), we get

$$\alpha = -\frac{8}{17},\ \beta = \frac{2}{17}.$$

22. Refer figure. Let $S = CP^2 = t^4 + \left(2\sqrt{2}t + 6\right)^2$

$$\frac{ds}{dt} = 4t^3 + 4\sqrt{2}\left(2\sqrt{2}t + 6\right)$$

$$= 4\left[t^3 + 4t + 6\sqrt{2}\right]$$

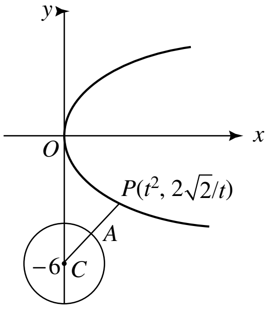


Fig. 18.27

For least value of S, $\dfrac{ds}{dt} = 0$

$$\Rightarrow t^3 + 4t + 6\sqrt{2} = 0$$

$$t^3 + \sqrt{2}t^2 - \sqrt{2}t^2 - 2t + 6t + 6\sqrt{2} = 0$$

$$\Rightarrow \left(t + \sqrt{2}\right)\left(t^2 - \sqrt{2}t + 6\right) = 0$$

$$\Rightarrow t = -\sqrt{2} \qquad [\because t \text{ is real}]$$

Also, $\dfrac{d^2s}{dt^2} = 4[3t^2 + 4]$

$$\Rightarrow \left.\frac{d^2s}{dt^2}\right]_{t=-\sqrt{2}} > 0$$

$\Rightarrow S$ is least when $t = -\sqrt{2}$. Thus, coordinates of P are $(2, -4)$.

An equation of required circle is

$$(x - 2)^2 + (y + 4)^2 = (0 - 2)^2 + (-6 + 4)^2$$
$$\text{or } x^2 + y^2 - 4x + 8y + 12 = 0$$

23. Let coordinates of a point P on $y = x^2 - 4$ is $(t, t^2 - 4)$.

Let d = distance of P from the origin, then

$$d^2 = (t - 0)^2 + (t^2 - 4 - 0)^2$$

$$= t^2 + t^4 + 16 - 8t^2$$

$$= t^4 - 7t^2 + 16$$

$= \left(t^2 - \frac{7}{2}\right)^2 + \frac{15}{4}$

Note that d is least if $t^2 = 7/2$. Least value of d is $\sqrt{15}/2$.

24. An equation of normal at $P(t)$ is $y = -tx + 2t + t^3$

It passes through $Q(t_1)$ i.e. $(t_1^2, 2t_1)$
if $2t_1 = -tt_1^2 + 2t + t^3$

$\Rightarrow 2(t_1 - t) = t(t^2 - t_1^2)$

$\Rightarrow 2 = -t(t + t_1)$

$\Rightarrow t_1 = -t - \frac{2}{t}$

$\Rightarrow t_1^2 = t^2 + \frac{4}{t^2} + 4 \geq \sqrt{t^2\left(\frac{4}{t^2}\right)} + 4 = 6$ [AM ≥ GM]

Previous Years' B-Architecture Entrance Examination Questions

1. Equation of the side AB of the triangle ABC is $y = \frac{1}{\sqrt{3}}x$ which meets the parabola $y^2 = 8x$ at the point $B\left(24, 8\sqrt{3}\right)$

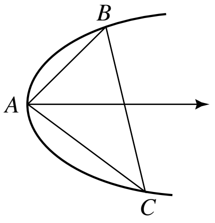


Fig. 18.28

$\Rightarrow (AB)^2 = (24)^2 + \left(8\sqrt{3}\right)^2 \Rightarrow AB = 16\sqrt{3}$

2. Tangent at $\left(\frac{1}{4}, \frac{1}{2}\right)$ to the parabola $y^2 = x$ is

$y\left(\frac{1}{2}\right) = \frac{1}{2}\left(x + \frac{1}{4}\right) \Rightarrow y = x + \frac{1}{4}$

Which is parallel to $y = x + 1$, so statement-2 is true.

Any point on the parabola $y^2 = x$ is $\left(\frac{1}{4}t^2, \frac{1}{2}t\right)$

whose distance from the line $y = x + 1$ is

$\left|\frac{\frac{1}{2}t - \frac{1}{4}t^2 - 1}{\sqrt{2}}\right| = \frac{|(t-1)^2 + 3|}{4\sqrt{2}}$

Which minimum for $t = 1$. So statement-1 is also true but does not follow from statement-2.

3. Equation of a tangent to the parabola $y^2 = 4x$ is $y = mx + \frac{1}{m}$ which passes through (1, 4) if $4 = m + \frac{1}{m} \Rightarrow m^2 - 4m + 1 = 0$

$\Rightarrow m_1 m_2 = -1$. So the required angle is $\frac{\pi}{2}$.

4. Tangent to the parabola $x^2 = 4y$ at (4, 4) and (–4, 4) are $x(4) = 2(y + 4)$ and $x(-4) = 2(y + 4)$ which intersect on $x = 0$, i.e. y-axis.

So statement-1 is true.

Extremities of the latus rectum are (±2, 1) and the tangents at these points are $2x = 2(y + 1)$ and $(-2)x = 2(y + 1)$.

Which again intersect on y-axis, the axis of the parabola and statement-2 is also true but does justify statement-1.

5. Coordinates of L are (4, 8).

Equation of the chord through L with slope –1, is
$x + y = 12$ (1)

Any point on the parabola is $(4t^2, 8t)$ which lies on (1) if

$4t^2 + 8t - 12 = 0 \Rightarrow t = 1, -3$.

$t = 1$ coordinates to L, so other end M of the chord is (36, – 24) and length of the chord is

$\sqrt{(36-4)^2 + (-24-8)^2} = 32\sqrt{2}$

6. Let (h, k) be the mid point of the chord of the parabola $x^2 = 4py$ with slope m then its equation is

$hx - 2p(y - k) = h^2 - 4pk$.

$\Rightarrow m = \frac{h}{2p} \Rightarrow h = 2pm$.

Locus of (h, k) is $x = 2pm$.

Which is parallel to y-axis at a distance $|2pm|$ from it.

7. Let coordinates of P be $(t^2, 2t)$, then coordinates of Q are $\left(\frac{1}{t^2}, \frac{-2}{t}\right)$. Centre of the circle with PQ as diameter is $\left(\frac{1}{2}\left(t^2+\frac{1}{t^2}\right), t-\frac{1}{t}\right)$.

   As it lies on $\sqrt{5}y + 4 = 0$,

   $$t - \frac{1}{t} = \frac{-4}{\sqrt{5}}$$

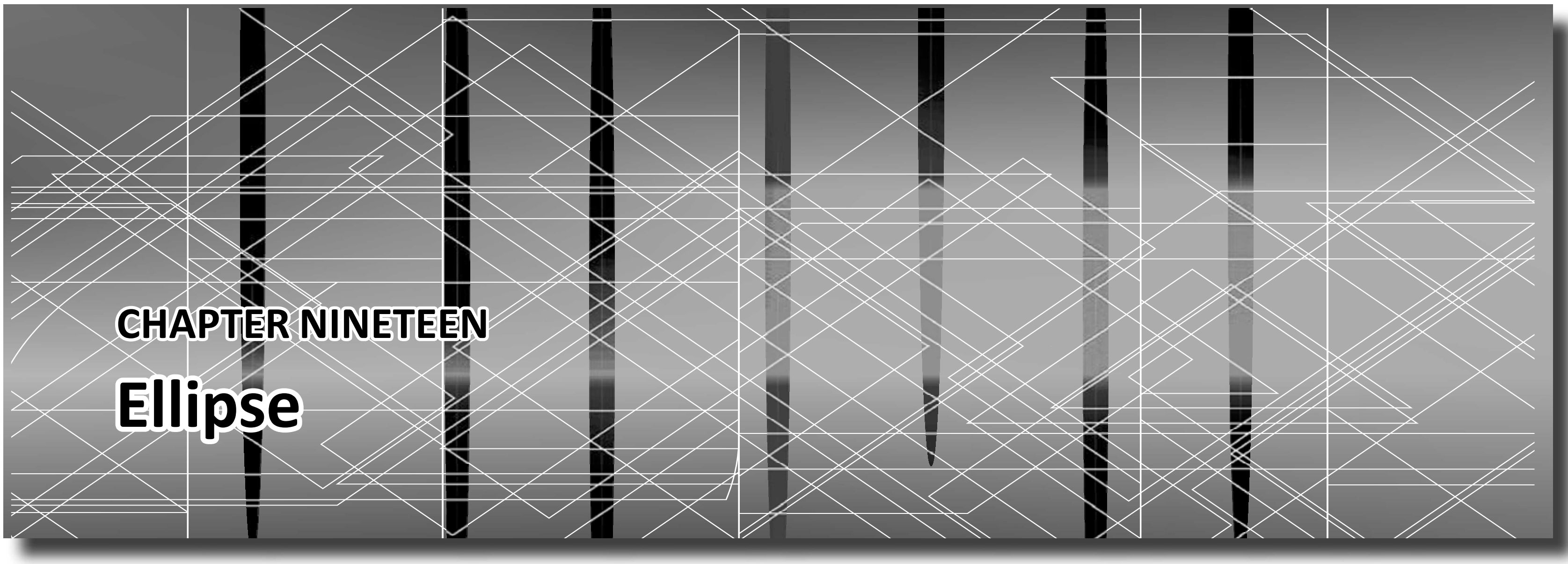
   Also, $PQ^2 = \left(t^2-\frac{1}{t^2}\right)^2 + 4\left(t+\frac{1}{t}\right)^2$

   $$= \left(t+\frac{1}{t}\right)^2\left[\left(t-\frac{1}{t}\right)^2 + 4\right]$$

   $$= \left[\left(t-\frac{1}{t}\right)^2 + 4\right]^2$$

   $$\Rightarrow PQ = \left(t-\frac{1}{t}\right)^2 + 4$$

   $$= \frac{16}{5} + 4 = \frac{36}{5}$$



CHAPTER NINETEEN

Ellipse

Definition 1:

An ellipse is the locus of a point which moves in a plane such that the sum of its distances from two fixed points is always constant and the constant is greater than the distance between the fixed points.

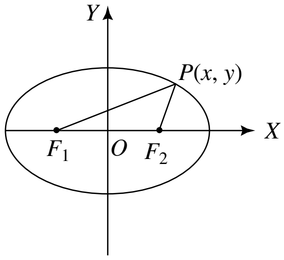


Fig. 19.1

(i) Two fixed points are called the *foci* of the ellipse.

(ii) Mid-point of the line joining the foci is called the *centre* of the ellipse.

(iii) The line segment through the foci of the ellipse is called the *major axis* of the ellipse.

(iv) The line segment through the centre and perpendicular to the major axis is called the *minor axis* of the ellipse.

(v) The end points of the major axis are called the *vertices* of the ellipse.

(vi) A line segment through a focus perpendicular to the major axis is called a *Latus rectum* of the ellipse.

(vii) Major axis and minor axis are called the *principal axis* of the ellipse.

Standard Equation of the Ellipse

Let F_1 $(-c, 0)$ and F_2 $(c, 0)$ be the foci of the ellipse. Then O $(0, 0)$, the mid-point of $F_1 F_2$ is the centre of the ellipse.

Let P (x, y) be any point on the ellipse. Then by definition $PF_1 + PF_2 = 2a$ (constant)

$$\Rightarrow \sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} = 2a. \ (2a > 2c)$$

Also $[(x+c)^2+y^2]-[(x-c)^2+y^2] = 4cx.$

$$\Rightarrow \sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2} = \frac{4cx}{2a} = \frac{2cx}{a}$$

$$\Rightarrow (x-c)^2+y^2 = \left(a-\frac{cx}{a}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2-c^2} = 1$$

Let $a^2 - c^2 = b^2$, then locus of $P(x, y)$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ where $a > b > 0$.

Definition 2:

An ellipse is the locus of a point which moves in a plane in such a way that the ratio of its distance from a fixed point in the plane to its distance from a fixed line in the plane (not passing through fixed point) is a constant and equal to e, where $0 < e < 1$.

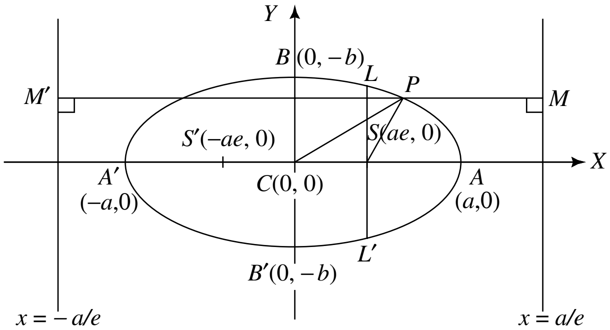
(i) Fixed point is called a *focus* of the ellipse.

(ii) Fixed line is called a *directrix* of the ellipse.

(iii) The constant ratio e is called the *eccentricity* of the ellipse.

Standard Equation of an Ellipse referred to its principle axes along the coordinate axes is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$,

$$b^2 = a^2\,(1-e^2)$$



Ellipse (e , – b)

Fig. 19.2

Let $P(x, y)$ be any point on the ellipse.

$S = (ae, 0)$ be a focus and $x = \frac{a}{e}$ be a directrix.

Then according to the definition

$$(x - ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

Since $e < 1$, $1 - e^2 > 0$ so let $a^2(1 - e^2) = b^2$ and the required equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

In the figure

(i) $A'A$ is the *major axis* of the ellipse along x-axis of length $2a$; $A(a, 0)$, $A'(-a, 0)$ are the vertices of the ellipse;

(ii) BB' is the minor axis of the ellipse along y-axis of length $2b$.

(iii) $O(0, 0)$ is the centre of the ellipse.

(iv) $e = \sqrt{\frac{a^2 - b^2}{a^2}}$ is the eccentricity of the ellipse.

From symmetry we observe that if

$S' = (-ae, 0)$ be taken as a focus and $x = \frac{-a}{e}$ is taken as a directrix, same ellipse is described. So

(v) $S'(-ae, 0)$ and $S(ae, 0)$ are the two foci of the ellipse.

(vi) $x = \frac{-a}{e}$ and $x = \frac{a}{e}$ are the two directrices of the ellipse.

(vii) $x = \pm ae$ are the two latera recta of the ellipse.

(viii) Latus rectum $x = ae$ meets the ellipse at point $\left(ae, \pm\frac{b^2}{a}\right)$, so length of each latus rectum is $\frac{2b^2}{a}$.

Illustration 1

Find the length and equations of the latera recta of the ellipse

$$\frac{x^2}{36} + \frac{y^2}{25} = 1$$

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

where $a^2 = 36$, $b^2 = 25 \Rightarrow e^2 = 1 - \frac{25}{36} = \frac{11}{36}$

Equations of latera recta are $x = \pm ae$

$$\Rightarrow x = \pm 6 \times \frac{\sqrt{11}}{6} = \pm\sqrt{11}$$

and length of each latus rectum $= \frac{2b^2}{a} = \frac{2 \times 25}{6}$

$= \frac{25}{3}$ units.

Illustration 2

Find the centre, foci, the length of the axis, eccentricity and the equation of the directrices of the ellipse.

$9x^2 + 16y^2 - 18x + 32y - 119 = 0$

Solution: The equation of the ellipse can be written as

$9(x - 1)^2 + 16(y + 1)^2 = 144$

$$\Rightarrow \frac{(x-1)^2}{16} + \frac{(y+1)^2}{9} = 1$$

Which can be written as $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$

where $X = x - 1$, $Y = y + 1$, $a = 4$, $b = 3$.

centre is $X = 0$, $Y = 0$ i.e. $(1, -1)$

foci are $X = \pm ae$, $Y = 0 \Rightarrow x - 1 = \pm 4\sqrt{1 - \frac{b^2}{a^2}} = \pm\sqrt{7}$

$\Rightarrow x = 1 \pm \sqrt{7}$; $y = -1$

Length of the major axis $= 2a = 8$

Length of the minor axis $= 2b = 6$.

eccentricity $= \sqrt{1 - \frac{b^2}{a^2}} = \frac{\sqrt{7}}{4}$

Equations of the directrices are $X = \pm\frac{a}{e}$.

$$\Rightarrow x - 1 = \pm\frac{4 \times 4}{\sqrt{7}} \Rightarrow x = 1 \pm \frac{16}{\sqrt{7}}$$

Illustration 3

Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, a focus is $(2, 3)$ and a directrix is $x = 7$. Find the length of the major and minor axes of the ellipse.

Solution: Equation of the ellipse by definition is

$$(x - 2)^2 + (y - 3)^2 = \frac{1}{4}(7 - x)^2$$

$\Rightarrow 3x^2 + 4y^2 - 2x - 24y + 3 = 0$

which can be written as

$$3\left(x - \frac{1}{3}\right)^2 + 4(y - 3)^2 = 3 \times \frac{1}{9} + 4 \times 9 - 3 = \frac{100}{3}$$

$$\text{or } \frac{\left(x - \frac{1}{3}\right)^2}{\frac{100}{9}} + \frac{(y-3)^2}{\frac{100}{12}} = 1$$

So length of the major axis = $2\sqrt{\dfrac{100}{9}} = \dfrac{20}{3}$.

Length of the minor axis = $2\sqrt{\dfrac{100}{12}} = \dfrac{20}{2\sqrt{3}} = \dfrac{10}{\sqrt{3}}$.

Some Properties and Standard Results for the Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

1. The parametric equations of the ellipse or the coordinates of any point on the ellipse are $x = a\cos\theta$, $y = b\sin\theta$. The point is denoted by "θ". θ is called the eccentric angle of the point.
2. An equation of the tangent at the above point "θ" is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \text{ and at } (x', y') \text{ is } \frac{xx'}{a^2}+\frac{yy'}{b^2} = 1.$$

Illustration 4

Find an equation of the tangent to the ellipse $\dfrac{x^2}{81}+\dfrac{y^2}{49} = 1$ at the point P whose eccentric angle is $\pi/6$. Also find the coordinates of P.

Solution: Coordinates of P are $\left(9\cos\dfrac{\pi}{6}, 7\sin\dfrac{\pi}{6}\right) = \left(\dfrac{9\sqrt{3}}{2}, \dfrac{7}{2}\right)$

An equation of the tangent at P to the ellipse is

$$\frac{x}{9}\cos\frac{\pi}{6}+\frac{y}{7}\sin\frac{\pi}{6} = 1$$

$$\Rightarrow \frac{\sqrt{3}x}{9\times 2}+\frac{y}{7\times 2} = 1 \Rightarrow 7\sqrt{3}x + 9y = 126.$$

3. An equation of the normal at $P(\theta)$ is

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta} = a^2 - b^2$$

and at (x', y') is

$$\frac{x-x'}{x'/a^2} = \frac{y-y'}{y'/b^2}$$

Illustration 5

$5\sqrt{3}x + 7y = 70$ is a tangent to the ellipse $\dfrac{x^2}{49}+\dfrac{y^2}{25} = 1$ at the point P. Find the coordinates of P and the equation of the normal at P.

Solution: Let the coordinates of P be $(7\cos\theta, 5\sin\theta)$

Equation of the tangent at P is

$$\frac{x}{7}\cos\theta + \frac{y}{5}\sin\theta = 1 \qquad (1)$$

Given equation can be written as

$$\frac{x\sqrt{3}}{14}+\frac{y}{10} = 1 \qquad (2)$$

Comparing (1) and (2) we get

$$\cos\theta = \frac{\sqrt{3}}{2}, \sin\theta = \frac{1}{2} \Rightarrow \theta = 30°$$

So the coordinates of P are $\left(\dfrac{7\sqrt{3}}{2}, \dfrac{5}{2}\right)$.

Equation of the normal at P is

$$\frac{7x}{\cos\theta}-\frac{5y}{\sin\theta} = 1$$

$$\Rightarrow 14x - 10\sqrt{3}\,y = \sqrt{3}$$

4. The condition that the line $y = mx + c$ is a tangent to the ellipse is $c^2 = a^2m^2 + b^2$. So equation of any tangent to the ellipse (not parallel to y-axis) can be written as $y = mx \pm \sqrt{a^2m^2+b^2}$

Illustration 6

If $y = mx + 5$ is a tangent to the ellipse $4x^2 + 25y^2 = 100$, then find the value of $100m^2$.

Solution: Equation of the ellipse is

$$\frac{x^2}{25}+\frac{y^2}{4} = 1$$

If $y = mx + 5$ touches the ellipse.

Then $(5)^2 = 25\,m^2 + 4 \Rightarrow 25m^2 = 21$

$\Rightarrow 100\,m^2 = 84.$

5. An equation of the *chord of contact* of the point (x', y') joining the points of contact of the tangents drawn from (x', y') to the ellipse is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2} = 1$$

Note

☐ This equation is same as the tangent at (x', y') to the ellipse. When $P\,(x', y')$ lies on the ellipse the two points of contact coincide with P and the tangent at P and the chord of contact of P are same).

---

6. An equation of the chord of the ellipse whose mid-point is (x', y') is $T = S'$, where

$$T \equiv \frac{xx'}{a^2}+\frac{yy'}{b^2} - 1 \text{ and } S' \equiv \frac{x'^2}{a^2}+\frac{y'^2}{b^2} = 1.$$

7. An equation of pair of tangents from a point $P(x', y')$ outside the ellipse to the ellipse is $SS' = T^2$, $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$.

Illustration 7

Find the equation of the chord of contact of the point (3, 1) to the ellipse $x^2 + 9y^2 = 9$. Also find the mid-point of this chord of contact.

Solution: Equation of the ellipse is

$$\frac{x^2}{9} + \frac{y^2}{1} = 1$$

Equation of the chord of contact of (3, 1) is

$$\frac{x(3)}{9} + \frac{y(1)}{1} = 1 \Rightarrow x + 3y = 3 \qquad (1)$$

If (h, k) is the mid-point of this chord, then its equation is

$$\frac{hx}{9} + \frac{ky}{1} - 1 = \frac{h^2}{9} + \frac{k^2}{1} - 1 \qquad (2)$$

From (1) and (2) we get

$$\frac{h}{1} = \frac{9k}{3} = \frac{h^2 + 9k^2}{3}$$

$$\Rightarrow h = \frac{3}{2}, k = \frac{1}{2}.$$

So the mid point of the chord of contact is $\left(\frac{3}{2}, \frac{1}{2}\right)$.

8. *Director circle* of the ellipse is the locus of the point of intersection of the tangents to the ellipse which intersect at right angles and its equation is $x^2 + y^2 = a^2 + b^2$.

   So a pair of tangent draw from any point on the director circle to the ellipse are at right angles.

9. *Auxiliary circle* of the ellipse is the circle on the major-axis as a diameter and its equation is $x^2 + y^2 = a^2$. If P is a point on the ellipse and Q is a point on the auxiliary circle such that Q lies on the ordinate produced of the point P. If C is the centre of the ellipse and CA is the semi-major axis of the ellipse then $ACQ = \theta$ is called the *eccentric angle* of the point P and the coordinate of P are $(a \cos \theta, b \sin \theta)$. Also $PQ = (a - b) \sin \theta$.

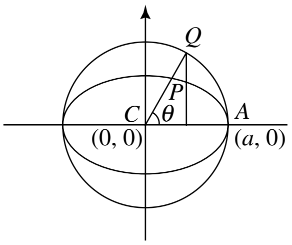


Fig. 19.3

Illustration 8

Find the equation of the ellipse whose auxiliary circle is the director circle of the ellipse $\frac{x^2}{36} + \frac{y^2}{13} = 1$ and the length of a latus rectum is 2 units.

Solution: Equation of the director circle of the ellipse is $x^2 + y^2 = 36 + 13 = 49$.

So if the required equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then auxiliary circle is $x^2 + y^2 = a^2 = 49 \Rightarrow a^2 = 49 \Rightarrow a = 7$.

Also length of a latus rectum $= \frac{2b^2}{a} = 2$

$\Rightarrow b^2 = a = 7$

and the required equation is

$$\frac{x^2}{49} + \frac{y^2}{7} = 1.$$

10. *A diameter* of an ellipse is the locus of the mid points of a system of parallel chords of the ellipse and its equation is

$$y = -\frac{b^2}{a^2 m} x,$$

   where m is the slope of the parallel chords of the ellipse which are bisected by it. This is a line through the centre of the ellipse. Two diameters of an ellipse are said to be *conjugate* when each bisects that chords parallel to the others. Thus two diameters $y = mx$ and $y = m'x$ of the ellipse are conjugate if $mm' = -\frac{b^2}{a^2}$.

Illustration 9

Find the equation of the diameters of the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ bisecting the chords parallel to the line $y = x - 1$. Also find the equation of the conjugate diameter bisecting the chords parallel of this diameter.

Solution: Slope of the given line is $m = 1$.

So equation of the diameter bisecting the chords with slope 1 is $y = -\frac{16}{36} x \Rightarrow 4x + 9y = 0$. Slope of the conjugate diameter is m' such that $-\frac{16}{36} m' = -\frac{16}{36} \Rightarrow m' = 1$ and the required equation of the conjugate diameter is $y = x$.

11. The tangent and normal at any point of an ellipse bisect the angle between the focal radii to that point.
12. The locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle.

13. If W, W' are the feet of the perpendiculars from the foci S and S' respectively on the tangent at any point P of an ellipse with centre C, then CW is parallel to $S'P$ and CW' is parallel to SP.

14. If the normal at any point P of the ellipse $x^2/a^2 + y^2/b^2 = 1$ meet the major and minor axes in G and g respectively and CF is perpendicular from the centre C on the normal then
$PF \,.\, PG = b^2$ and $PF.\, Pg = a^2$

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: Equation of a directrix of the ellipse $\frac{x^2}{36} + \frac{y^2}{4} = 1$ is

(a) $9x - 8 = 0$ (b) $8x - 9 = 0$
(c) $\sqrt{2}\,x + 9 = 0$ (d) $x + 9\sqrt{2} = 0$

Ans. (c)

Solution: Let $a^2 = 36$, $b^2 = 4$ then if e is the eccentricity of the ellipse

$$e^2 = \frac{36-4}{36} = \frac{32}{36} = \frac{8}{9} \Rightarrow e = \frac{2\sqrt{2}}{3}$$

Equation of the directrices are

$$x = \pm\,\frac{a}{e} = \pm\,\frac{6\times 3}{2\sqrt{2}} = \pm\,\frac{9}{\sqrt{2}}$$

$$\sqrt{2}x \pm 9 = 0$$

Example 2: The locus of a point whose distance from the point (3, 0) is 3/5 times its distance from the line $x = p$ is an ellipse with centre at the origin. The value of p is

(a) 5 (b) 7
(c) $\frac{25}{3}$ (d) $\frac{25}{9}$

Ans. (c)

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $e = \frac{3}{5}$ is the eccentricity Focus is $(3, 0) = (ae, 0)$ $\Rightarrow a = 5$

Equation of the directrix is $x = \frac{a}{e} = \frac{5\times 5}{3} = \frac{25}{3}$ which is same as $x = p \Rightarrow p = \frac{25}{3}$.

Example 3: An equation of the ellipse whose length of the major axis is 10 and foci are (± 2, 0) is

(a) $\frac{x^2}{25} + \frac{y^2}{21} = 1$ (b) $\frac{x^2}{25} + \frac{y^2}{4} = 1$
(c) $\frac{x^2}{25} + \frac{y^2}{16} = 1$ (d) $\frac{x^2}{29} + \frac{y^2}{25} = 1$

Ans. (a)

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
where $2a = 10 \Rightarrow a = 5$

$$ae = 2 \Rightarrow e = \frac{2}{5}$$

$$b^2 = a^2(1 - e^2) = 25\left(1 - \frac{4}{25}\right) = 21$$

Required equation of the ellipse is

$$\frac{x^2}{25} + \frac{y^2}{21} = 1$$

Example 4: If $F_1 = (3, 0)$, $F_2 = (-3, 0)$ and P is any point on the curve $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals

(a) 8 (b) 6
(c) 10 (d) 12

Ans. (c)

Solution: Equation of the curve is $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Here $a^2 = 25,\ b^2 = 16,\ e^2 = \frac{25-16}{25} = \frac{9}{25}$

Foci are $\left(\pm 5 \times \frac{3}{5}, 0\right) = (\pm 3, 0)$

$\Rightarrow$ F_1, F_2 are the foci of the ellipse.

So $PF_1 + PF_2 = e \times$ distance between the directrices

$$= e \times \frac{2a}{e} = 2a = 10$$

Example 5: Equation of a common tangent to the circle $x^2 + y^2 = 16$, parabola $x^2 = y - 4$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 0$ is

(a) $x = 4$ (b) $x = -4$
(c) $y = 4$ (d) $y = 5$

Ans. (c)

Solution: From the figure we find $y = 4$ is a common tangent to the three curves.

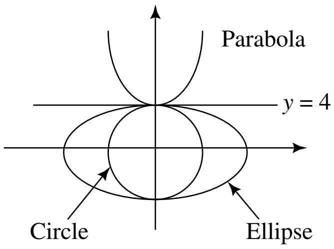


Fig. 19.4

Example 6: If $y = x + c$ is a normal to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, then c^2 is equal to

(a) $\frac{13}{25}$ (b) $\frac{25}{13}$
(c) $\frac{25}{9}$ (d) $\frac{13}{4}$

Ans. (b)

Solution: Any point on the ellipse is $P(3\cos\theta, 2\sin\theta)$ equation of the normal at P is
$3.x \sec\theta - 2.y \operatorname{cosec}\theta = 9 - 4$
Comparing it with $y = x + c$, we get

$$\frac{1}{3\sec\theta} = \frac{1}{2\operatorname{cosec}\theta} = \frac{-c}{5}$$

$$\Rightarrow \quad \cos\theta = \frac{-3c}{5}, \sin\theta = \frac{-2c}{5}.$$

$$\Rightarrow \quad 9c^2 + 4c^2 = 25 \text{ as } \cos^2\theta + \sin^2\theta = 1$$

$$\Rightarrow \quad c^2 = \frac{25}{13}$$

Example 7: If $F_1(-3, 4)$ and $F_2(2, 5)$ are the foci of an ellipse passing through the origin, then the eccentricity of the ellipse is

(a) $\frac{\sqrt{29}}{5+\sqrt{26}}$ (b) $\frac{\sqrt{21}}{5+\sqrt{29}}$
(c) $\frac{\sqrt{26}}{5+\sqrt{29}}$ (d) $\frac{\sqrt{29}}{5+\sqrt{21}}$

Ans. (c)

Solution: If $2a$ is the length of the major axis and e is the eccentricity of the ellipse
then $F_1F_2 = 2ae$

$$\Rightarrow \quad \sqrt{26} = 2ae \tag{1}$$

Also $OF_1 + OF_2 = 2a$, O being the origin.
(As O lies on the ellipse)

$$\Rightarrow \sqrt{25} + \sqrt{29} = 2a \tag{2}$$

From (1) and (2)

$$e = \frac{\sqrt{26}}{5+\sqrt{29}}$$

Example 8: E_1: $\frac{x^2}{81} + \frac{y^2}{b^2} = 1$ and E_2: $\frac{x^2}{b^2} + \frac{y^2}{49} = 1$ are two ellipses having the same eccentricity if b^2 is equal to

(a) 32 (b) 63
(c) 65 (d) 64

Ans. (b)

Solution: We have $\frac{81-b^2}{81} = \frac{b^2-49}{b^2}$

$$\Rightarrow \quad b^4 = 49 \times 81 \Rightarrow b^2 = 63.$$

Example 9: If the normal at any point P on the ellipse $\frac{x^2}{64} + \frac{y^2}{36} = 1$ meets the major axis at G_1 and the minor axis at G_2 then the ratio of PG_1 and PG_2 is equal to

(a) 6 : 8 (b) 8 : 6
(c) 9 : 16 (d) 16 : 9

Ans. (c)

Solution: Equation of the normal at $P(8\cos\theta, 6\sin\theta)$ is $8x\sec\theta - 6y\operatorname{cosec}\theta = 64 - 36 = 28$.

So G_1 is $\left(\frac{28}{8}\cos\theta, 0\right)$ and G_2 is $\left(0, \frac{-28}{6}\sin\theta\right)$

$$(PG_1)^2 = \left(8 - \frac{28}{8}\right)^2 \cos^2\theta + 36\sin^2\theta$$

$$= \frac{36}{64}(36\cos^2\theta + 64\sin^2\theta)$$

$$(PG_2)^2 = 64\cos^2\theta + \left(6 + \frac{28}{6}\right)^2 \sin^2\theta$$

$$= \frac{64}{36}(36\cos^2\theta + 64\sin^2\theta)$$

$$\left(\frac{PG_1}{PG_2}\right)^2 = \frac{36}{64} \times \frac{36}{64} \Rightarrow \frac{PG_1}{PG_2} = \frac{36}{64} = \frac{9}{16}.$$

Example 10: The tangent at the point $P\left(\frac{4}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$ to the ellipse $9x^2 + 16y^2 = 144$ meets the axis of x at A and the axis of y at B. If C is the centre of the ellipse, then area of the $\Delta\,ABC$ is (in sq. units)

(a) 12 (b) 16
(c) 9 (d) 24

Ans. (a)

Solution: Ellipse is $\frac{x^2}{16}+\frac{y^2}{9} = 1$.

Equation of the tangent at $P\left(\frac{4}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$ is

$$\frac{x\cdot 4}{16\sqrt{2}}+\frac{y\cdot 3}{9\sqrt{2}} = 1$$

coordinate of A are $(4\sqrt{2},0)$

and of B are $(0,3\sqrt{2})$

C is (0, 0)

Area of the triangle $ABC = \frac{1}{2}\times 4\sqrt{2}\times 3\sqrt{2} = 12$ sq. units

Example 11: A point on the ellipse $4x^2 + 9y^2 = 36$. Where the normal is parallel to the line $4x - 2y - 5 = 0$ is:

(a) $\left(\frac{9}{5},\frac{8}{5}\right)$ (b) $\left(\frac{8}{5},-\frac{9}{5}\right)$
(c) $\left(-\frac{9}{5},\frac{8}{5}\right)$ (d) $\left(\frac{8}{5},\frac{9}{5}\right)$

Ans. (a)

Solution: Slope of the normal is 2, so slope of the tangent is $-\frac{1}{2}$.

Equation of the tangent is $y = -\frac{1}{2}x \pm \sqrt{9\times\frac{1}{4}+4}$

$$\Rightarrow\quad x + 2y = \pm 5 \qquad (1)$$

If it touches at (x_1, y_1), then equation of the tangent at (x_1, y_1) to the ellipse is

$$\frac{xx_1}{9}+\frac{yy_1}{4} = 1 \qquad (2)$$

Comparing (1) and (2) we get

$$\frac{x_1}{9} = \frac{y_1}{8} = \pm\frac{1}{5}$$

$$\Rightarrow\quad (x_1, y_1) = \left(\frac{9}{5},\frac{8}{5}\right)$$

Example 12: The equation of the parabola whose vertex is at the centre of the ellipse $\frac{x^2}{25}+\frac{y^2}{16} = 1$ and the focus coincide with the focus of the ellipse on the positive side of the major axis of the ellipse is

(a) $y^2 = 3x$ (b) $y^2 = 4x$
(c) $y^2 = 5x$ (d) $y^2 = 12\,x$

Ans. (d)

Solution: Centre of the ellipse is the origin and focus is $(ae, 0) = \left(\sqrt{a^2-b^2},0\right) = (3, 0)$. Required equation of the parabola is

$$y^2 = 4\times 3x \Rightarrow y^2 = 12x.$$

Example 13: Consider the ellipses E_1: $\frac{x^2}{9}+\frac{y^2}{5} = 1$ and E_2: $\frac{x^2}{5}+\frac{y^2}{9} = 1$. Both the ellipses have

(a) the same foci (b) same major axis
(c) the same minor axis (d) the same eccentricity

Ans. (d)

Solution: Major axis and minor axis of E_1 are respectively the minor axis and major axis of E_2

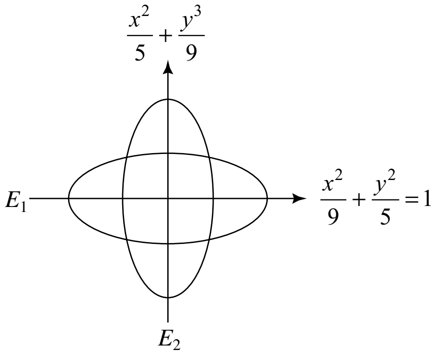

$\frac{x^2}{5}+\frac{y^3}{9}$
E_1
$\frac{x^2}{9}+\frac{y^2}{5}=1$
E_2

Fig. 19.5

Foci of E_1 are (±2, 0)
Foci of E_2 are (0, ±2)

Eccentricity of both $= \sqrt{\frac{9-5}{9}} = \frac{2}{3}$

Example 14: If the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is 4π, then the maximum area of a rectangle inscribed in the ellipse in sq. units is

(a) 4 (b) 8
(c) 16 (d) 32

Ans. (b)

Solution: Area of the ellipse

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{b}{a}\left|\int_0^a 4\sqrt{a^2 - x^2}\,dx\right|$$

$$= \frac{b}{a}\ \text{(Area of the circle of radius } a\text{)}$$

$$= \frac{b}{a} \times \pi a^2 = \pi ab = 4\pi\ \text{(given)}$$

$\Rightarrow ab = 4$

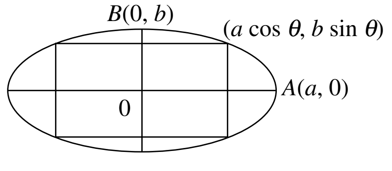


Fig. 19.6

Area of a rectangle inscribed in the ellipse is $4a \cos \theta \times b \sin \theta = 2ab \sin 2\theta$ which is maximum when $\sin 2\theta = 1$

Hence the maximum area of the rectangle is $2ab = 8$ (sq. units)

Example 15: Equation of a circle described on the Latus rectum of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ as a diameter with centre on the + ve x-axis is

(a) $25x^2 + 25y^2 - 150y - 31 = 0$
(b) $25x^2 + 25y^2 - 150x + 31 = 0$
(c) $25x^2 + 25y^2 - 150x - 31 = 0$
(d) $25x^2 + 25y^2 - 150y + 31 = 0$

Ans. (c)

Solution: Coordinates of the centre which is the focus of the ellipse on the +ve x-axis is $(\sqrt{25-16}, 0)$ i.e (3, 0) and radius of the circle is equal to the length of the semi-latus rectum i.e. $\frac{16}{5}$. So the required equation of the circle is

$$(x - 3)^2 + y^2 = \left(\frac{16}{5}\right)^2$$

or $25x^2 + 25y^2 - 150x - 31 = 0$

LEVEL 1

Straight Objective Type Questions

Example 16: The eccentricity of an ellipse with its centre at the origin is ½. If one of the directrix is $x = 4$, then the equation of the ellipse is

(a) $4x^2 + 3y^2 = 12$ (b) $3x^2 + 4y^2 = 12$
(c) $3x^2 + 4y^2 = 1$ (d) $4x^2 + 3y^2 = 1$

Ans. (b)

Solution: Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ then } \frac{a^2 - b^2}{a^2} = \frac{1}{4} = e^2$$

and directrix $x = a/e = 4 \Rightarrow a = 2, b^2 = 3$
and the ellipse is $x^2/4 + y^2/3 = 1$

Example 17: In an ellipse, the distance between the foci is 6 and minor axis is 8, then the eccentricity is

(a) $1/\sqrt{5}$ (b) 3/5
(c) 1/2 (d) 4/5

Ans. (b)

Solution: Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $b = 4$.

If e is the eccentricity then $2ae = 6$
$\Rightarrow a^2e^2 = 9$
$\Rightarrow a^2 - b^2 = 9 \Rightarrow a^2 = 25$
$\Rightarrow a = 5$ and $e = 3/5$

Example 18: The locus of the point of intersection of the tangents to the ellipse $x^2/a^2 + y^2/b^2 = 1$ which are at right angles is

(a) a circle (b) a parabola
(c) an ellipse (d) a hyperbola

Ans. (a)

Solution: Equation of the tangent to the given ellipse with slope m is

$$y = mx + \sqrt{a^2m^2 + b^2} \qquad (1)$$

and the equation of tangent perpendicular to (1) is

$$my + x = \sqrt{a^2 + b^2m^2} \qquad (2)$$

Squaring and adding (1) and (2) to eliminate m, we get

$$(y - mx)^2 + (my + x)^2 = a^2m^2 + b^2 + a^2 + b^2 m^2$$

$$\Rightarrow \quad (x^2 + y^2)(1 + m^2) = (a^2 + b^2)(1 + m^2)$$

$$\Rightarrow \quad x^2 + y^2 = a^2 + b^2$$

which is a circle.

Note

- ❑ The circle $x^2 + y^2 = a^2 + b^2$ is called the *Director Circle* of the ellipse $x^2/a^2 + y^2/b^2 = 1$, whose centre is at the centre of the ellipse and whose radius is equal to the length of the line joining the end points of the major axis minor axis of the ellipse.

◉ **Example 19:** The normal at an end of a latus rectum of the ellipse $x^2/a^2 + y^2/b^2 = 1$ passes through an end of the minor axis if

(a) $e^4 + e^2 = 1$ (b) $e^3 + e^2 = 1$

(c) $e^2 + e = 1$ (d) $e^3 + e = 1$

Ans. (a)

◎ **Solution:** Let an end of a latus rectum be $\left(ae, b\sqrt{1-e^2}\right)$,

then the equation of the normal at this end is

$$\frac{x - ae}{ae/a^2} = \frac{y - b\sqrt{1-e^2}}{b\sqrt{1-e^2}/b^2}$$

It will pass through the end $(0, -b)$ if

$$-a^2 = \frac{-b^2\left(1+\sqrt{1-e^2}\right)}{\sqrt{1-e^2}} \quad \text{or} \quad \frac{b^2}{a^2} = \frac{\sqrt{1-e^2}}{1+\sqrt{1-e^2}}$$

or $(1 - e^2)\left[1+\sqrt{1-e^2}\right] = \sqrt{1-e^2}$

or $\sqrt{1-e^2} + 1 - e^2 = 1$ or $e^4 + e^2 = 1$.

◉ **Example 20:** The locus of the middle points of the portions of the tangents of the ellipse $x^2/a^2 + y^2/b^2 = 1$ included between the axis is the curve.

(a) $x^2/a^2 + y^2/b^2 = 4$ (b) $a^2/x^2 + b^2/y^2 = 4$

(c) $a^2x^2 + b^2y^2 = 4$ (d) $b^2x^2 + a^2y^2 = 4$

Ans. (b)

◎ **Solution:** Equation of a tangent to the ellipse can be written as $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ which meets the axes at A $(a/\cos\theta, 0)$ and B $(0, b/\sin\theta)$. If (h, k) is the middle point of AB, then

$$h = a/2\cos\theta, \quad k = b/2\sin\theta$$

Eliminating θ we get $(a/2h)^2 + (b/2k)^2 = 1$

$\Rightarrow$ locus of P (h, k) is $a^2/x^2 + b^2/y^2 = 4$.

◉ **Example 21:** In a model, it is shown that an arc of a bridge in semi-elliptical with major axis horizontal. If the length of the base is 9 m and the highest part of the bridge is 3 m from the horizontal; the best approximation of the height of the arch at 2 m from the centre of the base is

(a) 11/4 *m* (b) 8/3 *m*

(c) 7/2 *m* (d) 2 *m*

Ans. (b)

◎ **Solution:** Let the equation of the semi elliptical arc be $x^2/a^2 + y^2/b^2 = 1$ $(y > 0)$.

Length of the major axis $= 2a = 9 \Rightarrow a = 9/2$

Length of the semi minor axis $= b = 3$.

So the equation of the arc becomes $\frac{4x^2}{81} + \frac{y^2}{9} = 1$

If $x = 2$ then $y^2 = \frac{65}{9} \Rightarrow y = \frac{1}{3}\sqrt{65} = \frac{8}{3}$ approximately.

◉ **Example 22:** The locus of the foot of the perpendicular drawn from the centre to any tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

(a) a circle (b) an ellipse

(c) a hyperbola (d) none of these

Ans. (d)

◎ **Solution:** Equation of any tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2} \tag{1}$$

Equation of the line through the centre (0, 0) perpendicular to (1) is

$$y = (-1/m)\,x \tag{2}$$

Eliminating m from (1) and (2) we get the required locus of the foot of the perpendicular

as $$y = -\frac{x^2}{y} + \sqrt{a^2\frac{x^2}{y^2} + b^2}$$

or $$(x^2 + y^2)^2 = a^2x^2 + b^2y^2.$$

which does not represent a circle, an ellipses or a hyperbola.

◉ **Example 23:** Sum of the focal distance of any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ is equal to the length of the

(a) major axis (b) minor axis

(c) latus rectum (d) none of these

Ans. (a)

◎ **Solution:** If P be any point on the ellipse, then distance of P from a focus is e times its distance from the corresponding directrix (by definition of ellipse). So that the sum of the focal distance is equal to e times the distance between the two directrices of the ellipse, $x = \pm a/e$. Hence the required distance $= e \times 2a/e = 2a$ = the length of the major axis of the ellipse.

◉ **Example 24:** The line passing through the extremity A of major axis and extremity B of the minor axes of the ellipse $9x^2 + 16y^2 = 144$ meets the circle $x^2 + y^2 = 16$ at the point P. Then the area of the triangle OAP, O being the origin (in square units) is

(a) 96/25 (b) 192/25

(c) 48/25 (d) 96/50

Ans. (b)

Solution: Coordinates of A are $(4, 0)$ and of B are $(0, 3)$. So the equation of AB is $\frac{x}{4}+\frac{y}{3} = 1$. Which meets the circle $x^2 + y^2 = 16$ at points whose y coordinate is given by

$$y^2 + \left(\frac{12-4y}{3}\right)^2 = 16$$

$\Rightarrow$ $y = 0$ or $y = 96/25$

$y = 0$ corresponds to the point A.

So the y-coordinate of P is 96/25

Area of the triangle OAP

$= (1/2)\ OA \times (y\text{-coordinate of } P)$

$= (1/2) \times 4 \times (96/25)$

$= 192/25.$

Example 25: The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point $(4, 0)$. Then equation of the ellipse is

(a) $4x^2 + 48y^2 = 48$ (b) $4x^2 + 6y^2 = 48$

(c) $x^2 + 16y^2 = 16$ (d) $x^2 + 12y^2 = 16$

Ans. (d)

Solution: Let the equation of the required ellipse by $x^2/a^2 + y^2/b^2 = 1$

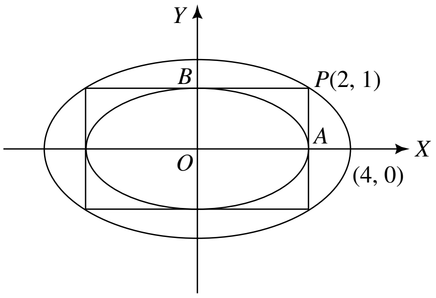


Fig. 19.7

It passes through $(4, 0)$ and $(2, 1)$ as $OB = 1$ and $OA = 2$ are respectively the lengths of the semi-minor axis and semi-major axis of the given ellipse, coordinates of P are $(2, 1)$

So, $\frac{16}{a^2} = 1$ and $\frac{4}{a^2}+\frac{1}{b^2} = 1$

$\Rightarrow$ $a^2 = 16$ and $b^2 = 4/3$.

and the required equation is

$$\frac{x^2}{16}+\frac{3y^2}{4} = 1$$

$\Rightarrow$ $x^2 + 12y^2 = 16.$

Example 26: A parabola has its latus rectum along PQ, where $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 > 0$, $y_2 > 0$ are the end points of the latus rectums of the ellipse $\frac{x^2}{4}+\frac{y^2}{1} = 1$. Coordinates of the focus of the parabola are

(a) $(0, -1/2)$ (b) $(0, 0)$

(c) $(0, 1/2)$ (d) $(0, 1)$

Ans. (c)

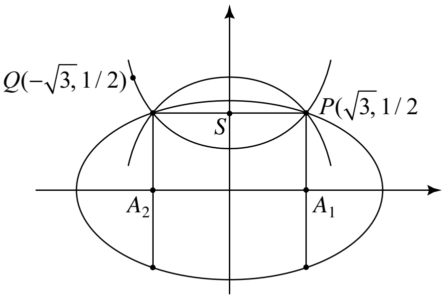


Fig. 19.8

Solution: Eccentricity of the ellipse is

$$e^2 = \frac{a^2-b^2}{a^2} = \frac{4-1}{4} = \frac{3}{4}$$

$\Rightarrow$ $e = \sqrt{3}/2$

$\Rightarrow$ coordinates of the foci of the ellipse are

$(\pm 2\times\sqrt{3}/2, 0) = (\pm\sqrt{3}, 0)$

Length of the latus rectum of the ellipse is $\frac{2\times 1}{2} = 1$. So the coordinates of P are $(\sqrt{3}, 1/2)$ and of Q are $(-\sqrt{3}, 1/2)$. Focus of the parabola is the mid point of PQ i.e., $(0, 1/2)$

Note

❑ There are two such parabola.

Example 27: Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9}+\frac{y^2}{4} = 1$ touching the ellipse at point A and B. The equation of the locus of the point whose distances from the point P and the line AB are equal is

(a) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

(b) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(c) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$

(d) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Ans. (a)

Solution: AB being the chord of contact of the ellipse from $P(3, 4)$ has its equation.

$$\frac{3x}{9}+\frac{4y}{4} = 1 \Rightarrow x + 3y = 3$$

If (h, k) is any point on the locus, then

$$\sqrt{(h-3)^2+(k-4)^2} = \left|\frac{h+3k-3}{\sqrt{1+9}}\right|$$

$\Rightarrow \quad 10(h^2 + k^2 - 6h - 8k + 25) = (h + 3k - 3)^2$

Locus of (h, k) is

$9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

Example 28: Equation of the ellipse whose axes are the axes of coordinates, which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{2/5}$ is

(a) $5x^2 + 3y^2 - 32 = 0$ (b) $3x^2 + 5y^2 - 32 = 0$
(c) $5x^2 + 3y^2 - 48 = 0$ (d) $3x^2 + 5y^2 - 15 = 0$

Ans. (b)

Solution: Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We have $b^2 = a^2(1 - e^2) = a^2(1 - 2/5) = 3a^2/5$

Since it passes through the point $(-3, 1)$

$$\frac{9}{a^2} + \frac{1}{b^2} = 1 \Rightarrow \frac{9}{a^2} + \frac{5}{3a^2} = 1$$

$$\Rightarrow \quad a^2 = 9 + 5/3 = 32/3. \Rightarrow b^2 = 32/5$$

So the required equation is

$$\frac{x^2}{32/3} + \frac{y^2}{32/5} = 1 \text{ or } 3x^2 + 5y^2 = 32$$

Example 29: An ellipse is drawn by taking a diameter of the circle $(x - 1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of circle $x^2 + (y - 2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is

(a) $4x^2 + y^2 = 8$ (b) $x^2 + 4y^2 = 16$
(c) $4x^2 + y^2 = 4$ (d) $x^2 + 4y^2 = 8$

Ans. (b)

Solution: Diameter of $(x - 1)^2 + y^2 = 1$ is $2 \times 1 = 2 = b$ and of $x^2 + (y - 2)^2 = 4$ is $2 \times 2 = 4 = a$

So the required equation of the ellipse is

$$\frac{x^2}{(4)^2} + \frac{y^2}{(2)^2} = 1 \Rightarrow x^2 + 4y^2 = 16$$

Example 30: The ellipse E_1: $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle R. The eccentricity of the ellipse E_2 is

(a) $\sqrt{2}/2$ (b) $\sqrt{3}/2$
(c) 1/2 (d) 3/4

Ans. (c)

Solution: Let the required ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ since it passes through $(0, 4) \Rightarrow b^2 = 16$.

It also passes through the point $(3, 2)$

$$\Rightarrow \frac{9}{a^2} + \frac{4}{16} = 1$$

$\Rightarrow a^2 = 12$

Equation of the ellipse is $\frac{x^2}{12} + \frac{y^2}{16} = 1$

So eccentricity of the ellipse is $\sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

Note Fig ia Similar to the Fig of Example 25.

Example 31: A rod of length 12 cm moves with its ends always touching the coordinate axes. The locus of a point P on the rod, which is 3 cm from the end in contact with x-axis is an ellipse whose eccentricity is:

(a) $\frac{2}{3}$ (b) $\frac{\sqrt{2}}{3}$
(c) $\frac{2\sqrt{2}}{3}$ (d) $\frac{2\sqrt{3}}{2}$

Ans. (c)

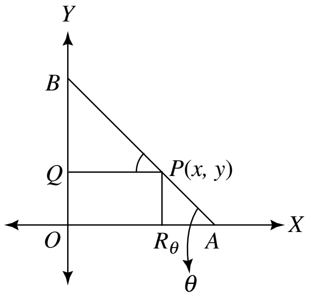


Fig. 19.9

Solution: Let AB be the rod making an angle θ with x-axis and $P(x, y)$ be any point on the rod such that $AP = 3$ cm.

$\Rightarrow PB = AB - AP = 12 - 3 = 9$ cm.

$$\cos\theta = \frac{PQ}{PB} = \frac{x}{9}$$

$$\sin\theta = \frac{PR}{PA} = \frac{y}{3}$$

so eliminating θ we get the required locus as $\frac{x^2}{81} + \frac{y^2}{9} = 1$

which is an ellipse with eccentricity $\sqrt{\frac{81 - 9}{81}} = \frac{2\sqrt{2}}{3}$.

Example 32: Eccentricity of the ellipse whose latus rectum is half of its major axis is:

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$
(c) $\frac{2}{3}$ (d) $\frac{1}{3}$

Ans. (b)

◎ **Solution:** Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Length of the latus rectum = $\frac{2b^2}{a}$ = a, half of major axis

$$\Rightarrow \qquad 2b^2 = a^2 \Rightarrow e^2 = \frac{a^2-b^2}{a^2} = \frac{1}{2}$$

$$\Rightarrow \qquad e = \frac{1}{\sqrt{2}}.$$

Example 33: If the equation $5[(x-2)^2 + (y-3)^2] = (\lambda^2 - 2\lambda + 1)(2x + y - 1)^2$ represents an ellipse then

(a) $\lambda \in (0, 2)$ (b) $\lambda \in (-1, 1)$

(c) $\lambda \in (0, 2) - \{1\}$ (d) $\lambda \in (-1, 1) - \{0\}$

Ans. (c)

◎ **Solution:** Given equation represents an ellipse with eccentricity $\sqrt{\lambda^2 - 2\lambda + 1} = e$

Since $0 < e < 1$

$$\Rightarrow \qquad 0 < |\lambda - 1| < 1$$

$$\Rightarrow \qquad \lambda \in (0, 2) \sim \{1\}$$

Example 34: If the line $lx + my + n = 0$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ at points whose eccentric angles differ by $\frac{\pi}{2}$, then the value of $\frac{a^2l^2 + b^2m^2}{n^2}$ is equal to

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) $\frac{3}{2}$.

Ans. (b)

◎ **Solution:** Let the line meet the ellipse at $P(a\cos\theta, b\sin\theta)$ and $Q\left(a\cos\left(\frac{\pi}{2}+\theta\right), b\sin\left(\frac{\pi}{2}+\theta\right)\right)$. Since they lie on the line $lx + my + n = 0$

$la\cos\theta + mb\sin\theta + n = 0$

and $-la\sin\theta + mb\cos\theta + n = 0$

Eliminating θ by squaring and adding we get

$a^2l^2 + b^2m^2 = 2n^2$

$$\Rightarrow \frac{a^2l^2 + b^2m^2}{n^2} = 2.$$

Example 35: An ellipse has OB as semi minor axis F and F' its foci and the angle FBF' is a right angle. The eccentricity of the ellipse is

(a) $\frac{1}{4}$ (b) $\frac{1}{\sqrt{3}}$

(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2}$

Ans. (c)

◎ **Solution:** Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$F(ae, 0)$, $F'(-ae, 0)$

BF is perpendicular to BF'

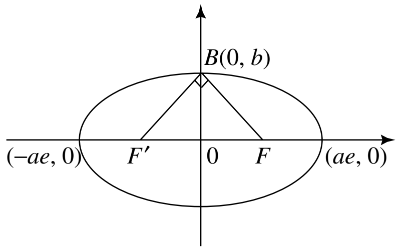


Fig. 19.10

$$\Rightarrow \frac{b}{-ae}\times\frac{b}{ae} = -1 \Rightarrow b^2 = a^2e^2$$

$$\Rightarrow a^2(1-e^2) = a^2e^2 \Rightarrow 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$$

Example 36: The locus of the foot of the perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent is

(a) $(x^2 - y^2)^2 = 6x^2 + 2y^2$

(b) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

(c) $(x^2 + y^2)^2 = 6x^2 + 2y^2$

(d) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

Ans. (c)

◎ **Solution:** Equation of any tangent to the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ is $y = mx \pm \sqrt{6m^2+2}$. If (h, k) is the foot of the perpendicular from (0, 0), the centre of the ellipse on this tangent then $k = mh \pm \sqrt{6m^2+2}$ and $\frac{k}{h}\times m = -1$

Eliminating m, we get $(h^2 + k^2)^2 = 6h^2 + 2k^2$

So, required locus is $(x^2 + y^2)^2 = 6x^2 + 2y^2$

Example 37: The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is:

(a) $\frac{27}{4}$ (b) 18

(c) $\frac{27}{2}$ (d) 27

Ans. (d)

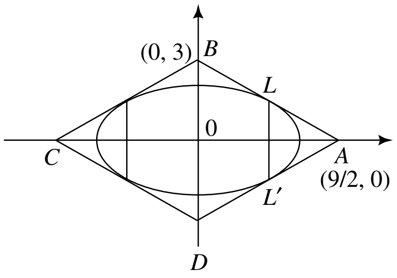


Fig. 19.11

◎ **Solution:** we have $a^2 = 9$, $b^2 = 5$

$$e^2 = 1 - \frac{5}{9} = \frac{4}{9}$$

$$\Rightarrow \qquad e = \frac{2}{3}$$

Coordinates of L are $\left(ae, \frac{b^2}{a}\right) = \left(2, \frac{5}{3}\right)$

Equation of the tangent at L is

$$\frac{2x}{9} + \frac{5}{3}\left(\frac{y}{5}\right) = 1 \Rightarrow 2x + 3y = 9.$$

It meets the axes at $A\left(\frac{9}{2}, 0\right)$ and $B(0, 3)$

From symmetry, we observe that the quadrilateral is a rhombus whose area is 4 × area of the $\Delta\, AOB$.

Hence the required area is $4 \times \frac{9}{2} \times 3 \times \frac{1}{2}$

= 27 sq. units.

◉ **Example 38:** If the distance between the foci of an ellipse is half the length of its latus rectum, then eccentricity of the ellipse is:

(a) $\frac{1}{2}$ (b) $\frac{2\sqrt{2}-1}{2}$

(c) $\sqrt{2}-1$ (d) $\frac{\sqrt{2}-1}{2}$

Ans. (c)

◎ **Solution:** Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then

$$2ae = \frac{1}{2}\left(\frac{2b^2}{a}\right)$$

$$\Rightarrow \qquad b^2 = 2a^2e$$

$$\Rightarrow \qquad a^2(1 - e^2) = 2a^2e$$

$$\Rightarrow \qquad e^2 + 2e = 1 \Rightarrow (e + 1)^2 = 2$$

$$\Rightarrow \qquad e = \sqrt{2} - 1.$$

◉ **Example 39:** The tangent to the ellipse $3x^2 + 16y^2 = 12$, at the point $\left(1, \frac{3}{4}\right)$ intersects the curves $y^2 + x = 0$ at:

(a) no point (b) exactly one point

(c) two distinct point (d) more than two points.

Ans. (b)

◎ **Solution:** Tangent at $\left(1, \frac{3}{4}\right)$ to the ellipse is $\frac{x \cdot 1}{4} + \frac{y\left(\frac{3}{4}\right)}{\frac{3}{4}}$ $= 1 \Rightarrow x + 4y = 4$

which intersects the curve $y^2 + x = 0$ at points for which

$y^2 + (4 - 4y) = 0$

$\Rightarrow (y - 2)^2 = 0 \Rightarrow y = 2$

and the point of intersection is $(-4, 2)$ i.e. exactly one point.

◉ **Example 40:** The line $2x + y = 3$ intersects the ellipse $4x^2 + y^2 = 5$ at two points. The tangents to the ellipse at these two points intersect at the point:

(a) $\left(\frac{5}{6}, \frac{5}{3}\right)$ (b) $\left(\frac{5}{6}, \frac{5}{6}\right)$

(c) $\left(\frac{5}{3}, \frac{5}{6}\right)$ (d) $\left(\frac{5}{3}, \frac{5}{3}\right)$

Ans. (a)

◎ **Solution:** Let the line intersect the ellipse at P and Q and the tangents at P and Q to the ellipse intersect at $R(h, k)$, then PQ is the chord of contact of R w.r.t. the ellipse; so its equation is $4hk + ky = 5$

But the equation of the chord of contact is given as $2x + 3y = 3$

$$\Rightarrow \qquad \frac{4h}{2} = \frac{k}{1} = \frac{5}{3}$$

$$\Rightarrow \qquad h = \frac{5}{6}, k = \frac{5}{3}$$

and the required point is $\left(\frac{5}{6}, \frac{5}{3}\right)$

Assertion-Reason Type Questions

Example 41: E: $2x^2 + 3y^2 = 6$, e: $x^2 + y^2 = 2x + 4y + 4 = 0$

Statement 1: The area of the ellipse E is more than the area of the circle C.

Statement 2: The length of the semi-major axis of E is more than the radius of C.
Ans. (b)

Solution: Equation of the ellipse is $\frac{x^2}{3} + \frac{y^2}{2} = 1$

whose area is $\pi\sqrt{3}\sqrt{2} = \pi\sqrt{6}$. sq. units.

Equation of the circle is
$(x - 1)^2 + (y + 2)^2 = 1$
whose area is π sq. units
So area of $E >$ area of C and thus Statement 1 is true.

Length of the semi major axis of E is $\sqrt{3} > 1$, the radius of C so statement 2 is also true but does not justify statement 1.

Example 42: Statement 1: The line $2x + 3y = 1$ intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{2} = 1$ at P and Q. The tangents to the ellipse at P and Q. intersect at the point (8, 6).

Statement 2: Equation of the chord of contact to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from an external point (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.
Ans. (a)

Solution: Statement 2 is true (see theory).
In statement 1, let the coordinates of the required point be (h, k), then $2x + 3y = 1$ is the chord of contact of the ellipse and by statement 2, its equation is $\frac{hx}{4} + \frac{ky}{2} = 1$. Comparing the two equations, we get $h = 8$, $k = 6$ and the statement 1 is also true.

Example 43: Statement 1: The locus of the centre of a variable circle touching two circles C_1: $(x-2)^2 + (y-1)^2 = 16$ and C_2: $(x - 4)^2 + (y - 1)^2 = 1$ is an ellipse.

Statement 2: If a circle $S_2 = 0$ lies completely inside the circle $S_1 = 0$, then the locus of the centre of a variable circle touching both the circles is an ellipse.

Ans. (a)

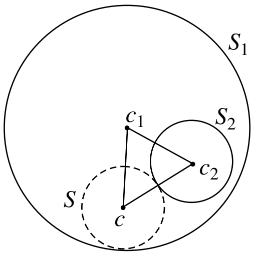


Fig. 19.12

Solution: Let C_1, C_2 be the centres and r_1, r_2 the radii of the circles S_1 and S_2 respectively. C be the centre and r the radius of the circles touching S_1 and S_2.
Then $CC_1 = r_1 - r$ and $CC_2 = r_2 + r$
$\Rightarrow CC_1 + CC_2 = r_1 + r_2$ (constant)
So distance of C from two fixed points C_1 and C is constant and hence by definition of the ellipse locus of C is an ellipse.
In statement 1, the circle C_2 lies inside the circle C_1 and hence by statement 2, the required locus is an ellipse and thus the statement 1 is also true.

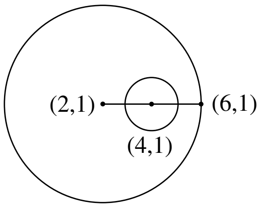


Fig. 19.13

Example 44: Statement 1: If the line $\frac{x}{a} + \frac{y}{b} = \sqrt{2}$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and θ is the eccentric angle of the point of contact, then $\sin^2\theta = \frac{1}{2}$.

Statement 2: If the length of the semi major axis of an ellipse is $\sqrt{2}$ times the length of the semi-minor axis, then the eccentricity e of the ellipse satisfies $2e^2 - 1 = 0$.
Ans. (b)

Solution: Let the coordinates of the point of contact in statement 1 be $(a \cos \theta, b \sin \theta)$. Equation of the tangent at this point to the ellipse is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Comparing with the given line

$\cos\theta = \sin\theta = \dfrac{1}{\sqrt{2}} \Rightarrow \sin^2\theta = \dfrac{1}{2}$

and the statement 1 is true.

In statement 2, let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ s.t $a = \sqrt{2}\, b$

$\Rightarrow a^2 = 2b^2$ and $e^2 = \dfrac{a^2 - b^2}{a^2} = \dfrac{1}{2}$ thus statement 2 is also true but does not justify statement 1.

Example 45: Statement 1: If L and M are the feet of the perpendiculars draw from the foci of the ellipse $\dfrac{x^2}{100} + \dfrac{y^2}{50} = 1$ on any tangent to it, then L and M lie on the circle $x^2 + y^2 = 150$.

Statement 2: The locus of feet of the perpendiculars from the foci upon any tangent is the auxiliary circle of the ellipse.
Ans. (d)

Solution: Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Equation of any tangent is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \tag{1}$$

Equation of the perpendiculars from (± ae, 0) to the tangent (1) is

$$\frac{x\sin\theta}{b} - \frac{y\cos\theta}{a} = \pm\frac{ae\sin\theta}{b} \tag{2}$$

Eliminating θ from (1) and (2) we get the required locus by squaring and adding

$$x^2\left[\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right] + y^2\left[\frac{\sin^2\theta}{b^2} + \frac{\cos^2\theta}{a^2}\right] = 1 + \frac{a^2e^2\sin^2\theta}{b^2}$$

$$\Rightarrow \quad (x^2 + y^2)\left(\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}\right)$$

$$= \left[\frac{b^2 + (a^2 - b^2)\sin^2\theta}{a^2b^2}\right]a^2$$

$$= \left[\frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2}\right]a^2$$

$$\Rightarrow \quad x^2 + y^2 = a^2$$

which is the auxiliary circle of the ellipse. Hence the statement 2 is true. Using it, we find statement 1 is false.

(Note $x^2 + y^2 = 150$ is the director circle of the ellipse $\dfrac{x^2}{100} + \dfrac{y^2}{50} = 1$)

Example 46: The tangent at a point $P(a\cos\alpha, b\sin\alpha)$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ meets the auxiliary circle at A and B such that $\underline{|AOB} = \dfrac{\pi}{2}$, O being the origin.

Statement 1: Equation of the directrices of the ellipse are $x = \pm\, a\sqrt{1 + \sin^2\alpha}$.

Statement 2: Eccentricity e of the ellipse satisfies $e^2\sin^2\alpha + e^2 - 1 = 0$.
Ans. (a)

Solution: Equation of the tangent at P to the ellipse is

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1 \tag{1}$$

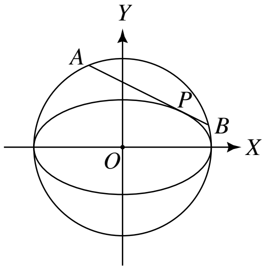


Fig. 19.14

Equation of the auxiliary circle is $x^2 + y^2 = a^2$ (2)

Equation of the pair of lines OA and OB is $x^2 + y^2 = a^2\left(\dfrac{x}{a}\cos\alpha + \dfrac{y}{b}\sin\alpha\right)^2$

(Making (2) homogeneous with the help of (1))
Since $OA \perp OB$

$$(1 - \cos^2\alpha) + \left(1 - \frac{a^2}{b^2}\sin^2\alpha\right) = 0$$

$\Rightarrow b^2 = (a^2 - b^2)\sin^2\alpha \Rightarrow 1 - e^2 = e^2\sin^2\alpha$
$\Rightarrow e^2\sin^2\alpha + e^2 - 1 = 0$
So statement 2 is true.

Using it we find the directrices of the ellipse are $x = \pm\dfrac{a}{e} = \pm a\sqrt{1 + \sin^2\alpha}$ and the statement 1 is also true.

Example 47: The tangent and normal at a point $P(x_1, y_1)$, $x_1 > 0$, $y_1 > 0$ on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ meets the major axis at T and N respectively.

Statement 1: *PN* is the internal bisector of $\underline{|F_1PF_2}$; F_1, F_2 being the foci of the ellipse.

Statement 2: *PT* is constant for all positions of the point *P*.

Ans. (c)

◎ **Solution:** Equation of the tangent at $P(x_1, y_1)$ is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}$ = 1 which meets the major axis at $\left(\frac{a^2}{x_1}, 0\right)$

$$\text{So} \qquad (PT)^2 = \left(x_1 - \frac{a^2}{x_1}\right)^2 + y_1^2$$

$$= \frac{(x_1^2 - a^2)^2}{x_1^2} + y_1^2$$

Which shows that *PT* depends on the position of *P* and thus the statement 2 is false.

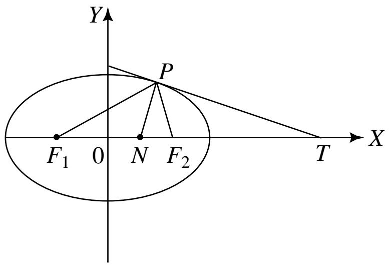


Fig. 19.15

For statement 1, equation of the normal at $P(x_1, y_1)$ is

$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$$

So coordinates of *N* are $(e^2x_1, 0)$

F_1 are $(-ae, 0)$, F_2 are $(ae, 0)$

$$\text{Also} \qquad PF_1 = e\left|\frac{a}{e} + x_1\right|$$

$$= a + ex_1$$

$$PF_2 = a - ex_1$$

$$NF_1 = e^2x_1 + ae = ePF_1$$

$$NF_2 = ae - e^2x_1 = e(PF_2)$$

$\Rightarrow \frac{PF_1}{PF_2} = \frac{NF_1}{NF_2}$ which shows that *PN* bisects the $\underline{|F_1PF_2}$ and the statement 1 is true.

◉ **Example 48: Statement 1:** Equations of the tangents drawn at the ends of the major axis of the ellipse $9x^2 + 5y^2 - 30y = 0$ is $y = 0$, $y = 6$.

Statement 2. The tangents drawn at the ends of the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ = 1 are always parallel to the *y*-axis.

Ans. (c)

◎ **Solution:** Equation of the ellipse in statement 1 is: $\frac{x^2}{5}+\frac{(y-3)^2}{9}$ = 1 whose centre is (0, 3) and the ends of the major axis are (0, 0) and (6, 0) as the length of semi major axis is 3.

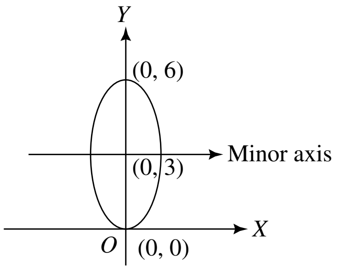


Fig. 19.16

The tangents at these ends are $y = 6$ and $y = 0$.

Statement 1: is true.

Statement 2: is false as the tangents at the ends of the major axis are parallel to *y*-axis only when $a > b$.

◉ **Example 49: Statement 1:** If the length of the latus rectum of an ellipse is $\frac{1}{3}$ of the major axis, then the eccentricity of the ellipse is $\sqrt{\frac{2}{3}}$.

Statement 2: If a focus of an ellipse is at the origin, directrix is the line $x = 4$ and the eccentricity is $\sqrt{\frac{2}{3}}$, then the length of the semi major axis is $4\sqrt{6}$.

Ans. (b)

◎ **Solution:** In statement 1, if the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ = 1, then $\frac{2b^2}{a} = \frac{1}{3}(2a)$

$\Rightarrow \frac{b^2}{a^2} = \frac{1}{3} = 1 - e^2$, *e* being the eccentricity

$\Rightarrow e = \sqrt{\frac{2}{3}}$ and the statement 1 is true

In statement 2, the equation of the ellipse is

$$x^2 + y^2 = \frac{2}{3}\ (4 - x)^2$$

$$\Rightarrow 3(x^2 + y^2) = 2(16 - 8x + x^2)$$

$$\Rightarrow x^2 + 16x + 3y^2 = 32$$

$$\Rightarrow \frac{(x+8)^2}{96} + \frac{y^2}{32} = 1$$

Length of the semi major axis = $\sqrt{96} = 4\sqrt{6}$.

Thus statement 2 is also true but does not justify statement 1.

Example 50: Statement 1: An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement 2: If the line $y = mx + 4\sqrt{3}/m$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$

Ans. (a)

Solution: Equation of a tangent to $y^2 = 16\sqrt{3}x$ is $y = mx + 4\sqrt{3}/m$.

And to $\frac{x^2}{2} + \frac{y^2}{4} = 1$ is $x = m_1 y + \sqrt{4m_1^2 + 2}$

or $$y = \frac{1}{m_1}x - \sqrt{4 + \frac{2}{m_1^2}}, m = \frac{1}{m_1}$$

and $$\left(\frac{4\sqrt{3}}{m}\right)^2 = \left[-\sqrt{4 + \frac{2}{m_1^2}}\right]^2$$

$$\Rightarrow \quad \frac{48}{m^2} = 4 + 2m^2$$

$$\Rightarrow \quad m^4 + 2m^2 - 24 = 0$$

$$\Rightarrow \quad m^2 = 4 \Rightarrow m = \pm 2.$$

Showing that both the statement are true and statement 2 is a correct explanation for statement 1.

LEVEL 2

Straight Objective Type Questions

Example 51: If p is the length of the perpendicular from a focus upon the tangent at any point P of the ellipse $x^2/a^2 + y^2/b^2 = 1$ and r is the distance of P from the focus, then $\frac{2a}{r} - \frac{b^2}{p^2} =$

(a) – 1 (b) 0
(c) 1 (d) 2

Ans. (c)

Solution: The equation of the tangent at $P(a \cos \theta, b \sin \theta)$ to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ length of the perpendicular from the focus $(ae, 0)$ on the line is

$$p = \left|\frac{e\cos\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right| = \left|\frac{ab(e\cos\theta - 1)}{\sqrt{b^2\cos^2\theta + a^2(1 - \cos^2\theta)}}\right|$$

$$= \left|\frac{ab(e\cos\theta - 1)}{\sqrt{a^2 - a^2e^2\cos^2\theta}}\right| = b\sqrt{\frac{1 - e\cos\theta}{1 + e\cos\theta}}$$

$$\Rightarrow \quad \frac{b^2}{p^2} = \frac{1 + e\cos\theta}{1 - e\cos\theta}$$

Now $$r^2 = (ae - a\cos\theta)^2 + b^2\sin^2\theta$$
$$= a^2[(e - \cos\theta)^2 + (1 - e^2)\sin^2\theta]$$
$$= a^2[e^2\cos^2\theta - 2e\cos\theta + 1]$$
$$= a^2(1 - e\cos\theta)^2$$
$$\Rightarrow \quad r = a(1 - e\cos\theta)$$

Now $\frac{2a}{r} - \frac{b^2}{p^2} = \frac{2}{1 - e\cos\theta} - \frac{1 + e\cos\theta}{1 - e\cos\theta} = 1$

Example 52: If $y = x$ and $3y + 2x = 0$ are the equations of a pair of conjugate diameters of an ellipse, then the eccentricity of the ellipse is

(a) $\sqrt{2/3}$ (b) $1/\sqrt{3}$
(c) $1/\sqrt{2}$ (d) $2/\sqrt{5}$

Ans. (b)

Solution: Let the equation of the ellipse be
$x^2/a^2 + y^2/b^2 = 1$
Slope of the given diameters are $m_1 = 1$, $m_2 = -2/3$.
$\Rightarrow m_1 m_2 = -2/3 = -b^2/a^2$
[using the condition of conjugacy of two diameters]
$3b^2 = 2a^2 \Rightarrow 3a^2(1 - e^2) = 2a^2$,

$1 - e^2 = 2/3 \Rightarrow e^2 = 1/3 \Rightarrow e = 1/\sqrt{3}$

Example 53: If α, β are the eccentric angles of the extremities of a focal chord of the ellipse $x^2/16 + y^2/9 = 1$, then $\tan(\alpha/2)\tan(\beta/2) =$

(a) $\frac{\sqrt{7}+4}{\sqrt{7}-4}$ (b) $-\frac{9}{23}$

(c) $\frac{\sqrt{5}-4}{\sqrt{5}+4}$ (d) $\frac{8\sqrt{7}-23}{9}$

Ans. (d)

Solution: The eccentricity $e = \sqrt{1-\frac{9}{16}} = \frac{\sqrt{7}}{4}$.

Let P ($4\cos\alpha$, $3\sin\alpha$) and Q ($4\cos\beta$, $3\sin\beta$) be a focal chord of the ellipse passing through the focus at $(\sqrt{7},0)$.

Then $$\frac{3\sin\beta}{4\cos\beta-\sqrt{7}} = \frac{3\sin\alpha}{4\cos\alpha-\sqrt{7}}$$

$$\Rightarrow \quad \frac{\sin(\alpha-\beta)}{\sin\alpha-\sin\beta} = \frac{\sqrt{7}}{4}$$

$$\Rightarrow \quad \frac{\cos[(\alpha-\beta)/2]}{\cos[(\alpha+\beta)/2]} = \frac{\sqrt{7}}{4}$$

$$\Rightarrow \quad \tan\left(\frac{\alpha}{2}\right)\tan\left(\frac{\beta}{2}\right) = \frac{\sqrt{7}-4}{\sqrt{7}+4} = \frac{23-8\sqrt{7}}{-9}$$

Example 54: If an ellipse slides between two perpendicular straight lines, then the locus of its centre is

(a) a parabola (b) an ellipse

(c) a hyperbola (d) a circle

Ans. (d)

Solution: Let $2a$, $2b$ be the length of the major and minor axes respectively of the ellipse. If the ellipse slides between two perpendicular lines, the point of intersection P of these lines being the point of intersection of perpendicular tangents lies on the Director circle of the ellipse. This means that the centre C of the ellipse is always at a constant distance $\sqrt{a^2+b^2}$ from P. Hence the locus of C is a circle.

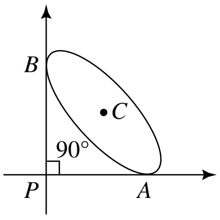


Fig. 19.17

Example 55. The locus of the points of intersection of the tangents at the extremities of the chords of the ellipse $x^2 + 2y^2 = 6$ which touch the ellipse $x^2 + 4y^2 = 4$ is

(a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 6$

(c) $x^2 + y^2 = 9$ (d) none of these

Ans. (c)

Solution: We can write $x^2 + 4y^2 = 4$ as $\frac{x^2}{4}+\frac{y^2}{1} = 1$ (i)

Equation of a tangent to the ellipse (i) is

$$\frac{x}{2}\cos\theta + y\sin\theta = 1 \qquad \text{(ii)}$$

Equation of the ellipse $x^2 + 2y^2 = 6$ can be written as

$$\frac{x^2}{6}+\frac{y^2}{3} = 1 \qquad \text{(iii)}$$

Suppose (ii) meets the ellipse (iii) at P and Q and the tangents at P and Q to the ellipse (iii) intersect at (h, k), then (ii) is the chord of contact of (h, k) with respect to the ellipse (iii) and thus its equation is $\frac{hx}{6}+\frac{ky}{3} = 1$ (iv)

Since (ii) and (iv) represent the same line

$$\frac{h/6}{(\cos\theta)/2} = \frac{k/3}{\sin\theta} = 1$$

$\Rightarrow$ $h = 3\cos\theta$, $k = 3\sin\theta$.

and the locus of (h, k) is $x^2 + y^2 = 9$

Example 56: The sum of the squares of the perpendiculars on any tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ from two points on the minor axis each at a distance $\sqrt{a^2-b^2}$ from the centre is

(a) $2a^2$ (b) $2b^2$

(c) $a^2 + b^2$ (d) $a^2 - b^2$

Ans. (a)

Solution: The eccentricity e of the given ellipse is given by $e^2 = 1 - b^2/a^2 \Rightarrow a^2 - b^2 = a^2e^2$. So the point on the minor axis, i.e. y-axis at a distance $\sqrt{a^2-b^2}$ from the centre (0, 0) of the ellipse are $(0, \pm ae)$.

The equation of the tangent at any point ($a\cos\theta$, $b\sin\theta$) on the ellipse is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

So the required sum is

$$\left[\frac{\frac{ae\sin\theta}{b}-1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}\right]^2 + \left[\frac{\frac{-ae\sin\theta}{b}-1}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}\right]^2$$

$$= \frac{(ae\sin\theta-b)^2+(ae\sin\theta+b)^2}{(b^2\cos^2\theta+a^2\sin^2\theta)} \times a^2$$

$$= \frac{2a^2(a^2e^2\sin^2\theta+b^2)}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$= \frac{2a^2[(a^2-b^2)\sin^2\theta + b^2)}{b^2\cos^2\theta + a^2\sin^2\theta} = 2a^2$$

Example 57. $y = mx + c$ is a normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ if c^2 is equal to

(a) $\dfrac{(a^2-b^2)^2}{a^2m^2+b^2}$ (b) $\dfrac{(a^2-b^2)^2}{a^2m^2}$

(c) $\dfrac{(a^2-b^2)^2m^2}{a^2+b^2m^2}$ (d) $\dfrac{(a^2-b^2)^2m^2}{a^2m^2+b^2}$

Ans. (c)

Solution: Equation of a normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$, if it represents, $mx - y = -c$, then

$$\frac{a}{m\cos\theta} = \frac{b}{\sin\theta} = \frac{-(a^2-b^2)}{c}$$

$$\Rightarrow \quad \cos\theta = \frac{ac}{m(a^2-b^2)},\ \sin\theta = -\frac{bc}{a^2-b^2}$$

$$\Rightarrow \quad \frac{c^2}{(a^2-b^2)^2}\left[\frac{a^2}{m^2} + b^2\right] = 1$$

$$\Rightarrow \quad c^2 = \frac{(a^2-b^2)^2m^2}{a^2+b^2m^2}$$

Example 58: A tangent at any point to the ellipse $4x^2 + 9y^2 = 36$ is cut by the tangent at the extremities of the major axis at T and T'. The circle on TT' as diameter passes through the point.

(a) $(0, \sqrt{5})$ (b) $(\sqrt{5}, 0)$

(c) $(2, 1)$ (d) $(0, -\sqrt{5})$

Ans. (b)

Solution: Any point on the ellipse is $P(3\cos\theta, 2\sin\theta)$.

Equation of the tangent at P is $\dfrac{x}{3}\cos\theta + \dfrac{y}{2}\sin\theta = 1$ which meets the tangents $x = 3$ and $x = -3$ at the extremities of the major axis at

$T\left(3, \dfrac{2(1-\cos\theta)}{\sin\theta}\right)$ and $T'\left(-3, \dfrac{2(1+\cos\theta)}{\sin\theta}\right)$

Equation of the circle on TT' as diameter is

$$(x-3)(x+3) + \left(y - \frac{2(1-\cos\theta)}{\sin\theta}\right)\left(y - \frac{2(1+\cos\theta)}{\sin\theta}\right) = 0$$

$\Rightarrow x^2 + y^2 - \dfrac{4}{\sin\theta}\, y - 5 = 0$, which passes through $(\sqrt{5}, 0)$

Example 59: P is a point on the ellipse E: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and P' be the corresponding point on the auxiliary circle C: $x^2 + y^2 = a^2$. The normal at P to E and at P' to C intersect on circle whose radius is

(a) $a + b$ (b) $a - b$

(c) $2a$ (d) $2b$.

Ans. (a)

Solution: $P(a\cos\theta, b\sin\theta)$, $P'(a\cos\theta, a\sin\theta)$

Equation of the normal at P is

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$ (1)

Equation of the normal at P' is

$x\sec\theta - y\operatorname{cosec}\theta = 0$ (2)

From (1) and (2) we get

$(a-b)\, y\operatorname{cosec}\theta = a^2 - b^2$

$\Rightarrow y = (a+b)\sin\theta.$

From (2) we get $x = (a+b)\cos\theta.$

So point of intersection of (1) and (2) lie on a circle of radius $a + b$.

Example 60: If the common tangent in the first quadrant of the circle $x^2 + y^2 = 16$ and the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ meets the coordinate axes at A and B, the length of AB is

(a) $7/\sqrt{3}$ units (b) $14/\sqrt{3}$ units

(c) $4\sqrt{7}$ units (d) 14 units

Ans. (b)

Solution: An equation of the tangent to the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$ is

$$y = mn + \sqrt{25m^2+4} \qquad (1)$$

As the tangent is in the first quadrant $m < 0$.

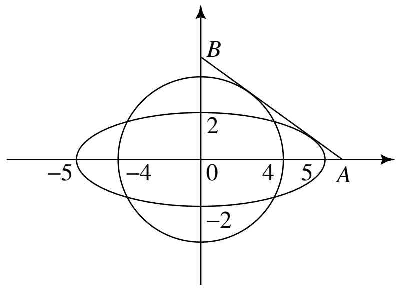


Fig. 19.18

The line will touch the circle $x^2 + y^2 = 16$ if

$$\frac{m(0) - 0 + \sqrt{25m^2+4}}{\sqrt{m^2+1}} = 4$$

$$\Rightarrow \quad 25m^2 + 4 = 16\,(m^2 + 1)$$

$$\Rightarrow \quad m^2 = \frac{4}{3} \Rightarrow m = \frac{-2}{\sqrt{3}}$$

So equation of the tangent is

$$y = \frac{-2}{\sqrt{3}}x + \sqrt{25\left(\frac{4}{3}\right) + 4}$$

$$\Rightarrow \quad 2x + \sqrt{3}y = 4\sqrt{7}$$

which meets the coordinate axes at $A\left(\frac{4\sqrt{7}}{2}, 0\right)$ and $B\left(0, \frac{4\sqrt{7}}{\sqrt{3}}\right)$

so that $(AB)^2 = \left(\frac{4\sqrt{7}}{2}\right)^2 + \left(\frac{4\sqrt{7}}{\sqrt{3}}\right)^2$

$$= (4\sqrt{7})^2 \times \frac{7}{12}$$

$$\Rightarrow \quad AB = \frac{14}{\sqrt{3}} \text{ units}$$

EXERCISE

Concept-based Straight Objective Type Questions

1. Equation of an ellipse with centre at the origin passing through (5, 0) and having eccentricity 2/3 is:
   (a) $4x^2 + 9y^2 = 100$ (b) $9x^2 + 5y^2 = 225$
   (c) $5x^2 + 9y^2 = 125$ (d) $6x^2 + 4y^2 = 150$

2. Equation of a tangent to the ellipse $\frac{x^2}{36} + \frac{y^2}{25} = 1$, passing through the point where a directrix of the ellipse meets the + ve x-axis is
   (a) $5y + \sqrt{11}x = 36$ (b) $6y + \sqrt{11}x = 25$
   (c) $6y + \sqrt{11}x = 36$ (d) $5y - \sqrt{11}x = 36$

3. Equation of a line joining a foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ to a foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$ is,
   (a) $x + y = 4$ (b) $x + y = 5$
   (c) $x + y = 3$ (d) $5x + 3y = 1$

4. If the eccentricity of the ellipse $\frac{x^2}{25} + \frac{y^2}{a^2} = 1$ and $\frac{x^2}{a^2} + \frac{y^2}{16} = 1$ is same, then the value of a^2 is:
   (a) 9 (b) 41
   (c) 15 (d) 20

5. If $y = mx + 4$ is a tangent to the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$, then $625m^4 + 25m^2 - 156$ is equal to
   (a) – 1 (b) 0
   (c) 1 (d) 4

6. Length of a latera recta of the ellipse $\frac{x^2}{81} + \frac{y^2}{63} = 1$ is (in units)
   (a) 9 (b) 7
   (c) 14 (d) 18

7. The normal at the point $\left(3, 2\sqrt{3}\right)$ on the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ meets the major axis of the ellipse at $(\alpha, 0)$, the value of α is:
   (a) $\frac{5}{3}$ (b) $\frac{3}{5}$
   (c) $\frac{2}{3}$ (d) $\frac{3}{2}$

8. In an ellipse, the distance between the foci is 6 and length of semi minor axis is 4, then eccentricity of the ellipse is
   (a) $\frac{2}{5}$ (b) $\frac{3}{5}$
   (c) $\frac{2}{3}$ (d) $\frac{1}{\sqrt{2}}$

9. If F_1 and F_2 are the foci of the ellipse $4x^2 + 9y^2 = 36$, P is a point on the ellipse such that $PF_1 : PF_2 = 2 : 1$; then area of $\Delta\, PF_1F_2$ is:
   (a) 1 sq. unit (b) 2 sq. unit
   (c) 3 sq. unit (d) 4 sq. unit

10. If the eccentric angles of two points P and Q on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ are α, β such that $\alpha + \beta = \frac{\pi}{2}$, then the locus of the point of intersection of the normals at P and Q is

(a) $ax + by = 0$ (b) $ax - by = 0$
(c) $x + y = 0$ (d) $x + y = a + b$

11. $P_1(\theta_1)$ and $P_2(\theta_2)$ are two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ such that $\tan\theta_1 \tan\theta_2 = \frac{-a^2}{b^2}$. The chord joining P_1 and P_2 of the ellipse subtends a right angle at the

(a) focus $(ae, 0)$ (b) focus $(-ae, 0)$
(c) centre $(0, 0)$ (d) vertex $(a, 0)$

12. Coordinates of the foci of the ellipse
$2x^2 + 3y^2 - 4x - 12y + 13 = 0$ are

(a) $\left(1 \pm \frac{1}{\sqrt{6}}, 2\right)$ (b) $\left(2, 1 \pm \frac{1}{\sqrt{6}}\right)$
(c) $(1, 2)$ (d) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$

13. C_1: $x^2 + y^2 = a^2$ and C_2: $x^2 + y^2 = b^2$ are two contentric circles. A line through the centre of these circles intersects them at P and Q respectively. If the lines through P and Q parallel to y-axis and x-axis respectively meet at the point R, then the locus of R is

(a) $x^2 + y^2 = a^2 + b^2$ (b) $x^2 - y^2 = a^2 - b^2$
(c) $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ (d) $\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1$

14. P is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ $(a > b)$ and Q is the point corresponding to P on the auxiliary circle $x^2 + y^2 = a^2$. N is the foot of the perpendicular from P on the major axis of the ellipse. $\frac{PN}{PQ}$ is equal to:

(a) $\frac{b}{a-b}$ (b) $\frac{a}{a-b}$
(c) $\frac{b}{a+b}$ (d) $\frac{a}{a+b}$

15. If the normal at any point P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ meets the axis of x at G and the axis of y at g, then $PG : pg$ is:

(a) $a^2 : b^2$ (b) $a : b$
(c) $b : a$ (d) $b^2 : a^2$

LEVEL 1

Straight Objective Type Questions

16. On the ellipse $4x^2 + 9y^2 = 1$, the point at which the tangent is parallel to the line $8x = 9y$ is

(a) $(2/5, 1/5)$ (b) $(-2/5, 1/5)$
(c) $(-2/5, -1/5)$ (d) none of these

17. The coordinates of the foci of the ellipse $4x^2 + 9y^2 = 1$ are

(a) $(\pm\sqrt{5}/3, 0)$ (b) $(\pm\sqrt{5}/6, 0)$
(c) $(0, \pm\sqrt{5}/3)$ (d) $(0, \pm\sqrt{5}/6)$

18. The distance of the point $(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$ on the ellipse from the centre of the ellipse is 2 if $\theta =$

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) none of these

19. An equation of the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ with eccentricity e at the positive end of the latus rectum is

(a) $x + ey + e^3 a = 0$ (b) $x - ey - ae^3 = 0$
(c) $x - ey + e^3 a = 0$ (d) $x + ey - e^3 a = 0$

20. The circle $x^2 + y^2 = c^2$ contains the ellipse $x^2/a^2 + y^2/b^2 = 1$ if

(a) $c < a$ (b) $c < b$
(c) $c > \max\{a, b\}$ (d) $c > b$

21. In an ellipse, if the lines joining a focus to the extremilites of the minor axis make an equilateral triangle with the minor axis, the eccentricity of the ellipse is

(a) 3/4 (b) $\sqrt{3}/2$
(c) 1/2 (d) 2/3

22. An equilateral triangle is inscribed in the ellipse $x^2 + 3y^2 = 3$ such that one vertex of the triangle is $(0, 1)$ and one altitude of the triangle is along the y-axis. The length of its side is

(a) $4\sqrt{3}/5$
(b) $3\sqrt{3}/5$
(c) $6\sqrt{3}/5$
(d) $2\sqrt{3}$

23. If the equation $\dfrac{x^2}{10-2a}+\dfrac{y^2}{4-2a} = 1$ represents an ellipse, then a lies in the interval
(a) $(-\infty, 5)$
(b) $(2, 5)$
(c) $(-\infty, 2)$
(d) $(5, \infty)$

24. The curve represented by $x = 3(\cos t + \sin t)$, $y = 4(\cos t - \sin t)$ is an ellipse whose eccentricity is e, such that $16e^2 + 7$ is equal to:
(a) 14
(b) 12
(c) 11
(d) 21

25. The eccentricity of an ellipse, with centre at the origin is 2/3. If one of directrices is $x = 6$, then equation of the ellipse is
(a) $\dfrac{x^2}{9}+\dfrac{y^2}{4} = 1$
(b) $\dfrac{x^2}{9}+\dfrac{y^2}{5} = 1$
(c) $5x^2 + 9y^2 = 80$
(d) $3x^2 + 2y^2 = 6$

26. The locus of the foot of the perpendicular from a focus of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$ on any tangent is
(a) $x^2 + y^2 = a^2 + b^2$
(b) $x^2 + y^2 = a^2$
(c) $x^2 + y^2 = b^2$
(d) $x^2 + y^2 = 2a^2$

27. If the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} =1$ and the circle $x^2 + y^2 = r^2$ where $b < r < a$ intersect in four points and the slope of a common tangent to the ellipse and the circle is $\dfrac{b}{a}$, then r^2 is equal to
(a) $\dfrac{2b^2}{a^2+b^2}$
(b) $\dfrac{2a^2b^2}{a^2+b^2}$
(c) $\dfrac{2a^2}{a^2+b^2}$
(d) $\dfrac{2a^2b^2}{a^2-b^2}$

28. The product of the perpendiculars from the foci of the ellipse $\dfrac{x^2}{144}+\dfrac{y^2}{100} = 1$ on any tangent is:
(a) 144
(b) 100
(c) 122
(d) 200

29. The locus of the foot of the perpendicular drawn from the centre on any tangent to the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16} = 1$ is:
(a) $25x^2 + 16y^2 = (x^2 + y^2)^2$
(b) $25x^2 + 16y^2 = 9$
(c) $16x^2 + 25y^2 = 1$
(d) $x^2 + y^2 = 41$

30. If the normal at $P\left(2, \dfrac{3\sqrt{3}}{2}\right)$ meets the major axis of the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9} = 1$ at Q, S_1 and S_2 are the foci of the ellipse, then $S_1Q : S_2Q$ is equal to:
(a) $\dfrac{8}{\sqrt{7}}$
(b) $\dfrac{7+\sqrt{8}}{7-\sqrt{8}}$
(c) $\dfrac{8+\sqrt{7}}{8-\sqrt{7}}$
(d) $\dfrac{\sqrt{7}}{4}$

31. Chord of the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16} = 1$ whose middle point is $\left(\dfrac{1}{2}, \dfrac{2}{5}\right)$ meets the minor axis at A and major axis at B, length of AB (in units) is:
(a) $\dfrac{\sqrt{41}}{5}$
(b) $\dfrac{2\sqrt{41}}{5}$
(c) $\dfrac{3\sqrt{41}}{5}$
(d) $\dfrac{7\sqrt{41}}{5}$

32. If B is an end of the minor axis of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$, F_1 and F_2 are the foci such that the triang BF_1F_2 is equilateral, then eccentricity of the ellipse is:
(a) $\dfrac{1}{2}$
(b) $\dfrac{1}{\sqrt{2}}$
(c) $\dfrac{1}{3}$
(d) $\dfrac{\sqrt{3}}{2}$

33. Length of the major axis of the ellipse $(5x - 10)^2 + (5y + 15)^2 = \dfrac{(3x-4y+7)}{4}$ is:
(a) $\dfrac{10}{3}$ units
(b) 5 units
(c) $\dfrac{20}{3}$ units
(d) $\dfrac{5}{3}$ units

34. A tangent to the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16} = 1$ at any point meets the line $x = 0$ at a point Q. R is the image of Q in the line $y = x$. The circle on QR as a diameter

passes through a fixed point whose coordinate are:

(a) (5, 5) (b) (4, 5)
(c) (1, 1) (d) (0, 0)

35. *P* is a point on a directrix of an ellipse *S* is the corresponding focus and *C* is the centre of the ellipse. The line *PC* meets the ellipse at *A*. The angle which the tangent at *A* makes with *PS* is α. If the eccentricity of the ellipse is *e*, then α is equal to

(a) $\tan^{-1} e$ (b) $\tan^{-1} (1/e)$
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

36. If the tangent with slope –2 of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is a normal to the circle $x^2 + y^2 - 4x + 1 = 0$, then the maximum value of *ab* is

(a) 1 (b) 2
(c) 3 (d) 4

37. If a line $3px + 2\sqrt{1-p^2}\ y = 1$ touches a fixed ellipse *E* for all $p \in [-1, 1]$, then equation of a directrix of the ellipse is:

(a) $x = \frac{3\sqrt{5}}{10}$ (b) $y = \frac{3\sqrt{5}}{10}$
(c) $x = \frac{\sqrt{5}}{3}$ (d) $y = \frac{\sqrt{5}}{3}$

38. If *a* and *b* are the natural numbers such that $a + b = ab$, then equation of the chord of the ellipse $x^2 + 4y^2 = 4$ with (*a*, *b*) as the mid point is:

(a) $x + 4y = 2$ (b) $x + 4y = 10$
(c) $4x + y = 2$ (d) $4x + y = 10$

39. If $\frac{x^2}{\sec^2\theta} + \frac{y^2}{\tan^2\theta} = 1$ represents an ellipse with eccentricity *e* and length of the major axis *l* then

(a) *e* is independent of θ
(b) *l* is independent of θ
(c) *el* is independent of θ
(d) $l^2 - e^2$ is independent of θ

40. $3x^2 + 4y^2 - 6x + 8y + k = 0$ represents an ellipse with eccentricity 1/2,

(a) for all value of *k* (b) for $k > 7$
(c) $k < 7$ (d) $k = 7$

Assertion-Reason Type Questions

41. The tangent at a point *P* on the ellipse, which is not an extremity of major axis meets a directrix at *T*.

Statement 1: The circle on *PT* as diameter passes through the focus of the ellipse corresponding to the directix on which *T* lies.

Statement 2: *PT* subtends a right angle at the focus of the ellipse corresponding to the directix on which *T* lies.

42. **Statement 1:** If the normal at an end of a latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis at *G*, *O* is the centre of the ellipse, then $OG = ae^3$, *e* being the eccentricity of the ellipse.

Statement 2: Equation of the normal at a point ($a\cos\theta$, $b\sin\theta$) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{ax}{\cos\theta} + \frac{by}{\sin\theta} = a^2 + b^2$

43. **Statement 1:** In a triangle *PQR*, if the base *QR* is fixed and perimeter of the triangle is constant, then vertex *P* traces an ellipse.

Statement 2: If the sum of the distances of a point *A* from two fixed points *B* and *C* is constant > *BC*, then *A* traces an ellipse.

44. **Statement 1:** The locus of a moving point (*x*, *y*) satisfying $\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 4$ is an ellipse.

Statement 2: The distance between (–2, 0) and (2, 0) in 4 units.

45. **Statement 1:** Product of the perpendiculars drawn from the foci on any tangent to the ellipse $\frac{x^2}{15} + \frac{y^2}{7} = 1$ is 7.

Statement 2: Foot of the perpendiculars drawn from the foci on any tangent lies on the circle $x^2 + y^2 = 15$.

46. **Statement 1:** Origin is the centre of the conic $x^2 + y^2 + xy = 1$.

    Statement 2: A point is the centre of a conic if all chords of the conic through this point are bisected at this point.

47. For all real p, the line $2\,px + y\sqrt{1-p^2} = 1$ touches a fixed ellipse whose axes are the coordinate axes.

    Statement 1: Equation of the director circle of the ellipse is $4x^2 + 4y^2 = 5$.

    Statement 2: Length of the major and minor axes of the ellipse are 2 and 1 units respectively.

48. **Statement 1:** If the extremities of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ $(a > b)$, having positive ordinates lies on the parabola $x^2 = -2(y - 2)$, then $a = 2$.

    Statement 2: If the length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is equal to the distance between the foci, then the eccentricity e of the ellipse satisfies $e^2 + e - 1 = 0$.

49. **Statement 1:** Two diameters $y = mx$ and $y = m'x$ $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ are conjugate if $mm' = -\frac{a^2}{b^2}$.

    Statement 2: Two diameters of an ellipse are said to be conjugate when each bisects the chords parallel to the other.

50. E: $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and $P : y^2 = 4bx$, $a > b$.

    Statement 1: The tangent at the positive end of the minor axis of the ellipse. E passes through the positive end of the latus rectum of the parabola P.

    Statement 2: If the latus rectum of the parabola P is same as that of the ellipse E, then eccentricity of E is $1/\sqrt{2}$.

LEVEL 2

Straight Objective Type Questions

51. If chords of contact of the tangent from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ are at right angles, then $\frac{x_1x_2}{a^2}\times\frac{y_1y_2}{b^2}$ is equal to

    (a) $\frac{a^2}{b^2}$ (b) $\frac{-b^2}{a^2}$

    (c) $\frac{-a^4}{b^4}$ (d) $\frac{b^4}{a^4}$

52. If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$ intersect it again at the point $Q(2\theta)$, then $\cos\theta =$

    (a) $\frac{-2}{3}$ (b) $\frac{2}{3}$

    (c) $\frac{-6}{7}$ (d) $\frac{6}{7}$

53. Let E be the ellipse $x^2/9 + y^2/4 = 1$ and C be the circle $x^2 + y^2 = 9$. Let P (1, 2) and Q (2, 1) be two points, then

    (a) Q lies inside C but outside E
    (b) Q lies outside both C and E
    (c) P lies inside both C and E
    (d) P lies inside C but out side E

54. If CF is perpendicular from the centre of the ellipse $\frac{x^2}{25}+\frac{y^2}{9} = 1$ to the tangent at P, G is the point where the normal at P meets the major axis, then $CF \cdot PG =$

    (a) 9 (b) 18
    (c) 25 (d) 34

55. If the tangent at $P(\theta)$ on the ellipse $16x^2 + 11y^2 = 256$ touches the circle $x^2 + y^2 + 2x - 15 = 0$, then $\theta =$

    (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

    (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

56. If $lx + my + n = 0$ is an equation of the line joining the extremities of a pair of semi-conjugate diameters of the ellipse $\frac{x^2}{9}+\frac{y^2}{4} = 1$, then $\frac{9l^2+4m^2}{n^2}$ is equal to

(a) -1　(b) 0
(c) 1　(d) 2

57. Let f be a strictly decreasing function defined on **R** such that $f(x) > 0\ \forall\ x \in$ **R**, and $\dfrac{x^2}{f(a^2+5a+3)} + \dfrac{y^2}{f(3a+15)} = 1$ represents an ellipse with major axis along the y-axis, then
(a) $a \notin (-\infty, -6)$　(b) $a \notin (2, \infty)$
(c) $a \notin (-6, 2)$　(d) $a > 0$

58. Number of points from which two perpendicular tangents can be drawn to both the ellipses E_1: $\dfrac{x^2}{a^2+2} + \dfrac{y^2}{b^2} = 1$ and E_2: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2+1} = 1$ is
(a) 0　(b) 1
(c) 2　(d) infinitely many

59. Let d be the perpendicular distance from the centre of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ to the tangent at a point P on the ellipse. If F_1 and F_2 are two foci of the ellipse then $(PF_1 - PF_2)^2 =$
(a) $4b^2\left(1-\dfrac{a^2}{d^2}\right)$　(b) $4d^2\left(1-\dfrac{b^2}{a^2}\right)$
(c) $4a^2\left(1-\dfrac{b^2}{d^2}\right)$　(d) $4a^2$

60. If from a point $(a, 0)$ three distinct chords of the ellipse $x^2 + 2y^2 = 1$ are drawn, which are bisected by the parabola $y^2 = 4x$, then
(a) $4 < a < 8$　(b) $8 < a < 4+\sqrt{17}$
(c) $0 < a < \sqrt{17}-4$　(d) $0 < a < 4$

Previous Years' AIEEE/JEE Main Questions

1. The eccentricity of an ellipse, with its centre at the origin is 1/2. If one of the directrix is $x = 4$, then the equation of the ellipse is
(a) $4x^2 + 3y^2 = 12$　(b) $3x^2 + 4y^2 = 12$
(c) $3x^2 + 4y^2 = 1$　(d) $4x^2 + 3y^2 = 1$　**[2004]**

2. Area of the greaterst rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ is
(a) $\sqrt{ab}$　(b) a/b
(c) $2ab$　(d) ab　**[2005]**

3. An ellipse has OB as semi minor axis. F and F' its foci and the angle FBF' is a right angle. The eccentricity of the ellipse is
(a) 1/4　(b) $1/\sqrt{3}$
(c) $1/\sqrt{2}$　(d) 1/2　**[2005]**

4. In an ellipse, the distance between foci is 6 and minor axis is 8. Then its eccentricity is
(a) $1/\sqrt{5}$　(b) 3/5
(c) $\dfrac{1}{2}$　(d) $\dfrac{4}{5}$　**[2006]**

5. A focus of an ellipse is at origin. The directrix is the line $x = 4$ and the eccentricity is 1/2. Then the length of the semi-major axis is
(a) 5/3　(b) 8/3
(c) 2/3　(d) 4/3　**[2008]**

6. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is
(a) $4x^2 + 48y^2 = 48$　(b) $4x^2 + 6y^2 = 48$
(c) $x^2 + 16y^2 = 16$　(d) $x^2 + 12y^2 = 16$　**[2009]**

7. Equation of the ellipse whose axes are the axes of coordinate, which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{2/5}$ is
(a) $5x^2 + 3y^2 - 32 = 0$　(b) $3x^2 + 5y^2 - 32 = 0$
(c) $5x^2 + 3y^2 - 48 = 0$　(d) $3x^2 + 5y^2 - 15 = 0$
[2011]

8. An ellipse is drawn by taking a diameter of the circle $(x-1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y-2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is at the origin and

its axes are the coordinate axes, then the equation of the ellipse is

(a) $4x^2 + y^2 = 8$ (b) $x^2 + 4y^2 = 16$

(c) $4x^2 + y^2 = 4$ (d) $x^2 + 4y^2 = 8$ **[2012]**

9. If a and c are positive real numbers and the ellipse $\frac{x^2}{4c^2} + \frac{y^2}{c^2} = 1$ has four distinct points common with circle $x^2 + y^2 = 9a^2$, then

(a) $9ac - 9a^2 - 2c^2 < 0$ (b) $6ac + 9a^2 - 2c^2 < 0$

(c) $9ac - 9a^2 - 2c^2 > 0$ (d) $6ac + 9a^2 - 2c^2 > 0$

[2013, online]

10. Let the equation of two ellipses be E_1: $\frac{x^2}{3} + \frac{y^2}{2} = 1$ and E_2: $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$. If the product of their eccentricities is 1/2, then the length of the minor axis of E_2 is:

(a) 8 (b) 9

(c) 4 (d) 2 **[2013, online]**

11. If the curves $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1$ and $y^2 = 16x$ intersect at right angles, then a value of α is

(a) 2 (b) 4/3

(c) 1/2 (d) 3/4 **[2013, online]**

12. A point on the ellipse $4x^2 + 9y^2 = 36$ where the normal is parallel to the $4x - 2y - 5 = 0$, is:

(a) $\left(\frac{9}{5}, \frac{8}{5}\right)$ (b) $\left(\frac{8}{5}, \frac{-9}{5}\right)$

(c) $\left(-\frac{9}{5}, \frac{8}{5}\right)$ (d) $\left(\frac{8}{5}, \frac{9}{5}\right)$ **[2013, online]**

13. The focus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent is:

(a) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (b) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

(c) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (d) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

[2014]

14. If OB is the semi-minor axis of an ellipse, F_1 and F_2 are its foci and the angle between F_1B and F_2B is a right angle, then the square of the eccentricity of the ellipse is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$

(c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{4}$ **[2014, online]**

15. The minimum area of the triangle formed by any tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{81} = 1$ and the coordinate - axes is:

(a) 12 (b) 18

(c) 26 (d) 36 **[2014, online]**

16. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is:

(a) $\frac{27}{4}$ (b) 18

(c) $\frac{27}{2}$ (d) 27 **[2015]**

17. If the distance between the foci of an ellipse is half the length of its latus rectum, then the eccentricity of the ellipse is:

(a) $\frac{1}{2}$ (b) $\frac{2\sqrt{2}-1}{2}$

(c) $\sqrt{2}-1$ (d) $\frac{\sqrt{2}-1}{2}$ **[2015, online]**

18. If the tangent at a point on the ellipse $\frac{x^2}{27} + \frac{y^2}{3} = 1$ meets the coordinate axes at A and B, and O is the origin, then the minimum area (in sq. units) of the triangle OAB is

(a) $3\sqrt{3}$ (b) $\frac{9}{2}$

(c) 9 (d) $9\sqrt{3}$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. In an ellipse, the distance between its directrices is four times the distance between its foci. If (– 2, 0) is one of its vertices, then the equation of the ellipse is

(a) $3x^2 + 4y^2 = 1$ (b) $4x^2 + 3y^2 = 12$

(c) $3x^2 + 4y^2 = 12$ (d) $4x^2 + 3y^2 = 1$ **[2007]**

2. The tangent to the ellipse $3x^2 + 16y^2 = 12$, at the point $(1, 3/4)$, intersects the curve $y^2 + x = 0$ at:
   (a) no point
   (b) exactly one point
   (c) two distinct points
   (d) more than two points **[2012]**

3. Let P be a point in the first quadrant lying on the ellipse $9x^2 + 16y^2 = 144$, such that the tangent at P to the ellipse is inclined at an angle 135° to the positive direction of x-axis. Then the coordinates of P are:
   (a) $\left(\frac{\sqrt{143}}{3}, \frac{1}{4}\right)$ (b) $\left(\frac{8}{9}, \frac{\sqrt{77}}{3}\right)$
   (c) $\left(\frac{4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ (d) $\left(\frac{16}{5}, \frac{9}{5}\right)$ **[2014]**

4. The line $2x + y = 3$ intersects the ellipse $4x^2 + y^2 = 5$ at two points. The tangents to the ellipse at these two points intersect at the point:
   (a) $\left(\frac{5}{6}, \frac{5}{3}\right)$ (b) $\left(\frac{5}{6}, \frac{5}{6}\right)$
   (c) $\left(\frac{5}{3}, \frac{5}{6}\right)$ (d) $\left(\frac{5}{3}, \frac{5}{3}\right)$ **[2015]**

Answers

Concept-based

1. (c)	**2.** (c)	**3.** (a)	**4.** (d)
5. (b)	**6.** (c)	**7.** (a)	**8.** (b)
9. (d)	**10.** (a)	**11.** (c)	**12.** (a)
13. (c)	**14.** (a)	**15.** (d)	

Level 1

16. (b)	**17.** (b)	**18.** (b)	**19.** (b)
20. (c)	**21.** (b)	**22.** (c)	**23.** (c)
24. (a)	**25.** (c)	**26.** (b)	**27.** (b)
28. (b)	**29.** (a)	**30.** (c)	**31.** (a)
32. (a)	**33.** (b)	**34.** (d)	**35.** (c)
36. (d)	**37.** (b)	**38.** (b)	**39.** (c)
40. (c)	**41.** (a)	**42.** (c)	**43.** (a)
44. (c)	**45.** (b)	**46.** (a)	**47.** (a)
48. (b)	**49.** (d)	**50.** (d)	

Level 2

51. (c)	**52.** (a)	**53.** (d)	**54.** (a)
55. (c)	**56.** (d)	**57.** (c)	**58.** (a)
59. (c)	**60.** (b)		

Previous Years' AIEEE/JEE Main Questions

1. (b)	**2.** (c)	**3.** (c)	**4.** (b)
5. (b)	**6.** (d)	**7.** (b)	**8.** (c)
9. (a)	**10.** (c)	**11.** (a)	**12.** (a)
13. (c)	**14.** (a)	**15.** (d)	**16.** (d)
17. (c)	**18.** (c)		

Previous Years' B-Architecture Entrance Examination Questions

1. (c)	**2.** (b)	**3.** (d)	**4.** (a)

Hints and Solutions

Concept-based

1. Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It passes through $(5, 0) \Rightarrow a^2 = 25$

   $$b^2 = a^2(1 - e^2) \Rightarrow b^2 = 25\left(1 - \frac{4}{9}\right) = \frac{125}{9}.$$

   Hence the required equation of the ellipse is $5x^2 + 9y^2 = 125$

2. Equation of a tangent to the ellipse is

   $$y = mx + \sqrt{36m^2 + 25} \qquad (1)$$

   Equation of the directrix is

   $$x = \frac{6 \times 6}{\sqrt{36 - 25}} = \frac{36}{\sqrt{11}}$$

   which meets the +ve x axis at $\left(\frac{36}{\sqrt{11}}, 0\right)$.

   The tangent (1) passes through this point if $\left[m\left(\frac{36}{\sqrt{11}}\right)\right]^2 = 36m^2 + 25$

   $$\Rightarrow m^2 = \frac{11}{36} \Rightarrow m = \pm\frac{\sqrt{11}}{6}.$$

   Since the tangent meets the directrix on positive x-axis, $m < 0$, and the required equation is $y = -\frac{\sqrt{11}}{6}x + \sqrt{36 \times \frac{11}{36} + 25}$

   $$6y + \sqrt{11}x = 36.$$

3. Foci of $\frac{x^2}{25} + \frac{y^2}{9} = 1$ is $(\pm 4, 0)$ and that of $\frac{x^2}{9} + \frac{y^2}{25} = 1$

is $(0, \pm 4)$. So the equation joining two foci $(4, 0)$ and $(0, 4)$ is $x + y = 4$.

4. $\dfrac{a^2 - 16}{a^2} = \dfrac{25 - a^2}{25} \Rightarrow a^4 = 16 \times 25$

$= a^2 = 20$.

5. $(4)^2 = 25m^2 + 4 \Rightarrow 25m^2 = 12$

So $625m^4 + 25m^2 - 156$

$= 144 + 12 - 156 = 0$

6. Length of the latera recta of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{2b^2}{a}$ so the required length is $\dfrac{2 \times 63}{9} = 14$ units

7. Equation of the normal is

$$\frac{\dfrac{x-3}{3}}{36} = \frac{\dfrac{y-2\sqrt{3}}{2\sqrt{3}}}{16}$$

$\Rightarrow 12\sqrt{3}\ (x - 3) = 8\ (y - 2\sqrt{3})$

which meets the major axis $y = 0$ at point for which

$12\sqrt{3}\ (x - 3) = -16\sqrt{3} \Rightarrow x = \dfrac{5}{3}$.

$\Rightarrow \alpha = \dfrac{5}{3}$.

8. $2ae = 6$ and $b = 4$. So $a^2e^2 = a^2 - b^2$

$\Rightarrow a^2 = 25$ and $e = \dfrac{3}{5}$.

9. Equation of the ellipse is

$\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$

$PF_1 + PF_2 = 2(3) = 6$

$\Rightarrow PF_1 = 4, PF_2 = 2$

$F_1F_2 = 2ae = 2\sqrt{9-4} = 2\sqrt{5}$

$(PF_1)^2 + (PF_2)^2 = 16 + 4 = (2\sqrt{5})^2 = (F_1F_2)^2$

So $\Delta\ PF_1F_2$ is right angled and hence its area is $\dfrac{1}{2}\ (PF_1)\ (PF_2) = 4$ sq. units.

10. Equations of the normal at P and Q are

$ax \sec \alpha - by \operatorname{cosec} \alpha = a^2 - b^2$

$ax \sec \beta - by \operatorname{cosec} \beta = a^2 - b^2$

If they intersect at (h, k).

$ah \sec \alpha - bk \operatorname{cosec} \alpha = ah \operatorname{cosec} \alpha - bk \sec \alpha$ $(\because \beta = \pi/2 - \alpha)$

$(ah + bk)(\sec \alpha - \operatorname{cosec} \alpha) = 0 \Rightarrow ah + bk = 0$

Locus of (h, k) is $ax + by = 0$

11. $P_1\ (a \cos \theta_1, b \sin \theta_1), P_2\ (a \cos \theta_2, b \sin \theta_2)$

Slope of $OP_1 = \dfrac{b \sin \theta_1}{a \cos \theta_1} = \dfrac{b}{a} \tan \theta_1$

Slope of $OP_2 = \dfrac{b \sin \theta_2}{a \cos \theta_2} = \dfrac{b}{a} \tan \theta_2$

Product of the slope $= \dfrac{b^2}{a^2} \tan \theta_1 \tan \theta_2$

$= \dfrac{b^2}{a^2}\left(\dfrac{-a^2}{b^2}\right) = -1$

So PQ subtends a right angle at $(0, 0)$, the centre.

12. Equation of the ellipse is

$$\frac{(x-1)^2}{\left(\dfrac{1}{\sqrt{2}}\right)^2} + \frac{(y-2)^2}{\left(\dfrac{1}{\sqrt{3}}\right)^2} = 1$$

i.e. $\dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} = 1$

where $X = x - 1$, $Y = y - 2$

$a = \dfrac{1}{\sqrt{2}}, b = \dfrac{1}{\sqrt{3}}$

If e is the eccentricity, then

$e = \sqrt{1 - \dfrac{2}{3}} = \dfrac{1}{\sqrt{3}}$.

Coordinates of the foci are

$X = \pm ae, Y = 0$

$\Rightarrow x - 1 = \pm \dfrac{1}{\sqrt{2}} \times \dfrac{1}{\sqrt{3}}, y - 2 = 0$

$\Rightarrow x = 1 \pm \dfrac{1}{\sqrt{6}}, y = 2$

Hence the required coordinate are $\left(1 \pm \dfrac{1}{\sqrt{6}}, 2\right)$

13. Coordinates of P are $(a \cos \theta, a \sin \theta)$ and of Q are $(b \cos \theta, b \sin \theta)$. Equation of the line through P parallel to y-axis is $x = a \cos \theta$ and of the line through Q parallel to x-axis is $y = b \sin \theta$. So the locus of the point of intersection is $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y}{b}\right)^2 = \cos^2\theta + \sin^2\theta = 1$

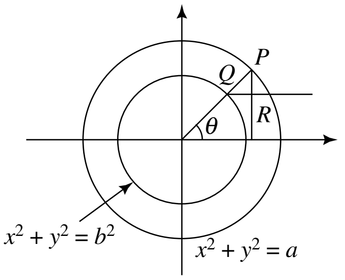


Fig. 19.19

14. Let the coordinates of P be (a cos θ, b sin θ) then coordinate of Q are (a cos θ, a sin θ)

    $PN = b \sin\theta$, $QN = a \sin\theta$

    $\Rightarrow PQ = (a - b) \sin\theta$

    and $\dfrac{PN}{PQ} = \dfrac{b}{a-b}$.

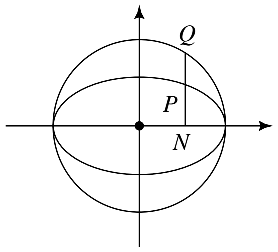


Fig. 19.20

15. Equation of the normal at P (a cos θ, b sin θ) on the ellipse is $ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2$.

    So coordinates of G are $\left(\dfrac{a^2-b^2}{a\sec\theta}, 0\right)$ and of g are $\left(0, -\dfrac{a^2-b^2}{a \operatorname{cosec}\theta}\right)$

    $$(PG)^2 = \left(a\cos\theta - \frac{a^2-b^2}{a}\cos\theta\right)^2 + b^2\sin^2\theta$$

    $$= \frac{b^2}{a^2}\,(b^2\cos^2\theta + a^2\sin^2\theta)$$

    $$\text{and } (Pg)^2 = \frac{a^2}{b^2}\,(b^2\cos^2\theta + a^2\sin^2\theta)$$

    So $PG : Pg = b^2 : a^2$

Level 1

16. Let the point be ((1/2) cos θ, (1/3) sin θ), then the slope of the tangent is $-\dfrac{1/3}{1/2}$ cot $\theta = \dfrac{8}{9}$

    $\Rightarrow \tan\theta = \dfrac{3}{4} \Rightarrow \sin\theta = \pm\dfrac{3}{5}$ and $\cos\theta = \pm\dfrac{4}{5}$ and the required point can be $\left(-\dfrac{4}{5}\times\dfrac{1}{2}, \dfrac{3}{5}\times\dfrac{1}{3}\right) = \left(-\dfrac{2}{5}, \dfrac{1}{5}\right)$

17. Coordinates of the focii are ($\pm\, ae$, 0) where $a^2 = 1/4$, $b^2 = 1/9$ and $a^2e^2 = a^2 - b^2 = 5/36$.
18. $6\cos^2\theta + 2\sin^2\theta = 4 \Rightarrow \cos^2\theta = 1/2$ so $\theta = \pi/4$.
19. Positive end of the latus rectum is (ae, b^2/a) equation of the normal is

    $y - b^2/a = (1/e)\,(x - ae)$

    $\Rightarrow ae\, y - ax + (a^2 - b^2)e = 0$

    $\Rightarrow ae\, y - ax + a^2e^3 = 0 \Rightarrow x - ey - ae^3 = 0$
20. Radius of the circle must be greater than the major axis of the ellipse.
21. $a^2e^2 + b^2 = (2b)^2 \Rightarrow a^2e^2 = 3a^2\,(1 - e^2)$

    $\Rightarrow e^2 = 3/4$.
22. Point C on the ellipse is $(\sqrt{3}\cos\theta, -\sin\theta)$,

    $\underline{|OBC} = \pi/6$

    $\Rightarrow \dfrac{1+\sin\theta}{-\sqrt{3}\cos\theta} = \tan(\pi/2 + \pi/6) = -\sqrt{3}$

    $\Rightarrow (1 + \sin\theta)^2 = 9\cos^2\theta$.

    $\Rightarrow \sin\theta = 4/5$, $\cos\theta = 3/5$, $BC = 6\sqrt{3}/5$.

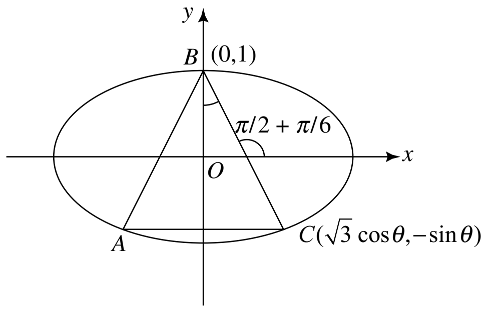


Fig. 19.21

23. $10 - 2a > 0$, $4 - 2a > 0$

    $\Rightarrow a < 5$, $a < 2$.

    $\Rightarrow a \in (-\infty, 2)$
24. $\left(\dfrac{x}{3}\right)^2 + \left(\dfrac{y}{4}\right)^2 = 2(\cos^2 t + \sin^2 t) = 2$

    $\Rightarrow \dfrac{x^2}{18} + \dfrac{y^2}{32} = 1$

    which is an ellipse with $e^2 = \dfrac{32-18}{32} = \dfrac{7}{16}$

    $\Rightarrow 16e^2 - 7 = 0$

    $\Rightarrow 16e^2 + 7 = 14$

25. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ then

$$e = \frac{2}{3},\ \frac{a}{e} = 6 \Rightarrow a = 4\ e^2 = 1-\frac{b^2}{a^2} \Rightarrow b^2 = \frac{80}{9}.$$

Hence the required equation is

$$\frac{x^2}{16}+\frac{9y^2}{80} = 1 \Rightarrow 5x^2 + 9y^2 = 80$$

26. Equation of a tangent is $y = mx + \sqrt{a^2m^2+b^2}$ (1)

Equation of the perpendicular from the focus $(ae, 0)$ on it is

$$y = -\frac{1}{m}\ (x - ae) \text{ or } my + x = ae \qquad (2)$$

Squaring and adding (1) and (2), eliminating m,

$(y - mx)^2 + (my + x)^2 = a^2m^2 + b^2 + a^2e^2$

$\Rightarrow (1 + m^2)\ (x^2 + y^2) = (1 + m^2)\ a^2$

$\Rightarrow x^2 + y^2 = a^2$ is the required locus

27. An equation of the tangent to the ellipse with slope $\frac{b}{a}$ is

$$y = \frac{b}{a}x \pm \sqrt{a^2 \times \frac{b^2}{a^2}+b^2}$$

As it touches the circle $x^2 + y^2 = r^2$

$$2b^2 = r^2\left(1+\frac{b^2}{a^2}\right)$$

$$\Rightarrow r^2 = \frac{2a^2b^2}{a^2+b^2}.$$

28. The product of perpendiculars from foci upon any tangent to the ellipse is equal to square of the semi-minor axis (see theory). So the required product is 100.

29. Let $P(h, k)$ be the foot of the perpendicular, O the centre then slope of $OP = \frac{k}{h}$ and the slope of the tangent is $-\frac{h}{k}$. Equation of the tangent is

$$y - k = -\frac{h}{k}\ (x - h)$$

$$\Rightarrow y = -\frac{h}{k}x+\frac{h^2+k^2}{k}$$

which touches the given ellipse

$$\text{if} \left(\frac{h^2+k^2}{k}\right)^2 = 25\left(\frac{h}{k}\right)^2 + 16$$

$\Rightarrow 25h^2 + 16k^2 = (h^2 + x^2)^2$

Locus of (h, k) is

$25x^2 + 16y^2 = (x^2 + y^2)^2$

30. $e^2 = \frac{7}{16} \Rightarrow e = \frac{\sqrt{7}}{4}$

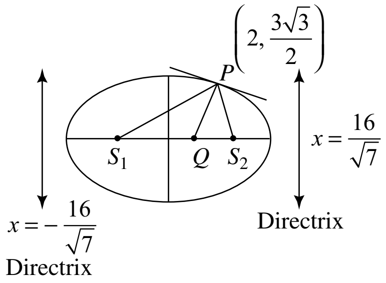

$\left(2, \frac{3\sqrt{3}}{2}\right)$
P
S_1
Q
S_2
$x = \frac{16}{\sqrt{7}}$
Directrix
$x = -\frac{16}{\sqrt{7}}$
Directrix

Fig. 19.22

Coordinates of S_1 are $(-\sqrt{7}, 0)$ and S_2 are $(\sqrt{7}, 0)$ PQ is the internal bisector of $\underline{|S_1PS_2}$

$$\text{So } \frac{S_1Q}{S_2Q} = \frac{PS_1}{PS_2} = \frac{\frac{16}{\sqrt{7}}+2}{\frac{16}{\sqrt{7}}-2}$$

$$= \frac{8+\sqrt{7}}{8-\sqrt{7}}$$

31. Equation of the chord is

$$\frac{1/4}{25}+\frac{4/25}{16}-1 = \frac{(1/2)x}{25}+\frac{(2/5)y}{16}-1 \quad (T = S_1)$$

$\Rightarrow 4x + 5y = 4$

which meets minor axis at $A\left(0, \frac{4}{5}\right)$ and major axis at B (1, 0).

So the length of $AB = \sqrt{1+\frac{16}{25}} = \frac{\sqrt{41}}{5}$ units.

32. $F_1\ (-ae, 0)$, $F_2\ (ae, 0)$, $B\ (0, b)$

$(F_1F_2)^2 = (B\ F_1)^2$

$\Rightarrow 4a^2e^2 = a^2e^2 + b^2$

$\Rightarrow 3a^2e^2 = a^2\ (1 - e^2)$

$\Rightarrow e^2 = \frac{1}{4} \Rightarrow e = \frac{1}{2}$

33. Equation of the ellipse is

$$(x - 2)^2 + (y + 3)^2 = \left(\frac{1}{2}\right)^2\left[\frac{3x-4y+7}{5}\right]^2$$

Centre is $(2, -3)$, eccentricity $e = \frac{1}{2}$ and a directrix is $3x - 4y + 7 = 0$.

Length of the perpendicular from the centre on the directrix is

$$\left|\frac{3\times 2-4(-3)+7}{5}\right| = 5$$

$$\Rightarrow \frac{a}{e} = 5 \Rightarrow 2a = 5$$

Hence the length of the major axis is 5 units.

34. The equation of the tangent to the ellipse at $P(5 \cos \theta, 4 \sin \theta)$ is

$$\frac{x\cos\theta}{5}+\frac{y\sin\theta}{4} = 1$$

which meets $x = 0$ at Q $(0, 4 \operatorname{cosec} \theta)$

Image of Q in $y = x$ is R $(4 \operatorname{cosec} \theta, 0)$

Equation of the circle on QR as a diameter is

$x(x - 4 \operatorname{cosec} \theta) + y(y - 4 \operatorname{cosec} \theta) = 0$

$\Rightarrow x^2 + y^2 - 4(x + y) \operatorname{cosec} \theta = 0$

which passes through the point of intersection of $x^2 + y^2 = 0$ and $x + y = 0$ i.e the point $(0, 0)$.

35. Let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$

A be the point $(a \cos \theta, b \sin \theta)$

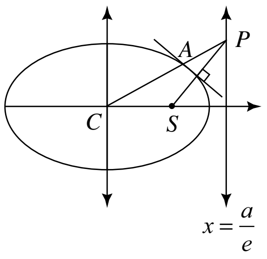


Fig. 19.23

Equation of AC is

$$y = \frac{b}{a} \tan \theta x.$$

which meets the directrix

$$x = \frac{a}{e} \text{ at } P\left(\frac{a}{e}, \frac{b}{e}\tan\theta\right)$$

slope of the tangent at $A = \dfrac{-b}{a\tan\theta}$.

S is $(ae, 0)$

$$\text{So slope of } PS = \frac{\frac{b}{e}\tan\theta}{\frac{a}{e}-ae} = \frac{b\tan\theta}{a(1-e^2)} = \frac{b\tan\theta}{a\times\frac{b^2}{a^2}}$$

$$= \frac{a}{b} \tan \theta.$$

So PS is perpendicular to the tangent at A. $\Rightarrow \alpha = \dfrac{\pi}{2}$

36. Equation of a tangent with slope-2 is

$$y = -2x \pm \sqrt{4a^2+b^2}$$

This is a normal to the circle $x^2 + y^2 - 4x + 1 = 0$ if it passes through the centre $(2, 0)$ of the circle.

$$0 = -4 \pm \sqrt{4a^2+b^2}$$

$\Rightarrow 4a^2 + b^2 = 16$

Using A.M. ≥ G. M.

$$\frac{4a^2+b^2}{2} \geq \sqrt{4a^2b^2}$$

$8 \geq 2ab.$

$\Rightarrow ab \leq 4.$

37. Let $p = \cos \theta$, $0 \leq \theta \leq \pi$. The equation of the line becomes $(3 \cos \theta)\, x + (2 \sin \theta) y = 1$

$$\text{or } \frac{x}{1/3}\cos\theta+\frac{y}{1/2}\sin\theta = 1$$

which touches the ellipse

$$\frac{x^2}{\frac{1}{9}}+\frac{y^2}{\frac{1}{4}} = 1$$

eccentricity of the ellipse is

$$\sqrt{\frac{\frac{1}{4}-\frac{1}{9}}{\frac{1}{4}}} = \frac{\sqrt{5}}{3}$$

and the equation of a directrix is

$$y = \frac{1/2}{\sqrt{5}/3} = \frac{3\sqrt{5}}{10}.$$

38. Since a and b are natural numbers such that $a + b = ab$, $a = b = 2$

Equation of the chord of the ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{1} = 1$ having (a, b) as the mid point is $\dfrac{x(2)}{4}+\dfrac{y(2)}{1}-1 = \dfrac{2^2}{4}+\dfrac{2^2}{1}-1$ $(T = S_1)$

$\Rightarrow x + 4y = 10.$

39. $l = 2 \sec \theta$, $e = \sqrt{\dfrac{\sec^2\theta-\tan^2\theta}{\sec^2\theta}} = \cos \theta$

$\Rightarrow el = 2$ which is independent of θ.

40. Given equation can be written as $3(x-1)^2+4(y+1)^2 = 7-k$ which represent an ellipse if $k<7$

$$\Rightarrow \frac{(x-1)^2}{\frac{7-k}{3}}+\frac{(y+1)^2}{\frac{7-k}{4}} = 1$$

So $\dfrac{\frac{7-k}{3}-\frac{7-k}{4}}{\frac{7-k}{3}} = e^2 = \dfrac{1}{4}$ for all $k<7$

41. Tangent at $P(a\cos\theta, b\sin\theta)$ be

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta = 1$$

It meets the directrix $x = \dfrac{a}{e}$ at the point T

$$\left(\frac{a}{e}, \frac{b(e-\cos\theta)}{e\sin\theta}\right).$$

Focus $S(ae, 0)$

Slope of $SP = \dfrac{b\sin\theta}{a(\cos\theta-e)}$,

Slope of $ST = \dfrac{b(e-\cos\theta)}{ae\sin\theta(1-\mathrm{e}^2)}$

Product of the slope $= -\dfrac{b^2}{a^2(1-e^2)} = -1$

$\Rightarrow$ Statement-2 is true using which statement-1 is also true.

42. **Statement-2** is false. Equation of the normal is $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta} = a^2-b^2$

In statement-1 $L(ae, b^2/a) = (a\cos\theta, b\sin\theta)$

$\Rightarrow \cos\theta = e.$

So normal at L, $\dfrac{ax}{e}-\dfrac{by}{\sqrt{1-e^2}} = a^2e^2$

which meets the major axis $y=0$ at $x = ae^3$ and the statement-1 is true.

43. By definition of the ellipse statement-2 is true and using it statement-1 is also true.

44. Statement-2 is true but statement-1 is false as the locus represents a straight line.

45. Statement-1 is true as $p_1p_2 = b^2 = 7$. Statement-2 is also true as the foot of the perpendicular lie on the auxiliary circle which is $x^2+y^2 = 15$ but does not justify statement-1.

46. Let $y = mx$ be a chord of the conic $x^2+y^2+xy = 1$ through (0, 0), which meets the conic at points for which $x^2+m^2x^2+mx^2-1 = 0$.

$\Rightarrow (1+m+m^2)x^2-1 = 0$

$\Rightarrow x_1+x_2 = 0 \Rightarrow y_1+y_2 = 0$

Showing that (0, 0) is the mid point of the chord and hence statement-1 is true using statement-2 which is true.

47. Let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$

Equation of any tangent is $y = mx \pm \sqrt{a^2m^2+b^2}$.

Comparing with given equation we get

$m = -\dfrac{2p}{\sqrt{1-p^2}}$ and $a^2m^2+b^2 = \dfrac{1}{1-p^2}$

$$\Rightarrow a^2\times\frac{4p^2}{1-p^2}+b^2 = \frac{1}{1-p^2}$$

$\Rightarrow p^2(4a^2-b^2)+b^2-1 = 0$

which is true for all p if $b^2 = 1$, $4a^2 = b^2 = 1$

so the equation of the ellipse is

$$\frac{x^2}{\frac{1}{4}}+\frac{y^2}{1} = 1$$

Length of the major axis $= 2b = 2$ and of minor axis $= 2a = 1$.

Thus statement-2 is true. Using it in statement-1 equation of the director circle is $x^2+y^2 = a^2+b^2$ or $x^2+y^2 = 1+\dfrac{1}{4} = \dfrac{5}{4} \Rightarrow 4x^2+4y^2 = 5$ and thus statement-1 is also true.

48. Extremities of the latus rectum are $\left(\pm ae, \dfrac{b^2}{a}\right)$ since they lie on the parabola $x^2 = -2(y-2)$

We have $a^2e^2 = -2\left(\dfrac{b^2}{a}-2\right)$, $b^2 = a^2(1-e^2)$

$\Rightarrow a^2e^2-2ae^2+2a-4 = 0$

$\Rightarrow (ae^2+2)(a-2) = 0 \Rightarrow a = 2$

Thus statement-1 is true.

In statement-2 $\dfrac{2b^2}{a} = 2ae \Rightarrow b^2 = a^2e$

$\Rightarrow a^2(1-e^2) = a^2e \Rightarrow e^2+e-1 = 0$

and the statement-2 is also true but does not justify statement-1.

49. Statement-2 is true by definition of conjugate diameters.

Let $y = mx$ and $y = m'x$ be two conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Let (h, k) be the mid point of chord whose step is m then

$$\frac{hx}{a^2}+\frac{ky}{b^2}-1=\frac{h^2}{a^2}+\frac{k^2}{b^2}-1$$

$$\Rightarrow m=-\frac{b^2}{a^2}\frac{h}{k} \Rightarrow \text{Locus of } (h, k) \text{ is } y=\frac{-b^2}{a^2m}x$$

$$\Rightarrow \frac{-b^2}{a^2m}=m' \Rightarrow m'm=\frac{-b^2}{a^2}. \text{ (using statement-2)}$$

and thus statement-1 is false.

50. Tangent at the positive end of the minor axis of E is $y = b$ which meets the parabola $y^2 = 4bx$ at the point $\left(\frac{b}{4}, b\right)$ which is not an end of the latus rectum of P.

so statement-1 is false.

In statement-2 if $x = b$ and $x = ae$ represent the same line then $b = ae$

$$\Rightarrow \frac{b}{a}=e \Rightarrow a^2(1-e^2)=a^2e^2$$

$$\Rightarrow e^2=\frac{1}{2} \Rightarrow e=\frac{1}{\sqrt{2}}.$$

Thus statement-2 is true.

Level 2

51. Chords of contacts are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \text{ and } \frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1$$

Since they are at right angles

$$\frac{-b^2}{a^2}\times\frac{x_2}{y_2}\times\frac{-b^2}{a^2}\times\frac{x_1}{y_1}=-1$$

$$\Rightarrow \frac{x_1x_2}{y_1y_2}=\frac{-a^4}{b^4}$$

52. Normal at $P\ (\sqrt{14}\cos\theta, \sqrt{5}\sin\theta)$ to the ellipse is

$$\frac{\sqrt{14}x}{\cos\theta}-\frac{\sqrt{5}y}{\sin\theta}=14-5=9.$$

Since it passes through $Q\ (2\theta)$

$$\frac{\sqrt{14}\times\sqrt{14}\cos 2\theta}{\cos\theta}-\frac{\sqrt{5}\times\sqrt{5}\sin 2\theta}{\sin\theta}=9$$

$$\Rightarrow \frac{14(2\cos^2\theta-1)}{\cos\theta}-\frac{5\times 2\sin\theta\cos\theta}{\sin\theta}=9$$

$\Rightarrow 18\cos^2\theta - 9\cos\theta - 14 = 0$

$\Rightarrow (3\cos\theta + 2)(6\cos\theta - 7) = 0$

$\Rightarrow \cos\theta = -2/3.$

53. $C: x^2 + y^2 - 9 = 0$

$E: x^2/9 + y^2/4 - 1 = 0$

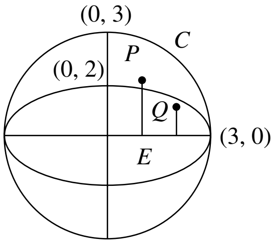


Fig. 19.24

For $P(1, 2)$, $C: 1 + 4 - 9 < 0$, $E: \frac{1}{9}+\frac{4}{4}-1>0$

So P lies outside E and inside C.

For $Q(2, 1)$, $C: 4 + 1 - 9 < 0$, $E: \frac{4}{9}+\frac{1}{4}-1<0$

So Q lies inside both C and E.

Note: We get the result from the figure.

54. Let $P(5\cos\theta, 3\sin\theta)$

Equation of the tangent at P is

$$\frac{x\cos\theta}{5}+\frac{y\sin\theta}{3}=1$$

$$\Rightarrow CF=\frac{1}{\sqrt{\frac{\cos^2\theta}{25}+\frac{\sin^2\theta}{9}}}=\frac{15}{\sqrt{9\cos^2\theta+25\sin^2\theta}}$$

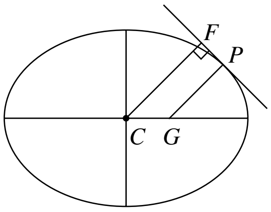


Fig. 19.25

Equation of the normal at P is

$$\frac{5x}{\cos\theta}-\frac{3y}{\sin\theta}=25-9=16$$

Coordinates of G are $\left(\frac{16\cos\theta}{5}, 0\right)$

$$\Rightarrow PG=\frac{3}{5}\sqrt{9\cos^2\theta+25\sin^2\theta}$$

Hence $CF \cdot PG = 9$.

55. Centre of the circle is (– 1, 0) and radius is 4. Equation of the tangent at $P(\theta)$ to the ellipse is

$$\frac{x}{4}\cos\theta + \frac{y}{16/\sqrt{11}}\sin\theta = 1$$

If this touches the circle, then

$$\frac{\frac{-\cos\theta}{4} - 1}{\sqrt{\frac{\cos^2\theta}{16} + \frac{\sin^2\theta}{256/11}}} = 4$$

$$\Rightarrow (\cos\theta + 4)^2 = 256\left(\frac{\cos^2\theta}{16} + \frac{11\sin^2\theta}{256}\right)$$

$\Rightarrow 4\cos^2\theta - 8\cos\theta - 5 = 0$

$\Rightarrow (2\cos\theta - 5)(2\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = -1/2 \Rightarrow \theta = 2\pi/3$

56. Let the extremities of a pair of semi-conjugate diameters of the ellipse be $P(3\cos\theta, 2\sin\theta)$ and $Q(3\cos\alpha, 2\sin\alpha)$, C being the centre.

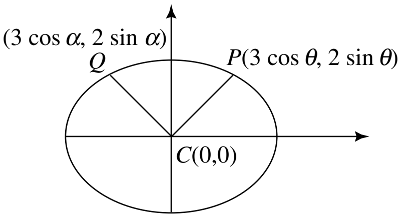


Fig. 19.26

Then slope of $CP \times$ slope of $CQ = -\frac{4}{9}$

$$\Rightarrow \frac{2}{3}\tan\theta \times \frac{2}{3}\tan\alpha = -\frac{4}{9}$$

$\Rightarrow 1 + \tan\theta\tan\alpha = 0 \Rightarrow \alpha - \theta = \pi/2$

$\Rightarrow$ coordinates of Q are $(-3\sin\theta, 2\cos\theta)$ and the equation of PQ is

$2(\sin\theta - \cos\theta)x - 3(\cos\theta + \sin\theta)y + 6 = 0$

which represents $lx + my + n = 0$, then

$$\frac{l}{2(\sin\theta - \cos\theta)} = \frac{m}{-3(\cos\theta + \sin\theta)} = \frac{n}{6}$$

So $\frac{9l^2 + 4m^2}{n^2} = (\sin\theta - \cos\theta)^2 + (\cos\theta + \sin\theta)^2 = 2.$

57. We must have $f(a^2 + 5a + 3) < f(3a + 15)$

$\Rightarrow a^2 + 5a + 3 > 3a + 15$ [$\because f$ is strictly decreasing]

$\Rightarrow a^2 + 2a - 12 > 0$

$\Rightarrow (a + 6)(a - 2) > 0$

$\Rightarrow a \notin (-6, 2)$.

58. The director circle of E_1 is $x^2 + y^2 = a^2 + 2 + b^2$ and of E_2 is $x^2 + y^2 = a^2 + b^2 + 1$.

Both are concentric circle such that one lies inside the other so that they have no point in common.

59. Equation of the tangent at $P(a\cos\theta, b\sin\theta)$ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

$$d = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}} \Rightarrow \frac{1}{d^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \quad (1)$$

$F_1(-ae, 0)$, $F_2(ae, 0)$

$$\Rightarrow PF_1 = e\left(a\cos\theta + \frac{a}{e}\right) = a(1 + e\cos\theta)$$

$PF_2 = a(1 - e\cos\theta)$

So $(PF_1 - PF_2)^2 = 4a^2e^2\cos^2\theta$

$$= 4(a^2 - b^2) \times \frac{\frac{1}{d^2} - \frac{1}{b^2}}{\frac{1}{a^2} - \frac{1}{b^2}}$$

$$= 4a^2\left(1 - \frac{b^2}{d^2}\right).$$

60. Let the middle point of the chord be $(t^2, 2t)$

It must lie inside the ellipse

So $t^4 + 8t^2 - 1 < 0$

$\Rightarrow t^2 \in (0, -4 + \sqrt{17})$

Equation of the chord of the ellipse with $(t^2, 2t)$ as the mid-point is

$t^2x + 4ty = t^4 + 8t^2$ $(T = S_1)$

Since it passes through $(a, 0)$

$at^2 = t^4 + 8t^2$

$\Rightarrow t^4 + (8 - a)t^2 = 0$

$\Rightarrow t^2 = 0$ or $t^2 = a - 8$

$\Rightarrow a = t^2 + 8$

$\Rightarrow a \in (8, 4 + \sqrt{17})$

Previous Years' AIEEE/JEE Main Questions

1. Equation one of directrix of ellipse with centre at the origin is $x = a/e$

$\therefore \frac{a}{e} = 4 \Rightarrow a = 2$ [$\because e = 1/2$]

Also, $b^2 = a^2(1 - e^2) = 4\left(1 - \frac{1}{4}\right) = 3$

Thus, equation of required ellipse is

$$\frac{x^2}{4}+\frac{y^2}{3}=1 \text{ or } 3x^2+4y^2=12$$

2. $AB = 2\,b\sin\theta$

   $AD = 2\,a\cos\theta$

   Area of rectangle = $4ab\sin\theta\cos\theta$

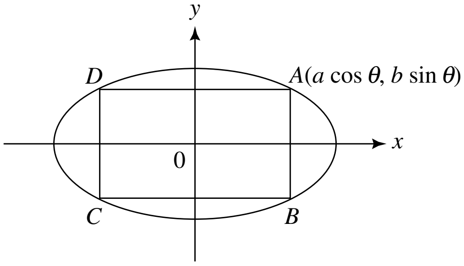


Fig. 19.27

   $= 2ab\sin 2\theta$

   Greatest area is obtained when $\theta = \pi/4$ and greatest area = $2ab$.

3. Slope of BF is $m_1 = \frac{-b}{ae}$

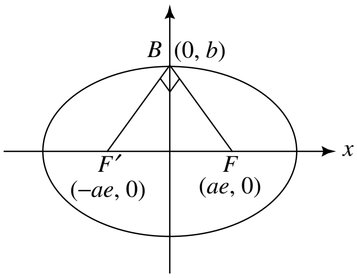


Fig. 19.28

   and that of BF' is $m_2 = \frac{-b}{-ae} = \frac{b}{ae}$

   As $BF \perp BF'$, $m_1 m_2 = -1$

   $$\Rightarrow \frac{b^2}{a^2e^2} = 1$$

   $$\Rightarrow a^2(1-e^2) = a^2e^2$$

   $$\Rightarrow 2e^2 = 1 \text{ or } e = \frac{1}{\sqrt{2}}$$

4. We have $2ae = 6$ and $2b = 8$

   $\Rightarrow ae = 3,\ b = 4$

   Now, $b^2 = a^2(1-e^2)$

   $\Rightarrow 16 = a^2 - 9 \Rightarrow a = 5$

   $\therefore e = 3/5$

5. Equation of the ellipse is

   $$x^2+y^2 = \frac{1}{4}(4-x)^2$$

   $$\Rightarrow 3x^2+8x+4y^2 = 16$$

   $$\Rightarrow \left(x+\frac{4}{3}\right)^2+\frac{4}{3}y^2 = \frac{16}{3}+\frac{16}{9} = \frac{64}{9}$$

   $$\Rightarrow \frac{\left(x+\frac{4}{3}\right)^2}{\frac{64}{9}}+\frac{y^2}{\frac{3}{4}\times\frac{64}{9}} = 1$$

   Length of the semi-major axis = $\sqrt{\frac{64}{9}} = \frac{8}{3}$.

6. The required ellipse passes through (4, 0) and (2, 1). Let equation of ellipse be

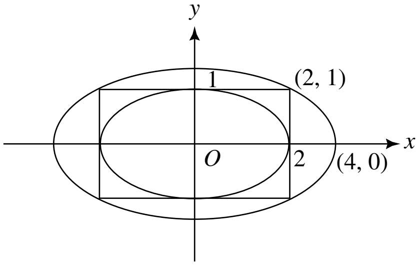


Fig. 19.29

   $$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

   As it pass through (4, 0) and (2, 1)

   $$\frac{16}{a^2} = 1 \text{ and } \frac{4}{a^2}+\frac{1}{b^2} = 1$$

   $\Rightarrow a^2 = 16$ and $b^2 = 4/3$

   Thus, required ellipse is

   $$\frac{x^2}{16}+\frac{y^2}{4/3} = 1 \Rightarrow x^2+12y^2 = 16$$

7. Let the equation of the ellipse be

   $$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

   Since it passes through (– 3, 1), we have

   $$\frac{9}{a^2}+\frac{1}{b^2} = 1, \text{ also } b^2 = a^2\left(1-\frac{2}{5}\right) = \frac{3}{5}a^2$$

   $$\Rightarrow 9+\frac{5}{3} = a^2 \Rightarrow a^2 = \frac{32}{3},\ b^2 = \frac{32}{5}.$$

   and the required equation is

   $$\frac{x^2}{\frac{32}{3}}+\frac{y^2}{\frac{32}{5}} = 1 \Rightarrow 3x^2+5y^2 = 32$$

8. Minor axis is along the x-axis and its length is 2(1) = 2. Major axis is along the y-axis and its length is 2(2) = 4.

Equation of the ellipse is

$$\frac{x^2}{1^2}+\frac{y^2}{2^2} = 1 \Rightarrow 4x^2 + y^2 = 4$$

9. We have $x^2 + 4y^2 = 4c^2$ as the ellipse.

Points of intersection with the circle $x^2 + y^2 = 9a^2$ are given by $9a^2 - y^2 + 4y^2 = 4c^2$

$\Rightarrow 3y^2 = 4c^2 - 9a^2$

$\Rightarrow 4c^2 > 9a^2 \Rightarrow 3a < 2c$

$\Rightarrow 9a^2 + 2c^2 < 6c^2, 9ac < 6c^2$

$\Rightarrow 9ac - (9a^2 + 2c^2) < 0$

10. $e_1^2 = \frac{3-2}{3} = \frac{1}{3},\ e_2^2 = \frac{16-b^2}{16}$

$$(e_1e_2)^2 = \frac{16-b^2}{48} = \left(\frac{1}{2}\right)^2$$

$\Rightarrow 16 - b^2 = 12 \Rightarrow b^2 = 4 \Rightarrow b = 2$

and the required length is 4.

11. Let $P(x_1, y_1)$ be a common point.

Tangents to the curves at Pare

$$yy_1 = 8(x + x_1) \text{ and } \frac{xx_1}{\alpha}+\frac{yy_1}{4} = 1$$

$$\text{Product of the slopes} = \frac{8}{y_1}\times\left(\frac{-4x_1}{\alpha y_1}\right) = -1$$

$$\Rightarrow y_1^2 = \frac{32}{\alpha}x_1 \qquad (1)$$

As P lies on the parabola $y^2 = 16x$.

$$\Rightarrow y_1^2 = 16x_1 \qquad (2)$$

From (1) and (2) we get $\alpha = 2$

12. Slope of the tangent is $-\frac{1}{2}$, so its equation is

$$y = -\frac{1}{2}x \pm \sqrt{9\times\frac{1}{4}+4} \Rightarrow x + 2y = \pm 5$$

If it meets the parabola $\frac{x^2}{9}+\frac{y^2}{4} = 1$ at (x_1, y_1)

its equation is $\frac{xx_1}{9}+\frac{yy_1}{4} = 1$

So $\frac{x_1}{9} = \frac{y_1}{8} = \pm\frac{1}{5}$

$\Rightarrow (x_1, y_1) = \left(\frac{9}{5},\frac{8}{5}\right)$ taking +ve sign.

13. Equation of any tangent to the ellipse $x^2 + 3y^2 = 6$ is $y = mx \pm\sqrt{6m^2+2}$

If (h, k) is the foot of the perpendicular from (0, 0), the centre of the ellipse on this tangent then

$$k = mh \pm\sqrt{6m^2+2} \text{ and } \left(\frac{k}{h}\right) m = -1$$

Eliminating m we get

$(h^2 + k^2)^2 = 6h^2 + 2k^2$

Required locus is $(x^2 + y^2)^2 = 6x^2 + 2y^2$

14. Coordinates of F_1, F_2 and B are $(ae, 0)$, $(-ae, 0)$ and $(0, b)$ respectively. Slope of $BF_1 = -b/ae$ and slope of BF_2 is b/ae.

As $F_1B \perp F_2B$,

$$\left(\frac{-b}{ae}\right)\left(\frac{b}{ae}\right) = -1 \Rightarrow b^2 = a^2e^2$$

$\Rightarrow a^2(1 - e^2) = a^2e^2 \Rightarrow e^2 = 1/2$

15. An equation of tangent to the ellipse

$\frac{x^2}{16}+\frac{y^2}{81} = 1$ is

$$\frac{x}{4}\cos\theta+\frac{y}{9}\sin\theta = 1$$

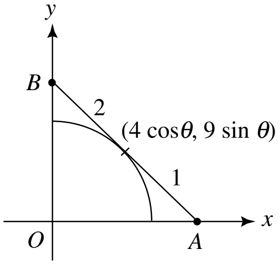


Fig. 19.30

Coordinates of A and B are $(4\sec\theta, 0)$ and $(0, 9\operatorname{cosec}\theta)$ respectively.

$$\text{Area of } \Delta OAB = \frac{1}{2}\frac{36}{|\sin\theta\cos\theta|} = \frac{36}{\sin 2\theta}$$

So minimum value of area 36.

16. We have $a^2 = 9$, $b^2 = 5$, $e^2 = 1-\frac{5}{9} = \frac{4}{9} \Rightarrow e = \frac{2}{3}$.

Coordinates of L are $(ae, b^2/a) = (2, 5/3)$

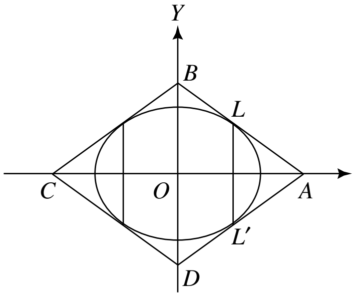


Fig. 19.31

Equation of tangent at L is

$$\frac{2x}{9}+\frac{5}{3}\left(\frac{y}{5}\right)=1$$

or $2x + 3y = 9$

It meets the axes at $A(9/2, 0)$ and $B(0, 3)$.

Area of quadrilateral

$= 4[\text{area of } \Delta OAB]$

$$= 4\left[\frac{1}{2}\left(\frac{9}{2}\right)(3)\right] = 27\ (\text{unit})^2$$

17. We have distance between the foci $= 2ae$

length of the latus rectum $= \dfrac{2b^2}{a}$

So $2\,ae = \dfrac{1}{2}\left(2\dfrac{b^2}{a}\right) \Rightarrow b^2 = 2\,a^2e$

$\Rightarrow a^2(1-e^2) = 2\,a^2e$

$\Rightarrow e^2 + 2e = 1 \Rightarrow (e+1)^2 = 2$

$\Rightarrow e = \sqrt{2}-1$

18. An equation of tangent at $(3\sqrt{3}\cos\theta, \sqrt{3}\sin\theta)$ to the ellipse $\dfrac{1}{27}x^2+\dfrac{1}{3}y^2 = 1$ is

$x\cos\theta + 3y\sin\theta = 3\sqrt{3}$.

It meets the axes in $A\,(3\sqrt{3}\sec\theta, 0)$, and

$B\ \ (0, \sqrt{3}\operatorname{cosec}\theta)$

Let A = area of ΔOAB is $\dfrac{1}{2}|(3\sqrt{3}\sec\theta)(\sqrt{3}\operatorname{cosec}\theta)|$

$$= \frac{9}{|\sin 2\theta|} \geq 9$$

Thus, least value of A is 9 which is attained when $\theta = \pi/4, 3\pi/4, 5\pi/4$, or $7\pi/4$.

Previous Years' B-Architecture Entrance Examination Questions

1. $2\,\dfrac{a}{e} = 4\times 2ae \Rightarrow e = \dfrac{1}{2}$

$a = 2$, so $b^2 = 4\left(1-\dfrac{1}{4}\right) = 3$

Hence the required equation is

$$\frac{x^2}{4}+\frac{y^2}{3} = 1 \Rightarrow 3x^2 + 4y^2 = 12$$

2. Tangent at $\left(1,\dfrac{3}{4}\right)$ to the ellipse $3x^2 + 16y^2 = 12$ is

$$\frac{x\cdot 1}{4}+\frac{y\left(\frac{3}{4}\right)}{\frac{3}{4}} = 1$$

$\Rightarrow x + 4y = 4$

which intersects the curve $y^2 + x = 0$ at points for which

$y^2 + (4 - 4y) = 0 \Rightarrow (y-2)^2 = 0 \Rightarrow y = 2$

and the points of intersection is $(-4, 2)$ exactly one point.

3. Equation of the tangent with slope tan 135° to the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{9} = 1$ is $y = -x + \sqrt{16(1)+9}$

$\Rightarrow x + y = 5$ which meets the given ellipse at $\left(\dfrac{16}{5},\dfrac{9}{5}\right)$

4. Suppose the tangent intersect at (h, k), then equation of chord of contact is

$4hx + ky = 5$.

But equation of chord of contact is:

$2x + y = 3$

$\therefore\ \dfrac{4h}{2} = \dfrac{k}{1} = \dfrac{5}{3}$

$\Rightarrow h = 5/6$, $k = 5/3$.

Thus, required point is $\left(\dfrac{5}{6},\dfrac{5}{3}\right)$.

CHAPTER TWENTY

Hyperbola

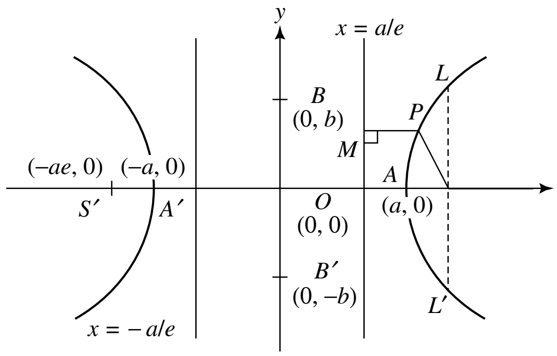
Definition A hyperbola is the locus of a point which moves in a plane in such a way that the ratio of its distance from a fixed point in the plane to its distance from a fixed line in the plane (not passing through the fixed point) is a constant and equal to e, where $e > 1$.

(i) Fixed point is called a *focus* of the hyperbola.

(ii) Fixed line is called a *directrix* of the hyperbola.

(iii) Constant ratio e, is called the eccentricity of the hyperbola.

Standard equation of a hyperbola referred to its principal axes along the coordinate axies is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $b^2 = a^2(e^2 - 1)$.



Hyperbola ($e > 1$)

Fig. 20.1

Let $P(x, y)$ be any point on the hyperbola.

$S(ae, 0)$ be a focus and $x = \frac{a}{e}$ be a directrix then according to the definition

$$(x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$$

Since $e > 1$, $e^2 - 1 > 0$ so $a^2(e^2 - 1) = b^2$ and the required equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

In the figure

(i) $A'A$ is the *transverse axis* of the hyperbola along the axis of x and is of length $2a$. $A(a, 0)$, $A^1(-a, 0)$ are the vertices of the hyperbola.

(ii) $b'b$ is the *conjugate axis* of the hyperbola along the axis of y and is of length $2b$.

(iii) $O(0, 0)$ is the centre of the hyperbola.

(iv) $e = \sqrt{\frac{a^2 + b^2}{a^2}}$ is the eccentricity of the hyperbola.

From symmetry we observe that if $S'(-ae, 0)$ be taken as the focus and $x = -\frac{a}{e}$ is taken as the directrix, same hyperbola is described. So

(v) $S'(-ae, 0)$ and $S(ae, 0)$ are the two foci of the hyperbola.

(vi) $x = -\frac{a}{e}$ and $x = \frac{a}{e}$ are the two directrices of the hyperbola.

(vii) $x = ae$ and $x = -ae$ are the two latera recta of the hyperbola.

(viii) Latus rectum $x = ae$ meets the hyperbola at the points $\left(ae, \pm\frac{b^2}{a}\right)$ so length of each latus rectum is $2b^2/a$.

Illustration 1

Find the eccentricity, length of a latus rectum, equations of the latera recta of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Solution Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a^2 = 16$, $b^2 = 9$.

$\Rightarrow e^2 = \frac{16 + 9}{16} \Rightarrow e = \frac{5}{4}$.

Lengths of a latus rectum = $\frac{2b^2}{a} = \frac{2\times 9}{4} = \frac{9}{2}$.

Equations of the latera recta are $x = \pm ae$

$\Rightarrow x = \pm\ 4\times\frac{5}{4} \Rightarrow x = \pm 5.$

Illustration 2

Find the centre, foci and the eccentricity of the hyperbola. $11x^2 - 25y^2 - 44x + 50y - 256 = 0.$

Solution: The equation of the hyperbola can be written as $11(x - 2)^2 - 25(y - 1)^2 = 275$

$$\Rightarrow \frac{(x-2)^2}{25} - \frac{(y-1)^2}{11} = 1$$

which can be written as $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

where $X = x - 2$, $Y = y - 1$, $a^2 = 25$, $b^2 = 11$.
So centre of the hyperbola is $(X = 0, Y = 0) = (2, 1)$

eccentricity $e = \sqrt{\frac{25+11}{25}}$

$\Rightarrow e = \frac{6}{5}$.

So the coordinates of the foci are
$(X = \pm ae, Y = 0)$
$= (x - 2 = \pm 6, y - 1 = 0)$
$= (8, 1)$ and $(-4, 1)$.

Illustration 3

e_1 is the eccentricity of the ellipse $\frac{x^2}{25} + \frac{y^2}{b^2} = 1$ and e_2 is the eccentricity of the hyperbola $\frac{x^2}{16} - \frac{y^2}{b^2} = 1$ such that $e_1e_2 = 1$.

Find the value of b.

Solution: $e_1 = \frac{25-b^2}{25}$ and $e_2 = \frac{16+b^2}{16}$

$e_1e_2 = 1 \Rightarrow (25 - b^2)(16 + b^2) = 25 \times 16.$
$\Rightarrow 9b^2 - b^4 = 0 \Rightarrow b^2 = 9 \Rightarrow b = \pm 3.$

Some Properties and Standard Results for the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

1. The parametric equation of the hyperbola or the coordinates of any point on the hyperbola are $x = a\sec\theta$, $y = b\tan\theta$. The point is denoted by 'θ'. θ is called the eccentric angle of the point.
2. An equation of the tangent at the above point 'θ' is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$ and at (x', y') is $\frac{x\,x'}{a^2} - \frac{y\,y'}{b^2} = 1$.
3. An equation of the normal at the point 'θ' is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ and at (x', y') is $\frac{x-x'}{x'/a^2} + \frac{y-y'}{y'/b^2} = 0$
4. The condition that the line $y = mx + c$ is a tangent to the hyperbola is $c^2 = a^2m^2 - b^2$.
   So equation of any tangent to the hyperbola (not parallel to y-axis) can be written as $y = mx \pm \sqrt{a^2m^2 - b^2}$

Illustration 4

If $y = mx + 1$ is a tangent to the hyperbola $4x^2 - 25y^2 = 100$, find the value of $25m^4 + 5m^2 + 1$.

Solution: Equation of the hyperbola is $\frac{x^2}{25} - \frac{y^2}{4} = 1$

$y = mx + 1$ is a tangent to the hyperbola
$\Rightarrow (1)^2 = 25m^2 - 4 \Rightarrow 5m^2 = 1$
$\Rightarrow 25m^4 + 5m^2 + 1 = 3.$

Illustration 5

Find the equation of the normal to the hyperbola $4x^2 - 9y^2 = 144$ at the point whose eccentric angle θ is $\pi/3$.

Solution: Equation of the hyperbola is $\frac{x^2}{36} - \frac{y^2}{16} = 1$

Coordinates of the point whose eccentric angle is $\pi/3$ are $(6\sec(\pi/3), 4\tan(\pi/3)) = (12, 4\sqrt{3})$

The required equation of the normal is $\frac{6x}{2} + \frac{4y}{\sqrt{3}} = 36 + 16$

$\Rightarrow 3\sqrt{3}\,x + 4y = 52\sqrt{3}$.

5. An equation of the *chord of contact* of the point (x', y') joining the points of contact of the tangents drawn from (x', y') to the hyperbola is $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$
6. An equation of the chord of the hyperbola whose mid-point is (x', y') is $T = S'$, where
   $T \equiv \frac{xx'}{a^2} - \frac{yy'}{b^2} - 1$ and $S' \equiv \frac{x'^2}{a^2} - \frac{y'^2}{b^2} - 1$.
7. An equation of pair of tangents drawn from a point $P(x', y')$ outside the hyperbola to the hyperbola is $SS' = T^2$, where.

$S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$

Illustration 6

Find the equation of the chord of contact of the point (5, 1) to the hyperbola, $x^2 - 4y^2 = 16$. Also find the mid-point of this chord.

Solution: Equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{4} = 1$

Equation of the chord of contact of (5, 1) is $\frac{5x}{16} - \frac{y(1)}{4} =$

$\Rightarrow 5x - 4y = 16.$ (1)

If (α, β) is the mid-point of this chord then its equation is

$$\frac{\alpha x}{16} - \frac{\beta y}{4} - 1 = \frac{\alpha^2}{16} - \frac{\beta^2}{4} - 1 \tag{2}$$

From (1) and (2) we get

$\alpha = \frac{80}{21}, \beta = \frac{16}{21}.$

So that coordinate of the mid-point of the chord are $\left(\frac{80}{21}, \frac{16}{21}\right)$.

8. *Director circle* of a hyperbola is the locus of the point of intersection of the tangents to the hyperbola which intersect at right angles and its equation is $x^2 + y^2 = a^2 - b^2$.
9. *A diameter* of a hyperbola is the locus of mid-points of a system of parallel chords of the hyperbola and its equation is $y = \frac{b^2}{a^2 m} x$ where m is the slope of the parallel chords of the hyperbola which are bisected by it.

   Two diameters of a hyperbola are said to be *conjugate* when each bisects the chords parallel to the other. Thus two diameters $y = mx$ and $y = m'x$ of the hyperbola are conjugate if $mm' = \frac{b^2}{a^2}$.

Illustration 7

Find the points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$ from which two perpendicular tangents can be drawn to the circle $x^2 + y^2 = a^2$.

Solution: Director circle of $x^2 + y^2 = a^2$ is $x^2 + y^2 = 2a^2$ or

$$\frac{x^2}{2a^2} + \frac{y^2}{2a^2} = 1 \tag{1}$$

Equation of the hyperbola can be written as

$$\frac{x^2}{2a^2} - \frac{y^2}{2b^2} = 1 \tag{2}$$

Since this represents the director circle of the given circle. Subtracting (2) from (1) we get

$$\left(\frac{1}{2a^2} + \frac{1}{2b^2}\right) y^2 = 0 \Rightarrow y = 0$$

So the required points are $\left(\pm\sqrt{2}a, 0\right)$.

Illustration 8

If $y = mx + 2\sqrt{5}$ is a tangent to the hyperbola $4x^2 - y^2 = 16$. Find the equation of a diameter of the hyperbola bisecting the chords parallel of the tangent.

Solution: Equation of the hyperbola can be written as $\frac{x^2}{4} - \frac{y^2}{16} = 1$

$y = mx + 2\sqrt{5}$ touches the hyperbola.

$\Rightarrow (2\sqrt{5})^2 = 4m^2 - 16 \Rightarrow m^2 = 9 \Rightarrow m = \pm 3$

So the required equation of a diameter is

$y = \frac{16}{4(\pm 3)} x \Rightarrow 4x \pm 3y = 0.$

10. *Asymptotes* If the length of the perpendicular let fall from a point on a hyperbola. to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola then the straight line is called an *asymptote* of the hyperbola.

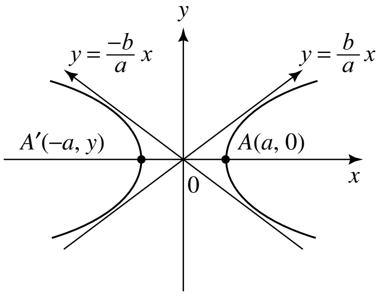


Fig. 20.2

(i) Equation of the asymptotes is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

i.e. $\frac{x}{a} \pm \frac{y}{b} = 0$

(ii) Angle between two asymptotes is $2\tan^{-1}\left(\frac{b}{a}\right)$.

Asymptotes are at right angle if and only if $a = b$.

(iii) If angle between two asymptotes is 2θ, then eccentricity of the hyperbola is $\sec\theta$.

11. *Rectangular hyperbola* A hyperbola in which the lengths of the transverse and conjugate axes are equal is called a rectangular hyperbola.

(i) Equation of a rectangular hyperbola is $x^2 - y^2 = a^2$.

(ii) Eccentricity of the rectangular hyperbola is $\sqrt{2}$.

(iii) Asymptotes of a rectangular hyperbola are at right angles.

(iv) Rotation of the system of coordinates through an angle $\pi/4$ in the clockwise direction (i.e. the axes are taken along the asymptotes of the hyperbola) gives another form of the equation of the rectangular hyperbola i.e. $xy = c^2$.

(v) Some facts about rectangular hyperbola $xy = c^2$

(a) Vertices (c, c) and $(-c, -c)$

(b) Foci $(\sqrt{2}c, \sqrt{2}c)$ and $(-\sqrt{2}c, -\sqrt{2}c)$

(c) Directrices $x + y = \pm\ \sqrt{2}\ c$

(d) length of the latus rectum = $2\sqrt{2}\ c$

(e) parametric equation: $x = ct$, $y = c/t$.

Illustration 9

If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = 25$ in four points, then find the product of the ordinates of these points.

Solution: The ordinates of the points of intersection of $x^2 + y^2 = a^2$ and the hyperbola $xy = 25$ are given by

$$\left(\frac{25}{y}\right)^2 + y^2 = a^2 \text{ or } y^4 - a^2y^2 + (25)^2 = 0$$

If y_1, y_2, y_3, y_4 are the roots of this equation then $y_1\,y_2\,y_3\,y_4 = (25)^2 = 625$.

12. *Conjugate hyperbola* Two hyperbolas such that the transverse and conjugate axes of one hyperbola are, respectively, the conjugate and transverse axis of the other are called conjugate hyperbolas of each other.

    (a) Equation of the hyperbola conjugate to $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $\dfrac{x^2}{b^2} - \dfrac{y^2}{a^2} = 1$

    (b) If e_1 and e_2 are the eccentricities of the hyperbola and its conjugate, then $\dfrac{1}{e_1^2} + \dfrac{1}{e_2^2} = 1$.

    (c) The foci of a hyperbola and its conjugate are concentric and form the vertices of a square.

13. *Similar hyperbolas* Two hyperbolas are said to be similar if they have the same eccentricity.
14. *Equal hyperbola* two hyperbolas are equal if they are similar and have the same latus rectum.
15. *Some more properties of the hyperbola $x^2/a^2 - y^2/b^2 = 1$*

    (i) If PN be the ordinate of a point P on the hyperbola and the tangent at P meets the tranverse axis in T, then $ON.\ OT = a^2$, O being the origin.

    (ii) If PM be drawn perpendiculars to the conjugate axis from a point P on the hyperbola and the tangent at P meets the conjugate axis in T, then $OM.\ OT = -b^2$; O, being the origin.

    (iii) If the normal at P on the hyperbola meets the transverse axis in G, then $SG = eSP$; S being a foci and e the eccentricity of the hyperbola.

    (iv) The tangent and normal at any point of a hyperbola bisect the angle between the focal radii to that point.

    (v) The locus of the feet of the perpendiculars from the foci on a tangent to a hyperbola is the auxillary circle.

    (vi) The product of the perpendiculars from the foci on any tangent to a hyperbola is constant and equal to b^2.

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: If e_1 and e_2 are respectively the eccentricities of the conics $\dfrac{x^2}{25} - \dfrac{y^2}{11} = 1$ and $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$ then e_1e_2 is equal to

(a) $\dfrac{10}{9}$ (b) $\dfrac{4}{3}$

(c) $\dfrac{9}{10}$ (d) $\dfrac{8}{5}$

Ans. (c)

Solution: $e_1 = \sqrt{\dfrac{25+11}{25}} = \dfrac{6}{5}$, $e_2 = \sqrt{\dfrac{16-7}{16}} = \dfrac{3}{4}$.

$$\Rightarrow e_1e_2 = \frac{6}{5} \times \frac{3}{4} = \frac{9}{10}$$

Example 2: If $x - \sqrt{5}\ y + c = 0$ is a tangent to the hyperbola $\dfrac{x^2}{25} - \dfrac{y^2}{4} = 1$, then the value of c is

(a) $\pm 3\sqrt{5}$ (b) $\pm\sqrt{5}$

(c) $\pm 2\sqrt{5}$ (d) none of these

Ans. (b)

Solution: Equation of the tangent is $y = \dfrac{1}{\sqrt{5}}x + \dfrac{c}{\sqrt{5}}$

so $\left(\dfrac{c}{\sqrt{5}}\right)^2 = 25\left(\dfrac{1}{\sqrt{5}}\right)^2 - 4 \Rightarrow c = \pm\sqrt{5}$

Example 3: If the tangent at the point $P(a \sec\theta, b\tan\theta)$ to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ passes through the point where a directrix of the hyperbola meets the positive side of the transverse axis, then θ is equal to

(a) $\cos^{-1}(1/e)$ (b) $\tan^{-1}(1/e)$
(c) $\cot^{-1}(1/e)$ (d) $\sec^{-1}(1/e)$

Where e is the eccentricity of the hyperbola.

Ans. (a)

◎ **Solution:** Equation of the tangent at $P(a\sec\theta, b\tan\theta)$ to the hyperbola is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \quad (1)$$

which passes through the point of intersections of the directrix $x = \frac{a}{e}$ and $y = 0$, the transverse axis of the hyperbola.

i.e. $\left(\frac{a}{e}, 0\right)$. So from (1) we have

$\frac{a}{a \times e}\sec\theta = 1 \Rightarrow \sec\theta = e \Rightarrow \cos\theta = \frac{1}{e}$

$\Rightarrow \theta = \cos^{-1}(1/e)$.

Example 4: The circle described on the line joining the foci of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ as a diameter passes through an end of the latus rectum of the parabola $y^2 = 4ax$, the length of the latus rectum of the parabola is

(a) $2\sqrt{5}$ units (b) 5 units
(c) $4\sqrt{5}$ units (d) $5\sqrt{5}$ units

Ans. (c)

◎ **Solution:** Eccentricity of the hyperbola is given by $e = \sqrt{\frac{16+9}{16}} = \frac{5}{4}$.

Coordinates of the foci are $\left(\pm 4 \times \frac{5}{4}, 0\right) = (\pm 5, 0)$.

So the circle described on the line joining the foci is $x^2 + y^2 = 25$.

If it passes through an end $(a, \pm 2a)$ of the latus rectum of the parabola $y^2 = 4ax$ then $a^2 + 4a^2 = 25 \Rightarrow a^2 = 5 \Rightarrow a = \sqrt{5}$.

Length of the latus rectum $= 4a = 4\sqrt{5}$.

Example 5: The curve represented by $x = 5\left(t + \frac{1}{t}\right)$, $y = \left(t - \frac{1}{t}\right)$, $t \neq 0$ is

(a) a point of straight lines
(b) an ellipse
(c) a hyperbola
(d) a rectangular hyperbola.

Ans. (c)

◎ **Solution:** $\left(\frac{x}{5}\right)^2 - y^2 = 4$

$\Rightarrow \frac{x^2}{100} - \frac{y^2}{4} = 1$

which is a hyperbola.

Example 6: If the distance between two directrices of a rectangular hyperbola is 15, then the distance between its foci in units is:

(a) $15\sqrt{2}$ (b) 30
(c) 60 (d) 45

Ans. We know that the eccentricity of a rectangular hyperbola is $\sqrt{2} = e$. Distance between the directricess $= \frac{2a}{e}$, where $2a$ is the length of the transverse axis.

So $\frac{2a}{e} = 15 \Rightarrow 2a = 15\sqrt{2}$

Distance between the foci $= 2ae = 15\sqrt{2} \times \sqrt{2} = 30$.

Example 7: If two perpendicular tangents can be drawn from the point (α, β) to the hyperbola $x^2 - y^2 = a^2$, then (α, β) lies on

(a) $y = \pm x$ (b) $x^2 + y^2 = a^2$
(c) $x^2 + y^2 = 2a^2$ (d) $y^2 = 4ax$

Ans. (a)

◎ **Solution:** An equation of a tangent to the hyperbola $x^2 - y^2 = a^2$ is $y = mx + a\sqrt{m^2 - 1}$ (1)

Equation of the tangent perpendicular to (1) is

$$y = -\frac{1}{m}x + a\sqrt{\frac{1}{m^2} - 1}$$

$$\Rightarrow my + x = a\sqrt{1 - m^2} \quad (2)$$

We must have $m^2 - 1 \geq 0$ and $1 - m^2 \geq 0$

$\Rightarrow m^2 = 1 \Rightarrow m = \pm 1$ and thus (α, β) lies on $y = \pm x$.

Example 8: The tangent at an extremity (in the first quadrant) of latus rectum of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ meets x-axis and y-axis at A and B respectively. Then $(OA)^2 - (OB)^2$, where O is the origin, is equal to

(a) $-\frac{20}{9}$ (b) $\frac{16}{9}$
(c) 4 (d) $-\frac{4}{3}$

Ans. (a)

◎ **Solution:** Coordinates of the extremity of latus rectum in the first quadrant is $P\left(ae, \frac{b^2}{a}\right) = \left(3, \frac{5}{2}\right)$. Equation of the tangent at P is $\frac{3x}{4} - \frac{5}{2}\left(\frac{y}{5}\right) = 1$ or $3x - 2y = 4$.

which meets x-axis at $A\left(\frac{4}{3}, 0\right)$ and y-axis at $B(0, -2)$.

So $(OA)^2 - (OB)^2 = \frac{16}{9} - 4 = -\frac{20}{9}$.

◉ **Example 9:** If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then eccentricity e of the hyperbola satisfies.

(a) $e = \frac{2}{\sqrt{3}}$ (b) $e = \frac{\sqrt{3}}{2}$

(c) $e > \frac{2}{\sqrt{3}}$ (d) $1 < e < \frac{2}{\sqrt{3}}$

Ans. (c)

◎ **Solution:** As $\underline{|POQ} = 60°$. OP makes an angle of 30° with the positive side of x-axis.

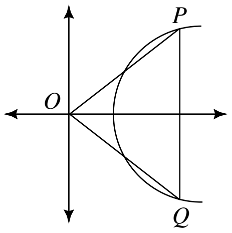


Fig. 20.3

Equation of OP is $y = \frac{1}{\sqrt{3}}x$, which meets the hyperbola at points for which

$$\frac{3y^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow (3b^2 - a^2)y^2 = a^2b^2$$

For real values of y, $3b^2 > a^2$

$$\Rightarrow 3a^2(e^2 - 1) > a^2 \Rightarrow 3e^2 > 4 \Rightarrow e > \frac{2}{\sqrt{3}}.$$

◉ **Example 10:** If the tangent and normal to the hyperbola $x^2 - y^2 = 4$ at a point cut off intercepts a_1 and a_2 respectively on the x-axis, and b_1 and b_2 respectively on the y-axis, then the value of $a_1a_2 + b_1b_2$ is

(a) –1 (b) 0

(c) 4 (d) 1

Ans. (b)

◎ **Solution:** Let the coordinates of a point P on the hyperbola be (x_1, y_1). Equation of the tangent at P to the hyperbola is

$$xx_1 - yy_1 = 4 \Rightarrow a_1 = \frac{4}{x_1},\ b_1 = -\frac{4}{y_1}.$$

Equation of the normal is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$$

$$y_1x + x_1y = 2x_1y_1$$

$\Rightarrow$ $a_2 = 2x_1$, $b_2 = 2y_1$

So that $a_1a_2 + b_1b_2 = 0$.

◉ **Example 11:** The rectangular hyperbola $xy = 16$ and the circle $x^2 + y^2 = 32$ meet at a point P in the first quadrant. Equation of the common tangent to two curves at P is

(a) $x + y - 4 = 0$ (b) $x + y + 4 = 0$

(c) $x + y - 8 = 0$ (d) $x + y + 8 = 0$

Ans. (c)

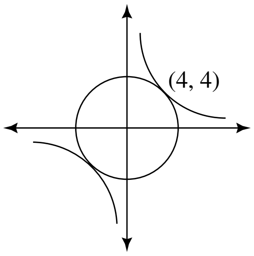


Fig. 20.4

◎ **Solution:** Any point on the hyperbola $xy = 16$ is $\left(4t, \frac{4}{t}\right)$ which meets the circle $x^2 + y^2 = 32$.

If $16t^2 + \frac{16}{t^2} = 32$

$\Rightarrow t^4 - 2t^2 + 1 = 0 \Rightarrow t^2 = 1, t = \pm 1$

Curves meet at the point (4, 4) in the first quadrant, Equation of the tangent at (4, 4) to both the curves is $4x + 4y = 32$ or $x + y = 8$.

◉ **Example 12:** If the hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ passes through the focus of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then eccentricity of the hyperbola is

(a) $\sqrt{2}$ (b) $\sqrt{3}$

(c) 3/2 (d) $\sqrt{3/2}$

Ans. (b)

◎ **Solution:** Let e_1 and e_2 be the eccentricities of the ellipse and hyperbola respectively. Since the hyperbola passes through the focus

$$(\pm ae_1, 0) \Rightarrow b = \pm ae_1$$

For the ellipse $b^2 = a^2(1 - e_1^2) \Rightarrow e_1 = 1/\sqrt{2}$

For the hyperbola $a^2 = b^2(e_2^2 - 1) \Rightarrow e_2 = \sqrt{3}$.

◉ **Example 13:** Which one of the following is independent of α $(0 < \alpha < \pi/2)$ for the hyperbola $\dfrac{x^2}{\cos^2\alpha} - \dfrac{y^2}{\sin^2\alpha} = 1$

(a) eccentricity (b) equation of a directrix
(c) abscissa of foci (d) abscissa of vertices

Ans. (c)

◎ **Solution:** Eccentricity

$$= \sqrt{\frac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha}} = \sqrt{\sec^2\alpha}$$

Equation of the directrix is $x = \pm\dfrac{\cos\alpha}{\sec\alpha} = \pm\cos^3\alpha$

Abscissa of foci = $\cos\alpha\,(\sec\alpha) = 1$
which is independent of α
abscissa of vertices = $\pm\cos\alpha$.

◉ **Example 14:** If e_1 and e_2 are the eccentricities of the hyperbolas $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$ and $\dfrac{y^2}{25} - \dfrac{x^2}{16} = 1$ respectively, then $\dfrac{1}{e_1^2} + \dfrac{1}{e_2^2}$ is equal to

(a) $\dfrac{5}{4}$ (b) $\dfrac{4}{5}$
(c) 1 (d) $\dfrac{1}{2}$

Ans. (c)

◎ **Solution:** $e_1^2 = \dfrac{25+16}{25}$, $e_2^2 = \dfrac{25+16}{16}$

$$\Rightarrow \frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$$

◉ **Example 15:** The normal at $P(x_1, y_1)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ meets the coordinate axes at A and B. If O, is the origin and e, the eccentricity of the hyperbola, then

(a) $OA = e^2x_1$ (b) $OB = e^2y_1$
(c) $OA = e^2y_1$ (d) $OB = e^2x_1$

Ans. (a)

◎ **Solution:** Equation of the normal at $P(x_1, y_1)$ to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is

$$\frac{x - x_1}{\dfrac{x_1}{a^2}} + \frac{y - y_1}{\dfrac{y_1}{b^2}} = 0.$$

$$\Rightarrow OA = b^2\left(\frac{x_1}{a^2}\right) + x_1 = \frac{a^2 + b^2}{a^2}x_1 = e^2x_1$$

$$OB = a^2\left(\frac{y_1}{b^2}\right) + y_1 = \frac{a^2 + b^2}{b^2}y_1 = \frac{a^2e^2}{b^2}y_1$$

LEVEL 1

Straight Objective Type Questions

◉ **Example 16:** If the latus rectum of a hyperbola subtend an angle of 60° at the other focus, then eccentricity of the hyperbola is

(a) 2 (b) $\dfrac{\sqrt{3}+1}{2}$
(c) $2\sqrt{3}$ (d) $\sqrt{3}$

Ans. (d)

◎ **Solution:** Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$F_1(-ae, 0)$, $F_2(ae, 0)$ be the foci, e being the eccentricity. Let LF_2L', the latus rectum subtend an angle of 60° at F_1, then slope of LF_1 is tan 30°.

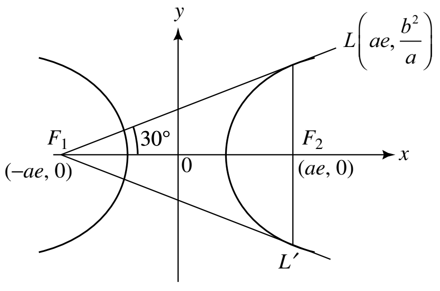


Fig. 20.5

So $\tan 30^\circ = \dfrac{b^2/a}{2ae}$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \frac{a^2(e^2-1)}{2a^2e} = \frac{e^2-1}{2e}$$

$$\Rightarrow \quad e^2 - 1 = \frac{2}{\sqrt{3}}e \Rightarrow \left(e - \frac{1}{\sqrt{3}}\right)^2 = 1 + \frac{1}{3}$$

$$\Rightarrow \quad e - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} \Rightarrow e = \sqrt{3}$$

◉ **Example 17:** The locus of the foot of the perpendicular drawn from the origin to any tangent to the hyperbola $\dfrac{x^2}{36} - \dfrac{y^2}{16} = 1$ is

(a) $(x^2 + y^2)^2 = 36x^2 - 16y^2$
(b) $(x^2 - y^2)^2 = 36x^2 - 16y^2$
(c) $(x^2 + y^2)^2 = 36x^2 + 16y^2$
(d) $(x^2 - y^2)^2 = 36x^2 + 16y^2$

Ans. (a)

◎ **Solution:** Any tangent to the hyperbola is

$$y = mx + \sqrt{36m^2 - 16} \qquad (1)$$

Equation of the line through the origin perpendicular to the tangent is

$$y = -\frac{1}{m}x. \qquad (2)$$

Eliminating m from (1) and (2) we get the required locus as

$$y = -\frac{x^2}{y} + \sqrt{36\frac{x^2}{y^2} - 16}$$

$\Rightarrow \quad (x^2 + y^2)^2 = 36x^2 - 16y^2.$

◉ **Example 18:** The locus of the middle points of the portions of the tangents of the hyperbola.

$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ included between the axes is

(a) $a^2x^2 - b^2y^2 = x^2y^2$
(b) $b^2x^2 - a^2y^2 = x^2y^2$
(c) $b^2x^2 - a^2y^2 + 4x^2y^2 = 0$
(d) $a^2x^2 - b^2y^2 + 4x^2y^2 = 0$

Ans. (c)

◎ **Solution:** Equation of a tangent to the hyperbola is $\dfrac{x}{a}\sec\theta - \dfrac{y}{b}\tan\theta = 1$ which meets the axes at $A\ (a\cos\theta, 0)$ and $B\ (0, -b\cot\theta)$.

If (h, k) is the midpoint of AB then $h = \dfrac{a\cos\theta}{2}$, $k = \dfrac{-b\cot\theta}{2}$.

Eliminating θ, we get the required locus

$$\left(\frac{a}{2h}\right)^2 - \left(\frac{-b}{2k}\right)^2 = 1$$

$\Rightarrow b^2x^2 - a^2y^2 + 4x^2y^2 = 0$

◉ **Example 19:** If the slope of a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $2\sqrt{2}$, then the eccentricity e of the hyperbola lies in the interval.

(a) $(1, \sqrt{2}]$ (b) $(1, 2\sqrt{2}]$
(c) $(1, 3]$ (d) $(1, 4]$

Ans. (c)

◎ **Solution:** Any tangent to the hyperbola is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

we must have $a^2m^2 - b^2 \geq 0$.

$$\Rightarrow \quad m^2 \geq \frac{b^2}{a^2}$$

$$\Rightarrow \quad m^2 \geq e^2 - 1$$

$$\Rightarrow \quad 1 < e \leq \sqrt{1 + m^2}$$

$$\Rightarrow \quad 1 < e \leq 3 \text{ when } m = 2\sqrt{2}.$$

◉ **Example 20:** If $\dfrac{x^2}{\lambda + 3} + \dfrac{y^2}{2 - \lambda} = 1$ represents a hyperbola, then

(a) $\lambda \notin (2, 3)$ (b) $\lambda \notin (-2, 3)$
(c) $\lambda \notin (-3, 2)$ (d) $\lambda \notin (2, \infty)$

Ans. (c)

◎ **Solution:** $\lambda + 3$ and $2 - \lambda$ must be of opposite signs
$\Rightarrow \lambda + 3 > 0$ and $2 - \lambda < 0$ or $\lambda + 3 < 0$ and $2 - \lambda > 0$
$\Rightarrow \lambda > -3$ and $\lambda > 2$ or $\lambda < -3$ and $\lambda < 2$
$\Rightarrow \lambda > 2$ and $\lambda < -3 \Rightarrow \lambda \notin (-3, 2)$

◉ **Example 21:** The normal to the curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is

(a) ellipse (b) parabola
(c) circle (d) hyperbola or ellipse

Ans. (d)

◎ **Solution:** Equation of the normal at (x, y) is $Y - y = -\dfrac{dx}{dy}(X - x)$ which meets the x-axis at $G\left(0, x + y\dfrac{dy}{dx}\right)$, then

$$x + y\frac{dy}{dx} = \pm 2x$$

$$\Rightarrow x + y\frac{dy}{dx} = 2x \Rightarrow y\,dy - x\,dx = 0 \Rightarrow x^2 - y^2 = c$$

or $y\,dy = -3x\,dx \Rightarrow 3x^2 + y^2 = c$

Thus the curve is either hyperbola or ellipse.

◉ **Example 22:** If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$ $R(x_3, y_3)$, $S(x_4, y_4)$, then which of the following need not hold.

(a) $x_1 + x_2 + x_3 + x_4 = 0$
(b) $x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = c^4$
(c) $y_1 + y_2 + y_3 + y_4 = 0$
(d) $x_1 + y_2 + x_3 + y_4 = 0$

Ans. (d)

◎ **Solution:** The abcissaes of the points of intersection of the given curves are connected by

$$x^2 + \frac{c^4}{x^2} = a^2 \qquad \Rightarrow \quad x^4 - a^2x^2 + c^4 = 0$$

As x_1, x_2, x_3, x_4 are the roots of this equation
$x_1 + x_2 + x_3 + x_4 = 0$, $x_1 x_2 x_3 x_4 = c^4$
Similarly $y_1 + y_2 + y_3 + y_4 = 0$, $y_1 y_2 y_3 y_4 = c^4$

◉ **Example 23:** If the normal to the rectangular hyperbola $xy = c^2$ at the point $(ct, c/t)$ meets the curve again at $(ct', c/t')$, then

(a) $t^3t' = 1$ (b) $t^3 t' = -1$
(c) $t t' = 1$ (d) $t t' = -1$

Ans. (b)

◎ **Solution:** Equation of the tangent at $(ct, c/t)$ to the hyperbola $xy = c^2$ is

$$\frac{cx}{t} + cty = 2c^2$$

Slope of the tangent $= -1/t^2$ and slope of the normal $= t^2$
Equation of the normal at $(ct, c/t)$ is $y - c/t = t^2(x - ct)$
If it passes through $(ct', c/t')$, then

$$\left(\frac{c}{t'} - \frac{c}{t}\right) = t^2(ct' - ct)$$

$$\Rightarrow \qquad (t - t') = t^3 t'(t' - t) \Rightarrow t^3t' = -1$$

◉ **Example 24:** If the normal at P to the rectangular hyperbola $x^2 - y^2 = 4$ meets the axes of x and y in G and g respectively and C is the centre of the hyperbola, then $2PC =$

(a) PG (b) Pg
(c) Gg (d) none of these

Ans. (c)

◎ **Solution:** Let $P(x_1, y_1)$ be any point on the hyperbola $x^2 - y^2 = 4$ then equation of the normal at P is

$$y - y_1 = \frac{-y_1}{x_1}(x - x_1)$$

$$\Rightarrow \qquad x_1y + y_1x = 2x_1y_1.$$

Then coordinates of G are $(2x_1, 0)$ and of g are $(0, 2y_1)$

so that $PG = \sqrt{(2x_1 - x_1)^2 + y_1^2} = \sqrt{x_1^2 + y_1^2} = PC$

$$Pg = \sqrt{x_1^2 + (2y_1 - y_1)^2} = \sqrt{x_1^2 + y_1^2} = PC$$

and $Gg = \sqrt{(2x_1)^2 + (2y_1)^2} = 2\sqrt{x_1^2 + y_1^2} = 2\,PC$

◉ **Example 25:** If e_1, e_2 are the eccentricities of the hyperbola $2x^2 - 2y^2 = 1$ and the ellipse $x^2 + 2y^2 = 2$ respectively. Then

(a) $e_1 + e_2 = 1$ (b) $e_1e_2 = 1$
(c) $e_1^2 + e_2^2 = 1$ (d) none of these

Ans. (b)

◎ **Solution:** Equation of the hyperbola is

$$\frac{x^2}{1/2} - \frac{y^2}{1/2} = 1$$

So $\qquad e_1^2 = \dfrac{a^2 + b^2}{a^2}$ where $a^2 = b^2 = \dfrac{1}{2}$

$\Rightarrow \qquad e_1 = \sqrt{2}$ (Note: It is a rectangular hyperbola)

Equation of the ellipse is

$$\frac{x^2}{2} + \frac{y^2}{1} = 1$$

So $\qquad e_2^2 = \dfrac{a^2 - b^2}{a^2} = \dfrac{2 - 1}{2} = \dfrac{1}{2}$

$\Rightarrow \qquad e_2 = 1/\sqrt{2}$ hence $e_1e_2 = 1$.

◉ **Example 26:** The line $2x + y = 1$ touches a hyperbola and passes through the point of intersection of a directrix and the x-axis. The equation of the hyperbola is

(a) $\dfrac{x^2}{1} - \dfrac{y^2}{3} = 1$ (b) $\dfrac{x^2}{1} - \dfrac{y^2}{3} = 2$

(c) $\dfrac{x^2}{3} - \dfrac{y^2}{1} = 1$ (d) $\dfrac{x^2}{3} - \dfrac{y^2}{1} = 2$

Ans. (a)

◎ **Solution:** Let the equation of the hyperbola be $x^2/a^2 - y^2/b^2 = 1$. As the line $2x + y = 1$ touches the hyperbola $(2)^2 a^2 - b^2 = 1$

$\Rightarrow \qquad 4a^2 - b^2 = 1$

$\Rightarrow \qquad 4a^2 + a^2(1 - e^2) = 1$, e being the eccentricity.

$\Rightarrow \qquad 5a^2 - a^2e^2 = 1$

As the line passes through $(a/e, 0)$.

$$2\,a/e + 0 = 1 \Rightarrow a = e/2.$$

So that $5\dfrac{e^2}{4} - \dfrac{e^4}{4} = 1$

$\Rightarrow \qquad e^4 - 5e^2 + 4 = 0$

$\Rightarrow \qquad e^2 = 1$ or $e^2 = 4$. $\Rightarrow e = 2$ as $e^2 \neq 1$

Hence $a = 1$ and $b^2 = 3$ and the required equation is

$$\frac{x^2}{1} - \frac{y^2}{3} = 1.$$

◉ **Example 27:** The equation of the hyperbola whose foci are (–2, 0) and (2, 0) and eccentricity is 2 is given by

(a) $x^2 - 3y^2 = 3$ (b) $3x^2 - y^2 = 3$
(c) $-x^2 + 3y^2 = 3$ (d) $-3x^2 + y^2 = 3$

Ans. (b)

◎ **Solution:** Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$ae = 2 \text{ and } e = 2 \Rightarrow a = 1$$

$$b^2 = a^2(e^2 - 1) = 1(4 - 1) = 3$$

so the equation of the hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{3} = 1$$

or $$3x^2 - y^2 = 3$$

◉ **Example 28:** The eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then

(a) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$

(b) a focus of the hyperbola is $(\sqrt{3}, 0)$

(c) the eccentricity of the hyperbola is $\sqrt{5/3}$

(d) the equation of the hyperbola is $x^2 - 3y^2 = 3$

Ans. (d)

◎ **Solution:** Eccentricity of the ellipse = $\sqrt{1 - \frac{1}{4}} = \sqrt{3}/2$

Foci of the ellipse = $(\pm\sqrt{3}, 0)$

Eccentricity of the hyperbola = $\sqrt{1 + \frac{b^2}{a^2}} = \frac{2}{\sqrt{3}}$.

$$\Rightarrow \quad \frac{b}{a} = \frac{1}{\sqrt{3}}$$

Since the hyperbola passes through the focus of the ellipse.

$$\frac{3}{a^2} - 0 = 1 \Rightarrow a^2 = 3 \text{ and } b^2 = 1$$

and the equation of the hyperbola is

$$\frac{x^2}{3} - \frac{y^2}{1} = 1 \text{ or } x^2 - 3y^2 = 3$$

Focus of the hyperbola is (± 2, 0).

◉ **Example 29:** Let P (6, 3) be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal at the point P intersects the x-axis at (9, 0) then the eccentricity of the hyperbola is

(a) $\sqrt{5/2}$ (b) $\sqrt{3/2}$

(c) $\sqrt{2}$ (d) $\sqrt{3}$

Ans. (b)

◎ **Solution:** The equation of the normal at $P(6, 3)$

$$\frac{a^2 x}{6} + \frac{b^2 y}{3} = a^2 + b^2$$

As it passes through (9, 0)

$$\frac{9a^2}{6} = a^2 + b^2 \Rightarrow a^2 = 2b^2$$

Eccentricity of the hyperbola = $\sqrt{1 + \frac{b^2}{a^2}} = \sqrt{\frac{3}{2}}$

◉ **Example 30:** If both the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then the value of b^2 is

(a) 9 (b) 7
(c) 41 (d) 12

Ans. (b)

◎ **Solution:** Foci of the ellipse are $\left(\pm\sqrt{16 - b^2}, 0\right)$

and Foci of the hyperbola are $\left(\pm\sqrt{\frac{144}{25} + \frac{81}{25}}, 0\right) = (\pm 3, 0)$

Since they coincide

$$16 - b^2 = 9 \Rightarrow b^2 = 7.$$

◉ **Example 31:** If the tangent at the point ($a \sec \alpha$, $b \tan \alpha$) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at T, then the distance of T from a focus of the hyperbola is

(a) $a(e - \cos \alpha)$ (b) $b(e + \cos \alpha)$

(c) $a(e + \cos \alpha)$ (d) $\sqrt{a^2 e^2 + b^2 \cot^2 \alpha}$

Ans. (a)

◎ **Solution:** Equation of the tangent is $\frac{x}{a} \sec \alpha - \frac{y}{b} \tan \alpha = 1$

Coordinates of T are ($a \cos \alpha$, 0) and of a focus are (ae, 0).
Distance of T from the focus ($ae - a \cos \alpha$)

$$= a\,(e - \cos \alpha)$$

(Note $e > \cos \alpha$).

◉ **Example 32:** The distance between the tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$, parallel to the line $y = x + 2$ is

(a) 2 (b) $2\sqrt{2}$

(c) $\sqrt{2}$ (d) 1

Ans. (c)

◎ **Solution:** Equations of the tangents are

$y = mx \pm \sqrt{4m^2 - 3}$ where $m = 1$

$$\Rightarrow \qquad y = x \pm 1$$

Distance between them $= \dfrac{2}{\sqrt{1+1}} = \sqrt{2}$

◉ **Example 33:** The locus of the middle points of the normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is

(a) $(y^2 - x^2)^3 = 4a^2x^2y^2$

(b) $(y^2 - x^2)^2 = 4a^2x^2y^2$

(c) $(y^2 + x^2)^3 = 4a^2x^2y^2$

(d) $(y^2 + x^2)^2 = 4a^2x^2y^2$

Ans. (a)

◎ **Solution:** Equation of the normal at $(a\sec\theta, a\tan\theta)$ to the hyperbola $x^2 - y^2 = a^2$ is

$$\frac{ax}{\sec\theta} + \frac{ay}{\tan\theta} = a^2 + a^2 = 2a^2 \qquad (1)$$

Let (h, k) be the middle point of this normal then its equation is

$$hx - ky - a^2 = h^2 - k^2 - a^2 \qquad (T = S_1)$$

$$\Rightarrow \qquad hx - ky = h^2 - k^2 \qquad (2)$$

Comparing (1) and (2) we get

$$h\sec\theta = -k\tan\theta = \frac{h^2 - k^2}{2a}$$

$$\Rightarrow \qquad \sec\theta = \frac{h^2 - k^2}{2ah}, \tan\theta = \frac{h^2 - k^2}{-2ak}$$

So $$1 = \sec^2\theta - \tan^2\theta = \frac{(h^2 - k^2)^2}{4a^2}\left(\frac{1}{h^2} - \frac{1}{k^2}\right)$$

$$\Rightarrow \qquad (h^2 - k^2)^3 + 4a^2h^2k^2 = 0$$

Locus of (h, k) is $(y^2 - x^2)^3 = 4a^2x^2y^2$

◉ **Example 34:** If $y = mx + 6$ is a tangent to the hyperbola $\dfrac{x^2}{100} - \dfrac{y^2}{49} = 1$ and the parabola $y^2 = 4ax$, then the length of the latus rectum of the parabola is

(a) $6\sqrt{\dfrac{17}{20}}$ (b) $4\sqrt{\dfrac{17}{20}}$

(c) $24\sqrt{\dfrac{17}{20}}$ (d) $\sqrt{\dfrac{17}{20}}$

Ans. (c)

◎ **Solution:** Since $y = mx + 6$ is a tangent to the hyperbola, $(6)^2 = 100m^2 - 49$.

$$\Rightarrow \qquad m^2 = \frac{17}{20} \Rightarrow m = \pm\sqrt{\frac{17}{20}}$$

Since $y = mx + 6$ is a tangent to the parabola

$$y^2 = 4ax;\ 6 = \frac{a}{m} \Rightarrow a = 6m = \pm 6\sqrt{\frac{17}{20}}$$

Length of the latus rectum of the parabola

$$= 4|a| = 24\sqrt{\frac{17}{20}}.$$

◉ **Example 35:** *P* is a point on the hyperbola $\dfrac{x^2}{81} - \dfrac{y^2}{49} = 1$. The tangent at *P* meets the transverse axis at *T*, *N* is the foot of the perpendicular from *P* to the transverse axis. If *O* is the origin, then *ON.OT* is equal to.

(a) 81 (b) 49

(c) –81 (d) – 49.

Ans. (a)

◎ **Solution:** Let the coordinate of *P* be $(9\sec\theta, 7\tan\theta)$ then $ON = 9\sec\theta$.

Equation of the tangent at *P* is $\dfrac{x}{9}\sec\theta - \dfrac{y}{7}\tan\theta = 1$ which meets the transverse axis at $(9\cos\theta, 0) \Rightarrow OT = 9\cos\theta$.

Then $ON.OT = 9\sec\theta \times 9\cos\theta = 81$

◉ **Example 36:** The product of the perpendiculars from the foci on any tangent to the hyperbola $\dfrac{x^2}{64} - \dfrac{y^2}{9} = 1$ is

(a) 8 (b) 9

(c) 16 (d) 18.

Ans. (b)

◎ **Solution:** Equation of a tangent at $(8\sec\theta, 3\tan\theta)$ to the hyperbola is

$$\frac{x}{8}\sec\theta - \frac{y}{3}\tan\theta = 1$$

If *e* is the eccentricity of the hyperbola; product of the perpendiculars

$$= \frac{(e\sec\theta - 1)(-e\sec\theta - 1)}{\dfrac{\sec^2\theta}{64} + \dfrac{\tan^2\theta}{9}}$$

$$= \frac{(1 - e^2\sec^2\theta) \times (64 \times 9)}{9\sec^2\theta + 64\tan^2\theta}$$

$$= \frac{\left[1 - \dfrac{64 - 9}{64}\sec^2\theta\right](64 \times 9)}{9\sec^2\theta + 64\tan^2\theta}$$

$$= \frac{64\tan^2\theta + 9\sec^2\theta}{9\sec^2\theta + 64\tan^2\theta} \times 9 = 9.$$

Example 37: If the normal at P on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meets the transverse axis at G, S is a foci and e the eccentricity of the hyperbola then $SG : SP$ is equal to

(a) a (b) b

(c) e (d) $1/e$.

Ans. (c)

Solution: Equation of the normal at $P(a\sec\theta, b\tan\theta)$ is

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$$

Which meets the transverse axis at

$G\left(\frac{(a^2+b^2)\sec\theta}{a},0\right)$ whose distance from S $(ae, 0)$ is

$$SG=\frac{(a^2+b^2)\sec\theta}{a}-ae$$

$$SP=e\left[a\sec\theta-\frac{a}{e}\right]=a[e\sec\theta-1]$$

$$\frac{SG}{SP}=\frac{[a^2+a^2(e^2-1)]\sec\theta-a^2e}{a\times a[e\sec\theta-1]}$$

$$=\frac{e^2\sec\theta-e}{e\sec\theta-1}=e.$$

Example 38: If the chords of contacts of the tangents from the points (x_1, y_1) and (x_2, y_2) to the hyperbola $2x^2-3y^2=6$ are at right angle, then $4x_1x_2+9y_1y_2$ is equal to

(a) -1 (b) 0

(c) 6 (d) -12

Ans. (b)

Solution: Equation of the chords of contacts are

$$2xx_1-3yy_1=6,\ 2xx_2-3yy_2=6$$

Since they are at right angles $\frac{2x_1}{3y_1}\times\frac{2x_2}{3y_2}=-1$

$$\Rightarrow \quad 4x_1x_2+9y_1y_2=0.$$

Example 39: Consider a branch of the hyperbola $x^2-2y^2-2\sqrt{2}x-4\sqrt{2}y-6=0$ with vertex at the point A. let B be one of the end points of its latus rectum. If C is the focus of the hyperbola near A, then area of the ΔABC is:

(a) $1-\sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{3}{2}}-1$

(c) $1+\sqrt{\frac{2}{3}}$ (d) $\sqrt{\frac{3}{2}}+1$.

Ans. (b)

Solution: Equation of the hyperbola is

$$\frac{(x-\sqrt{2})^2}{4}-\frac{(y+\sqrt{2})^2}{2}=1$$

Which can be written as

$$\frac{X^2}{4}-\frac{Y^2}{2}=1 \text{ where } x=X+\sqrt{2},\ y=Y-\sqrt{2}.$$

Eccentricity of the hyperbola is $\sqrt{\frac{4+2}{4}}=\sqrt{\frac{3}{2}}$

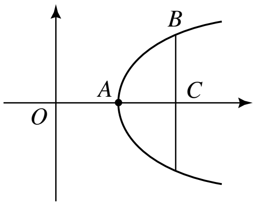


Fig. 20.6

Area of the $\Delta ABC=\frac{1}{2}(AC)(BC)$

$$=\frac{1}{2}(ae-a)\times\frac{b^2}{a} \text{ where } a^2=4,\ b^2=2$$

$$=\frac{1}{2}\left[2\sqrt{\frac{3}{2}}-2\right]\times\frac{2}{2}=\sqrt{\frac{3}{2}}-1$$

Example 40: Normal at point (5, 3) to the rectangular hyperbola $xy-y-2x-2=0$ meets the curve at the point whose coordinates are

(a) $(0, -2)$ (b) $(-1, 0)$

(c) $\left(\frac{1}{4},\frac{-10}{3}\right)$ (d) $\left(\frac{3}{4},-14\right)$.

Ans. (d)

Solution: Equation of the hyperbola can be written as $(x-1)(y-2)=4$

or $$y=2+\frac{4}{x-1}$$

$$\frac{dy}{dx}=-\frac{4}{(x-1)^2}$$

Any point on the hyperbola is

$$\left(1+2t,2+\frac{2}{t}\right)=(5,3)\Rightarrow t=2$$

So slope of the normal at (5, 3) is

$$-\left.\frac{dx}{dy}\right|_{(5,3)}=4$$

Equation of the normal at (5, 3) is $y-3=4(x-5)$ which will pass through $\left(1+2t,2+\frac{2}{t}\right)$ if $2+\frac{2}{t}=4(1+2t)-17$.

$$\Rightarrow 8t^2-15t-2=0\Rightarrow t=2,\ -\frac{1}{8}.$$

For $t=-\frac{1}{8}$, we get the required point as

$$\left(1-\frac{2}{8},2-\frac{2}{1/8}\right)=\left(\frac{3}{4},-14\right)$$

Assertion-Reason Type Questions

Example 41: Statement-1 An ellipse passes through the foci of the hyperbola $9x^2 - 4y^2 = 36$ and its major and minor axes lie along the transverse and conjugate axes respectively of the hyperbola. If the product of the eccentricities of the two conics is $\frac{1}{2}$, then the ellipse does not pass through the point $\left(\frac{1}{2}\sqrt{i3}, \frac{\sqrt{3}}{2}\right)$.

Statement-2 Product of the eccentricities of the hyperbola $9x^2 - 4y^2 = 36$ and the ellipse $\frac{x^2}{65} - \frac{y^2}{60} = 1$ is $\frac{1}{2}$.

Ans. (b)

Solution: Eccentricity e_1 of the hyperbola $9x^2 - 4y^2 = 36$ or $\frac{x^2}{4} - \frac{y^2}{9} = 36$ is $\sqrt{\frac{4+9}{4}} = \frac{\sqrt{13}}{2}$.

Foci of the hyperbola $\left(\pm\sqrt{13}, 0\right)$

If e_2 is the eccentricity of the ellipse then

$$e_1 e_2 = \frac{1}{2} \Rightarrow e_2 = \frac{1}{\sqrt{13}}.$$

Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since it passes through $\left(\pm\sqrt{13}, 0\right)$

$$\frac{13}{a^2} = 1 \Rightarrow a^2 = 13 \text{ and } b^2 = a^2(1 - e_2^2) = 12$$

Equation of the ellipse is $\frac{x^2}{13} + \frac{y^2}{12} = 1$

Which clearly does not pass through $\left(\frac{1}{2}\sqrt{13}, \frac{\sqrt{3}}{2}\right)$

So statement-1 is true.

In statement-2 if e_3 is the eccentricity of the ellipse, then

$$e_3 = \sqrt{\frac{65-60}{65}} = \frac{1}{\sqrt{13}}.$$

$\Rightarrow e_1 e_3 = \frac{1}{2}$, so statement-2 is also true but does not justify statement-1.

Example 42: Statement-1 If a tangent to the hyperbola $2x^2 - 4y^2 = 8$ meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that $OPRQ$ is a rectangle where O is the origin then locus of R is a hyperbola.

Statement-2 The curve described parametrically by $x = \frac{2}{3} + 4t$ and $y = -\frac{4}{7} + \frac{4}{t}$ represents a hyperbola.

Ans. (d)

Solution: Tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ at a point (x_1, y_1) is $\frac{xx_1}{4} - \frac{yy_1}{2} = 1$

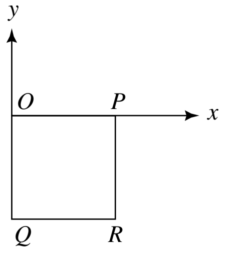


Fig. 20.7

Coordinates of P are $\left(\frac{4}{x_1}, 0\right)$ of Q are $\left(0, \frac{-2}{y_1}\right)$ and of R are $\left(\frac{4}{x_1}, \frac{-2}{y_1}\right) = (h, k)$

Also $\frac{x_1^2}{4} - \frac{y_1^2}{2} = 1$ as (x_1, y_1) lies on the hyperbola

$$\Rightarrow \frac{4}{h^2} - \frac{2}{k^2} = 1$$

Locus of (h, k) is $\frac{4}{x^2} - \frac{2}{y^2} = 1$

Which is not a hyperbola and thus statemnet-1 is false.

In statement-2 we have $\left(x - \frac{2}{3}\right)\left(y + \frac{4}{7}\right) = 16$ which represents a rectangular hyperbola and thus is true.

Example 43: $P(2, 3)$ and $Q(2\tan\theta, 3\sec\theta)$; $0 < \theta < \frac{\pi}{2}$ are two given points. R divides the line segment PQ externally in the ratio 2:3.

Statement-1: Locus of R is a hyperbola length of whose transverse axis is 12.
Statement-2: Locus of Q is a hyperbola length of whose transverse axis is 6.
Ans. (b)

◎ **Solution:** If coordinates of R are (x, y) then

$$x = \frac{3(2) - 2(2\tan\theta)}{3-2},\ y = \frac{3(3) - 2(3\sec\theta)}{3-2}$$

$$\Rightarrow \tan\theta = \frac{1}{4}(6 - x),\ \sec\theta = \frac{1}{6}(9 - y)$$

Eliminating θ, we get the locus of R as

$$\frac{1}{36}(y-9)^2 - \frac{1}{16}(x-6)^2 = 1$$

Which is a hyperbola, length of whose transverse axis is $2(6) = 12$

So statement-1 is true.

Locus of $Q(2\tan\theta, 3\sec\theta)$ is $\left(\frac{y}{3}\right)^2 - \left(\frac{x}{2}\right)^2 = 1$

$$\Rightarrow \frac{y^2}{9} - \frac{x^2}{4} = 1$$

Which is a hyperbola, length of whose transverse axis is $2(3) = 6$.

So statement-2 is also true but does not justify statement-1.

◉ **Example 44: Statement-1:** The line $bx - ay = 0$ does not meet the hyperbola $x^2/a^2 - y^2/b^2 = 1$.
Statement-2: The line $y = mx + c$ does not meet the hyperbola $x^2/a^2 - y^2/b^2 = 1$ if $c^2 = a^2m^2 - b^2$.
Ans. (c)

◎ **Solution:** Statement-1 is true, as solving $y = (b/a)x$ and $x^2/a^2 - y^2/b^2 = 1 \Rightarrow 0 = 1$ which is not possible.
Statement-2 is false as the line is a tangent to the hyperbola and hence meets the hyperbola at one point only. In statement-1, the line satisfies the condition of being a tangent, so it must touch the hyperbola at a point infinity.

◉ **Example 45: Statement-1:** The rectangular hyperbola $xy = 4^2$ does not intersect the parabola $y^2 = 4x$ in real point.

Statement-2: If hyperbola $\frac{x^2}{b^2} - \frac{y^2}{a^2} = 1$ passes through the focus of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then eccentricity of the hyperbola is $\sqrt{3}$.

Ans. (d)

◎ **Solution:** Statement-1 is false as the point (4, 4) lies on both the curves.

In statmetn-2 if e is the eccentricity of the ellipse, then $\frac{a^2e^2}{b^2} = 1 \Rightarrow e^2 = \frac{b^2}{a^2} \Rightarrow \frac{a^2 - b^2}{a^2} = \frac{b^2}{a^2} \Rightarrow a^2 = 2b^2$

eccentricity of the hyperbola is $\sqrt{\frac{b^2 + a^2}{b^2}} = \sqrt{3}$ and thus the statement-2 is true.

◉ **Example 46 Statement-1:** If a hyperbola does not have mutually perpendicular tangents, then its eccentricity is greater than $\sqrt{2}$.
Statement-2: Locus of a point from which two perpendicular tangents can be drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x^2 + y^2 = a^2 - b^2$.
Ans. (a)

◎ **Solution:** Statement-2 is true as $x^2 + y^2 = a^2 - b^2$ is the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ using it in statement-1, if the hyperbola does not have mutually perpendicular tangents then the points on $x^2 + y^2 = a^2 - b^2$ are not real.

$\Rightarrow b^2 > a^2 \Rightarrow e^2 = \frac{a^2 - b^2}{a^2} > 2 \Rightarrow e > \sqrt{2}$ and thus statement-1 is also true.

◉ **Example 47: Statement-1:** The angle between a pair of tangents drawn to the curve $7x^2 - 12y^2 = 84$ from $M(1, 2)$ is $\frac{\pi}{2}$.
Statement-2: Equation of the chord of contact of the point $M(1, 2)$ with respect to the hyperbola $7x^2 - 12y^2 = 84$ is $7x - 24y = 84$.
Ans. (b)

◎ **Solution:** Equation of the hyperbola is $\frac{x^2}{12} - \frac{y^2}{7} = 1$.

Equation of its director circle is $x^2 + y^2 = 12 - 7 = 5$ $M(1, 2)$ lies on it. So the tangents drawn from M to the hyperbola are at right angles. Thus statement-1 is true.
Equation of the chord of contact of $M(1, 2)$ is

$$\frac{x \times 1}{12} - \frac{y(2)}{7} = 1.$$

$\Rightarrow 7x - 24y = 84$. So statement-2 is also true but does not justify statement-1.

◉ **Example 48:** Let H be the rectangular hyperbola $xy = c^2$.
Statement-1: If the sum of the slopes of the normal from a point P to the hyperbola H is $\frac{1}{c^2}$, then P lies on a pair of lines parallel to y-axis.

Statement-2: Four normals can be drawn to the hyperbola H from any point P.

Ans. (c)

◎ **Solution:** Equation of the normal at any point $\left(ct, \frac{c}{t}\right)$ on the hyperbola H is

$$y - \frac{c}{t} = t^2(x - ct) \qquad (1)$$

If it passes through $P(c, c)$, then $1 - \frac{1}{t} = t^2(1 - t)$

$\Rightarrow (t - 1)(t^3 + 1) = 0.$

Which gives only two real values of t, 1 and -1.

So from the point P only two normals can be drawn and the statement-2 is false.

Note: Point $P(c, c)$ lies on the hyperbola. If the point P does not lie on the hyperbola, then there will be four normals from P.

For statement-1, let $P(\alpha, \beta)$ be a point such that sum of the slopes of the normals from P to H is $\frac{1}{c^2}$, then from (1)

$$\beta - \frac{c}{t} = t^2(\alpha - ct)$$

$$\Rightarrow \qquad ct^4 - \alpha t^3 + \beta t - c = 0 \qquad (2)$$

If t_1, t_2, t_3, t_4 are the roots of (2). Then

$$\sum_{i=1}^{4} t_i = \frac{\alpha}{c}, \ \Sigma t_1 t_2 = 0$$

Sum of the slopes of the normals

$$= \sum_{i=1}^{4} t_i^2 = \left(\sum_{i=1}^{4} ti\right)^2 - 2\Sigma t_1 t_2 = \frac{\alpha^2}{c^2} = \frac{1}{c^2}$$

$\Rightarrow \alpha = \pm 1 \Rightarrow P$ lies on the lines $x = \pm 1$ and thus statement-1 is true.

◉ **Example 49:** H: $\frac{x^2}{4} - \frac{y^2}{16} = 1$ is a hyperbola.

Statement-1 Every line which cuts the hyperbola H at two distinct points has slope lying in $(-2, 2)$.

Statement-2 The slope of the tangents of the hyperbola lies in $(-\infty, -2) \cup (2, \infty)$.

Ans. (a)

◎ **Solution:** Equation of the tangent with slope m of the hyperbola H is $y = mx \pm \sqrt{4m^2 - 16}$.

This will represent a real tangent if $4m^2 - 16 > 0$

$\Rightarrow m^2 > 4 \Rightarrow m \in (-\infty, -2) \cup (2, \infty)$

Showing that statement-2 is true.

Using it in statement-1 if any line cuts the hyperbola H at two distinct points, it is not a tangent to H and hence its slope lies in $(-2, 2)$ thus statement-1 is also true.

◉ **Example 50: Statement-1:** If l is the length of the chord of the hyperbola $x^2 - y^2 = 8$ whose mid-point is (4, 2) then $3l^2 = 80$.

Statement-2: Length of the chord of contact of the point $\left(\frac{8}{3}, \frac{4}{3}\right)$ with respect to the hyperbola $x^2 - y^2 = 8$ is $4\sqrt{\frac{5}{3}}$.

Ans. (b)

◎ **Solution:** Equation of the chord whose mid-point is (4, 2) of the hyperbola $x^2 - y^2 = 8$ is

$$4x - 2y = 4^2 - 2^2 = 12$$

$$\Rightarrow \qquad 2x - y = 6$$

Which meets the hyperbola $2x - y = 6$ at points for which.

$$x^2 - (2x - 6)^2 = 8$$

$$\Rightarrow \qquad 3x^2 - 24x + 44 = 0$$

Let (x_1, y_1) and (x_2, y_2) be the end points of the chord, then $x_1 + x_2 = 8$, $x_1 x_2 = \frac{44}{3}$.

$$y_1 = 2x_1 - 6, \ y_2 = 2x_2 - 6$$

$$\begin{aligned} l^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (x_2 - x_1)^2 + 4(x_2 - x_1)^2 \\ &= 5[(x_1 + x_2)^2 - 4x_1 x_2] \\ &= 5\left[8^2 - 4\left(\frac{44}{3}\right)\right] = \frac{80}{3}. \end{aligned}$$

Thus statement-1 is true.

In statement-2, equation of the chord of contact of $\left(\frac{8}{3}, \frac{4}{3}\right)$ to the hyperbola $x^2 - y^2 = 8$ is $x \times \frac{8}{3} - y \times \frac{4}{3} = 8$

$$\Rightarrow \qquad 2x - y = 6.$$

Which is same as the chord in statement-1 and hence the required length is $\sqrt{\frac{80}{3}} = 4\sqrt{\frac{5}{3}}$.

Thus statement-2 is also true but does not justify statement-1.

LEVEL 2

Straight Objective Type Questions

Example 51: If $(a \sec \alpha, b \tan \alpha)$ and $(a \sec \beta, b \tan \beta)$ be the ends of a chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, passing through the focus $(ae, 0)$ then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ is equal to:

(a) $\frac{1+e}{1-e}$ (b) $\frac{e+1}{e-1}$

(c) $\frac{1-e}{1+e}$ (d) $\frac{e-1}{e+1}$

Ans. (c)

Solution: Equation of the chord joining the given points is

$$\begin{vmatrix} x & y & 1 \\ a\sec\alpha & b\tan\alpha & 1 \\ a\sec\beta & b\tan\beta & 1 \end{vmatrix} = 0$$

$bx(\tan\alpha - \tan\beta) - ay(\sec\alpha - \sec\beta) + ab(\sec\alpha \tan\beta - \sec\beta \tan\alpha) = 0$

$\Rightarrow bx \sin(\alpha - \beta) - ay (\cos\beta - \cos\alpha) + ab(\sin\beta - \sin\alpha) = 0$

$$\Rightarrow bx\cos\frac{\alpha-\beta}{2} - ay\sin\frac{\alpha+\beta}{2} - ab\cos\frac{\alpha+\beta}{2} = 0$$

$$\Rightarrow \frac{x}{a}\cos\frac{\alpha-\beta}{2} - \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha+\beta}{2}$$

If it passes through $(0e, 0)$ $e\cos\frac{\alpha-\beta}{2} = \cos\frac{\alpha+\beta}{2}$

$$\Rightarrow \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} = e$$

$$\Rightarrow \frac{\cos\frac{\alpha+\beta}{2} - \cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha-\beta}{2}} = \frac{e-1}{e+1}$$

$$\Rightarrow -\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{e-1}{e+1}$$

$$\Rightarrow \tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1-e}{1+e}$$

Example 52: Equations of the asymptotes of the hyperbola $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$ is

(a) $3x^2 + 10xy + 8y^2 + 14x + 22y + 11 = 0$
(b) $3x^2 + 10xy + 8y^2 - 14x - 22y + 13 = 0$
(c) $3x^2 - 10xy + 8y^2 + 14x + 22y + 15 = 0$
(d) $3x^2 + 10xy + 8y^2 + 14x + 22y + 15 = 0$

Ans. (d)

Solution: Since the equation of a hyperbola differs from the combined equation of the asymptotes by a constant. Let the equation of the asymptotes be.

$$3x^2 + 10xy + 8y^2 + 14x + 22y + \lambda = 0$$

This will represent a pair of straight lines if

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

where $a = 3, b = 8, c = \lambda, f = 11, g = 7, h = 5$

$\Rightarrow$ $3 \times 8 \times \lambda + 2 \times 11 \times 7 \times 5 - 3 \times 121 - 8 \times 49 - \lambda \times 25 = 0$

$\Rightarrow$ $\lambda = 15.$

Example 53: The product of the lengths of the perpendiculars drawn from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its assumption is

(a) $\frac{a^2b^2}{a^2+b^2}$ (b) $\frac{1}{a^2} + \frac{1}{b^2}$

(c) $\frac{1}{a^2} - \frac{1}{b^2}$ (d) a^2b^2

Ans. (a)

Solution: Let $P(a \sec \alpha, b \tan \alpha)$ be a point on the hyperbola.

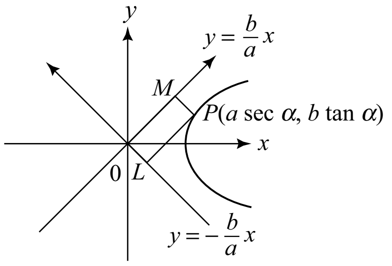


Fig. 20.8

Equation of the asymptotes is $y = \pm\frac{b}{a}x.$

Required product = $PL.PM$

$$= \frac{a^2b^2(\sec\alpha + \tan\alpha)(\sec\alpha - \tan\alpha)}{a^2+b^2}$$

$$= \frac{a^2b^2}{a^2+b^2}$$

◉ **Example 54:** If (5, 12) and (24, 7) are the foci of a hyperbola passing though the origin then the eccentricity of the hyperbola is

(a) $\sqrt{386}/12$ (b) $\sqrt{386}/13$

(c) $\sqrt{386}/25$ (d) $\sqrt{386}/38$

Ans. (a)

◎ **Solution:** Let $S(5, 12)$ and $S'(24, 7)$ be the two foci and $P(0, 0)$ be a point on the conic

then $SP = \sqrt{25+144} = \sqrt{169} = 13;$

$$S'P = \sqrt{(24)^2+7^2} = \sqrt{625} = 25$$

and $SS' = \sqrt{(24-5)^2+(7-12)^2} = \sqrt{19^2+5^2}$

$$= \sqrt{386}$$

since the conic is a hyperbola, $S'P - SP = 2a$, the length of transverse axis and $SS' = 2ae$, e being the eccentricity.

$$\Rightarrow \qquad e = \frac{SS'}{S'P - SP} = \frac{\sqrt{386}}{12}.$$

◉ **Example 55:** An equation of a tangent to the hyperbola. $16x^2 - 25y^2 - 96x + 100y - 356 = 0$ which makes an angle $\pi/4$ with the transverse axis is

(a) $y = x + 2$ (b) $y = x + 4$

(c) $x = y + 3$ (d) $x + y + 2 = 0$

Ans. (a)

◎ **Solution:** Equation of the hyperbola can be written as

$$X^2/5^2 - Y^2/4^2 = 1 \qquad (1)$$

where $X = x - 3$ and $Y = y - 2$.

Equation of a tangent which makes an angle $\pi/4$. With the transverse axis $X = 0$ of (1) is

$$Y = \tan\frac{\pi}{4}X \pm \sqrt{25\tan^2\frac{\pi}{4} - 16}$$

$$\Rightarrow \qquad y - 2 = x - 3 \pm \sqrt{25-16}$$

$$\Rightarrow \qquad y = x + 2 \text{ or } y = x - 4$$

◉ **Example 56:** Let $P(a\sec\theta, b\tan\theta)$ and $Q(a\sec\phi, b\tan\phi)$ where $\theta + \phi = \pi/2$, be two points on the hyperbola $x^2/a^2 - y^2/b^2 = 1$. If (h, k) is the point of intersection of normals at P and Q, then k is equal to

(a) $\dfrac{a^2+b^2}{a}$ (b) $-\left[\dfrac{a^2+b^2}{a}\right]$

(c) $\dfrac{a^2+b^2}{b}$ (d) $-\left[\dfrac{a^2+b^2}{b}\right]$

Ans. (d)

◎ **Solution:** Equation of the tangent at $P(a\sec\theta, b\tan\theta)$ is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1.$$

Therefore equation of the normal at P is

$$y - b\tan\theta = -\frac{a}{b}\sin\theta\,(x - a\sec\theta)$$

$$\Rightarrow \qquad ax + b\operatorname{cosec}\theta\, y = (a^2+b^2)\sec\theta \qquad \text{(i)}$$

Similarly the equation of the normal at $Q(a\sec\phi, b\sec\phi)$ is $ax + b\operatorname{cosec}\phi\, y = (a^2+b^2)\sec\phi$ (ii)

Subtracting (ii) from (i) we get

$$y = \frac{a^2+b^2}{b}\cdot\frac{\sec\theta - \sec\phi}{\operatorname{cosec}\theta - \operatorname{cosec}\phi}$$

So that $k = y = \dfrac{a^2+b^2}{b}\,\dfrac{\sec\theta - \sec(\pi/2-\theta)}{\operatorname{cosec}\theta - \operatorname{cosec}(\pi/2-\theta)}$

$$[\because\ \theta + \phi = \pi/2]$$

$$= \frac{a^2+b^2}{b}\,\frac{\sec\theta - \operatorname{cosec}\theta}{\operatorname{cosec}\theta - \sec\theta} = -\left[\frac{a^2+b^2}{b}\right]$$

◉ **Example 57:** If P is a point on the rectangular hyperbola $x^2 - y^2 = a^2$, C is its centre and S, S' are the two foci, then $SP.S'P =$

(a) 2 (b) $(CP)^2$

(c) $(CS)^2$ (d) $(SS')^2$

Ans. (b)

◎ **Solution:** Let the coordinate of P be (x, y).

The coordinates of the centre are (0, 0).

The eccentricity of the hyperbola is $\sqrt{1+\dfrac{a^2}{b^2}} = \sqrt{2}$

So the coordinates of the foci are $S(a\sqrt{2}, 0)$ and $S(-a\sqrt{2}, 0)$. (Fig 19.9)

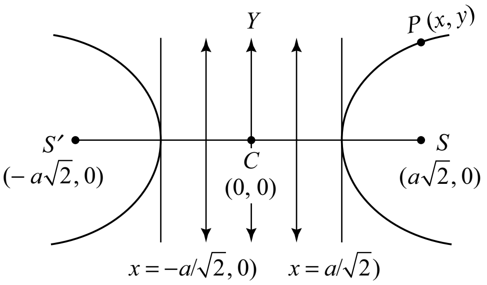


Fig. 20.9

Equation of the corresponding directrices are $x = a/\sqrt{2}$ and $S = --a/\sqrt{2}$.

By definition of the hyperbola

$$SP = e(\text{distance of } P \text{ from } x = a/\sqrt{2})$$

$$= \sqrt{2}\,|x - a/\sqrt{2}|$$

Similarly $S'P = \sqrt{2}\,|x + a/\sqrt{2}|$

So that $SP.S'.P = 2|x^2 - a^2/2|$

$$= 2x^2 - a^2 = x^2 + y^2 = (CP)^2$$

$$(\because P \text{ lies on the hyperbola } x^2 - y^2 = a^2)$$

◉ **Example 58:** If P is the length of the perpendicular from a focus upon the tangent at any point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and r is the distance of P from that focus, then

$$\frac{b^2}{p^2} - \frac{2a}{r} =$$

(a) –1 (b) 0
(c) 1 (d) 2

Ans. (c)

◎ **Solution:** Equation of the tangent at $P(a \sec\theta, b \tan\theta)$ to the hyperbola is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

Length of the perpendicular from the focus $S(ae, 0)$ to this tangent is

$$P = \left|\frac{e\sec\theta - 1}{\sqrt{\frac{\sec^2\theta}{a^2} + \frac{\tan^2\theta}{b^2}}}\right| = \frac{ab(e\sec\theta - 1)}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$$

$$= \frac{ab(e\sec\theta - 1)}{\sqrt{a^2(e^2 - 1)\sec^2\theta + a^2\tan^2\theta}}$$

$$= \frac{ab(e\sec\theta - 1)}{a\sqrt{e^2\sec^2\theta - 1}} = b\sqrt{\frac{e\sec\theta - 1}{e\sec\theta + 1}}$$

$$\Rightarrow \quad \frac{b^2}{p^2} = \frac{e\sec\theta + 1}{e\sec\theta - 1}$$

Now
$$r^2 = (ae - a\sec\theta)^2 + b^2\tan^2\theta$$
$$= a^2[e^2 - 2e\sec\theta + \sec^2\theta] + a^2(e^2 - 1)\tan^2\theta.$$
$$= a^2[e^2\sec^2\theta - 2e\sec\theta + 1]$$
$$= a^2(e\sec\theta - 1)^2$$

$$\Rightarrow \quad \frac{r^2}{a^2} = (e\sec\theta - 1)^2 \Rightarrow \frac{a}{r} = \frac{1}{e\sec\theta - 1}$$

So
$$\frac{b^2}{p^2} - \frac{2a}{r} = \frac{e\sec\theta + 1}{e\sec\theta - 1} - \frac{2}{e\sec\theta - 1} = 1.$$

◉ **Example 59:** A hyperbola having transverse axis of length 2 $\sin\theta$ is confocal with the ellipse.
$3x^2 + 4y^2 = 12$, then its equation is

(a) $x^2 \operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$
(b) $x^2\sec^2\theta - y^2\operatorname{cosec}^2\theta = 1$
(c) $x^2\sin^2\theta - y^2\cos^2\theta = 1$
(d) $x^2\cos^2\theta - y^2\sin^2\theta = 1$

Ans. (a)

◎ **Solution:** Foci of the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ is (± 1, 0)

Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Such that $2a = 2\sin\theta \Rightarrow a = \sin\theta$.
and $a^2 + b^2 = a^2e^2 = 1 \Rightarrow b = \cos\theta$.
So the required equation is

$$\frac{x^2}{\sin^2\theta} - \frac{y^2}{\cos^2\theta} = 1$$

or
$$x^2\operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$$

◉ **Example 60:** If a hyperbola passing through the origin has $3x + 4y - 1 = 0$ and $4x - 3y - 6 = 0$ as its assumptions, then the equation of the transverse axis of the hyperbola is

(a) $x + y - 5 = 0$ (b) $x - 7y - 5 = 0$
(c) $x - y - 1 = 0$ (d) $x + y - 1 = 0$

Ans. (b)

◎ **Solution:** Assumptions are perpendicular so the hyperbola is a rectangle hyperbola.

Axes are the bisectors of the angles between the asymptotes.

Equation of the transverse axis is the bisector of the of the angle which contains the origin and is given by

$$\frac{3x + 4y - 1}{5} = \frac{4x - 3y - 6}{5}$$

$$\Rightarrow \quad x - 7y - 5 = 0$$

EXERCISE

Concept-based Straight Objective Type Questions

1. If e is the eccentricity of the hyperbola $\frac{x^2}{18} - \frac{y^2}{6} = 1$ and r is the radius of the circle $x^2 + y^2 - 6x - 18y + 87 = 0$, then the value of er is equal to:

(a) $2\sqrt{3}$ (b) $2/\sqrt{3}$
(c) 2 (d) 3

2. If m_1 and m_2 are two values of m for which the line $y = mx + 2\sqrt{5}$ is a tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{16} = 1$, then the value of $\left|m_1 + \frac{1}{m_2}\right|$ is equal to

   (a) $\frac{8}{3}$ (b) $\frac{10}{3}$

   (c) 0 (d) 9.

3. Distance between the directrices of the hyperbola $\frac{x^2}{49} - \frac{y^2}{16} = 1$ is

   (a) $\frac{\sqrt{65}}{7}$ (b) $\frac{49}{\sqrt{65}}$

   (c) $\frac{\sqrt{33}}{4}$ (d) $\frac{98}{\sqrt{65}}$

4. If *A* line with slope m touches the hyperbola $\frac{x^2}{25} - \frac{y^2}{4} = 1$ and the parabola $y^2 = 20x$. Then the value of $25m^4 - 4m^2$ is equal to

   (a) 29 (b) 21

   (c) 25 (d) 4

5. If length of the transverse axis of a hyperbola is 8 and its eccentricity is $\sqrt{5}/2$ then the length of a latus rectum of the hyperbola is

   (a) 1 (b) 2

   (c) $2\sqrt{5}$ (d) $8/\sqrt{5}$

6. A tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$, meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that $OPRQ$ is a rectangle (where O is the origin), then R lies on

   (a) $\frac{4}{x^2} + \frac{2}{y^2} = 1$ (b) $\frac{2}{x^2} - \frac{4}{y^2} = 1$

   (c) $\frac{2}{x^2} + \frac{4}{y^2} = 1$ (d) $\frac{4}{x^2} - \frac{2}{y^2} = 1$

7. If $P(3\sec\theta, 2\tan\theta)$ and $Q(3\sec\phi, 2\tan\phi)$ where $\theta + \phi = \frac{\pi}{2}$, be two distinct points on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, then the ordinate of the point of intersection of the normals at P and Q is

   (a) 11/3 (b) –11/3

   (c) 13/2 (d) –13/2

8. A common tangent to $x^2 - 2y^2 = 18$ and $x^2 + y^2 = 9$ is

   (a) $y = 2x + 3\sqrt{5}$ (b) $y = \sqrt{2}x + 3\sqrt{3}$

   (c) $y = 2x + 3\sqrt{7}$ (d) $y = \sqrt{2}x + 3\sqrt{5}$

9. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then b^2 equals

   (a) 5 (b) 7

   (c) 9 (d) 1

10. Tangent drawn from the point (c, d) to the hyperbola $9x^2 - 64y^2 = 576$ make angle α and β with the x-axis. If $\tan\alpha \tan\beta = 1$, then the value of $c^2 - d^2$ is equal to

    (a) 73 (b) 55

    (c) 64 (d) 9

11. If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then the ratio of the square of its conjugate axis to the square of half the distance between the foci is

    (a) 4/3 (b) $4/\sqrt{3}$

    (c) $2/\sqrt{3}$ (d) 3/4

12. If the chords of contact of the tangents from two points (x_1, y_1) and (x_2, y_2) to the hyperbola $4x^2 - 5y^2 = a^2$ are at right angles. Then $16x_1x_2 + 25y_1y_2$ is equal to

    (a) –1 (b) 0

    (c) a^2 (d) 1

13. If two perpendicular tangents are drawn from a point (α, β) to the hyperbola $x^2 - y^2 = 16$, then the locus of (α, β) is

    (a) a pair of straight line (b) a circle

    (c) a parabola (d) an ellipse

14. If $\frac{x^2}{a+7} + \frac{y^2}{5-a} = 1$ represents a hyperbola, then

    (a) $a > 5$ (b) $a < -7$

    (c) $a < 5$ (d) $a < -7$ or $a > 5$

15. If e_1 is the eccentricity of the hyperbola $\frac{x^2}{49} - \frac{y^2}{36} = 1$ and e_2 is the eccentricity of the hyperbola $\frac{x^2}{36} - \frac{y^2}{49} = 1$, then

    (a) $e_1e_2 = 1$ (b) $\frac{e_1}{e_2} = 1$

    (c) $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$ (d) $e_1^2 + e_2^2 = 1$

LEVEL 1

Straight Objective Type Questions

16. The line $4\sqrt{2}x - 5y = 40$ touches the hyperbola $\frac{x^2}{100} - \frac{y^2}{64} = 1$ at the point

(a) $(10, 8\sqrt{2})$
(b) $(10\sqrt{2}, 8)$
(c) $(20, 8\sqrt{3})$
(d) $\left(\frac{20}{\sqrt{3}}, \frac{8}{\sqrt{3}}\right)$

17. The line $9\sqrt{3}x + 12y = 234\sqrt{3}$ is a normal to the hyperbola $\frac{x^2}{81} - \frac{y^2}{36} = 1$ at the points.

(a) $(18, 6\sqrt{3})$
(b) $(9\sqrt{2}, 6)$
(c) $(9\sqrt{3}, 6)$
(d) $\left(\frac{18}{\sqrt{3}}, \frac{6}{\sqrt{3}}\right)$

18. Length of the latus rectum of the hyperbola $x^2 \tan\alpha - y^2 \cot\alpha = 1$, $(0 < \alpha < \pi/4)$ is

(a) $2\sqrt{\cot\alpha}$
(b) $2\sqrt{\tan\alpha}$
(c) $2(\tan\alpha)^{3/2}$
(d) $2(\cot\alpha)^{3/2}$

19. If the angle between the asymptotes of the hyperbola $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$ is 2θ and e is its eccentricity then $\sin\theta$ is equal to

(a) $\frac{b}{a}$
(b) $\frac{1}{e}$
(c) $\frac{a}{be}$
(d) $\sqrt{1 - \frac{1}{e^2}}$

20. Axymptotes of the hyperbola $2xy = 4x + 3y$ pass through the point

(a) $(2, 3/2)$
(b) $(2, 3)$
(c) $(3/2, 2)$
(d) $(0, 0)$

21. The curve described parametrically by $x = t^2 + t + 1$, $y = t^2 - t + 1$ represents

(a) a pair of straight lines
(b) an ellipse
(c) a parabola
(d) a hyperbola

22. The point $(at^2, 2bt)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ for

(a) all real values of t
(b) $t^2 = 2 + \sqrt{5}$
(c) $t^2 = 2 - \sqrt{5}$
(d) no real value of t

23. If the coordinates of four concyclic point on the rectangular hyperbola $xy = c^2$ are $(ct_i, c/t_i)$ $i = 1, 2, 3, 4$ then

(a) $t_1 t_2 t_3 t_4 = -1$
(b) $t_1 t_2 t_3 t_4 = 1$
(c) $t_1 t_3 = t_2 t_4$
(d) $t_1 + t_2 + t_3 + t_4 = c^2$

24. The eccentricity of a rectangular hyperbola is

(a) 2
(b) $\sqrt{2}$
(c) $2 + \sqrt{2}$
(d) none of these

25. If e and e_1 are the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e^2} + \frac{1}{e_1^2}$ is equal to

(a) -1
(b) 0
(c) 1
(d) none of these

26. Foci of the rectangular hyperbola are $(\pm 7, 0)$, the equation of the hyperbola is

(a) $x^2 - y^2 = 49$
(b) $x^2 - y^2 = 98$
(c) $2x^2 - 2y^2 = 49$
(d) none of these

27. P is a point on the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, Q is the point where the right directrix meets the axis of x and S is the right focus of the hyperbola. If the area of the triangle PQS is 27/10 sq. units, coordinates of P are

(a) $(4\sqrt{2}, 3)$
(b) $(4\sqrt{3}, 3\sqrt{2})$
(c) $(4\sqrt{5}, 6)$
(d) none of these

28. The normal at a point P to the parabola $y^2 = 4x$ is parallel to the tangent at Q $(\sqrt{2}, 2)$ to the hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ and meets the axis of the parabola at R. If S is the focus of the parabola, area of the triangle PSR in sq. units is

(a) $9\sqrt{2}$
(b) $10\sqrt{2}$
(c) $18\sqrt{2}$
(d) $20\sqrt{2}$

29. The difference between the length $2a$ of the transverse axis of a hyperbola of eccentricity e and the length of its latus rectum is

(a) $2a|3 - e^2|$
(b) $2a|2 - e^2|$
(c) $2a|e^2 - 1|$
(d) $a|2e^2 - 1|$

30. The locus of the point of intersection of the tangents to the hyperbola $\frac{x^2}{16} - \frac{y^2}{36} = 1$ which are at right angles is

(a) $x^2 + y^2 = 20$ (b) $x^2 - y^2 = 20$
(c) $x^2 + y^2 = 52$ (d) none of these

31. If the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{36} = 1$ are perpendicular to the asymptotes of the hyperbola $\frac{x^2}{49} - \frac{y^2}{b^2} = 1$ then
(a) $7a \pm 6b = 0$ (b) $6a \pm 7b = 0$
(c) $a^2 - b^2 = 1$ (d) $a - b = 1$

32. P and Q are two points on the rectangular hyperbola $xy = c^2$ such that the abscissa of P and Q are the roots of the equations $x^2 - 6x - 16 = 0$. Equation of the chord joining P and Q is
(a) $16x - c^2y = 6c^2$ (b) $c^2x - 16y = c^2$
(c) $c^2x - 16y = 6c^2$ (d) $c^2x - 6y = 16c^2$

33. Normal at (3, 4) to the rectangular hyperbola $xy - y - 2x - 2 = 0$ meets the curve again at the points
(a) (1, 2) (b) (2, 3)
(c) (–1, 0) (d) none of these

34. Locus of the mid-points of the chords of the circle $x^2 + y^2 = 16$ which touch the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ is
(a) $(x^2 - y^2)^2 = 16x^2 - 9y^2$
(b) $(x^2 + y^2)^2 = 9x^2 - 16y^2$
(c) $(x^2 + y^2)^2 = 16x^2 + 9y^2$
(d) $(x^2 - y^2)^2 = 16x^2 + 9y^2$

35. If the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\sqrt{5}$ and the distance between the foci is 12, then $b^2 - a^2$ is equal to (3/5) k^2 where k is equal to
(a) 5 (b) 3
(c) 2 (d) 6

36. If the extremities of the latus rectum of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with positive coordinates lie on the parabola $x^2 = 3(y + 3)$, then length of the latus rectum of the hyperbola when its eccentricity is $\sqrt{3}$ is
(a) 3 (b) 6
(c) 12 (d) none of there

37. If the locus of the point of intersection of two straight lines $\sqrt{3}\ \alpha x + \alpha y - 4\sqrt{3} = 0$ and $\sqrt{3}x - y - 4\sqrt{3}\ \alpha = 0$ is a hyperbola with eccentricity e; for different values of α, then $e^2 + 6e - 9$ is equal to
(a) 0 (b) 7
(c) 3 (d) 4

38. Locus of the mid-points of the chords of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ that are parallel to $3x - 4y = 1$ is
(a) $4x - 3y = 0$ (b) $3x - 4y = 0$
(c) $3x + 4y = 0$ (d) $4x + 3y = 0$

39. The parametric equation:
$x = a(\sec\theta + \tan\theta)$, $y = b(\sec\theta - \tan\theta)$ represents
(a) a parabola (b) an ellipse
(c) a hyperbola (d) a rectangular hyperbola

40. If a normal to the hyperbola $x^2 - 4y^2 = 4$ having equal positive intercepts on the axes is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the distance between the foci of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is
(a) $\frac{10}{\sqrt{3}}$ (b) $\frac{5}{\sqrt{3}}$
(c) $10\sqrt{3}$ (d) $5\sqrt{3}$

Assertion-Reason Type Questions

41. A line $y = x + 4$ meets the hyperbola $xy = 16$ at A and B.

Statement 1: A circle on AB as diameter passes through the points (4, 4) and (– 4, – 4).

Statement 2: A circle on AB as diameter passes through the intersection of the circle $x^2 + y^2 = 32$ and the line $y = x$.

42. P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ S and S' are its foci.

Statement-1: Product of the lengths of the perpendiculars from S and S' on the tangent at P is equal to b^2.

Statement-2: $|PS - PS'| = 2a$.

43. **Statement-1:** Two tangents drawn from any point on the hyperbola $x^2 - y^2 = a^2 - b^2$ to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ make complementary angles with the axis of the ellipse.

Statement-2: If two lines make complementary angles with the axis of x then the product of their slopes is 1.

44. **Statement-1:** If the foci of a hyperbola are at the points (4, 1) and (–6, 1), eccentricity is 5/4 then the length of the transverse axis is 8.

    Statement-2: Distance between the foci of a hyperbola is equal to the product of its eccentricity and the length of the transverse axis.

45. **Statement-1:** If the angle between two asymptotes of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\frac{\pi}{3}$, its eccentricity is $2\sqrt{3}$.

    Statement 2: Angles between the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $2\tan^{-1}\left(\frac{b}{a}\right)$ or $\pi-2\tan^{-1}\left(\frac{b}{a}\right)$.

46. **Statement-1:** Equation of a circle on the ends of a latus rectum of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ as a diameter is $16x^2+16y^2\pm 160x+319=0$

    Statement-2: Focus of the parabola $y^2=20x$ coincides with a focus of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$

47. Let $P(x, y)$ be a variable point such that $\left|\sqrt{(x-3)^2+(y-2)^2}-\sqrt{(x-6)^2+(y+2)^2}\right|=3$.

    Statement-1: P traces a hyperbola whose eccentricity is $\frac{5}{3}$.

    Statement-2: P traces a hyperbola such that the equation of its conjugate axis is $6x-8y=27$.

48. **Statement 1:** A normal to the hyperbola with eccentricity 3, meets the transverse axis and conjugate axis at P and Q respectively. The locus of the mid-point of PQ is a hyperbola with eccentricity $\frac{3}{2\sqrt{2}}$.

    Statement 2: Eccentricity of the hyperbola $8x^2-y^2=8a^2$ is $\frac{3}{2\sqrt{2}}$.

49. **Statement-1:** The locus of the point of intersection of the tangents that are at right angles to the hyperbola $\frac{x^2}{36}-\frac{y^2}{16}=1$ is the circle $x^2+y^2=52$.

    Statement-2: Perpendicular tangents to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ interest on the director circle $x^2+y^2=a^2-b^2$ $(a^2>b^2)$ of the hyperbola.

50. A hyperbola H: $\frac{x^2}{9}-\frac{y^2}{4}=1$ intersects the circle, C: $x^2+y^2-8x=0$ at the points A and B.

    Statement-1: $2x-\sqrt{5}\,y+4=0$ is a common tangent to both C and H.

    Statement-2: Circle on AB as a diameter passes through the centre of the hyperbola H.

LEVEL 2

Straight Objective Type Questions

51. Locus of the mid-point of the chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ which is a tangent to the circle $x^2+y^2=c^2$ is

    (a) $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2=c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)$

    (b) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=c^2\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)$

    (c) $\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$

    (d) $\left(\frac{x^2}{a^4}-\frac{y^2}{b^4}\right)=c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2$

52. $H_1 : xy=c^2$ and $H_2 : xy=k^2$ are two different hyperbolas. From a point on H_1, tangents are drawn to H_2. Area of the triangle formed by the chord of contact and the asymptote to H_2 is

    (a) $\frac{k^2}{c^2}$ (b) $\frac{k^4}{c^2}$

    (c) $\frac{2k^4}{c^2}$ (d) none of these

53. e_1, e_2 are respectively the eccentricities of the hyperbola $x^2 - y^2 \operatorname{cosec}^2\theta = 5$ and the ellipse $x^2\operatorname{cosec}^2\theta + y^2 = 5$. If $0 < \theta < \pi/2$ and $e_1 = \sqrt{7}\, e_2$, then θ is equal to

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{6}$

(c) $\dfrac{\pi}{3}$ (d) none of these

54. If θ is an angle between the two asymptotes of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, then $\cos\left(\dfrac{\theta}{2}\right)$ is equal to

(a) $\dfrac{b}{\sqrt{a^2+b^2}}$ (b) $\dfrac{ab}{\sqrt{a^2+b^2}}$

(c) $\dfrac{a}{\sqrt{a^2+b^2}}$ (d) $\sqrt{\dfrac{a-b}{a+b}}$.

55. A and B are two points on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, O is the centre. If OA is perpendicular to OB then $\dfrac{1}{(OA)^2} + \dfrac{1}{(OB)^2}$ is equal to

(a) $\dfrac{1}{a^2} + \dfrac{1}{b^2}$ (b) $\dfrac{1}{a^2} - \dfrac{1}{b^2}$

(c) $\dfrac{1}{b^2} - \dfrac{1}{a^2}$ (d) $a^2 + b^2$

56. The coordinates of a point common to a directrix and an asymptote of the hyperbola $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$ are

(a) $\left(\dfrac{25}{\sqrt{41}}, \dfrac{20}{3}\right)$ (b) $\left(\dfrac{20}{\sqrt{41}}, \dfrac{-25}{\sqrt{41}}\right)$

(c) $\left(\dfrac{25}{3}, \dfrac{20}{3}\right)$ (d) $\left(\dfrac{-25}{\sqrt{41}}, \dfrac{20}{\sqrt{41}}\right)$

57. If the normals at P, Q, R on the rectangular hyperbola $xy = c^2$ intersect at a point S on the hyperbola, then centroid of the triangle PQR is at

(a) an extremity of the latus rectum
(b) the centre
(c) a focus
(d) the point S on the hyperbola.

58. If a diameter of a hyperbola meets the hyperbola in real points then

(a) it meets the conjugate hyperbola in imaginary points.
(b) the conjugate diameter meets the given hyperbola in real points.
(c) the conjugate diameter meets the conjugate hyperbola in imaginary points.
(d) none of these.

59. An ellipse has eccentricity 1/2 and a focus at the point $P(1/2, 1)$. One of its directrix is the common tangent near to the point P, to the circle $x^2 + y^2 = 1$ and the hyperbola $x^2 - y^2 = 1$, the equation of the ellipse is

(a) $3x^2 + 4y^2 - 6x - 8y + 4 = 0$
(b) $3x^2 + 4y^2 - 2x - 8y + 4 = 0$
(c) $4x^2 + 3y^2 - 8x - 6y + 4 = 0$
(d) $4x^2 + 3y^2 - 8x - 2y + 4 = 0$

60. The asymptotes of $xy = hx + ky$ are

(a) $x = h, y = k$ (b) $x = -h, y = -k$

(c) $x = k, y = h$ (d) $x = -k, y = -h$

Previous Years' AIEEE/JEE Main Questions

1. The foci of the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{b^2} = 1$ and the hyperbola $\dfrac{x^2}{144} - \dfrac{y^2}{81} = \dfrac{1}{25}$ coincide then, the value of b^2 is:

(a) 5 (b) 7

(c) 9 (d) 1 **[2003]**

2. The locus of the point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is:

(a) a parabola (b) a hyperbola

(c) an ellipse (d) a circle **[2005]**

3. The normal to curve at $P(x, y)$ meets the x-axis at G. If the distance of G from the origin is twice the abscissa of P, then the curve is

(a) ellipse (b) parabola

(c) circle (d) hyperbola **[2007]**

4. For the hyperbola $\dfrac{x^2}{\cos^2\alpha} - \dfrac{y^2}{\sin^2\alpha} = 1$, which of the following remains constant when α varies?

(a) eccentricity
(b) directrix
(c) abscissae of vertices
(d) abscissae of foci. **[2007]**

5. The equation of the hyperbola whose foci are (–2, 0) and (2, 0) and eccentricity is 2 is given by
(a) $x^2 - 3y^2 = 3$ (b) $3x^2 - y^2 = 3$
(c) $-x^2 + 3y^2 = 3$ (d) $-3x^2 + y^2 = 3$ **[2011]**

6. A tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that $OPRQ$ is a rectangle (where O is the origin) then R lies on
(a) $\frac{4}{x^2} + \frac{2}{y^2} = 1$ (b) $\frac{2}{x^2} - \frac{4}{y^2} = 1$
(c) $\frac{2}{x^2} + \frac{4}{y^2} = 1$ (d) $\frac{4}{x^2} - \frac{2}{y^2} = 1$
[2013, online]

7. A common tangent to the conic $x^2 = 6y$ and $2x^2 - 4y^2 = 9$ is
(a) $x - y = \frac{3}{2}$ (b) $x + y = 1$
(c) $x + y = \frac{9}{2}$ (d) $x - y = 1$ **[2013, online]**

8. If $P(3 \sec\theta, 2 \tan\theta)$ and $Q(3 \sec \phi, 2 \tan \phi)$ where $\theta + \phi = \frac{\pi}{2}$, be two distinct points on the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$. Then the ordinate of the points of intersection of the normals at P and Q is
(a) $\frac{11}{3}$ (b) $-\frac{11}{3}$
(c) $\frac{13}{2}$ (d) $-\frac{13}{2}$ **[2014, online]**

9. The tangent at an extremity (in the first quadrant) of latus rectum of the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ meets x-axis and y-axis at A and B respectively. Then $(OA)^2 - (OB)^2$, where O is the origin, equals
(a) $-\frac{20}{9}$ (b) $\frac{16}{9}$
(c) 4 (d) $-\frac{4}{3}$ **[2014, online]**

10. An ellipse passes through the foci of the hyperbola, $9x^2 - 4y^2 = 36$ and it major and minor axes lie along the transvers and conjugate axes of the hyperbola respectively. If the product of eccentricities of the two conics is $\frac{1}{2}$, then which of the following points does not lie on the ellipse?
(a) $(\sqrt{13}, 0)$ (b) $\left(\frac{\sqrt{39}}{2}, \sqrt{3}\right)$
(c) $\left(\frac{1}{2}\sqrt{13}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(\frac{\sqrt{13}}{2}, \sqrt{6}\right)$
[2015, online]

11. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is
(a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$
(c) $\frac{2}{\sqrt{3}}$ (d) $\sqrt{3}$ **[2016]**

12. A hyperbola whose transverse axis is along the major axis of the conic, $\frac{x^2}{3} + \frac{y^2}{4} = 4$ and has vertices at the foci of this conic. If the eccentricity of the hyperbola is $\frac{3}{2}$, then which of the following points does NOT lie on it?
(a) $(\sqrt{5}, 2\sqrt{2})$ (b) (0, 2)
(c) $(5, 2\sqrt{3})$ (d) $(\sqrt{10}, 2\sqrt{3})$
[2016, online]

13. Let a and b respectively be the semi-transverse and semi-conjugate axes of a hyperbola whose eccentricity satisfies the equation $9e^2 - 18e + 5 = 0$. If $S(5, 0)$ is a focus and $5x = 9$ is corresponding directrix of this hyperbola, then $a^2 - b^2$ is equal to:
(a) – 7 (b) – 5
(c) 5 (d) 7 **[2016, on line]**

Previous Years' B-Architecture Entrance Examination Questions

1. If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola, then the eccentricity e of hyperbola satisfies
   (a) $e = \frac{2}{\sqrt{3}}$ (b) $e = \frac{\sqrt{3}}{2}$
   (c) $e > \frac{2}{\sqrt{3}}$ (d) $1 < e < \frac{2}{\sqrt{3}}$ **[2006]**
2. If the tangent and the normal to the hyperbola $x^2 - y^2 = 4$ at a point cut off intercepts a_1 and a_2 respectively on the x-axis, and b_1 and b_2 respectively on the y-axis, then the value of $a_1 a_2 + b_1 b_2$ is
   (a) -1 (b) 0
   (c) 4 (d) 1 **[2010]**
3. A common tangent to $x^2 - 2y^2 = 18$ and $x^2 + y^2 = 9$ is
   (a) $y = 2x + 3\sqrt{5}$ (b) $y = \sqrt{2}x + 3\sqrt{3}$
   (c) $y = 2x + 3\sqrt{7}$ (d) $y = \sqrt{2}x + 3\sqrt{5}$ **[2013]**
4. If the point R divides the line segment joining the point (2, 3) and ($2 \tan \theta$, $3 \sec \theta$); $0 < \theta < \frac{\pi}{2}$, externally in the ratio 2 : 3, then the locus of R is
   (a) an ellipse length of whose major axis is 12.
   (b) an ellipse length of whose major axis is 8.
   (c) a hyperbola length of whose transverse axis is 12.
   (d) a hyperbola length of whose transverse axis is 8. **[2015]**
5. The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$. If the eccentricity of the hyperbola is 2, then the equation of the tangent to this hyperbola passing through the point (4, 6) is
   (a) $3x - 2y = 0$ (b) $2x - 3y + 10 = 0$
   (c) $x - 2y + 8 = 0$ (d) $2x - y - 2 = 0$ **[2016]**

Answers

Concept-based

1. (c) **2.** (a) **3.** (d) **4.** (c)
5. (b) **6.** (d) **7.** (d) **8.** (b)
9. (b) **10.** (a) **11.** (d) **12.** (b)
13. (a) **14.** (d) **15.** (c)

Level 1

16. (b) **17.** (a) **18.** (c) **19.** (d)
20. (c) **21.** (c) **22.** (b) **23.** (b)
24. (b) **25.** (c) **26.** (c) **27.** (a)
28. (c) **29.** (c) **30.** (d) **31.** (a)
32. (c) **33.** (c) **34.** (c) **35.** (d)
36. (c) **37.** (b) **38.** (b) **39.** (d)
40. (a) **41.** (a) **42.** (b) **43.** (a)
44. (a) **45.** (c) **46.** (b) **47.** (b)
48. (c) **49.** (d) **50.** (a)

Level 2

51. (a) **52.** (c) **53.** (c) **54.** (c)
55. (b) **56.** (d) **57.** (b) **58.** (a)
59. (b) **60.** (c)

Previous Years' AIEEE/JEE Main Questions

1. (b) **2.** (b) **3.** (a) **4.** (d)
5. (b) **6.** (d) **7.** (a) **8.** (d)
9. (a) **10.** (c) **11.** (c) **12.** (c)
13. (a)

Previous Years' B-Architecture Entrance Examination Questions

1. (c) **2.** (b) **3.** (b) **4.** (c)
5. (d)

Hints and Solutions

Concept-based

1. $e^2 = \frac{6+18}{18} = \frac{4}{3}$, $r^2 = 9 + 81 - 87 = 3$
   $\Rightarrow er = 2$
2. $(2\sqrt{5})^2 = 4m^2 - 16 \Rightarrow m^2 = 9, m_1 = 3, m_2 = -3$
   $\Rightarrow m_1 m_2 = -9 \Rightarrow \left| m_1 + \frac{1}{m_2} \right| = \left| \frac{-9+1}{\pm 3} \right| = \frac{8}{3}$
3. The required distance is $\frac{2a}{e} = \frac{2a^2}{\sqrt{a^2+b^2}} = \frac{98}{\sqrt{65}}$.

4. Let the equation of the line be $y = mx + c$, then

$c^2 = 25m^2 - 4$ and $c = \frac{5}{m}$.

$\Rightarrow 25 = (25m^2 - 4)m^2$

$\Rightarrow 25m^4 - 4m^2 = 25.$

5. Length of the L.R. = $\frac{2b^2}{a} = 2a(e^2 - 1) = 8\left(\frac{5}{4} - 1\right) = 2$

6. Let the equation of the tangent be $\frac{xx_1}{4} - \frac{yy_1}{2} = 1$

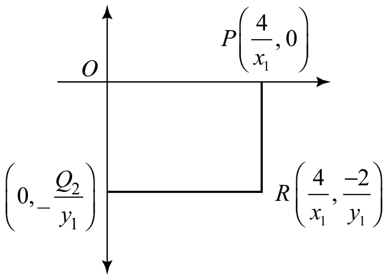
Coordinates of P are $\left(\frac{4}{x_1}, 0\right)$

and of Q are $\left(0, \frac{-2}{y_1}\right)$

$\Rightarrow$ Coordinates of R are $\left(\frac{4}{x_1}, \frac{-2}{y_1}\right) = (h, k)$ (x_1, y_1) lies on the hyperbola.

$\Rightarrow \frac{x_1^2}{4} - \frac{y_1^2}{2} = 1$

$\Rightarrow \frac{4}{h^2} - \frac{2}{k^2} = 1 \Rightarrow$ Locus of R is $\frac{4}{x^2} - \frac{2}{y^2} = 1$



O
$P\left(\frac{4}{x_1}, 0\right)$
$\left(0, \_\frac{Q_2}{y_1}\right)$
$R\left(\frac{4}{x_1}, \frac{-2}{y_1}\right)$

7. An equation of the normal at P is

$3x \cos\theta + 2y \cot\theta = 9 + 4$

$\Rightarrow 3x + 2y \operatorname{cosec}\theta = 13 \sec\theta.$

An equation of the normal at ϕ is

$3x + 2y \operatorname{cosec}\phi = 13 \sec\phi$

Subtracting we get

$2y (\operatorname{cosec}\theta - \operatorname{cosec}\phi) = 13 (\sec\theta - \sec\phi)$

$\Rightarrow 2y (\operatorname{cosec}\theta - \sec\theta) = 13 (\sec\theta - \operatorname{cosec}\theta)$

$\Rightarrow y = -\frac{13}{2}.$

8. Let $y = mx + c$ be a common tangent

then $c^2 = 18m^2 - 9$ as it touches the hyperbola.

$= 9 (1 + m^2)$ as it touches the circle.

$\Rightarrow m^2 = 2$, $c = 3\sqrt{3}$ and the required equation is

$y = \sqrt{2}x + 3\sqrt{3}$.

9. Foci of the ellipse are $\left(\pm\sqrt{16 - b^2}, 0\right)$

and of the hyperbola are $\left(\pm\sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{9}{5}\right)^2}, 0\right)$

If they coincide $16 - b^2 = \frac{144 + 81}{25} \Rightarrow b^2 = 7$

10. Any tangent to the hyperbola is

$y = mx + \sqrt{64m^2 - 9}$. If passes through (c, d)

$\Rightarrow (d - mc)^2 = 64\, m^2 - 9$

$\Rightarrow (c^2 - 64)\, m^2 - 2cd\, m + d^2 + 9 = 0.$

Product of the roots = $\frac{d^2 + 9}{c^2 - 64} = \tan\alpha \tan\beta = 1$

$\Rightarrow c^2 - d^2 = 64 + 9 = 73$

11. Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Centre is $(0, 0)$, vertex $(a, 0)$, focus $(ae, 0)$

So $ae = 2a \Rightarrow e = 2$.

Conjugate axis = b, distance between the foci = $2\,ae$

So $\frac{b^2}{(ae)^2} = \frac{e^2 - 1}{e^2} = \frac{3}{4}$

12. Equations of the chords of contact are

$4xx_1 - 5yy_1 = a^2$ and $4xx_2 - 5yy_2 = a^2$

If they are at right angles

$\frac{4x_1}{5y_1} \times \frac{4x_2}{5y_2} = -1 \Rightarrow 16x_1x_2 + 25y_1y_2 = 0.$

13. An equation of a tangent to the hyperbola is

$y = mx + 4\sqrt{m^2 - 1}$

and equation of the tangent perpendicular to it is

$y = -\frac{1}{m}x + 4\sqrt{\frac{1}{m^2} - 1}$

$\Rightarrow my + x = 4\sqrt{1 - m^2}$.

We must have $m^2 - 1 \geq 0$, $1 - m^2 \geq 0 \Rightarrow m^2 = 1$

As (α, β) lies on both, locus of (α, β) is $y = \pm x$, which is a pair of straight lines.

14. We must have $\left.\begin{array}{l} a + 7 > 0 \text{ and } 5 - a < 0 \\ a + 7 < 0 \text{ and } 5 - a > 0 \end{array}\right\} \Rightarrow \begin{array}{c} a > 5 \\ \text{or} \\ a < -7 \end{array}$

15. $e_1^2 = \frac{49 + 36}{49}$ and $e_2^2 = \frac{49 + 36}{36}$.

$\Rightarrow \frac{1}{e_1^2} + \frac{1}{e_2^2} = 1.$

Level 1

16. Equation of the tangent at (10 sec θ, 8 tan θ) on the hyperbola is $\frac{x}{10}\sec\theta - \frac{y}{8}\tan\theta = 1$

    Comparing with the given equation

    $\frac{x\sqrt{2}}{10} - \frac{y}{8} = 1$, we get

    $\sec\theta = \sqrt{2}$, $\tan\theta = 1$

    so the required point is $(10\sqrt{2}, 8)$

17. Equation of the normal at (9 sec θ, 6 tan θ) on the hyperbola is $\frac{9x}{\sec\theta} + \frac{6y}{\tan\theta} = 117$.

    Comparing with the given equation

    $\frac{9x}{2} + \frac{6y}{\sqrt{3}} = 117$

    we get $\sec\theta = 2$, $\tan\theta = \sqrt{3}$

    and the required point is $(18, 6\sqrt{3})$.

18. Length of the latus rectum is $\frac{2(\tan\alpha)}{\sqrt{\cot\alpha}} = 2(\tan\alpha)^{3/2}$.

19. Equation of the asymptotes is $\frac{x}{a} \pm \frac{y}{b} = 0$.

    $$\text{So } \tan 2\theta = \frac{\frac{b}{a} - \left(-\frac{b}{a}\right)}{1 + \left(\frac{b}{a}\right)\left(-\frac{b}{a}\right)}$$

    $$\Rightarrow 2\theta = \tan^{-1}\frac{2(b/a)}{1-(b/a)^2} = 2\tan^{-1}\left(\frac{b}{a}\right)$$

    $$\Rightarrow \theta = \tan^{-1}\frac{b}{a} \Rightarrow \tan\theta = \frac{b}{a}.$$

    $$\Rightarrow \sin\theta = \frac{b}{a} \times \frac{1}{\sqrt{1+\frac{b^2}{a^2}}} = \frac{b}{ae} = \sqrt{\frac{a^2(e^2-1)}{a^2e^2}}$$

    $$= \sqrt{1 - \frac{1}{e^2}}$$

20. Given hyperbola is $xy = 2x + \frac{3}{2}y$

    or $\left(x - \frac{3}{2}\right)(y-2) = 3$.

    Equations of the asymptotes are $x - \frac{3}{2} = 0$, $y - 2 = 0$

    Both of them pass through the point $\left(\frac{3}{2}, 2\right)$

21. Eliminating t, we get $x = \left(\frac{x-y}{2}\right)^2 + \frac{x-y}{2} + 1$

    $\Rightarrow (x-y)^2 = 2(x+y-2)$ which is a parabola.

22. $\frac{a^2t^4}{a^2} - \frac{4b^2t^2}{b^2} = 1 \Rightarrow t^4 - 4t^2 - 1 = 0$

    $\Rightarrow t^2 = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$ But $t^2 \neq 2 - \sqrt{5}$

    so $t^2 = 2 + \sqrt{5}$.

23. Equation of a circle be $x^2 + y^2 + 2gx + 2fy + k = 0$.
    If the point $(ct, c/t)$ lies on it, then
    $c^2t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$
    which gives four values of t say t_1, t_2, t_3, t_4 and the four concyclic points such that $t_1 t_2 t_3 t_4 = 1$.

24. For a rectangular hyperbola $b^2 = a^2$ so
    $b^2 = a^2(e^2 - 1) \Rightarrow e^2 - 1 = 1$

    $\Rightarrow e^2 = 2 \Rightarrow e = \sqrt{2}$

25. $e^2 = \frac{a^2+b^2}{a^2}$ and $e_1^2 = \frac{a^2+b^2}{b^2} = \frac{1}{e^2} + \frac{1}{e_1^2} = 1$

26. Equation of the hyperbola is $x^2 - y^2 = a^2$

    Foci = ($\pm$ ae, 0) = ($\pm$ 7, 0) where $e = \sqrt{2}$.

27. Eccentricity $e = 5/4$, coordinates of $Q(16/5, 0)$, $S(5, 0)$, $QS = 9/5$
    If $P(x, y)$, then area of the triangle PQS
    $= (1/2)(9/5)|y| = 27/10 \Rightarrow y = 3$, $x = 4\sqrt{2}$

28. Equation of the tangent at Q $(\sqrt{2}, 2)$ is $y = 2\sqrt{2}x - 2$

    so equation of the normal at P is

    $y = 2\sqrt{2}x - 2 \times 2\sqrt{2} - (2\sqrt{2})^3$

    or $y = 2\sqrt{2}x - 20\sqrt{2}$

    $S(1, 0)$, $R(10, 0)$, $P(8, -4\sqrt{2})$

    Area of the triangle $\frac{1}{2} \times |10-1||-4\sqrt{2}|$

    $= 18\sqrt{2}$

29. Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

    $b^2 = a^2(e^2 - 1)$

    length of the latus rectum $= 2b\sqrt{e^2 - 1}$
    $= 2a(e^2 - 1)$

30. The locus is the circle $x^2 + y^2 = 16 - 36$ which is not possible. So there is no pair of tangents that are at right angles.

31. Asymptotes to the two hyperbola are

$$\frac{x}{a} \pm \frac{y}{6} = 0 \text{ and } \frac{x}{7} \pm \frac{y}{b} = 0$$

If they are at right angles then $\left(\pm\frac{6}{a}\right)\left(\pm\frac{b}{7}\right) = -1$

$\Rightarrow 7a \pm 6b = 0$

32. Let $P(x_1, c^2/x_1)$ and $Q(x_2, c^2/x_2)$ such that $x_1 + x_2 = 6$, $x_1x_2 = -16$.
Equation of the chord PQ is

$$y - \frac{c^2}{x_1} = \frac{\frac{c^2}{x_2} - \frac{c^2}{x_1}}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow x - x_1 = -\frac{x_1x_2}{c^2}\left(y - \frac{c^2}{x_1}\right)$$

$\Rightarrow c^2x + x_1x_2\, y = c^2(x_1 + x_2)$

$\Rightarrow c^2x - 16y = 6c^2$

33. Equation of the hyperbola is $(x - 1)(y - 2) = 4$
or $y = 2 + \frac{4}{x-1}$

Any point on this hyperbola is $\left(1 + 2t, 2 + \frac{2}{t}\right)$

$$\frac{dy}{dx} = -\frac{4}{(x-1)^2}$$

Slope of the normal at $\left(1 + 2t, 2 + \frac{2}{t}\right)$ is t^2.

Now $(3, 4) = \left(1 + 2t, 2 + \frac{2}{t}\right) \Rightarrow t = 1$

Slope of the normal at (3, 4) is 1.
So equation of the normal at (3, 4) is $x - y + 1 = 0$

Point $\left(1 + 2t, 2 + \frac{2}{t}\right)$ lies on this normal.

If $1 + 2t - 2 - \frac{2}{t} + 1 = 0 \Rightarrow t^2 = 1 \Rightarrow t = \pm 1$

So the required point is for $t = -1$, i.e. $(-1, 0)$.

34. Let (h, k) be the mid-point, equation of the chord is $hx + ky = h^2 + k^2$ which touches the hyperbola at $(4 \sec \theta, 3 \tan \theta)$.

So its equation is $\frac{x}{4} \sec \theta - \frac{y}{3} \tan \theta = 1$.

Comparing the two equations we get

$$\frac{\sec\theta}{4} = \frac{h}{h^2 + k^2} \text{ and } \frac{-\tan\theta}{3} = \frac{k}{h^2 + k^2}$$

Eliminating θ, we get

$(h^2 + k^2)^2 (\sec^2 \theta - \tan^2 \theta) = (4h)^2 - (3k)^2$

$\Rightarrow (h^2 + k^2)^2 = 16h^2 - 9k^2$.

and the required locus is $(x^2 + y^2)^2 = 16x^2 - 9y^2$

35. We have $e = \sqrt{5}$, $2ae = 12$

$$b^2 - a^2 = a^2e^2 - 2a^2 = 36 - 2\left(\frac{36}{5}\right) = \frac{3}{5}(6)^2$$

36. As $\left(ae, \frac{b^2}{a}\right), \left(-ae, \frac{b^2}{a}\right)$ lies on the parabola

$x^2 = 3(y + 3)$, we have $a^2e^2 = 3\left(\frac{b^2}{a} + 3\right)$

$= 3(a(e^2 - 1) + 3)$

$\Rightarrow (a - 3)(ae^2 + 3) = 0 \Rightarrow a = 3$.

So length of the latus rectum of the hyperbola

$$= \frac{2b^2}{a} = \frac{2a^2(e^2 - 1)}{a} = 2 \times 3 \times 2 = 12.$$

37. Point of intersection is (x, y)

$$= \left(2\left(\alpha + \frac{1}{\alpha}\right), 2\sqrt{3}\left(\frac{1}{\alpha} - \alpha\right)\right)$$

$$\Rightarrow \left(\frac{x}{2}\right)^2 - \left(\frac{y}{2\sqrt{3}}\right)^2 = 4$$

$$\Rightarrow \frac{x^2}{16} - \frac{y^2}{48} = 1$$

So $e^2 = 1 + \frac{48}{16} = 4 \Rightarrow e = 2$

and hence $e^2 + 6e - 9 = 7$

38. Equation of the chord with mid-point (h, k) is

$\frac{hx}{16} - \frac{ky}{9} = \frac{h^2}{16} - \frac{k^2}{9}$ whose slope is $\frac{9h}{16k}$

so $\frac{9h}{16k} = \frac{3}{4} \Rightarrow 3h - 4k = 0$

Locus of (h, k) is $3x - 4y = 0$.

39. We have $bx + ay = 2ab \sec \theta$

$bx - ay = 2ab \tan \theta$

Eliminating θ we get

$$\left(\frac{bx + ay}{2ab}\right)^2 - \left(\frac{bx - ay}{2ab}\right)^2 = 1$$

$\Rightarrow 4abxy = 4a^2b^2$

$\Rightarrow xy = ab$

which represents a rectangular hyperbola.

40. Equation of the normal to the hyperbola at (2 sec θ, tan θ) is $\frac{2x}{\sec\theta}+\frac{y}{\tan\theta} = 5$.

 Intercepts on axes are $\frac{5}{2}$ sec θ and 5 tan θ

 $\frac{5}{2}$ sec $\theta = 5$ tan $\theta \Rightarrow$ sin $\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

 and the equation of the normal is

 $\sqrt{3}x+\sqrt{3}y = 5$

 $\Rightarrow y = -x + \frac{5}{\sqrt{3}}$

 which will touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

 if $\left(\frac{5}{\sqrt{3}}\right)^2 = a^2 (-1)^2 + b^2$

 $\Rightarrow a^2 + b^2 = \left(\frac{5}{\sqrt{3}}\right)^2$

 Distance between the foci of the hyperbola

 $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ is $2ae = 2\sqrt{a^2+b^2}$

 $= 2 \times \frac{5}{\sqrt{3}} = \frac{10}{\sqrt{3}}$

41. Any point on the hyperbola $xy = 16$ is $\left(4t, \frac{4}{t}\right)$

 Let the coordinate of A and B be $\left(4t_1, \frac{4}{t_1}\right)$ and $\left(4t_2, \frac{4}{t_2}\right)$ respectively. So slope of $AB = -\frac{1}{t_1t_2} = 1$

 $\Rightarrow t_1t_2 = -1$.

 Equation of the circle on AB as diameter is

 $(x - 4t_1)(x - 4t_2) + \left(y-\frac{4}{t_1}\right)\left(y-\frac{4}{t_2}\right) = 0$

 $\Rightarrow (x - 4t_1)\left(x+\frac{4}{t_1}\right)+\left(y-\frac{4}{t_1}\right)(y + 4t_1) = 0$

 $\Rightarrow x^2 + y^2 - 32 + 4\left(\frac{1}{t_1}-t_1\right)(x - y) = 0$

 which shows that the circle passes through the points of intersection of the circle $x^2 + y^2 = 32$ and the line $x - y = 0$ i.e. $y = x$ and thus statement-2 is true.

 Using it the required points of intersection are (4, 4) and (– 4, – 4).

 Hence statement-1 is also true.

42. Let $P(a \sec\theta, b \tan\theta)$, equation of the tangent at P is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$.

 Product of the lengths of the perpendiculars from $S(ae, 0)$ and $S'(-ae, 0)$ is

 $$\left|\frac{(e\sec\theta-1)(e\sec\theta+1)}{\frac{\sec^2\theta}{a^2}+\frac{\tan^2\theta}{b^2}}\right|$$

 $$= \frac{a^2b^2(e^2\sec^2\theta-1)}{a^2[(e^2-1)\sec^2\theta+\tan^2\theta]} = b^2$$

 So statement-1 is true.

 For statement-2, by definition of hyperbola distance of P from a focus is e times its distance from the corresponding directrix.

 So $PS = e\left(x-\frac{a}{e}\right)$ and $PS' = e\left(x+\frac{a}{e}\right)$.

 $\Rightarrow |PS - PS'| = 2a$.

 Thus statement-2 is also true but does not justify statement-1.

43. If the angle in statement-2 are α and β such that $\alpha + \beta = \pi/2$ then the product of the slopes is $\tan\alpha\tan\beta = 1$, and the statement-2 is true.

 In statement-1, any tangent to the ellipse is $y = mx + \sqrt{a^2m^2+b^2}$ which passes through

 $\left(\sqrt{a^2-b^2}\sec\theta, \sqrt{a^2-b^2}\tan\theta\right)$

 $\Rightarrow (a^2 - b^2)(\tan\theta - m\sec\theta)^2 = a^2m^2 + b^2$

 Product of the slopes

 $$= \frac{(a^2-b^2)\tan^2\theta-b^2}{(a^2-b^2)\sec^2\theta-a^2} = \frac{a^2\sin^2\theta-b^2}{a^2\sin^2\theta-b^2} = 1$$

 So by statement-2, statement-1 is also true.

44. Statement-2 is true, as the distance between the foci of $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ is $2ae$ where $2a$ is the length of the transverse axis and e is the eccentricity.

 Using it in statement-1, length of the transverse axis is $\frac{\sqrt{(4+6)^2}}{5/4} = \frac{10\times 4}{5} = 8$

 and thus statement-1 is also true.

45. Equation of the asymptotes is $\frac{x}{a} \pm \frac{y}{b} = 0$

And if θ is an angle between the asymptotes $\tan\theta =$

$$\frac{2\frac{b}{a}}{1+\frac{b^2}{a^2}} = \frac{2\tan\phi}{1+\tan^2\phi} \quad \text{where} \quad \phi = \tan^{-1}\frac{b}{a}$$

$= \tan 2\phi$

$\Rightarrow \theta = 2\phi = 2\tan^{-1}\frac{b}{a}$ and the statement-1 is true.

Using in statement-1.

If $\frac{\pi}{3} = 2\tan^{-1}\frac{b}{a} \Rightarrow \tan\frac{\pi}{6} = \frac{b}{a}$

$\Rightarrow 3b^2 = a^2 \Rightarrow 3(e^2 - 1) = 1 \Rightarrow e = \frac{2}{\sqrt{3}}$

If $\frac{\pi}{3} = \pi - 2\tan^{-1}\frac{b}{a} \Rightarrow \frac{b}{a} = \tan\frac{\pi}{3} = \sqrt{3}$

$\Rightarrow b^2 = 3a^2 \Rightarrow e^2 - 1 = 3 \Rightarrow e = 2$

Showing that statement-1 is false.

46. Foci ($\pm 5, 0$), length of the latus rectum $= \frac{2\times 9}{4}$

So the equation of a required circle is $(x \pm 5)^2 + y^2 = \left(\frac{9}{4}\right)^2$

$\Rightarrow 16(x^2 + y^2 \pm 10x + 25) = 81$

or $16x^2 + 16y^2 \pm 160x + 319 = 0$

and statement-1 is true, statement-2 is also true as the focus of $y^2 = 20x$ is $(5, 0)$ but does not justify statement-1.

47. Distance of P from the fixed points $F_1(3, 2)$ and $F_2(6, -2)$ is constant 3.

So by definition, locus of P is a hyperbola with foci F_1 and F_2, length of the transverse axes is 3. If e is the eccentricity then $F_1F_2 = 2ae$ where $2a = 3$.

$\Rightarrow e = \frac{1}{3}\sqrt{9+16} = \frac{5}{3}$ and the statement-1 is true.

In statement-2 slope of the conjugate axis is $-\frac{6-3}{-2-2}$

$= \frac{3}{4}$ and it passes through the mid-point of F_1F_2.

So its equation is

$$y = \frac{3}{4}\left(x - \frac{9}{2}\right)$$

$\Rightarrow 6x - 8y = 27$

Thus statement-2 is also true but does not lead to statement-1.

48. Equation of the normal at ($a\sec\theta$, $b\tan\theta$) to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$$

It meets the axes at $L\left(\frac{a^2+b^2}{a\cos\theta}, 0\right)$, $M\left(0, \frac{a^2+b^2}{b\cot\theta}\right)$

If (h, k) is the mid-point of LM

then $h = \frac{a^2+b^2}{2a\cos\theta}$, $k = \frac{a^2+b^2}{2b\cot\theta}$

$\Rightarrow \sec\theta = \frac{2h}{9a}, \tan\theta = \frac{2bk}{9a^2}$ $[\because a^2 + b^2 = a^2e^2 = 9a^2]$

Eliminating θ, we get

$$\frac{4h^2}{81a^2} - \frac{4b^2k^2}{81a^4} = 1$$

Locus of (h, k) is $\frac{x^2}{\frac{81a^2}{4}} - \frac{y^2}{\frac{81a^4}{4b^2}} = 1$

which is a hyperbola and its eccentricity

$$\sqrt{1+\frac{81a^4}{4b^2}\times\frac{4}{81a^2}} = \sqrt{1+\frac{a^2}{b^2}} = \sqrt{1+\frac{1}{e^2-1}} = \sqrt{1+\frac{1}{8}}$$

$= \frac{3}{2\sqrt{2}}$

So statement-1 is true. In statement-2, the eccentricity is

$$\sqrt{1+\frac{8a^2}{a^2}} = 3$$

and the statement-2 is false.

49. Equation of a tangent to the hyperbola is $y = mx + \sqrt{a^2m^2 - b^2}$ and the equation of the tangent perpendicular to it is $y = -\frac{1}{m}x + \sqrt{\frac{a^2}{m^2} - b^2}$.

Eliminating m we get the required locus as $(y - mx)^2 + (x + my)^2 = a^2m^2 - b^2 + a^2 - b^2m^2$

$\Rightarrow x^2 + y^2 = a^2 - b^2$ and the statement-2 is true using which statement-1 is false.

50. Equation of a tangent with slope $\frac{2}{\sqrt{5}}$ to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is

$$y = \frac{2}{\sqrt{5}}x + \sqrt{9 \times \frac{4}{5} - 4} \Rightarrow 2x - \sqrt{5}\,y + 4 = 0$$

Next $2x - \sqrt{5}\,y + 4 = 0$ touches the circle $(x - 4)^2 + y^2 = 16$ if the length of the perpendicular from (4, 0) on the line is 4 which is true. Hence statement-1 is true.

In statement-2, let A be $(3 \sec\theta, 2 \tan\theta)$

A lies on the circle $x^2 + y^2 - 8x = 0$

$\Rightarrow 13\sec^2\theta - 24 \sec\theta - 4 = 0$

$\Rightarrow \sec\theta = 2 \Rightarrow \tan\theta = \pm \sqrt{3}$

So the coordinate of A are $(6, 2\sqrt{3})$ and of B are $(6, -2\sqrt{3})$ and equation of the circle on AB as diameter is $(x - 6)(x - 6) + (y - 2\sqrt{3})(y + 2\sqrt{3}) = 0$.

$\Rightarrow x^2 + y^2 - 12x + 24 = 0$ which does not pass through the centre (0, 0) of the hyperbola. Thus statement-2 is false.

Level - 2

51. Let (h, k) be the mid-point of the chord, then its equation is $\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$

Since it touches the circle $x^2 + y^2 = c^2$

$$\frac{\frac{h^2}{a^2} - \frac{k^2}{b^2}}{\sqrt{\left(\frac{h}{a^2}\right)^2 + \left(\frac{k}{b^2}\right)^2}} = \pm c$$

$$\Rightarrow \left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)^2 = c^2\left(\frac{h^2}{a^4} + \frac{k^2}{b^4}\right)$$

Required locus is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^2 = c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)$$

52. Equation of the chord of contact from $\left(ct, \frac{c}{t}\right)$ on H_1 to H_2 is $(ct)x + \left(\frac{c}{t}\right)y = 2k^2$.

Asymptote of H_2 are $x = 0$ and $y = 0$ and the chord meets these asymptotes at $A\left(\frac{2k^2}{ct}, 0\right)$ and $B\left(0, \frac{2k^2 t}{c}\right)$

Area of the required triangle is $\frac{1}{2}\left|\frac{2k^2}{ct}\right|\left|\frac{2k^2 t}{c}\right| = \frac{2k^4}{c^2}$.

53. $e_1^2 = \frac{5 + 5\sin^2\theta}{5} = 1 + \sin^2\theta$.

$e_2^2 = \frac{5 - 5\sin^2\theta}{5} = \cos^2\theta$.

$e_1 = \sqrt{7}\ e_2 \Rightarrow 1 + \sin^2\theta = 7 - 7\sin^2\theta$

$\Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$.

54. Slopes of the asymptotes are $\pm\frac{b}{a}$

$$\Rightarrow \tan\theta = \frac{\frac{b}{a} - \left(-\frac{b}{a}\right)}{1 + \left(\frac{b}{a}\right)\left(-\frac{b}{a}\right)}$$

$$\Rightarrow \frac{2\tan\left(\frac{\theta}{2}\right)}{1 - \tan^2\left(\frac{\theta}{2}\right)} = \frac{2(b/a)}{1 - \left(\frac{b}{a}\right)^2}$$

$$\Rightarrow \tan\left(\frac{\theta}{2}\right) = \frac{b}{a} \text{ as } \tan\left(\frac{\theta}{2}\right) > 0$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{1 + \frac{b^2}{a^2}}} = \frac{a}{\sqrt{a^2 + b^2}}$$

55. Let $OA = r_1$ and the coordinates of A be $(r_1 \cos\alpha, r_1 \sin\alpha)$.

If $OB = r_2$ then coordinates of B are

$$\left(r_2 \cos\left(\alpha + \frac{\pi}{2}\right), r_2 \sin\left(\alpha + \frac{\pi}{2}\right)\right)$$

As A, B lie on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$r_1^2\left(\frac{\cos^2\alpha}{a^2} - \frac{\sin^2\alpha}{b^2}\right) = 1$$

$$r_2^2\left(\frac{\sin^2\alpha}{a^2} - \frac{\cos^2\alpha}{b^2}\right) = 1 \Rightarrow \frac{1}{(OA)^2} + \frac{1}{(OB)^2}$$

$$= \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{1}{a^2} - \frac{1}{b^2}.$$

56. $e^2 = \frac{25 + 16}{25} \Rightarrow e = \frac{\sqrt{41}}{5}$

Directrix is $x = \pm\frac{25}{\sqrt{41}}$ $\left(x = \pm\frac{a}{e}\right)$

Asymptotes are $\frac{x^2}{25} - \frac{y^2}{16} = 0$

So the required coordinates are $\left(-\frac{25}{\sqrt{41}}, \frac{20}{\sqrt{41}}\right)$

57. Normal at any point $(ct, c/t)$ on the hyperbola $xy = c^2$ is $y - (c/t) = t^2 (x - ct)$ or $t^3 x - ty + c - ct^4 = 0$

If it passes through another points $(c\alpha, c/\alpha)$ on the hyperbola, then

$$t^3 \times c\alpha - t \times \frac{c}{\alpha} + c - ct^4 = 0$$

$\Rightarrow t^3\alpha^2 - t + \alpha - \alpha t^4 = 0$

$\Rightarrow (t^3\alpha + 1)(\alpha - t) = 0$

$\Rightarrow t^3\alpha + 1 = 0$ as $\alpha \neq t$

which gives three values of t say t_1, t_2, t_3 and hence three points P, Q, R on the hyperbola the normals at which pass through S.

We have $t_1 + t_2 + t_3 = 0 = \Sigma t_1t_2$ and $t_1t_2t_3 = -1/\alpha$.

Centroid of the triangle PQR is

$$\left(\frac{c(t_1+t_2+t_3)}{3}, \frac{c(1/t_1+1/t_2+1/t_3)}{3}\right) = (0, 0)$$

which is the centre of hyperbola.

58. Equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, so equation of the conjugate hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

Equation of a diameter of the hyperbola is $y = (b^2/a^2m)\,x$, which meets the hyperbola at points given by

$$\frac{x^2}{a^2} - \frac{b^4x^2}{b^2a^4m^2} = 1 \Rightarrow x^2 = \frac{a^2m^2}{m^2 - b^2}$$

These points are real if $m^2 > b^2$ and for this value of m, the diameter will meet the conjugate hyperbola at point for which $x^2 = \frac{-a^2m^2}{m^2 - b^2}$ and these points are imaginary.

Note: Conjugate diameter will meet the conjugate hyperbola in real points and the hyperbola in imaginary points.

59. Equation of a common tangent is $x = \pm 1$, nearer to P (1/2, 1) is $x = 1$.

So the directrix of the ellipse is $x = 1$ and the focus is P (1/2, 1), $e = 1/2$.

Thus the required equation is

$(x - 1/2)^2 + (y - 1)^2 = 1/4(1 - x)^2$

$\Rightarrow 3x^2 + 4y^2 - 2x - 8y + 4 = 0$

60. Equation of the asymptotes differ from the equation of the hyperbola by a constant so the equation of the asymptotes is $xy - hx - ky + \lambda = 0$ which represents a pair of lines if $\lambda = hk$ and the required asymptotes are $x = k$ and $y = h$.

Previous Years' AIEEE/JEE Main Questions

1. Foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ are $(\pm\sqrt{16 - b^2}, 0)$

and of the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ are $\pm\sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{9}{5}\right)^2}$

If the foci of the two conics coincide then $16 - b^2 = \frac{144 + 81}{25} = 9 \Rightarrow b^2 = 7$

2. Equation of tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = mx + \sqrt{a^2m^2 - b^2}$

Comparing it with $y = \alpha x + \beta$, we get

$m = \alpha$, $\beta = \sqrt{a^2m^2 - b^2}$

$\Rightarrow \beta^2 = a^2\alpha^2 - b^2$

$\Rightarrow a^2\alpha^2 - \beta^2 = b^2$

$\Rightarrow$ Locus of (α, β) is $a^2x^2 - y^2 = b^2$

which is a hyperbola.

3. Equation of normal at point (x, y) is

$$Y - y = -\frac{dx}{dy}(X - x)$$

It meets the x-axis at $G\left(x + y\frac{dx}{dy}, 0\right)$

We are given

$$\left(x + y\frac{dx}{dy}\right) = 2|x|$$

$$\Rightarrow x + y\frac{dy}{dx} = \pm 2x$$

$$\Rightarrow x + y\frac{dy}{dx} = 2x \text{ or } x + y\frac{dy}{dx} = -2x$$

$\Rightarrow y\,dy = x\,dx$

or $y\,dy = -3x\,dx$

$\Rightarrow x^2 - y^2 = c$

or $3x^2 + y^2 = c$

Thus curve is either a hyperbola or an ellipse.

4. $b^2 = a^2(e^2 - 1)$

$\Rightarrow \sin^2\alpha = \cos^2\alpha\,(e^2 - 1)$

$\Rightarrow e^2 - 1 = \tan^2\alpha \Rightarrow e = \sec\alpha$

Coordinates of foci are $(\pm ae, 0)$

$= (\pm 1, 0)$

$\therefore$ abscissae of foci remains constant.

5. Let the equation of the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$,

$\pm ae = \pm 2$ and $e = 2 \Rightarrow a = 1$ and $b^2 = 1(4 - 1) \Rightarrow b^2 = 3$

and the required equation is $x^2 - \dfrac{y^2}{3} = 1$

or $3x^2 - y^2 = 3$.

6. Equation of the tangent at a point (x_1, y_1) on the hyperbola $\dfrac{x^2}{4} - \dfrac{y^2}{2} = 1$ is $\dfrac{xx_1}{4} - \dfrac{yy_1}{2} = 1$

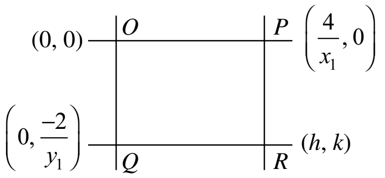

(0, 0)
O
P
$\left(\frac{4}{x_1}, 0\right)$
$\left(0, \frac{-2}{y_1}\right)$
Q
R
(h, k)

Fig. 20.10

So coordinates of P are $\left(\dfrac{4}{x_1}, 0\right)$ and of Q are $\left(0, \dfrac{-2}{y_1}\right)$

Let the coordinates of R be (h, k)

$h = \dfrac{4}{x_1},\ k = \dfrac{-2}{y_1}$

(x_1, y_1) lies on the hyperbola.

So $\dfrac{x_1^2}{4} - \dfrac{y_1^2}{2} = 1$

$\Rightarrow \dfrac{4}{h^2} - \dfrac{2}{k^2} = 1$

Locus of (h, k) is $\dfrac{4}{x^2} - \dfrac{2}{y^2} = 1$

7. Equation of a tangent to $x^2 = 6y$ is $x = my + \dfrac{3}{2m}$

$\Rightarrow y = \dfrac{1}{m}x - \dfrac{3}{2m^2}$

which touches the hyperbola $\dfrac{x^2}{\frac{9}{2}} - \dfrac{y^2}{\frac{9}{4}} = 1$

if $\left(\dfrac{-3}{2m^2}\right)^2 = \dfrac{9}{2} \times \dfrac{1}{m^2} - \dfrac{9}{4} \Rightarrow m^2 = 1$

So equation of a common tangent is $x = \pm y \pm \dfrac{3}{2}$ and the required equation is $x - y = \dfrac{3}{2}$.

8. An equation of the normal at P is $3x\cos\theta + 2y\cot\theta = 9 + 4$

$\Rightarrow 3x + 2y\,\mathrm{cosec}\,\theta = 13\sec\theta \qquad (1)$

An equation of normal at Q is

$3x + 2y\,\mathrm{cosec}\,\phi = 13\sec\phi \qquad (2)$

Subtracting (2) from (1), we get

$2y[\mathrm{cosec}\,\theta - \mathrm{cosec}\,\phi] = 13(\sec\theta - \sec\phi)$

$\Rightarrow 2y[\mathrm{cosec}\,\theta - \sec\theta] = 13(\sec\theta - \mathrm{cosec}\,\theta)$

$[\because\ \phi = \pi/2 - \theta]$

$\Rightarrow y = -\dfrac{13}{2}$

9. Coordinates of P, an extremity of latus rectum in the first quadrant are $\left(3, \dfrac{5}{2}\right)$. An equation of tangent at P is $\dfrac{3}{4}x - \dfrac{5}{2}\left(\dfrac{y}{5}\right) = 1$ or $3x - 2y = 4$

Coordinates of A are $\left(\dfrac{4}{3}, 0\right)$ and that of B are $(0, -2)$.

Now, $OA^2 - OB^2 = \dfrac{16}{9} - 4 = -\dfrac{20}{9}$.

10. Equation of hyperbola is

$\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

If e_1 is the eccentricity of the hyperbola, then

$9 = 4(e_1^2 - 1) \Rightarrow e_1 = \sqrt{13}/2$

Foci of the hyperbola are $(\pm 2e_1, 0) = (\pm\sqrt{13}, 0)$

If e_2 is the eccentricity of the ellipse, then

$e_1e_2 = 1/2 \Rightarrow e_2 = 1/\sqrt{13}$.

Let the equation of ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

As it passes through $(\pm\sqrt{3}, 0)$,

$13/a^2 = 1 \Rightarrow a^2 = 13$.

Also, $b^2 = a^2(1 - e_2^2) = 13\left(1 - \dfrac{1}{13}\right) = 12$.

$\therefore$ equation of ellipse is $\dfrac{x^2}{13} + \dfrac{y^2}{12} = 1$.

$\left(\dfrac{1}{2}\sqrt{13}, \dfrac{\sqrt{3}}{2}\right)$ does not lie on it.

11. $\dfrac{2b^2}{a} = 8 \Rightarrow b^2 = 4a$

Also, $2b = \dfrac{1}{2}(2ae) \Rightarrow 2b = ae$

$\therefore \left(\dfrac{1}{2}ae\right)^2 = 4a \Rightarrow ae^2 = 16$

$\Rightarrow a\left(1+\dfrac{b^2}{a^2}\right) = 16$

$\Rightarrow a + \dfrac{b^2}{a} = 16 \Rightarrow a + 4 = 16$

$\Rightarrow a = 12$

$\therefore b^2 = 4a \Rightarrow a^2(e^2 - 1) = 4a$

$\Rightarrow a(e^2 - 1) = 4$

$\Rightarrow 12(e^2 - 1) = 4 \Rightarrow e^2 = 1 + \dfrac{1}{3}$

$\Rightarrow e = \dfrac{2}{\sqrt{3}}$

12. Equation of ellipse is

$\dfrac{x^2}{12}+\dfrac{y^2}{16} = 1$

Its major axis is along y-axis, and its foci are (0, 2) and (0, –2).

Let equation of hyperbola be $\dfrac{y^2}{a^2}-\dfrac{x^2}{b^2} = 1$, then $a = 2$ and $b^2 = a^2(e^2 - 1)$

$\Rightarrow b^2 = 4\left(\dfrac{9}{4}-1\right) = 5$

Thus, equation of hyperbola is

$$\frac{y^2}{4}-\frac{x^2}{5} = 1 \qquad (1)$$

Note that $(\sqrt{5}, 2\sqrt{2})$, (0, 2) and $(\sqrt{10}, 2\sqrt{3})$ lie on (1)

13. As $9e^2 - 18e + 5 = 0$

$\Rightarrow 9e^2 - 15e - 3e + 5 = 0$

$\Rightarrow 3e(3e - 5) - (3e - 5) = 0$

$\Rightarrow (3e - 1)(3e - 5) = 0 \Rightarrow e = 1/3, 5/3$

As $e > 1$, $e = 5/3$

We have $ae = 5 \Rightarrow a = 3$

Also, $a^2 - b^2 = a^2 - a^2(e^2 - 1)$

$= 9(2 - 25/9) = -7$

Previous Years' B-Architecture Entrance Examination Questions

1. Equation of OP is $y = \dfrac{1}{\sqrt{3}}x$ which meets the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2} = 1$ at points for which $\dfrac{3y^2}{a^2}-\dfrac{y^2}{b^2} = 1$

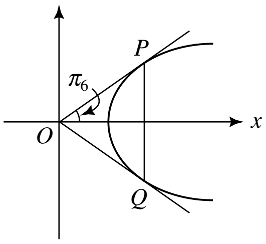


Fig. 20.11

$\Rightarrow (3b^2 - a^2)y^2 = a^2b^2$

For real values of y, $3b^2 > a^2$

$\Rightarrow 3a^2(e^2 - 1) > a^2 \Rightarrow 3e^2 > 4 \Rightarrow e > \dfrac{2}{\sqrt{3}}$

2. Equation of the tangent at (x_1, y_1) to the hyperbola $x^2 - y^2 = 4$ is

$xx_1 - yy_1 = 4$

then $a_1 = \dfrac{4}{x_1}$, $b_1 = \dfrac{-4}{y_1}$.

Equation of the normal at (x_1, y_1) is $y_1x + x_1y = 2x_1y_1$

then $a_2 = 2x_1$, $b_2 = 2y_1$.

Hence $a_1a_2 + b_1b_2 = 0$.

3. Let $y = mx + c$ be a common tangent to $x^2 - 2y^2 = 18$ and $x^2 + y^2 = 9$.

As it touches the circle, $c^2 = 9(1 + m^2)$ and as it touches the hyperbola, $c^2 = 18m^2 - 9$

So $18m^2 - 9 = 9 + 9m^2 \Rightarrow m^2 = 2$, $c = 3\sqrt{3}$ and the required equation is $y = \sqrt{2}x + 3\sqrt{3}$.

4. If coordinates of R are (x, y), then

$x = \dfrac{(3)(2)-2(2\tan\theta)}{3-2}$, $y = \dfrac{3(3)-2(3\sec\theta)}{3-2}$

$\Rightarrow \tan\theta = \dfrac{1}{4}(6 - x)$, $\sec\theta = \dfrac{1}{6}(9 - y)$

As $\sec^2\theta - \tan^2\theta = 1$, we get

$\dfrac{1}{36}(y - 9)^2 - \dfrac{1}{16}(x - 6)^2 = 1$

It is a hyperbola, length of whose transverse axis is 2(6) = 12.

5. Foci of $\frac{x^2}{25}+\frac{y^2}{9}=1$ are $(\sqrt{25-9}, 0)$ and $(-\sqrt{25-9}, 0)$ i.e. (4, 0) and (–4, 0).

   If a is the length of transverse axis of the hyperbola, then

   $ae = 4 \Rightarrow a = 2$

Also, $b^2 = a^2(e^2 - 1) = 4(4 - 1) = 12$

$\therefore$ Equation of hyperbola is $\frac{x^2}{4}-\frac{y^2}{12}=1$

An equation of tangent at (4, 6) is $\frac{4x}{4}-\frac{6y}{12}=1$

or $2x - y = 2$.

CHAPTER TWENTY ONE

Three Dimensional Geometry

COORDINATES

Distance Formula

Distance between $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Section Formulae

The coordinates of the point P which divides the join of $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

By taking $m = n$, we find the coordinates of the mid-point of AB as

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

Coordinates of any point on the join of two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}, \frac{\lambda z_2 + z_1}{\lambda + 1}\right)$ where $\lambda \neq -1$. If λ takes only positive real values, we get the coordinates of the points on the segment AB (except A and B).

Volume of a Tetrahedron

If $A_i(x_i, y_i, z_i)$, $i = 1, 2, 3, 4$ are the vertices of a tetrahedron, its volume is equal to

$$\frac{1}{6}\left|\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}\right|$$

Direction-Cosines

If α, β, γ are the angles that a given directed line makes with the positive directions $X'OX$, $Y'OY$, $Z'OZ$ of the coordinate axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the *direction-cosines* of the line.

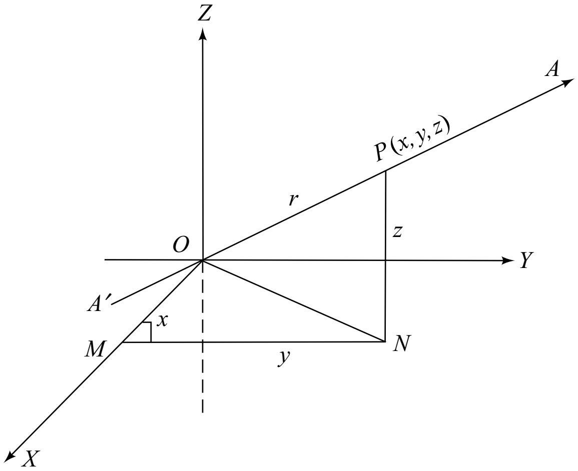


Fig. 21.1

Let $P(x, y, z)$ be any point on $A'OA$, and let the measure of OP be r. Let PN be the perpendicular from P on the plane XOY, and NM be the perpendicular from N on OX. Then $PN = z$, $NM = y$, $OM = x$

$$x = r\cos\alpha,\ y = r\cos\beta,\ z = r\cos\gamma.$$

and $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Direction-Ratios

If l, m, n be the direction-cosines of a line OP and a, b, c be numbers proportional to l, m, n respectively, then a, b, c are called the direction ratios of OP. If l, m, n be the direction-cosines of a line and k ($\neq 0$) be any number, then kl, km, kn are the direction-ratios of OP.

If a, b, c are the direction-ratios of a line, the actual direction-cosines $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are obtained from the relations

$$\frac{\cos\alpha}{a} = \frac{\cos\beta}{b} = \frac{\cos\gamma}{c}$$

$$= \pm\frac{\sqrt{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}}{\sqrt{a^2 + b^2 + c^2}} = \frac{\pm 1}{\sqrt{a^2 + b^2 + c^2}}$$

If P be the point (a, b, c), and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the directed line OP, then

$$\cos\alpha = \frac{a}{\sqrt{(a^2+b^2+c^2)}},\ \cos\beta = \frac{b}{\sqrt{(a^2+b^2+c^2)}},$$

$$\cos\gamma = \frac{c}{\sqrt{(a^2+b^2+c^2)}}$$

The direction-cosines of PO are

$$-\frac{a}{\sqrt{(a^2+b^2+c^2)}}, -\frac{b}{\sqrt{(a^2+b^2+c^2)}}, -\frac{c}{\sqrt{(a^2+b^2+c^2)}}$$

Illustration 1

Find the direction cosines of the line whose direction ratios are 12, 4, – 18.

Solution: Direction cosines of the given line are

$$\frac{12}{\sqrt{(12)^2+(4)^2+(-18)^2}},\ \frac{4}{\sqrt{(12)^2+(4)^2+(-18)^2}},\ \frac{-18}{\sqrt{(12)^2+(4)^2+(-18)^2}}$$

or $\frac{12}{22}, \frac{4}{22}, \frac{-18}{22}$

or $\frac{6}{11}, \frac{2}{11}, \frac{-9}{11}$.

Angle Between Two Lines

(a) If θ is an angle between two lines whose direction-cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) then

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2.$$

The expressions for $\sin\theta$, $\tan\theta$ are given below:

$$\sin^2\theta = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}^2$$

and $\tan\theta = \pm[\Sigma(l_1m_2 - l_2m_1)^2]^{1/2}/(l_1l_2 + m_1m_2 + n_1n_2)$

The lines are parallel to each other if and only if

$$l_1/l_2 = m_1/m_2 = n_1/n_2.$$

Also the lines are perpendicular to each other provided

$$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

(b) Angle θ between two lines whose direction-ratios are a_1, b_1, c_1 and a_2, b_2, c_2 is given by

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\sin\theta = \pm\frac{[(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2]^{1/2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\tan\theta = \pm\frac{[(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2]^{1/2}}{a_1a_2+b_1b_2+c_1c_2}$$

The lines are parallel to each other if and only if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Also, the lines are perpendicular to each other if and only if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Illustration 2

Direction ratios of a line L_1, are 1, –1, 1 and of L_2 are 1, 0, λ. Find the value of λ so that L_1 and L_2 are perpendicular to each other.

Solution: The line L_1 and L_2 are perpendicular if the angle θ between them is 90°

$$\Rightarrow \cos\theta = \frac{1(1)+(-1)(0)+(1)(\lambda)}{\sqrt{(1)^2+(-1)^2+(1)^2}\sqrt{1^2+0+\lambda^2}} = \cos 90° = 0$$

$$\Rightarrow \lambda = -1$$

THE PLANE

Equation of a Plane

A plane is represented by an equation of the first degree i.e. by $ax + by + cz + d = 0$. Conversely, every equation of the first degree in x, y, z represents a plane.

(a) Equation of the yz-plane is $x = 0$.
(b) Equation of the zx-plane is $y = 0$.
(c) Equation of the xy-plane is $z = 0$.
(d) If l, m, n be the direction cosines of the normal to a plane and p be the length of the perpendicular from the origin on the plane, then an equation of the plane is $lx + my + nz = p$.

Illustration 3

Find the direction cosines of the normal to the plane $4x - 3y + 5z = 25$ and the length of the perpendicular from the origin on this plane

Solution: Direction ratios of the normal to the plane are 4, –3, 5. So direction cosines are

$$l = \frac{4}{\sqrt{4^2+(-3)^2+5^2}},\ m = \frac{-3}{\sqrt{4^2+(-3)^2+5^2}},\ n = \frac{5}{\sqrt{4^2+(-3)^2+5^2}}$$

or $l = \frac{4}{5\sqrt{2}}, m = \frac{-3}{5\sqrt{2}}, n = \frac{5}{5\sqrt{2}}$

We can write the equation of the plane as

$$\frac{4}{5\sqrt{2}}x - \frac{3}{5\sqrt{2}}y + \frac{5}{5\sqrt{2}}z = \frac{25}{5\sqrt{2}}$$

or $lx + my + nz = p$ where $p = \dfrac{25}{5\sqrt{2}} = \dfrac{5}{\sqrt{2}}$

is the required distance of the plane from the origin.

(e) If a plane makes intercepts a, b, c on the axes of x, y, z respectively its equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Systems of Planes

An equation of a plane contains three independent constants, as such a plane is uniquely determined by three independent conditions. If we consider a plane satisfying just two given conditions, its equation will contain one arbitrary constant. If we consider a plane satisfying one given condition, its equation will contain two arbitrary constants.

We give below the equations of some well-known systems of planes:

(i) The equation $ax + by + cz + k = 0$ represents a system of planes parallel to the plane $ax + by + cz + d = 0$, k being a parameter.

(ii) The equation $ax + by + cz + k = 0$, represents a system of planes perpendicular to the line $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

(iii) The equation $(a_1x + b_1y + c_1z + d_1) + k\,(a_2x + b_2y + c_2z + d_2) = 0$ represents a system of planes passing through the intersection of the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, k being a parameter.

Illustration 4

Find the equation of the plane passing through the intersection of the planes $3x + 4y + 7z + 5 = 0$ and $2x - 3y + 4z - 6 = 0$ and making an intercept of 1 unit on the axis of x.

Solution: Equation of any plane through the intersection of the given planes is

$3x + 4y + 7z + 5 + \lambda\,(2x - 3y + 4z - 6) = 0$

Intercept on x-axis $= \dfrac{6\lambda - 5}{3 + 2\lambda} = 1 \Rightarrow \lambda = 2$

So the required equation of the plane is

$7x - 2y + 15z - 7 = 0$

(iv) The equation $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$ represents a system of planes passing through the point (x_1, y_1, z_1), the ratios of A, B, C being the two parameters.

It may be noted that (i)–(iii) above are examples of one-parameter family of planes and (iv) is an example of a two-parameter family of planes.

Angle Between Two Planes

Angle θ between the planes

$a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$, is given by

$$\cos\theta = \pm\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\,\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

Parallelism and Perpendicularity of Two Planes

The planes $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ are parallel if and only if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2};$$

and perpendicular if and only if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Two Sides of a Plane

Two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on the same or different sides of the plane $ax + by + cz + d = 0$ according as $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of the same or of different signs.

Length of the Perpendicular from a Point to a Plane

The perpendicular distance of the point (x_1, y_1, z_1) from the plane $lx + my + nz = p$ is $|p - lx_1 - my_1 - nz_1|$, where l, m, n are direction cosines of the normal to the plane and p is the length of the perpendicular to the plane from the origin.

The perpendicular distance of the point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is

$$\frac{|\,ax_1 + by_1 + cz_1 + d\,|}{\sqrt{\left(a^2 + b^2 + c^2\right)}}$$

Bisectors of the Angles between Two Planes

(A) Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be the equations of two planes, written in such a way that d_1 and d_2 are both positive. Then

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{\left(a_1^2 + b_1^2 + c_1^2\right)}} = \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{\left(a_2^2 + b_2^2 + c_2^2\right)}}$$

is the equation of the plane bisecting that angle between the given planes which contains the origin, and

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{\left(a_1^2 + b_1^2 + c_1^2\right)}} = -\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{\left(a_2^2 + b_2^2 + c_2^2\right)}}$$

is the equation of the plane bisecting that angle between the given planes which does not contain the origin.

Illustration 5

Find the angle between the planes $x + y + z + 1 = 0$ and $x - y + z - 1 = 0$. Also find the equation of the plane bisecting the angle between these planes which contain the origin.

Solution: If θ is the angle between these planes, then

$$\Rightarrow \cos\theta = \frac{(1)\times(1)+(1)(-1)+(1)(1)}{\sqrt{1^2+1^2+1^2}\sqrt{(1)^2+(-1)^2+(1)^2}} = \frac{1}{3}$$

$$\Rightarrow \theta = \cos^{-1}(1/3)$$

Equation of the required plane bisecting the angle between the given planes is

$$\frac{x+y+z+1}{\sqrt{1^2+1^2+1^2}} = \frac{-x+y-z+1}{\sqrt{(1)^2+(-1)^2+(1)^2}}$$

$$\Rightarrow x+z=0$$

(B) The equations of the planes bisecting the angles between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{\left(a_1^2+b_1^2+c_1^2\right)}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{\left(a_2^2+b_2^2+c_2^2\right)}}$$

Of these two planes one makes angle less than 45° with both the given planes and the other makes angles greater than 45° with both the given planes.

The bisector of the acute angle between the given planes is the one which makes angles less than 45° with both the given planes.

THE STRAIGHT LINE

Equations of a Line

An equation of the first degree represents a plane. Two equations of the first degree are satisfied by the coordinates of every point on the line of intersection of the planes represented by them. Therefore, the two equations together represent that line. Therefore $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ together represent a straight line.

The Equations to the Axes of Coordinates

(a) Equations to the x-axis are $y = 0$, $z = 0$.
(b) Equations to the y-axis are $x = 0$, $z = 0$.
(c) Equations to the z-axis are $x = 0$, $y = 0$.

Symmetrical form of the Equation of a Line

If a straight line passes through a given point (x_1, y_1, z_1) and has direction cosines l, m, n, then the coordinates of any point on it satisfy the equations

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}\ (=r)$$

These equations enable us to write down the coordinates of any point (x, y, z) on the line in terms of its distance r from the point (x_1, y_1, z_1) in the form:

$$x = x_1 + lr,\ y = y_1 + mr,\ z = z_1 + nr.$$

This form is called the *distance form* of the equation of a line.

Illustration 6

P and Q are two points on a line whose direction ratios are 3, 5, 1. If the coordinates of P are (1, – 4, 2) and $(PQ)^2 = 35$. Find the coordinates of Q.

Solution: Direction cosines of the line are

$$\frac{3}{\sqrt{3^2+5^2+1^2}},\ \frac{5}{\sqrt{3^2+5^2+1^2}},\ \frac{1}{\sqrt{3^2+5^2+1^2}}$$

or $\dfrac{3}{\sqrt{35}}, \dfrac{5}{\sqrt{35}}, \dfrac{1}{\sqrt{35}}$

So the equation of the line can be written as

$$\frac{x-1}{3/\sqrt{35}} = \frac{y+4}{5/\sqrt{35}} = \frac{z-2}{1/\sqrt{35}} = r = \sqrt{35} = PQ.$$

and the coordinates of Q are.
(3 + 1, 5 – 4, 1 + 2) = (4, 1, 3)

If l, m, n are only direction-ratios of the line and not the actual direction cosines, even then the equations represent a straight line, but in this case r is only a parameter, it is *not* the distance of (x, y, z) from (x_1, y_1, z_1).

Conversely, the equations

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n},$$

represent a straight line passing through the point (a, b, c) and having direction ratios l, m, n.

Equation of Line through two Given Points

If $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ be two given points, an equation of the line AB is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

The coordinates of a variable point on AB can be expressed in terms of a parameter λ in the form

$$x = \frac{\lambda x_2 + x_1}{\lambda+1},\ y = \frac{\lambda y_2 + y_1}{\lambda+1},\ z = \frac{\lambda z_2 + z_1}{\lambda+1},$$

λ being any real number different from – 1. In fact, (x, y, z) are the coordinates of the point which divides the join of A and B in the ratio $\lambda : 1$.

Changing unsymmetrical form to symmetrical form

The unsymmetrical form of a line

$$ax + by + cz + d = 0,$$
$$a'x + b'y + c'z + d' = 0.$$

can be changed to symmetrical form as follows:

$$\frac{x - \dfrac{bd'-b'd}{ab'-a'b}}{bc'-b'c} = \frac{y - \dfrac{da'-d'a}{ab'-a'b}}{ca'-c'a} = \frac{z}{ab'-a'b}$$

Illustration 7

Find the direction ratios of the line $x + 2y - z = 3$, $2x - y + z = 1$ and its equation in a symmetrical form.

Solution: Direction ratios of the line are

$$\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix},\ -\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix},\ \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

or 1, –3, –5

For any point on the line, let $x = 0$, then $x = y = 1$ and the equation of the line in the symmetrical form is

$$\frac{x-1}{1} = \frac{y-1}{-3} = \frac{z-0}{-5} \qquad (1)$$

Verification Any point on the line is

$r + 1, -3r + 1, -5r$

$r + 1 + 2(-3r + 1) - (-5r) = 3$ for all r

and $2(r + 1) - (-3r + 1) + (-5r) = 1$ for all r.

showing that each point on the line (1)

satisfies the given equations of the line.

Number of Constants in the Equation of a Line

The equation $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$,

of a line can be written as $x = (l/n)\, z + a - (lc/n)$,

$$y = (m/n)\, z + b - (mc/n),$$

which are of the form $\quad x = Az + B,\ y = Cz + D$

Therefore the *equation of a line contains four arbitrary constants.*

A Plane and a Straight Line

Let the equations

$$ax + by + cz + d = 0,\ \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

represent a given plane and a straight line respectively.

(i) The line is perpendicular to the plane if and only if

$$\frac{a}{l} = \frac{b}{m} = \frac{c}{n},$$

(ii) The line is parallel to the plane if and only if $al + bm + cn = 0$.

(iii) The line lies in the plane if and only if

$$al + bm + cn = 0 \text{ and } a\alpha + b\beta + c\gamma + d = 0.$$

Angle between a Line and a Plane

The angle θ between the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

and the plane $ax + by + cz + d = 0$, is given by

$$\sin\theta = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(l^2 + m^2 + n^2)}}$$

Coplanar Lines

The lines $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$,

and $\dfrac{x-\alpha'}{l'} = \dfrac{y-\beta'}{m'} = \dfrac{z-\gamma'}{n'}$

are coplanar if and only if

$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$$

In case the above condition is satisfied, the equation of the plane containing the lines is

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0.$$

Illustration 8

Find the value of λ, so that the lines

$\dfrac{x-1}{1} = \dfrac{y-2}{2} = \dfrac{z+\lambda}{3}$ and

$\dfrac{x+1}{2} = \dfrac{y-1}{3} = \dfrac{z-3}{1}$ are coplanar

Also find the equation of the plane containing them.

Solution: Given lines are coplanar if

$$\begin{vmatrix} 1+1 & 2-1 & -\lambda-3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} = 0 \Rightarrow \lambda = 6$$

and the equation of the plane containing the coplanar lines is

$$\begin{vmatrix} x-1 & y-2 & z+6 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix} = 0$$

or $7x - 5y + z + 9 = 0$

General Equation of a Plane Containing a Line

The general equation of the plane containing the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

is $\qquad A(x - \alpha) + B(y - \beta) + C(z - \gamma) = 0,$

where $\quad Al + Bm + Cn = 0.$

Length of the Perpendicular from a Point to a Line

The length p of the perpendicular from a given point $P(x_1,\ y_1,\ z_1)$ to a given line $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$

(l, m, n are direction cosines of the line), is given by $p^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - [l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)]^2$.
Another expression for p^2 is given by

$$p^2 = \begin{vmatrix} x_1 - \alpha & y_1 - \beta \\ l & m \end{vmatrix}^2 + \begin{vmatrix} y_1 - \beta & z_1 - \gamma \\ m & n \end{vmatrix}^2$$

$$+ \begin{vmatrix} z_1 - \gamma & x_1 - \alpha \\ n & l \end{vmatrix}^2$$

Illustration 9

Find the perpendicular distance of the point (2, 1, 3) from the line

$$\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$$

Solution: Let $P(2, 1, 3)$ be the given point $A(1, 3, 4)$ is a point on the given line $p=PM$, be the perpendicular from P on the line.

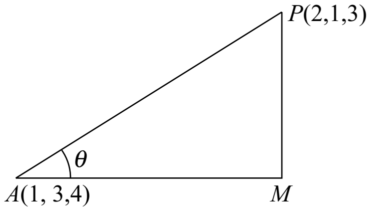


Fig. 21.2

Let $\angle PAM = \theta$ then

$p^2 = (PM)^2 = (AP)^2 - (AM)^2 = (AP)^2 - (AP \cos\theta)^2$

$\Rightarrow p^2 = (AP)^2 \sin^2\theta$

Direction ratios of AP are $2 - 1, 1 - 3, 3 - 4$
or $1, -2, -1$ and of the given line are $3, 1, -5$

$$\text{so } \cos\theta = \frac{(1)(3) + (-2)(1) + (-1)(-5)}{\sqrt{(1)^2 + (-2)^2 + (-1)^2}\sqrt{(3)^2 + (1)^2 + (-5)^2}}$$

$$= \frac{6}{\sqrt{6}\sqrt{35}} = \sqrt{\frac{6}{35}}$$

$(AP)^2 = (2 - 1)^2 + (1 - 3)^2 + (3 - 4)^2 = 6$

Hence $p^2 = 6 \times \frac{29}{35} \Rightarrow p = \sqrt{\frac{174}{35}}$

Equation of a Line through the Intersection of Given Lines

If $L_1 = 0 = L_2$ and $L_3 = 0 = L_4$, where $L_K = a_k x + b_k y + c_k z + d_k$, $k = 1, 2, 3, 4$; are equations of two intersecting lines then for all values of λ and μ the equation $L_1 + \lambda L_2 = 0 = L_3 + \mu L_4$ represents a line through their intersection.

VECTORIAL EQUATIONS

1. Parametric vectorial equation of the line through a point with position vector **a** and parallel to the vector **b** is
   $\mathbf{r} = \mathbf{a} + t\mathbf{b}$
   where t, is a scalar parameter.
2. Parametric vectorial equation of the line through two points with position vectors **a** and **b** is
   $\mathbf{r} = (1 - t)\mathbf{a} + t\mathbf{b}$
   where t, is a scalar parameter.
3. Parametric vectorial equation of a plane which passes through the point with position vector **a** and which is parallel to the vectors **b** and **c** is
   $\mathbf{r} = \mathbf{a} + t\mathbf{b} + p\mathbf{c}$
   where, t and p are scalar parameters.
4. Parametric vectorial equation of a plane passing through two given points with position vectors **a** and **b** and parallel to the vectors **c** is
   $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a}) + p\mathbf{c} = (1 - t)\,\mathbf{a} + t\mathbf{b} + p\mathbf{c}$
   where t and p are scalar parameters.
5. Equation of the plane passing through three points with position vectors **a**, **b**, **c** is
   $\mathrm{r} \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.
6. *Normal form of the vector equation of a plane* $\mathbf{r} \cdot \mathbf{n} = p$ is vector equation of a plane, such that **n** is the unit vector normal to the plane and p is the length of the perpendicular from the origin to the plane.
7. $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{N} = 0$ is vector equation of a plane normal to the vector **N** and passing through a point with position vector **a**.

Illustration 10

Find the vectorial equation of the plane passing through the point (2, 5, –3) and perpendicular to the line with direction ratios 3, 7, 5.

Solution: Let $\mathbf{a} = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

and $\mathbf{N} = 3\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}$

Required equation of the plane is

$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{N} = 0$

or $[\mathbf{r} - (2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})]\,(3\mathbf{i} + 7\mathbf{j} + 5\mathbf{k}) = 0$

8. *Angle between two planes* $\mathbf{r} \cdot \mathbf{N}_1 = p_1$ and $\mathbf{r} \cdot \mathbf{N}_2 = p_2$ is $\cos^{-1}\frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{|\mathbf{N}_1||\mathbf{N}_2|}$
9. *Angle between a line and a plane* whose vectorial equations are $\mathbf{r} = \mathbf{a} + \mathbf{b}t$ and $\mathbf{r} \cdot \mathbf{N} = q$ respectively is $\sin^{-1}\frac{\mathbf{N} \cdot \mathbf{b}}{|\mathbf{N}|\,|\mathbf{b}|}$.
10. *The perpendicular distance* of a point with position vector **a** from the plane $\mathbf{r} \cdot \mathbf{N} = q$ is $\frac{|q - \mathbf{a} \cdot \mathbf{N}|}{|\mathbf{N}|}$.

11. Equation of the plane containing two coplanar lines $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + p\mathbf{d}$ is

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} \times \mathbf{d} = 0$$

$$\Rightarrow \qquad \mathbf{r} \cdot \mathbf{b} \times \mathbf{d} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{d}.$$

12. Condition for the lines $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + p\mathbf{d}$ to be *coplanar* is $[\mathbf{cbd}] = [\mathbf{abd}]$.
13. The line *LM* of shortest distance between the lines $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + p\mathbf{d}$ is parallel to the vector $\mathbf{b} \times \mathbf{d}$.

Length of *LM* is the projection of the line joining the points with position vectors **a** and **c** on **LM**.

$$\text{Thus} \quad LM = \frac{|(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d})|}{|\mathbf{b} \times \mathbf{d}|} = \frac{|[\mathbf{cbd}] - [\mathbf{abd}]|}{|\mathbf{b} \times \mathbf{d}|}$$

Also, the line of shortest distance is the line of intersection of the two planes through the given lines and the line *LM*. The equations of these planes are
$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} \times (\mathbf{b} \times \mathbf{d}) = 0$,
$(\mathbf{r} - \mathbf{c}) \cdot \mathbf{d} \times (\mathbf{b} \times \mathbf{d}) = 0$
The line of intersection of these planes is the required line of shortest distance between the given lines.

Illustration 11

Find an equation of the line of shortest distance between the lines

$\mathbf{r} = \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $\mathbf{r} = (\mathbf{i} - \mathbf{j}) + \mu(-2\mathbf{j} + \mathbf{k})$

In the vectorial notation and Cartesian notation. Also find the shortest distance between them.

Solution: $\mathbf{a} = \mathbf{0}, \quad \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$\mathbf{c} = \mathbf{i} - \mathbf{j}, \quad \mathbf{d} = -2\mathbf{j} + \mathbf{k}$

Then equation of the line of shortest distance is

$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{b} \times (\mathbf{b} \times \mathbf{d}) = 0, (\mathbf{r} - \mathbf{c}) \cdot \mathbf{d} \times (\mathbf{b} \times \mathbf{d}) = 0$

$(\mathbf{b} \times \mathbf{d}) = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$

$\mathbf{b} \times (\mathbf{b} \times \mathbf{d}) = 3\mathbf{i} + 3\mathbf{j}$

$\mathbf{d} \times (\mathbf{b} \times \mathbf{d}) = 5\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \mathbf{c} \cdot [\mathbf{d} \times (\mathbf{b} \times \mathbf{d})] = 4$

So equation of the required line is

$\mathbf{r} \cdot (3\mathbf{i} + 3\mathbf{j}) = 0, \mathbf{r} \cdot (5\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 4$

$\mathbf{r} = (\mathbf{i} - \mathbf{j})$. satisfies both the equations.

So the line of shortest distance is

$\mathbf{r} = (\mathbf{i} - \mathbf{j}) + \lambda(\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

$$\text{Or} \quad \frac{x-1}{1} = \frac{y+1}{-1} = \frac{z}{-2}$$

$$\text{Required shortest distance is } \frac{|(\mathbf{c} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{d})|}{|\mathbf{b} \times \mathbf{d}|}$$

$$= \frac{|(\mathbf{i} - \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j} - 2\mathbf{k})|}{\sqrt{(1)^2 + (-1)^2 + (-2)^2}} = \frac{2}{\sqrt{6}}$$

SOLVED EXAMPLES

Concept-based Straight Objective Type Questions

Example 1: The acute angle between the lines such that the direction cosines l, m, n of each of them satisfy the equations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$ is

(a) 15° (b) 30°
(c) 60° (d) 45°

Ans. (c)

Solution: From the given relations $l^2 + m^2 = n^2$
$\Rightarrow (l + m)^2 - 2lm = n^2 \Rightarrow lm = 0$ as $l + m = -n$
if $l = 0$, $m = -n$, if $m = 0$, $l = -n$
So direction ratios of the lines are
0, 1, – 1 and 1, 0, – 1
and the angle θ between them is given by

$$\cos\theta = \frac{(0)(1) + (1)(0) + (-1)(-1)}{\sqrt{1^2 + (-1)^2}\,\sqrt{(1)^2 + (-1)^2}} = \frac{1}{2}$$

$\Rightarrow \theta = 60°$

Example 2: If the projections of a line segment on the x, y, and z-axes in 3-dimensional space are 2, 3 and 6 respectively, then the length of the line segment is

(a) 12 (b) 7
(c) 9 (d) 6.

Ans. (b)

Solution: Let r be the length of the line segment which makes angles α, β, γ respectively with x, y and z-axes, then

$$r\cos\alpha = 2, r\cos\beta = 3, r\cos\gamma = 6$$

$$\Rightarrow \qquad r^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2^2 + 3^2 + 6^2 = 49$$

$$\Rightarrow \qquad r = 7 \text{ as } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Example 3: The equation of a plane through the line of intersection of the planes $x + 2y = 3$, $y - 2z + 1 = 0$ and perpendicular to the first plane is :

(a) $2x - y - 10z = 9$ (b) $2x - y + 7z = 11$
(c) $2x - y + 10z = 11$ (d) $2x - y - 9z = 10$

Ans. (c)

◎ **Solution:** Equation of any plane through the intersection of the given planes is

$$x + 2y - 3 + \lambda (y - 2z + 1) = 0$$

or $$x + (2 + \lambda) y - 2\lambda z + (\lambda - 3) = 0$$

Since it is perpendicular to the first plane.

$(1)(1) + 2(2 + \lambda) + 0(-2\lambda) = 0 \Rightarrow \lambda = -5/2$

and the required equation of the plane is $2x - y + 10z = 11$

◉ **Example 4:** The image of the line $\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$ in the plane $2x - y + z + 3 = 0$ is the line

(a) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$ (b) $\frac{x+3}{-3} = \frac{y-5}{-1} = \frac{z-2}{5}$

(c) $\frac{x-3}{3} = \frac{y+5}{1} = \frac{z-2}{-5}$ (d) $\frac{x-3}{-3} = \frac{y+5}{-1} = \frac{z-2}{5}$

Ans. (a)

◎ **Solution:** Any point P on the given line is $(3r + 1, r + 3, -5r + 4)$

Image of P in the given plane will be the point P' at a distance equal to twice the distance of P from the plane and on the line through P perpendicular to plane.

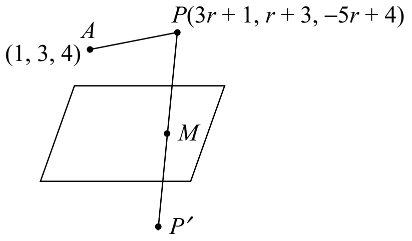


Fig. 21.3

Equation of PP' is

$$\frac{x-(3r+1)}{-2} = \frac{y-(r+3)}{1} = \frac{z-(-5r+4)}{-1}$$

Note

Direction of PP' is opposite to the direction of the normal to the plane.

Length of the perpendicular of P from the plane is

$$\frac{2(3r+1)-(r+3)+(-5r+4)+3}{\sqrt{2^2+(-1)^2+1^2}} = \sqrt{6}$$

So the coordinates of P' are given by

$$\frac{x-(3r+1)}{-2} = \frac{y-(r+3)}{1}$$

$$= \frac{z-(-5r+4)}{-1} = \frac{2\sqrt{6}}{\sqrt{(-2)^2+(1)^2+(-1)^2}} = 2$$

$\Rightarrow x = 3r + 1 - 4,\ y = 2 + r + 3,\ z = -2 - 5r + 4$

$\Rightarrow \frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$ is the required equation of the line.

◉ **Example 5:** Equation of the plane which passes through the point of intersection of the lines

$\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2}$ and $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

and has largest distance from the origin is:

(a) $7x + 2y + 4z = 54$ (b) $3x + 4y + 5z = 49$
(c) $4x + 3y + 5z = 50$ (d) $5x + 4y + 3z = 57$

Ans. (c)

◎ **Solution:** Any point on the first line is $(3r + 1, r + 2, 2r + 3)$

which lies on the second line if

$$\frac{3r+1-3}{1} = \frac{r+2-1}{2} = \frac{2r+3-2}{3} \Rightarrow r = 1$$

So the point of intersection of the given lines is (4, 3, 5)

Which lies on the planes given in (*a*), (*b*), (*c*).

Their distances from the origin are

$$P_1 = \frac{54}{\sqrt{69}},\ P_2 = \frac{49}{\sqrt{50}},\ P_3 = \frac{50}{\sqrt{50}}$$

$P_1 < 7,\ P_2, P_3 > 7 \qquad P_2 < P_3$

Hence the required plane is $4x + 3y + 5z = 50$

◉ **Example 6:** Let ABC be a triangle with vertices at points $A(2, 3, 5)$, $B(-1, 3, 2)$ and $C(\lambda, 5, \mu)$ in three dimensional space. If the median through A is equally inclined with the axes then (λ, μ) is equal to:

(a) (10, 7) (b) (7, 5)
(c) (7, 10) (d) (5, 7)

Ans. (c)

◎ **Solution:** Coordinates of D, the mid point of BC are $\left(\frac{\lambda-1}{2}, 4, \frac{\mu+2}{2}\right)$. So direction ratios of AD are

$$\frac{\lambda-1}{2} - 2, , 4 - 3, \frac{\mu+2}{2} - 5 \text{ or } \frac{\lambda-5}{2}, 1, \frac{\mu-8}{2}.$$

Since AD, is equally inclined to the axes.

$\frac{\lambda-5}{2} = 1 = \frac{\mu-8}{2} \Rightarrow \lambda = 7, \mu = 10$ and $(\lambda, \mu) = (7, 10)$

◉ **Example 7:** The plane containing the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and parallel to the line $\frac{x}{1} = \frac{y}{1} = \frac{z}{4}$ passes through the point

(a) (1, –2, 5) (b) (1, 0, 5)
(c) (0, 3, – 5) (d) (–1, –3, 0)

Ans. (b)

◎ **Solution:** Equation of the plane containing the first line is $a(x - 1) + b(y - 2) + c(z - 3)$ where

$a + 2b + 3c = 0$, as it is parallel to the second line

$a + b + 4c = 0$ solving, we get

$$\frac{a}{+5}=\frac{b}{-1}=\frac{c}{-1}$$

Equation of the required plane is

$5x - y - z = 0$ which passes through (1, 0, 5)

◉ **Example 8:** The distance between the point (–1, –5, –10) and the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ = with the plane $x - y + z = 5$ is

(a) 13 (b) 15

(c) 16 (d) 12

Ans. (a)

◎ **Solution:** Any point on the given plane is $(3r + 2, 4r - 1, 12r + 2)$ which lies on the given plane if $3r + 2 - (4r - 1) + 12r + 2 = 5 \Rightarrow r = 0$

So the point of intersection of the line and the plane is (2, –1, 2) and the required distance is

$$\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2} = 13$$

◉ **Example 9:** Let α, β, γ be the angles made by a line with the positive directions of the axes of reference in three dimensions. If θ is the acute angle given by

$$\cos\theta=\frac{\cos^2\alpha+\cos^2\beta+\cos^2\gamma}{\sin^2\alpha+\sin^2\beta+\sin^2\gamma}$$, then θ equal.

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

Ans. (b)

◎ **Solution:** $$\cos\theta=\frac{\cos^2\alpha+\cos^2\beta+\cos^2\gamma}{3-(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)}$$

$$=\frac{1}{3-1}=\frac{1}{2}$$

$$\Rightarrow \qquad \theta=\frac{\pi}{3}$$

◉ **Example 10:** The reflection point of the point (0, 3, –2) in the line $\frac{1-x}{2} = 2 - y = z + 1$ is

(a) (1, 2, –1) (b) (2, 1, 4)

(c) (2, 1, 0) (d) (0, 0, 1)

Ans. (c)

◎ **Solution:** Equation of the line can be written as

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z+1}{-1}$$

Let Q (α, β, γ) be the required point of reflection of the given point P (0, 3, –2). Then PQ is perpendicular to the given line and the mid point of PQ lies on the line.

So $2(\alpha - 0) + (\beta - 3) - (\gamma + 2) = 0 \Rightarrow 2\alpha + \beta - \gamma - 5 = 0$(1)

and $$\frac{\frac{\alpha}{2}-1}{2}=\frac{\frac{\beta+3}{2}-2}{1}=\frac{\frac{\gamma-2}{2}+1}{-1}$$

$$\Rightarrow \frac{\alpha-2}{2}=\frac{\beta-1}{1}=\frac{\gamma}{-1} = r\ (say) \qquad (2)$$

From (1) and (2) we get

$2(2r + 2) + r + 1 + r - 5 = 0 \Rightarrow r = 0$

and the required point is (2, 1, 0)

◉ **Example 11:** A variable plane passes through a fixed point (1, –2, 3) and meets the coordinate axes at points A, B, C then the point of intersection of the planes through A, B, C parallel to the coordinate planes lies on

(a) $xy - \frac{1}{2}yz + \frac{1}{3}zx = 6$

(b) $yz - 2zx + 3xy = xyz$

(c) $xy - 2yz + 3zx = 3xyz$

(d) $xy - \frac{1}{2}yz - \frac{1}{3}zx = 6$

Ans. (b)

◎ **Solution:** Let the coordinates of A be $(a, 0, 0)$, B be $(0, b, 0)$ and C be $(0, 0, c)$. Then equation of the plane is

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$. As it passes through (1, –2, 3)

$\frac{1}{a}-\frac{2}{b}+\frac{3}{c} = 1$

or $bc - 2ac + 3ab = abc$ (1)

Equations of the planes through A, B, C parallel to the coordinate planes are $x = a$, $y = b$, $z = c$

So their point of intersection is (a, b, c)

From (1), it lies on $yz - 2zx + 3xy = xyz$

◉ **Example 12:** Let L_1 be the line $2x - 2y + 3z - 2 = 0 = x - y + z + 1$ and L_2 be the line $x + 2y - z - 3 = 0 = 3x - y + 2z - 1 = 0$. The distance of the origin from the plane through L_1 and L_2 is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2\sqrt{2}}$

(c) $\frac{1}{3\sqrt{2}}$ (d) $\frac{1}{\sqrt{15}}$

Ans. (c)

◎ **Solution:** The plane through L_1 is

$2x - 2y + 3z - 2 + \lambda(x - y + z + 1) = 0$

and the plane through L_2 is

$x + 2y - z - 3 + \mu(3x - y + 2z - 1) = 0$

Two planes are same

so $\dfrac{2+\lambda}{1+3\mu} = \dfrac{2+\lambda}{\mu-2} = \dfrac{3+\lambda}{2\mu-1} = \dfrac{2-\lambda}{3+\mu}$

$\Rightarrow \mu = -\dfrac{3}{2}$ and $\lambda = 5$ and the equation of the plane is $7x - 7y + 8z + 3 = 0$.

Distance of this plane from the origin is

$$\frac{3}{\sqrt{7^2+(-7)^2+(8)^2}} = \frac{1}{3\sqrt{2}}$$

Example 13: Foot of the perpendicular drawn from the origin to the plane passing through (1, 0, 0), (0, 1, 0) and (0, 0, 1) is

(a) (3, 3, 3) (b) $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

(c) $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ (d) (2, 2, 2)

Ans. (c)

Solution: Equation of the plane is $x + y + z = 1$ only the point given in (c) lies on it. Also the line joining the origin to $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is parallel to the normal to the plane.

Example 14: The equation of the plane which contains the line of intersection of the planes $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}). = 0$, $\mathbf{r} \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k}) + 5 = 0$ and which is perpendicular to the plane. $\mathbf{r} \cdot (5\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) + 8 = 0$ is

(a) $\mathbf{r} \cdot (33\mathbf{i} + 45\mathbf{j} + 50\mathbf{k}) - 41 = 0$

(b) $\mathbf{r} \cdot (13\mathbf{i} + 15\mathbf{j} + 25\mathbf{k}) - 14 = 0$

(c) $\mathbf{r} \cdot (24\mathbf{i} + 45\mathbf{j} + 30\mathbf{k}) - 17 = 0$

(d) $\mathbf{r} \cdot (45\mathbf{i} + 33\mathbf{j} + 50\mathbf{k}) - 41 = 0$

Ans. (a)

Solution: The equation of the line through the intersection of the given planes is

$$\mathbf{r} \cdot [(1 + 2\lambda)\mathbf{i} + (2 + \lambda)\mathbf{j} + (3 - \lambda)\mathbf{k} + (-4 + 5\lambda)] = 0$$

If it perpendicular to the plane $\mathbf{r} \cdot (5\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) + 8 = 0$

then $(1 + 2\lambda)(5) + (2 + \lambda)(3) + (3 - \lambda)(-6) = 0$

$\Rightarrow \lambda = \dfrac{7}{19}$ and the required equation of the place in $\mathbf{r} \cdot (33\mathbf{i} + 45\mathbf{j} + 50\mathbf{k}) - 40 = 0$.

Example 15: The points *A*, *B* and *C* with position vectors $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$, respectively are

(a) Collinear

(b) Vertices of an equilateral triangle

(c) Vertices of a right angled triangle

(d) none of these

Ans. (c)

Solution: $\mathbf{AB} = (2 - 3)\mathbf{i} + (-1 + 4)\mathbf{j} + (1 + 4)\mathbf{k}$

$$= -\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$$

$$\mathbf{BC} = -\mathbf{i} - 2\mathbf{j} - 6\mathbf{k},\ \mathbf{CA} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$$

Since $\mathbf{AB} + \mathbf{BC} + \mathbf{CA} = 0$; A, B, C are the Vertices of a triangle.

$$\mathbf{AB} \cdot \mathbf{CA} = 0 \Rightarrow \angle BCA = 90°$$

So the triangle *ABC* is right angled.

LEVEL 1

Straight Objective Type Questions

Example 16: The coordinates of a point which divide the line joining the points *P*(2, 3, 1) and *Q* (5, 0, 4) in the ratio 1 : 2 are

(a) (7/3, 1, 5/3) (b) (4, 1, 3)

(c) (3, 2, 2) (d) (1, –1, 1)

Ans. (c)

Solution: The coordinates of the required point are

$$\left(\frac{2\times2+5\times1}{2+1}, \frac{2\times3+0\times1}{2+1}, \frac{2\times1+4\times1}{2+1}\right) \quad \text{or} \quad (3, 2, 2)$$

Example 17: If a line *OP* through the origin O makes angles α, 45° and 60° with *x*, *y* and *z* axis respectively then the direction cosines of *OP* are.

(a) $1/\sqrt{2}$, 1/2, 1/2 (b) 1/2, 1/2, $1/\sqrt{2}$

(c) 1/2, $1/\sqrt{2}$, 1/2 (d) none of these

Ans. (c)

Solution: Direction cosines of *OP* are $\cos\alpha$, cos 45°, cos 60° and $\cos^2\alpha + \cos^2 45° + \cos^2 60° = 1$

$\Rightarrow \quad \cos^2\alpha + (1/2) + (1/4) = 1$

$$\Rightarrow \qquad \cos^2 \alpha = 1/4$$

$$\Rightarrow \qquad \cos \alpha = 1/2$$

Example 18: The direction cosines of the line joining the points (1, 2, – 3) and (– 2, 3, 1) are.

(a) – 3, 1, 4 (b) – 1, 5, – 2

(c) $\frac{-3}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}$ (d) $\frac{-1}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{-2}{\sqrt{30}}$

Ans. (c)

Solution: The direction cosines of the given line are proportional to – 2 – 1, 3 – 2, 1 – (– 3) i.e. – 3, 1, 4. Therefore, the actual direction cosines are $\frac{-3}{\sqrt{9+1+16}}$,

$$\frac{1}{\sqrt{9+1+16}}, \frac{4}{\sqrt{9+1+16}} \text{ i.e } \frac{-3}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}.$$

Example 19: An equation of z-axis is

(a) $z = 0, x = 0$ (b) $z = 0, y = 0$

(c) $x = 0, y = 0$ (d) $x = k, y = -k.,\ (k \neq 0)$

Ans. (c)

Solution: See theory

Example 20: The ratio in which the yz plane divides the segment joining the points (–2, 4, 7) and (3, – 5, 8) is

(a) 2 : 3 (b) 3 : 2

(c) 4 : 5 (d) – 7 : 8

Ans. (a)

Solution: Let yz plane divide the segment joining (– 2, 4, 7) and (3, – 5, 8) in the ratio $\lambda : 1$. Then

$$\Rightarrow \quad \frac{3\lambda - 2}{\lambda + 1} = 0 \Rightarrow \lambda = \frac{2}{3} \text{ and the required ratio is } 2 : 3.$$

Example 21: The coordinates of a point equidistant from the points (a, 0, 0), (0, a, 0), (0, 0, a) and (0, 0, 0) are.

(a) (a/3, a/3, a/3) (b) (a/2, a/2, a/2)

(c) (a, a, a) (d) (2a, 2a, 2a)

Ans. (b)

Solution: Let the coordinates of the required point be (x, y, z) then $x^2 + y^2 + z^2 = (x - a)^2 + y^2 + z^2$
$= x^2 + (y - a)^2 + z^2 = x^2 + y^2 + (z - a)^2$
$\Rightarrow \quad x = a/2 = y = z$. Hence the required point is (a/2, a/2, a/2).

Example 22: If O is the origin and the line OP of length r makes an angle α with x-axis and lies in the xz plane, then the coordinate of P are

(a) ($r \cos \alpha$, 0, $r \sin \alpha$) (b) (0, 0, $r \sin \alpha$)

(c) (0, 0, $r \cos \alpha$) (d) ($r \cos \alpha$, 0, 0)

Ans. (a)

Solution: Let the coordinate of P be (x, y, z).
Since OP lies in xz plane and makes an angle α with the x-axis, it makes angle $\pi/2 - \alpha$ with z-axis and $\pi/2$ with Y-axis. so, $x = r \cos \alpha$, $y = r \cos \pi/2$, $z = r \cos (\pi/2 - \alpha)$ are the required coordinates and therefore are ($r \cos \alpha$, 0, $r \sin \alpha$).

Example 23: If a line is equally inclined with the coordinate axes, then the angle of inclination is

(a) $\cos^{-1} (1/2)$ (b) $\cos^{-1} (1/\sqrt{2})$

(c) $\cos^{-1} (1/\sqrt{3})$ (d) $\cos^{-1} (\sqrt{3}/2)$

Ans. (c)

Solution: Let the line be inclined at an angle α with each of the three coordinates axes, then the direction cosines of the line are $\cos \alpha$, $\cos \alpha$, $\cos \alpha$ and $3 \cos^2\alpha = 1$

$$\Rightarrow \cos \alpha = 1/\sqrt{3} \Rightarrow \alpha = \cos^{-1}(1/\sqrt{3})$$

Example 24: A line makes an angle of 60° with each of x and y axis, the angle which it makes with z axis is

(a) 30° (b) 45°

(c) 60° (d) none of these

Ans. (b)

Solution: Let α be the angle which the line makes with z-axis, thus the direction cosines of the line are cos 60°, cos 60°, $\cos \alpha$.

$$\Rightarrow \qquad \cos^2 60° + \cos^2 60° + \cos^2 \alpha = 1$$

$$\Rightarrow \qquad \cos^2 \alpha = 1 - 1/4 - 1/4 = 1/2$$

$$\Rightarrow \qquad \cos \alpha = \pm\, 1/\sqrt{2} \text{ so } \alpha = 45°.$$

Example 25: The projections of a line segment on x, y and z axes are respectively 3, 4 and 12. The length of the line segment is

(a) 5 (b) $4\sqrt{10}$

(c) $3\sqrt{17}$ (d) 13

Ans. (d)

Solution: The length of the given segment is

$$\sqrt{3^2 + 4^2 + 12^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13.$$

Example 26: If P (x, y, z) is a point in the space at a distance r from the origin O, then the direction cosines of the line OP are

(a) $\frac{r}{x}, \frac{r}{y}, \frac{r}{z}$ (b) rx, ry, rz

(c) $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ (d) none of these

Ans. (c)

Solution: Let α, β, γ be the angle which OP makes with the coordinates axes, then

$$x = r \cos \alpha,\ y = r \cos \beta \text{ and } z = r \cos \gamma.$$

$\Rightarrow \quad \cos\alpha = x/r, \cos\beta = y/r$ and $\cos r = z/r$.
So the required direction cosines are x/r, y/r, z/r.

Example 27: Let N be the foot of the perpendicular of length p from the origin to a plane and l, m, n be the direction cosines of ON, the equation of the plane is

(a) $px + my + nz = l$ (b) $lx + py + nz = m$
(c) $lx + my + pz = n$ (d) $lx + my + nz = p$.

Ans. (d)

Solution: The coordinates of N are (pl, pm, pn) and let $P(x, y, z)$ be any point on the plane. The direction cosines of PN are proportional to $x - pl$, $y - pm$, $z - pn$. Since ON is perpendicular to the plane, it is perpendicular to PN hence
$l(x - pl) + m(y - pm) + n(z - pn) = 0$

$$\Rightarrow \quad lx + my + nz = p(l^2 + m^2 + n^2) = p$$

which is the locus of P and hence is the required equation of the plane.

Example 28: An equation of the plane passing through the point (1, 2, 3), the direction cosines of the normal to which are l, m, n is

(a) $lx + my + nz = l + 2m + 3n$

(b) $\dfrac{x-1}{l} + \dfrac{y-2}{m} + \dfrac{z-3}{n} = 0$

(c) $lx + my + nz = 1$

(d) $\dfrac{lx}{1} + \dfrac{my}{2} + \dfrac{nz}{3} = 0$

Ans. (a)

Solution: Equation of the required plane is

$$l(x - 1) + m(y - 2) + n(z - 3) = 0$$

$$\Rightarrow \quad lx + my + nz = l + 2m + 3n.$$

Example 29: If a plane meets the co-ordinate axes in A, B, C such that the centroid of the triangle ABC is the point $(1, r, r^2)$, then equation of the plane is

(a) $x + ry + r^2z = 3r^2$ (b) $r^2x + ry + z = 3r^2$
(c) $x + ry + r^2z = 3$ (d) $r^2x + ry + z = 3$

Ans. (b)

Solution: Let an equation of the required plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

This meets the coordinates axes in

$A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

So that the coordinates of the centroid of the triangle ABC are $(a/3, b/3, c/3) = (1, r, r^2)$ (given)
$\Rightarrow \quad a = 3, b = 3r, c = 3r^2$
and the required equation of the plane is

$$\frac{x}{3} + \frac{y}{3r} + \frac{z}{3r^2} = 1 \quad \text{or} \quad r^2x + ry + z = 3r^2.$$

Example 30: Algebraic sum of the intercepts made by the plane $x + 3y - 4z + 6 = 0$ on the axes is

(a) $-13/2$ (b) $19/2$
(c) $-22/3$ (d) $26/3$

Ans. (a)

Solution: Equation of the plane can be written as

$$\frac{x}{-6} + \frac{y}{-2} + \frac{z}{3/2} = 1$$

So the intercepts on the coordinate axes are -6, -2, $3/2$ and the required sum is $-6 - 2 + 3/2 = -13/2$.

Example 31: An equation of the plane passing through the origin and containing the lines whose direction cosines are proportional to 1, –2, 2 and 2, 3, –1 is

(a) $x - 2y + 2z = 0$ (b) $2x + 3y - z = 0$
(c) $x + 5y - 3z = 0$ (d) $4x - 5y - 7z = 0$

Ans. (d)

Solution: Let the direction cosines of the normal to the plane be l, m, n. Since the normal is perpendicular to both the lines lying in the plane, whose direction cosines are proportional to 1, – 2, 2 and 2, 3, –1; we have

$$l - 2m + 2n = 0 \quad \text{and} \quad 2l + 3m - n = 0$$

$$\Rightarrow \quad \frac{l}{-4} = \frac{m}{5} = \frac{m}{7}$$

Since the plane passes through the origin its equation is

$$lx + my + nz = 0 \text{ or } 4x - 5y - 7z = 0$$

Example 32: An equation of the plane passing through the point (1, – 1, 2) and parallel to the plane $3x + 4y - 5z = 0$ is

(a) $3x + 4y - 5z + 11 = 0$ (b) $3x + 4y - 5z = 11$
(c) $6x + 8y - 10z = 1$ (d) $3x + 4y - 5z = 2$

Ans. (a)

Solution: Equation of any plane parallel to the plane $3x + 4y - 5z = 0$ is $3x + 4y - 5z = K$
If it passes through (1, – 1, 2), then

$$3 - 4 - 5(2) = K \quad \Rightarrow \quad K = -11$$

So the required equation is $3x + 4y - 5z + 11 = 0$.

Example 33: Vector equation of the line
$6x - 2 = 3y + 1 = 2z - 2$ is

(a) $\mathbf{r} = \mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

(b) $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda\left(\dfrac{1}{3}\mathbf{i} - \dfrac{1}{3}\mathbf{j} + \mathbf{k}\right)$

(c) $\mathbf{r} = \dfrac{1}{3}\mathbf{i} - \dfrac{1}{3}\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

(d) $\mathbf{r} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$; λ, being a parameter

Ans. (c)

◎ **Solution:** Equation of the given line can be written in the symmetrical form as

$$\frac{x-1/3}{1}=\frac{y+1/3}{2}=\frac{z-1}{3}$$

So the line passes through the point (1/3, –1/3, 1) and the direction cosines of the line are proportionate to 1, 2, 3 and hence its vector equation is

$$\mathbf{r}=\frac{1}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}+\mathbf{k}+\lambda(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$$

◉ **Example 34:** Equations of a line passing through (2, –1, 1) and parallel to the line whose equations are

$$\frac{x-3}{2}=\frac{y+1}{7}=\frac{z-2}{-3}\quad\text{are}$$

(a) $\frac{x-2}{3}=\frac{y+1}{-1}=\frac{z-1}{2}$ (b) $\frac{x-2}{2}=\frac{y+1}{7}=\frac{z-1}{-3}$

(c) $\frac{x-2}{2}=\frac{y-7}{-1}=\frac{z+3}{1}$ (d) $\frac{x-3}{2}=\frac{y+1}{-1}=\frac{z-2}{1}$

Ans. (b)

◎ **Solution:** The required line passes through (2, – 1, 1) and its direction cosines are proportional to 2, 7, –3 so its equation is

$$\frac{x-2}{2}=\frac{y+1}{7}=\frac{z-1}{-3}$$

◉ **Example 35:** If *M* denotes the mid-point of the line joining *A* (4 **i** + 5**j** – 10**k**) and *B* (– **i** + 2**j** + **k**), then equation of the plane through *M* and perpendicular to *AB* is

(a) $\mathbf{r}\cdot(-5\,\mathbf{i}-3\,\mathbf{j}+11\mathbf{k})+135/2=0$

(b) $\mathbf{r}\cdot\left(\frac{3}{2}\mathbf{i}+\frac{7}{2}\mathbf{j}-\frac{9}{2}\mathbf{k}\right)+\frac{135}{2}=0$

(c) $\mathbf{r}\cdot(4\mathbf{i}+5\mathbf{j}-10\mathbf{k})+4=0$

(d) $\mathbf{r}\cdot(-\,\mathbf{i}+2\mathbf{j}+\mathbf{k})+4=0$

Ans. (a)

◎ **Solution:** Middle point *M* of *AB* is

$$M\left(\frac{1}{2}(4\mathbf{i}+5\mathbf{j}-10\mathbf{k}-\mathbf{i}+2\mathbf{j}+\mathbf{k})\right)$$

$$=M\left(\frac{3}{2}\mathbf{i}+\frac{7}{2}\mathbf{j}-\frac{9}{2}\mathbf{k}\right)$$

Also $\mathbf{AB}=-\mathbf{i}+2\mathbf{j}+\mathbf{k}-(4\mathbf{i}+5\mathbf{j}-10\mathbf{k})$

$$=-5\mathbf{i}-3\mathbf{j}+11\mathbf{k}$$

So the plane passing through *M* and perpendicular to the direction **AB** is

$$\left[\mathbf{r}-\left(\frac{3}{2}\mathbf{i}+\frac{7}{2}\mathbf{j}-\frac{9}{2}\mathbf{k}\right)\right]\cdot(-5\mathbf{i}-3\mathbf{j}+11\mathbf{k})=0$$

$$\mathbf{r}\cdot(-5\,\mathbf{i}-3\,\mathbf{j}+11\,\mathbf{k})+135/2=0$$

◉ **Example 36:** Equation of the plane through (3, 4, –1) which is parallel to the plane **r** · (2**i** – 3**j** + 5 **k**) + 7 = 0 is

(a) $\mathbf{r}\cdot(2\mathbf{i}-3\mathbf{j}+5\mathbf{k})+11=0$

(b) $\mathbf{r}\cdot(3\mathbf{i}+4\mathbf{j}-\mathbf{k})+11=0$

(c) $\mathbf{r}\cdot(3\mathbf{i}+4\mathbf{j}-\mathbf{k})+7=0$

(d) $\mathbf{r}\cdot(2\mathbf{i}-3\mathbf{j}+5\mathbf{z})-7=0$

Ans. (a)

◎ **Solution:** Equation of any plane parallel to the given plane is

$$\mathbf{r}\cdot(2\mathbf{i}-3\mathbf{j}+5\mathbf{k})+\lambda=0.$$

If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, we get

$$2x-3y+5z+\lambda=0$$

This plane passes through the point (3, 4, –1)

if $2\times3-3\times4+5(-1)+\lambda=0$ or if $\lambda=11$

and hence the equation of the required plane is

$$\mathbf{r}\cdot(2\mathbf{i}-3\mathbf{j}+5\mathbf{k})+11=0$$

◉ **Example 37:** The ratio in which the plane 2*x* – 1 = 0 divides the line joining (– 2, 4, 7) and (3, – 5, 8) is

(a) 2 : 3 (b) 4 : 5

(c) 7 : 8 (d) 1 : 1

Ans. (d)

◎ **Solution:** Let the required ratio be *k* : 1, then the co-ordinates of the point which divides the join of the points (– 2, 4, 7) and (3, – 5, 8) in this ratio are given by

$$\left(\frac{3k-2}{k+1},\frac{-5k+4}{k+1},\frac{8k+7}{k+1}\right)$$

As this point lies on the plane 2*x* – 1 = 0.

$$\Rightarrow\quad\frac{3k-2}{k+1}=\frac{1}{2}$$

⇒ *k* = 1 and thus the required ratio is 1 : 1.

◉ **Example 38:** A line passes through the points (6, – 7, – 1) and (2, – 3, 1). If the angle α which the line makes with the positive direction of *x*-axis is acute, the direction cosines of the line are

(a) 2/3, – 2/3, – 1/3 (b) 2/3, 2/3, – 1/3

(c) 2/3, – 2/3, 1/3 (d) 2/3, 2/3, 1/3

Ans. (a)

◎ **Solution:** The direction cosines of the given line are proportional to

2 – 6, – 3 + 7, 1 + 1, i.e. – 2, 2, 1

the direction cosines are therefore $=\frac{\mp2}{3},\frac{\pm2}{3},\frac{\pm1}{3}$.

Since the angle α which the line makes with positive direction of *x*-axis is acute, $\cos\alpha>0\Rightarrow\ \cos\alpha=2/3$

Thus, required direction cosines are 2/3, –2/3, –1/3.

Example 39: The d.r. of normal to the plane through (1, 0, 0), (0, 1, 0) which makes an angle $\pi/4$ with the plane $x + y = 3$ are

(a) $1, \sqrt{2}, 1$ (b) $1, 1, \sqrt{2}$

(c) $1, 1, 2$ (d) $\sqrt{2}, 1, 1$

Ans. (b)

Solution: Equation of a plane through (1, 0, 0) is $A(x - 1) + By + Cz = 0$.
Since it passes through the point (0, 1, 0), $-A + B = 0$
$\Rightarrow$ $A = B$ and as it makes an angle $\pi/4$ with $x + y = 3$

$$\frac{A+B}{\sqrt{1+1}\sqrt{A^2+B^2+C^2}} = \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad (A+B)^2 = A^2 + B^2 + C^2$$

$$\Rightarrow \quad 2A^2 = C^2$$

So $$\frac{A}{1} = \frac{B}{1} = \frac{C}{\pm\sqrt{2}}$$

and the required d.r. are $1, 1, \sqrt{2}$.

Example 40: A plane which passes through the point (3, 2, 0) and the line $\dfrac{x-4}{1} = \dfrac{y-7}{5} = \dfrac{z-4}{4}$ is

(a) $x - y + z = 1$ (b) $x + y + z = 5$
(c) $x + 2y - z = 1$ (d) $2x - y + z = 5$

Ans. (a)

Solution: Let the equation of the plane be

$$a(x - 3) + b(y - 2) + cz = 0$$

Since it contains the given line.

$$a(4 - 3) + b(7 - 2) + 4c = 0$$

$$\Rightarrow \quad a + 5b + 4c = 0$$

which is satisfied by the d.r 1, – 1, 1 of the plane in (a).

Example 41: The lines $\dfrac{x-2}{1} = \dfrac{y-3}{1} = \dfrac{z-4}{-k}$ and $\dfrac{x-1}{k} = \dfrac{y-4}{2} = \dfrac{z-5}{1}$ are coplanar if

(a) $k = 1$ or -1 (b) $k = 0$ or -3
(c) $k = 3$ or -3 (d) $k = 0$ or -1

Ans. (b)

Solution: The lines are coplanar if

$$\begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1-k \\ k & k+2 & k+1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(k + 1) - (1 - k)(k + 2) = 0 \quad \Rightarrow \quad k = 0 \text{ or } -3$$

Example 42: Two system of rectangular axes have the same origin. If a plane cuts them at distances a, b, c and a', b', c' from the origin, then

(a) $\dfrac{1}{a^2} + \dfrac{1}{b^2} - \dfrac{1}{c^2} + \dfrac{1}{a'^2} + \dfrac{1}{b'^2} - \dfrac{1}{c'^2} = 0$

(b) $\dfrac{1}{a^2} - \dfrac{1}{b^2} - \dfrac{1}{c^2} + \dfrac{1}{a'^2} - \dfrac{1}{b'^2} - \dfrac{1}{c'^2} = 0$

(c) $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} - \dfrac{1}{a'^2} - \dfrac{1}{b'^2} - \dfrac{1}{c'^2} = 0$

(d) $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2} + \dfrac{1}{a'^2} + \dfrac{1}{b'^2} + \dfrac{1}{c'^2} = 0$

Ans. (c)

Solution: Let the equation of the plane in the two systems be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

and $$\frac{X}{a'} + \frac{Y}{b'} + \frac{Z}{c'} = 1$$

Since the origin is same, the length of the perpendicular from the origin on the plane is same in both the systems.

So $$\left|\frac{-1}{\sqrt{\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}}}\right| = \left|\frac{-1}{\sqrt{\dfrac{1}{a'^2} + \dfrac{1}{b'^2} + \dfrac{1}{c'^2}}}\right|$$

$$\Rightarrow \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}.$$

Example 43: Two lines $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ will be perpendicular, if and only if

(a) $aa' + bb' + cc' = 0$
(b) $(a + a')(b + b')(c + c') = 0$
(c) $aa' + cc' + 1 = 0$
(d) $aa' + bb' + cc' + 1 = 0$

Ans. (c)

Solution: Lines can be written as

$$\frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c} \text{ and } \frac{x-b'}{a'} = \frac{y}{1} = \frac{z-d'}{c'}$$

which will be perpendicular if and only if

$$aa' + 1 + cc' = 0$$

Example 44: Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is

(a) 7/2 (b) 5/2
(c) 3/2 (d) 9/2

Ans. (a)

Solution: Required distance is

$$\left|\frac{5/2+8}{\sqrt{4+1+4}}\right| = \frac{7}{2}$$

Example 45: A tetrahedron has vertices at $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. Then the angle between the faces OAB and ABC will be

(a) $\cos^{-1}(17/31)$ (b) $30°$
(c) $90°$ (d) $\cos^{-1}(19/35)$

Ans. (d)

Solution: Let the equation of the face OAB be
$ax + by + cz = 0$
where $a + 2b + c = 0$ and $2a + b + 3c = 0$

$$\Rightarrow \quad \frac{a}{5} = \frac{b}{-1} = \frac{c}{-3}$$

Similarly, the equation of the face ABC be
$a'(x + 1) + b'(y - 1) + c'(z - 2) = 0$
where $2a' + b' - c' = 0$ and $3a' + c' = 0$

$$\Rightarrow \quad \frac{a'}{1} = \frac{b'}{-5} = \frac{c'}{-3}$$

If θ is the required angle, then

$$\cos\theta = \frac{5\times 1 + (-1)(-5) + (-3)(-3)}{\sqrt{25+1+9}\,\sqrt{1+25+9}} = \frac{19}{35}.$$

Example 46: A line makes the same angle θ with each of the x and z axis. If the angle β, which it makes with y-axis, is such that $\sin^2\beta = 3\sin^2\theta$, then $\cos^2\theta$ equals

(a) 3/5 (b) 1/5
(c) 2/3 (d) 2/5

Ans. (a)

Solution: We have $\cos^2\theta + \cos^2\theta + \cos^2\beta = 1$

$$\Rightarrow \quad 2\cos^2\theta = \sin^2\beta = 3\sin^2\theta = 3(1 - \cos^2\theta)$$

$$\Rightarrow \quad 5\cos^2\theta = 3 \quad \Rightarrow \quad \cos^2\theta = 3/5.$$

Example 47: A line with directional cosines proportional to 2, 1, 2 meets each of the lines $x = y + a = z$ and $x + a = 2y = 2z$. The coordinates of each of the points of intersection are given by

(a) $(3a, 2a, 3a), (a, a, 2a)$
(b) $(3a, 2a, 3a), (a, a, a)$
(c) $(3a, 3a, 3a), (a, a, a)$
(d) $(2a, 3a, 3a), (2a, a, a)$

Ans. (b)

Solution: Let the points be $P(r, r - a, r)$ on the first line and $Q(2r' - a, r', r')$ on the second line then

$$\frac{r - 2r' + a}{2} = \frac{r - a - r'}{1} = \frac{r - r'}{2}$$

$\Rightarrow$ $r = 3a$, $r' = a$, so the required points are

$P(3a, 2a, 3a)$ and $Q(a, a, a)$

Example 48: If the straight lines.
$x = 1 + s$, $y = -3 - \lambda s$, $z = 1 + \lambda s$ and $x = t/2$, $y = 1 + t$, $z = 2 - t$, with parameters s and t are coplanar, then λ equals

(a) $-1/2$ (b) -1
(c) -2 (d) 0

Ans. (c)

Solution: Equation of the lines can be written as $\frac{x-1}{1} = \frac{y+3}{-\lambda} = \frac{z-1}{\lambda}$ and $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{-2}$. Since these are coplanar

$$\begin{vmatrix} 1 & -4 & -1 \\ 1 & 2 & -2 \\ 1 & -\lambda & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & -4 & - \\ 0 & 6 & -1 \\ 0 & -\lambda+4 & \lambda+1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 6(\lambda + 1) + (4 - \lambda) = 0 \quad \Rightarrow \quad \lambda = -2$$

Example 49: The angles between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is

(a) $45°$ (b) $30°$
(c) $0°$ (d) $90°$

Ans. (d)

Solution: Lines can be written as

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{-6} \text{ and } \frac{x}{2} = \frac{y}{-12} = \frac{z}{-3}$$

Since $3 \times 2 + 2 \times (-12) + (-6)(-3) = 0$
The lines are perpendicular.

Example 50: If the angle θ between the lines $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$ and the plane $2x - y + \sqrt{\lambda}\,z + 4 = 0$ is such that $\sin\theta = 1/3$, then the value of λ is

(a) 3/4 (b) –4/3
(c) 5/3 (d) –3/5.

Ans. (c)

Solution: Since the line makes an angle θ with the plane, it makes an angle $\pi/2 - \theta$ with the normal to the plane.

$$\therefore \quad \cos\left(\frac{\pi}{2} - \theta\right) = \frac{2(1) + (-1)(2) + (\sqrt{\lambda})(2)}{\sqrt{1+4+4} \times \sqrt{4+1+\lambda}}$$

$$\Rightarrow \quad \frac{1}{3} = \frac{2\sqrt{\lambda}}{3\sqrt{\lambda+5}} \quad \Rightarrow \quad \lambda + 5 = 4\lambda$$

$$\Rightarrow \quad \lambda = 5/3.$$

Example 51: The distance between the line $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ and the plane $\mathbf{r}.(\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 5$ is

(a) 3/10 (b) 10/3
(c) 10/9 (d) $10/3\sqrt{3}$

Ans. (d)

◎ **Solution:** Direction ratios of the line are 1, – 1, 4 and of the normal to the plane are 1, 5,1

Since $1 \times 1 + (-1)(5) + 4 \times 1 = 0$

Normal to the plane is perpendicular to the line and thus the line is parallel to the plane and the distance between them is the distance of any point on the line from the plane and is equal to

$$\left|\frac{2(1)+(-2)(5)+3\times 1-5}{\sqrt{1+25+1}}\right| = \frac{10}{\sqrt{27}} = \frac{10}{3\sqrt{3}}$$

◉ **Example 52:** The image of the point (– 1, 3, 4) in the plane $x - 2y = 0$ is

(a) (8, 4, 4) (b) (– 17/3, – 19/3, 4)

(c) (15, 11, 4) (d) none of these

Ans. (d)

◎ **Solution:** Required image lies on the line through $A(-1, 3, 4)$ and perpendicular to $x - 2y = 0$ that is on the line

$$\frac{x+1}{1} = \frac{y-3}{-2} = \frac{z-4}{0} = t\text{(say)}$$

Let the coordinates of image point be $B(t-1, -2t+3, 4)$

Mid point M of AB is $\left(\frac{t-2}{2}, \frac{-2t+6}{2}, 4\right)$ which lies on the plane $x - 2y = 0$

$$\Rightarrow \quad \frac{1}{2}t - 1 + 2t - 6 = 0 \quad \Rightarrow \quad t = \frac{14}{5}$$

Required image is $\left(\frac{9}{5}, -\frac{13}{5}, 4\right)$

◉ **Example 53:** Let L be the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$. If L makes an angle α with positive x-axis, then $\cos\alpha$ equals

(a) $1/\sqrt{3}$ (b) 1/2

(c) 1 (d) $1/\sqrt{2}$

Ans. (a)

◎ **Solution:** If l, m, n are the direction ratios of the line L then

$$2l + 3m + n = 0 \text{ and } l + 3m + 2n = 0$$

$$\Rightarrow \quad \frac{l}{3-6} = \frac{m}{4-1} = \frac{m}{3-6}$$

$$\Rightarrow \quad \frac{l}{-1} = \frac{m}{1} = \frac{n}{-1}$$

and $$\cos\alpha = \frac{l}{\sqrt{l^2+m^2+n^2}} = \frac{-1}{\pm\sqrt{1+1+1}} = \pm\frac{1}{\sqrt{3}}$$

◉ **Example54:** If a line makes an angle $\pi/4$ with the positive dtrection of each of x-axis and y-axis and passes through the origin, equation of the line is

(a) $x - y = 0 = z$ (b) $x - z = 0 = y$

(c) $y - z = 0 = x$ (d) $x = y = z$

Ans. (a)

◎ **Solution:** If θ is the angle which the line makes with the positive direction of z-axis, then $\cos^2 \pi/4 + \cos^2 \pi/4 + \cos^2\theta = 1$

$\Rightarrow \quad \cos^2\theta = 0 \quad \Rightarrow \quad \cos\theta = 0 \Rightarrow \quad \theta = \pi/2$ and the equation of the line is

$$\frac{x}{1/\sqrt{2}} = \frac{y}{1/\sqrt{2}} = \frac{z}{0}$$

Which can be written as $x - y = 0 = z$

◉ **Example 55:** Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = x\mathbf{i} + (x-2)\,\mathbf{j} - \mathbf{k}$. If the vector **c** lies in the plane of **a** and **b**, then x equals

(a) 0 (b) 1

(c) – 4 (d) – 2

Ans. (d)

◎ **Solution:** Since the three vectors are coplanar

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x-2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ x & -2 & -1-x \end{vmatrix} = 0$$

$$\Rightarrow \quad -2(-1-x) + 2 = 0$$

$$\Rightarrow \quad x = -2$$

◉ **Example 56:** If the angle between the line $x = \frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane $x + 2y + 3z = 4$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals :

(a) 5/3 (b) 2/3

(c) 3/2 (d) 2/5

Ans. (b)

◎ **Solution:** If θ is the angle between the line and the plane then

$$\sin\theta = \frac{(1)(1)+(2)(2)+(3)(\lambda)}{\sqrt{1+4+\lambda^2}\sqrt{1+4+9}} = \frac{5+3\lambda}{\sqrt{14}\sqrt{\lambda^2+5}}$$

since $\cos\theta = \sqrt{\frac{5}{14}} \Rightarrow \sin\theta = \frac{3}{\sqrt{14}}$

so $$\frac{5+3\lambda}{\sqrt{14}\sqrt{\lambda^2+5}} = \frac{3}{\sqrt{14}}$$

$$\Rightarrow \quad 9(\lambda^2+5) = 9\lambda^2 + 30\lambda + 25$$

$$\Rightarrow \quad \lambda = 2/3$$

Example 57: The distance of the point (1, –5, 9) from the plane $x - y + z = 5$ measured along a straight line $x = y = z$ is

(a) $10\sqrt{3}$ (b) $5\sqrt{3}$
(c) $3\sqrt{10}$ (d) $3\sqrt{5}$

Ans. (a)

Solution: Equation of line through (1, –5, 9) parallel to the line $x = y = z$ is

$$\frac{x-1}{1}=\frac{y+5}{1}=\frac{z-9}{1} = r \text{ (say)}$$

Any point Pon this line is $(r+1, r-5, r+9)$
which lies on the given plane if
$r+1-(r-5)+r+9=5 \Rightarrow r=-10$
coordinate of P are (– 9, –15,–1)
Distances of P from (1, –5, 9)

$$= \sqrt{(-9-1)^2+(+5-15)^2-(-9-1)^2} = 10\sqrt{3}$$

Example 58: The length of the perpendicular drawn from the point (3, –1, 11) to the line

$$\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ is}$$

(a) $\sqrt{29}$ (b) $\sqrt{33}$
(c) $\sqrt{53}$ (d) $\sqrt{66}$

Ans. (c)

Solution: Any point on the given line is
$P\,(2r, 3r+2, 4r+3)$ and the given point be A
Direction ratios of AP are
$2r-3, 3r+2+1, 4r+3-11$
or $2r-3, 3r+3, 4r-8$
AP is perpendicular to the given line
if $2(2r-3)+3(3r+3)+4(4r-8)=0$
or if $r = 1$, so coordinates of point (2, 5, 7)
and $AP = \sqrt{(2-3)^2+(5+1)^2+(7-11)^2} = \sqrt{53}$

Example 59: An equation of a plane parallel to the plane $x - 2y + 2z - 5 = 0$ and at a unit distance from the origin is

(a) $x-2y-2z-1=0$ (b) $x-2y+2z+5=0$
(c) $x-2y+2z-3=0$ (d) $x-2y+2z+1=0$

Ans. (c)

Solution: Let the equation of the plane be
$x-2y+2z+k= 0$

Its Distance from the origin is

$$\frac{k}{\sqrt{1+4+4}} = \pm 1 \Rightarrow k = \pm 3$$

Example 60: If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then k is equal to

(a) 9/2 (b) 0
(c) –1 (d) 2/9

Ans. (a)

Solution: Any point on the first line is
$(2r_1+1, 3r_1-1, 4r_1+1)$
and on the second line is
$(r_2+3, 2r_2+k, r_2)$
The lines will intersect when
$2r_1+1 = r_2+3, 3r_1-1 = 2r_2+k, 4r_1+1 = r_2$
$\Rightarrow$ $2r_1 - r_2 = 2,\quad 4r_1 - r_2 = -1$
$\Rightarrow$ $r_1 = -3/2,\ r_2 = -5.$
and $k = 3r_1 - 2r_2 - 1 = -9/2 + 10 - 1 = 9/2.$

Example 61: If the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$. Then (α, β) equals.

(a) (5, – 15) (b) (– 5, 5)
(c) (6, – 17) (d) (–6, 7)

Ans. (d)

Solution: As the line lies in the plane. The point (2, 1, – 2) lies in the plane and normal to the plane is perpendicular to the line, so $2 + 3(1) - \alpha(-2) + \beta = 0$
and $(3)(1) + (3)(-5) - \alpha(2) = 0 \Rightarrow \alpha = -6, \beta = 7.$

Example 62: The projection of a vector on the three coordinate axis are 6, – 3, 2 respectively. The direction cosines of the vector are

(a) 6/7, – 3/7, 2/7 (b) – 6/7, – 3/7, 2/7
(c) 6, – 3, 2 (d) 6/5, – 3/5, 2/5

Ans. (*a*)

Solution: $\sqrt{6^2+(-3)^2+(2)^2} = \pm 7$

So the required direction cosines are

$$\frac{\pm 6}{7}, \frac{\mp 3}{7}, \frac{\pm 2}{7}$$

Example 63: A line AB in three dimensional space makes angles 45° and 120° with the positive x-axis and positive y-axis respectively. If AB makes an acute angle θ with the positive z-axis, then θ equals

(a) 60° (b) 75°
(c) 30° (d) 45°

Ans. (*a*)

Solution: $\cos^2 45° + \cos^2 120° + \cos^2\theta = 1$
$\Rightarrow$ $\cos^2\theta = 1 - (1/2) - (-1/2)^2 = 1/4$
$\Rightarrow$ $\cos\theta = \pm 1/2 \Rightarrow \theta = 60°$

Example 64: A plane L passes through the point $(-1, -2, -1)$ and normal to it is perpendicular to the lines $\frac{x+1}{3}=\frac{y+2}{1}=\frac{z+1}{2}$ and $\frac{x-2}{1}=\frac{y+2}{2}=\frac{z-3}{3}$

The distance of the point $(1, 1, 1)$ from the plane L is

(a) $2/\sqrt{75}$ (b) $7/\sqrt{75}$

(c) $13/\sqrt{75}$ (d) $23/\sqrt{75}$

Ans. (c)

Solution: Direction ratios of the normal to the plane be l, m, n; then $3l + m + 2n = 0$, $l + 2m + 3n = 0$

$$\Rightarrow \quad \frac{l}{-1}=\frac{m}{-7}=\frac{n}{5}$$

Equation of the plane L is

$$-1(x+1)-7(y+2)+5(z+1)=0$$

$\Rightarrow \quad x + 7y - 5z + 10 = 0$. Its distance from $(1, 1, 1)$ is

$$\frac{1+7-5+10}{\sqrt{1+7^2+(-5)^2}}=\frac{13}{\sqrt{75}}$$

Example 65: A line with positive direction cosines passes through the point $P(2, -1, 2)$ and makes equal angles with the coordinate axes. The line meet the plane $2x + y + z = 9$ at point Q. The length of the line segment PQ equals.

(a) 1 (b) $\sqrt{2}$

(c) $\sqrt{3}$ (d) 2

Ans. (c)

Solution: Equation of the line is

$$\frac{x-2}{1}=\frac{y+1}{1}=\frac{z-2}{1}=r$$

$$\Rightarrow \quad x = r + 2,\ y = r - 1,\ z = r + 2$$

The point $(r + 2, r - 1, r + 2)$ lies on the plane $2x + y + z = 9$ is

$$2(r+2)+r-1+r+2=9 \Rightarrow r=1$$

So, the coordinates of Q are $(3, 0, 3)$ and thus PQ

$$= \sqrt{(3-2)^2+(0+1)^2+(3-2)^2}=\sqrt{3}\,.$$

Example 66: The vertices of a triangle are $A(1, 0, 0)$, $B(0, 2, 0)$, $C(0, 0, 3)$. If the direction ratios of the line joining the orthoceutre and circumcentre of the triangle are $a, b, -111$, the $a + b$ is equal to

(a) 5 (b) 10

(c) 15 (d) 25

Ans. (c)

Solution: Equation of the plane containing the triangle ABC is $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$

Let $H(\alpha, \beta, \gamma)$ be the orthocenter, then

$$AH \perp BC \Rightarrow (\alpha-1)\times 0+2\beta-3\gamma=0$$

and $\quad BH \perp CA \Rightarrow \alpha \,.\, 1+(\beta-2)\times 0-3\gamma=0$

we get $\alpha = 2\beta = 3\gamma$ and as H lies on the plane ABC, $\frac{\alpha}{1}+\frac{\alpha}{4}+\frac{\alpha}{9}=1 \Rightarrow \alpha=\frac{36}{49}$.

$$\Rightarrow \quad H=\left(\frac{36}{49},\frac{18}{49},\frac{12}{49}\right)$$

Since the circumcentre, orthocenter and the centroid are collinear,

If G be the centroid, its coordinates are $\left(\frac{1}{3},\frac{2}{3},1\right)$ and the direction ratios of the line HG are

$$\frac{36}{49}-\frac{1}{3},\frac{18}{49}-\frac{2}{3},\frac{12}{49}-1, \quad \text{or} \quad 59, -44, -111$$

so $\quad a = 59,\ b = -44 \Rightarrow a + b = 15$

Example 67: The equation of the plane which contains two parallel lines $\frac{x+1}{3}=\frac{y-2}{2}=\frac{z}{1}$ and $\frac{x-3}{3}=\frac{y+4}{2}=\frac{z-1}{1}$ is

(a) $3x + 2y + z = 14$ (b) $8x + y - 26z + 6 = 0$

(c) $4x - 6y + z = 53$ (d) none of these

Ans. (b)

Solution: Let the equation of the plane containing the first line be

$$a(x+1)+b(y-2)+cz=0$$

where $3a + 2b + c = 0$ other line will also lie on the plane if the point $(3, -4, 1)$ lies on the plane

$$a(3+1)+b(-4-2)+c=0$$

$$\Rightarrow \quad 4a-6b+c=0$$

Solving, we get $\frac{a}{8}=\frac{b}{1}=\frac{c}{-26}$ and the required equation is $8x + y - 26z + 6 = 0$.

Example 68: The length of projection of the line segment joining the points $(1, -1, 0)$ and $(-1, 0, 1)$ to the plane $2x + y + 6z = 1$ is equal to

(a) $\sqrt{255/41}$ (b) $\sqrt{237/41}$

(c) $\sqrt{137/41}$ (d) $\sqrt{155/41}$

Ans. (b)

Solution: Direction ratios of the line joining $A(1, -1, 0)$ and $B(-1, 0, 1)$ are $2, -1, -1$. If θ is the acute angle which the line AB makes with the normal to the plane, then

$$\cos\theta = \left|\frac{2\times 2+(1)(-1)+6(-1)}{\sqrt{(4+1+1)}\,\sqrt{4+1+36}}\right|=\frac{3}{\sqrt{246}}\,.$$

Length of segment $AB = \sqrt{6}$.

Length of the projection $= \sqrt{6}\ \sin\theta = \frac{\sqrt{6}\times\sqrt{237}}{\sqrt{246}}=\sqrt{\frac{237}{41}}\,.$

Example 69: The point P is the intersection of the straight line joining the points Q (2, 3, 5) and R (1, –1, 4) with the plane $5x - 4y - z = 1$. If S is the foot of the perpendicular drawn from the point T (2, 1, 4) to QR, then length of the line segment PS is

(a) $1/\sqrt{2}$ (b) $\sqrt{2}$

(c) 2 (d) $2\sqrt{2}$

Ans. (a)

Solution: Equation of QR is

$$\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-5}{1}$$

Let the coordinates of P be $(r + 2, 4r + 3, r + 5)$ As P lies on the plane $5x - 4y - z = 1$

$5(r + 2) - 4(4r + 3) - (r + 5) = 1$

$\Rightarrow r = -2/3 \Rightarrow P\left(\frac{4}{3}, \frac{1}{3}, \frac{13}{3}\right)$

Let the coordinates of S be $(t + 2, 4t + 3, t + 5)$

As TS is perpendicular to QR

$1(t + 2 - 2) + 4(4t + 3 - 1) + 1(t + 5 - 4) = 0$

$$\Rightarrow \quad t = -\frac{1}{2} \Rightarrow S\left(\frac{3}{2}, 1, \frac{9}{2}\right)$$

$$\text{Hence } PS = \sqrt{\left(\frac{3}{2}-\frac{4}{3}\right)^2 + \left(1-\frac{1}{3}\right)^2 + \left(\frac{9}{2}-\frac{13}{3}\right)^2}$$

$$= \sqrt{\frac{9}{18}} = \frac{1}{\sqrt{2}}$$

Example 70: The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $2/\sqrt{3}$ from the point (3, 1, –1) is

(a) $5x - 11y + z = 17$ (b) $\sqrt{2}\,x + y = 3\sqrt{2} - 1$

(c) $x + y + z = \sqrt{3}$ (d) $x + \sqrt{2}\,y = 1 - \sqrt{2}$

Ans.

Solution: Let the equation of the plane be

$(x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$

$\Rightarrow \quad (1 + \lambda)x + (2 - \lambda)y + (3 + \lambda)z - (2 + 3\lambda) = 0$

Since its distance from the point (3, 1, –1) is $\frac{2}{\sqrt{3}}$

$$\frac{3(1+\lambda)+(2-\lambda)-(3+\lambda)-(2+3\lambda)}{\sqrt{(\lambda+1)^2+(2-\lambda)^2+(3+\lambda)^2}} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{4\lambda^2}{3\lambda^2+4\lambda+14} = \frac{4}{3} \Rightarrow \lambda = \frac{-7}{2}$$

So the required equation of the plane is

$5x - 11y + z = 17$

Assertion-Reason Type Questions

Example 71: Statement-1: The point $A(3, 1, 6)$ is the mirror image of the point $B(1, 3, 4)$ in the plane $x - y + z = 5$.

Statement-2: The plane $x - y + z = 5$ bisects the line segment joining $A(3, 1, 6)$ and $B(1, 3, 4)$

Ans. (*b*)

Solution: Direction ratios of AB are $1 - 3, 3 - 1, 4 - 6$ or $1, -1, 1$

$\Rightarrow AB$ is normal to the plane $x - y + z = 6$. Also the mid point (2, 2, 5) of AB lies on $x - y + z = 5$. Thus the plane bisects the line segment AB so statement-2 is true.

As the plane is the perpendicular bisector of the segment AB, A is the mirror image of B in the plane and statement-1 is also true but does not follow from statement-2 alone.

Example 72: Statement-1: If the distance of the point $P(1, -2, 1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$, is 5, then the foot of the perpendicular from P to the plane is (8/3, 4/3/, –7/3)

Statement-2: A line through $P(1, -2, 1)$ and perpendicular to the plane $x + 2y - 2z = \alpha$ intersects the plane at Q. If $PQ = 5$ then $\alpha = 10$

Ans. (a)

Solution: Equation of the line through P perpendicular to the plane is

$$\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-1}{-2} = \frac{r}{\sqrt{1+4+4}} = \frac{r}{3}.$$

Coordinates of a point on this line at a distance r from P are

$$\left(\frac{r}{3}+1, \frac{2r}{3}-2, \frac{-2r}{3}+1\right)$$

If this represents the point Q then it lies on the plane.

$$\frac{r}{3}+1+2\left(\frac{2r}{3}-2\right)-2\left(\frac{-2r}{3}+1\right) = \alpha$$

Since $PQ = 5$, $r = 5$

$$\Rightarrow \quad \alpha = \frac{5}{3}+1+\frac{4\times 5}{3}-4+\frac{4\times 5}{3}-2 = 10$$

So the statement-2 is true. Q is now the foot of the perpendicular from P to the plane and its coordinates are (8/3, 4/3, –7/3) and hence statement-1 is also true.

Example 73: Consider the plane $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Statement-1: The parametric equations of the line of intersection of the given planes are $x = 3 + 14t$, $y = 1 + 2t$, $z = 15t$; t being the parameter

Statement-2: The vector $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of the given planes.

Ans. (d)

Solution: Equation of the line in statement-1 can be written as $\frac{x-3}{14} = \frac{y-1}{2} = \frac{z-0}{15}$ = t. This is the line of intersection of the planes, then the point (3, 1, 0) which lies on the line must be on both the planes which is not true and hence the statement-1 is false. Direction ratios of the line of intersection of the given planes is

$(-6)(-2) - (-2)(1), (-2)(2) - (-2)(3), 3(1) - (2)(-6)$

i.e. 14, 2, 15; showing that the vector in statement-2 is parallel to the line of intersection of the planes and thus statement-2 is True.

Example 74: **Statement-1:** The point A (1, 0, 7) is the mirror image of the point (1, 6, 3) in the line

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$$

Statement-2: The line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ bisects the line segment joining A (1, 0, 7) and B (1, 6, 3).

Ans. (b)

Solution: Mid point of the segment AB is

$$\left(\frac{1+1}{2}, \frac{0+6}{2}, \frac{7+3}{2}\right) = (1, 3, 5)$$

which lies on the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ so Statement-2 is true.

Next, direction ratios of AB are 0,6,–4 and the given line are 1, 2, 3.

Since $0 \times 1 + 6 \times 2 + (-4) \times 3 = 0$, the line AB is perpendicular to the given line showing than Statement-1 is also true, but statement-2 is not a correct explanation for Statement-1.

Example 75: L_1: $\frac{x-1}{1} = \frac{y}{-1} = \frac{z+1}{1}$

$$L_2 : \frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{3}$$

Statement-1: L_1 and L_2 are coplanar and the equation of the plane containing them is $5x + 2y - 3z - 8 = 0$

Statement-2: L_1 and L_2 intersect at a point.

Ans. (b)

Solution: Since $\begin{vmatrix} 1-2 & 0+1 & -1-0 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$, the lines L_1 and L_2 are coplanar and the equation of the plane containing them is $\begin{vmatrix} x-1 & y & z+1 \\ 1 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

$= -(5x + 2y - 3z - 8) = 0$ so statement-1 is true.

L_1 and L_2 are not parallel as $\frac{1}{1} \neq \frac{-1}{2} \neq \frac{1}{3}$ and are coplanar so they intersect at a point and the statement-2 is also true but does not lead to statement-1.

LEVEL 2

Straight Objective Type Questions

Example 76: The reflection of the point A (1, 0, 0) in the line

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} \text{ is}$$

(a) (3, – 4, – 2) (b) (5, – 8, – 4)

(c) (1, –1, – 10) (d) (2, – 3, 8)

Ans. (b)

Solution: Any point P on the given line is $(2r + 1, -3r - 1, 8r - 10)$

So the direction ratios of AP are $2r$, $-3r - 1$, $8r - 10$. Now AP is perpendicular to the given line if

$$2(2r) - 3(-3r - 1) + 8(8r - 10) = 0$$

$$\Rightarrow \quad 77r - 77 = 0 \quad \Rightarrow \quad r = 1$$

and thus the coordinates of P, the foot of the perpendicular from A on the line are (3, – 4, – 2).

Let $B(a, b, c)$ be the reflection of A in the given line. Then P is the mid-point of AB

$$\frac{a+1}{2} = 3,\ \frac{b}{2} = -4,\ \frac{c}{2} = -2$$

$$\Rightarrow \quad a = 5, b = -8, c = -4$$

Thus the coordinates of required point are (5, – 8, – 4).

Example 77: An equation of the plane passing through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ and passing through (1, 1, 1) is

(a) $2x+3y+4z=9$ (b) $x+y+z=3$
(c) $x+2y+3z=6$ (d) $20x+23y+26z=69$.

Ans. (d)

◎ **Solution:** Equation of any plane through the line of intersection of the given planes is

$$2x+3y+4z+5+\lambda(x+y+z-6)=0$$

It passes through (1, 1, 1)

if $(2+3+4+5)+\lambda(1+1+1-6)=0$

$\Rightarrow \lambda = 14/3$ and the required equation is therefore,

$$20x+23y+26z=69.$$

● **Example 78:** Equation of the plane through three points A, B, C with position vectors $-6\mathbf{i}+3\mathbf{j}+2\mathbf{k}, 3\mathbf{i}-2\mathbf{j}+4\mathbf{k}, 5\mathbf{i}+7\mathbf{j}+3\mathbf{k}$ is

(a) $\mathbf{r}.(\mathbf{i}-\mathbf{j}+7\mathbf{k})+23=0$

(b) $\mathbf{r}.(\mathbf{i}+\mathbf{j}+7\mathbf{k})=23$

(c) $\mathbf{r}.(\mathbf{i}+\mathbf{j}-7\mathbf{k})+23=0$

(d) $\mathbf{r}.(\mathbf{i}-\mathbf{j}-7\mathbf{k})=23$

Ans. (a)

◎ **Solution:** Equation of the plane passing through three points A, B, C with position vectors **a**, **b**, **c** is

$$\mathbf{r}.(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})=\mathbf{a}\,.\,\mathbf{b}\times\mathbf{c}$$

So that if **a**, **b**, **c** represent the given vectors, then

$(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -6 & 3 & 2\\ 3 & -2 & 4\end{vmatrix}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -2 & 4\\ 5 & 7 & 3\end{vmatrix}+\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5 & 7 & 3\\ -6 & 3 & 2\end{vmatrix}$$

$=\mathbf{i}\,(12+4-6-28+14-9)-\mathbf{j}\,(-24-6+9-20+10+18)+\mathbf{k}\,(12-9+21+10+15+42)$

$=-13\mathbf{i}+13\mathbf{j}-91\mathbf{k}$

and $$\mathbf{a}.\mathbf{b}\times\mathbf{c}=\begin{vmatrix}-6 & 3 & 2\\ 3 & -2 & 4\\ 5 & 7 & 3\end{vmatrix}=299$$

So the required equation of the plane is

$$\mathbf{r}.\ (-13\mathbf{i}+13\mathbf{j}-91\mathbf{k})=299$$

or $$\mathbf{r}.\ (\mathbf{i}-\mathbf{j}+7\mathbf{k})+23=0$$

● **Example 79:** The lines whose vector equations are

$$\mathbf{r}=\mathbf{a}+t\mathbf{b},\ \mathbf{r}=\mathbf{c}+t'\mathbf{d}$$

are coplanar if

(a) $(\mathbf{a}-\mathbf{b})\,.\,\mathbf{c}\times\mathbf{d}=0$ (b) $(\mathbf{a}-\mathbf{c})\,.\,\mathbf{b}\times\mathbf{d}=0$
(c) $(\mathbf{b}-\mathbf{c})\,.\,\mathbf{a}\times\mathbf{d}=0$ (d) $(\mathbf{b}-\mathbf{d})\,.\,\mathbf{a}\times\mathbf{c}=0$

Ans. (b)

◎ **Solution:** The given lines are coplanar, if the normal to the plane containing these lines is perpendicular to both of them. Since the given lines are parallel to the vectors b and d, the normal to the plane is parallel to b × d, which is perpendicular to the line joining the points on the plane with position vectors **a** and **c** $\Rightarrow$ $(\mathbf{a}-\mathbf{c})$. $\mathbf{b}\times\mathbf{d}=0$, which is the required condition for the given lines to be coplanar.

● **Example 80:** The shortest distance between the skew lines

$l_1 : \mathbf{r}=\mathbf{a}_1+\lambda\,\mathbf{b}_1$ and $l_2 : \mathbf{r}=\mathbf{a}_2+\mu\,\mathbf{b}_2$ is

(a) $\dfrac{|(\mathbf{a}_2-\mathbf{a}_1)\cdot\mathbf{b}_1\times\mathbf{b}_2|}{|\mathbf{b}_1\times\mathbf{b}_2|}$ (b) $\dfrac{|(\mathbf{a}_1-\mathbf{b}_1)\cdot\mathbf{a}_2\times\mathbf{b}_2|}{|\mathbf{b}_1\times\mathbf{b}_2|}$

(c) $\dfrac{|(\mathbf{a}_2-\mathbf{b}_2)\cdot\mathbf{a}_1\times\mathbf{b}_1|}{|\mathbf{b}_1\times\mathbf{b}_2|}$ (d) $\dfrac{|(\mathbf{a}_1-\mathbf{b}_2)\cdot\mathbf{b}_1\times\mathbf{a}_2|}{|\mathbf{b}_1\times\mathbf{a}_2|}$

Ans. (a)

◎ **Solution:** Let **PQ** be the shortest distance vector between l_1 and l_2. Now l_1 passes through A_1 $(\mathbf{a}_1)$ and is parallel to $\mathbf{b}_1$ and l_2 passes through A_2 $(\mathbf{a}_2)$ and is parallel to $\mathbf{b}_2$. Since PQ is perpendicular to both l_1 and l_2 it is parallel to $\mathbf{b}_1\times\mathbf{b}_2$. Let **n** be the unit vector along **PQ**.

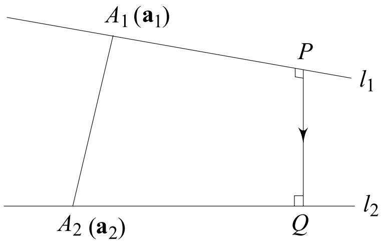


Fig. 21.4

Then $\mathbf{n}=\dfrac{\mathbf{b_1}\times\mathbf{b_2}}{|\mathbf{b_1}\times\mathbf{b_2}|}$.

Let d be the shortest distance between the given lines l_1 and l_2

$$|\,\mathbf{PQ}|=d \text{ and } \mathbf{PQ}=d\mathbf{n}\,d\,\hat{n}.$$

Next PQ being the line of shortest distance between l_1 and l_2, is the projection of the line joining the points A_1 $(\mathbf{a}_1)$ and A_2 $(\mathbf{a}_2)$ on **n**.

$$|\mathbf{PQ}|=|\mathbf{A}_1\mathbf{A}_2\,.\,\mathbf{n}|\quad\Rightarrow\quad d=\left|\frac{(\mathbf{a_2}-\mathbf{a_1}).\,\mathbf{b_1}\times\mathbf{b_2}}{|\mathbf{b_1}\times\mathbf{b_2}|}\right|$$

● **Example 81:** The length of the shortest distance between the lines $\mathbf{r}=3\mathbf{i}+5\mathbf{j}+7\mathbf{k}+\lambda\,(\mathbf{i}-2\mathbf{j}+\mathbf{k})$ and $\mathbf{r}=-\mathbf{i}-\mathbf{j}-\mathbf{k}+\mu\,(7\mathbf{i}-6\mathbf{j}+\mathbf{k})$ is

(a) 83 (b) $\sqrt{6}$

(c) $\sqrt{3}$ (d) $2\sqrt{29}$

Ans. (d)

◎ **Solution:** Taking $\mathbf{a}_1=3\mathbf{i}+5\mathbf{j}+7\mathbf{k}$, $\mathbf{b}_1=\mathbf{i}-2\mathbf{j}+\mathbf{k}$

$$\mathbf{a}_2 = -\mathbf{i}-\mathbf{j}-\mathbf{k}\ ,\ \mathbf{b}_2 = 7\mathbf{i}-6\mathbf{j}+\mathbf{k}$$

We get $|\,\mathbf{b}_1 \times \mathbf{b}_2| = |4\mathbf{i}+6\mathbf{j}+8\mathbf{k}|$

$$= \sqrt{4^2+6^2+8^2} = \sqrt{116}$$

and $|(\mathbf{a}_2 - \mathbf{a}_1)\ .\ \mathbf{b}_1 \times \mathbf{b}_2| = |-4 \times 4 - 6 \times 6 - 8 \times 8|$

$$= 116$$

hence the length of shortest distance is $\dfrac{116}{\sqrt{116}} = 2\sqrt{29}$.

● **Example 82:** The angle between the lines whose direction cosines are given by the equations $l^2 + m^2 - n^2 = 0$, $l + m + n = 0$ is

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

Ans. (c)

◎ **Solution:** We are given $l^2 + m^2 - n^2 = 0$ and $l + m + n = 0$ also we have $l^2 + m^2 + n^2 = 1$

So that $2n^2 = 1 \Rightarrow n = \pm 1/\sqrt{2}$

and $l + m = -n$

$\Rightarrow$ $(l+m)^2 = n^2 = l^2 + m^2 \Rightarrow 2lm = 0$

$\Rightarrow$ Either $l = 0$ or $m = 0$ if $l = 0$, $m + n = 0$

$\Rightarrow$ $m = -n = \pm 1/\sqrt{2}$

So the direction cosines of one of the lines are, $0, \mp 1/\sqrt{2}, \pm 1/\sqrt{2}$ and if $m = 0$, $l + n = 0$ $\Rightarrow$ $l = -n$ $= \pm 1/\sqrt{2}$ and the direction cosines of the other line are $\mp 1/\sqrt{2}, 0, \pm 1/\sqrt{2}$.

Hence the required angle is

$$\cos^{-1}\left[0 \times \mp\frac{1}{\sqrt{2}} + \mp\frac{1}{\sqrt{2}} \times 0 + \left(\pm\frac{1}{\sqrt{2}}\right)\left(\pm\frac{1}{\sqrt{2}}\right)\right]$$

$$= \cos^{-1}\frac{1}{2} = \frac{\pi}{3}.$$

● **Example 83:** The volume of the tetrahedron included between the plane. $3x + 4y - 5z - 60 = 0$ and the coordinate planes is

(a) 60 (b) 600

(c) 720 (d) none of these

Ans. (b)

◎ **Solution:** Equation of the given plane can be written as

$$\frac{x}{20} + \frac{y}{15} + \frac{z}{-12} = 1$$

which meets the coordinates axes in points A (20, 0, 0), B (0, 15, 0) and C (0, 0, $-$ 12) and the coordinates of the origin are (0, 0, 0).

$\therefore$ the volume of the tetrahedron $OABC$ is

$$\frac{1}{6}\left\|\begin{matrix} 0 & 0 & 0 & 1 \\ 20 & 0 & 0 & 1 \\ 0 & 15 & 0 & 1 \\ 0 & 0 & -12 & 1 \end{matrix}\right\|$$

$$= \left|\frac{1}{6} \times 20 \times 15 \times (-12)\right| = 600.$$

● **Example 84:** A line segment has length 63 and direction ratios are 3, – 2, 6. If the line makes an obtuse angle with x-axis, the components of the line vector are

(a) 27, –18, 54 (b) – 27, 18, – 54

(c) – 27, 18, – 54 (c) 27, –18, –54

Ans. (b)

◎ **Solution:** Let the components of the line segment vector be a, b, c, then

$$a^2 + b^2 + c^2 = (63)^2 \qquad (1)$$

also $\dfrac{a}{3} = \dfrac{b}{-2} = \dfrac{c}{6} = \lambda$ (say) then

$a = 3\lambda$, $b = -2\lambda$ and $c = 6\lambda$

and from (1) we have

$$9\lambda^2 + 4\lambda^2 + 36\lambda^2 = (63)^2 \Rightarrow 49\lambda^2 = (63)^2$$

$\Rightarrow$ $\lambda = \pm\dfrac{63}{7} = \pm 9.$

Since $a = 3\lambda < 0$ as the line makes an obtuse angle with x-axis, $\lambda = -9$ and the required components are, -27, 18, -54.

● **Example 85:** Equation of a plane bisecting the angle between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ is

(a) $5x - y - 4z - 45 = 0$

(b) $5x - y - 4z - 3 = 0$

(c) $23x + 13y + 32z - 45 = 0$

(d) $23x - 13y + 32z + 5 = 0$

Ans. (b)

◎ **Solution:** Equations of the planes bisecting the angle between the given planes are

$$\frac{2x - y + 2z + 3}{\sqrt{2^2 + (-1)^2 + 2^2}} = \pm\frac{3x - 2y + 6z + 8}{\sqrt{3^2 + (-2)^2 + 6^2}}$$

$\Rightarrow$ $7(2x - y + 2z + 3) = \pm 3\,(3x - 2y + 6z + 8)$

$\Rightarrow$ $5x - y - 4z - 3 = 0$ taking the + ve sign, and $23x - 13y + 32z + 45 = 0$ taking the $-$ ve sign.

● **Example 86:** An equation of the line passing through $3\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$ and perpendicular to the plane $3x - 4y + 5z = 8$ is

(a) $\dfrac{x-3}{3} = \dfrac{y+5}{-4} = \dfrac{z-7}{5}$

(b) $\frac{x-3}{3} = \frac{y+4}{-5} = \frac{z-5}{7}$

(c) $\mathbf{r} = 3\mathbf{i} + 5\mathbf{j} - 7\mathbf{k} + \lambda\,(3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k})$

(d) $\mathbf{r} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k} + \mu\,(3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k})$

(λ, μ are parameter)

Ans. (a)

Solution: Equation of the given plane can be written as

$$(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})\cdot(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 8.$$

So that the normal to the given plane is $3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ and the required line being perpendicular to the plane is parallel to this normal and since it passes through $3\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$, its equation is

$$\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} + 7\mathbf{k} + \lambda\,(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$$

where λ is a parameters.

Since this line passes through the vector $3\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$ i.e. the point (3, –5, 7) and is parallel to $3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$, its direction ratios are 3, – 4, 5

its cartesian equation is $\frac{x-3}{3} = \frac{y+5}{-4} = \frac{z-7}{5}$

Example 87: If the perpendicular distance of a point *P* other than the origin from the plane $x + y + z = p$ is equal to the distance of the plane from the origin, then the coordinate of *P* are

(a) $(p, 2p, 0)$ (b) $(0, 2p - p)$

(c) $(2p, p, -p)$ (d) $(2p, -p, 2p)$

Ans. (c)

Solution: The perpendicular distance of the origin (0, 0, 0) from the plane $x + y + z = p$ is $\left|\frac{-p}{\sqrt{1+1+1}}\right| = \frac{|p|}{\sqrt{3}}$.

If the coordinates of *P* are (*x, y, z*), then we must have

$$\left|\frac{x+y+z-p}{\sqrt{3}}\right| = \frac{|p|}{\sqrt{3}}$$

$$\Rightarrow \quad |x+y+z-p| = |p|$$

which is satisfied by (c)

Example 88: If p_1, p_2, p_3 denote the distances of the plane $2x - 3y + 4z + 2 = 0$ from the planes $2x - 3y + 4z + 6 = 0$, $4x - 6y + 8z + 3 = 0$ and $2x - 3y + 4z - 6 = 0$ respectively, then

(a) $p_1 + 8p_2 - p_3 = 0$ (b) $p_3^2 = 16p_2^2$

(c) $8p_2^2 = p_1^2$ (d) $p_1 + 2p_2 + 3p_3 = \sqrt{29}$

Ans. (a)

Solution: Since the planes are all parallel planes,

$$p_1 = \frac{|2-6|}{\sqrt{2^2+3^2+4^2}} = \frac{4}{\sqrt{4+9+16}} = \frac{4}{\sqrt{29}}$$

Equation of the plane $4x - 6y + 8z + 3 = 0$ can be written as $2x - 3y + 4z + 3/2 = 0$

So $$p_2 = \frac{|2-3/2|}{\sqrt{2^2+3^2+4^2}} = \frac{1}{2\sqrt{29}}$$

and $$p_3 = \frac{|2+6|}{\sqrt{2^2+3^2+4^2}} = \frac{8}{\sqrt{29}}$$

$$\Rightarrow \quad p_1 + 8p_2 - p_3 = 0$$

Example 89: Equation of the plane containing the lines.

$$\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda\,(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$$

and $$\mathbf{r} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} + \mu\,(\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \text{ is}$$

(a) $\mathbf{r}\cdot(7\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = 0$

(b) $7(x-1) - 4(y-1) - (z+3) = 0$

(c) $\mathbf{r}\cdot(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 0$

(d) $\mathbf{r}\cdot(\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = 0$

Ans. (a)

Solution: Since both the given lines pass through the point with position vector $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, the required plane also passes through $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and normal to the plane is perpendicular to the vectors $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. If $\mathbf{d} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is normal to the required plane, then $a + 2b - c = 0$ and $a + b + 3c = 0$.

$$\Rightarrow \quad \frac{a}{7} = \frac{b}{-4} = \frac{c}{-1}$$

$\Rightarrow$ $\mathbf{d} = 7\mathbf{i} - 4\mathbf{j} - \mathbf{k}$.

So the required plane passes through $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and the normal to plane is $7\mathbf{i} - 4\mathbf{j} - \mathbf{k}$, hence required equation is

$$[\mathbf{r} - (\mathbf{i} + 2\mathbf{j} - \mathbf{k})]\cdot(7\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = 0$$

$$\mathbf{r}\cdot(7\mathbf{i} - 4\mathbf{j} - \mathbf{k}) = 1 \times 7 + 2\,(-4) + (-1)\,(-1) = 0$$

Alternately since the required plane passes through $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, i.e. the point (1, 2, –1) and the direction ratios of the normal to the plane are 7, – 4, –1, the equation of the plane in cartesian form can be written as

$$7\,(x-1) - 4\,(y-2) - (z+1) = 0$$

Note: Use the result number 11 given in vectorial equations.

Example 90: The foot of the perpendicular from the origin to the join of *A* (– 9, 4, 5) and *B* (11, 0, – 1) divides *AB* in the ratio

(a) 2 : 3 (b) 3 : 2

(c) 1 : 1 (d) none of these

Ans. (c)

◎ **Solution:** Let D be the foot of the perpendicular from the origin to the join of A and B and divide AB in the ratio $k : 1$, then the coordinates of D are

$$\left(\frac{11k-9}{k+1}, \frac{0k+4}{k+1}, \frac{-k+5}{k+1}\right)$$

So that the direction cosines of OD are proportional to

$$11k-9, 4, 5-k$$

and direction cosines of AB are proportional to

$11 + 9, 0 - 4, - 1 - 5$

i.e. $20, - 4, - 6.$ or $10, - 2, - 3.$

Since OD is perpendicular to AB.

$$10\,(11k-9) - 2\,(4) - 3\,(5-k) = 0$$

$\Rightarrow$ $110k - 90 - 8 - 15 + 3k = 0$

$\Rightarrow$ $113k = 113 \quad \Rightarrow \quad k = 1$

◉ **Example 91:** Cosine of the angle between the lines whose vector equations are $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda\,(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 5\mathbf{i} - 2\mathbf{k} + \mu\,(3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$; λ, μ being parameters, is

(a) $-1/3\sqrt{29}$ (b) $3/7\sqrt{29}$

(c) $23/29$ (d) $19/21$

Ans. (d)

◎ **Solution:** Direction vectors of the given lines are

$$\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \text{ and } 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}.$$

The angle θ between the given lines is equal to the angle between these vectors.

Hence $$\cos\theta = \frac{(\mathbf{i}+2\mathbf{j}+2\mathbf{k})}{\sqrt{1+4+4}} \cdot \frac{(3\mathbf{i}+2\mathbf{j}+6\mathbf{k})}{\sqrt{9+4+36}}$$

$$= \frac{3+4+12}{3\times 7} = \frac{19}{21}$$

◉ **Example 92:** The cartesian equation of the plane passing through the line of intersection of the planes $\mathbf{r}\cdot(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 1$ and $\mathbf{r}\cdot(\mathbf{i} - \mathbf{j}) + 4 = 0$ and perpendicular to the plane $\mathbf{r}\cdot(2\mathbf{i} - \mathbf{j} + \mathbf{k}) + 8 = 0$ is

(a) $3x - 4y + 4z = 5$

(b) $x - 2y + 4z = 3$

(c) $5x - 2y - 12z + 47 = 0$

(d) $2x + 3y + 4 = 0$

Ans. (c)

◎ **Solution:** Equation of any plane passing through the intersection of the planes $\mathbf{r}\cdot(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) = 1$

and $\mathbf{r}\cdot(\mathbf{i} - \mathbf{j}) + 4 = 0$ **is**

$$2x - 3y + 4z - 1 + \lambda\,(x - y + 4) = 0$$

or $$(2 + \lambda)x - (3 + \lambda)y + 4z + 4\lambda - 1 = 0$$

This plane is perpendicular to the plane $\mathbf{r}\cdot(2\mathbf{i} - \mathbf{j} + \mathbf{k}) + 8 = 0$ if

$\Rightarrow$ $2\,(2 + \lambda) + (3 + \lambda) + 4 = 0.$

$\Rightarrow$ $11 + 3\lambda = 0 \quad \Rightarrow \quad \lambda = -11/3.$

and the required equation of the plane is

$$3(2x - 3y + 4z - 1) - 11\,(x - y + 4) = 0$$

$\Rightarrow$ $5x - 2y - 12z + 47 = 0$

◉ **Example 93:** The shortest distances between the lines $\dfrac{x-3}{2} = \dfrac{y+15}{-7} = \dfrac{z-9}{5}$ and $\dfrac{x+1}{2} = \dfrac{y-1}{1} = \dfrac{z-9}{-3}$ is

(a) $4\sqrt{5}$ (b) $4\sqrt{17}$

(c) $4\sqrt{3}$ (d) $8\sqrt{2}$

Ans. (c)

◎ **Solution:** Let l, m, n be the direction cosines of the line of shortest distance, then as it is perpendicular to the given lines $2l - 7m + 5n = 0$; $2l + m - 3n = 0$

$$\Rightarrow \quad \frac{l}{16} = \frac{m}{16} = \frac{n}{16}$$

$$\Rightarrow \quad \frac{l}{1} = \frac{m}{1} = \frac{n}{1} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+1+1}}$$

$\Rightarrow$ $l = m = n = 1/\sqrt{3}$.

Now the shortest distance between the given lines is the projection of the join of the points $(3, -15, 9)$ and $(-1, 1, 9)$ on the line of shortest distance. Hence the required distance $= \dfrac{1}{\sqrt{3}}\,|(3 + 1) + (-15 - 1) + 9 - 9|$ $= 4\sqrt{3}$.

◉ **Example 94:** Equation of the line through the point $(2, -1, 1)$ and the intersection of the lines $2x - y - 4 = 0 = y + 2z$, $x + 3z - 4 = 0 = 2x + 5z - 8$ is

(a) $x + y + z = 2$, $x + 2y = 0$

(b) $x + y + z = 2$, $x + 2z = 4$

(c) $x + 2y + z = 1$, $x + 2z = 4$

(d) $x + 2y + z = 1$, $x + 2y = 0$

Ans. (b)

◎ **Solution:** Equation of a line through the intersection of the given lines is

$$2x - y - 4 + \lambda\,(y + 2z) = 0$$
$$= x + 3z - 4 + \mu\,(2x + 5z - 8)$$

for all values of λ and μ.

If it passes through the point $(2, -1, 1)$

if $2 \times 2 - 1 - 4 + \lambda\,(-1 + 2) = 0$

$= 2 + 3 - 4 + \mu\,(4 + 5 - 8)$

$\Rightarrow$ $\lambda = 1, \mu = -1$

and the required equation is therefore ; $x + y + z = 2$, $x + 2z = 4$.

◉ **Example 95:** The planes $bx - ay = n$, $cy - bz = l$, $az - cx = m$ intersect in a line if

(a) $al - bm + cn = 1$ (b) $al + bm + cn = 0$

(c) $al - bm - cn + 1 = 0$ (d) $al + bm + cn = 1$

Ans. (b)

Solution: Eliminating x, y, z from the given equations, we get the required condition. So multiplying the given equations by c, a, b respectively and adding we get the required condition is $al + bm + cn = 0$.

Example 96: The value of k for which the line

$\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies in the plane $2x - 4y + z = 7$ is

(a) 7 (b) 6

(c) no real value (d) –7

Ans. (a)

Solution: Since (2) (1) + (–4) (1) + (1) (2) = 0

the line is parallel to the plane.It will lie in the plane if the point (4,2, k) lies on the plane.

$$\Rightarrow \quad 2(4) - 4(2) + k = 7 \Rightarrow k = 7$$

Example 97: A variable plane at a distance of 1 unit from the origin cuts the coordiante axes at A, B and C. If the centroid G (x, y, z) of the triangle ABC satisfies the relation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$, then the value of k is

(a) 9 (b) 3

(c) 1 (d) 1/3

Ans. (a)

Solution: Let the coordinates of A, B, C be $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$

then $x = a/3$, $y = b/3$, $z = c/3$

Equation of the plane is

$$\frac{X}{a} + \frac{Y}{b} + \frac{Z}{c} = 1$$

Since it is at a unit distance from the origin.

$$\frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = 1$$

$$\Rightarrow \quad \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = 1$$

$$\Rightarrow \quad \frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2} = 1$$

$$\Rightarrow \quad \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = 9 = k$$

Example 98: A plane π is perpendicular to the two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$ and passes through the point (1, –2, 1). The distance of π from the point (1, 2, 2) is

(a) 0 (b) 1

(c) $\sqrt{2}$ (d) $2\sqrt{2}$

Ans. (d)

Solution: Let the equation of the plane be

$$a(x - 1) + b(y + 2) + c(z - 1) = 0.$$

Since it is perpendicular to the given planes

$$2a - 2b + c = 0, \qquad a - b + 2c = 0,$$

$\Rightarrow$ $c = 0$, $a = b$,

$\Rightarrow$ Equation of the plane is $x + y + 1 = 0$

Its distance from the point (1, 2, 2) is

$$\left|\frac{1+2+1}{\sqrt{1+1}}\right| = 2\sqrt{2}$$

Example 99: A line with positive direction cosines passes through the point P(2, –1, 2) and makes equal angles with the coordinate axes. The line meets the plane $2x + y + z = 9$ at point Q. The length of the line segment PQ equals.

(a) 1 (b) $\sqrt{2}$

(c) $\sqrt{3}$ (d) 2

Ans. (c)

Solution: The equation of the line is

$$\frac{x-2}{1} = \frac{y+1}{1} = \frac{z-2}{1} = k$$

Let the coordinates of the point Q on this line be

$Q(k + 2, k - 1, k + 2)$. Since Q lies on the plane

$2(k + 2) + (k - 1) + (k + 2) = 9$

$$\Rightarrow \quad k = 1 \text{ and } PQ = \sqrt{(3-2)^2 + (0+1)^2 + (3-2)^2} = \sqrt{3}$$

Example 100: Let P (3, 2, 6) be a point in space and Q be a point on the line

$$\mathbf{r} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu(-3\mathbf{i} + \mathbf{j} + 5\mathbf{k})$$

Then the value of μ for which the vector **PQ** is parallel to the plane $x - 4y + 3z - 1$ is

(a) 1/4 (b) –1/4

(c) 1/8 (d) –1/8

Ans. (a)

Solution: As Q lies on the given line

Let $\mathbf{OQ} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu(-3\mathbf{i} + \mathbf{j} + 5\mathbf{k})$

Also $\mathbf{OP} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

As **PQ** is parallel to the plane $x - 4y + 3z = 1$

$$\mathbf{PQ} \cdot (\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) = 0$$

$$\Rightarrow \quad (\mathbf{OQ} - \mathbf{OP}) \cdot (\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}) = 0$$

$$\Rightarrow \quad (1 + 4 + 6) + \mu(-3 - 4 + 15) - (3 - 8 + 18) = 0$$

$$\Rightarrow \quad 11 + 8\mu - 13 = 0 \Rightarrow \mu = 1/4$$

◉ **Example 101:** Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane containing the straight lines.

$\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$

(a) $x + 2y - 2z = 0$ (b) $3x + 2y - 2z = 0$

(c) $x - 2y + z = 0$ (d) $5x + 2y - 4z = 0$

Ans. (c)

◎ **Solution:** Let a, b, c be the direction ratios of the normal to the plane containing the lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$

then $3a + 4b + 2c = 0$

$4a + 2b + 3c = 0$

$$\Rightarrow \quad \frac{a}{8} = \frac{b}{-1} = \frac{c}{-10}$$

Let α, β, γ be the direction ratios of the normal to the required plane.

then $8\alpha - \beta - 10\gamma = 0$

$2\alpha + 3\beta + 4\gamma = 0$

$$\Rightarrow \quad \frac{\alpha}{26} = \frac{\beta}{-52} = \frac{\gamma}{26}$$

$$\Rightarrow \quad \frac{\alpha}{1} = \frac{\beta}{-2} = \frac{\gamma}{1}$$

As the required plane contains the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$, the point (0, 0, 0) lying on it also lies on the plane and hence equation of the required plane is $x - 2y + z = 0$

◉ **Example 102:** If the distance of the point P (1, –2, 1) from the plane $x + 2y - 2z = \alpha$, where, $\alpha > 0$, is 5, then the foot of the perpendicalar from P to the plane is

(a) $\left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$ (b) $\left(\frac{4}{3}, \frac{-4}{3}, \frac{1}{3}\right)$

(c) $\left(\frac{1}{3}, \frac{2}{3}, \frac{10}{3}\right)$ (d) $\left(\frac{2}{3}, \frac{-1}{3}, \frac{5}{2}\right)$

Ans. (a)

◎ **Solution:** Let Q (x, y, z) be the foot of the perpendicular from P then equation of PQ is $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-1}{-2} = r$ (say)

Let the coordinates of Q be $(r + 1, 2r - 2, -2r + 1)$ Since it lies on the plane, $r + 1 + 2(2r - 2) - 2(-2r + 1) = \alpha$

$\Rightarrow \quad 9r - 5 = \alpha \quad \Rightarrow \quad r = (1/9)(5 + \alpha)$

Now $PQ = 5$

$\Rightarrow \quad r^2 + 4r^2 + 4r^2 = 25 \Rightarrow r = \pm 5/3$

So $\quad \frac{\alpha+5}{9} = r = \frac{\pm 5}{3} \Rightarrow \alpha = 10 \qquad (\because \alpha > 0)$

and $r = 5/3$

Hence the coordinates of Q are $\left(\frac{8}{3}, \frac{4}{3}, \frac{-7}{3}\right)$

◉ **Example 103:** The distance of the point (1, 1, 1) from the plane passing through the point (–1, –2, –1) and whose normal is perpendicular to the lines $\frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$ and $\frac{x-2}{1} = \frac{y+2}{2} = \frac{z+3}{3}$ is

(a) $2/\sqrt{75}$ (b) $7/\sqrt{75}$

(c) $13/\sqrt{75}$ (d) $23/\sqrt{75}$

Ans. (c)

◎ **Solution:** Let the equation of the plane through (–1, –2, –1) be

$$a(x + 1) + b(y + 2) + c(z + 1) = 0$$

As the normal to the plane is perpendicular to the given lines

$3a + b + 2c = 0,$

$a + 2b + 3c = 0$

$$\Rightarrow \quad \frac{a}{-1} = \frac{b}{-7} = \frac{c}{5}$$

$\Rightarrow$ Equation the plane is $x + 7y - 5z + 10 = 0$

Distance of (1, 1,1) from the plane is

$$\left|\frac{1+7-5+10}{\sqrt{1+49+25}}\right| = \frac{13}{\sqrt{75}}$$

◉ **Example 104:** Mirror image of the point P (2, 1, 6) in the plane passing through the points. (2, 1, 0), (5, 0, 1) and (4, 1, 1) is the point.

(a) (–2, – 1, –6) (b) (6, 5, –2)

(c) (1, 1, 4) (d) (2, 5, 4)

Ans. (b)

◎ **Solution:** Equation of the plane through the given points is

$$\begin{vmatrix} x-2 & y-1 & z-0 \\ 5-2 & 0-1 & 1-0 \\ 4-2 & 1-1 & 1-0 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x-2 & y-1 & z-0 \\ 3 & -1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad x + y - 2z = 3$

Equation of the line through P perpendicular to this plane is

$$\frac{x-2}{1} = \frac{y-1}{1} = \frac{z-6}{-2} = r \text{ say}$$

Let Q $(r + 2, r + 1, -2r + 6)$ be the image of P in this plane, then the mid-point $R\left(\frac{r}{2} + 2, \frac{r}{2} + 1, -r + 6\right)$ of PQ lies on the plane.

So $\quad \frac{r}{2}+2+\frac{r}{2}+1-2(-r+6)=3$

$\Rightarrow \quad 3r = 12 \quad \Rightarrow r = 4$

and the coordinates of Q are (6, 5, –2)

Example 105: If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is

(a) 3 (b) 4

(c) 5 (d) 6

Ans. (d)

Solution: Equation of the plane containing the given lines is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$\Rightarrow \quad x - 2y + z = 0$

Since this plane must be parallel to the plane $Ax - 2y + z = d$, we get $A = 1$

As distance between the planes is $\sqrt{6}$

we get $\frac{|d|}{\sqrt{1+4+1}} = \sqrt{6} \Rightarrow |d| = 6$

EXERCISE

Concept-based Straight Objective Type Questions

1. If Q be the foot of the perpendicular from the origin to the plane $4x - 3y + z + 13 = 0$ and R be the point (–1, 1, –6). Then length QR is

(a) $\sqrt{14}$ (b) $\sqrt{19/2}$

(c) $3\sqrt{7/2}$ (d) $3/\sqrt{2}$

2. If the lines $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z+1}{3}$ and $\frac{x+2}{2}=\frac{y-k}{3}=\frac{z}{4}$ are coplanar, then the value of k is

(a) 11/2 (b) – 11/2

(c) 9/2 (d) – 9/2

3. A line in 3-dimensional space makes an angle θ $(0 < \theta \le \pi/2)$ with both x and y-axis. Then the set of all values of θ is the interval:

(a) $\left(0, \frac{\pi}{4}\right]$ (b) $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

(c) $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$ (d) $\left(\frac{\pi}{6}, \frac{\pi}{2}\right]$

4. A symmetric form of the line of intersection of the planes $x = ay + b$ and $z = cy + d$ is

(a) $\frac{x-b}{a}=\frac{y-1}{1}=\frac{z-d}{c}$

(b) $\frac{x-b-a}{a}=\frac{y-1}{1}=\frac{z-d-c}{c}$

(c) $\frac{x-a}{b}=\frac{y-0}{1}=\frac{z-c}{d}$

(d) $\frac{x-b-a}{b}=\frac{y-1}{0}=\frac{z-d-c}{d}$

5. If the angle between the line $2(x + 1) = y = z + 4$ and the plane $2x - y + \sqrt{\lambda}z + 4$ is $\pi/6$, then the value of λ is

(a) 135/7 (b) 45/11

(c) 45/7 (d) 135/7

6. If the centroid of the triangle with vertices $(3c + 2, 2, 0)$, $(2c, -1, -1)$ and $(c + 2, 3c + 1, c + 3)$ lies in the plane $z = c$, then the coordinates of the centroid are:

(a) $\left(-\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}\right)$ (b) $\left(\frac{10}{3}, \frac{5}{3}, 1\right)$

(c) $\left(\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right)$ (d) $\left(\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}\right)$

7. A plane passes through a fixed point (α, β, γ). The locus of the foot of the perpendiculars to the plane from the origin is

(a) a plane inclined at an angle $\frac{\pi}{3}$ with the given plane

(b) a straight line

(c) a plane perpendicular to the given plane

(d) none of these

8. Let $(a, b, c) \neq (0, 0, 0)$. The pair of equations which does not represent a straight line is

   (a) $ax - by + cz + d = 0,\ ax + b'y + cz + d = 0\ (b \neq b')$
   (b) $ax - by + cz + d = 0,\ ax + by + c'z + d = 0\ (c \neq c')$
   (c) $ax + by + cz + d = 0,\ ax + by + cz - d' = 0\ (d \neq d')$
   (d) $ax + by + cz + d = 0,\ a'x + by + cz + d = 0\ (a \neq a')$

9. Shortest distance between z-axis and the line $\frac{x-2}{3} = \frac{y-5}{2} = \frac{z+1}{-5}$ is

   (a) $1/\sqrt{13}$ (b) $11/\sqrt{13}$
   (c) $\sqrt{11}/13$ (d) $11/\sqrt{13}$

10. If the lines $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-\lambda}{3}$ and $\frac{x}{1} = \frac{y+z}{2} = \frac{z}{4}$ intersect each other, then λ lies in the interval

    (a) $(-5, -3)$ (b) $(13, 15)$
    (c) $(11, 13)$ (d) $(9, 11)$

11. The distance between the point (6,5,9) and the plane determined by the points A (3, –1, 2), B (5, 2, 4) and C (–1, –1, 6) in units is

    (a) $3/\sqrt{34}$ (b) $6/\sqrt{34}$
    (c) $2/\sqrt{17}$ (d) $3/\sqrt{17}$

12. Let L: $x - 2y + 4z = 9$ be a plane, $P(2, 1, -1)$ be a point and O, be the Origin. Q is the foot of the perpendicular from P on the plane L, then $(OP)^2 + (PQ)^2$ is equal to

    (a) $\frac{190}{21}$ (b) $\frac{43}{21}$
    (c) $\frac{295}{21}$ (d) $\frac{211}{21}$

13. The algebraic sum of the intercepts made by the plane $x - 2y + 3z = 24$ on the coordinate axes is

    (a) 28 (b) 20
    (c) 4 (d) 44

14. L_1 : $2(x - 1) + 3(y + 2) + (z - 7) = 0$ and L_1 : $2(x + 1) - 3(y + 2) + (z + 7) = 0$ are two planes.

    (a) L_1 and L_2 are parallel
    (b) L_1 and L_2 are perpendicular
    (c) L_1 and L_2 are equidistant from the origin
    (d) Equation of a plane through the intersection of L_1 and L_2 is $y = 0$

15. If the points $(1, 1, p)$ and $(-3, 0, 1)$ are equidistant from the plane $\mathbf{r}\,(3\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}) + 13 = 0$ then the value of p is

    (a) –1 (b) 0
    (c) 7/3 (d) 3/7

LEVEL 1

Straight Objective Type Questions

16. The coordinate of the middle point of the line joining the points (–1, –1, 1) and (–1, 1, –1) are

    (a) (0, 0, 0) (b) (– 1, 0, 0)
    (c) (0, –1, 1) (d) (0, 1, –1).

17. If l, m, n and l', m', n' be the direction cosines of two lines which include an angle θ, then

    (a) $\cos\theta = ll' + mm' + nn'$
    (b) $\sin\theta = ll' + mm' + nn'$
    (c) $\cos\theta = mn' + m'n + nl' + n'l + lm' + l'm$.
    (d) $\sin\theta = mn' + m'n + nl' + n'l + lm' + l'm$.

18. An equation of the XOY plane is

    (a) $x = 0$ (b) $y = 0$
    (c) $z = 0$ (d) $z = c, c \neq 0$

19. The coordinate of the foot of the perpendicular from the point (a, b, c) on z-axis is

    (a) $(a, 0, 0)$ (b) $(0, b, 0)$
    (c) $(0, 0, c)$ (d) $(a, b, 0)$

20. $l = m = n = 1$ represents the direction ratios of

    (a) x-axis (b) y-axis
    (c) z-axis (d) none of these

21. If l_1, m_1, n_1 and l_2, m_2, n_2 are direction cosines of two perpendicular lines, then

    (a) $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$
    (b) $l_1 l_2 + m_1 m_2 + n_1 n_2 = 1$
    (c) $\frac{l_1}{l_2} + \frac{m_1}{m_2} + \frac{n_1}{n_2} = 1$
    (d) $\frac{l_1}{l_2} + \frac{m_1}{m_2} + \frac{n_1}{n_2} = 0$

22. P (0, 5, 6), Q (1, 4, 7), R (2, 3, 7) and S (3, 4, 6) are four points in the space. The point farthest from the origin O (0, 0, 0) is

    (a) P (b) Q
    (c) R (d) S

23. P (1, 1, 1) and Q (λ, λ, λ) are two points in the space such that $PQ = \sqrt{27}$, the value of λ can be

(a) – 4 (b) – 3
(c) 2 (d) 4

24. The points (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) are the vertices of

(a) a right angled isosceles triangle
(b) equilateral triangle
(c) an isosceles triangle
(d) an obtuse angled triangle

25. If α, β, γ are the angles which a line makes with the coordinate axes, then

(a) $\sin^2\alpha + \cos^2\beta + \cos^2\gamma = 0$
(b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 0$
(c) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
(d) $\sin^2\alpha + \sin^2\beta = 1 - \cos^2\gamma$

26. If the vector $2\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$ is inclined at angles α, β, γ with the coordinate axes, then

(a) $3\cos\alpha = 2/\sqrt{62}$
(b) $2\cos\beta = -3/\sqrt{62}$
(c) $\cos\gamma = 7/\sqrt{62}$
(d) $2\cos\alpha = -3\cos\beta = 7\cos\gamma$

27. An equation of a line through A (3, 4, – 7) and B (1, – 1, 6) is

(a) $\mathbf{r} = 3\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} + \lambda(2\mathbf{i} - 5\mathbf{j} - 13\mathbf{k})$
(b) $\mathbf{r} = \mathbf{i} - \mathbf{j} + 6\mathbf{k} - \lambda(2\mathbf{i} - 5\mathbf{j} - 13\mathbf{k})$
(c) $\dfrac{x-3}{2} = \dfrac{y-4}{-5} = \dfrac{z+7}{-13}$
(d) $\dfrac{x-1}{-2} = \dfrac{y+1}{-5} = \dfrac{z-6}{13}$

28. The lines $\dfrac{x-1}{3} = \dfrac{y-1}{-1} = \dfrac{z+1}{0}$ and

$$\frac{x-4}{2} = \frac{y+0}{0} = \frac{z+1}{3}$$

(a) do not intersect (b) intersect
(c) intersect at (4, 0, 4) (d) intersect at (1, 1, – 1)

29. The sine of the angle between the lines $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} + (\mathbf{i} + \mathbf{j} + \mathbf{k})\,t$ and the plane $\mathbf{r}.\,(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}) = q$ is

(a) $2\sqrt{6}/15$ (b) $2\sqrt{3}/15$
(c) $\sqrt{201}/15$ (d) none of these

30. Equation of a plane passing through (1, 1, 1) and containing x-axis is

(a) $x - y = 0$ (b) $x - z = 0$
(c) $y - z = 0$ (d) $x + y + z = 3$

31. The coordinates of the foot of the perpendicular from the point (3, –1, 11) on the line $\dfrac{x}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ are

(a) (2, 5, 7) (b) (– 2, –1, –1)
(c) (0, 2, 3) (d) (2, 3, 4)

32. The length of the line segment whose projections on the coordinate axes are 12, 4, 3 is

(a) 19 (b) 13
(c) 11 (d) none of these

33. The number of lines which are equally inclined to the three coordinate axes is

(a) 3 (b) 4
(c) 6 (d) 8

34. An equation of the plane which passes through the point with position vector **a** and is parallel to the vectors **b** and **c** is

(a) [**r b c**]= [**a b c**] (b) [**r c a**[= [**b c a**]
(c) [**r a b**] = [**c a b**] (d) none of these

35. If the lines $\mathbf{r} = (\mathbf{i} + \mathbf{j} - \mathbf{k}) + \lambda(3\mathbf{i} - \mathbf{j})$ and $\mathbf{r} = (4\mathbf{i} - \mathbf{k}) + \mu\,(2\mathbf{i} + 3\mathbf{k})$ intersect at the point (p, 0, p – 5) then

(a) $p = 0$ (b) $p = -1$
(c) $p = 4$ (d) $p = 5$

36. The line of intersection of the planes $\mathbf{r}.\,(3\mathbf{i} - \mathbf{j} + \mathbf{k}) = 1$ and $\mathbf{r}.\,(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 2$ is parallel to the vector

(a) $-2\mathbf{i} + 7\mathbf{j} + 13\mathbf{k}$ (b) $2\mathbf{i} + 7\mathbf{j} - 13\mathbf{k}$
(c) $-2\mathbf{i} + 7\mathbf{j} - 13\mathbf{k}$ (d) $2\mathbf{i} + 7\mathbf{j} + 13\mathbf{k}$

37. An equation of the plane containing the line $\dfrac{x+1}{-3} = \dfrac{y-3}{2} = \dfrac{z+2}{1}$ and the point (0, 7, – 7) is

(a) $x + y + z = 0$ (b) $x + 2y - 3z = 35$
(c) $3x - 2y + 3z + 35 = 0$ (d) $3x - 2y - z = 21$

38. A line passes through two points A (2, – 3, – 1) and B (8, – 1, 2), the coordinates of a point on this line nearer to the origin at a distance of 14 units from A are

(a) (14, 1, 5) (b) (– 10, – 7, – 7)
(c) (10, 7, 7) (d) (– 14, – 1, – 5)

39. The plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ meets the coordinate axes at A, B, C respectively. D and E are the mid-points of AB and AC respectively. Coordinates of the mid-point of DE are

(a) (a, b/4, c/4) (b) (a/4, b, c/4)
(c) (a/4, b/4, c) (d) (a/2, b/4, c/4)

40. The lines $\mathbf{r} = \mathbf{a} + \lambda\,(\mathbf{b} \times \mathbf{c})$ and $\mathbf{r} = \mathbf{b} + \mu\,(\mathbf{c} \times \mathbf{a})$ will intersect if

(a) $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$
(b) $\mathbf{a.c} = \mathbf{b.c}$
(c) $\mathbf{b} \times \mathbf{a} = \mathbf{c} \times \mathbf{a}$
(d) none of these

41. The shortest distance between the lines

$\mathbf{r} = (4\mathbf{i} - \mathbf{j}) + \lambda\,(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ and

$\mathbf{r} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) + \mu\,(2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$ is

(a) 6
(b) $\sqrt{5}$
(c) $6/\sqrt{5}$
(d) $6\sqrt{5}$

42. The coordinates of a point on the line $x = 4y + 5$, $z = 3y - 6$ at a distance $3\sqrt{26}$ from the point $(5, 0, -6)$ are

(a) (17, 3, 3)
(b) (– 7, 3, – 15)
(c) (– 17, – 3, – 3)
(d) (7, – 3, 15)

43. If θ denotes the acute angle between the line

$\mathbf{r} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \lambda\,(\mathbf{i} - \mathbf{j} + \mathbf{k})$ and the plane $\mathbf{r}.(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 4$, then $\sin\theta + \sqrt{2}\cos\theta =$

(a) $1/\sqrt{2}$
(b) 1
(c) $\sqrt{2}$
(d) $1 + \sqrt{2}$

44. The plane containing the lines

$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$ passes through

(a) (0, 0, 0)
(b) (1, 0, 1)
(c) (1, – 1, 1)
(d) (– 1, 1, 0)

45. Equation of a plane bisecting an angle between the plane $\mathbf{r}\,.\,(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 19$ and $\mathbf{r}.\,(4\mathbf{i} - 3\mathbf{j} + 12\mathbf{k}) + 3 = 0$, passing through the point with position vector $\mathbf{i} + 7\mathbf{j} - \mathbf{k}$ is

(a) $\mathbf{r}.(\mathbf{i} + 35\mathbf{j} - 10\mathbf{k}) - 256 = 0$
(b) $\mathbf{r}.(25\mathbf{i} + 17\mathbf{j} + 62\mathbf{k}) - 238 = 0$
(c) $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) - 13 = 0$
(d) $\mathbf{r}.(4\mathbf{i} - 3\mathbf{j} + 12\mathbf{k}) + 29 = 0$

46. If the line $\frac{x-1}{2} = \frac{y-3}{a} = \frac{z+1}{3}$ lies in the plane $bx + 2y + 3z - 4 = 0$, then

(a) $a = 11/2, b = 1$
(b) $a = -5/2, b = -7$
(c) $a = -11/2, b = 1$
(d) $a = 1, b = -11/2$

47. Equation of the plane passing through the points $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ is

(a) $\mathbf{r}\,.\,(4\mathbf{i} + 2\mathbf{j}) = 20$
(b) $\mathbf{r}\,.\,(9\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 14$
(c) $\mathbf{r}.\,(9\mathbf{i} + 3\mathbf{j} - \mathbf{k}) = 6$
(d) none of these

48. The line of shortest distance between the lines $\frac{x-1}{2} = \frac{y+8}{-7} = \frac{z-4}{5}$ and $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-6}{-3}$ intersects the first line at the point

(a) (1, 1, 1)
(b) (– 1, – 1, – 1)
(c) (3, 3, 3)
(d) (– 3, – 3, – 3)

49. If $\boldsymbol{r}.\,\boldsymbol{n} = q$ is the equation of a plane normal to the vector $\boldsymbol{n}$, the length of the perpendicular from the origin on the plane is

(a) q
(b) $|\mathbf{n}|$
(c) $q|\mathbf{n}|$
(d) $\frac{q}{|\mathbf{n}|}$

50. If the foot of the perpendicular from the origin to a plane is (a, b, c), then the equation of the plane is

(a) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$
(b) $ax + by + cz = 1$
(c) $ax + by + cz = a^2 + b^2 + c^2$
(d) $ax + by + cz = 0$

51. The plane passing through the points (– 2, – 2, 2) and containing the line joining the points (1, 1, 1) and (1, – 1, 2) makes intersects on the coordinates axes, the sum of whose length is

(a) 3
(b) 4
(c) 6
(d) 12

52. Equation of a line passing through the point whose position vector is $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and in the direction of the vector $3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ is

(a) $4x + 3y = 17, 5y - 4z = 1$
(b) $4x - 3y = 17, 5y + 4z = 1$
(c) $4x + 5y = 12, 3y + 4z = 1$
(d) $5y + 4z = 1,\ \ 4x + 3z = 1$

53. The lines $\mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{k})$ and

$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mu(\mathbf{i} + \mathbf{j} - \mathbf{k})$

(a) intersect each others
(b) do not intersect
(c) intersect at $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$
(d) are parallel.

54. The points (1, – 2, 3), (2, 3, – 4) (0, – 7, 10) are the vertices of

(a) a right angled triangles
(b) isosceles triangle
(c) equilateral triangle
(d) none of these

55. The foot of the perpendicular from (a, b, c) on the line $x = y = z$ is the point (r, r, r) where

(a) $r = a + b + c$ (b) $r = 3(a + b + c)$
(c) $3r = a + b + c$ (d) none of these

56. Equation of a plane which passes through the line $x + py + q = 0 = rz + s$ and makes equal intercepts on y and z axes is $x + py + q + \lambda(rz + s) = 0$ where λ is equal to

(a) q/s (b) p/r
(c) r/s (d) p/q

57. Parametric form of the equation of the line $3x - 6y - 2z - 15 = 2x + y - 2z - 5 = 0$ is

(a) $\frac{x-5}{14} = \frac{y}{2} = \frac{z}{15}$ (b) $\frac{x-1}{14} = \frac{y-5}{2} = \frac{z-1}{15}$
(c) $\frac{x-3}{14} = \frac{y+1}{2} = \frac{z}{15}$ (d) none of these

58. A line with direction ratios 2, 7, – 5 is drawn to intersect the lines

$$\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1} \text{ and } \frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-6}{4}$$

at P and Q respectively. Length of PQ is

(a) $\sqrt{78}$ (b) $\sqrt{77}$
(c) $\sqrt{54}$ (d) $\sqrt{74}$

59. YZ-plane divides the line joining the points (3, 5, – 7) and (– 2, 1, 8) in the ratio

(a) 1 : 2 (b) 2 : 3
(c) 3 : 2 (d) 2 : 1

60. The angle between the lines

$$\frac{x-2}{3} = \frac{y+1}{-2},\ z = 2 \text{ and}$$

$$\frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2} \text{ is}$$

(a) $\pi/3$ (b) $\pi/6$
(c) $\pi/4$ (d) $\pi/2$

61. If the plane $\frac{x}{a} + \frac{y}{b} + \frac{y}{c} = 3$ meets the coordinate axes in A, B, C, and the centroid of the triangle ABC is at $P(2, 4, 8)$, then a, b, c are in

(a) A.P (b) G.P
(c) H.P (d) none of these

62. The variable plane $(2\lambda + 1)x + (3 - \lambda)y + z = 4$ always passes through the line

(a) $\frac{x}{2} = \frac{y}{1} = \frac{z+4}{1}$ (b) $\frac{x}{1} = \frac{y}{2} = \frac{z}{-3}$
(c) $\frac{x}{1} = \frac{y}{2} = \frac{z-4}{-7}$ (d) none of these

63. Algebraic sum of the intercepts by the plane $3x - 4y + 7z = 84$ on the axes is

(a) 6 (b) 14
(c) 19 (d) 61

Assertion-Reason Type Questions

64. Vertices of a triangle ABC are A (1, 1, 0), B(1, 0, 1) and C(0, 1, 1)

Statement-1: The radius of the circumcircle of the triangle ABC is $\sqrt{2/3}$.

Statement-2: The centre of the circumcircle of the triangle ABC lies on the plane $x + y + z - 2 = 0$.

65. Consider the line $L : \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and the plane π: $x + y + z = 0$

Staement-1: If P is a point on L at a distance $\sqrt{14}$ from the origin O and N is the foot of the perpendicular from P to the plane π, then $ON = \sqrt{2}$

Staement-2: If R is a point on L such that the perpendicular distance of R from the plane π is $\sqrt{3}$ then the coordinates of R are (1, 2, 3)

66. $L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$,

$L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$.

Statement-1: The unit vector perpendicular to both L_1 and L_2 is $\frac{-\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}}{5\sqrt{3}}$.

Statement-2: The distance of the point (1, 1, 1) from the plane passing through the point

(– 1, – 2, – 1) and whose normal is perpendicular to both the lines L_1 and L_2 is $23/5\sqrt{3}$.

67. **Statement-1:** The distance between the line $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ and the plane $\mathbf{r}.(\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 5$ is $10/3\sqrt{3}$.

    Statement-2: If a line is parallel to a plane, then the distance between the line and the plane is equal to the length of the perpendicular from any point on the line to the plane.

68. **Statement-1:** The direction cosines of the line $6x - 2 = 3y + 1 = 2z - 2$ are same as the direction cosines of the normal to the plane $2x + 3y + z = 14$.

    Statement-2: The direction angles of a normal to the plane are $\pi/4$, $\pi/4$, $\pi/2$ and the length of the perpendicular from the origin on the plane is $\sqrt{2}$, equation of the plane is $x + y = 2$.

69. Consider two lines in three-dimensional plane.

    L_1: $\dfrac{x-2}{1} = \dfrac{y-2}{1} = \dfrac{z-2}{1}$

    L_2: $\dfrac{x-1}{0} = \dfrac{y}{1} = \dfrac{z+1}{2}$

    Statement-1: The shortest distance between L_1 and L_2 is zero.

    Statement-2: The line L_1 and L_2 are coplanar.

70. Let L be the line $x - 4 = y - 2 = \dfrac{z-7}{2}$. and P be the plane $2x - 4y + z = 7$.

    Statement-1: The line L lies in the plane P.

    Statement-2: The direction ratios of L are $l_1 = 1$, $m_1 = 1$, $n_1 = 2$ and that of the normal to the plane are $l_2 = 2$, $m_2 = -4$, $n_2 = 1$; and $l_1l_2 + m_1m_2 + n_1n_2 = 0$ holds.

LEVEL 2

Straight Objective Type Questions

71. The lines $\dfrac{x-a}{a'} = \dfrac{y-b}{b'} = \dfrac{z-c}{c'}$ and $\dfrac{x-a'}{a} = \dfrac{y-b'}{b} = \dfrac{z-c'}{c}$ intersect at the point

    (a) $(a - a', b - b', c - c')$ (b) $(a + a', b + b', c + c')$
    (c) (a, b, c) (d) (a', b', c')

72. The mirror image of the point (3, – 2, 1) in the plane $3x - y + 4z = 2$ is

    (a) (0, 1, – 3) (b) (– 1, 0, – 3)
    (c) (0, – 1, – 3) (d) (0, – 1, 3)

73. The shortest distance between the lines $x + a = 2y = -12z$ and $x = y + 2a = 6z - 6a$ is

    (a) a (b) $2a$
    (c) $4a$ (d) $6a$

74. The $P(u, v, w)$ is a point whose distance from the line $x = y = z$ is twice its distance from the plane $x + y + z = 1$ and $uv + vw + wu = 0$ then $u^2 + v^2 + w^2 - 4(u + v + w)$ is equal to

    (a) – 2 (b) 0
    (c) 2 (d) 4

75. The angle between the two diagonals of a cube is

    (a) 30° (b) 45°
    (c) $\cos^{-1}(1/\sqrt{3})$ (d) $\cos^{-1}(1/3)$

76. The mid-points of the sides of a triangle are (2, 3, – 1), (0, 8, 5) and (5, 7, 11). The distance of the origin from the vertex of the triangle which is farthest from it is

    (a) $\sqrt{74}$ (b) $2\sqrt{19}$
    (c) $\sqrt{78}$ (d) $\sqrt{442}$

77. If α, β, γ, δ be the angles which a line makes with the four diagonals of a cube, then the value of $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is

    (a) 1 (b) 1/3
    (c) 2/3 (d) 4/3

78. The volume of the tetrahedron included between the plane $3x + 4y - 5z - 60 = 0$ and the coordinate planes in cubic units is

    (a) 60 (b) 600
    (c) 720 (d) none of these

79. A triangle ABC is placed so that the mid-points of the sides are on the x, y, z axes. Lengths of the intercepts made by the plane containing the triangle on these axes are respectively α, β, γ. Coordinates of the centroid of the triangle ABC are

    (a) $(-\alpha/3, \beta/3, \gamma/3)$ (b) $(\alpha/3, -\beta/3, \gamma/3)$
    (c) $(\alpha/3, \beta/3, -\gamma/3)$ (d) $(\alpha/3, \beta/3, \gamma/3)$

80. A point moves so that the sum of the squares of its distances from the six faces $x = \pm 1$, $y = \pm 1$, $z = \pm 1$ of a cube is 10 units. The locus of the point is

    (a) $x^2 + y^2 + z^2 = 1$ (b) $x^2 + y^2 + z^2 = 2$
    (c) $x + y + z = 1$ (d) $x + y + z = 2$.

81. The plane $2x - y + 3z + 5 = 0$ is rotated through 90° about its line of intersection with the plane $5x - 4y - 2z + 1 = 0$. The equation of the plane in the new position is
(a) $6x - 9y - 29z - 31 = 0$
(b) $27x - 24y - 26z - 13 = 0$
(c) $43x - 32y - 2z + 27 = 0$
(d) $26x - 43y - 151z - 165 = 0$

82. A plane passes through $(1, -2, 1)$ and is perpendicular to two planes $2x - 2y + z = 0$ and $x - y + 2z = 4$. The distance of the plane from the point $(1, 2, 2)$ is
(a) 0 (b) 1
(c) $\sqrt{2}$ (d) $2\sqrt{2}$

83. A variable plane at a unit distance from the origin cuts the coordinates axes at points A, B, C. If the centroid (x, y, z) of ΔABC satisfies the equation $x^{-2} + y^{-2} + z^{-2} = k$, then the value of k is
(a) 9 (b) 3
(c) 1 (d) 1/3

84. If t is a parameter, then the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ is
(a) $x = 3 + 14t, \quad y = 1 + 2t, \quad z = 15t$
(b) $x = 3 + 14t, \quad y = -1 + 2t, z = 15t$
(c) $x = -3 + 14t, y = 1 + 2t, \quad z = 15t$
(d) none of these.

85. Projection of the line $8x - y - 7z = 8$, $x + y + z = 1$ on the plane $5x - 4y - z = 5$ is
(a) $\frac{x-1}{1} = \frac{y-0}{2} = \frac{z-0}{-3}$ (b) $\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-0}{-3}$
(c) $\frac{x-0}{1} = \frac{y-0}{2} = \frac{z-1}{-3}$ (d) none of these

Previous Years' AIEEE/JEE Main Questions

1. The d.r. of normal to the plane through $(1, 0, 0)$ $(0, 1, 0)$ which makes an angle $\pi/4$ with plane $x + y = 3$ are
(a) $1, \sqrt{2}, 1$ (b) $1, 1, \sqrt{2}$
(c) $1, 1, 2$ (d) $\sqrt{2}, 1, 1$ **[2002]**

2. A plane which passes through the point $(3, 2, 0)$ and the line $\frac{x-4}{1} = \frac{y-7}{5} = \frac{z-4}{4}$ is
(a) $x - y + z = 2$
(b) $x + y + z = 5$
(c) $x + 2y - z = 1$
(d) none of these **[2002]**

3. Two system of rectangular axes have the same origin. If a plane cuts them at distance a, b, c and a', b', c' from the origin, then
(a) $\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2} + \frac{1}{a'^2} + \frac{1}{b'^2} - \frac{1}{c'^2} = 0$
(b) $\frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} + \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$
(c) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$
(d) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2} = 0$ **[2003]**

*4. The shortest distance from the plane $12x + 4y + 3z = 327$ to the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z = 155$ is
(a) $11\frac{3}{4}$ (b) 13
(c) 39 (d) 26 **[2003]**

5. The two lines $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d$ will be perpendicular, if and only if
(a) $aa' + bb' + cc' = 0$
(b) $(a + a')(b + b')(c + c') = 0$
(c) $aa' + cc' + 1 = 0$
(d) $aa' + bb' + c' + 1 = 0$ **[2003]**

6. The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar if
(a) $k = 1$ or -1 (b) $k = 0$ or -3
(c) $k = 3$ or -3 (d) $k = 0$ or -1 **[2003]**

*7. The radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$ is cut by the plane $x + 2y + 2z + 7 = 0$ is
(a) 2 (b) 3
(c) 4 (d) 1 **[2003]**

8. A tetrahedron has vertices at $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. Then the angle between the faces OAB and ABC will be

(a) $\cos^{-1}(17/31)$ (b) 30°
(c) 90° (d) $\cos^{-1}(19/35)$ **[2003]**

9. A line makes the same angle θ with each of the x and z-axis. If the angle β, which it makes with y-axis, is such that $\sin^2\beta = 3\sin^2\theta$, then $\cos^2\theta$ equals
(a) 3/5 (b) 1/5
(c) 2/3 (d) 2/5 **[2004]**

10. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is
(a) 7/2 (b) 5/2
(c) 3/2 (d) 9/2 **[2004]**

11. A line with directional cosines proportional to 2, 1, 2 meets each of the lines $x = y + a = z$ and $x + a = 2y = 2z$. The coordinates of each of the points of intersection are given by
(a) $(3a, 2a, 3a)$, $(a, a, 2a)$
(b) $(3a, 2a, 3a)$, (a, a, a)
(c) $(3a, 3a, 3a)$, (a, a, a)
(d) $(2a, 3a, 3a)$, $(2a, a, a)$ **[2004]**

12. If the straight lines $x = 1 + s$, $y = -3 - \lambda s$, $z = 1 + \lambda s$ and $x = t/2$, $y = 1 + t$, $z = 2 - t$, with parameter s and t are coplanar, then λ equals.
(a) $-1/2$ (b) -1
(c) -2 (d) 0 **[2004]**

*13. The intersection of the spheres $x^2 + y^2 + z^2 + 7x - 2y - z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$ is the same as the intersection of one of the sphere and the plane
(a) $x - y - 2z = 1$
(b) $x - 2y - z = 1$
(c) $x - y - z = 1$
(d) $2x - y - z = 1$ **[2004]**

14. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is
(a) 45° (b) 30°
(c) 0° (d) 90° **[2005]**

15. If the angle θ between the lines $\dfrac{x+1}{1} = \dfrac{y-1}{2} = \dfrac{z-2}{2}$ and the plane $2x - y + \sqrt{\lambda}\, z + 4 = 0$ is such that $\sin\theta = 1/3$, the value of λ is
(a) 3/4 (b) $-4/3$
(c) 5/3 (d) $-3/5$ **[2005]**

16. The distance between the line $\mathbf{r} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 4\mathbf{k})$ and the plane $\mathbf{r} \cdot (\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = 5$ is
(a) 3/10 (b) 10/3
(c) 10/9 (d) $10/3\sqrt{3}$ **[2005]**

*17. If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the mid-point of the line joining the centres of the spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$, then a equals
(a) -2 (b) 2
(c) -1 (d) 1 **[2005]**

*18. The plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z - 2 = 0$ in a circle of radius
(a) 2 (b) $\sqrt{2}$
(c) 3 (d) 1 **[2005]**

19. The two lines $x = ay + b$, $z = cy + d$; and $x = a'y + b'$, $z = c'y + d'$ are perpendicular to each other if
(a) $\dfrac{a}{a'} + \dfrac{c}{c'} = 1$ (b) $aa' + cc' = -1$
(c) $aa' + cc' = 1$ (d) $\dfrac{a}{a'} + \dfrac{c}{c'} = -1$ **[2006]**

20. The image of the point $(-1, 3, 4)$ in the plane $x - 2y = 0$ is
(a) $(-17/3, -19/3, 4)$ (b) $(15, 11, 4)$
(c) $(-17/3, -19/3, 1)$ (d) None
[2006]

21. Let L be the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = z$. If L makes an angle α with the positive x-axis, then $\cos\alpha$ equals
(a) $1/\sqrt{3}$ (b) 1/2
(c) 1 (d) $1/\sqrt{2}$ **[2007]**

22. If a line makes an angle of $\pi/4$ with the positive direction of each of x-axis and y-axis, then the angle which the line makes with the positive direction of z-axis is
(a) $\pi/6$ (b) $\pi/3$
(c) $\pi/4$ (d) $\pi/2$ **[2007]**

*23. If (2, 3, 5) is one end of a diameter of the sphere $x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$, then the coordinate of the other end of the diameter are
(a) $(4, 9, -3)$ (b) $(4, -3, 3)$
(c) $(4, 3, 5)$ (d) $(4, 3, -3)$ **[2007]**

24. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = x\mathbf{i} + (x - 2)\mathbf{j} - \mathbf{k}$. If the vector $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$, the x equals
(a) 0 (b) 1
(c) -4 (d) -2 **[2007]**

25. The line passing through the points $(5, 1, a)$ and $(3, b, 1)$ crosses the yz-plane at the point $(0, 17/2, -13/2)$ then
(a) $a = 8, b = 2$ (b) $a = 2, b = 8$
(c) $a = 4, b = 6$ (d) $a = 6, b = 4$ **[2008]**

26. If the straight lines $\frac{x-1}{k}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x-2}{3}=\frac{y-3}{k}=\frac{z-1}{2}$ intersect at a point, then the integer k is equal to

(a) -2 (b) -5

(c) 5 (d) 2 **[2008]**

27. Let the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z+2}{2}$ lie in the plane $x + 3y - \alpha z + \beta = 0$.

Then (α, β) equals

(a) $(5, -15)$ (b) $(-5, 5)$

(c) $(6, -17)$ (d) $(-6, 7)$ **[2009]**

28. The projections of a vector on the three coordinate axis are 6, – 3, 2 respectively. The direction cosines of the vector are

(a) $\frac{6}{7}, \frac{-3}{7}, \frac{2}{7}$ (b) $\frac{-6}{7}, \frac{-3}{7}, \frac{2}{7}$

(c) $6, -3, 2$ (d) $\frac{6}{5}, \frac{-3}{5}, \frac{2}{5}$ **[2009]**

29. A line AB in three dimensional space makes angles 45° and 120° with the positive x-axis and positive y-axis respectively. If AB makes an acute angle θ with the positive z-axis, then θ equals

(a) 60° (b) 75°

(c) 30° (d) 45° **[2010]**

30. **Statement-1:** The point $A(3, 1, 6)$ is the mirror image of the point $B(1, 3, 4)$ in the plane $x - y + z = 5$.

Statement-2: The plane $x - y + z = 5$ bisects the line segment joining $A(3, 1, 6)$ and $B(1, 3, 4)$

(a) Statement-1 is true, statement-2 is false

(b) Statement-1 is false, Statement-2 is true

(c) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1.

(d) Statement-1 is true, Statement-2 is true. Statement-2 is *not* a correct explanation for Statement-1. **[2010]**

31. **Statement-1 :** The point A (1, 0, 7) is the mirror image of the point (1, 6, 3) in the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$.

Statement-2 : The line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ bisects the line segment joining A (1, 0, 7) and B (1, 6, 3). **[2011]**

32. The angle between the line $x = \frac{y-1}{2}=\frac{z-3}{\lambda}$ and the plane $x + 2y + 3z = 4$ is $\cos^{-1}\left(\sqrt{5/14}\right)$, then λ equals.

(a) 5/3 (b) 2/3

(c) 3/2 (d) 2/5 **[2011]**

33. The distance of the point (1, –5, 9) from the plane $x - y + z = 5$ measured along a straight line $x = y = z$ is

(a) $10\sqrt{3}$ (b) $5\sqrt{3}$

(c) $3\sqrt{10}$ (d) $3\sqrt{5}$ **[2011]**

34. The length of the perpendicular drawn from the point (3, –1, 11) to the line $\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is

(a) $\sqrt{29}$ (b) $\sqrt{33}$

(c) $\sqrt{53}$ (d) $\sqrt{66}$ **[2011]**

35. An equation of a plane parallel to the plane $x - 2y + 2z - 5 = 0$ and at a unit distance from the origin is

(a) $x - 2y + 2z - 1 = 0$

(b) $x - 2y + 2z + 5 = 0$

(c) $x - 2y + 2z - 3 = 0$

(d) $x - 2y + 2z + 1 = 0$ **[2012]**

36. If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then k is equal to

(a) 9/2 (b) 0

(c) –1 (d) 2/9 **[2012]**

37. If the lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar, then k can have

(a) exactly three values (b) any value

(c) exactly one value (d) exactly two values **[2013]**

38. Distance between the parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is

(a) 5/2 (b) 7/2

(c) 9/2 (d) 3/2 **[2013]**

39. If the lines $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z+1}{3}$ and $\frac{x+2}{2}=\frac{y-k}{3}=\frac{z}{4}$ are coplanar, then the value of k is

(a) 11/2 (b) – 11/2

(c) 9/2 (d) – 9/2 **[2013, online]**

40. Let Q be the foot of the perpendicular from the origin to the plane $4x - 3y + z + 13 = 0$ and R be the point $(-1, 1, -6)$ on the plane. Then length QR is:

(a) $\sqrt{14}$ (b) $\sqrt{19/2}$

(c) $3\sqrt{7/2}$ (d) $3/\sqrt{2}$ **[2013, online]**

41. The acute angle between two lines such that the direction cosines l, m, n of each of them satisfy the equations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$ is:

(a) 15° (b) 30°

(c) 60° (d) 45° **[2013, online]**

42. If the lines L_1 and L_2 in space, are defined by $L_1 = \{x = \sqrt{\lambda}\, y + (\sqrt{\lambda} - 1)\ z = (\sqrt{\lambda} - 1)y + \sqrt{\lambda}\}$ and $L_2 = \{x = \sqrt{\mu}\, y + (1 - \sqrt{\mu}),\ z = (1 - \sqrt{\mu})y + \sqrt{\mu}\}$ then L_1 is perpendicular to L_2, for all non-negative reals λ and μ, such that:

(a) $\sqrt{\lambda} + \sqrt{\mu} = 1$ (b) $\lambda \neq \mu$

(c) $\lambda + \mu = 0$ (d) $\lambda = \mu$ **[2013, online]**

43. If the projections of a line segment on the x, y and z-axes in 3 dimensional space are 2, 3, and 6 respectively, then the length of the line segment is:

(a) 12 (b) 7

(c) 9 (d) 6 **[2013, online]**

44. Let ABC be a triangle with vertices at points $A(2, 3, 5)$, $B(-1, 3, 2)$ and $(\lambda, 5, \mu)$ in three dimensional space. If the median through A is equally inclined with the axes, then (λ, μ) is equal to

(a) (10, 7) (b) (7, 5)

(c) (7, 10) (d) (5, 7) **[2013, online]**

45. The equation of a plane through the line of intersection of the planes $x + 2y = 3$, $y - 2z + 1 = 0$ and perpendicular to the first plane is:

(a) $2x - y - 10z = 9$ (b) $2x - y + 7z = 11$

(c) $2x - y + 10z = 11$ (d) $2x - y - 9z = 10$

[2013, online]

46. The image of the line $\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$ in the plane $2x - y + z + 3 = 0$ is the line

(a) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$ (b) $\frac{x+3}{-3} = \frac{y-5}{-1} = \frac{z+2}{5}$

(c) $\frac{x-3}{3} = \frac{y+5}{1} = \frac{z-2}{-5}$ (d) $\frac{x-3}{-3} = \frac{y+5}{-1} = \frac{z-2}{5}$

[2014]

47. The angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$ and $l^2 = m^2 + n^2$ is

(a) $\pi/3$ (b) $\pi/4$

(c) $\pi/6$ (d) $\pi/2$ **[2014]**

48. Equation of the plane which passes through the point of intersection of the lines $\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2}$ and $\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and has the largest distance from the origin is:

(a) $7x + 2y + 4z = 54$ (b) $3x + 4y + 5z = 49$

(c) $4x + 3y + 5z = 50$ (d) $5x + 4y + 3z = 57$

[2014, online]

49. A line in 3-dimensional space makes an angle θ $(0 < \theta \leq \pi/2)$ with both the x and y axes. Then the set of all values of θ is the interval.

(a) $\left(0, \frac{\pi}{4}\right]$ (b) $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

(c) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (d) $\left(\frac{\pi}{3}, \frac{\pi}{2}\right]$ **[2014, online]**

50. Let $A(2, 3, 5)$, $B(-1, 3, 2)$ and $C(\lambda, 5, \mu)$ be the vertices of a $\Delta\, ABC$. If the median through A is equally inclined to the coordinate axes, then:

(a) $5\lambda - 8\mu = 0$ (b) $8\lambda - 5\mu = 0$

(c) $10\lambda - 7\mu = 0$ (d) $7\lambda - 10\mu = 0$

[2014, online]

51. The plane containing the line $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and parallel to the line $\frac{x}{1} = \frac{y}{1} = \frac{z}{4}$ passes through the point:

(a) (1, –2, 5) (b) (1, 0, 5)

(c) (0, 3, –5) (d) (–1, –3, 0)

[2014, online]

52. A symmetrical form of the line of intersection of the planes $x = ay + b$ and $z = cy + d$ is

(a) $\frac{x-b}{a} = \frac{y-1}{1} = \frac{z-d}{c}$

(b) $\frac{x-b-a}{a} = \frac{y-1}{1} = \frac{z-d-c}{c}$

(c) $\frac{x-a}{b} = \frac{y-0}{1} = \frac{z-c}{d}$

(d) $\frac{x-b-a}{b} = \frac{y-1}{0} = \frac{z-d-c}{d}$ **[2014, online]**

53. If the distance between the planes $4x - 2y - 4z + 1 = 0$ and $4x - 2y - 4z + d = 0$ is 7, then d is

(a) 41 or – 42 (b) 42 or – 43

(c) – 41 or 43 (d) – 42 or 44

[2014, online]

54. Equation of the line of shortest distance between the lines $\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$ and $\frac{x-1}{0}=\frac{y+1}{-2}=\frac{z}{1}$ is:

(a) $\frac{x}{1}=\frac{y}{-1}=\frac{z}{-2}$ (b) $\frac{x-1}{1}=\frac{y+1}{-1}=\frac{z}{-2}$

(c) $\frac{x-1}{1}=\frac{y+1}{1}=\frac{z}{1}$ (d) $\frac{1}{8}$ **[2014, online]**

55. If the angle between the line $2(x + 1) = y = z + 4$ and the plane $2x - y + \sqrt{\lambda}\, z + 4$ is $\frac{\pi}{6}$, then the value of λ is

(a) $\frac{135}{7}$ (b) $\frac{45}{11}$

(c) $\frac{45}{7}$ (d) $\frac{135}{11}$ **[2014, online]**

56. The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x - y + z = 16$ is:

(a) $2\sqrt{14}$ (b) 8

(c) $3\sqrt{21}$ (d) 13 **[2015]**

57. The equation of the plane containing the line $2x - 5y + z = 3$; $x + y + 4z = 5$, and parallel is the plane, $x + 3y + 6z = 1$, is:

(a) $2x + 6y + 12z = 13$ (b) $x + 3y + 6z = -7$

(c) $x + 3y + 6z = 7$ (d) $2x + 6y + 12z = -13$. **[2015]**

58. The shortest distance between the z-axis and the line $x + y + 2z - 3 = 0 = 2x + 3y + 4z - 4$, is:

(a) 1 (b) 2

(c) 3 (d) 4 **[2015 online]**

59. A plane containing the point (3, 2, 0) and the line $\frac{x-1}{1}=\frac{y-2}{5}=\frac{z-3}{4}$ also contains the point:

(a) (0, –3, 1) (b) (0, 7, 10)

(c) (0, 7, –10) (d) (0, 3, 1) **[2015, online]**

60. If the points (1, 1, λ) and (– 3, 0, 1) are equidistant from the plane, $3x + 4y - 12z + 13 = 0$, then λ satisfies the equation:

(a) $3x^2 - 10x + 7 = 0$ (b) $3x^2 + 10x + 7 = 0$

(c) $3x^2 + 10x - 13 = 0$ (d) $3x^2 - 10x + 21 = 0$ **[2015, online]**

61. If the shortest distance between the lines $\frac{x-1}{\alpha}=\frac{y+1}{-1}=\frac{z}{1}$, $(\alpha \neq -1)$ and $x + y + z + 1 = 0 = 2x - y + z + 3$ is $\frac{1}{\sqrt{3}}$, then a value of α is:

(a) $-\frac{16}{19}$ (b) $-\frac{19}{16}$

(c) $\frac{32}{19}$ (d) $\frac{19}{32}$ **[2015, online]**

62. The distance of the point (1, –5, 9) from the plane $x - y + z = 5$ measured along the line $x = y = z$ is:

(a) $3\sqrt{10}$ (b) $10\sqrt{3}$

(c) $\frac{10}{\sqrt{3}}$ (d) $\frac{20}{3}$ **[2016]**

63. If the line, $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$ lies in the plane, $lx + my - z = 9$, then $l^2 + m^2$ is equal to:

(a) 26 (b) 18

(c) 5 (d) 2 **[2016]**

64. The shortest distance between the lines $\frac{x}{2}=\frac{y}{2}=\frac{z}{1}$ and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$ lies in the interval:

(a) (3, 4] (b) (2, 3]

(c) [1, 2) (d) [0, 1) **[2016, online]**

65. The distance of the point (1, –2, 4) from the plane passing through the point (1, 2, 2) and perpendicular to the planes $x - y + 2z = 3$ and $2x - 2y + z = 0$, is

(a) 2 (b) $\sqrt{2}$

(c) $2\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$ **[2016, online]**

66. *ABC* is a triangle in a plane with vertices $A(2, 3, 5)$, $B(-1, 3\ 2)$ and $C(\lambda, 5, \mu)$. If the median through A is equally inclined to the coordinate axes, then the value $(\lambda^3 + \mu^3 + 5)$ is:

(a) 1130 (b) 1348

(c) 1077 (d) 676 **[2016, online]**

67. The number of distinct real values of λ for which the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z+3}{\lambda^2}$ and $\frac{x-3}{1}=\frac{y-2}{\lambda^2}=\frac{z-1}{2}$ are coplanar is:

(a) 2 (b) 4

(c) 3 (d) 1 **[2016, online]**

Note: The questions with * are not in the present syllabus.

Previous Years' B-Architecture Entrance Examination Questions

1. If the centroid of the triangle with vertices $(3c+2, 2, 0)$, $(2c, -1, -1)$ and $(c+2, 3c+1, c+3)$ coincides with the centre of the sphere $x^2 + y^2 + z^2 + 5ax - 4by - 2cz = 13$, then
   (a) $c = 1$ (b) $c = 2$
   (c) $c = 3$ (d) $c = 0$ **[2006]**

2. A plane passes through a fixed point (p, q, r). The locus of the foot of the perpendicular to the plane from the origin is
   (a) A plane inclined at an angle of $\pi/4$ with the given plane
   (b) A straight line
   (c) A sphere
   (d) A plane perpendicular to the given plane. **[2006]**

3. If the section of the sphere $x^2 + y^2 + z^2 = a^2$ by the plane $z = a/2$ is a circle of radius $\sqrt{27}$ units, then its section by the plane $z = a/6$ is a circle of radius
   (a) $\sqrt{35}$ units (b) $\sqrt{40}$ units
   (c) $\sqrt{45}$ units (d) $\sqrt{30}$ units **[2007]**

4. The distance between the point $(-1, -5, -10)$ and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ with plane $x - y + z = 5$ is
   (a) 13 (b) 15
   (c) 16 (d) 12 **[2007]**

5. Let L_1 be a line with direction ratios $(-2, -1, 2)$ and L_2 be the line joining the points $(1, 2, 3)$ and $(3, 2, 1)$. If θ is the angle between the lines L_1 and L_2 then $|\sin \theta|$ equals:
   (a) $1/3$ (b) $1/4$
   (c) $1/3\sqrt{2}$ (d) $1/2$ **[2007]**

6. Let $(a, b, c) \neq (0, 0,0)$. The pair of equations which does not represent a straight line is:
   (a) $ax - by + cz + d = 0,\ ax + b'y + cz + d = 0\ (b \neq b')$
   (b) $ax - by + cz + d = 0,\ ax + by + c'z + d = 0\ (c \neq c')$
   (c) $ax + by + cz + d = 0,\ ax + by + cz - d' = 0\ (d \neq d')$
   (d) $ax + by + cz + d = 0,\ a'x + by + cz + d = 0\ (a \neq a')$ **[2009]**

7. Consider two lines in three-dimensional plane
   L_1: $\frac{x-2}{1} = \frac{y-2}{1} = \frac{z-2}{1}$
   L_2: $\frac{x-1}{0} = \frac{y}{1} = \frac{z+1}{2}$

   Statement-1: The shortest distance between L_1 and L_2 is zero.

   Statement-2: The lines L_1 and L_2 are coplanar.
   (a) Statement-1 is true, statement-2 is true; Statement-2 is **not** a correct explanation for statement-1
   (b) Statement-1 is true, statement-2 is false;
   (c) Statement-1 is false, statement-2 is true;
   (d) Statement-1 is true, statement-2 is true; Statement-2 is a correct explanation for statement-1 **[2010]**

8. Let α, β and γ be the angles made by a line with the positive directions of the axes of reference in three dimensions. If θ is the acute angle given by $\cos\theta = \frac{\cos^2\alpha + \cos^2\beta + \cos^2\gamma}{\sin^2\alpha + \sin^2\beta + \sin^2\gamma}$, then θ equals.
   (a) $\pi/6$ (b) $\pi/3$
   (c) $\pi/2$ (d) $\pi/4$ **[2010]**

9. If a plane meets the coordinate axes at A, B, C, and ΔABC has centroid at the point $G\,(a/2, b/2, c/2)$, then the equation of the plane is
   (a) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{3}{2}$ (b) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{2}{3}$
   (c) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{1}{2}$ (d) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{1}{3}$ **[2011]**

10. Shortest distance between z-axis and the line $\frac{x-2}{3} = \frac{y-5}{2} = \frac{z+1}{-5}$ is
    (a) $1/\sqrt{13}$ (b) $11/13$
    (c) $\sqrt{11}/13$ (d) $11/\sqrt{13}$ **[2011]**

11. Let L be the line $x - 4 = y - 2 = \frac{z-7}{2}$ and P be the plane $2x - 4y + z = 7$

    Statement-1: The line L lies in the plane P.

    Statement-2: The direction ratios of the line L are $l_1 = 1$, $m_1 = 1$, $n_1 = 2$ and that of the normal to the plane P are $l_2 = 2$, $m_2 = -4$, $n_2 = 1$, and $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ holds

(a) Statement-1 is true, statement-2 is true; Statement-2 is a correct explanation for statement-1.
(b) Statement-1 is true, statement-2 is true; Statement-2 is **not** a correct explanation for statement-1.
(c) Statement-1 is true, statement-2 is false;
(d) Statement-1 is false, statement-2 is true. **[2012]**

12. The reflection point of the point (0, 3, –2) on the line $\frac{1-x}{2} = 2 - y = z + 1$ is

(a) (1, 2, –1) (b) (2, 1, 4)
(c) (2, 1, 0) (d) (0, 0, 1) **[2012]**

13. A variable plane passes through a fixed point (1, –2, 3) and meets the coordinate axes at points *A*, *B*, *C* then the point of intersection of the planes through *A*, *B*, *C* parallel to the coordinate planes lies on:

(a) $xy - (1/2)\,yz + (1/3)\,zx = 6$
(b) $yz - 2zx + 3xy = xyz$
(c) $xy - 2yz + 3zx = 3xyz$
(d) $xy + (1/2)\,yz - (1/3)\,zx = 6$ **[2013]**

14. The angle between the lines $2x = 3y = -z$ and $-6x = y = 4z$ is:

(a) 30° (b) 45°
(c) 90° (d) 0° **[2013]**

15. If the lines $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-\lambda}{3}$ and $\frac{x}{1} = \frac{y+2}{2} = \frac{z}{4}$ intersect each other, then λ lies in the interval

(a) (–5, –3) (b) (13, 15)
(c) (11, 13) (d) (9, 11) **[2014]**

16. A variable plane is at a constant distance *p* from the origin *O* and meets the set of rectangular axes *OX* (*i* = 1, 2, 3) at points *Ai* (*i* = 1, 2, 3), respectively. If planes are drawn through A_1 A_2, A_3, which are parallel to the coordinate planes, then the locus of their point of intersection is:

(a) $\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = p$ (b) $\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} = \frac{1}{p^2}$

(c) $\frac{1}{x_1^3} + \frac{1}{x_2^3} + \frac{1}{x_3^3} = \frac{1}{p^3}$ (d) $x_1^2 + x_2^2 + x_3^2 = p^2$

[2014]

17. Let *P*(*x*, *y*, *z*) be the foot of the perpendicular from the point *Q*(0, 2, 3) on the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$. If *R* denotes the point (–3, 1, –4) and *A* denotes the area of the ΔPQR, then $2A^2$ is equal to:

(a) 798 (b) 418
(c) 399 (d) 378 **[2015]**

18. The set of all non-zero real values of *k*, for which the lines $\frac{x-4}{2} = \frac{y-6}{2} = \frac{z-8}{-2k^2}$ and $\frac{x-2}{2k^2} = \frac{y-8}{4} = \frac{z-10}{2}$ are coplanar:

(a) is an empty set.
(b) is a singleton.
(c) contains two points.
(d) contains more than two points. **[2015]**

19. The plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ and parallel to *y*-axis, also passes through the point

(a) (3, 0, 1) (b) (–3, 0, 1)
(c) (3, 0, –1) (d) (–3, 0, –1) **[2016]**

Answers

Concept-based

1. (c)	**2.** (a)	**3.** (c)	**4.** (b)
5. (c)	**6.** (b)	**7.** (d)	**8.** (c)
9. (d)	**10.** (d)	**11.** (b)	**12.** (c)
13. (b)	**14.** (c)	**15.** (c)	

Level 1

16. (b)	**17.** (a)	**18.** (c)	**19.** (c)
20. (d)	**21.** (a)	**22.** (b)	**23.** (d)
24. (a)	**25.** (c)	**26.** (c)	**27.** (d)
28. (b)	**29.** (a)	**30.** (c)	**31.** (a)
32. (b)	**33.** (b)	**34.** (a)	**35.** (c)
36. (a)	**37.** (a)	**38.** (b)	**39.** (d)
40. (b)	**41.** (c)	**42.** (a)	**43.** (c)
44. (a)	**45.** (a)	**46.** (c)	**47.** (b)
48. (b)	**49.** (d)	**50.** (c)	**51.** (d)
52. (b)	**53.** (b)	**54.** (b)	**55.** (c)
56. (b)	**57.** (c)	**58.** (a)	**59.** (c)
60. (d)	**61.** (b)	**62.** (c)	**63.** (c)
64. (b)	**65.** (c)	**66.** (c)	**67.** (a)
68. (d)	**69.** (b)	**70.** (b)	

Level 2

71. (b)	**72.** (c)	**73.** (b)	**74.** (a)
75. (d)	**76.** (d)	**77.** (d)	**78.** (b)
79. (d)	**80.** (b)	**81.** (b)	**82.** (d)
83. (a)	**84.** (b)	**85.** (a)	

Previous Years' AIEEE/JEE Main Questions

1. (a)	**2.** (d)	**3.** (c)	**4.** (b)
5. (c)	**6.** (b)	**7.** (b)	**8.** (d)
9. (a)	**10.** (a)	**11.** (b)	**12.** (c)
13. (d)	**14.** (d)	**15.** (c)	**16.** (d)
17. (a)	**18.** (d)	**19.** (b)	**20.** none
21. (a)	**22.** (d)	**23.** (a)	**24.** (d)
25. (d)	**26.** (b)	**27.** (d)	**28.** (a)
29. (a)	**30.** (d)	**31.** (b)	**32.** (b)
33. (a)	**34.** (c)	**35.** (c)	**36.** (a)
37. (d)	**38.** (b)	**39.** (a)	**40.** (c)
41. (c)	**42.** (d)	**43.** (b)	**44.** (c)
45. (c)	**46.** (a)	**47.** (a)	**48.** (c)
49. (c)	**50.** (c)	**51.** (b)	**52.** (b)
53. (c)	**54.** (b)	**55.** (c)	**56.** (d)
57. (c)	**58.** (b)	**59.** (b)	**60.** (a)
61. (c)	**62.** (a)	**63.** (d)	**64.** (c)
65. (c)	**66.** (b)	**67.** (c)	

Previous Years' B-Architecture Entrance Examination Questions

1. (a)	**2.** (c)	**3.** (a)	**4.** (a)
5. (a)	**6.** (c)	**7.** (a)	**8.** (b)
9. (a)	**10.** (d)	**11.** (b)	**12.** (c)
13. (b)	**14.** (c)	**15.** (c)	**16.** (b)
17. (c)	**18.** (a)	**19.** (a)	

Hints and Solutions

Concept-based

1. Let the coordinates of Q be (α, β, γ). OQ − to the plane. So $\frac{\alpha}{4} = \frac{\beta}{-3} = \frac{\gamma}{1} = k$. As Q lies on the plane.

   $4(4k) - 3(-3k) + 1(k) + 13 = 0 \Rightarrow k = -\frac{1}{2}$

   so coordinates of Q are $\left(-2, \frac{3}{2}, -\frac{1}{2}\right)$

   and $(QR)^2 = (-2+1)^2 + \left(\frac{3}{2}-1\right)^2 + \left(-\frac{1}{2}+6\right)^2$

   $= \frac{63}{2} \Rightarrow QR = 3\sqrt{\frac{7}{2}}$.

2. $\begin{vmatrix} -2+1 & k-1 & 0+1 \\ 2 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0 \Rightarrow k = \frac{11}{2}$

3. Let the line make an angle ϕ with the z-axis, then

   $\cos^2\theta + \cos^2\theta + \cos^2\phi = 1 \Rightarrow 2\cos^2\theta = \sin^2\phi \le 1$

   $\Rightarrow 0 \le \cos\theta \le \frac{1}{\sqrt{2}} \Rightarrow \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

4. Equation of the line is

   $$\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$$

   $$\Rightarrow \frac{x-b}{a} - 1 = \frac{y-0}{1} - 1 = \frac{z-d}{c} - 1$$

   $$\Rightarrow \frac{x-b-a}{a} = \frac{y-1}{1} = \frac{z-d-c}{c}$$

5. Equation of the line is $\frac{x+1}{1} = \frac{y}{2} = \frac{z+4}{2}$

   $$\sin\frac{\pi}{6} = \frac{2(1) + (-1)(2) + (\sqrt{\lambda})(2)}{\sqrt{1+4+4}\sqrt{4+1+\lambda}}$$

   $\Rightarrow 9(5+\lambda) = 4\left(2\sqrt{\lambda}\right)^2 \Rightarrow \lambda = \frac{45}{7}$

6. Coordinates of the centroid are $\left(\frac{6c+4}{3}, \frac{3c+2}{3}, \frac{c+2}{3}\right)$ which lies on $z = c$ so $\frac{c+2}{3} = c \Rightarrow c = 1$ and the required coordinates are $\left(\frac{10}{3}, \frac{5}{3}, 1\right)$

7. Let (p, q, r) be the foot of the perpendicular then equation of the plane is

   $p(x-\alpha) + q(y-\beta) + r(z-\gamma) = 0$

   since (p, q, r) lies on it

   $p(p-\alpha) + q(q-\beta) + r(r-\gamma) = 0$

   $\Rightarrow p^2 + q^2 + r^2 - \alpha p - \beta q - \gamma r = 0$

   Locus of (p, q, r) is

   $x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0$

   which represents neither a line nor a plane.

8. Planes given in (c) are parallel, so they do not represent a straight line.

9. Equation of the given line is $\mathbf{r} = (2\mathbf{i} + 5\mathbf{j} - \mathbf{k}) + \mathbf{t}(3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$ and of z-axis is $\mathbf{r} = p\mathbf{k}$. So the required shortest distance is

   $$\left|\frac{(2\mathbf{i}+5\mathbf{j}-\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}-5\mathbf{k})\times\mathbf{k}}{|(3\mathbf{i}+2\mathbf{j}-5\mathbf{k})\times\mathbf{k}|}\right|$$

   $$= \left|\frac{(2\mathbf{i}+5\mathbf{j}-\mathbf{k})\cdot(2\mathbf{i}-3\mathbf{j})}{\sqrt{4+9}}\right| = \frac{11}{\sqrt{13}}$$

10. Lines must be coplanar, so

$$\begin{vmatrix} 4-0 & 2+2 & \lambda-0 \\ 1 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 0 \Rightarrow \lambda = 12 \in (11,\ 13)$$

11. Equation of the plane is $a(x-3) + b(y+1) + c(z-2) = 0$

where $2a + 3b + 2c = 0$, $-4a + 0b + 4c = 0$

$\Rightarrow \dfrac{a}{3} = \dfrac{b}{-4} = \dfrac{c}{3}$. So the equation of the plane becomes $3x - 4y + 3z = 19$ and its distance from $(6, 5, 9)$ is

$$\frac{(3)(6)-(4)(5)+3(9)-19}{\sqrt{9+16+9}} = \frac{6}{\sqrt{34}} \text{ units}$$

12. $(OP)^2 = 4 + 1 + 1 = 6$ and

$$(PQ)^2 = \frac{[2(1)+1(-2)+(-1)(4)-9]^2}{1+4+16} = \frac{169}{21}$$

$$(OP)^2 + (PQ)^2 = \frac{295}{21}.$$

13. Equation of the plane is $\dfrac{x}{24} + \dfrac{y}{-12} + \dfrac{z}{8} = 1$

Sum of the intercepts = 24–12+ 8 = 20.

14. L_1: $2x + 3y + z - 3 = 0$, L_2: $2x - 3y + z + 3 = 0$

Distance of L_1 from the origin $= \left|\dfrac{-3}{\sqrt{2^2+3^2+1^2}}\right|$

Distance of L_2 from the origin $= \left|\dfrac{3}{\sqrt{2^2+(-3)^2+1^2}}\right|$

15. Equation of the plane is

$3x + 4y - 12z + 13 = 0$

$|3 + 4 - 12p + 13| = |-9 - 12 + 13|$

$\Rightarrow 20 - 12p = \pm 8 \Rightarrow p = 1, \dfrac{7}{3}$

Level 1

16. Coordinates are $\left(\dfrac{-1-1}{2}, \dfrac{-1+1}{2}, \dfrac{1-1}{2}\right) = (-1, 0, 0)$
17. See the text.
18. In this plane z coordinates is 0.
19. For a point on z-axis; $x = 0$, $y = 0$.
20. Direction ratios of x, y, z axes are respectively 1, 0, 0; 0, 1, 0 and 0, 0, 1.
21. $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2 = 0$ if $\theta = \pi/2$.
22. $OP^2 = 0 + 5^2 + 6^2 = 61$, $OQ^2 = 66$, $OR^2 = 62$, $OS^2 = 61$.
23. $3(\lambda - 1)^2 = 27 \Rightarrow \lambda = -2$ or 4.
24. Length of the sides are $\dfrac{(8!)(3!)}{10!} = \dfrac{1}{15}$.
25. See the text.
26. $\cos\alpha = 2/\sqrt{62}$, $\cos\beta = -3/\sqrt{62}$, $\cos\gamma = 7/\sqrt{62}$
27. Equation of the line is $\dfrac{x-1}{3-1} = \dfrac{y+1}{4+1} = \dfrac{z-6}{-7-6}$
28. Any point on the first line is $(3r + 1, -r + 1, -1)$ and a point on the second line is $(2k + 4, 0, 3k - 1)$. The two points are same if $r = 1$ and $k = 0$ and the point of intersection is $(4, 0, -1)$.
29. $\sin\theta = \dfrac{(3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}).(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\sqrt{3^2+(-4)^2+(5)^2}\left(\sqrt{1+1+1}\right)}$

$= \dfrac{3-4+5}{\sqrt{150}} = \dfrac{2\sqrt{6}}{15}$.

30. Equation of a plane containing x-axis whose equation is $\dfrac{x-0}{1} = \dfrac{y-0}{0} = \dfrac{z-0}{0}$ is

$A(x-0) + B(y-0) + C(z-0) = 0$

where $A \times 1 + B \times 0 + C \times 0 = 0$

$\Rightarrow By + Cz = 0$, since it passes through (1, 1,1)

$B + C = 0$

$\Rightarrow$ equation of the plane is $y - z = 0$

31. Let the foot of the perpendicular be $(2r, 3r + 2, 4r + 3)$ direction ratios of the perpendicular are $2r - 3$, $3r + 3$, $4r - 8$ so $2(2r - 3) + 3(3r + 3) + 4(4r - 8) = 0 \Rightarrow r = 1$ and the foot is $(2, 5, 7)$.
32. The required length is $\sqrt{(12)^2 + (4)^2 + (3)^2} = 13$.
33. $3\cos^2\theta = 1 \Rightarrow \cos\theta = \pm 1\sqrt{3}$. So the direction ratios of the required lines are $(1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$ which gives four lines.
34. Equation of the plane is $\mathbf{r} = \mathbf{a} + t\mathbf{b} + p\mathbf{c}$.

$\Rightarrow \mathbf{r}.\mathbf{b}\times\mathbf{c} = \mathbf{a}.\mathbf{b}\times\mathbf{c} + t\mathbf{b}.\mathbf{b}\times\mathbf{c} + p\mathbf{c}.\mathbf{b}\times\mathbf{c}$

$\Rightarrow [\mathbf{r}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

35. The lines are $\dfrac{x-1}{3} = \dfrac{y-1}{-1} = \dfrac{z+1}{0}$ and

$\dfrac{x-4}{2} = \dfrac{y-0}{0} = \dfrac{z+1}{3}$ For all points on the first line

$z = -1$, $\Rightarrow p - 5 = -1 \Rightarrow p = 4$.

36. Line of intersection is parallel to the vector $l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ where $3l - m + n = 0$, $l + 4m - 2n = 0$

$\Rightarrow \dfrac{l}{-2} = \dfrac{m}{7} = \dfrac{n}{13}$.

37. Equation of the plane is $A(x + 1) + B(y - 3) + C(z + 2) = 0$, where $-3A + 2B + C = 0$ and $A + B(7 - 3) + C(-7 + 2) = 0$.

38. Equation of the line is $\dfrac{x-2}{8-2}=\dfrac{y+3}{-1+3}=\dfrac{z+1}{2+1}$. Any point on this at a distance r from A is $((6/7)\ r+2,\ (2/7)\ r-3,\ (3/7)\ r-1)$, so the required point is for $r=\pm 14$, and we get $(14, 1, 5)$ or $(-10, -7, -7)$, the one nearer the origin is $(-10, -7, -7)$.
39. $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$, $D(a/2, b/2, 0)$, $E(a/2, 0, c/2)$ so mid-point of DE is $(a/2, b/4, c/4)$.
40. Lines intersect if $\mathbf{a}+\lambda\ (\mathbf{b}\times\mathbf{c})=\mathbf{b}+\mu\ (\mathbf{c}\times\mathbf{a})$ or if $\mathbf{a.c}+\lambda\ (\mathbf{b}\times\mathbf{c})\ .\ \mathbf{c}=\mathbf{b.c}+\mu\ (\mathbf{c}\times\mathbf{a})\ .\ \mathbf{c}$ or if $\mathbf{a.c}=\mathbf{b.c}$.
41. $d=$

$$\frac{[(4\mathbf{i}-\mathbf{j})-(\mathbf{i}-\mathbf{j}+2\mathbf{k})].(\mathbf{i}+2\mathbf{j}-3\mathbf{k})\times(2\mathbf{i}+4\mathbf{j}-5\mathbf{k})}{|(\mathbf{i}+2\mathbf{j}-3\mathbf{k})\times(2\mathbf{i}+4\mathbf{j}-5\mathbf{k})|}$$

$$=\frac{(3\mathbf{i}-2\mathbf{k}).(2\mathbf{i}-\mathbf{j})}{|2\mathbf{i}-\mathbf{j}|}=\frac{6}{\sqrt{5}}.$$

42. Line is $\dfrac{x-5}{4/\sqrt{26}}=\dfrac{y}{1/\sqrt{26}}=\dfrac{z+6}{3/\sqrt{26}}$. A point on this line at a distance $3\sqrt{26}$ from $(5, 0, -6)$ is $(5\pm(3\times 4), \pm 3, -6\pm 9)=(17, 3, 3)$ or $(-7, -3, -15)$.
43. Direction ratios of the line are $1, -1, 1$ and that of the normal to the plane are $2, -1, 1$,

so $\sin\theta=\dfrac{2+1+1}{\sqrt{3}\sqrt{6}}=\dfrac{4}{\sqrt{18}}$, $\cos\theta=\dfrac{\sqrt{2}}{\sqrt{18}}$.

$\sin\theta+\sqrt{2}\ \cos\theta=\dfrac{4}{\sqrt{18}}+\dfrac{2}{\sqrt{18}}=\dfrac{6}{\sqrt{18}}=\sqrt{2}$

44. Let the equation of the plane be $lx+my+mz+d=0$, $3l+5m+7n=0$ and $l+4m+7n=0$.
$\Rightarrow$ $l/1=m/-2=n/1$ and the equation of the plane is $x-2y+z+d=0$ since it passes through $(-1, -3, -5)$ and $(2, 4, 6)$, $d=0$ and the plane passes through $(0, 0, 0)$.
45. Equation of the plane is $\dfrac{x+2y+2z-19}{\sqrt{1+4+4}}$ $=\pm\dfrac{4x-3y+12z+3}{\sqrt{16+9+144}}$ since it passes through the vector $\mathbf{i}+7\mathbf{j}-\mathbf{k}$ or the point $(1, 7, -1)$, $\dfrac{1+14-2-19}{3}$ $=\pm\dfrac{4-21-12+3}{13}$ which is true for the + ve sign.

Hence the required plane is $x+35y-10z-256=0$ or $\mathbf{r}.(\mathbf{i}+35\mathbf{j}-10\mathbf{k})-256=0$.
46. $b\times 1+2\times 3+3\times(-1)-4=0$
$\Rightarrow b=1$
$2b+2a+9=0$
$\Rightarrow a=-11/2$.
47. Let the plane be $\mathbf{r}.(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=d$.
$x+y-2z=2x-y+z=x+2y+z=d$.
$\Rightarrow$ $\dfrac{x}{9}=\dfrac{y}{3}=\dfrac{z}{-1}$ and $d=14$.
48. Only the point $(-1, -1, -1)$ in (b) lies on the first line.
49. Equation of the plane is $\mathbf{r}.\dfrac{\mathbf{n}}{|\mathbf{n}|}=\dfrac{q}{|\mathbf{n}|}$, $\dfrac{\mathbf{n}}{|\mathbf{n}|}$ is a unit vector, So the required length $=q/|\mathbf{n}|$.
50. Let $P(a, b, c)$, d.r. of the normal to the plane are a, b, c and as it passes through (a, b, c) its equation is $a(x-a)+b(y-b)+c(z-c)=0$
51. Equation of the plane be $a(x+2)+b(y+2)+c(z-2)=0$. As it passes through $(1, 1, 1)$ and $(1, -1, 2)$, $\dfrac{a}{1}=\dfrac{b}{-3}=\dfrac{c}{-6}$. Equation of the plane is $\dfrac{x}{-8}+\dfrac{y}{8/3}+\dfrac{z}{8/6}=1$ and the required sum $=12$.
52. Equation of the line is $\mathbf{r}=2\mathbf{i}-3\mathbf{j}+4\mathbf{k}-\lambda\ (3\mathbf{i}+4\mathbf{j}-5\mathbf{k})$ so the line passes through $(2, -3, 4)$ and the direction ratios are $3, 4, -5$ and so its equation is $\dfrac{x-2}{3}=\dfrac{y+3}{4}=\dfrac{z-4}{-5}$
$\Rightarrow$ $4x-3y=17$, $5y+4z=1$
53. Since $\begin{vmatrix}2-1 & -1+1 & 0\\ 2 & 0 & 1\\ 1 & 1 & -1\end{vmatrix}=-1\neq 0$, the lines do not intersect.
54. Lengths of the sides are $\sqrt{(2-1)^2+(3+2)^2+(-4-3)^2}=\sqrt{75}$, $\sqrt{75}$ and $\sqrt{300}$. So the triangle is isosceles.
55. Direction ratios of the perpendicular are $r-a$, $r-b$, $r-c$ and those of the line are $1, 1, 1$. So $1.(r-a)+1.(r-b)+1.(r-c)=0$
$\Rightarrow$ $3r=a+b+c$.
56. $\dfrac{-\lambda s-q}{p}=\dfrac{-\lambda s-q}{\lambda r}$ $\Rightarrow$ $p=\lambda r\Rightarrow\lambda=p/r$.
57. If l, m, n are the direction ratios of the line then $3l-6m-2n=0$ and $2l+m-2n=0$
$\Rightarrow$ $\dfrac{l}{14}=\dfrac{m}{2}=\dfrac{n}{15}$

Equation of the line is

$\dfrac{x-\alpha}{14}=\dfrac{y-\beta}{2}=\dfrac{z-\gamma}{15}$

where (α, β, γ) lies on the line. only $(\alpha, \beta, \gamma)=(3, -1, 0)$ satisfies the condition.

58. Let $P(3r_1 + 5, -r_1 + 7, r_1 - 2)$ and $Q(-3r_2 - 3, 2r_2 + 3, 4r_2 + 6)$, direction ratios of PQ are 2, 7, – 5

$$\text{So } \frac{3r_1+3r_2+8}{2} = \frac{-r_1-2r_2+4}{7} = \frac{r_1-4r_2-8}{-5}$$

$$\Rightarrow \qquad r_1 = r_2 = -1$$

$P(2, 8, -3), Q(0, 1, 2) \Rightarrow PQ = \sqrt{78}$.

59. Let the ratio be $m : n$, then x-coordinate of the point of division is zero

$$\Rightarrow \qquad \frac{-2m+3n}{m+n} = 0 \Rightarrow \frac{m}{n} = \frac{3}{2}.$$

60. Direction ratios of the lines are 3, –2, 0 and 1, 3/2, 2; $3 \times 1 - 2 \times \frac{3}{2} + 0 \times 2 = 0$

61. Coordinate of A are $(3a, 0, 0)$ $B(0, 3b, 0)$ and $C(0, 0, 3c)$ so the centroid of the triangle ABC is $(a, b, c) = (2, 4.\ 8)$

62. Plane passes through the line

$$x + 3y + z - 4 = 2x - y = 0$$

$$\text{or } \frac{x}{1} = \frac{y}{2} = \frac{z-4}{-7}$$

63. Equation of the plane is

$$\frac{x}{28} - \frac{y}{21} + \frac{z}{12} = 1$$

Required sum = 28 – 21 + 12 = 19.

64. As the triangle ABC is equilateral the circumcentre is (2/3, 2/3, 2/3), the centroid of the triangle and hence the radius $= \sqrt{(1-2/3)^2+(1-2/3)^2+(2/3)^2}$ $= \sqrt{2/3}$

Statement-1 is thus true.

Statement-2 is also true as the centre of the circumcircle lies on the plane $x + y + z - 2 = 0$ through A, B, C.

65. Any point on L is $P(r, 2r, 3r)$

$$(OP)^2 = 14 = r^2 + 4r^2 + 9r^2 \Rightarrow r^2 = 1$$

$$(PN)^2 = \left(\frac{r+2r+3r}{\sqrt{1+1+1}}\right)^2 = \frac{36}{6} = 12$$

$$(ON)^2 = OP^2 - PN^2 = 2.$$

⇒ Statement-1 is true.

Let $R(r, 2r, 3r)$

The $\left|\frac{r+2r+3r}{\sqrt{1+1+1}}\right| = \sqrt{3} \Rightarrow r = \pm 1/2$

So statement-2 is false.

66. L_1 and L_2 are parallel to the vectors $\mathbf{a} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ respectively. The vector perpendicular to both L_1 and L_2 is $\mathbf{a} \times \mathbf{b}$ and the required unit vector is $\frac{-\mathbf{i} - 7\mathbf{j} + 5\mathbf{k}}{\sqrt{1+49+25}}$, so statement-1 is true. In statement-2, equation of the plane is $-(x + 1) - 7(y + 2) + 5(z + 1) = 0$ whose distance from (1, 1, 1) is $13/5\sqrt{3}$, so the statement-2 is false.

67. Statement-2 is true, in statement-1, line is parallel to the vector $\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and the normal to the plane is parallel to $\mathbf{i} + 5\mathbf{j} + \mathbf{k}$. Since $1 - 5 + 4 = 0$, the two vectors are perpendicular so the line is parallel to the plane and using. Statement-2, required distance is $\left|\frac{2\times1+2\times5+3\times1-5}{\sqrt{1+25+1}}\right| = \frac{10}{3\sqrt{3}}$ and the statement-1 is also true.

68. Direction ratios of the line in statement-1 are 1, 2, 3 and of the plane are 2, 3, 1 so their direction cosines are different and the statement-1 is false. In statement-2, direction cosines of the normal to the plane are cos $\pi/4$, cos $\pi/4$,cos $\pi/2$ i.e. $1/\sqrt{2}$, $1/\sqrt{2}$, 0; so the equation of the plane is $\left(1/\sqrt{2}\right)x + \left(1/\sqrt{2}\right)y + 0z = \sqrt{2}$ or $x + y = 2$, statement-2 is true.

69. $\begin{vmatrix} 2-1 & 2-0 & 2+1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 0 \Rightarrow L_1$ and L_2 are coplanar so statement-2 is true.

L_1 and L_2 intersect at (1, 1, 1) so the shortest distance is zero and the statement-1 is also true but statement-2 does not explain the truth of statement-1.

70. Any point on L is $(r + 4, r + 2, 2r + 7)$

since $2(r + 4) - 4(r + 2) + (2r + 7) = 7$ for all values of r. L lies in the plane P. so statement-1 is true

Statement-2 is also true, which shows the line L is parallel to the plane P, so does not explain statement-1.

Level 2

71. Any point on the first line is

$(r_1a' + a, r_1b' + b, r_1c' + c)$

and on the second line is

$(r_2a + a', r_2b + b', r_2c + c')$

The two points are same if $r_1 = r_2 = 1$

72. Let the mirror image of $A(3, -2, 1)$ be $B(\alpha, \beta, \gamma)$ then AB is perpendicular to the given plane and the mid point of AB lies on the plane.

So $3\alpha - \beta + 4\gamma + 11 = 0$

and $\dfrac{\alpha-3}{3} = \dfrac{\beta+2}{-1} = \dfrac{\gamma-1}{4} = r$ (say)

Solving we get $r = -1$ and the required point is $(0, -1, -3)$

73. Equations of the lines in the polar form are

$\mathbf{r} = -\,a\mathbf{i} + \lambda\,(12\mathbf{i} + 6\,\mathbf{j} - \mathbf{k})$

and $\mathbf{r} = (-\,2a\mathbf{j} + a\mathbf{k}) + t(6\mathbf{i} + 6\mathbf{j} + \mathbf{k})$

Required shortest distance is

$$\frac{(a\,\mathbf{i} - 2\,a\,\mathbf{j} + a\,\mathbf{k})\cdot(12\,\mathbf{i} + 6\,\mathbf{j} - \mathbf{k})\times(6\,\mathbf{i} + 6\,\mathbf{j} + \mathbf{k})}{|(12\,\mathbf{i} + 6\,\mathbf{j} - \mathbf{k})\times(6\,\mathbf{i} + 6\,\mathbf{j} + \mathbf{k})|}$$

(using the shortest distance between the lines

$\mathbf{r} = \mathbf{a} + t\,\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \lambda\,\mathbf{d}$ is $\dfrac{(\mathbf{c}-\mathbf{a})\cdot\mathbf{b}\times\mathbf{d}}{|\mathbf{b}\times\mathbf{d}|}$)

$$= \begin{vmatrix} a & -2a & a \\ 12 & 6 & -1 \\ 6 & 6 & 1 \end{vmatrix} = 12a - 18\,(-2a) + 36a = 84a$$

and $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 6 & -1 \\ 6 & 6 & 1 \end{vmatrix} = |12\mathbf{i} - 18\mathbf{j} + 36\mathbf{k}|$

$= 6\sqrt{4+9+36} = 42$

Hence the required distance is $2a$.

74. $(PM)^2 = (OP)^2 - (OM)^2 = u^2 + v^2 + w^2 - 3\alpha^2$

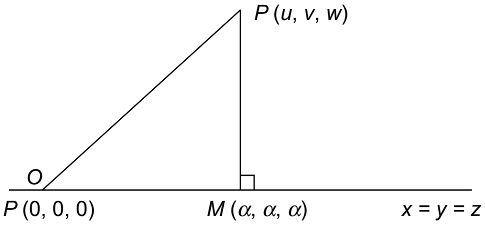


Fig. 21.5

Also PM is perpendicular to the line $x = y = z$

so $u - \alpha + v - \alpha + w - \alpha = 0 \Rightarrow u + v + w = 3\alpha.$

Now $u^2 + v^2 + w^2 - 3\alpha^2 = \dfrac{4}{3}\,[u + v + w - 1]^2$

$\Rightarrow 3(u^2 + v^2 + w^2) - (u + v + w)^2$

$= 4[(u + v + w)^2 - 2\,(u + v + w) + 1]$

$\Rightarrow 2(u^2 + v^2 + w^2) - 8(u + v + w) + 4 = 0$

$\Rightarrow u^2 + v^2 + w^2 - 4(u + v + w) = -2.$

75. Let the coordinates of the vertices of the square of side 1 unit be $A(0, 0, 0)$, $B(1, 0, 0)$

$C(1, 1, 0)$, $D(0, 1, 0)$

$A'(0, 0, 1)$, $B'(1, 0, 1)$

$C'(1, 1, 1)$, $D'(0, 1, 1)$

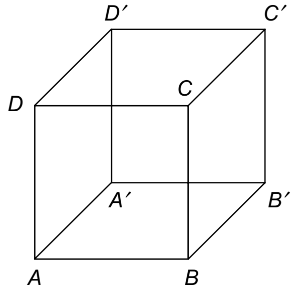


Fig. 21.6

Direction ratios of the diagonals AC' and BD' are 1, 1, 1 and –1, 1, 1

The angle θ between these diagonals is given by

$\cos\theta = \dfrac{1\,(-1) + (1)\,(1) + 1\,(1)}{\sqrt{1+1+1}\,\sqrt{1+1+1}} = \dfrac{1}{3}$

$\Rightarrow \theta = \cos^{-1}(1/3)$

76. Let the vertices of the triangle be

$A_i\ (x_i, y_i, z_i),\ i = 1, 2, 3$

then $\dfrac{x_1 + x_2}{2} = 2,\ \dfrac{x_2 + x_3}{2} = 0,\ \dfrac{x_3 + x_1}{2} = 5$

$\Rightarrow x_1 + x_2 + x_3 = 7$

$\Rightarrow x_1 = 7,\ x_2 = -\,3,\ x_3 = 3$

Similarly $y_1 = 2,\ y_2 = 4,\ y_3 = 12$

$z_1 = 5,\ z_2 = -7,\ z_3 = 17$

The point farthest from the origin is (3, 12, 17) and the required square of the distance is

$3^2 + 12^2 + 17^2 = 442.$

77. Let the direction ratios of the line be l, m, n. The direction ratios of the four diagonals are 1, ± 1, ± 1.

So $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$= \dfrac{1}{3\,(l^2 + m^2 + n^2)}\ [(l + m + n)^2 + (-\,l + m + \mathrm{n})^2$
$+ (l - m + n)^2 + (l + m - n)^2]$

$= \dfrac{4\,(l^2 + m^2 + n^2)}{3\,(l^2 + m^2 + n^2)} = \dfrac{4}{3}.$

78. Vertices of the tetrahedron are

(0, 0, 0), (20, 0, 0), (0, 15, 0) and (0, 0, 12) so its volume is $\frac{1}{6}\left|\begin{vmatrix}0 & 0 & 0 & 1\\ 20 & 0 & 0 & 1\\ 0 & 15 & 0 & 1\\ 0 & 0 & 12 & 1\end{vmatrix}\right| = 600$

cubic units.

79. Equation of the plane containing the triangle is $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{r} = 1$ which meets the coordinates axes at $P(\alpha, 0, 0,)$, $Q(0, \beta, 0)$, $r(0, 0, \gamma)$

P, Q, R are the midpoints of the sides of the triangle ABC. Centroid of the ΔABC is the centroid of the ΔPQR and hence the required coordinates are $(\alpha/3, \beta/3, \gamma/3)$

80. Let the coordinates of the moving point P be (x, y, z), then the distances of P from the six faces $x = \pm 1$, $y = \pm 1$, $z = \pm 1$ are

$1 + x, 1 - x, 1 + y, 1 - y, 1 + z, 1 - z$

$\Rightarrow 2(1 + x^2 + 1 + y^2 + 1 + z^2) = 10$

$\Rightarrow x^2 + y^2 + z^2 = 2.$

81. Let the equation of the required plane be

$(5x - 4y - 2z + 1) + \lambda(2x - y + 3z + 5) = 0$

Since this is perpendicular to the plane

$2x - y + 3z + 5 = 0$

$2(5 + 2\lambda) + (4 + \lambda) -3(2 - 3\lambda) = 0$

$\Rightarrow \lambda = - 4/7$ and the required equation is

$27x - 24y - 26z - 13 = 0$

82. Let the equation of the plane be

$l(x -1) + m(y + 2) + n(z - 1) = 0$

so $2l - 2m + n = 0$, $l - m + 2n = 0$

$\Rightarrow \frac{l}{-3} = \frac{m}{-3} = \frac{n}{0}$ and the equation of the plane is $x + y + 1 = 0$ and the required distance is $\frac{1+2+1}{\sqrt{1+1}} = 2\sqrt{2}$

83. Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$

so $\left|\frac{-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| = 1 \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = 1$

Centroid of the $\Delta ABC = (a/3, b/3, c/3) = (x, y, z)$

$\Rightarrow$ so $x^{-2} + y^{-2} + z^{-2} = k \Rightarrow k = 9.$

84. (See Text) The line of intersection of the given planes is

$$\frac{x-\frac{30+15}{3+12}}{12+2} = \frac{y-\frac{-30+15}{3+12}}{-4+6} = \frac{z}{3+12} = t \text{ (say)}$$

$\Rightarrow x = 14t + 3$, $y = 2t -1$, $z = 15t$

85. Equation of a plane through the intersection of the planes $8x - y - 7z = 8$ and $x + y + z - 1 = 0$ is $x + y + z - 1 + \lambda(8x - y - 7z - 8) = 0$

If this plane is perpendicular to the plane

$5x - 4y - z = 5$, then

$5(1 + 8\lambda) + (1 - \lambda)(- 4) - (1 - 7\lambda) = 0 \Rightarrow \lambda = 0$

and the equation of the plane is $x + y + z - 1 = 0$

The required line is now the line of intersection of the planes $x + y + z - 1 = 0$ and $5x - 4y - z - 5 = 0$

Let the equation of this line be

$$\frac{x-1}{l} = \frac{y-0}{m} = \frac{z-0}{n}$$

Then $l + m + n = 0$

$5l - 4m - n = 0$

$\Rightarrow \frac{l}{-1+4} = \frac{m}{5+1} = \frac{n}{-4-5}$

and the required line is

$$\frac{x-1}{1} = \frac{y-0}{2} = \frac{z-0}{-3}$$

Previous Years' AIEEE/JEE Main Questions

1. Let the equation of the plane be $lx + my + nz = 1$, l, m, n being the d. r. of the normal to the plane. Since it passes through (1, 0, 0) and (0, 1, 0), $l = m = 1$.

Also it makes are an angle $\pi/4$ with the plane $x + y = 3$

$\Rightarrow \frac{l+m}{\sqrt{l^2+m^2+n^2}\sqrt{2}} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\Rightarrow l^2 + m^2 + n^2 = (l + m)^2$

$\Rightarrow n^2 = 2 \Rightarrow n = \pm \sqrt{2}$

So the required d. r. are 1, 1, $\sqrt{2}$.

2. Planes given in (*a*) and (c) do not pass through (3, 2, 0) and the plane in (*b*) does not contain the line with direction ratios 1, 5, 4.

3. See Example 42 page 20.14.

4. Centre of the sphere is $(-2, 1, 3)$ and radius is 13.

   Distance of the centre from the plane is $\left|\dfrac{-24+4+9-327}{\sqrt{144+16+9}}\right| = 26.$

   Hence the shortest distance of the sphere from the plane is $26 - 13 = 13$.

5. Equations of the given lines can be written as

   $$\frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c} \text{ and } \frac{x-b'}{a'} = \frac{y}{1} = \frac{z-d'}{c'}$$

   Which are perpendicular if

   $aa' + 1 + cc' = 0$

6. Lines are coplanar if

   $$\begin{vmatrix} 2-1 & 3-4 & 4-5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

   $\Rightarrow k^2 + 3k = 0$

   $\Rightarrow k = 0$ or -3

7. Centre of the sphere is $(-1, 1, 2)$ and radius $= 5$

   Perpendicular distance of the plane from the centre is $\dfrac{-1+2+4+7}{\sqrt{1+4+4}} = 4.$

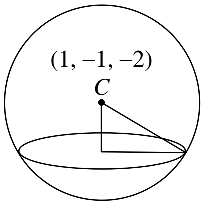


Fig. 21.7

   Radius of the required circle $= \sqrt{5^2 - 4^2} = 3$

8. Let l, m, n be the d. r. of the face OAB and its equation is

   $lx + my + nz = p.$

   Since it passes through O $(0, 0, 0)$, $p = 0$ and $A(1, 2, 1)$ and B $(2, 1, 3)$ lie on it.

   So $l + 2m + n = 0$

   $2l + m + 3n = 0$

   $$\Rightarrow \frac{l}{5} = \frac{m}{-1} = \frac{n}{-3}$$

   Next, let l', m', n', be the d. r. of the face ABC and its equation is $l'x + m'y, + n'z = p'$

   It passes through A, B, C.

   So $l' + 2m' + n' = p'$

   $2l' + m' + 3n' = p'$

   $-l' + m' + 2n' = p'.$

   Solving we get,

   $$\frac{l'}{1} = \frac{m'}{-5} = \frac{n'}{-3}$$

   If θ is the required angle

   $$\text{then } \cos\theta = \frac{ll' + mm' + nn'}{\sqrt{l^2+m^2+n^2}\sqrt{l'^2+m'^2+n'^2}}$$

   $$= \frac{5+5+9}{5^2+1^2+3^2} = \frac{19}{35}$$

9. Direction cosines of the line are

   $\cos\theta$, $\cos\beta$, $\cos\theta$.

   $\therefore \cos^2\theta + \cos^2\beta + \cos^2\theta = 1$

   $\Rightarrow 2\cos^2\theta + 1 - \sin^2\beta = 1$

   $\Rightarrow 2\cos^2\theta - 3\sin^2\theta = 0 \Rightarrow 5\cos^2\theta = 3$

   $\Rightarrow \cos^2\theta = 3/5$

10. Distance between two parallel planes

    $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$

    is $\dfrac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$

    Thus, distance between $2x + y + 2z - 8 = 0$ and $2x + y + 2z + 5/2 = 0$ is

    $$\frac{|-8-5/2|}{\sqrt{2^2+1^2+2^2}} = \frac{7}{2}$$

11. Coordinate of point A on L_1 are given by $(r_1, r_1 - a, r_1)$ and B on L_2 is $(2r_2 - a, r_2, r_2)$. Direction ratio of AB are $2r_2 - a - r_1$, $r_2 - r_1 + a$, $r_2 - r_1$

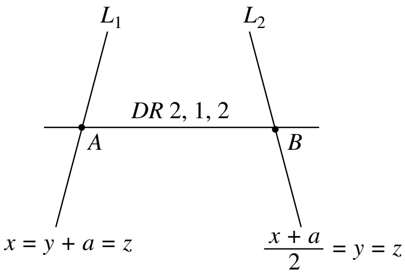


Fig. 21.8

Thus,

$$\frac{2r_2 - r_1 - a}{2} = \frac{r_2 - r_1 + a}{1} = \frac{r_2 - r_1}{2}$$

$\therefore\ r_2 = a$ and $r_1 = 3a$.

Hence, coordinate of the point of intersection are $(3a, 2a, 3a)$ and (a, a, a).

12. As the straight lines

$$\frac{x-1}{1} = \frac{y+3}{-\lambda} = \frac{z-1}{\lambda}$$

and $\dfrac{x-0}{1/2} = \dfrac{y-1}{1} = \dfrac{z-2}{-1}$

are coplanar

$$\begin{vmatrix} 1-0 & -3-1 & 1-2 \\ 1 & -\lambda & \lambda \\ 1/2 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & -5 & -1 \\ 1 & 0 & \lambda \\ 1/2 & 0 & -1 \end{vmatrix} = 0$$

$$\Rightarrow 5\begin{vmatrix} 1 & \lambda \\ 1/2 & -1 \end{vmatrix} = 0 \Rightarrow -1 - \frac{\lambda}{2} = 0$$

$\Rightarrow \lambda = -2$

13. The required plane is

$S_1 - S_2 = 0$

$\Rightarrow -10x + 5y + 5z = -5$

or $2x - y - z = 1$

14. The lines can be written as

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{-6}$$

and

$$\frac{x}{2} = \frac{y}{-12} = \frac{z}{-3}$$

Since $(3)(2) + (2)(-12) + (-6)(-3) = 0$,

we get the angle between two lines is 90°.

15. We have

$$\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Rightarrow \frac{1}{3} = \frac{(1)(2) + (2)(-1) + (2)(\sqrt{\lambda})}{\sqrt{1+4+4}\sqrt{4+1+\lambda}}$$

$$\Rightarrow \sqrt{5+\lambda} = 2\sqrt{\lambda}$$

$\Rightarrow 5 + \lambda = 4\lambda \Rightarrow \lambda = 5/3$.

16. As line is parallel to the plane, distance of line from the plane

= distance of a point from the plane

$$= \frac{|(2\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k}) - 5|}{\sqrt{1+25+1}}$$

$$= \frac{|2-10+3-51|}{\sqrt{27}} = \frac{10}{3\sqrt{3}}$$

17. Centres of two spheres are $C_1\ (-3, 4, 1)$ and $C_2(5, -2, 1)$. Mid-point of C_1C_2 is $M(1, 1, 1)$.

As M lies on $2ax - 3ay + 4az + 6 = 0$, we get $2a - 3a + 4a + 6 = 0$ or $a = -2$.

18. Centre of sphere is $C\ (1/2, 0, -1/2)$ and radius is given by

$$\sqrt{\frac{1}{4} + 0 + \frac{1}{4} + 2} = \sqrt{\frac{5}{2}}$$

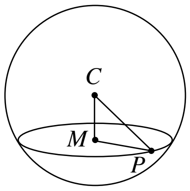


Fig. 21.9

CM = perpendicular distance of the plane $x + 2y - z - 4 = 0$ from $C(1/2, 0, -1/2)$

$$= \frac{\left|\frac{1}{2} + 2(0) + \frac{1}{2} - 4\right|}{\sqrt{1+4+1}} = \frac{3}{\sqrt{6}} = \sqrt{\frac{3}{2}}$$

Now, $MP^2 = CP^2 - CM^2$

$$= \frac{5}{2} - \frac{3}{2} = 1$$

$\Rightarrow$ radius of circle = 1.

19. See Question No. 5.

20. Note: None of the given choices is correct. Required image lies on the line through $A(-1, 3, 4)$ and perpendicular to $x - 2y = 0$, that is on the line

$$\frac{x+1}{1} = \frac{y-3}{-2} = \frac{z-4}{0} = t\text{(say)}$$

$\Rightarrow x = t - 1,\ y = 3 - 2t,\ z = 4$.

Let the coordinates of the image point be $B(t - 1, 3 - 2t, 4)$. Mid-point M of AB is $\left(\frac{1}{2}t - 1, 3 - t, 4\right)$ and it must lie on the plane $x - 2y = 0$.

$\Rightarrow \frac{1}{2}t - 1 - 6 + 2t = 0 \Rightarrow t = \frac{14}{5}$

Required image is $\left(\frac{9}{5}, -\frac{13}{5}, 4\right)$

None of the answer matches with the given choices.

However, if equation of the plane is taken as $x + 2y = 0$, then the answer matches with choice (c).

21. Let l, m, n be the direction cosines of the line, then

$2l + 3m + n = 0$

and $l + 3m + 2n = 0$

$\Rightarrow \frac{l}{6-3} = \frac{m}{1-4} = \frac{n}{6-3}$

$\Rightarrow l = -m = n \Rightarrow l : m : n = \frac{1}{\sqrt{3}} : -\frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}}$

$\therefore \cos\alpha = \frac{1}{\sqrt{3}}$

22. $l = \frac{1}{\sqrt{2}}$, $m = \frac{1}{\sqrt{2}}$

$l^2 + m^2 + n^2 = 1 \Rightarrow n = 0 \Rightarrow \gamma = \frac{\pi}{2}$ (the required angle)

23. Centre of sphere is (3, 6, 1)

Let coordinates of other end point be (a, b, c), then

$\frac{a+2}{2} = 3, \frac{b+3}{2} = 6, \frac{c+5}{2} = 1$

$\Rightarrow a = 4, b = 9, c = -3$

Thus coordinates of other end point are (4, 9, –3).

24. As $\vec{a}, \vec{b}, \vec{c}$ are coplanar

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x-2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ x & -2 & -1-x \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -2 & 1 \\ -2 & -1-x \end{vmatrix} = 0$$

$\Rightarrow x = -2$

25. Equation of the line is

$\frac{x-5}{5-3} = \frac{y-1}{1-b} = \frac{z-a}{a-1}$

which passes through $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$

So $\frac{-5}{2} = \frac{15/2}{1-b} = \frac{-\frac{13}{2} - a}{a-1}$

$\Rightarrow b = 4$ and $a = 6$.

26. Any point on the two lines are respectively ($kr + 1$, $2r + 2$, $3r + 3$), ($3r' + 2$, $kr' + 3$, $2r' + 1$)

If they represent the same point.

$kr + 1 = 3r' + 2$

$2r + 2 = kr' + 3$

$3r + 3 = 2r' + 1$

Eliminating r and r^1 we get

$(k + 5)\, r = (k + 5)\, r'$

$\Rightarrow k = -5$, is the integral value of k.

27. As $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$, the point (2, 1, –2) lies in the plane and normal to the plane is perpendicular to the line.

Thus,

$2 + 3(1) - \alpha(-2) + \beta = 0$

and $(3)(1) + (3)(-5) + (-\alpha)(2) = 0$

$\Rightarrow 2\alpha + \beta = -5, 2\alpha = -12$

$\Rightarrow \alpha = -6, \beta = 7$

28. Here $a = 6$, $b = -3$, $c = 2$

$\therefore$ direction cosines are given by

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$= \pm\frac{6}{7}, \mp\frac{3}{7}, \pm\frac{2}{7}$$

29. Here $\alpha = 45°$, $\beta = 120°$ and $\gamma = \theta$

Since

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow (\cos 45°)^2 + [\cos(180° - 60°)]^2 + \cos^2\theta = 1$

$\Rightarrow \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{2}\right)^2 + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$

$\Rightarrow \cos\theta = \frac{1}{2}$ [$\because \theta$ is acute]

$\Rightarrow \theta = 60°$

30. Direction ratios of AB are

$1 - 3,\ 3 - 1,\ 4 - 6$ or $1, -1, 1$

$\Rightarrow AB$ is normal to the plane $x - y + z = 5$.

Also, the mid point $M\,(2, 2, 5)$ of AB lies on $x - y + z = 5$ since $2 - 2 + 5 = 5$

Thus, the plane $x - y + z = 5$ bisects the line segment AB.

As $x - y + z = 5$ is the perpendicular bisector of the segment AB $A(3, 1, 6)$ is the mirror image of $B(1, 3, 4)$ in the plane $x - y + z = 5$.

31. Mid point of the segment AB is

$\left(\frac{1+1}{2}, \frac{0+6}{2}, \frac{7+3}{2}\right) = (1, 3, 5).$

It lies on $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Direction ratios of AB are

$1 - 1,\ 6 - 0,\ 3 - 7.$ or $0, 6, -4,$

As $(1)(0) + (2)(6) + (3)(-4) = 0,$

we get AB is perpendicular to the straight line

$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

Although both statement-1 and statement-2 are true, statement-2 is not a correct explanation for the statement-1.

32. If θ is angle between the line and the plane, then

$$\sin\theta = \frac{(1)(1) + (2)(2) + 3(\lambda)}{\sqrt{1+4+\lambda^2}\sqrt{1+4+9}} = \frac{5+3\lambda}{\sqrt{14}\sqrt{5+\lambda^2}}$$

As $\cos\theta = \sqrt{\frac{5}{14}}$, we get $\sin\theta = \frac{3}{\sqrt{14}}$

Thus, $\frac{3}{\sqrt{14}} = \frac{5+3\lambda}{\sqrt{14}\sqrt{5+\lambda^2}} \Rightarrow 3\sqrt{5+\lambda^2} = 5 + 3\lambda$

$\Rightarrow 5 + \lambda^2 = \frac{25}{9} + \frac{10}{3}\lambda + \lambda^2 \Rightarrow \frac{20}{9} = \frac{10}{3}\lambda$

$\Rightarrow \lambda = 2/3.$

33. Equation of line through $(1, -5, 9)$ and parallel to $x = y = z$ is

$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = r(\text{say})$

$\Rightarrow x = r + 1,\ y = r - 5,\ z = r + 9$

The point $(r + 1, r - 5, r + 9)$ will lie on $x - y + z = 5$ if

$(r + 1) - (r - 5) + (r + 9) = 5 \Rightarrow r = -10$

$\therefore$ the point on the plane is $(-9, -15, -1)$.

Its distance from $(1, -5, 9)$ is

$\sqrt{(-9-1)^2 + (-15+5)^2 + (9+1)^2} = \sqrt{300}$

$= 10\sqrt{3}$

34. Any point on

$\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ (1)

is given by $P(2r, 3r + 2, 4r + 3)$

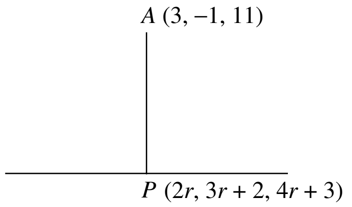


Fig. 21.10

Direction ratios of AP are

$2r - 3,\ 3r + 2 + 1,\ 4r + 3 - 11$ or $2r - 3,\ 3r + 3,\ 4r - 8$

AP will be perpendicular to (1) if

$2(2r - 3) + 3(3r + 3) + 4(4r - 8) = 0$

$\Rightarrow 29r - 29 = 0 \Rightarrow r = 1$

$\therefore$ Coordinates of P are $(2, 5, 7)$.

Thus, $AP^2 = (2 - 3)^2 + (5 + 1)^2 + (7 - 11)^2$

$= 1 + 36 + 16 = 53$

$\Rightarrow AP = \sqrt{53}$

35. Let the required plane be $x - 2y + 2z + k = 0$.

Its distance from the origin equals $\left|\frac{0 - 2(0) + 2(0) + k}{\sqrt{1+4+4}}\right|$

$= 1 \Rightarrow k = \pm 3.$

Thus, an equation of required plane is

$x - 2y + 2z - 3 = 0.$

36. As the given lines intersect, the lines are coplanar.

Thus

$$\begin{vmatrix} 3-1 & k-(-1) & 0-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2 & k+1 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow 2(3 - 8) - (k + 1)\,(2 - 4) - (4 - 3) = 0$

$\Rightarrow -10 + 2(k + 1) - 1 = 0 \Rightarrow k = 9/2$

37. The two lines will be coplanar if

$$\begin{vmatrix} 1-2 & 4-3 & 5-4 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

Using $C_3 \to C_3 - C_2$ and $C_2 \to C_2 + C_1$, we get

$$\begin{vmatrix} -1 & 0 & 0 \\ 1 & 2 & -k-1 \\ k & 2+k & -1 \end{vmatrix} = 0$$

$\Rightarrow -2 + (k + 1)\,(k + 2) = 0.$

$\Rightarrow k = 0, -3.$

38. One of the points on $2x + y + 2z = 8$ is $(4, 0, 0)$. It distance from $4x + 2y + 4z + 5 = 0$ is $\dfrac{|4(4)+5|}{\sqrt{16+4+16}}$

$= \dfrac{21}{6} = \dfrac{7}{2}$

Alternately The required distance between the planes $4x + 2y + 4z = 16$ and $4x + 2y + 4z + 5 = 0$ is

$$\frac{16+5}{\sqrt{4^2+2^2+4^2}} = \frac{21}{6} = \frac{7}{2}$$

39. If the lines are coplanar, then

$$\begin{vmatrix} -2+1 & k-1 & 0+1 \\ 2 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0 \Rightarrow k = \frac{11}{2}$$

40. Let the coordinates Q be (α, β, γ), O the origin.

OQ is perpendicular to the plane

$\Rightarrow \dfrac{\alpha}{4} = \dfrac{\beta}{-3} = \dfrac{\gamma}{1} = k$

Q lies on the plane

$\Rightarrow 4 \times 4\alpha - 3(-3)k + k + 13 = 0 \Rightarrow k = -\dfrac{1}{2}$

$Q(\alpha, \beta, \gamma) = Q\left(-2, \dfrac{3}{2}, -\dfrac{1}{2}\right)$

$(QR)^2 = (-2 + 1)^2 + \left(\dfrac{3}{2}-1\right)^2 + \left(-\dfrac{1}{2}+6\right)^2 = \dfrac{63}{2}$

$\Rightarrow QR = 3\sqrt{7/2}$

41. $l^2 + m^2 = n^2$

$\Rightarrow (l + m)^2 - 2lm = n^2$

$\Rightarrow n^2 - 2lm = n^2 \Rightarrow lm = 0$

Either $l = 0$ or $m = 0$

If $l = 0$, $m = -n$ and if $m = 0$, $l = -n$.

So the direction ratios are 0, 1, –1 and 1, 0, –1 and the angle between them is $\cos^{-1} \dfrac{0\times1+1\times0+(-1)(-1)}{\sqrt{2}\sqrt{2}}$

$= \cos^{-1} \dfrac{1}{2} = 60°$

42. Equation of the lines can be written as

$$\frac{x-(\sqrt{\lambda}-1)}{\sqrt{\lambda}} = \frac{y}{1} = \frac{z-\sqrt{\lambda}}{\sqrt{\lambda}-1}$$

$$\frac{x-(\sqrt{\mu}-1)}{\sqrt{\mu}} = \frac{y}{1} = \frac{z-\sqrt{\mu}}{-\sqrt{\mu}+1}$$

If they are perpendicular, then

$\sqrt{\lambda}\sqrt{\mu} + 1 + (\sqrt{\lambda}-1)(1-\sqrt{\mu}) = 0$

$\Rightarrow \sqrt{\lambda} + \sqrt{\mu} = 0 \Rightarrow \lambda = \mu.$

43. Required length $= \sqrt{2^2+3^2+6^2} = 7$

44. Coordinates of D are $\left(\dfrac{\lambda-1}{2}, 4, \dfrac{\mu+2}{2}\right)$

$D.R.$ of AD are $\dfrac{\lambda-1}{2} - 2$, $4 - 3$, $\dfrac{\mu+2}{2} - 5$

$= \dfrac{\lambda-5}{2}$, 1, $\dfrac{\mu-8}{2}$

AD is equally inclined with the axes

So d.r. of AD are 1, 1, 1

$\Rightarrow \dfrac{\lambda-5}{2} = 1$, $\dfrac{\mu-8}{2} = 1 \Rightarrow (\lambda, \mu) = (7, 10)$

45. Equation of a plane through the intersection of the given planes is

$x + 2y - 3 + \lambda(y - 2z + 1) = 0$

$\Rightarrow x + (2 + \lambda)\,y - 2\lambda z + (\lambda - 3) = 0$

As it is perpendicular to the first plane

$1 + 2(2 + \lambda) - 2\lambda(0) = 0 \Rightarrow \lambda = -\frac{5}{2}$.

and the equation of the required plane is
$2x - y + 10z = 11$

46. Any point on the given line is

$P(3k + 1, k + 3, -5k + 4)$

Its image Q in the plane $2x - y + z + 3 = 0$ is on the line through this point, perpendicular to the plane, so is given by

$$\frac{x-(3k+1)}{2} = \frac{y-(k+3)}{-1} = \frac{z-(-5k+4)}{1}$$

$$= \frac{-2(6k+2-k-3-5k+4+3)}{6} = -2$$

$\Rightarrow x = 3k - 3, y = k + 5, z = 2 - 5k$

And the equation of the required line is

$$\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$$

Note: Distance of the image Q from the point is equal to double the distance of point P from the plane.

47. See Question No. 41.

48. A point on

$$\frac{x-1}{3} = \frac{y-2}{1} = \frac{z-3}{2}$$

is $(3r + 1, r + 2, 2r + 3)$.

This point will lie on

$$\frac{x-3}{1} = \frac{y-1}{2} = \frac{z-2}{3}.$$

We get

$$\frac{3r+1-3}{1} = \frac{r+2-1}{2} = \frac{2r+3-2}{3}$$

$$\Rightarrow 3r - 2 = \frac{r+1}{2} = \frac{2r+1}{3}$$

$\Rightarrow r = 1$

Thus, point of intersection of two lies is (4, 3, 5).

This point lies on

$7x + 2y + 4z - 54 = 0$, $3x + 4y + 5z - 49 = 0$ and $4x + 3y + 5z - 50 = 0$

Their distance from the origin are

$$p_1 = \frac{54}{\sqrt{69}}, p_2 = \frac{49}{5\sqrt{2}} \text{ and } p_3 = \frac{50}{5\sqrt{2}}$$

Note that $p_1 < 7$, $p_2, p_3 > 7$ and $p_2 < p_3$.

Thus, $4x + 3y + 5z = 50$ is the required plane.

49. Suppose the line makes an angle ϕ with the z-axis. Direction cosines of the line are $\cos\theta$, $\cos\theta$, $\cos\phi$. We have

$\cos^2\theta + \cos^2\theta + \cos^2\phi = 1$

$\Rightarrow 2\cos^2\theta = \sin^2\phi \le 1$

$\Rightarrow 0 \le \cos\theta \le 1/\sqrt{2} \Rightarrow \theta \in [\pi/4, \pi/2]$

50. Mid point of BC is $M\left(\frac{\lambda-1}{2}, 4, \frac{\mu+2}{2}\right)$

Direction ratios of median AM are

$$\frac{\lambda-5}{2}, 1, \frac{\mu-8}{2}$$

As median AM is equally inclined to the axes, we get

$$\frac{\lambda-5}{2} = 1 = \frac{\mu-8}{2} \Rightarrow \lambda = 7, \mu = 10$$

$\therefore 10\lambda - 7\mu = 0$

51. An equation of plane containing the line

$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ is

$a(x - 1) + b(y - 2) + c(z - 3) = 0$ (1)

where $a + 2b + 3c = 0$ (2)

As $\frac{x}{1} = \frac{y}{1} = \frac{z}{4}$ is parallel to (1), we get

$a + b + 4c = 0$ (3)

From (2) and (3), we get

$$\frac{a}{-5} = b = c$$

Thus, equation of plane (1) is

$-5(x - 1) + (y - 2) + (z - 3) = 0$

or $-5x + y + z = 0$

It clearly passes through (1, 0, 5)

52. We can write the line as

$$\frac{x-b}{a} = y = \frac{z-d}{c}$$

or $\frac{x-b}{a} - 1 = y - 1 = \frac{z-d}{c} - 1$

or $\frac{x-b-a}{a} = \frac{y-1}{1} = \frac{z-d-c}{c}$

53. Distance between planes

$$= \frac{|d-1|}{\sqrt{16+4+16}} = 7$$

$\Rightarrow |d - 1| = 42 \Rightarrow d = -41, 43$

54. Suppose the line of shortest distance meets the lines at A and B.

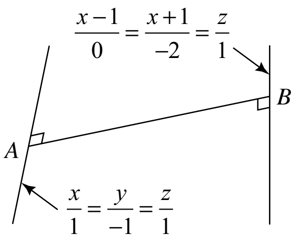

$\frac{x-1}{0}=\frac{x+1}{-2}=\frac{z}{1}$
B
A
$\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}$

Fig. 21.11

Let coordinates of A be $(a, -a, a)$ and that of B be $(1, -2b - 1, b)$

Direction ratios of AB are $1 - a, -2b + a - 1, b - a$

As AB is perpendicular to both the lines

$1(1 - a) + (-1)(-2b + a - 1) + 1(b - a) = 0$

and $0(1 - a) + (-2)(-2b + a - 1) + 1(b - a) = 0$

$\Rightarrow 3a - 3b = 2$ and $3a - 5b = 2$

Solving we get $a = \frac{2}{3}$, $b = 0$.

$\therefore$ Direction ratio of AB are

$\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}$ or $1, -1, -2$

Equation of the shortest distance is

$$\frac{x-1}{1} = \frac{y+1}{-1} = \frac{z}{-2}$$

55. Equation of line is

$$\frac{x+1}{1} = \frac{y}{2} = \frac{z+4}{2} \qquad (1)$$

As angle between (1) and

$2x - y + \sqrt{\lambda}\, z + 4 = 0$ is $\frac{\pi}{6}$, we get

$$\sin\left(\frac{\pi}{6}\right) = \frac{|(2)(1)+(-1)(2)+2\sqrt{\lambda}|}{\sqrt{1+4+4}\sqrt{4+1+\lambda}}$$

$\Rightarrow 9(5 + \lambda) = 4(2 - 2 + 2\sqrt{\lambda})^2$

$45 = 7\lambda \Rightarrow \lambda = \frac{45}{7}$

56. A point of $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ is of the form $(3r + 2, 4r - 1, 12r + 2)$ will lie on $x - y + z = 16$ if $(3r + 2) - (4r - 1) + (12r + 2) = 16$

$\Rightarrow r = 1$

Thus, point of intersection is (5, 3, 14) and its distance from (1, 0, 2) is $\sqrt{4^2+3^2+12^2} = 13$.

57. A plane passing through the intersection of $2x - 5y + z = 3$ and $x + y + 4z = 5$ is

$(2x - 5y + z - 3) + k(x + y + 4z - 5) = 0$

or $(2 + k)x + (-5 + k)y + (1 + 4k)z - 3 - 5k = 0$

It will be parallel to $x + 3y + 6z = 1$ if

$$\frac{2+k}{1} = \frac{-5+k}{3} = \frac{1+4k}{6}$$

$\Rightarrow k = -11/2$

Thus, equation of required plane is

$$-\frac{7}{2}x - \frac{21}{2}y - \frac{42}{2}z + \frac{49}{2} = 0$$

or $x + 3y + 6z - 7 = 0$

58. Let direction ratios of the line

$$x + y + 2z - 3 = 0 = 2x + 3y + 4z - 4 \qquad (1)$$

be a, b, c. Then

$a + b + 2c = 0$

and $2a + 3b + 4c = 0$

$$\Rightarrow \frac{a}{2} = \frac{b}{0} = \frac{c}{-1}$$

To obtain a point on the line put $z = 0$, so that

$x + y = 3$ and $2x + 3y = 4$

$\Rightarrow x = 5, y = -2$.

Thus line (1) is given by

$\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + \lambda(2\mathbf{i} - \mathbf{k})$ and z-axis is given by

$\mathbf{r} = \mu\mathbf{k}$

Here $\mathbf{d}_1 \times \mathbf{d}_2 = (2\mathbf{i} - \mathbf{k}) \times \mathbf{k} = -2\mathbf{j}$

$$\therefore \text{ s.d.} = \frac{|(5\mathbf{i}-2\mathbf{j})\cdot(\mathbf{d}_1\times\mathbf{d}_2)|}{|\mathbf{d}_1\times\mathbf{d}_2|}$$

$$= \frac{4}{2} = 2.$$

59. An equation of plane containing the line

$$\frac{x-1}{1} = \frac{y-2}{5} = \frac{z-3}{4}$$

is $a(x-1) + b(y-2) + c(z-3) = 0$

where

$a + 5b + 4c = 0.$ (1)

It will pass through (3, 2, 0) if

$2a - 3c = 0.$ (2)

From (1) and (2), we get

$$\frac{a}{15} = \frac{b}{-11} = \frac{c}{10}$$

thus, equation of plane is

$15(x-1) - 11(y-2) + 10(z-3) = 0$

which passes through (0, 7, 10)

60. $$\frac{|3+4-12\lambda+13|}{\sqrt{3^2+4^2+12^2}} = \frac{|3(-3)+4(0)-12+13|}{\sqrt{3^2+4^2+12^2}}$$

$\Rightarrow |20 - 12\lambda| = 8 \Rightarrow (3\lambda - 5)^2 = 4$

$\Rightarrow 9\lambda^2 - 30\lambda + 21 = 0$

$\Rightarrow 3\lambda^2 - 10\lambda + 7 = 0$

$\therefore$ λ satisfies the equation $3x^2 - 10x + 7 = 0$

61. Shortest distance between two lines L_1 and L_2 is equal to distance between L_1 and a plane through L_2 and parallel to L_1.

Equation of any plane through the intersection of

$x + y + z + 1 = 0$ and $2x - y + z + 3 = 0$ is

$x + y + z + 1 + \lambda(2x - y + z + 3) = 0$

or $(1 + 2\lambda)x + (1 - \lambda)y + (1 + \lambda)z + (1 + 3\lambda) = 0$ (1)

This will be parallel to

$$\frac{x-1}{\alpha} = \frac{y+1}{-1} = \frac{z}{1} \qquad (2)$$

if $(1 + 2\lambda)\alpha + (1 - \lambda)(-1) + (1 + \lambda)(1) = 0$

$$\Rightarrow \alpha = -\frac{2\lambda}{1+2\lambda}$$

A point on (2) is (1, –1, 0). Its distance from (1) is

$$\frac{|(1+2\lambda)(1)+(1-\lambda)(-1)+(1+3\lambda)|}{\sqrt{(1+2\lambda)^2+(1-\lambda)^2+(1+\lambda)^2}} = \frac{1}{\sqrt{3}}$$

$\Rightarrow (6\lambda + 1)^2 (3) = 6\lambda^2 + 4\lambda + 3$

$\Rightarrow (36\lambda^2 + 12\lambda + 1)(3) = 6\lambda^2 + 4\lambda + 3$

$\lambda = 0, -16/51$

$\therefore$ $\alpha = 0, 32/19$

62. An equation of line through $A(1, -5, 9)$ and parallel to $x = y = z$ is

$x - 1 = y + 5 = z - 9 = r$ (say)

A point on this line is $P(r + 1, r - 5, r + 9)$.

This point will lie on $x - y + z = 5$.

If $(r + 1) - (r - 5) + (r + 9) = 5$

$\Rightarrow r = -10.$

Distance between A and P is

$= AP = \sqrt{3}\,|r| = 3\sqrt{10}$

63. As the line

$$\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$$

lies on $lx + my + z = 9$,

$2l + (-1)m - 3z = 0$

and $3l + (-2)m - (-4) = 9$

$\Rightarrow 2l - m = 3,\ 3l - 2m = 5$

$\Rightarrow l = 1,\ m = -1$

$\Rightarrow l^2 + m^2 = 2$

64. Here $A_1 \equiv (0, 0, 0)$, $A_2 \equiv (-2, 4, 5)$

$\vec{d}_1 = 2\hat{i}+2\hat{j}+\hat{k}, \vec{d}_2 = -\hat{i}+8\hat{j}+4\hat{k}$

We have

$$\vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ -1 & 8 & 4 \end{vmatrix}$$

$= 0\hat{i} + (-9)\hat{j} + 18\hat{k}$

and $\overrightarrow{A_1A_2} = -2\hat{i} + 4\hat{j} + 5\hat{k}$

Now $\overrightarrow{A_1A_2} \cdot (\vec{d}_1 \times \vec{d}_2) = 0 - 36 + 90 = 54$

and $|\vec{d}_1 \times \vec{d}_2| = \sqrt{9^2 + 18^2} = 9\sqrt{5}$

$\therefore$ shortest distance

$$= \frac{54}{9\sqrt{5}} = \frac{6}{\sqrt{5}} = 1.2\sqrt{5}$$

65. An equation of plane through (1, 2, 2) is

$a(x - 1) + b(y - 2) + c(z - 2) = 0$ (1)

This plane will be perpendicular to the $x - y + 2z = 3$ and $2x - 2y + z + 12 = 0$ if

$a - b + 2c = 0$

$2a - 2b + c = 0$

$a = b,\ c = 0$

Thus, (1) becomes

$x - 1 + y - 2 = 0$

or $x + y - 3 = 0$ (2)

Length of perpendicular from (1, –2, 4) to (1)

$$= \frac{|1-2-3|}{\sqrt{1+1}} = 2\sqrt{2}$$

66. Coordinates of mid-point D of BC are $\left(\frac{\lambda-1}{2}, 4, \frac{2+\mu}{2}\right)$

Direction ratios of AD are

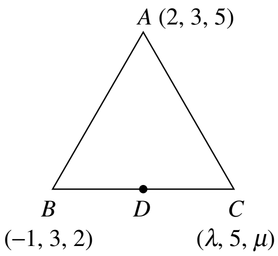


Fig. 21.12

$$\frac{\lambda-1}{2} - 2,\ 4 - 3,\ \frac{2+\mu}{2} - 5$$

or $\lambda - 5,\ 2,\ \mu - 8$

As AD is equally inclined to the axes,

$\lambda - 5 = 2 = \mu - 8$

$\Rightarrow \lambda = 7,\ \mu = 10$

Thus, $\lambda^3 + \mu^3 + 5 = 7^3 + 10^3 + 5 = 1348$

67. As the given lines are coplanar

$$\Delta = \begin{vmatrix} 3-1 & 2-2 & 1+3 \\ 1 & 2 & \lambda^2 \\ 1 & \lambda^2 & 2 \end{vmatrix} = 0$$

$\Rightarrow 2[4 - \lambda^4) + 4(\lambda^2 - 2) = 0$

$\Rightarrow 2(\lambda^2 - 2)(-\lambda^2) = 0$

$\Rightarrow \lambda = \pm\sqrt{2}, 0$

Thus, there three such values.

Previous Years' B-Architecture Entrance Examination Questions

1. Equating the z-coordinates of the centroid and the centre of the sphere we get

$$\frac{0-1+c+3}{3} = c \Rightarrow c = 1$$

2. Let (α, β, γ) be the foot of the perpendicular from the origin on the place through (p, q, r)

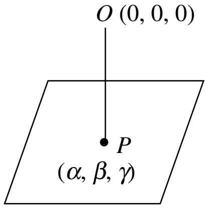


Fig. 21.13

Then α, β, γ are the direction ratios of the normal to the plane and its equation is $\alpha(x - p) + \beta(y - q) + \gamma(z - r) = 0$

Since the plane passes through (α, β, γ)

$\alpha(\alpha - p) + \beta(\beta - q) + \gamma(\gamma - r) = 0$

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 - p\alpha - q\beta - r\gamma = 0$

$\Rightarrow$ Locus of (α, β, γ) is a sphere.

3. $OR = a/2,\ OP = a$

$PR = \sqrt{27}$

So $a^2 = 27 + \dfrac{a^2}{4}$

$\Rightarrow a^2 = 36 \Rightarrow a = 6$

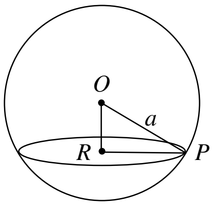


Fig. 21.14

Let R be the radius of the section of the sphere by the plane $z = a/6$.

Then $R^2 = a^2 - \left(\dfrac{a}{6}\right)^2 \Rightarrow R = \sqrt{35}$.

4. Any point on $\dfrac{x-2}{3} = \dfrac{y+1}{4} = \dfrac{z-2}{12}$ is $(3r + 2, 4r - 1, 12r + 2)$ which lies on the plane

$3r + 2 - 4r + 1 + 12r + 2 = 5$

$\Rightarrow r = 0$ and the point of intersection is (2, –1, 2) and the required distance is

$$\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2} = 13$$

5. Direction ratios of L_2 are 2, 0, –2

So $\cos\theta = \dfrac{(-2)(2)+(-1)(0)+(-2)(2)}{\sqrt{4+1+4}\sqrt{4+4+1}} = \dfrac{-\sqrt{8}}{3}$

$\Rightarrow \sin\theta = \dfrac{1}{3}$.

6. (c) represents a pair of parallel planes, so does not represent a line.

7. $\begin{vmatrix} 2-1 & 2-0 & 2+1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 0 \Rightarrow$ lines are coplanar

Also the lines intersect at the point (1, 1, 1).

So the shortest distance between the lines is zero coplanarity only $\Rightarrow$ shortest distance is zero.

8. We have $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \cos\theta = \dfrac{1}{3-(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)} = \dfrac{1}{2}$

$\Rightarrow \theta = \pi/3$

9. Let the equation of the plane be

$\dfrac{x}{\alpha}+\dfrac{y}{\beta}+\dfrac{z}{\gamma} = 1$, then $\dfrac{\alpha}{3}=\dfrac{a}{2} \Rightarrow \alpha = \dfrac{3}{2}a$

Similarly $\beta = \dfrac{3}{2}b$, $\gamma = \dfrac{3}{2}c$ and the required equation is $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c} = \dfrac{3}{2}$.

10. Equation of the given line is

$\vec{r} = (2\hat{i}+5\hat{j}-\hat{k})+t(3\hat{i}+2\hat{j}-5\hat{k}) = \vec{a}+t\vec{b}$

Equation of z-axis is $\dfrac{x}{0} = \dfrac{y}{0} = \dfrac{z}{1}$

$\Rightarrow \vec{r} = p(\hat{k}) = c + p\vec{d}$

Shortest distance $= \dfrac{(\vec{a}-\vec{c})\cdot(\vec{b}\times\vec{d})}{|\vec{b}\times\vec{d}|}$

$= \dfrac{(2\hat{i}+5\hat{j}-\hat{k})\cdot(2\hat{i}-3\hat{j})}{\sqrt{4+9}}$

$= \left|\dfrac{4-15}{\sqrt{13}}\right| = \dfrac{11}{\sqrt{13}}$.

11. Any point on the given line L is $(r+4, r+2, 2r+7)$

Since $2(r+4) - 4(r+2) + 2r + 7 = 7$ for all r.

The line L lies in the plane $\Rightarrow$ Statement-1 is true. Statement-2 is also true but only explains that the line is perpendicular to the normal to the plane. It can lie in a plane parallel to the plane P. So it is not a correct explanation for statement-1.

12. Equation of the lines $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z+1}{-1}$

Let $P(\alpha, \beta, \gamma)$ be the reflection of the point $Q(0, 3, -2)$

then $2(\alpha - 0) + 1(\beta - 3) - 1(\gamma + 2) = 0$

$\Rightarrow 2\alpha + \beta - \gamma - 5 = 0$ (1)

and $\dfrac{\frac{\alpha}{2}-1}{2} = \dfrac{\frac{\beta+3}{2}-2}{1} = \dfrac{\frac{\gamma-2}{2}+1}{-1}$

(mid point of PQ lies of the line)

$\Rightarrow \dfrac{\alpha-2}{2} = \dfrac{\beta-1}{1} = \dfrac{\gamma}{-1} = r$ (say) (2)

From (1) and (2) we have

$2(2r+2) + r + 1 + r - 5 = 0 \Rightarrow r = 0$

and the required point is (2, 1, 0)

13. Let the equation of the plane be

$\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c} = 1$

Since it passes through (1, –2, 3)

$\dfrac{1}{a}-\dfrac{2}{b}+\dfrac{3}{c} = 1$ (1)

Let A $(a, 0, 0)$, plane through A parallel to x-axis is $x = a$ and through B and C parallel to the axes are $y = b$ and $z = c$.

Point of intersection of these planes is (a, b, c)

From (1) we have $bc - 2ac + 3ab = abc$.

$\Rightarrow (a, b, c)$ lies on $yz - 2zx + 3xy = xyz$.

14. Given lines are $\dfrac{x}{3} = \dfrac{y}{2} = \dfrac{z}{-6}$ and $\dfrac{x}{-2} = \dfrac{y}{12} = \dfrac{z}{3}$

Since $3(-2) + 2(12) + (-6)(3) = 0$

The lines are perpendicular.

15. Lines must be coplanar

$\Rightarrow \begin{vmatrix} 4-0 & 2+2 & \lambda-0 \\ 1 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 0 \Rightarrow \lambda = 12$

16. $A_1(x, 0, 0)$, $A_2(0, x_2, 0)$, $A_3(0, 0, x_3)$

Equation of the plane is

$$\frac{x}{x_1} + \frac{y}{x_2} + \frac{z}{x_3} = 1 \qquad (1)$$

Plane through A_1, A_2, A_3 parallel to coordinate axes intersect at the point (x_1, x_2, x_3). Distance of (1) from the origin is p.

$$\Rightarrow \frac{1}{\sqrt{\frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2}}} = \pm p$$

$$\Rightarrow \frac{1}{x_1^2}+\frac{1}{x_2^2}+\frac{1}{x_3^2} = \frac{1}{p^2}$$

17. Let coordinate of P be

$(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$

Direction ratios of QP are

$5\lambda - 3 - 0, 2\lambda + 1 - 2, 3\lambda - 4 - 3$

or $5\lambda - 3, 2\lambda - 1, 3\lambda - 7$

As PQ is perpendicular to

$$\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$$

$5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0$

$\Rightarrow 38\lambda - 38 = 0 \Rightarrow \lambda = 1.$

$\Rightarrow$ coordinate of P are $(2, 3, -1)$

Area of ΔPQR is given by

$$A = \frac{1}{2}|\overrightarrow{PQ}\times\overrightarrow{PR}|$$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -1 & 4 \\ -5 & -2 & -3 \end{vmatrix}$$

$$= \frac{1}{2}|11\hat{i} - 26\hat{j} - \hat{k}|$$

$$\Rightarrow 2A^2 = \frac{1}{2}\ (121 + 676 + 1)$$

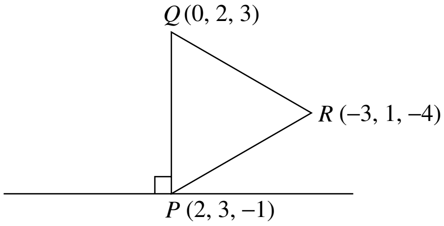


Fig. 21.15

$$= \frac{1}{2}(798) = 399.$$

18. As the lines are coplanar,

$$\Rightarrow \begin{vmatrix} 2-4 & 8-6 & 10-8 \\ 2 & 2 & -2k^2 \\ 2k^2 & 4 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & -2k^2 \\ 2k^2 & 4 & 2 \end{vmatrix} = 0$$

$\Rightarrow k^2(k^2 + 3) = 0$

But $k \neq 0$, therefore, $k^2(k^2 + 3) \neq 0$.

Thus, there is no non-zero real values of k for which the given lines are coplanar.

19. An equation of plane through intersection of given planes is

$x + y + z - 1 + k(2x + 3y - z + 4) = 0$

or $(1 + 2k)x + (1 + 3k)y + (1 - k)z - 1 + 4k = 0$

This will be parallel to y-axis if $1 + 3k = 0$

or $k = -1/3$

Thus, equation of plane becomes

$$\frac{1}{3}x+\frac{4}{3}z-\frac{7}{3} = 0$$

or $x + 4z - 7 = 0$

which passes through $(3, 0, 1)$

CHAPTER TWENTY TWO

Vectors

A vector is a quantity having both magnitude and direction. For instance displacement, velocity, force and acceleration are vector quantities. Any portion of a straight line, where the two end-points are distinguished as *initial* and *terminal*, is called a *directed line segment*. The directed line segment with initial point A and terminal point B is denoted by the symbol **AB** or $\overrightarrow{AB}$ and the length of the vector **AB** by |**AB**|. Graphically, a vector is represented by a directed line segment. We denote it by a letter with an arrow over it, as in $\vec{a}$, or in bold type, as in **a**. A scalar is a quantity having magnitude only but no direction, such as mass, length, time, temperature and any real number. A vector whose initial and terminal points are the same is called a zero vector, **0**.

Two vectors are equal if they have the same magnitude; they lie on the same line or on parallel lines and they have the same direction.

Let **a** be any vector and α a scalar then $\alpha\mathbf{a}$ is a vector whose length is equal to $|\alpha|\,|\mathbf{a}|$, **a** is a vector on the same line or on a line parallel to the line on which **a** lies. Moreover, direction of $\alpha\mathbf{a}$ is same as that of **a** if $\alpha > 0$ and is opposite to that of **a** if $\alpha < 0$.

If **a** = **AB** and **b** = **BC** then **a** + **b** is the vector **AC**.

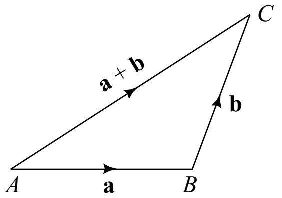


Fig. 22.1

The following are some elementary properties of addition and scalar multiplication:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (addition is commutative) (Fig. 22.2)
2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (addition is associative) (Fig. 22.3)
3. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$
4. $(\alpha + \beta)\,\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$

   $0\mathbf{a} = 0$, $1\mathbf{a} =$ and $(-1)\,\mathbf{a} = -\,\mathbf{a}$

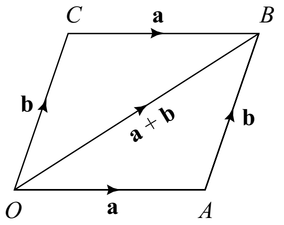


Fig. 22.2

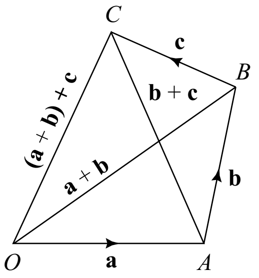


Fig. 22.3

The position vector **r** of any point P with respect to the origin of reference O is the vector **OP**. Two vectors **a** and **b** are said to be collinear if they are supported on the same or parallel lines. For such vectors, $\mathbf{b} = x\mathbf{a}$ for some scalar x. A set of vectors is said to be *coplanar* if they lie in the same plane, or the planes in which the different vectors lie are all parallel to the same plane. Three vectors **a**, **b** and **c** are coplanar if and only if $\mathbf{c} = x\mathbf{a} + y\mathbf{b}$ for some scalars x and y.

LINEAR COMBINATIONS

The vector $\mathbf{r} = \alpha_1\,\mathbf{a}_1 + \alpha_2\,\mathbf{a}_2 + \cdots + \alpha_n\,\mathbf{a}_n$, where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are scalars, is called a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$. The following results are useful in determining coplanar and collinear vectors:

1. If **a** and **b** are non-collinear vectors, then

   $$x\mathbf{a} + y\mathbf{b} = x'\mathbf{a} + y'\mathbf{b} \quad \Leftrightarrow \quad x = x',\ y = y'$$
2. *Fundamental theorem in plane*. If **a** and **b** are non-collinear vectors, then any vector **r**, coplanar with **a** and **b**, can be expressed uniquely as a linear combination of **a** and **b**. That is, there exist unique x and $y \in \mathbf{R}$ such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$.
3. If **a**, **b** and **c** are *non-coplanar* vectors, then

   $$x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = x'\mathbf{a} + y'\mathbf{b} + z'\mathbf{c}$$
   $$\Leftrightarrow \quad x = x',\ y = y',\ z = z'$$
4. *Fundamental theorem in space*. If **a**, **b** and **c** are non-coplanar vectors in space, then any vector **r**

can be uniquely expressed as a linear combination of **a**, **b** and **c**. That is, there exist unique $x, y, z \in \mathbf{R}$ such that $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$. If **i**, **j** and **k** are three unit vectors along x-axis, y-axis and z-axis respectively, then any vector **r** can be represented uniquely as $\mathbf{r} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, where a_1, a_2 and a_3 are the *coordinates of* ***r***.

5. *Section formula*. The position vector of a point P which divides the line joining the points A and B with position vectors **a** and **b** respectively in the ratio $m : n$, is
$$\frac{n\mathbf{a} + m\mathbf{b}}{m+n} \quad (m \neq -n)$$
The position vector of mid-point M of AB, is (1/2) $(\mathbf{a} + \mathbf{b})$. The point A with position vector **a** is written $A(\mathbf{a})$. If $A(\mathbf{a})$, $B(\mathbf{b})$ and $C(\mathbf{c})$ are the vertices of a triangle ABC, then the centroid of this triangle is (1/3) $(\mathbf{a} + \mathbf{b} + \mathbf{c})$.
6. *Test of collinearity*. Three points $A(\mathbf{a})$, $B(\mathbf{b})$, and $C(\mathbf{c})$ are collinear if and only if there exist scalars x, y and z, not all zero, such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$, where $x + y + z = 0$.
7. *Test of coplanarity*. Four points $A(\mathbf{a})$, $B(\mathbf{b})$, $C(\mathbf{c})$ and $D(\mathbf{d})$ are coplanar if and only if there exist scalars x, y, z and w, not all zero, such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + w\mathbf{d} = \mathbf{0}$, $x + y + z + w = 0$.

SCALAR OR DOT PRODUCT

The scalar product of two vectors **a** and **b** is given by $|\mathbf{a}|\,|\mathbf{b}| \cos\theta$, where θ $(0 \le \theta \le \pi)$ is the angle between the vectors **a** and **b**. It is denoted by $\mathbf{a} \cdot \mathbf{b}$.

Properties of the scalar product

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 = \mathbf{a}^2$.
2. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.
3. Two non-zero vectors **a** and **b** make an acute angle if $\mathbf{a} \cdot \mathbf{b} > 0$, an obtuse angle if $\mathbf{a} \cdot \mathbf{b} < 0$ and are inclined at a right angle if $\mathbf{a} \cdot \mathbf{b} = 0$.
4. $\mathbf{a} \cdot \mathbf{b}$ = (projection of **a** on **b**)$|\mathbf{b}|$.
5. $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a}^2 - \mathbf{b}^2$; $(\mathbf{a} + \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} \cdot \mathbf{b}$,
$(\mathbf{a} - \mathbf{b})^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a} \cdot \mathbf{b}$.
6. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, then
$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3,\ |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
and
$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$
7. Vector component of **u** orthogonal to a vector **a** is $\mathbf{u} - \text{proj}_{\mathbf{a}}\,\mathbf{u}$ where $\text{proj}_{\mathbf{a}}\mathbf{u} = \dfrac{\mathbf{u}.\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$ and $\text{proj}_{\mathbf{a}}\,\mathbf{u}$ is vector component of **u** along **a.**

VECTOR OR CROSS PRODUCT

The vector product of two vectors **a** and **b**, denoted $\mathbf{a} \times \mathbf{b}$, is the vector **c** with $|\mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, where θ is the angle between **a** and **b**, with $0 \le \theta \le \pi$. **c** is supported by the line perpendicular to **a** and **b**, and the direction of **c** is such that **a**, **b** and **c** form a right-handed system.

Properties of the vector product

1. $\mathbf{a} \times \mathbf{b} = -\,\mathbf{b} \times \mathbf{a}$.
2. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.
4. $(\mathbf{a} \times \mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})^2$.
5. Two non-zero vectors **a** and **b** are collinear if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.
6. If **i**, **j** and **k** are unit vectors along positive x-axis, y-axis and z-axis then $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, $\mathbf{j}\times\mathbf{k} = \mathbf{i}$, $\mathbf{k}\times\mathbf{i} = \mathbf{j}$.
7. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then
$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$
8. The area of the parallelogram whose adjacent sides are represented by the vectors $\mathbf{OA} = \mathbf{a}$ and $\mathbf{OB} = \mathbf{b}$, is $|\mathbf{a} \times \mathbf{b}|$, and the area of the triangle OAB is (1/2) $|\mathbf{a} \times \mathbf{b}|$. The vector area of the above parallelogram is $\mathbf{a} \times \mathbf{b}$.
9. A unit vector perpendicular to the plane of **a** and **b** is
$$\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$
and a vector of magnitude λ perpendicular to the plane of **a** and **b** is
$$\pm\,\frac{\lambda(\mathbf{a} \times \mathbf{b})}{|\mathbf{a}\times\mathbf{b}|}$$

SCALAR TRIPLE PRODUCT

The scalar triple product of the three vectors **a**, **b** and **c** is denoted by $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$, and is defined as $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.

Properties of the scalar triple product

1. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.
2. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{c}\ \mathbf{a}] = [\mathbf{c}\ \mathbf{a}\ \mathbf{b}] = -[\mathbf{b}\ \mathbf{a}\ \mathbf{c}] = -[\mathbf{c}\ \mathbf{b}\ \mathbf{a}] = -[\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$.
3. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then
$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
In particular $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

4. The volume of the parallelopiped whose adjacent sides are represented by the vectors **a**, **b** and **c**, is $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$.
5. The volume of a tetrahedron *ABCD* is equal to $\frac{1}{6}\,|\mathbf{AB} \times \mathbf{AC} \cdot \mathbf{AD}|$.
6. Any three vectors **a**, **b** and **c** are coplanar if and only if $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$.
7. $[\mathbf{a} + \mathbf{b}\ \mathbf{c}\ \mathbf{d}] = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}] + [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]$.
8. Three vectors **a**, **b** and **c** form a right-handed or left-handed system, according as $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] >$ or < 0.
9. $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$
10. $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \begin{vmatrix} \mathbf{a}\cdot\mathbf{u} & \mathbf{b}\cdot\mathbf{u} & \mathbf{c}\cdot\mathbf{u} \\ \mathbf{a}\cdot\mathbf{v} & \mathbf{b}\cdot\mathbf{v} & \mathbf{c}\cdot\mathbf{v} \\ \mathbf{a}\cdot\mathbf{w} & \mathbf{b}\cdot\mathbf{w} & \mathbf{c}\cdot\mathbf{w} \end{vmatrix}$
11. Four points with position vectors **a**, **b**, **c** and **d** will be coplanar if
$$[\mathbf{d}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$
12. $[\mathbf{a} + \mathbf{b}\quad \mathbf{b} + \mathbf{c}\quad \mathbf{c} + \mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
13. $[\mathbf{b} \times \mathbf{c}\quad \mathbf{c} \times \mathbf{a}\quad \mathbf{a} \times \mathbf{b}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

VECTOR TRIPLE PRODUCT

The vector triple product of three vectors **a**, **b** and **c** is the vector $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$,

Since $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is coplanar with **b** and **c** and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is coplanar with **a** and **b**, we have

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

Clearly, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ in general. In fact, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ if and only if the vectors **a** and **c** are collinear.

If **a**,**b**,**c** and **d** lie in the same plane then $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$

GEOMETRICAL AND PHYSICAL APPLICATIONS

Bisector of an angle If **a** and **b** are unit vectors along the sides of an angle, then $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are respectively the vectors along the internal and external bisectors of the angle. The bisectors of the angles between the lines $\mathbf{r} = x\mathbf{a}$ and $\mathbf{r} = y\mathbf{b}$ are given by

$$\mathbf{r} = \lambda \left(\frac{\mathbf{a}}{|\mathbf{a}|} \pm \frac{\mathbf{b}}{|\mathbf{b}|} \right) \qquad (\lambda \in \mathbf{R})$$

Equation of a line passing through a point with position vector **a** and parallel to a vector **b** is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$

Reciprocal systems of vectors Let **a**, **b** and **c** be a system of three non-coplanar vectors. Then the system **a′**, **b′** and **c′**, which satisfies

$$\mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1$$

and $\mathbf{a} \cdot \mathbf{b}' = \mathbf{a} \cdot \mathbf{c}' = \mathbf{b} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{c}' = \mathbf{c} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{b}' = 0$

is called the reciprocal system to the vectors **a**, **b** and **c**.

Equation of a plane. The equation of a plane passing through the point with position vector **a** and parallel to the plane containing **b** and **c**, is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c} \text{ or } [\mathbf{r} - \mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0.$$

λ and μ being parameters. The equation of a plane through three points whose position vectors are **a**, **b** and **c** is

$$\mathbf{r} = (1 - \lambda - \mu)\mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

or $\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

λ and μ being parameters.

Equation of a plane which is at a distance *d* from the origin having a unit normal **n** is $\mathbf{r} . \mathbf{n} = d$. Equation of a plane passing through a point with position vector **a** having a unit normal **n** is $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$.

Work done. If a force **F** acts at a point *A* and displaces it to the point *B*, then the work done by the force **F** is $\mathbf{F} \cdot \mathbf{AB}$. The moment of a force **F** applied at *B* about the point *A* is the vector $\mathbf{AB} \times \mathbf{F}$.

SOLVED EXAMPLES

Concept Based

Straight Objective Type Questions

Example 1: If in a triangle *OAC*, *B* is the mid point of *AC* and $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ then

(a) $\mathbf{OC} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ (b) $\mathbf{OC} = 2\mathbf{b} - 2\mathbf{a}$

(c) $\mathbf{OC} = 2\mathbf{b} - \mathbf{a}$ (d) $\mathbf{OC} = 3\mathbf{b} - 2\mathbf{a}$

Ans. (c)

Solution: Let *O* be the origin of reference. The position vector of *A* is **a** and that *B* is **b**. If position vector of *C* is **c**, then

$$\mathbf{b} = \frac{\mathbf{a} + \mathbf{c}}{2} \quad \Rightarrow \quad \mathbf{c} = 2\mathbf{b} - \mathbf{a}$$

Example 2: The angle between the vectors $\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is

(a) 45° (b) 60°

(c) 90° (d) 135°

Ans. (c)

Solution: $(\mathbf{i} - \mathbf{j} + \mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = -1 - 1 + 2 = 0$

So the angle is 90°.

Example 3: A unit vector **c** perpendicular to **a** = **i** – **j** and coplanar with **a** and **b** = **i** + **k** is

(a) $\frac{1}{\sqrt{6}}(\mathbf{i}+\mathbf{j}+2\mathbf{k})$ (b) $\frac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}+\mathbf{k})$

(c) $\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}-\mathbf{k})$ (d) $\frac{1}{\sqrt{6}}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$

Ans. (a)

Solution: $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$

$$= \lambda(\mathbf{i} - \mathbf{j}) + \mu(\mathbf{i} + \mathbf{k})$$

$$= (\lambda + \mu)\mathbf{i} - \lambda\mathbf{j} + \mu\mathbf{k}$$

since **c** is a unit vector so $(\lambda + \mu)^2 + \lambda^2 + \mu^2 = 1$ (i)

Also $\mathbf{a}.\mathbf{c}. = 0$, but $\mathbf{a}.\mathbf{c} = \lambda\, |a|^2 + \mu\, \mathbf{a}.\mathbf{b}$

$\Rightarrow$ $0 = \lambda.\ 2 + \mu\ .1$

$\Rightarrow$ $\mu = -2\lambda$

(i) $\Rightarrow$ $(1 - 2)^2\, \lambda^2 + \lambda^2 + 4\lambda^2 = 1$

$6\lambda^2 = 1$

$\lambda^2 = \frac{1}{6}$

$\Rightarrow \mathbf{c} = \pm\frac{1}{\sqrt{6}}\ (\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

Example 4: If **a** and **b** are two non-parallel vectors satisfying |**a**| = |**b**|, then the vector (**a** + **b**) × (**a** × **b**) is parallel to

(a) **a** (b) **a** – **b**

(c) **a** + **b** (d) **b**

Ans. (b)

Solution: $(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$

$$= \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{a} \times \mathbf{b})$$

$$= (\mathbf{a}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{a})\mathbf{b} + (\mathbf{b}\cdot\mathbf{b})\mathbf{a} - (\mathbf{b}\cdot\mathbf{a})\mathbf{b}$$

$$= (\mathbf{a}\cdot\mathbf{b})\ (\mathbf{a} - \mathbf{b}) + |\mathbf{a}|^2\, (\mathbf{a} - \mathbf{b})$$

$$= (\mathbf{a}\cdot\mathbf{b} + |\mathbf{a}|^2)\ (\mathbf{a} - \mathbf{b})$$

So (**a** + **b**) × (**a** × **b**) is parallel to **a** – **b**.

Example 5: If **a** = **i** + **j** + **k** and **b** = **i** – **j** + 2**k** then the projection of **a** on **b** is given by

(a) $\frac{1}{2}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$ (b) $\frac{1}{3}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

(c) $\frac{1}{3}(\mathbf{i}-\mathbf{j}-\mathbf{k})$ (d) $\frac{1}{3}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$

Ans. (d)

Solution: $\text{Proj}_{\mathbf{b}}\, \mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\ \mathbf{b}$

$$= \frac{2}{6}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$$

$$= \frac{1}{3}(\mathbf{i}-\mathbf{j}+2\mathbf{k})$$

Example 6: If **a**, **b**, **c** are unit vectors such that **a** – **b** + **c** = **0** then **c** · **a** is equal to

(a) $\frac{3}{2}$ (b) $-\frac{1}{2}$

(c) $\frac{1}{3}$ (d) $-\frac{1}{3}$

Ans. (b)

Solution: Taking dot product with **a**, **b**, **c** in the relation **a** – **b** + **c** = 0, we have

$$|\mathbf{a}|^2 - \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} = 0$$

$$\mathbf{a}\cdot\mathbf{b} - |\mathbf{b}|^2 + \mathbf{b}\cdot\mathbf{c} = 0$$

$$\mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{c} + |\mathbf{c}|^2 = 0$$

Adding, we get $2\,\mathbf{a}\cdot\mathbf{c} = -(|\mathbf{a}|^2 - |\mathbf{b}|^2 + |\mathbf{c}|^2)$

$$= -1$$

$$\mathbf{a}\cdot\mathbf{c} = -\frac{1}{2}$$

Example 7: The non-zero vectors **a**, **b** and **c** are related as **b** = 5**a** and **c** = –2**b**. The angle between **a** and **c** is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$

(c) π (d) $\frac{\pi}{3}$

Ans. (c)

Solution: cosine of angle between **a** and **c** is

$$= \frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}||\mathbf{c}|} = \frac{\frac{1}{5}\mathbf{b}\cdot -2\mathbf{b}}{\frac{1}{5}|\mathbf{b}|\ 2|\mathbf{b}|}$$

$$= -1$$

Hence the angle is π.

Example 8: A vector **b** collinear with $\mathbf{a} = 2\sqrt{2}\,\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ of length 10 is given by

(a) $3(2\sqrt{2}\mathbf{i}-\mathbf{j}+4\mathbf{k})$ (b) $2(2\sqrt{2}\mathbf{i}+\mathbf{j}-4\mathbf{k})$

(c) $2(2\sqrt{2}\mathbf{i}+\mathbf{j}+4\mathbf{k})$ (d) $2(2\sqrt{2}\mathbf{i}-\mathbf{j}+4\mathbf{k})$

Ans. (d)

Solution: $\mathbf{b} = \lambda\mathbf{a}$ and $10 = |\mathbf{b}| = |\lambda||\mathbf{a}|$

but $|\mathbf{a}|^2 = 8 + 1 + 16 = 25 \quad \Rightarrow \quad |\mathbf{a}| = 5$

Thus $|\lambda| = 2.$

So $\mathbf{b} = \pm 2(2\sqrt{2}\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

Example 9: The vector **p** = (**a**·**c**) **b** – (**a**·**b**) **c** is perpendicular to

(a) **c** (b) **b**

(c) **a** (d) **c** + **b**

Ans. (a)

Solution: **p** = **a** × (**b** × **c**) which is perpendicular to **a**.

Example 10: The angle between **a** + 2**b** and **a** – 3**b** if |**a**| = 1, |**b**| = 2 and angle between **a** and **b** is 60° is

(a) an acute angle (b) $\cos^{-1}\dfrac{-24}{\sqrt{21}\sqrt{31}}$

(c) $\cos^{-1}\dfrac{24}{\sqrt{21}\sqrt{31}}$ (d) $\cos^{-1}-\dfrac{1}{3}$

Ans. (b)

Solution: $|\mathbf{a}+2\mathbf{b}|^2 = |\mathbf{a}|^2 + 4|\mathbf{b}|^2 + 4\mathbf{a}\cdot\mathbf{b}$

$$= 1 + 16 + 4\,|\mathbf{a}||\mathbf{b}|\cos 60°$$

$$= 1 + 16 + 4.1.2.\frac{1}{2} = 21$$

$$|\mathbf{a}-3\mathbf{b}|^2 = |\mathbf{a}|^2 + 9|\mathbf{b}|^2 - 6\mathbf{a}\cdot\mathbf{b}$$

$$= 1 + 36 - 6 = 31$$

$$(\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}-3\mathbf{b}) = |\mathbf{a}|^2 - 6|\mathbf{b}|^2 - \mathbf{a}\cdot\mathbf{b}$$

$$= 1 - 24 - 1.2.\frac{1}{2} = -24$$

The angle between **a** + 2**b** and **a** – 3**b**

$$= \cos^{-1}\frac{(\mathbf{a}+2\mathbf{b})\cdot(\mathbf{a}-3\mathbf{b})}{|\mathbf{a}+2\mathbf{b}|\,|\mathbf{a}-3\mathbf{b}|}$$

$$= \cos^{-1}\frac{-24}{\sqrt{21}\sqrt{31}}.$$

LEVEL 1

Straight Objective Type Questions

Example 11: Let L_1: $\mathbf{r} = (\mathbf{i}+5\mathbf{j}+5\mathbf{k})+t(4\mathbf{i}-4\mathbf{j}+5\mathbf{k})$ and L_2: $\mathbf{r} = (2\mathbf{i}+4\mathbf{j}+5\mathbf{k})+t\,(8\mathbf{i}-3\mathbf{j}+\mathbf{k})$ be two lines then

(a) L_1 is parallel to L_2

(b) L_1 is perpendicular to L_2

(c) L_1 is not parallel to L_2

(d) none of these

Ans. (c)

Solution: The line L_1 is parallel to the vector $4\mathbf{i}-4\mathbf{j}+5\mathbf{k}$ and the line L_2 is parallel to the vector $8\mathbf{i}-3\mathbf{j}+\mathbf{k}$. These vectors are not parallel since neither is a scalar multiple of the other. Also $(4\mathbf{i}-4\mathbf{j}+5\mathbf{k}).(8\mathbf{i}-3\mathbf{j}+\mathbf{k})$ = 32 + 12 + 5 = 49 ≠ 0. So L_1 is not perpendicular to L_2.

Example 12: The angle between a diagonal of a cube and one of its edges is

(a) $\cos^{-1}(1/\sqrt{3})$ (b) $\pi/4$

(c) $\pi/6$ (d) $\pi/3$

Ans. (a)

Solution: Let $\mathbf{a} = a_1\,\mathbf{i}$, $\mathbf{b} = a_1\,\mathbf{j}$ and $\mathbf{c} = a_1\mathbf{k}$. Then the vector $\mathbf{d} = a_1(\mathbf{i}+\mathbf{j}+\mathbf{k})$ is a diagonal of the cube. The angle θ between **d** and **a** is given by

$$\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{d}}{|\mathbf{a}|\,|\mathbf{d}|} = \frac{a_1^2}{a_1\left(\sqrt{3a_1^2}\right)} = \frac{1}{\sqrt{3}}.$$

Example 13: Let $\mathbf{u} = 2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $\mathbf{a} = 4\mathbf{i}-\mathbf{j}+2\mathbf{k}$. The vector component of **u** orthogonal to **a** is

(a) $-\dfrac{1}{7}(6\mathbf{i}+2\mathbf{j}-11\mathbf{k})$ (b) $\dfrac{1}{7}(-6\mathbf{i}+2\mathbf{j}-11\mathbf{k})$

(c) $-\dfrac{1}{7}(6\mathbf{i}-2\mathbf{j}+11\mathbf{k})$ (d) $-\dfrac{1}{7}(-6\mathbf{i}+2\mathbf{j}+11\mathbf{k})$

Ans. (a)

Solution: $\text{Proj}_{\mathbf{a}}\,\mathbf{u} = \dfrac{\mathbf{u.a}}{|\mathbf{a}|^2}\mathbf{a} = \dfrac{5}{7}(4\mathbf{i}-\mathbf{j}+2\mathbf{k})$ and vector component of **u** orthogonal to **a** is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\,\mathbf{u} = (2\mathbf{i}-\mathbf{j}+3\mathbf{k}) - \frac{5}{7}(4\mathbf{i}-\mathbf{j}+2\mathbf{k})$$

$$= -\frac{6}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{11}{7}\mathbf{k}.$$

Example 14: Volume of the tetrahedron with vertices P(– 1, 2, 0), Q (2, 1, – 3), R (1, 0, 1) and S (3, – 2, 3) is

(a) 1/3 (b) 2/3

(c) 1/4 (d) 3/4

Ans. (b)

Solution: $\mathbf{PQ} = 3\mathbf{i}-\mathbf{j}-3\mathbf{k}$, $\mathbf{PR} = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $\mathbf{PS} = 4\mathbf{i}-4\mathbf{j}+3\mathbf{k}$ so the required volume

$$= \frac{1}{6}|\mathbf{PQ}\cdot(\mathbf{PR}\times\mathbf{PS})|$$

$$= \frac{1}{6}\left|\begin{vmatrix} 3 & -1 & -3 \\ 2 & -2 & 1 \\ 4 & -4 & 3 \end{vmatrix}\right| = \frac{|-4|}{6} = \frac{2}{3}.$$

Example 15: The distance between a point P whose position vector is $5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and the line $\mathbf{r} = (3\mathbf{i} + 7\mathbf{j} + \mathbf{k}) + t(\mathbf{j} + \mathbf{k})$ is

(a) 3 (b) 4

(c) 5 (d) 6

Ans. (d)

Solution: Let $\mathbf{u} = \mathbf{j} + \mathbf{k}$ and the point Q whose position vector is $3\mathbf{i} + 7\mathbf{j} + \mathbf{k}$ is on the line, so

$$\begin{aligned} \mathbf{v} &= \mathbf{QP} = (5 - 3)\mathbf{i} + (1 - 7)\mathbf{j} + (3 - 1)\mathbf{k} \\ &= 2\mathbf{i} - 6\mathbf{j} + 2\mathbf{k} \\ d &= |\mathbf{v}| \sin\theta \\ &= \frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{u}|} = \frac{|8\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}|}{|\mathbf{j}+\mathbf{k}|} = \frac{\sqrt{64+4+4}}{\sqrt{2}} = 6 \end{aligned}$$

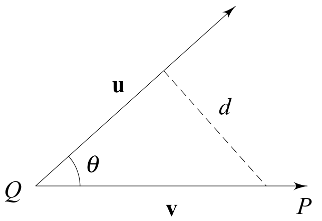


Fig. 22.4

Example 16: Let **a**, **b**, **c** be the three vectors such that $\mathbf{a}\cdot(\mathbf{b} + \mathbf{c}) = \mathbf{b}\cdot(\mathbf{c} + \mathbf{a}) = \mathbf{c}\cdot(\mathbf{a} + \mathbf{b}) = 0$ and $|\mathbf{a}| = 1$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 8$, then $|\mathbf{a} + \mathbf{b} + \mathbf{c}| =$

(a) 13 (b) 81

(c) 9 (d) 5

Ans. (c)

Solution: $|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = (\mathbf{a} + \mathbf{b} + \mathbf{c})\cdot(\mathbf{a} + \mathbf{b} + \mathbf{c})$

Adding $\mathbf{a}\cdot(\mathbf{b} + \mathbf{c}) = \mathbf{b}\cdot(\mathbf{c} + \mathbf{a}) = \mathbf{c}\cdot(\mathbf{a} + \mathbf{b}) = 0$,

we get $2(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{a}\cdot\mathbf{c}) = 0$

$$\begin{aligned} \Rightarrow\quad |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 &= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 \\ &\quad + 2(\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{c}) \\ &= 1 + 16 + 64 = 81. \text{ Hence } |\mathbf{a} + \mathbf{b} + \mathbf{c}| = 9. \end{aligned}$$

Example 17: If $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{j}$, then t such that $\mathbf{a} + t\,\mathbf{b}$ is at right angle to **c** will be equal to

(a) 5 (b) 4

(c) 6 (d) 2

Ans. (a)

Solution: Since $\mathbf{a} + t\mathbf{b}$ is at right angle to **c** so $(\mathbf{a} + t\mathbf{b})\cdot\mathbf{c} = 0$. But $\mathbf{a}\cdot\mathbf{c} = 5$ and $\mathbf{b}\cdot\mathbf{c} = -1$ so $5 - t = 0 \;\Rightarrow\; t = 5$.

Example 18: If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and $|\mathbf{a}\times\mathbf{b}| = 8$ then $|\mathbf{a}\cdot\mathbf{b}|$ is equal to

(a) 4 (b) 6

(c) 5 (d) none of these

Ans. (b)

Solution: Since $\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \dfrac{8}{10} = \dfrac{4}{5}$ so $\cos\theta = \pm 3/5$

Thus $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = 10\cdot(\pm 3/5) = \pm 6$.

Example 19: If $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{c} = \mathbf{c}\cdot\mathbf{a} = 0$, then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ is equal to

(a) 0 (b) 1

(c) – 1 (d) $|\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$

Ans. (d)

Solution: Since $\mathbf{a}\cdot\mathbf{b} = \mathbf{c}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{c} = 0$ so **a**, **b** and **c** are mutually perpendicular. Thus **a** is collinear with $\mathbf{b}\times\mathbf{c}$. Hence $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = |\mathbf{a}|\,|\mathbf{b}\times\mathbf{c}|$

$$= |\mathbf{a}||\mathbf{b}||\mathbf{c}|.$$

Example 20: Let **a**, **b** and **c** be three non-coplanar vectors, and let **p**, **q** and **r** be the vectors defined by the relations

$$\mathbf{p} = \frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\quad \mathbf{q} = \frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} \text{ and } \mathbf{r} = \frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Then the value of the expression $(\mathbf{a} + \mathbf{b})\cdot\mathbf{p} + (\mathbf{b} + \mathbf{c})\cdot\mathbf{q} + (\mathbf{c} + \mathbf{a})\cdot\mathbf{r}$ is equal to

(a) 0 (b) 1

(c) 2 (d) 3

Ans. (d)

Solution: $\mathbf{a}\cdot\mathbf{p} = \dfrac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \dfrac{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 1 = \mathbf{b}\cdot\mathbf{q} = \mathbf{c}\cdot\mathbf{r}$

$$\mathbf{b}\cdot\mathbf{p} = \frac{\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = \frac{0}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]} = 0 = \mathbf{c}\cdot\mathbf{q} = \mathbf{a}\cdot\mathbf{r}$$

Therefore, the given expression is equal to 1 + 0 + 1 + 0 + 1 + 0 = 3.

Example 21: The volume of the parallelopiped whose sides are given by $\mathbf{OA} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{OB} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{OC} = 3\mathbf{i} - \mathbf{k}$ is

(a) 4/13 (b) 4

(c) 2/7 (d) none of these

Ans. (b)

Solution: The volume of the parallelopiped

$$= |[\mathbf{OA}\ \mathbf{OB}\ \mathbf{OC}]| = \left|\begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}\right| = 4.$$

Example 22: The points with position vectors $60\mathbf{i} + 3\mathbf{j}$, $40\mathbf{i} - 8\mathbf{j}$, $a\mathbf{i} - 52\mathbf{j}$ are collinear if

(a) $a = -40$ (b) $a = 40$

(c) $a = 20$ (d) none of these

Ans. (a)

Solution: Denoting **a**, **b**, **c** by the given vectors respectively. These vectors will be collinear if there is some constant K such that $\mathbf{c} - \mathbf{a} = K(\mathbf{b} - \mathbf{a})$

$$\begin{aligned} \Rightarrow\quad a - 60 &= -20K \text{ and } -55 = -11K \\ \Rightarrow\quad a &= -100 + 60 = -40. \end{aligned}$$

Example 23: If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$ $|\mathbf{c}| = 4$ and $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ then the value of $\mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b}$ is equal to

(a) 19/2 (b) –19/2

(c) 29/2 (d) –29/2

Ans. (d)

◎ **Solution:** $0 = |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2$
$+ 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c})$

$\Rightarrow \quad 29 + 2\,(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = 0$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}) = -\,29/2.$

◉ **Example 24:** If *A*, *B*, *C*, *D* are four points in a space and $|\mathbf{AB} \times \mathbf{CD} + \mathbf{BC} \times \mathbf{AD} + \mathbf{CA} \times \mathbf{BD}| = \lambda$ (area of the triangle *ABC*). Then the value of λ is

(a) 1 (b) 2
(c) 3 (d) 4

Ans. (d)

◎ **Solution:** Let *D* be the origin of reference and $\mathbf{DA} = \mathbf{a}$, $\mathbf{DB} = \mathbf{b}$, $\mathbf{DC} = \mathbf{c}$

$$|\,\mathbf{AB}\times\mathbf{CD}+\mathbf{BC}\times\mathbf{AD}+\mathbf{CA}\times\mathbf{BD}\,|$$

$$= |\,(\mathbf{b}-\mathbf{a})\times(-\,\mathbf{c})+(\mathbf{c}-\mathbf{b})\times(-\,\mathbf{a})+(\mathbf{a}-\mathbf{c})\times(-\,\mathbf{b})\,|$$

$$= |\,\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{b}\,|$$

$$= 2\,|\,\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}\,|$$

$= 2$ (2 area of $\Delta\,ABC$) = 4 area of $\Delta\,ABC$.

Hence $\lambda = 4$.

◉ **Example 25:** Given $\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{b} = -\,\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = -\,\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. A unit vector perpendicular to both $\mathbf{a} + \mathbf{b}$ and $\mathbf{b} + \mathbf{c}$ is

(a) $\dfrac{2\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{6}}$ (b) $\mathbf{j}$

(c) $\mathbf{k}$ (d) $\dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$

Ans. (c)

◎ **Solution:** $(\mathbf{a} + \mathbf{b}) \times (\mathbf{b} + \mathbf{c}) = 3\mathbf{j}\times(-2\mathbf{i}+4\mathbf{j}) = 6\mathbf{k}$

Hence the required unit vector is $\mathbf{k}$.

◉ **Example 26:** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit coplanar vectors, then the scalar triple product $[2\mathbf{a} - \mathbf{b}\ \ 2\mathbf{b} - \mathbf{c}\ \ 2\mathbf{c} - \mathbf{a}] =$

(a) 0 (b) 1
(c) $-\sqrt{3}$ (d) $\sqrt{3}$

Ans. (a)

◎ **Solution:** As $\mathbf{a}\ \mathbf{b}, \mathbf{c}$ are coplanar vectors, $2\mathbf{a} - \mathbf{b}$, $2\mathbf{b} - \mathbf{c}$ and $2\mathbf{c} - \mathbf{a}$ are also coplanar vectors. Thus $[2\mathbf{a} - \mathbf{b}\ \ 2\mathbf{b} - \mathbf{c}\ \ 2\mathbf{c} - \mathbf{a}] = 0$.

◉ **Example 27:** If the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ form the sides *BC*, *CA* and *AB* respectively, of a triangle *ABC*, then

(a) $\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{c}+\mathbf{c}\cdot\mathbf{a}=0$

(b) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$

(c) $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$

(d) $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}=\mathbf{0}$

Ans. (b)

◎ **Solution:** $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{BC}+\mathbf{CA}+\mathbf{AB}=\mathbf{BA}+\mathbf{AB}$

So $\quad \mathbf{a}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{0}=\mathbf{0}$

$\Rightarrow \quad \mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=\mathbf{0} \quad\Rightarrow\quad \mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$

Similarly, $\quad \mathbf{b}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=\mathbf{0}$

$\Rightarrow \quad \mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{c}=\mathbf{0} \quad\Rightarrow\quad \mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}$. Thus

$\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}=\mathbf{b}\times\mathbf{c}$.

◉ **Example 28:** If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors such that $\mathbf{a} + 2\mathbf{b}$ and $5\mathbf{a} - 4\mathbf{b}$ are perpendicular to each other then the angle between $\mathbf{a}$ and $\mathbf{b}$ is

(a) 45º (b) 60º

(c) $\cos^{-1}(1/\sqrt{3})$ (d) $\cos^{-1}(2/7)$

Ans. (b)

◎ **Solution:** According to the given conditions $|\mathbf{a}| = |\mathbf{b}| = 1$ and $(\mathbf{a} + 2\mathbf{b})\cdot(5\mathbf{a} - 4\mathbf{b}) = 0$ so

$$5|\mathbf{a}|^2 - 8|\mathbf{b}|^2 + 6\mathbf{a}\cdot\mathbf{b} = 0$$

$\Rightarrow \quad \mathbf{a}\cdot\mathbf{b} = 3/6 = 1/2$

$\Rightarrow \quad |\mathbf{a}|\,|\mathbf{b}| \cos\theta = 1/2$

$\Rightarrow \quad \cos\theta = 1/2 \quad\Rightarrow\quad \theta = 60^\circ$.

◉ **Example 29:** Let $\mathbf{v} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{w} = \mathbf{i}+3\mathbf{k}$. If $\mathbf{u}$ is a unit vector, then the maximum value of the scalar triple product $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ is

(a) $-\,1$ (b) $\sqrt{10}+\sqrt{16}$

(c) $\sqrt{59}$ (d) $\sqrt{60}$

Ans. (c)

◎ **Solution:** $\mathbf{v}\times\mathbf{w} = 3\mathbf{i}-7\mathbf{j}-\mathbf{k}$

Now $\quad [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u}\cdot(3\mathbf{i}-7\mathbf{j}-\mathbf{k})$

$= |\mathbf{u}|\ |3\mathbf{i}-7\mathbf{j}-\mathbf{k}|\ \cos\theta$ (where θ is the angle between $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$)

$= \sqrt{59}\ \cos\theta$

Thus $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$ is maximum if $\cos\theta = 1$ i.e. $\theta = 0$ or take $\mathbf{u} = \dfrac{1}{\sqrt{59}}(3\mathbf{i}-7\mathbf{j}-\mathbf{k})$. Hence the maximum value is $\sqrt{59}$.

◉ **Example 30:** A vector $\mathbf{c}$ perpendicular to the vectors $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ satisfying $\mathbf{c}.(2\mathbf{i}-\mathbf{j}+\mathbf{k})=-6$ is

(a) $-2\mathbf{i}+\mathbf{j}-\mathbf{k}$ (b) $2\mathbf{i}-\mathbf{j}-\dfrac{4}{3}\mathbf{k}$

(c) $-3\mathbf{i}+3\mathbf{j}+3\mathbf{k}$ (d) $3\mathbf{i}-3\mathbf{j}+3\mathbf{k}$

Ans. (c)

◎ **Solution:** Vector $\mathbf{c}$ perpendicular to $\mathbf{a}=2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ and $\mathbf{b} = \mathbf{i}-2\mathbf{j}+3\mathbf{k}$ is of the form $\alpha\,(\mathbf{a}\times\mathbf{b}) = \alpha\,(7\mathbf{i} - 7\mathbf{j} - 7\mathbf{k})$. Since $\mathbf{c}\cdot(2\mathbf{i}-\mathbf{j}+\mathbf{k}) = -\,6$ so $\alpha\,[14+7-7] = -\,6 \Rightarrow \alpha = -\,3/7$. Hence $\mathbf{c} = -3\mathbf{i}+3\mathbf{j}+3\mathbf{k}$.

◉ **Example 31:** If **a**, **b**, **c** be three units vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (1/2)\,\mathbf{b}$; **b** and **c** being non-parallel then

(a) the angle between **a** and **c** is $\pi/3$
(b) the angle between **a** and **c** is $\pi/2$
(c) the angle between **a** and **b** is $\pi/3$
(d) the angle between **a** and **b** is $\pi/6$

Ans. (a)

◎ **Solution:** $(1/2)\ \mathbf{b} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

$$\Rightarrow \qquad \left(\mathbf{a} \cdot \mathbf{c} - \frac{1}{2}\right)\mathbf{b} = (\mathbf{a} \cdot \mathbf{b})\ \mathbf{c}.$$

The last relation is possible only if $\mathbf{a} \cdot \mathbf{c} = 1/2$ and $\mathbf{a} \cdot \mathbf{b} = 0$ as **b** and **c** are non-parallel. Hence the angle between **a** and **c** is $\pi/3$ and between **a** and **b** it is $\pi/2$.

◉ **Example 32:** A vector **a** of magnitude 50 is collinear with the vector $6\mathbf{i} - 8\mathbf{j} - (15/2)\mathbf{k}$ making on obtuse angle with the *z*-axis is

(a) $24\mathbf{i} - 32\mathbf{j} - 30\mathbf{k}$ (b) $-24\mathbf{i} + 32\mathbf{j} + 30\mathbf{k}$
(c) $24\mathbf{i} + 32\mathbf{j} - 30\mathbf{k}$ (d) none of these

Ans. (a)

◎ **Solution:** A unit vector along $\mathbf{b} = 6\mathbf{i} - 8\mathbf{j} - \dfrac{15}{2}\mathbf{k}$, is $\pm\dfrac{2}{25}\left(6\mathbf{i} - 8\mathbf{j} - \dfrac{15}{2}\mathbf{k}\right)$, so a vector of length 50 along **b** is $\pm 4\left(6\mathbf{i} - 8\mathbf{j} - \dfrac{15}{2}\mathbf{k}\right)$. Since **a** makes obtuse angle with *z*-axis so we must have $\mathbf{a}\cdot\mathbf{k} < 0$. Thus $\mathbf{a} = 24\mathbf{i} - 32\mathbf{j} - 30\mathbf{k}$.

◉ **Example 33:** Let there be two points *A*, *B* on the curve $y = x^2$ in the plane *OXY* satisfying $\mathbf{OA} \cdot \mathbf{i} = 1$ and $\mathbf{OB} \cdot \mathbf{i} = -2$ then the length of the vector $2\mathbf{OA} - 3\mathbf{OB}$ is

(a) $\sqrt{14}$ (b) $2\sqrt{51}$
(c) $3\sqrt{41}$ (d) none of these

Ans. (d)

◎ **Solution:** Let $\mathbf{OA} = x_1\,\mathbf{i} + y_1\,\mathbf{j}$ and $\mathbf{OB} = x_2\,\mathbf{i} + y_2\,\mathbf{j}$. Since $1 = \mathbf{OA}.\,\mathbf{i} = x_1$ and $-2 = \mathbf{OB}.\,\mathbf{i} = x_2$. Moreover, $y_1 = x_1^2 = 1$ and $y_2 = x_2^2 = 4$, so
$\mathbf{OA} = \mathbf{i} + \mathbf{j}$ and $\mathbf{OB} = -2\,\mathbf{i} + 4\,\mathbf{j}$. Hence $|2\mathbf{OA} - 3\mathbf{OB}| = |8\mathbf{i} - 10\mathbf{j}| = \sqrt{164} = 2\sqrt{41}$.

◉ **Example 34:** If *A*, *B*, *C*, *D* are four points in space satisfying $\mathbf{AB}\cdot\mathbf{CD} = K\left[|\mathbf{AD}|^2 + |\mathbf{BC}|^2 - |\mathbf{AC}|^2 - |\mathbf{BD}|^2\right]$ then the value of *K* is

(a) 2 (b) 1/3
(c) 1/2 (d) 1

Ans. (c)

◎ **Solution:** Let *A* be the origin of reference and the position vector of *B*, *C*, *D* be **b**, **c**, **d**, w.r.t. *A*. So $\mathbf{AB} = \mathbf{b}$, $\mathbf{CD} = \mathbf{d} - \mathbf{c}$, $\mathbf{AD} = \mathbf{d}$, $\mathbf{BC} = \mathbf{c} - \mathbf{b}$, $\mathbf{AC} = \mathbf{c}$ and $\mathbf{BD} = \mathbf{d} - \mathbf{b}$. The L.H.S. is equal to $\mathbf{b}\cdot(\mathbf{d} - \mathbf{c})$ .The R.H.S. is

$$K\left[|\mathbf{d}|^2 + |\mathbf{c} - \mathbf{b}|^2 - |\mathbf{c}|^2 - |\mathbf{d} - \mathbf{b}|^2\right]$$
$$= K\left[\mathbf{d}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{b} - 2\mathbf{c}\cdot\mathbf{b} - \mathbf{c}\cdot\mathbf{c} - \mathbf{d}\cdot\mathbf{d} - \mathbf{b}\cdot\mathbf{b} + 2\mathbf{d}\cdot\mathbf{b}\right]$$
$$= 2K\left[\mathbf{b}\cdot(\mathbf{d} - \mathbf{c})\right].$$

Hence $K = 1/2$.

◉ **Example 35:** The distance of the point *B* with position vector $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ from the line passing through the point *A* with position vector $4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and parallel to the vector $2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$ is

(a) $\sqrt{10}$ (b) $\sqrt{5}$
(c) $\sqrt{6}$ (d) none of these

Ans. (a)

◎ **Solution:** $\mathbf{AB} = -3\mathbf{i} + \mathbf{k}$. Since $\mathbf{AB}\cdot(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) = -6 + 6 = 0$. Hence **AB** is perpendicular to the given line. Thus required distance is equal to $|\mathbf{AB}| = \sqrt{9+1} = \sqrt{10}$.

◉ **Example 36:** If **a**, **b** and **c** are unit vectors then $|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2$ does not exceed.

(a) 4 (b) 9
(c) 8 (d) 6

Ans. (b)

◎ **Solution:** Since $|\mathbf{a} + \mathbf{b} + \mathbf{c}| \geq 0$

$$\Rightarrow \qquad 0 \leq |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b})$$
$$= 3 + 2\,(\boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{b})$$
$$\Rightarrow \qquad \boldsymbol{b}.\boldsymbol{c} + \boldsymbol{c}.\boldsymbol{a} + \boldsymbol{a}.\boldsymbol{b} \geq -3/2$$
$$|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{a}|^2$$
$$= 2\left(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 - \mathbf{b}.\mathbf{c} - \mathbf{c}.\mathbf{a} - \mathbf{a}.\mathbf{b}\right)$$
$$\leq 2\,(1 + 1 + 1) + 3 = 9$$

The maximum value is attained e.g. when

$$\boldsymbol{a} = \mathbf{i},\ \boldsymbol{b} = \frac{1}{2}\left(-\mathbf{i} + \sqrt{3}\mathbf{j}\right) \text{ and } \mathbf{c} = \frac{1}{2}\left(-\mathbf{i} - \sqrt{3}\mathbf{j}\right).$$

◉ **Example 37:** The value of *a* so that the volume of the parallelepiped formed by $\boldsymbol{i} + a\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{j} + a\boldsymbol{k}$ and $a\,\boldsymbol{i} + \boldsymbol{k}$ becomes minimum is

(a) –3 (b) 3
(c) $1/\sqrt{3}$ (d) $\sqrt{3}$

Ans. (c)

◎ **Solution:** Volume of the parallelepiped is

$$V(a) = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = a^3 - a + 1$$

$$V'(a) = 3a^2 - 1 = 0 \text{ if } a = a \pm \frac{1}{\sqrt{3}}$$

$$V''(a) = 6a > 0 \text{ if } a = \frac{1}{\sqrt{3}}$$

Thus $V(a)$ is minimum when $a = \dfrac{1}{\sqrt{3}}$.

Example 38: If $\mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}$, $\mathbf{a} \cdot \mathbf{b} = 1$ and $\mathbf{a} \times \mathbf{b} = -\mathbf{j} + \mathbf{k}$, then **k** is equal to

(a) $\mathbf{i} + \mathbf{j} - \mathbf{k}$ (b) $-2\mathbf{j} + \mathbf{k}$

(c) $\mathbf{i}$ (d) $-2\mathbf{i} + \mathbf{k}$

Ans (c)

Solution: let $\mathbf{b} = \alpha\,\mathbf{i} + \beta\,\mathbf{j} + \gamma\mathbf{k}$ then $\mathbf{a} \cdot \mathbf{b} = 1$

$$\Rightarrow \quad \alpha - \beta - \gamma = 1. \qquad (1)$$

And $\mathbf{a} \times \mathbf{b} = \mathbf{j} + \mathbf{k}$ $\Rightarrow$ $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ \alpha & \beta & \gamma \end{vmatrix} = \mathbf{j} + \mathbf{k}$

$$\Rightarrow \quad (\beta - \gamma)\,\mathbf{i} - (\gamma + \alpha)\mathbf{j} + (\beta + \gamma)\,\mathbf{k} = -\mathbf{j} + \mathbf{k}$$

$$\Rightarrow \quad \beta - \gamma = 0,\ \gamma + \alpha = 1,\ \beta + \alpha = 1$$

$$\Rightarrow \quad \beta = \gamma,\ \alpha = 1 - \gamma$$

Putting there values in (1)

$$1 - \gamma - \gamma - \gamma = 1 \Rightarrow \gamma = 0 \text{ so } \alpha = 1;\ \beta = 0$$

Thus $\quad \mathbf{b} = \mathbf{i}.$

Example 39: The unit vector which is orthogonal to the vector $5\,\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ and is coplanar with the vectors $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ is

(a) $\frac{1}{\sqrt{41}}(2\mathbf{i} - 6\mathbf{j} + \mathbf{k})$ (b) $\frac{1}{\sqrt{29}}(2\mathbf{i} - 5\mathbf{j})$

(c) $\frac{1}{\sqrt{10}}(3\mathbf{j} - \mathbf{k})$ (d) $\frac{1}{\sqrt{69}}(2\mathbf{i} - 8\mathbf{j} + \mathbf{k})$

Ans (c)

Solution: A vector coplanar with $2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ is of the form

$$\mathbf{a} = \alpha\,(2\mathbf{i} + \mathbf{j} + \mathbf{k}) + \beta\,(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$= (2\alpha + \beta)\,\mathbf{i} + (\alpha - \beta)\,\mathbf{j} + (\alpha + \beta)\mathbf{k}$$

This vector will be orthogonal to $5\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$ if

$$5(2\alpha + \beta) + 2(\alpha - \beta) + 6(\alpha + \beta) = 0$$

$$\Rightarrow \quad 18\alpha + 9\beta = 0 \quad \Rightarrow \quad \beta = -2\alpha$$

So **a** is of the form $\alpha\,(3\mathbf{j} - \mathbf{k})$

Thus, a required unit vector is

$$\frac{1}{\sqrt{10}}(3\mathbf{j} - \mathbf{k}).$$

Example 40: Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. A vector in the plane of **a** and **b** whose projection on **c** is $1/\sqrt{3}$ is

(a) $4\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ (b) $3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

(c) $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ (d) $4\mathbf{i} + \mathbf{j} - 4\mathbf{k}$

Ans. (c)

Solution: A vector in the plane of **a** and **b** is given by

$$\mathbf{d} = \alpha\,\mathbf{a} + \beta\,\mathbf{b}$$

$$= (\alpha + \beta)\,\mathbf{i} + (2\alpha - \beta)\mathbf{j} + (\alpha + \beta)\mathbf{k}$$

Length of projection of **d** on **c** is

$$\frac{\mathbf{d} \cdot \mathbf{c}}{|\mathbf{c}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{(\alpha + \beta) + (2d - \beta) - (d + \beta)}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}$$

This is satisfied when $\alpha = 1$, $\beta = 1$.

Example 41: If $|\mathbf{a}| = 1$, $|\mathbf{b}| = 2$ and $|\mathbf{a} - 2\mathbf{b}| = 4$ then $|\mathbf{a} + 3\mathbf{b}|$ is equal to

(a) 8 (b) $\sqrt{\frac{51}{2}}$

(c) $\frac{\sqrt{19}}{2}$ (d) $\sqrt{\frac{77}{2}}$

Ans (d)

Solution: $16 = |\mathbf{a} - 2\mathbf{b}|^2 = |\mathbf{a}|^2 + 4\,|\mathbf{b}|^2 - 4\,\mathbf{a} \cdot \mathbf{b}$

$$= 1 + 16 - 4\,\mathbf{a} \cdot \mathbf{b}$$

$$4\,\mathbf{a} \cdot \mathbf{b} = 1 \quad \Rightarrow \quad \mathbf{a} \cdot \mathbf{b} = \frac{1}{4}$$

$$|\mathbf{a} + 3\mathbf{b}|^2 = |\mathbf{a}|^2 + 9\,|\mathbf{b}|^2 + 6\,\mathbf{a} \cdot \mathbf{b}$$

$$= 1 + 36 + \frac{6}{4} = \frac{77}{2}.$$

Example 42: If $|\mathbf{a}|^2 = 8$ and $\mathbf{a} \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \mathbf{0}$ then the value of $\mathbf{a} \cdot (-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ is

(a) $\frac{4}{\sqrt{3}}$ (b) $\frac{16}{\sqrt{3}}$

(c) $\frac{8}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{3}}$

Ans (b)

Solution: Since $\mathbf{a} \times (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = \mathbf{0}$ so **a** is parallel to $\mathbf{i} + \mathbf{j} + 2\mathbf{k} \Rightarrow \mathbf{a} = \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

but $\quad 8 = |\mathbf{a}|^2 = \lambda^2 \cdot 6 \Rightarrow \lambda^2 = \frac{4}{3}.$

$$\mathbf{a}.(-\mathbf{i} + \mathbf{j} + 4\mathbf{k}) = \lambda(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$$

$$= \lambda\,(-1 + 1 + 8) = 8\lambda = \frac{8.2}{\sqrt{3}} = \frac{16}{\sqrt{3}}$$

Example 43: If **a**, **b**, **c** are unit vectors, then the maximum value of $|\mathbf{a} + 2\mathbf{b}|^2 + |\mathbf{b} + 3\mathbf{c}|^2 + |\mathbf{c} + 4\mathbf{a}|^2$ is

(a) 28 (b) 21

(c) 48 (d) 58

Ans (b)

Solution: $|\mathbf{a} + 2\mathbf{b}|^2 + |\mathbf{b} + 3\mathbf{c}|^2 + |\mathbf{c} + 4\mathbf{a}|^2$

$$= |\mathbf{a}|^2 + 4|\mathbf{b}|^2 + 2 \times \mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 + 9\,|\mathbf{c}|^2 + 6\,\mathbf{b} \cdot \mathbf{c}$$

$$+ |\mathbf{c}|^2 + 16\,|\mathbf{a}|^2 + 8\,\mathbf{a} \cdot \mathbf{c}$$

$$\le 1 + 4 + 2 + 1 + 9 + 6 + 1 + 16 + 8 = 48$$

Example 44: Let $\mathbf{a} = 2\,\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j}$. If **c** is a vector such that $\mathbf{a} \cdot \mathbf{c} = |\mathbf{c}|$ and $|\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$ and angle between $\mathbf{a} \times \mathbf{b}$ and **c** is 30°, then $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}|$ equals:

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$

(c) 2 (d) $\frac{\sqrt{3}}{2}$

Ans. (a)

Solution: $|\mathbf{a}|^2 = 4 + 1 + 4 = 9$, $|\mathbf{b}|^2 = 1 + 1 = 2$.

Also, $|\mathbf{c} - \mathbf{a}|^2 = 8 \Rightarrow |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\,\mathbf{c}.\mathbf{a} = 8$

$\Rightarrow \quad |\mathbf{c}|^2 + 9 - 2|\mathbf{c}| = 8$

$\Rightarrow \quad |\mathbf{c}|^2 - 2|\mathbf{c}| + 1 = 0 \Rightarrow (|\mathbf{c}| - 1)^2 = 0 \Rightarrow |\mathbf{c}| = 1.$

Also, $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\Rightarrow \quad |\mathbf{a} \times \mathbf{b}| = 3$

Also, $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = |\mathbf{a} \times \mathbf{b}|\,|\,\mathbf{c}| \sin 30° = (3)\,(1)\,(1/2) = 3/2.$

Example 45: The non-zero vectors **a**, **b** and **c** are related by **a** = 8**b** and **c**= – 7**b**. Then the angle between **a** and **c** is

(a) 0 (b) $\pi/4$

(c) $\pi/2$ (d) π

Ans. (d)

Solution: $\mathbf{a} = 8\mathbf{b} = -\frac{8}{7}\mathbf{c}$ so **a** and **c** are parallel and **a** and **c** have opposite direction so angle between **a** and **c** is π.

Example 46: If **u**, **v**, **w** are non-coplanar vectors and p, q are real numbers, then the equality $[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}] - [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}] = 0$ holds for

(a) more than two but not all values of (p, q)

(b) all values of (p, q)

(c) exactly one values of (p, q)

(d) exactly two values of (p, q)

Ans (c)

Solution: $0 = [3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}] - [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}]$

$= (3p^2 - pq + 2q^2)\,[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$

As **u**, **v**, **w** are non – coplanar so $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] \neq 0$, so

$3p^2 - pq + 2q^2 = 0$

$\Rightarrow \quad 2\left(q - \frac{1}{4}p\right)^2 + \frac{23}{8}p^2 = 0$

$\Rightarrow \quad p = 0, q = \frac{1}{4}p = 0$

Thus there is exactly one value of (p, q).

Example 47: If the vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \lambda\mathbf{i} + \mathbf{j} + \mu\mathbf{k}$ are mutually orthogonal then $(\lambda, \mu) =$

(a) (–2, 3) (b) (3, –2)

(c) (–3, 2) (d) (2, –3)

Ans. (c)

Solution: As **a**, **b** and **c** are mutually orthogonal,

$\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$

$\Rightarrow \quad \lambda - 1 + 2\mu = 0$ and $2\lambda + 4 + \mu = 0$

$\Rightarrow \quad \lambda + 2\mu = 1$ and $2\lambda + \mu = -4$

$\Rightarrow \quad \lambda = -3, \mu = 2.$

Example 48: Let $\mathbf{a} = \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Then the vector **b** satisfying $(\mathbf{a} \times \mathbf{b}) + \mathbf{c} = \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{b} = 3$ is

(a) $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ (b) $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(c) $-\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ (d) $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Ans (c)

Solution: Let $\mathbf{b} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$.

We have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ \alpha & \beta & \gamma \end{vmatrix} = (\beta + \gamma)\mathbf{i} - \alpha\mathbf{j} - \alpha\mathbf{k}$$

As $(\mathbf{a} \times \mathbf{b}) + \mathbf{c} = \mathbf{0}$, we get

$(\beta + \gamma + 1)\mathbf{i} - (\alpha + 1)\mathbf{j} - (\alpha + 1)\,\mathbf{k} = \mathbf{0}$

$\Rightarrow \quad \beta + \gamma + 1 = 0, \alpha = -1$

Also, as $\mathbf{a} \cdot \mathbf{b} = 3$, we get $\beta - \gamma = 3$

Thus $\alpha = -1$, $b = 1$, $\gamma = -2$

Hence, $\mathbf{b} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Example 49: If $\mathbf{a} = \frac{1}{\sqrt{10}}(3\mathbf{i} + k)$ and $\mathbf{b} = \frac{1}{7}(2\mathbf{i} + 3\mathbf{j} - 6k)$, then the value of $(2\mathbf{a} - \mathbf{b}).\,[(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$ is:

(a) 3 (b) –5

(c) –3 (d) 5

Ans. (b)

Solution: we have $\mathbf{a}\,.\,\mathbf{a} = 1$, $\mathbf{b}.\,\mathbf{b} = 1$ and

$\mathbf{a}.\,\mathbf{b} = \frac{1}{7\sqrt{10}} = ((3)\,(2) + (1)\,(-6)) = 0 = \mathbf{b}\,.\,\mathbf{a}$

Now, $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})$

$= (\mathbf{a}\,.\,(\mathbf{a} + 2\mathbf{b}))\,\mathbf{b} - (\mathbf{b}\,.\,(\mathbf{a} + 2\mathbf{b}))\,\mathbf{a}$

$= (\mathbf{a}\,.\,\mathbf{a})\,\mathbf{b} - 2\,(\mathbf{b}\,.\,\mathbf{b})\mathbf{a}\ (\mathbf{a}\,.\,\mathbf{b} = 0)$

$= \mathbf{b} - 2\mathbf{a}.$

Thus $(2\mathbf{a} - \mathbf{b})\,.\,[(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$

$= (2\mathbf{a} - \mathbf{b})\,.\,(\mathbf{b} - 2\mathbf{a}) = -\,|2\mathbf{a} - \mathbf{b}|^2$

$= -\left[4|\mathbf{a}|^2 + |\mathbf{b}|^2 - 4\mathbf{a}.\mathbf{b}\right] = -5.$

Example 50: The vectors **a** and **b** are not perpendicular and **c** and **d** are the vectors satisfying $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ and $\mathbf{a}\,.\,\mathbf{d} = 0$. Then the vector **d** is equal to :

(a) $\mathbf{c} - \left(\frac{\mathbf{a.c}}{\mathbf{a.b}}\right)\mathbf{b}$ (b) $\mathbf{b} - \left(\frac{\mathbf{b.c}}{\mathbf{a.b}}\right)\mathbf{c}$

(c) $\mathbf{c} + \left(\frac{\mathbf{a.c}}{\mathbf{a.b}}\right)\mathbf{b}$ (d) $\mathbf{b} + \left(\frac{\mathbf{b.c}}{\mathbf{a.b}}\right)\mathbf{c}$

And. (a)

Solution: $\mathbf{b}\times\mathbf{c}=\mathbf{b}\times\mathbf{d}\Rightarrow\mathbf{b}\times(\mathbf{c}-\mathbf{d})=\mathbf{0}$

$\Rightarrow\quad \mathbf{c}-\mathbf{d}\parallel\mathbf{b}$

$\Rightarrow\quad \mathbf{c}-\mathbf{d}=\alpha\mathbf{b}$ for some $\alpha\in\mathbf{R}$

$\Rightarrow\quad \mathbf{d}=\mathbf{c}-\alpha\,\mathbf{b}$

Also, $\mathbf{a}\cdot\mathbf{d}=\mathbf{a}\,.\,\mathbf{c}-\alpha\,\mathbf{a}\,.\,\mathbf{b}$

$\Rightarrow\quad 0=\mathbf{a}\cdot\mathbf{c}-\alpha\mathbf{a}\,.\,\mathbf{b}$

$$\Rightarrow\quad \alpha=\frac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}$$

Thus, $\quad \mathbf{d}=\mathbf{c}-\dfrac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\,\mathbf{b}.$

Example 51: If vectors p$\mathbf{i}$ + q$\mathbf{j}$ + $\mathbf{k}$, $\mathbf{i}$ + q$\mathbf{j}$ + $\mathbf{k}$ and $\mathbf{i}$ + $\mathbf{j}$ + r$\mathbf{k}$ ($p\neq q\neq r\neq 1$) are coplanar, then the value of $p\,q\,r-(p+q+r)$ is

(a) 2 (b) 0

(c) –1 (d) –2

Ans. (d)

Solution: Since the given vectors are coplanar so

$$\begin{vmatrix} p & 1 & 1\\ 1 & q & 1\\ 1 & 1 & r\end{vmatrix}=0$$

$$\Rightarrow\quad p\begin{vmatrix} q & 1\\ 1 & r\end{vmatrix}-\begin{vmatrix} 1 & 1\\ 1 & r\end{vmatrix}+\begin{vmatrix} 1 & q\\ 1 & 1\end{vmatrix}=0$$

$$\Rightarrow\quad p(qr-1)-(r-1)+(1-q)=0$$

$$\Rightarrow\quad pqr-p-q-r=-2$$

Example 52: Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three non-zero vectors which are pairwise non-collinear. If $\mathbf{a}+3\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b}+2\mathbf{c}$ is collinear with $\mathbf{a}$, then $\mathbf{a}+3\mathbf{b}+6\mathbf{c}$ is

(a) $\mathbf{a}$ (b) $\mathbf{b}$

(c) $\mathbf{0}$ (d) $\mathbf{a}+\mathbf{c}$

Ans. (c)

Solution: We are given that

$$\mathbf{a}+3\mathbf{b}=\alpha\mathbf{c}$$

and $\quad \mathbf{b}+2\mathbf{c}=\beta\mathbf{a}$

for some $\alpha,\beta\in\mathbf{R}$.

We have

$$\mathbf{a}+3\mathbf{b}+6\mathbf{c}=\alpha\mathbf{c}+6\mathbf{c}=(6+\alpha)\mathbf{c}$$

and $\quad \mathbf{a}+3\mathbf{b}+6\mathbf{c}=\mathbf{a}+3(\mathbf{b}+2\mathbf{c})$

$$=\mathbf{a}+3\beta\mathbf{a}=(1+3\beta)\mathbf{b}$$

Thus, $\quad (6+\alpha)\mathbf{c}=(1+3\beta)\mathbf{a}$

If $6+\alpha\neq 0$, we get $\mathbf{c}=\dfrac{1+3\beta}{6+\alpha}\,\mathbf{a}$, so $\mathbf{c}$ is collinear with $\mathbf{a}$.

Thus $6+\alpha=0$, therefore, $\mathbf{a}+3\mathbf{b}+6\mathbf{c}=0$.

Example 53: Let $\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b}=\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{c}=\mathbf{i}-\mathbf{j}-\mathbf{k}$ be three vectors. A vector $\mathbf{v}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$, whose projection on $\mathbf{c}$ is $1/\sqrt{3}$, , is given by

(a) $\mathbf{i}-3\mathbf{j}+3\mathbf{k}$ (b) $-3\mathbf{i}-3\mathbf{j}-\mathbf{k}$

(c) $3\mathbf{i}-\mathbf{j}+\mathbf{k}$ (d) $\mathbf{i}+3\mathbf{j}-3\mathbf{k}$

Ans. (c)

Solution: $\mathbf{v}=\alpha\,\mathbf{a}+\beta\mathbf{b}$, where $\alpha,\beta\in\mathbf{R}$

$\Rightarrow\quad \mathbf{v}=(\alpha+\beta)\mathbf{i}+(\alpha-\beta)\mathbf{j}+(\alpha+\beta)\,\mathbf{k}$

The projection of $\mathbf{v}$ on $\mathbf{c}$ is $1/\sqrt{3}$, so

$$\frac{|\mathbf{c}\cdot\mathbf{v}|}{|\mathbf{c}|}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow\quad \frac{|(\alpha+\beta)-(\alpha+\beta)-(\alpha+\beta)|}{\sqrt{3}}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow\quad |\alpha-\beta|=1\Rightarrow\alpha-\beta=\pm 1.$$

The only possible answer is (c).

Example 54: Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors. If the vectors $\mathbf{c}=\mathbf{a}+2\mathbf{b}$ and $\mathbf{d}=5\mathbf{a}-4\mathbf{b}$ are perpendicular to each other, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is

(a) $\pi/2$ (b) $\pi/3$

(c) $\pi/4$ (d) $\pi/6$

Ans. (b)

Solution: $\mathbf{c}\,.\,\mathbf{d}=0\Rightarrow(\mathbf{a}+2\mathbf{b})\,.\,(5\mathbf{a}-4\mathbf{b})=0$

$\Rightarrow\quad 5\,|\mathbf{a}|^2+10\mathbf{b}\,.\,\mathbf{a}-4\,\mathbf{a}\,.\,\mathbf{b}-8\,|\mathbf{b}|^2=0$

$\Rightarrow\quad 5+6\,\mathbf{a}\,.\,\mathbf{b}-8=0$

$\Rightarrow\quad \mathbf{a}\,.\,\mathbf{b}=1/2$

$\Rightarrow\quad |\mathbf{a}|\,|\mathbf{b}|\cos\theta=1/2$

$\Rightarrow\quad \cos\theta=1/2\Rightarrow\theta=\pi/3.$

Example 55: Let *ABCD* be a parallelogram such that $AB=\mathbf{q}$, $AD=\mathbf{p}$ and *BAD* be an acute angle. If $\mathbf{r}$ is the vector that coincides with the altitude directed from the vertex *B* to the side *AD*, then $\mathbf{r}$ is given by :

(a) $\mathbf{r}=-\mathbf{q}+\dfrac{(\mathbf{p}\cdot\mathbf{q})}{\mathbf{p}\cdot\mathbf{p}}\,\mathbf{p}$

(b) $\mathbf{r}=\mathbf{q}-\dfrac{(\mathbf{p}\cdot\mathbf{q})}{\mathbf{p}\cdot\mathbf{p}}\,\mathbf{p}$

(c) $\mathbf{r}=-3\mathbf{q}+\dfrac{3(\mathbf{p}\cdot\mathbf{q})}{\mathbf{p}\cdot\mathbf{p}}\,\mathbf{p}$

(d) $\mathbf{r}=3\mathbf{q}-\dfrac{3(\mathbf{p}\cdot\mathbf{q})}{\mathbf{p}\cdot\mathbf{p}}\,\mathbf{p}$

Ans. (a)

Solution: Let *M* be the foot of perpendicular from B to AD.

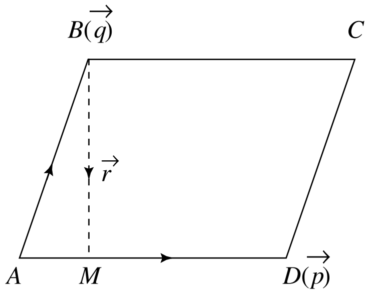


Fig. 22.5

Suppose $\quad \mathbf{AM} = \alpha\mathbf{p}$

Also, $\quad \mathbf{AM} = \mathbf{AB} + \mathbf{BM} = \mathbf{q} + \mathbf{r}$

As $\quad \mathbf{q} + \mathbf{r} = \alpha\mathbf{p}$, we get

$$(\mathbf{q} + \mathbf{r}).\mathbf{p} = \alpha\, \mathbf{p}.\mathbf{p}$$

$$\Rightarrow \quad \mathbf{q}.\mathbf{p} + \mathbf{r}.\mathbf{p} = \alpha\ \mathbf{p}.\mathbf{p} \Rightarrow \alpha = \frac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}} \quad [\because\ \mathbf{r}.\mathbf{p} = 0]$$

Thus, $$\mathbf{r} = -\,\mathbf{q} + \frac{\mathbf{q}\cdot\mathbf{p}}{\mathbf{p}\cdot\mathbf{p}}\ \mathbf{p}.$$

Assertion-Reason Type Questions

◉ **Example 56: Statement-1:** If vectors **a** and **c** are non-collinear then the lines

$$\mathbf{r} = 6\mathbf{a} - \mathbf{c} + \lambda(2\mathbf{c} - \mathbf{a})$$
$$\mathbf{r} = \mathbf{a} - \mathbf{c} + \mu(\mathbf{a} + 3\mathbf{c}) \text{ are coplanar}$$

Statement-2: There exist λ and μ such that the two values of **r** become same

Ans. (a)

◎ **Solution:** If lines have a common point then there exists λ and μ such that

$$6 - \lambda = 1 + \mu \text{ and } -1 + 2\lambda = -1 + 3\mu$$

$\Rightarrow \quad \lambda = 3, \mu = 2.$

◉ **Example 57:** Given that **a**, **b**, **c** are the position vectors of the vertices of a ΔABC

Statement-1: The area of ΔABC is $\frac{1}{2}[\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}]$

Statement-2: Cross product is distributive over addition of vectors

Ans. (b)

◎ **Solution:** Required area $= \frac{1}{2}(\mathbf{AB}\times\mathbf{AC})$

$$= \frac{1}{2}((\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}))$$

$$= \frac{1}{2}[\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}].$$

◉ **Example 58:** Let A, B, C be three points with position vectors $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $2\mathbf{i} + 3\mathbf{k}$, $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

Statement-1: The angle between **AB** and **AC** is acute

Statement-2: If θ is the angle between **AB** and **AC** then

$$\cos\theta = \frac{17}{\sqrt{21}\sqrt{22}}$$

Ans. (c)

◎ **Solution:** $\mathbf{AB} = \mathbf{OB} - \mathbf{OA} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

$\mathbf{AC} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$

$$\cos\theta = \frac{\mathbf{AB}.\mathbf{AC}}{|\mathbf{AB}|\,|\mathbf{AC}|} = \frac{20}{\sqrt{21}\sqrt{22}}$$

◉ **Example 59: Statement-1:** $((\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})).\ \mathbf{d} = \mathbf{b}.\mathbf{d}\ [\mathbf{abc}]$

Statement-2: $(\mathbf{a}\times\mathbf{b}).\mathbf{c} = \mathbf{a}.(\mathbf{b}\times\mathbf{c})$

Ans. (a)

◎ **Solution:** $((\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c}))\ .\ \mathbf{d}$

$= (\mathbf{a}\times\mathbf{b})\ .\ ((\mathbf{a}\times\mathbf{c})\times\mathbf{d})$

$= (\mathbf{a}\times\mathbf{b})\ .\ [(\mathbf{a}\ .\ \mathbf{d})\ \mathbf{c} - (\mathbf{c}\ .\ \mathbf{d})\ \mathbf{a}]$

$= (\mathbf{a}\ .\ \mathbf{d})\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ (the last scalar product is zero)

◉ **Example 60:** If **a**, **b**, **c** be three unit vectors such that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \frac{1}{2}\,\mathbf{b}$, **b**, **c** being non-parallel.

Statement-1: The angle between **a** and **b** is $\pi/2$.

Statement-2: The angle between **a** and **c** is $\frac{\pi}{3}$.

Ans. (b)

◎ **Solution:** $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \frac{1}{2}\ \mathbf{b}$

$\Rightarrow (\mathbf{a}\ .\ \mathbf{c})\ \mathbf{b} - (\mathbf{a}\ .\ \mathbf{b})\ \mathbf{c} = \frac{1}{2}\ \mathbf{b}$

$\Rightarrow (\mathbf{a}\ .\ \mathbf{c} - 1/2)\ \mathbf{b} - (\mathbf{a}\ .\ \mathbf{b})\ \mathbf{c} = 0$

$\Rightarrow \mathbf{a}\ .\ \mathbf{c} = 1/2$ and $\mathbf{a}\ .\ \mathbf{b} = 0$

Cosine of angle between **a** and **c** = 1/2 $\quad\Rightarrow\quad$ angle between **a** and **c** is $\pi/3$ and angle between **a** and **b** is $\pi/2$.

◉ **Example 61:** Let **u**, **v**, **w** be three non-coplanar vectors, then $(\mathbf{v} - \mathbf{w}).\ [(\mathbf{w} - \mathbf{u})\times(\mathbf{u} - \mathbf{v})] = 0$

Statement-1: $\mathbf{v} - \mathbf{w} = \lambda(\mathbf{w} - \mathbf{u}) + \mu(\mathbf{u} - \mathbf{v})$ for some $\lambda, \mu \in \mathbf{R}$.

Statement-2: The vectors $\mathbf{v} - \mathbf{w}$, $\mathbf{w} - \mathbf{u}$, $\mathbf{u} - \mathbf{v}$ are coplanar.

Ans. (a)

◎ **Solution:** The given condition implies $\mathbf{v} - \mathbf{w}$, $\mathbf{w} - \mathbf{u}$, and $\mathbf{u} - \mathbf{v}$ are coplanar. So one vector is linear combination of other two.

◉ **Example 62:** Let $\mathbf{r}$ be a vector such that $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$.

Statement-1: $\mathbf{r}$ is linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Statement-2: $\mathbf{r} = \mathbf{a}$

Ans. (c)

◎ **Solution:** $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b} \quad \Rightarrow \quad (\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$

$\Rightarrow \quad \mathbf{r} - \mathbf{a}$ is parallel to $\mathbf{b}$ so

$$\mathbf{r} - \mathbf{a} = \lambda\, \mathbf{b}, \text{ for some } \lambda \in \mathbf{R}.$$

$$\Rightarrow \qquad \mathbf{r} = \mathbf{a} + \lambda\, \mathbf{b}.$$

◉ **Example 63:** If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar vectors then

Statement-1: $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$, $\mathbf{a} \times \mathbf{b}$ are non-coplanar.

Statement-2: $[\mathbf{b} \times \mathbf{c} \quad \mathbf{c} \times \mathbf{a} \quad \mathbf{a} \times \mathbf{b}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

Ans. (c)

◎ **Solution:** $[\mathbf{b} \times \mathbf{c} \quad \mathbf{c} \times \mathbf{a} \quad \mathbf{a} \times \mathbf{b}]$

$$= ((\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})) \cdot (\mathbf{a} \times \mathbf{b})$$
$$= ((\mathbf{b}.(\mathbf{c} \times \mathbf{a}))\,\mathbf{c} - (\mathbf{c}.(\mathbf{c} \times \mathbf{a})\mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b})$$
$$= [\mathbf{a}.\ \mathbf{b}.\ \mathbf{c}]\ \mathbf{c}.(\mathbf{a} \times \mathbf{b})\ (\mathbf{c}.(\mathbf{c} \times \mathbf{a}) = 0)$$
$$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$$

Since R.H.S. is non-zero so is L.H.S. Hence $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$, $\mathbf{a} \times \mathbf{b}$ are non-coplanar.

LEVEL 2

Straight Objective Type Questions

◉ **Example 64:** The vector $\mathbf{c}$, directed along the internal bisector of the angle between the vectors $\mathbf{a} = 7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, with $|\mathbf{c}| = 5\sqrt{6}$ is

(a) $\pm (5/3)\ (\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$

(b) $(5/3)\ (5\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$

(c) $\pm\ (5/3)\ (\mathbf{i} + 7\mathbf{j} + 2\mathbf{k})$

(d) $(5/3)\ (-5\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$

Ans. (a)

◎ **Solution:** The required vector $\mathbf{c}$ is given by

$$\lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right)$$

Now $\quad \dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{1}{9}(7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k})$ and $\dfrac{\mathbf{b}}{|\mathbf{b}|} = \dfrac{1}{3}(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$$\Rightarrow \quad \mathbf{c} = \lambda\left(\frac{1}{9}\mathbf{i} - \frac{7}{9}\mathbf{j} + \frac{2}{9}\mathbf{k}\right) \quad \Rightarrow \quad |\mathbf{c}|^2 = \lambda^2 \times \frac{54}{81}$$

$$\Rightarrow \quad \lambda^2 = 225 \Rightarrow \lambda = \pm 15. \Rightarrow \mathbf{c} = \pm (5/3)\ (\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$$

◉ **Example 65:** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors, no two of which are collinear. If the vectors $\mathbf{a} + 2\mathbf{b}$ is collinear with $\mathbf{c}$, and $\mathbf{b} + 3\mathbf{c}$ is collinear with $\mathbf{a}$, then $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c}$ is equal to, (λ being some non-zero scalar)

(a) $\lambda\,\mathbf{a}$ (b) $\lambda\,\mathbf{b}$

(c) $\lambda\,\mathbf{c}$ (d) $\mathbf{0}$

Ans. (d)

◎ **Solution:** Let $\mathbf{a} + 2b = x\mathbf{c}$ and $\mathbf{b} + 3\mathbf{c} = y\mathbf{a}$. Then $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = (x + 6)\mathbf{c}$ and also, $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = (1 + 2y)\mathbf{a}$. So $(x + 6)\,\mathbf{c} = (1 + 2y)\mathbf{a}$. Since $\mathbf{a}$ and $\mathbf{c}$ are non-zero and non-collinear, we have $x + 6 = 0$ and $1 + 2y = 0$, i.e., $x = -6$ and $y = -1/2$. In either case, we have $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = \mathbf{0}$.

◉ **Example 66:** The value of k for which the points $A(1, 0, 3)$, $B(-1, 3, 4)$, $C(1, 2, 1)$ and $D(k, 2, 5)$ are coplanar, are

(a) 1 (b) 2

(c) 0 (d) –1

Ans. (d)

◎ **Solution:** Let $\mathbf{a} = (1, 0, 3)$, $\mathbf{b} = (-1, 3, 4)$, $\mathbf{c} = (1, 2, 1)$ and $\mathbf{d} = (k, 2, 5)$. Since A, B, C and D are coplanar, we have

$$[\mathbf{d}\ \mathbf{b}\ \mathbf{c}] + [\mathbf{d}\ \mathbf{c}\ \mathbf{a}] + [\mathbf{d}\ \mathbf{a}\ \mathbf{b}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$$\Rightarrow \quad \begin{vmatrix} k & 2 & 5 \\ -1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} k & 2 & 5 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} + \begin{vmatrix} k & 2 & 5 \\ 1 & 0 & 3 \\ -1 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 3 \\ -1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad -5k - 15 + 6k - 14 - 9k + 1 = -20$$

$$\Rightarrow \quad -8k - 28 = -20 \quad \Rightarrow \quad k = -1$$

Alternate Solution

A, B, C, D are coplanar $\Leftrightarrow$

$$[\mathbf{AB}\ \mathbf{AC}\ \mathbf{AD}] = 0$$

$$\begin{vmatrix} -2 & 3 & 1 \\ 0 & 2 & -2 \\ k-1 & 2 & 2 \end{vmatrix} = 0 \Leftrightarrow k = -1$$

◉ **Example 67:** Let a, b, c be distinct non-negative numbers. If the vectors $a\mathbf{i} + a\mathbf{j} + c\mathbf{k}$, $\mathbf{i} + \mathbf{k}$ and $c\mathbf{i} + c\mathbf{j} + b\mathbf{k}$ lie in a plane, then c is

(a) the arithmetic mean of a and b

(b) the geometric mean of a and b
(c) the harmonic mean of a and b
(d) equal to zero

Ans. (b)

◎ **Solution:** The three vectors are coplanar if and only if

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 0 & a & c \\ 1 & 0 & 1 \\ 0 & c & b \end{vmatrix} = 0. \ (C_1 \to C_1 - C_2)$$

$$\Leftrightarrow \qquad -1\,(ab - c^2) = 0 \quad \Leftrightarrow \quad ab = c^2.$$

◉ **Example 68:** Let $\mathbf{p}, \mathbf{q}, \mathbf{r}$ be three mutually perpendicular vectors of the same magnitude. If a vector $\mathbf{x}$ satisfies the equation

$$\mathbf{p} \times ((\mathbf{x} - \mathbf{q}) \times \mathbf{p}) + \mathbf{q} \times ((\mathbf{x} - \mathbf{r}) \times \mathbf{q}) + \mathbf{r} \times ((\mathbf{x} - \mathbf{p}) \times \mathbf{r}) = \mathbf{0}$$

then $\mathbf{x}$ is given by

(a) $(1/2)(\mathbf{p} + \mathbf{q} - 2\mathbf{r})$ (b) $(1/2)(\mathbf{p} + \mathbf{q} + \mathbf{r})$
(c) $(1/3)(\mathbf{p} + \mathbf{q} + \mathbf{r})$ (d) $(1/3)(2\mathbf{p} + \mathbf{q} - \mathbf{r})$

Ans. (b)

◎ **Solution:** Let $|\mathbf{p}| = |\mathbf{q}| = |\mathbf{r}| = K$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be unit vectors along $\mathbf{p}, \mathbf{q}, \mathbf{r}$ respectively. Clearly $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are mutually perpendicular vectors, so any vector $\mathbf{x}$ can be written as $a_1\,\mathbf{a} + a_2\,\mathbf{b} + a_3\,\mathbf{c}$.

$$\begin{aligned} \mathbf{p} \times ((\mathbf{x} - \mathbf{q}) \times \mathbf{p}) &= (\mathbf{p}\cdot\mathbf{p})\,(\mathbf{x} - \mathbf{q}) - (\mathbf{p}\cdot(\mathbf{x} - \mathbf{q}))\,\mathbf{p} \\ &= K^2\,(\mathbf{x} - \mathbf{q}) - (\mathbf{p}\cdot\mathbf{x})\,\mathbf{p}\ (\mathbf{p}\cdot\mathbf{q} = 0) \\ &= K^2\,(\mathbf{x} - \mathbf{q}) - K\,\mathbf{a}\cdot(a_1\,\mathbf{a} + a_2\,\mathbf{b} + a_3\,\mathbf{c})\,K\,\mathbf{a} \\ &= K^2\,(\mathbf{x} - \mathbf{q} - a_1\,\mathbf{a}) \end{aligned}$$

Similarly $\mathbf{q} \times ((\mathbf{x} - \mathbf{r}) \times \mathbf{q}) = K^2(\mathbf{x} - \mathbf{r} - a_2\mathbf{b})$

and $\quad \mathbf{r} \times ((\mathbf{x} - \mathbf{p}) \times \mathbf{r}) = K^2(\mathbf{x} - \mathbf{p} - a_3\mathbf{c})$

According to the given condition

$$K^2\,(\mathbf{x} - \mathbf{q} - a_1\,\mathbf{a} + \mathbf{x} - \mathbf{r} - a_2\,\mathbf{b} + \mathbf{x} - \mathbf{p} - a_3\,\mathbf{c}) = 0$$

$$\Rightarrow \qquad \{3\mathbf{x} - (\mathbf{p} + \mathbf{q} + \mathbf{r}) - (a_1\,\mathbf{a} + a_2\,\mathbf{b} + a_3\,\mathbf{c})\} = 0$$

$$\Rightarrow \qquad [2\mathbf{x} - (\mathbf{p} + \mathbf{q} + \mathbf{r})] = \mathbf{0}$$

$$\Rightarrow \ x = (1/2)\,(\mathbf{p} + \mathbf{q} + \mathbf{r}).$$

◉ **Example 69:** If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are unit vectors, then $|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c} - \mathbf{d}|^2 + |\mathbf{d} - \mathbf{a}|^2 + |\mathbf{c} - \mathbf{a}|^2 + |\mathbf{b} - \mathbf{d}|^2$ does not exceed

(a) 4 (b) 12
(c) 8 (d) 16

Ans. (b)

◎ **Solution:** We have

$$0 \le |\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2 + 2\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a}$$
$$+ \mathbf{a}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{d}$$
$$= 4 + 2\,\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{d}$$

So $\quad \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{d} + \mathbf{c}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{d} \ge -2.$

Now

$$|\mathbf{a} - \mathbf{b}|^2 + |\mathbf{b} - \mathbf{c}|^2 + |\mathbf{c}\cdot\mathbf{d}|^2 + |\mathbf{d} - \mathbf{a}|^2 + |\mathbf{c} - \mathbf{a}|^2 + |\mathbf{b} - \mathbf{d}|^2$$
$$= 2(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + |\mathbf{d}|^2 - (\mathbf{a}\cdot\mathbf{b} + \mathbf{c}\cdot\mathbf{d} + \mathbf{d}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{c} + \mathbf{b}\cdot\mathbf{d} + \mathbf{b}\cdot\mathbf{c})$$
$$\le 2\,(4 + 2) = 12.$$

◉ **Example 70:** Let $\mathbf{a} = \mathbf{i} - \mathbf{k}, \mathbf{b} = x\mathbf{i} + \mathbf{j} + (1 - x)\mathbf{k}$ and $\mathbf{c} = y\mathbf{i} + x\mathbf{j} + (1 + x - y)\mathbf{k}$. Then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ depends on

(a) only x
(b) only y
(c) neither x nor y
(d) both x and y

Ans. (c)

◎ **Solution:** $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1 - x \\ y & x & 1 + x - y \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1 + x \end{vmatrix} \ (\text{using } C_3 \to C_3 + C_1)$$

$= (1 + x) - x = 1$, which depends neither on x nor on y.

◉ **Example 71:** Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$. If $\mathbf{c}$ is a vector such that $\mathbf{a}\cdot\mathbf{c} = |\mathbf{c}|, |\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$ and the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ is 30°, then $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| =$

(a) 2/3 (b) 3/2
(c) 2 (d) 3

Ans. (b)

◎ **Solution:** $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$$\Rightarrow \qquad |\mathbf{a} \times \mathbf{b}| = \sqrt{4 + 4 + 1} = 3\ \text{. Also } |\mathbf{c} - \mathbf{a}|^2 = 8$$

$$\text{so} \qquad |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{a}.\mathbf{c} = 8 \quad \Rightarrow \quad |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2|\mathbf{c}| = 8$$

$$\Rightarrow \qquad |\mathbf{c}|^2 + 9 - 2|\mathbf{c}| = 8 \ \Rightarrow \ |\mathbf{c}|^2 - 2|\mathbf{c}| + 1 = 0$$

$$\Rightarrow \qquad (|\mathbf{c}| - 1)^2 \qquad \Rightarrow \qquad |\mathbf{c}| = 1$$

$$\text{Now} \qquad |(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = |\mathbf{a} \times \mathbf{b}|\,|\mathbf{c}|\ \sin 30^\circ = 3.1.\frac{1}{2} = \frac{3}{2}.$$

◉ **Example 72:** A tangent is drawn to the curve $y = 8/x^2$ in xy-plane at the point $A\,(x_0, y_0)$, where $x_0 = 2$, and the tangent cuts the x-axis at a point B. Then $\mathbf{AB}\cdot\mathbf{OB}$ is equal to

(a) 2 (b) 1
(c) 0 (d) 3

Ans. (d)

◎ **Solution:** Since $x_0 = 2$ so $y_0 = 8/4 = 2$

Equation of tangent is $Y - 2 = \dfrac{8(-2)}{2^3}\,(X - 2)$ i.e. $Y = -2X + 6$.

This tangent cuts x-axis at (3, 0)

Hence $B = (3, 0)$. Therefore **AB** = (3 – 2)**i** + (0 – 2)**j** and OB = 3**i**, thus $\mathbf{AB}\cdot\mathbf{OB} = 3$.

◉ **Example 73:** Let P, Q, R be points with position vectors $\mathbf{r}_1 = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$, $\mathbf{r}_2 = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{r}_3 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ relative to an origin 0. The distance of P from the plane OQR is (magnitude)

(a) 2 (b) 3
(c) 1 (d) $11/\sqrt{3}$

Ans. (b)

◎ **Solution:** Equation of the plane OQR is $\mathbf{r} = \lambda\mathbf{r}_2 + \mu\mathbf{r}_3$, i.e. $\mathbf{r}\cdot(\mathbf{r}_2\times\mathbf{r}_3) = 0$

So the distance of P from the plane OQR is $\left|\dfrac{\mathbf{r}_1\cdot(\mathbf{r}_2\times\mathbf{r}_3)}{|\mathbf{r}_2\times\mathbf{r}_3|}\right|$.

Since $\mathbf{r}_2\times\mathbf{r}_3 = -10\,\mathbf{i} + 10\,\mathbf{j} - 5\mathbf{k}$ so $|\mathbf{r}_2\times\mathbf{r}_3| = 15$.

$$\text{Required distance} = \frac{1}{15}\left|\begin{vmatrix} 3 & -2 & -1 \\ 1 & 3 & 4 \\ 2 & 1 & -2 \end{vmatrix}\right| = 3.$$

◉ **Example 74:** For unit vectors **b** and **c** and any non zero vector **a**, the value of $\{\{(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{c})\}\times(\mathbf{b}\times\mathbf{c})\}\cdot(\mathbf{b}+\mathbf{c})$ is

(a) $|\mathbf{a}|^2$ (b) $2\,|\mathbf{a}|^2$
(c) $3\,|\mathbf{a}|^2$ (d) none of these

Ans. (d)

◎ **Solution:** The given expression

$$\begin{aligned}
&= \{\{\mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{b}\times\mathbf{c}\}\times(\mathbf{b}\times\mathbf{c})\}\,.\,(\mathbf{b}+\mathbf{c}) \\
&= \{(\mathbf{a}\times\mathbf{c})\times(\mathbf{b}\times\mathbf{c}) + (\mathbf{b}\times\mathbf{a})\times(\mathbf{b}\times\mathbf{c})\}\,.\,(\mathbf{b}+\mathbf{c}) \\
&= [(\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}))\mathbf{c} - (\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c}))\mathbf{a} + (\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c}))\,\mathbf{a} \\
&\qquad - (\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\mathbf{b})]\ .\ (\mathbf{b}+\mathbf{c}) \\
&= [(\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}))\,(\mathbf{c}-\mathbf{b})\cdot(\mathbf{b}+\mathbf{c})] \\
&= (\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}))\left[|\mathbf{c}|^2 - |\mathbf{b}|^2\right] = 0
\end{aligned}$$

$[\because\ |\mathbf{b}| = |\mathbf{c}| = 1]$.

◉ **Example 75:** Three non-coplanar vector **a** , **b** and **c** are drawn from a common initial point. The angle between the plane passing through the terminal points of these vectors and the vector $\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}$ is

(a) $\pi/4$ (b) $\pi/2$
(c) $\pi/3$ (d) none of these

Ans. (b)

◎ **Solution:** Let the terminal points be A, B, C and the common initial point be the origin of reference so that **AB** = **b** – **a** and **AC** = **c** – **a**. The vector **AB**× **AC** is perpendicular to the plane ABC. **AB** × **AC** = (**b** – **a**) × (**c** – **a**) = **b** × **c** + **c** × **a** + **a** × **b**. Hence the angle between the plane and the given vector is $\pi/2$.

◉ **Example 76:** A unit tangent vector at $t = 2$ on the curve $x = t^2 + 2$, $y = 4t^3 - 5$, $z = 2t^2 - 6t$ is

(a) $\dfrac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ (b) $\dfrac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})$
(c) $\dfrac{1}{\sqrt{6}}(2\mathbf{i}+\mathbf{j}+\mathbf{k})$ (d) none of these

Ans. (d)

◎ **Solution:** The position vector of any point at t is

$$\mathbf{r} = (2 + t^2)\,\mathbf{i} + (4t^3 - 5)\,\mathbf{j} + (2t^2 - 6t)\,\mathbf{k}$$

$$\therefore\quad \frac{d\mathbf{r}}{dt} = 2t\,\mathbf{i} + 12t^2\,\mathbf{j} + (4t - 6)\mathbf{k},\ \left.\frac{d\mathbf{r}}{dt}\right|_{t=2} = 4\mathbf{i} + 48\mathbf{j} + 2\mathbf{k}$$

$$\left|\frac{d\mathbf{r}}{dt}\right|_{t=2} = \sqrt{16 + 2304 + 4} = \sqrt{2320}.$$

Hence the required unit tangent vector at $t = 2$ is $(1/\sqrt{580})$ (2**i** + 24**j** + **k**).

◉ **Example 77:** A particle moves along a curve so that its coordinates at time t are $x = t$, $y = \dfrac{1}{2}t^2$, $z = \dfrac{1}{3}t^3$. The acceleration at $t = 1$ is

(a) **j** + 2**k** (b) **j** + **k**
(c) 2**j** + **k** (d) none of these

Ans. (a)

◎ **Solution:** Let $\mathbf{r} = t\mathbf{i} + \dfrac{1}{2}t^2\,\mathbf{j} + \dfrac{1}{3}t^3\,\mathbf{k}$ then the velocity $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \mathbf{i} + t\,\mathbf{j} + t^2\,\mathbf{k}$ and the acceleration $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \mathbf{j} + 2t\,\mathbf{k}$. Hence $\mathbf{a}\,|_{(t=1)} = \mathbf{j} + 2\,\mathbf{k}$.

◉ **Example 78:** Consider the parallelopipped with sides **a** = 3**i** + 2**j** + **k**, **b** = **i** + **j** + 2**k** and **c** = **i** + 3**j** + 3**k** then the angle between **a** and the plane containing the face determined by **b** and **c** is

(a) $\sin^{-1}(1/3)$ (b) $\cos^{-1}(9/14)$
(c) $\sin^{-1}(9/14)$ (d) $\sin^{-1}(2/3)$

Ans. (c)

◎ **Solution:** $\mathbf{b}\times\mathbf{c} = -3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. If θ is the angle between **a** and the plane containing **b** and **c**

$$\text{then. } \cos(90^\circ - \theta) = \left|\frac{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}{|\mathbf{a}|\,|\mathbf{b}\times\mathbf{c}|}\right|$$

$$= \frac{1}{\sqrt{14}}\,\frac{1}{\sqrt{14}}\,|(-9 - 2 + 2)| = \frac{9}{14}.$$

Hence $\theta = \sin^{-1}(9/14)$.

◉ **Example 79:** A unit vector **n** perpendicular to the plane determined by the points A (0, –2, 1), B (1, – 1, –2) and C (–1, 1, 0)

(a) $\dfrac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k})$ (b) $1/4\sqrt{6}\,(8\mathbf{i}+4\mathbf{j}+4\mathbf{k})$
(c) $\dfrac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}+\mathbf{k})$ (d) $\dfrac{1}{\sqrt{14}}(3\mathbf{i}+\mathbf{j}+2\mathbf{k})$

Ans. (b)

◎ **Solution:** $\mathbf{AB} = \mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$ and $\mathbf{AC} = -\,\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{AB} \times \mathbf{AC} = 8\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$. Hence $\mathbf{n} = (1/4\sqrt{6})\,(8\mathbf{i} + 4\,\mathbf{j} + 4\,\mathbf{k})$.

◉ **Example 80:** If the vectors $\mathbf{AB} = -\,3\mathbf{i} + 4\mathbf{k}$ and $\mathbf{AC} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ are the sides of a triangle ABC. Then the length of the median through A is

(a) $\sqrt{14}$ (b) $\sqrt{18}$

(c) $\sqrt{29}$ (d) none of these

Ans. (b)

◎ **Solution:** Let A be the origin of reference so that the position vectors of B and C are $-\,3\mathbf{i} + 4\mathbf{j}$ and $-\,\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ respectively. The position vector of mid point of BC is $\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. Thus the length of the median is $\sqrt{1+1+16} = \sqrt{18}$.

◉ **Example 81:** If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and $|\mathbf{c}| = 7$ then the angle between **a** and **b** is

(a) $\pi/6$ (b) $2\pi/3$

(c) $\pi/3$ (d) $5\pi/3$

Ans. (c)

◎ **Solution:** $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0} \Rightarrow \mathbf{a} + \mathbf{b} = -\,\mathbf{c} \Rightarrow (\mathbf{a} + \mathbf{b})\cdot(\mathbf{a} + \mathbf{b}) = |\mathbf{c}|^2$.

Thus $|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\,|\mathbf{a}||\mathbf{b}| \cos\theta = |\mathbf{c}|^2$, where θ is the angle between **a** and **b**. Therefore, $\cos\theta = \dfrac{49-9-25}{2\cdot 3\cdot 5} = \dfrac{1}{2} \Rightarrow \theta = \pi/3$.

◉ **Example 82:** The vector $((\mathbf{i}-\mathbf{j})\times(\mathbf{j}-\mathbf{k}))\times(\mathbf{i}+5\mathbf{k})$ is equal to

(a) $5\mathbf{i} - 4\mathbf{j} - \mathbf{k}$ (b) $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$

(c) $4\mathbf{i} - 5\mathbf{j} - \mathbf{k}$ (d) $5\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

Ans. (a)

◎ **Solution:** The given expression

$$((\mathbf{i}-\mathbf{j})\cdot(\mathbf{i}+5\mathbf{k}))\,(\mathbf{j}-\mathbf{k}) - ((\mathbf{j}-\mathbf{k})\cdot(\mathbf{i}+5\mathbf{k}))\,(\mathbf{i}-\mathbf{j})$$

$$= 1\,(\mathbf{j}-\mathbf{k}) + 5(\mathbf{i}-\mathbf{j}) = 5\mathbf{i} - 4\mathbf{j} - \mathbf{k}.$$

◉ **Example 83:** The position vector of a point P is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $x, y, z \in \mathbf{N}$ and $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. If $\mathbf{r}\cdot\mathbf{u} = 10$, then the number of possible positions of P is

(a) 72 (b) 36

(c) 60 (d) 108

Ans. (b)

◎ **Solution:** $\mathbf{r}\cdot\mathbf{u} = 10 \Rightarrow x + y + z = 10$. The number of positive integral solution of this equation is ${}^9C_2 = 36$ (see Chapter 6).

◉ **Example 84:** If **a** and **b** are unit vectors and θ is the angle between **a** and **b** then $\sin(\theta/2)$ is equal to

(a) $\dfrac{1}{2}|\mathbf{a}-\mathbf{b}|$ (b) 1

(c) $\dfrac{1}{2}|\mathbf{a}+\mathbf{b}|$ (d) 0

Ans. (a)

◎ **Solution:** $\left|\dfrac{\mathbf{a}-\mathbf{b}}{2}\right|^2 = \dfrac{1}{4}\left[|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\mathbf{a}.\mathbf{b}\right]$

$$= \frac{1}{2}[1 - |\mathbf{a}||\mathbf{b}|\cos\theta] = \frac{1}{2}[1-\cos\theta] = \sin^2\frac{\theta}{2}$$

Hence $\sin\left(\dfrac{\theta}{2}\right) = \left(\dfrac{1}{2}\right)|\mathbf{a}-\mathbf{b}|$.

◉ **Example 85:** The vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + m\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} + (m+1)\,\mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} + m\mathbf{k}$ are coplanar if m is equal to

(a) 1 (b) 4

(c) 3 (d) none of these

Ans. (d)

◎ **Solution:** **a**, **b**, **c** are coplanar if

$$\begin{vmatrix} 1 & 1 & m \\ 1 & 1 & m+1 \\ 1 & -1 & m \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & m \\ 1 & 0 & m+1 \\ 1 & -2 & m \end{vmatrix} = 0$$

$\Rightarrow$ $2 = 0$.

So there no value of m for which the vectors are coplanar.

◉ **Example 86:** If **a**, **b**, **c** are non-coplanar unit vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \dfrac{\mathbf{b}+\mathbf{c}}{\sqrt{2}}$, then the angle between **a** and **b** is

(a) $3\pi/4$ (b) $\pi/4$

(c) $\pi/2$ (d) π

Ans. (a)

◎ **Solution:** The given equality implies

$$(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} = \frac{\mathbf{b}+\mathbf{c}}{\sqrt{2}}$$

$$\Rightarrow \left(\mathbf{a}\cdot\mathbf{c} - \frac{1}{\sqrt{2}}\right)\mathbf{b} = \left(\frac{1}{\sqrt{2}} + \mathbf{a}\cdot\mathbf{b}\right)\mathbf{c}$$

$\Rightarrow$ **b** and **c** are collinear thus **a**, **b**, **c** are coplanar if $\mathbf{a}\cdot\mathbf{c} \neq \dfrac{1}{\sqrt{2}}$ and $\mathbf{a}\cdot\mathbf{b} \neq -\dfrac{1}{\sqrt{2}}$. Hence $\mathbf{a}\cdot\mathbf{b} = -1/\sqrt{2}$

$\Rightarrow$ $\cos\theta = -1/\sqrt{2}$, where θ is the angle between **a** and **b**.

So $\theta = 3\pi/4$.

EXERCISE

Concept Based Straight Objective Type Questions

1. If M is the mid point of AB and O is any point, then

(a) $\mathbf{OM} = \mathbf{OA} + \mathbf{MA}$

(b) $\mathbf{OM} = \mathbf{OA} - \mathbf{MA}$

(c) $\mathbf{OM} = \frac{1}{2}(\mathbf{OA} - \mathbf{OB})$

(d) $\mathbf{OM} = \frac{1}{2}(\mathbf{OB} + \mathbf{OA})$

2. The angle between $3\mathbf{i} + 4\mathbf{j}$ and $2\mathbf{j} - 5\mathbf{k}$ is

(a) $\frac{\pi}{2}$ (b) $\cos^{-1}\frac{8}{5\sqrt{29}}$

(c) $\frac{\pi}{6}$ (d) $\cos^{-1}\frac{1}{3}$

3. A unit vector **c** perpendicular to **a** and coplanar with **a** and **b**, $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$ is given by

(a) $\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k})$ (b) $\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$

(c) $\frac{1}{\sqrt{2}}(\mathbf{j}+\mathbf{k})$ (d) $\frac{1}{\sqrt{2}}(-\mathbf{j}+\mathbf{k})$

4. A vector **b**, which is collinear with vector $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and satisfies $\mathbf{a}\cdot\mathbf{b} = 2$ is given by

(a) $\frac{1}{2}(2\mathbf{i}+\mathbf{j}-\mathbf{k})$ (b) $\frac{1}{3}(2\mathbf{i}+\mathbf{j}-\mathbf{k})$

(c) $\frac{1}{4}(2\mathbf{i}+\mathbf{j}-\mathbf{k})$ (d) $\frac{1}{2}(-2\mathbf{i}-\mathbf{j}+\mathbf{k})$

5. If $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ then the magnitude of projection of $\mathbf{u} \times \mathbf{v}$ on **w** is given by

(a) $\sqrt{\frac{1}{2}}$ (b) $\sqrt{\frac{1}{3}}$

(c) $\sqrt{\frac{3}{4}}$ (d) $\sqrt{\frac{3}{2}}$

6. If **a** and **b** are non-collinear vectors, then the value of λ for which

$\mathbf{u} = (\lambda + 2)\,\mathbf{a} + \mathbf{b}$ and

$\mathbf{v} = (1 + 4\lambda)\,\mathbf{a} - 2\mathbf{b}$ are collinear is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$

(c) $\frac{3}{4}$ (d) $\frac{1}{3}$

7. The area of the triangle formed by A (1, 0, 0), B (0, 1, 0), C (1, 1, 1) is

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{4}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{4}$

8. A unit vector perpendicular to $3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{i} - \mathbf{j} + \mathbf{k}$ is

(a) $\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$

(b) $\frac{1}{\sqrt{14}}(\mathbf{i}-2\mathbf{j}+3\mathbf{k})$

(c) $\frac{1}{\sqrt{74}}(4\mathbf{i}+3\mathbf{j}-7\mathbf{k})$

(d) $\frac{1}{\sqrt{74}}(4\mathbf{i}-3\mathbf{j}-7\mathbf{k})$

9. The value of scalar triple product $\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{j} + \mathbf{k}$ is

(a) 12 (b) 10

(c) 14 (d) 16

10. The vector $[(\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (2\mathbf{i} - 3\mathbf{j} - \mathbf{k})] \times [(-3\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (2\mathbf{j} + \mathbf{k})]$ is given by

(a) $3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

(b) $-5(3\mathbf{i} - 5\mathbf{j} - 3\mathbf{k})$

(c) $5(3\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$

(d) $(15\mathbf{i} - 25\mathbf{j} + 15\mathbf{k})$

LEVEL 1

Straight Objective Type Questions

11. Let $|\mathbf{a}| = 3$ and $|\mathbf{b}| = 4$. The value of μ for which the vectors $\mathbf{a} + \mu\mathbf{b}$ and $\mathbf{a} - \mu\mathbf{b}$ will be perpendicular is
    (a) $3/4$ (b) $2/3$
    (c) $-5/2$ (d) $-2/3$

12. The value of α for which the vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} + \alpha\mathbf{k}$ and $3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ are coplanar is
    (a) 3 (b) –3
    (c) 2 (d) none of these

13. The area of a parallelogram having diagonals $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ is
    (a) $5\sqrt{3}$ (b) $2\sqrt{3}$
    (c) $4\sqrt{3}$ (d) none of these

14. If $\mathbf{r}$ satisfies the equation $\mathbf{r} \times (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = \mathbf{i} - \mathbf{k}$, then for any scalar t, $\mathbf{r}$ is equal to
    (a) $\mathbf{i} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
    (b) $\mathbf{j} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
    (c) $\mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$
    (d) $\mathbf{i} - \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

15. The vectors **a**, **b** and **c** are equal in length and taken pairwise, make equal angles. If $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{j} + \mathbf{k}$ and **c** make an obtuse angle with the base vector **i**, then **c** is equal to
    (a) $\mathbf{i} + \mathbf{k}$ (b) $-\mathbf{i} + 4\mathbf{j} - \mathbf{k}$
    (c) $-\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$ (d) $\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$

16. The vectors $\mathbf{AB} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{BC} = -\mathbf{i} - 2\mathbf{k}$ are the adjacent sides of a parallelogram. An angle between its diagonals is
    (a) $\pi/4$ (b) $\pi/3$
    (c) $\pi/2$ (d) $2\pi/3$

17. Let the unit vectors **a** and **b** be perpendicular and the unit vector **c** be inclined at an angle θ to both **a** and **b**. If $\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b})$ then
    (a) $\alpha = 2\beta$ (b) $\gamma^2 = 1 + 2\alpha^2$
    (c) $\gamma^2 = \cos 2\theta$ (d) $\beta^2 = \dfrac{1 + \cos 2\theta}{2}$

18. If the unit vectors **a** and **b** are inclined at an angle 2θ and $|\mathbf{a} - \mathbf{b}| < 1$, then if $0 \le \theta \le \pi$, θ lies in the interval
    (a) $\left[0, \frac{\pi}{6}\right)$ (b) $\left(\frac{5\pi}{6}, 2\pi\right]$
    (c) $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ (d) $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$

19. For non-coplanar vectors **a**, **b** and **c**, $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a}|\,|\mathbf{b}|\,|\mathbf{c}|$ holds if and only if
    (a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$
    (b) $\mathbf{a} \cdot \mathbf{b} = 0 = \mathbf{b} \cdot \mathbf{c}$
    (c) $\mathbf{a} \cdot \mathbf{b} = 0 = \mathbf{c} \cdot \mathbf{a}$
    (d) $\mathbf{b} \cdot \mathbf{c} = 0 = \mathbf{c} \cdot \mathbf{a}$

20. The volume of the tetrahedron whose vertices are the points with position vectors $\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$, $-\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$, $5\mathbf{i} - \mathbf{j} + \lambda\mathbf{k}$ and $7\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$ is 11 cubic units if the value of λ is
    (a) –1 (b) 1
    (c) –7 (d) 5

21. The vectors $(x, x + 1, x + 2)$, $(x + 3, x + 4, x + 5)$ and $(x + 6, x + 7, x + 8)$ are coplanar for
    (a) only finite number of values of x
    (b) $x < 0$
    (c) Only $x = z$
    (d) none of these

22. A vector of length $\sqrt{7}$ which is perpendicular to $2\mathbf{j} - \mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and makes an obtuse angle with y-axis is
    (a) $(1/\sqrt{5})\left(4\mathbf{i} - \mathbf{j} + \sqrt{18}\mathbf{k}\right)$
    (b) $(1/\sqrt{3})(4\mathbf{i} - \mathbf{j} - 2\mathbf{k})$
    (c) $(1/\sqrt{3})(-4\mathbf{i} + \mathbf{j} + 2\mathbf{k})$
    (d) $(1/\sqrt{3})(-4\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

23. Let $|\mathbf{a}| = |\mathbf{b}| = 2$ and $\mathbf{p} = \mathbf{a} + \mathbf{b}$, $\mathbf{q} = \mathbf{a} - \mathbf{b}$. If $|\,\mathbf{p} \times \mathbf{q}\,| = 2(k - (\mathbf{a}\,.\,\mathbf{b})^2)^{1/2}$ then
    (a) $k = 16$ (b) $k = 8$
    (c) $k = 4$ (d) $k = 1$

24. If $\mathbf{r} \cdot \mathbf{a} = 0$, $\mathbf{r} \cdot \mathbf{b} = 0$ and $\mathbf{r} \cdot \mathbf{c} = 0$ for some non-zero vector **r**, then the value of $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}\ ]$ is
    (a) 0 (b) $1/2$
    (c) 1 (d) 2

25. The position vectors of three consecutive vertices of a parallelogram are $\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ and $7\mathbf{i} + 9\mathbf{j} + 11\mathbf{k}$. The position vector of the fourth vertex is

(a) $6(\mathbf{i} + \mathbf{j} + \mathbf{k})$
(b) $7(\mathbf{i} + \mathbf{j} + \mathbf{k})$
(c) $2\mathbf{j} - 4\mathbf{k}$
(d) $6\mathbf{i} + 8\mathbf{j} + 10\mathbf{k}$.

26. The volume of the parallelopiped whose sides are given by $\mathbf{OA} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{OB} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{OC} = 3\mathbf{i} - \mathbf{k}$ is

(a) $4/13$
(b) 4
(c) $2/7$
(d) none of these.

27. The value of $|\mathbf{a} \times \mathbf{i}|^2 + |\mathbf{a} \times \mathbf{j}|^2 + |\mathbf{a} \times \mathbf{k}|^2$ is

(a) $\mathbf{a}^2$
(b) $2\mathbf{a}^2$
(c) $3\mathbf{a}^2$
(d) none of these.

28. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three vectors, then $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ if and only if

(a) **b** and **c** are collinear
(b) **a** and **c** are collinear
(c) **a** and **b** are collinear
(d) none of these.

29. The value of $\mathbf{i} \times (\mathbf{a} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{a} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{a} \times \mathbf{k})$ is

(a) $\mathbf{a}$
(b) $2\mathbf{a}$
(c) $\mathbf{0}$
(d) $3\mathbf{a}$

30. The value of $[\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}]$ is

(a) $2\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
(b) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$
(c) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$
(d) 0

31. Given vectors $\mathbf{a} = (3, -1, 5)$ and $\mathbf{b} = (1, 2, -3)$. A vector $\mathbf{c}$ which is perpendicular to z-axis and satisfying $\mathbf{c} \cdot \mathbf{a} = 9$ and $\mathbf{c} \cdot \mathbf{b} = -4$ is

(a) $(2, -2, 0)$
(b) $(4, -2, 0)$
(c) $(2, -3, 0)$
(d) $(1, 2, 4)$

32. Area of the parallelogram on the vectors $\mathbf{a} + 3\mathbf{b}$ and $3\mathbf{a} + \mathbf{b}$ if $|\mathbf{a}| = |\mathbf{b}| = 1$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\pi/6$ is

(a) 2
(b) 4
(c) 8
(d) 16

33. If $\mathbf{a} = x\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ are coplanar then the value of x is

(a) 1
(b) -2
(c) -1
(d) none of these

34. If $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 3$ then

(a) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -3$
(b) $\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -3$
(c) $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = 3$
(d) $(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b} = 3$

35. Let $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b}$ be another vector such that $\mathbf{a} \cdot \mathbf{b} = 14$ and $\mathbf{a} \times \mathbf{b} = 3\mathbf{i} + \mathbf{j} - 8\mathbf{k}$ then the vector $\mathbf{b}$ is equal to

(a) $5\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
(b) $5\mathbf{i} - \mathbf{j} - 2\mathbf{k}$
(c) $5\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
(d) $3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

36. ABCDEF is a regular hexagon with centre at the origin such that $\mathbf{AD} + \mathbf{EB} + \mathbf{FC} = \lambda\,\mathbf{ED}$. Then λ equals

(a) 2
(b) 4
(c) 6
(d) 3.

37. A non-zero vector $\mathbf{a}$ is parallel to the line of intersection of the plane determined by the vectors $\mathbf{i}$, $\mathbf{i} + \mathbf{j}$ and the plane determined by the vectors $\mathbf{i} - \mathbf{j}$, $\mathbf{i} + \mathbf{k}$. The angle between $\mathbf{a}$ and $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is:

(a) $\dfrac{\pi}{2}$
(b) $\dfrac{\pi}{3}$
(c) $\dfrac{\pi}{6}$
(d) $\dfrac{\pi}{4}$

38. Let P, Q, R and S be the points on the plane with position vectors $-2\mathbf{i} - \mathbf{j}$, $4\mathbf{i}$, $3\mathbf{i} + 3\mathbf{j}$ and $-3\mathbf{i} + 2\mathbf{j}$ respectively. The quadrilateral $PQRS$ must be a

(a) parallelogram, which is neither rhombus nor a rectangle
(b) square
(c) rectangle but not a square
(d) rhombus, but not a square

39. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are unit vectors such that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 1$ and $\mathbf{a} \cdot \mathbf{c} = 1/2$. Then

(a) **a** , **b** , **c** are non-coplanar
(b) **b**, **c**, **d** are non-coplanar
(c) **b**, **d** are non-parallel
(d) **a**, **d** are parallel and **b**, **c** are parallel

40. The edges of a paralllelopiped are of unit length and parallel to non-coplanar unit length and are parallel to non-coplanar unit vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 1/2$. Then the volume of the parallelopiped is

(a) $1/\sqrt{2}$
(b) $1/2\sqrt{2}$
(c) $\sqrt{3}/2$
(d) $1/\sqrt{3}$

41. The set of all $a \in \mathbf{R}$ for which, the vector $a\mathbf{i} + 2a\mathbf{j} - 3a\mathbf{k}$, $(2a + 1)\,\mathbf{i} + (2a + 3)\,\mathbf{j} + (a + 1)\mathbf{k}$, $(3a + 5)\mathbf{i} + (a + 5)\,\mathbf{j} + (a + 2)\mathbf{k}$ are coplanar is

(a) $\{0\}$
(b) $(0, \infty)$
(c) $(-\infty, 1)$
(d) $(1, \infty)$

42. If $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = p\mathbf{a} + q\mathbf{b}$, where **a**, **b** are non-collinear and **c**, **d** are also non-collinear, then

(a) $p = [\mathbf{c\ b\ d}]$ (b) $p = [\mathbf{a\ c\ d}]$
(c) $p = [\mathbf{a\ b\ d}]$ (d) $p = [\mathbf{a\ b\ c}]$

43. The unit vectors **a** and **b** are perpendicular. Suppose that a unit vector **c** which is equal to $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b})$ and is equally inclined to **a** and **b**. Then

(a) $\alpha = \beta$ (b) $\alpha = 2\beta$
(c) $\alpha = \dfrac{\beta}{2}$ (d) $\beta^2 - \dfrac{1+\alpha}{2}$

Assertion-Reason Type Questions

44. If $\mathbf{a} \,.\, \mathbf{c} = 3/2$, $\mathbf{b} \,.\, \mathbf{d} = 2$, $\mathbf{a} \,.\, \mathbf{d} = 3$ and $\mathbf{b} \,.\, \mathbf{c} = 1/2$
Statement-1: $\mathbf{a} \times \mathbf{b}$, **c**, **d** are non-coplanar
Statement-2: $(\mathbf{a} \times \mathbf{b}) \,.\, (\mathbf{c} \times \mathbf{d}) = (\mathbf{b} \,.\, \mathbf{c})(\mathbf{a} \,.\, \mathbf{d}) - (\mathbf{a} \,.\, \mathbf{c})(\mathbf{b} \,.\, \mathbf{d})$

45. Suppose that **a**, **b**, **c** are non-coplanar such $[\mathbf{a\ b\ c}] = 8$
Statement-1: $\mathbf{a} \times \mathbf{b}$, $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$ are coplanar
Statement-2: $[\mathbf{a} \times \mathbf{b}\ \ \mathbf{b} \times \mathbf{c}\ \ \mathbf{c} \times \mathbf{a}] = 64$

46. Let $|\mathbf{a} + \mathbf{b}| = 2\,|\mathbf{a} - \mathbf{b}| = 1$
Statement-1: The angle between **a** and **b** is acute
Statement-2: $4\mathbf{a} \,.\, \mathbf{b} = |\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2$

47. **Statement-1:** Unit vectors perpendicular to $\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ are $\pm\dfrac{1}{\sqrt{4}}(\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})$
Statement-2: Unit vectors perpendicular to **a** and **b** are $\pm\dfrac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$.

48. If **a**, **b**, **c** are non-coplanar vectors then
Statement-1: $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$, $\mathbf{c} + \mathbf{a}$ are non-coplanar
Statement-2: $[\mathbf{a} + \mathbf{b}\ \ \mathbf{b} + \mathbf{c}\ \ \mathbf{c} + \mathbf{a}] = [\mathbf{a\ b\ c}]$

LEVEL 2

Straight Objective Type Questions

49. A line makes angles α, β, γ and δ with diagonals of a cube. The value of $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$ is

(a) 1 (b) 1/3
(c) 8/3 (d) 4/3

50. If $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a\ b\ d}]\,\mathbf{c} + k\,\mathbf{d}$ then the value of k is

(a) $[\mathbf{b\ a\ c}]$ (b) $[\mathbf{a\ b\ c}]$
(c) $[\mathbf{b\ c\ d}]$ (d) $[\mathbf{c\ b\ d}]$

51. The one of the value of x for which the angle between $\mathbf{c} = x\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{d} = \mathbf{i} + x\mathbf{j} + \mathbf{k}$ is $\pi/3$ is

(a) $1 + \sqrt{2}$ (b) $2 + \sqrt{2}$
(c) $3 + \sqrt{2}$ (d) none of these

52. The line $x = -2$, $y = 4 + 2t$, $z = -3 + t$ intersect

(a) the xy-plane
(b) the xz-plane in $(-2, 0, -4)$
(c) the yz-plane
(d) none of these

53. Let $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. The vector component of **u** orthogonal to **a** is

(a) $(1/7)\,(20\mathbf{i} - 5\mathbf{j} + 10\mathbf{k})$
(b) $(1/7)\,(4\mathbf{i} + 24\mathbf{j} + 4\mathbf{k})$
(c) $(1/7)\,(11\mathbf{i} + 2\mathbf{j} + 6\mathbf{k})$
(d) $(-1/7)\,(6\mathbf{i} + 2\mathbf{j} - 11\mathbf{k})$

54. If **a, b, c, d** lie in the same plane then $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$ is equal to

(a) $\mathbf{c} + \mathbf{d}$ (b) $\mathbf{0}$
(c) $[\mathbf{a}, \mathbf{b}, \mathbf{c}]\,\mathbf{a} + 2\mathbf{b}$ (d) $[\mathbf{b}, \mathbf{c}, \mathbf{d}]\,\mathbf{c} + \mathbf{d}$

55. If $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) + k\,(\mathbf{a}.\mathbf{d})(\mathbf{b} \,.\, \mathbf{c})$ then the value of k is

(a) 1 (b) 0
(c) -2 (d) -1

56. The distance between (5, 1, 3) and the line $x = 3$, $y = 7 + t$, $z = 1 + t$ is

(a) 4 (b) 2
(c) 6 (d) 8

57. The distance between the lines
$x = 1 - 4t$, $y = 2 + t$, $z = 3 + 2t$ and $x = 1 + s$, $y = 4 - 2s$, $z = -1 + s$ is

(a) 8
(b) $16/\sqrt{90}$
(c) $8/\sqrt{5}$
(d) $16/\sqrt{110}$

58. A triangle ABC is defined by the coordinates of vertices $A(1, -2, 2)$, $B\ (1, 4, 0)$ and $C\ (-4, 1, 1)$. The vector **BM**, where M is the foot of the altitude drawn from B to AC is

(a) $-\dfrac{20}{3}\mathbf{i}-10\mathbf{j}+\dfrac{10}{3}\mathbf{k}$
(b) $-\dfrac{10}{7}\mathbf{i}-\dfrac{30}{7}\mathbf{j}+\dfrac{10}{7}\mathbf{k}$
(c) $\dfrac{20}{7}\mathbf{i}+5\mathbf{j}-\dfrac{10}{7}\mathbf{k}$
(d) $-\dfrac{20}{7}\mathbf{i}-\dfrac{30}{7}\mathbf{j}+\dfrac{10}{7}\mathbf{k}$

59. If **a, b, c,** are non-coplanar vectors such that $(2h + k)\ \mathbf{a} + (3 - 4h + l)\mathbf{b} + (1 + h + k)\ \mathbf{c} = h\mathbf{a} + k\mathbf{b} + l\mathbf{c}$ then

(a) $h = 1, k = -4/3, l = 4/3$
(b) $h = 4/3, k = -4/3, l = 1$
(c) $h = 1/3, k = -1/3, l = 2/3$
(d) none of these.

60. The angle between two diagonals of a cube is

(a) $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$
(b) $\cos^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$
(c) $\cos^{-1}\left(\dfrac{1}{3}\right)$
(d) $\cos^{-1}\left(\dfrac{2}{3}\right)$

61. The point of intersection of the lines $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$, $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ is

(a) $\mathbf{a}$
(b) $\mathbf{b} - \mathbf{a}$
(c) $\mathbf{a} - \mathbf{b}$
(d) $\mathbf{a} + \mathbf{b}$

62. If $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\mathbf{k}$; $\mathbf{b} = b_1\,\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$; $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$; $\mathbf{d} = d_1\mathbf{i} + d_2\mathbf{j} + d_3\mathbf{k}$ and

$$k\,(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} -\mathbf{a} & -\mathbf{b} & \mathbf{c} & \mathbf{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

(formal expression) then

(a) $k = 1$
(b) $k = 2$
(c) $k = 4$
(d) none of these

63. The value of $(\mathbf{b} \times \mathbf{c})\cdot(\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a})\cdot(\mathbf{b} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{b})\cdot(\mathbf{c} \times \mathbf{d})$ is

(a) $[\mathbf{a}, \mathbf{b}, \mathbf{c}] - [\mathbf{b}, \mathbf{c}, \mathbf{d}]$
(b) $[\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{b}, \mathbf{c}, \mathbf{d}]$
(c) 0
(d) none of these

64. The lines $\mathbf{r} = \mathbf{b} - 2\mathbf{c} + \lambda\,(\mathbf{a} + \mathbf{b})$ and $\mathbf{r} = 2\mathbf{b} - \mathbf{c} + \mu\,(\mathbf{b} + \mathbf{c})$ intersect at the point.

(a) $\mathbf{b} - 2\mathbf{c}$
(b) $\mathbf{b} + 2\mathbf{c}$
(c) $\mathbf{b} + \mathbf{c}$
(d) $\mathbf{c} - \mathbf{b}$

65. If $\mathbf{a} = \mathbf{i}+2\mathbf{j}-3\mathbf{k}, \mathbf{b}=2\mathbf{i}+\mathbf{j}-\mathbf{k}$ then the vector **v** satisfying $\mathbf{a} \times \mathbf{v} = \mathbf{a} \times \mathbf{b}$ and $\mathbf{a}\cdot\mathbf{v} = 0$ is $\mathbf{b} + \boldsymbol{t}\,\mathbf{a}$, t being a scalar for

(a) all values of t
(b) for no value of t
(c) finite number of values of t
(d) $t = -1/4$

66. The value of

$$|\,\mathbf{a}\times(\mathbf{i}\times\mathbf{j})|^2+|\mathbf{a}\times(\mathbf{j}\times\mathbf{k})|^2+|\mathbf{a}\times(\mathbf{k}\times\mathbf{i})|^2 \text{ is}$$

(a) $|\,\mathbf{a}\,|^2$
(b) $2|\,\mathbf{a}\,|^2$
(c) $3\,|\,\mathbf{a}\,|^2$
(d) none of these

67. The locus of a point equidistant from two points whose position vectors are **a** and **b** is

(a) $(\mathbf{r}-(\mathbf{a}+\mathbf{b}))\cdot\mathbf{b} = 0$
(b) $\left(\mathbf{r}-\dfrac{1}{2}(\mathbf{a}+\mathbf{b})\right)\cdot\mathbf{a} = 0$
(c) $\left(\mathbf{r}-\dfrac{1}{2}(\mathbf{a}+\mathbf{b})\right)\cdot(\mathbf{a}-\mathbf{b}) = 0$
(d) $\left(\mathbf{r}-\dfrac{1}{2}(\mathbf{a}+\mathbf{b})\right)\cdot(\mathbf{a}+\mathbf{b}) = 0$

68. A vector $\mathbf{a} = (x, y, z)$ of length $2\sqrt{3}$ which makes equal angles with the vectors $\mathbf{b} = (y, -2z, 3x)$ and $\mathbf{c} = (2z, 3x, -y)$ and is perpendicular to $\mathbf{d} = (1, -1, 2)$ and makes an obtuse angle with y-axis is

(a) $(-2, 2, 2)$
(b) $\left(1, 1, \sqrt{10}\right)$
(c) $(2, -2, -2)$
(d) none of these

69. If $\mathbf{a} \times \mathbf{b} = \mathbf{c}$ and $\mathbf{b} \times \mathbf{c} = \mathbf{a}$, then

(a) **a**, **b**, **c** are orthogonal in pairs but $|\mathbf{a}| \neq |\mathbf{c}|$
(b) **a**, **b**, **c** are orthogonal in pairs but $|\mathbf{b}| \neq 1$
(c) **a**, **b**, **c** are not orthogonal to each other in pairs
(d) **a**, **b**, **c** are orthogonal in pairs and $|\mathbf{a}| = |\mathbf{c}|$, $|\mathbf{b}| = 1$

70. The acute angle between the lines $x = -2 + 2t$, $y = 3 - 4t$, $z = -4 + t$ and $x = -2 - t$, $y = 3 + 2t$, $z = -4 + 3t$ is

(a) $\sin^{-1}\dfrac{1}{\sqrt{3}}$
(b) $\cos^{-1}\dfrac{1}{\sqrt{6}}$
(c) $\cos^{-1}\dfrac{1}{\sqrt{5}}$
(d) $\cos^{-1} 2/3$

71. Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, $b = b_1\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$ and $c = c_1\mathrm{i} + c_2\mathbf{j} + c_3\mathbf{k}$ be three non zero vectors such that $\boldsymbol{c}$ is unit vector perpendicular to both the vectors **a** and **b**. If the angle between **a** and **b** is $\pi/6$, then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$ is equal to

(a) 0

(b) 1

(c) $\left(\frac{1}{4}\right)\left(\sum_{i=1}^{3} a_i^2\right)\left(\sum_{i=1}^{3} b_i^2\right)$

(d) none of these

72. A vector **a** has components $2p$ and 1 with respect to a rectangular cartesian system. This system is rotated through a certain angle about the origin in the counterclockwise sense. If, with respect to the new system, **a** has components $p + 1$ and 1, then

(a) $p = 0$

(b) $p = 1$ or $p = 1/3$

(c) $p = 1$ or $p = 1/3$

(d) $p = 1$ or $p = -1$

73. $\mathbf{a}\cdot((\mathbf{b}\times\mathbf{c})\times(\mathbf{a}+(\mathbf{b}\times\mathbf{c}))$ is equal to

(a) 0 (b) $2\,[\mathbf{a\,b\,c}]$

(c) $[\mathbf{a\,b\,c}]$ (d) none of these

74. If $\mathbf{X}\cdot\mathbf{A} = \mathbf{X}\cdot\mathbf{B} = \mathbf{X}\cdot\mathbf{C}$ for some non-zero vector **X**, then $[\mathbf{A\,B\,C}]$ is equal to

(a) $|\mathbf{A}||\mathbf{B}|\mathbf{C}|$ (b) 0

(c) $2\,|\mathbf{A}||\mathbf{B}|\mathbf{C}|$ (d) none of these

75. Given the vectors $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. A vector **c** which is perpendicular to the z-axis and satisfies $\mathbf{c}\cdot\mathbf{a} = 9$ and $\mathbf{c}\cdot\mathbf{b} = -4$ is

(a) $2\mathbf{i} - 3\mathbf{j}$ (b) $-2\mathbf{i} + 3\mathbf{j}$

(c) $-4\mathbf{i} - 4\mathbf{j}$ (d) $\mathbf{i} - \mathbf{j} + \mathbf{k}$

76. A unit vector in XZ plane making angles $\pi/4$ and $\pi/3$ respectively with $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{j} - \mathbf{k}$ is

(a) $\frac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{k})$ (b) $\frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$

(c) $-\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{k})$ (d) none of these

77. The vector $\mathbf{a} + \mathbf{b}$ bisects the angle between **a** and **b** if

(a) $|\mathbf{a}| = 2|\mathbf{b}|$ (b) $|\mathbf{a}| + |\mathbf{b}|^2 = |\mathbf{a} + \mathbf{b}|^2$

(c) $|\mathbf{a}| = |\mathbf{b}|$ (d) $|\mathbf{a}| - |\mathbf{b}| = |\mathbf{a} - \mathbf{b}|$

78. The vectors $\mathbf{AB} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{AC} = 5\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$ are the sides of the triangle ABC. The length of the median **AM** is

(a) $\sqrt{5}$ (b) $\sqrt{14}$

(c) $\sqrt{17}$ (d) $\sqrt{18}$

79. $ABCD$ is a quadrilateral such that $\mathbf{AB} = \mathbf{b}$, $\mathbf{AD} = \mathbf{d}$, $\mathbf{AC} = m\mathbf{b} + p\mathbf{d}$ where $m, p > 0$. The area of the quadrilateral $ABCD$ is

(a) $\frac{1}{2}\,(m + p)\,|\mathbf{b} \times \mathbf{d}|$ (b) $(m + p)\,|\mathbf{b} \times \mathbf{d}|$

(c) $2(m + p)\,|\mathbf{b} \times \mathbf{d}|$ (d) $\frac{1}{2}\,|\,m - p|\,|\mathbf{b} \times \mathbf{d}|$

80. If $\mathbf{u} = \mathbf{a} + \mathbf{b}$ and $\mathbf{v} = \mathbf{a} - \mathbf{b}$ and $|\mathbf{a}| = |\mathbf{b}| = k$ then $|\mathbf{u} \times \mathbf{v}|$ is equal to

(a) $2(k^2 - (\mathbf{a}\,.\,\mathbf{b})^2)$ (b) $2(k^4 - (\mathbf{a}\,.\,\mathbf{b})^2)^{1/2}$

(c) $(k^4 + (\mathbf{a}\,.\,\mathbf{b})^2)^{1/2}$ (d) $(k^4 + (\mathbf{a}\,.\,\mathbf{b})^2)^{1/2}$

Previous Years' AIEEE/JEE Main Questions

1. If $|\mathbf{a}| = 4$, $|\mathbf{b}| = 2$ and the angle between **a** and **b** is $\frac{\pi}{6}$ then $(\mathbf{a} \times \mathbf{b})^2$ is equal to

(a) 48 (b) 16

(c) 9 (d) none of these **[2002]**

2. If **a**, **b**, **c** are vectors such that $[\mathbf{a\;b\;c}] = 4$ then $[\mathbf{a}\times\mathbf{b}\quad \mathbf{b}\times\mathbf{c}\quad \mathbf{c}\times\mathbf{a}] =$

(a) 16 (b) 64

(c) 4 (d) 8 **[2002]**

3. If **a**, **b**, **c** are vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $|\mathbf{a}| = 7$, $|\mathbf{b}| = 5$, $|\mathbf{c}| = 3$ then the angle between vector **b** and **c** is

(a) 60° (b) 30°

(c) 45° (d) 90° **[2002]**

4. If $|\mathbf{a}| = 5$, $|\mathbf{b}| = 4$, $|\mathbf{c}| = 3$ then the value of $(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{c} + \mathbf{c}\cdot\mathbf{a})$ given that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

(a) 25 (b) 50
(c) –25 (d) –50 **[2002]**

5. $\mathbf{a} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j}$ are two vectors and $\mathbf{c}$ is a vector such that $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ then $|\mathbf{a}|:|\mathbf{b}|:|\mathbf{c}|$

(a) $\sqrt{34} : \sqrt{45} : \sqrt{39}$ (b) $\sqrt{34} : \sqrt{45} : 39$
(c) 34 : 39 : 45 (d) 39 : 35 : 34 **[2002]**

6. Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$ and $\boldsymbol{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.
If $\mathbf{n}$ is a unit vector such that $\mathbf{u.n} = 0$ then $|\mathbf{w.n}|$ is equal to

(a) 1 (b) 2
(c) 3 (d) 0 **[2003]**

7. A particle acted by constant forces $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is displaced from the point $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ to the point $5\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. The total work done by the forces is

(a) 30 units (b) 40 units
(c) 50 units (d) 20 units **[2003, 04]**

8. The vector $\mathbf{AB} = 3\mathbf{i} + 4\mathbf{k}$ and $\mathbf{AC} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ are the sides of a triangle ABC. The length of the median through A is

(a) $\sqrt{72}$ (b) $\sqrt{33}$
(c) $\sqrt{288}$ (d) $\sqrt{18}$ **[2003]**

9. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar vectors and λ is a real number, then the vectors $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\lambda\mathbf{b} + 4\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ are non-coplanar for

(a) all except two values of λ
(b) all except one value of λ
(c) for all values of λ
(d) no value of λ **[2004]**

10. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be such that $|\mathbf{u}| = 1$, $|\mathbf{v}| = 2$, $|\mathbf{w}| = 3$. If the projection $\mathbf{v}$ along $\mathbf{u}$ is equal to that of $\mathbf{w}$ along $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{w}$ are perpendicular to each other than $|\mathbf{u} - \mathbf{v} + \mathbf{w}|$ equals

(a) $\sqrt{14}$ (b) $\sqrt{7}$
(c) 2 (d) 14 **[2004]**

11. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be non-zero vectors such that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3} |\mathbf{b}| |\mathbf{c}| \mathbf{a}$. If θ is acute angle between the vectors $\mathbf{b}$ and $\mathbf{c}$, then $\sin\theta$ equals

(a) 2/3 (b) $\sqrt{2}/3$
(c) 1/3 (d) $2\sqrt{2}/3$ **[2004]**

12. If C is the mid point of AB and P is any point outside AB, then

(a) $\mathbf{PA} + \mathbf{PB} + 2\mathbf{PC} = \mathbf{0}$
(b) $\mathbf{PA} + \mathbf{PB} + \mathbf{PC} = \mathbf{0}$
(c) $\mathbf{PA} + \mathbf{PB} = 2\mathbf{PC}$
(d) $\mathbf{PA} + \mathbf{PB} = \mathbf{PC}$ **[2005]**

13. For any vector $\mathbf{a}$, the value of $(\mathbf{a} \times \mathbf{i})^2 + (\mathbf{a} \times \mathbf{j})^2 + (\mathbf{a} \times \mathbf{k})^2$ is equal to

(a) $2\mathbf{a}^2$ (b) $4\mathbf{a}^2$
(c) $3\mathbf{a}^2$ (d) $\mathbf{a}^2$ **[2005]**

14. Let a, b and c be distinct non-negative numbers. If the vectors $a\mathbf{i} + a\mathbf{j} + c\mathbf{k}$, $\mathbf{i} + \mathbf{k}$ and $c\mathbf{i} + c\mathbf{j} + b\mathbf{k}$ lie in a plane, then $\mathbf{c}$ is

(a) equal to zero
(b) the harmonic mean of a and b
(c) the geometric mean of a and b
(d) the arithmetic mean of a and b. **[2005]**

15. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar vectors and λ is a real number then

$$[\lambda(\mathbf{a} + \mathbf{b})\ \lambda^2\mathbf{b}\ \lambda\mathbf{c}] = [\ \mathbf{a}\ \mathbf{b} + \mathbf{c}\ \mathbf{b}] \text{ for}$$

(a) exactly three values of λ
(b) exactly two values of λ
(c) exactly one value of λ
(d) no value of λ **[2005]**

16. Let $\mathbf{a} = \mathbf{i} - \mathbf{k}$, $\mathbf{b} = x\mathbf{i} + \mathbf{j} + (1 - x)\mathbf{k}$, $\mathbf{c} = y\mathbf{i} + x\mathbf{j} + (1 + x - y)\mathbf{k}$ then $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ depends on

(a) both x and y (b) neither on x nor on y
(c) only y (d) only x **[2005]**

17. If $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three vectors such that $\mathbf{a}, \mathbf{b} \neq 0$, $\mathbf{b}\ .\ \mathbf{c} \neq \mathbf{0}$, then $\mathbf{a}$ and $\mathbf{c}$ are

(a) parallel
(b) inclined at an angle of $\pi/3$ between then
(c) inclined at an angle of $\pi/6$ between them
(d) perpendicular **[2006]**

18. The value of a, for which the points A, B, C with position vectors $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $a\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ respectively are vertices of a right-angled triangle with $C = \pi/2$ are

(a) 2 and – 1 (b) 2 and 1
(c) – 2 and – 1 (d) – 2 and 1 **[2007]**

19. Let $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = \mathrm{x}\mathbf{i} + (x - 2)\mathbf{j} - \mathbf{k}$. If the vector $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{x}$ equals

(a) 0 (b) 1
(c) – 4 (d) – 2 **[2007]**

20. If $\mathbf{u}$ and $\mathbf{v}$ are unit vectors and θ is the angle between them, then $2\mathbf{u} \times 3\mathbf{v}$ is a unit vector for

(a) exactly two values of θ
(b) more than two value of θ
(c) no value of θ
(d) Exactly one value of θ **[2007]**

21. The vector $\mathbf{a} = \alpha\mathbf{i} + 2\mathbf{j} + \beta\mathbf{k}$ lies in the plane of the vectors $\mathbf{b} = \mathbf{i} + \mathbf{j}$ and $\mathbf{c} = \mathbf{j} + \mathbf{k}$ and bisects the angle between **b** and **c**. Then which one of the following gives possible values of α and β

(a) $\alpha = 2, \beta = 2$ (b) $\alpha = 1, \beta = 2$
(c) $\alpha = 2, \beta = 1$ (d) $\alpha = 1, \beta = 1$ **[2008]**

22. The non-zero vectors **a**, **b** and **c** are related by $\mathbf{a} = 8\mathbf{b}$ and $\mathbf{c} = -7\mathbf{b}$. Then the angle between **a** and **c** is

(a) 0 (b) $\pi/4$
(c) $\pi/2$ (d) π **[2008]**

23. If **u**, **v**, **w** are non-coplanar vectors and p, q are real numbers, then the equality

$$[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}] - [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}]$$

holds for

(a) more than two but not all values of (p, q)
(b) all value of (p, q)
(c) exactly one value of (p, q)
(d) exactly two values of (p, q) **[2009]**

24. If the vectors $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ and $\mathbf{c} = \lambda\mathbf{i} + \mathbf{j} + \mu\mathbf{k}$ are mutually orthogonal, then $(\lambda, \mu) =$

(a) $(-2, 3)$ (b) $(3, -2)$
(c) $(-3, 2)$ (d) $(2, -3)$ **[2010]**

25. Let $\mathbf{a} = \mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Then the vector **b** satisfying $\mathbf{a} \times \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $\mathbf{a} \,.\, \mathbf{b} = 3$ is

(a) $\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ (b) $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
(c) $-\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ (d) $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ **[2010]**

26. If $\mathbf{a} = \dfrac{1}{\sqrt{10}}(3\mathbf{i} + \mathbf{k})$ and $\mathbf{b} = \dfrac{1}{7}(2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$, then the value of $(2\mathbf{a} - \mathbf{b}) \,.\, [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$ is

(a) 3 (b) -5
(c) -3 (d) 5 **[2011]**

27. The vectors **a** and **b** are not perpendicular and **c** and **d** are vectors satisfying $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ and $\mathbf{a} \,.\, \mathbf{d} = 0$. Then the vector **d** is equal to

(a) $\mathbf{c} - \left(\dfrac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{b}$ (b) $\mathbf{b} - \left(\dfrac{\mathbf{b}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{c}$

(c) $\mathbf{c} + \left(\dfrac{\mathbf{a}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{b}$ (d) $\mathbf{b} + \left(\dfrac{\mathbf{b}\cdot\mathbf{c}}{\mathbf{a}\cdot\mathbf{b}}\right)\mathbf{c}$ **[2011]**

28. If the vectors $p\mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{i} + q\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + r\mathbf{k}$ $(p \neq q \neq r \neq 1)$ are coplanar, then the value of $pqr - (p + q + r)$ is

(a) 2 (b) 0
(c) -1 (d) -2 **[2011]**

29. Let **a**, **b**, **c** be three non-zero vectors which are pairwise non-collinear. If $\mathbf{a} + 3\mathbf{b}$ is collinear with **c** and $\mathbf{b} + 2\mathbf{c}$ is collinear with **a**, then $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}$ is

(a) **a** (b) **c**
(c) **0** (d) $\mathbf{a} + \mathbf{c}$ **[2011]**

30. Let **a** and **b** be two unit vectors. If the vectors $\mathbf{c} = \mathbf{a} + 2\mathbf{b}$ and $\mathbf{d} = 5\mathbf{a} - 4\mathbf{b}$ are perpendicular to each other, then the angle between **a** and **b** is

(a) $\pi/2$ (b) $\pi/3$
(c) $\pi/4$ (d) $\pi/6$ **[2012]**

31. Let ABCD be a parallelogram such that $\mathbf{AB} = \mathbf{q}$, $\mathbf{AD} = \mathbf{p}$ and $\angle$BAD be an acute. If **r** is the vector that coincides with the attitude directed from the vertex B to the side AD, then **r** is given by

(a) $\mathbf{r} = -\mathbf{q} + \left(\dfrac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$

(b) $\mathbf{r} = \mathbf{q} - \left(\dfrac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$

(c) $\mathbf{r} = -3\mathbf{q} + 3\left(\dfrac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$

(d) $\mathbf{r} = 3\mathbf{q} - 3\left(\dfrac{\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{p}}\right)\mathbf{p}$ **[2012]**

32. If the vectors $\mathbf{AB} = 3\mathbf{i} + 4\mathbf{k}$ and $\mathbf{AC} = 5\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ are the sides of a triangle ABC, then the length of the median through A is

(a) $\sqrt{72}$ (b) $\sqrt{33}$
(c) $\sqrt{45}$ (d) $\sqrt{18}$ **[2013]**

33. If **a** and **b** are non-collinear vectors, then the value of α for which the vectors $\mathbf{u} = (\alpha - 2)\,\mathbf{a} + \mathbf{b}$ and $\mathbf{v} = (2 + 3\alpha)\,\mathbf{a} - 3\mathbf{b}$ are collinear is:

(a) $\dfrac{3}{2}$ (b) $\dfrac{2}{3}$

(c) $-\dfrac{3}{2}$ (d) $-\dfrac{2}{3}$ **[2013, online]**

34. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{c} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$ be three vectors. A vector of the type $\mathbf{b} + \lambda\mathbf{c}$ for some scalar λ, whose projection on **a** is of magnitude $\sqrt{\dfrac{2}{3}}$ is

(a) $2\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ (b) $2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$
(c) $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ (d) $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$
[2013, online]

35. If $[\mathbf{a}\times\mathbf{b}\ \ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}] = \lambda\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$ then λ is equal to

(a) 2 (b) 3
(c) 0 (d) 1 **[2014]**

36. If $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$ and $|2\mathbf{a} - \mathbf{b}| = 5$, then $|2\mathbf{a} + \mathbf{b}|$ equals
    (a) 17 (b) 7
    (c) 5 (d) 1 **[2014, online]**

37. If $|\mathbf{c}|^2 = 60$ and $\mathbf{c} \times (\mathbf{i} + \mathbf{j} + 5\mathbf{k}) = \mathbf{0}$, then a value of $\mathbf{c}\cdot(-7\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ is:
    (a) $4\sqrt{2}$ (b) 12
    (c) 24 (d) $12\sqrt{2}$ **[2014, online]**

38. If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are three unit vectors in three dimensional space, then the minimum value of $|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{y} + \mathbf{z}|^2 + |\mathbf{z} + \mathbf{x}|^2$ is
    (a) $\frac{3}{2}$ (b) 3
    (c) $3\sqrt{3}$ (d) 6 **[2014, online]**

39. If $\mathbf{x} = 3\mathbf{i} - 6\mathbf{j} - \mathbf{k}$, $\mathbf{y} = \mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ and $\mathbf{z} = 3\mathbf{i} - 4\mathbf{j} - 12\mathbf{k}$, then the magnitude of the projection of $\mathbf{x} \times \mathbf{y}$ on $\mathbf{z}$ is
    (a) 12 (b) 15
    (c) 14 (d) 13 **[2014, online]**

40. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non zero vectors such that no two of them are collinear and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}|\,|\mathbf{c}|\,\mathbf{a}$. If θ is the angle between vectors $\mathbf{b}$ and $\mathbf{c}$, then a value of $\sin\theta$ is:
    (a) $\frac{2\sqrt{2}}{3}$ (b) $\frac{-\sqrt{2}}{3}$
    (c) $\frac{2}{3}$ (d) $\frac{-2\sqrt{3}}{3}$ **[2015]**

41. In a parallelogram *ABCD*, $|\mathbf{AB}| = a$, $|\mathbf{AD}| = b$ and $|\mathbf{AC}| = c$, then $\mathbf{DB}.\,\mathbf{AB}$ has the value: (choices modified from original)
    (a) $\frac{1}{2}\left(3a^2 + b^2 - c^2\right)$ (b) $\frac{1}{4}\left(a^2 + b^2 - c^2\right)$
    (c) $\frac{1}{3}\left(b^2 + c^2 - a^2\right)$ (d) $\frac{1}{2}\left(a^2 + b^2 + c^2\right)$
    [2015, online]

42. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors such that $|\mathbf{a} + \mathbf{b}| = \sqrt{3}$. If
    $\mathbf{c} = \mathbf{a} + 2\mathbf{b} + 3(\mathbf{a} \times \mathbf{b})$, then $2|\mathbf{c}|$ is equal to:
    (a) $\sqrt{55}$ (b) $\sqrt{51}$
    (c) $\sqrt{43}$ (d) $\sqrt{37}$ **[2015, online]**

43. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three unit vectors such that
    $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{\sqrt{3}}{2}(\mathbf{b} \times \mathbf{c})$.
    If $\mathbf{b}$ is not parallel to $\mathbf{c}$ then the angle between $\mathbf{a}$ and $\mathbf{b}$ is:
    (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$
    (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$ **[2016]**

44. In a triangle ABC, right angled at the vertex A, if the position vectors of A, B and C are respectively $3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$, $-\boldsymbol{i} + 3\boldsymbol{j} + p\boldsymbol{k}$ and $5\boldsymbol{i} + q\boldsymbol{j} - 4\boldsymbol{k}$, then the point (p, q) lies on a line:
    (a) making an obtuse angle with the positive direction of *x*-axis.
    (b) parallel to *x*-axis.
    (c) parallel to *y*-axis.
    (d) making an acute angle with the positive direction of *x*-axis. **[2016, online]**

45. Let ABC be a triangle whose circumcentre is at P. If the positive vectors of A, B, C and P are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ respectively, then the position vector of the orthocentre of this triangle, is:
    (a) $-\frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ (b) $\mathbf{a} + \mathbf{b} + \mathbf{c}$
    (c) $\frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ (d) $\mathbf{0}$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors such that
   $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$
   If $|\mathbf{u}| = 3$, $|\mathbf{v}| = 4$ and $|\mathbf{w}| = 5$, then
   $\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} + \mathbf{w}\cdot\mathbf{u}$ is
   (a) –25 (b) 0
   (c) 25 (d) 47 **[2006]**

2. If $\mathbf{a}$ and $\mathbf{b}$ are two non-parallel vectors having equal magnitude, then the vector $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$ is parallel to
   (a) $\mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$
   (c) $\mathbf{a} + \mathbf{b}$ (d) $\mathbf{a}$ **[2007]**

3. If numbers **a**, **b**, **c** are distinct and non-negative. If three vectors $a\mathbf{i} + a\mathbf{j} + c\mathbf{k}$, $c\mathbf{i} + c\mathbf{j} + b\mathbf{k}$ and $\mathbf{i} + \mathbf{k}$ are coplanar, then c is
   (a) geometric mean of a, b
   (b) harmonic mean of a, b
   (c) equal to zero
   (d) arithmetic mean of a, b **[2007]**

4. Let **x**, **y** and **z** be unit vectors such that
   $|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{y} - \mathbf{z}|^2 + |\mathbf{z} - \mathbf{x}|^2 = 9$
   Then $|\mathbf{x} + \mathbf{y} - \mathbf{z}|^2 - 4\mathbf{x}.\mathbf{y} =$
   (a) 1 (b) 4
   (c) 6 (d) 8 **[2008]**

5. If **a**, **b** and **c** are three unit vectors satisfying $2\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{c} = \mathbf{0}$ then the acute angle between **a** and **b** is
   (a) $\dfrac{\pi}{5}$ (b) $\dfrac{\pi}{4}$
   (c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{6}$ **[2009]**

6. If $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = \mathbf{j} + 2\mathbf{k}$ and **a** is a unit vector, then the maximum value of the scalar triple product [**a b c**] is
   (a) $\sqrt{30}$ (b) $\sqrt{29}$
   (c) $\sqrt{26}$ (d) $\sqrt{60}$ **[2009]**

7. **Statement-1:** If **a** and **b** are two vectors such that $|\mathbf{a}| = 2$, $|\mathbf{b}| = 3$, $|2\mathbf{a} - \mathbf{b}| = 5$ then $|2\mathbf{a} + \mathbf{b}| = 5$

   Statement-2: For any two vectors **c** and **d**, $|\mathbf{c} - \mathbf{d}| = |\mathbf{c} + \mathbf{d}|$ **[2010]**

8. If **a**, **b** and **c** are non-zero vectors such that $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, $\mathbf{b} \times \mathbf{c} = \mathbf{a}$ and $\mathbf{c} \times \mathbf{a} = \mathbf{b}$ then
   (a) $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$
   (b) $\mathbf{a} = \mathbf{b} = \mathbf{c}$
   (c) $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$
   (d) $|\mathbf{a}| + |\mathbf{b}| - |\mathbf{c}| = 0$ **[2010]**

9. Let $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = 2\mathbf{b} + 10\mathbf{a}$ and $\mathbf{OC} = \mathbf{b}$ where O is the origin. If p is the area of the quadrilateral $OABC$ and q is the area of the parallelogram with OA and OC as adjacent sides then p is equal to
   (a) q^6 (b) $6q$
   (c) $q/6$ (d) $6 - q$ **[2011]**

10. If **a** and **b** are two vectors such that $2\mathbf{a} + \mathbf{b} = \mathbf{e}_1$ and $\mathbf{a} + 2\mathbf{b} = \mathbf{e}_2$, where $\mathbf{e}_1 = (1, 1, 1)$ and $\mathbf{e}_2 = (1, 1, -1)$, then the angle between **a** and **b** is
    (a) $\cos^{-1}\left(\dfrac{7}{9}\right)$ (b) $\cos^{-1}\left(\dfrac{7}{11}\right)$
    (c) $\cos^{-1}\left(-\dfrac{7}{11}\right)$ (d) $\cos^{-1}\left(-\dfrac{7}{9}\right)$ **[2012]**

11. If **u**, **v**, **w** are unit vectors satisfying $2\mathbf{u} + 2\mathbf{v} + 3\mathbf{w} = \mathbf{0}$, then $|\mathbf{u} - \mathbf{v}|$ equals
    (a) $\dfrac{7}{4}$ (b) $\dfrac{\sqrt{5}}{2}$
    (c) $\dfrac{\sqrt{7}}{2}$ (d) $\dfrac{5}{4}$ **[2012]**

12. Let $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 3\mathbf{k}$. If **u** is a unit vector, then the maximum value of the scalar triple product [**u v w**] is
    (a) $\sqrt{6}$ (b) $\sqrt{10} + \sqrt{6}$
    (c) $\sqrt{59}$ (d) $\sqrt{60}$ **[2013]**

13. Unit vectors **a**, **b**, **c** are coplanar. A unit vector **d** is perpendicular to them. If
    $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \dfrac{1}{6}\mathbf{i} - \dfrac{1}{3}\mathbf{j} + \dfrac{1}{3}\mathbf{k}$
    and the angle between **a** and **b** is 30°, then **c** is/are
    (a) $\pm\dfrac{1}{3}(-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$ (b) $\dfrac{1}{3}(2\mathbf{i} + \mathbf{j} - \mathbf{k})$
    (c) $\pm\dfrac{1}{3}(-\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ (d) $\dfrac{1}{3}(-2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ **[2014]**

14. Let $\mathbf{x} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and $\mathbf{y} = \boldsymbol{i} + \boldsymbol{j}$. If **z** is a vector such that $\mathbf{x}.\mathbf{z} = |\mathbf{z}|$, $|\mathbf{z} - \mathbf{x}| = 2\sqrt{2}$ and the angle between $\mathbf{x} \times \mathbf{y}$ and **z** is 30°, then the magnitude of the vector $(\mathbf{x} \times \mathbf{y}) \times \mathbf{z}$ is:
    (a) $\dfrac{\sqrt{3}}{2}$ (b) $\dfrac{3}{2}$
    (c) $\dfrac{1}{2}$ (d) $\dfrac{3\sqrt{3}}{2}$ **[2015]**

15. From a point A with position vector $p(\mathbf{i} + \mathbf{j} + \mathbf{k})$, AB and AC are drawn perpendicular to the lines $\mathbf{r} = \mathbf{k} + \lambda(\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = -\mathbf{k} + \mu(\mathbf{i} - \mathbf{j})$ respectively. A value of p is equal to
    (a) -1 (b) $\sqrt{2}$
    (c) 2 (d) -2 **[2016]**

Answers

Concept Based

1. (d) **2.** (b) **3.** (d) **4.** (b)
5. (d) **6.** (b) **7.** (c) **8.** (d)
9. (a) **10.** (b)

Level 1

11. (a) **12.** (b) **13.** (a) **14.** (b)
15. (c) **16.** (a) **17.** (d) **18.** (a)
19. (a) **20.** (b) **21.** (b) **22.** (b)

23. (a) **24.** (a) **25.** (b) **26.** (b)
27. (b) **28.** (b) **29.** (b) **30.** (c)
31. (c) **32.** (b) **33.** (d) **34.** (b)
35. (a) **36.** (b) **37.** (d) **38.** (a)
39. (c) **40.** (a) **41.** (a) **42.** (a)
43. (a) **44.** (c) **45.** (d) **46.** (b)
47. (a) **48.** (c)

Level 2

49. (d) **50.** (a) **51.** (d) **52.** (a)
53. (d) **54.** (b) **55.** (d) **56.** (c)
57. (d) **58.** (d) **59.** (b) **60.** (c)
61. (d) **62.** (b) **63.** (c) **64.** (a)
65. (c) **66.** (b) **67.** (c) **68.** (c)
69. (d) **70.** (b) **71.** (c) **72.** (c)
73. (a) **74.** (b) **75.** (a) **76.** (b)
77. (c) **78.** (d) **79.** (a) **80.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (b) **2.** (a) **3.** (a) **4.** (c)
5. (b) **6.** (c) **7.** (b) **8.** (b)
9. (a) **10.** (a) **11.** (d) **12.** (c)
13. (a) **14.** (c) **15.** (d) **16.** (b)
17. (a) **18.** (b) **19.** (d) **20.** (d)
21. (d) **22.** (d) **23.** (c) **24.** (c)
25. (c) **26.** (b) **27.** (a) **28.** (d)
29. (c) **30.** (b) **31.** (a) **32.** (b)
33. (b) **34.** (b) **35.** (d) **36.** (c)
37. (d) **38.** (b) **39.** (c) **40.** (a)
41. (a) **42.** (a) **43.** (d) **44.** (d)
45. (c)

Previous Years' B-Architecture Entrance Examination Questions

1. (a) **2.** (c) **3.** (a) **4.** (c)
5. (d) **6.** (a) **7.** (c) **8.** (c)
9. (b) **10.** (c) **11.** (c) **12.** (c)
13. (c) **14.** (b) **15.** (a) (b) (c) (d)

Hints and Solutions

Concept Based

1. Let O be the origin of reference. Then the position vector of A, B, M are **OA**, **OB** and **OM** respectively

   so $\mathbf{OM} = \frac{1}{2}(\mathbf{OA} + \mathbf{OB})$

2. $(3\mathbf{i} + 4\mathbf{j}) \cdot (2\mathbf{j} - 5\mathbf{k}) = 8$, $|3\mathbf{i} + 4\mathbf{j}| = 5$, $|2\mathbf{j} - 5\mathbf{k}| = \sqrt{29}$. Hence the required angle is

   $$\cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \cos^{-1}\frac{8}{5\sqrt{29}}.$$

3. $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b} = (\lambda + \mu)\mathbf{i} + (\lambda + 2\mu)\mathbf{j} + \lambda\mathbf{k}$

   $$0 = \mathbf{a}.\mathbf{c} = \lambda\,|\mathbf{a}|^2 + \mu\,\mathbf{a}.\mathbf{b}$$
   $$= 3\lambda + 3\mu \quad \Rightarrow \quad \lambda + \mu = 0$$

   Also $(\lambda + \mu)^2 + (\lambda + 2\mu)^2 + \lambda^2 = 1$

   $$\Rightarrow \quad 2\lambda^2 = 1 \quad \Rightarrow \quad \lambda = \pm\frac{1}{\sqrt{2}}$$

   Hence $\mathbf{c} = -\lambda\mathbf{j} + \lambda\mathbf{k} = \pm\frac{1}{\sqrt{2}}(-\mathbf{j} + \mathbf{k})$

4. $\mathbf{b} = \lambda\mathbf{a} = \lambda(2\mathbf{i} + \mathbf{j} - \mathbf{k})$. so
   $\lambda\,(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \cdot (2\mathbf{i} + \mathbf{j} - \mathbf{k}) = 2$

   $$\Rightarrow \quad 6\lambda = 2 \quad \Rightarrow \quad \lambda = \frac{1}{3}$$

   Hence $\mathbf{b} = \frac{1}{3}(2\mathbf{i} + \mathbf{j} - \mathbf{k})$

5. $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k}$$

   $$\text{so} \quad \text{Proj}_{\mathbf{w}}\,\mathbf{u} \times \mathbf{v} = \frac{(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}}{|\mathbf{w}|^2}\mathbf{w}$$
   $$= \frac{2 - 3 - 2}{6}(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$
   $$= -\frac{1}{2}(\mathbf{i} + \mathbf{j} + 2\mathbf{k})$$

   $$|\text{Proj}_{\mathbf{w}}\,\mathbf{u} \times \mathbf{v}|^2 = \frac{1}{4} \cdot 6 = \frac{3}{2}$$

6. $\mathbf{u} = t\mathbf{v} \Rightarrow (\lambda + 2)\mathbf{a} + \mathbf{b} = t(1 + 4\lambda)\mathbf{a} - 2t\,\mathbf{b}$

   $\Rightarrow \quad (\lambda + 2 - t - 4t\,\lambda)\mathbf{a} = (-2t - 1)\mathbf{b}$

   Since **a** and **b** are non-collinear, so

   $t = -\frac{1}{2}$ and $\lambda + 2 - t - 4 + \lambda = 0$

   $$\Rightarrow \quad \lambda + 2 - \frac{1}{2} - 2\lambda = 0$$

   $$\Rightarrow \quad -\lambda + \frac{3}{2} = 0 \quad \Rightarrow \quad \lambda = \frac{3}{2}.$$

7. $\mathbf{AB} = -\mathbf{i} + \mathbf{j}$, $\mathbf{BC} = \mathbf{i} + \mathbf{k}$

   Area of triangle $ABC = \frac{1}{2}|\mathbf{AB} \times \mathbf{BC}|$

   $$= \frac{1}{2}\left|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}\right| = \frac{1}{2}|\mathbf{i} + \mathbf{j} - \mathbf{k}| = \frac{\sqrt{3}}{2}$$

8. $(3\mathbf{i} + 4\mathbf{j}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 1 & -1 & 1 \end{vmatrix}$

$$= 4\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$$

The length of this vector $\sqrt{16+9+49} = \sqrt{74}$.

A unit vector perpendicular to given vectors is

$$\frac{1}{\sqrt{74}}(4\mathbf{i} - 3\mathbf{j} - 7\mathbf{k})$$

9. $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 1 \times 2 - 2 \times (-2) + 3 \times 2$

$$= 2 + 4 + 6 = 12$$

10. $\mathbf{u} = (\mathrm{i} - \mathrm{j} + \mathrm{k}) \times (2\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & -3 & -1 \end{vmatrix}$

$$= 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

$\mathbf{v} = (-3\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (2\mathbf{j} + \mathbf{k})$

$$= -\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$$

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -1 \\ -1 & 3 & -6 \end{vmatrix} = -15\mathbf{i} + 25\mathbf{j} + 15\mathbf{k}$

Level 1

11. $0 = (\mathbf{a} + \mu\mathbf{b}) \,.\, (\mathbf{a} - \mu\mathbf{b}) = |\,\mathbf{a}\,|^2 - \mu^2\,|\,\mathbf{b}\,|^2$

$$\Rightarrow 9 - 16\mu^2 = 0 \quad \Rightarrow \quad \mu = \pm\, 3/4.$$

12. $0 = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & \alpha \\ 3 & -4 & 5 \end{vmatrix} \quad \Rightarrow \quad 15 + 5\alpha = 0$

$$\Rightarrow \alpha = -\,3.$$

13. If **p** and **q** are adjacent sides of the parallelogram, their sum gives one of the diagonals and their difference gives the other that is

$$\mathbf{p} + \mathbf{q} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k} \text{ and } \mathbf{p} - \mathbf{q} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$$

$$\Rightarrow \mathbf{p} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} \text{ and } \mathbf{q} = \mathbf{i} + 2\mathbf{k} - 3\mathbf{k}$$

Required area

$= |\,\mathbf{p} \times \mathbf{q}\,| = |\,\mathbf{i}(3-2) - \mathbf{j}(-6-1) + \mathbf{k}(4+1)\,|$

$$= \sqrt{1^2 + 7^2 + 5^2} = 5\sqrt{3}.$$

14. Put $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. The given equation implies $(y - 2z)\mathbf{i} + (z - x)\mathbf{j} + (2x - y)\mathbf{k} = \mathbf{i} - \mathbf{k}$

That is $y - 2z = 1$, $z - x = 0$ and $2x - y = -1$, from which we get $x = z = t$ (say), so that $y = 1 + 2t$. Substituting these, we get

$\mathbf{r} = t\mathbf{i} + (1 + 2t)\mathbf{j} + t\mathbf{k} = \mathbf{j} + t(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

15. $|\,\mathbf{a}\,| = \sqrt{2} = |\,\mathbf{b}\,| = |\,\mathbf{c}\,|$. Let $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$, since **c** makes an obtuse angle with **i**, we must have $\mathbf{c} \,.\, \mathbf{i} = c_1 < 0$. It is also given that the angles between the vectors are equal i.e.,

$$\cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \cos^{-1}\frac{1}{2} = \cos^{-1}\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \cos^{-1}\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|}$$

But $\mathbf{a}.\,\mathbf{b} = 1$, $\mathbf{a} \,.\, \mathbf{c} = c_1 + c_2$ and $\mathbf{b} \,.\, \mathbf{c} = c_2 + c_3$

This gives $c_1 + c_2 = 1$ and $c_2 + c_3 = 1 \Rightarrow c_3 = c_1$ and $c_2 = 1 - c_1$. Putting in $c_1^{\,2} + c_2^{\,2} + c_3^{\,2} = 2$, we get $c_1 = 1, -\,1/3$. Since $c_1 < 0$ so $c_1 = -\,1/3 = c_3$ and $c_2 = 4/3$. Hence

$$\mathbf{c} = -\frac{1}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

16. The two diagonals are

$\mathbf{AB} - \mathbf{BC} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{AB} + \mathbf{BC} = 2\mathbf{i} - 2\mathbf{j}$

These vectors have magnitudes 6 and $2\sqrt{2}$ respectively, and their dot product is 12. Therefore, the angle between them is

$$\cos^{-1}\frac{12}{(6)\,(2\sqrt{2})} = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

17. $|\,\mathbf{a}\,| = |\,\mathbf{b}\,| = 1 = |\,\mathbf{c}\,|$. It is also given that the angle θ between **a** and **c** equals that between **b** and **c**

$$\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \mathbf{a} \,.\, \mathbf{c} = \cos\theta = \frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|} = \mathbf{b} \,.\, \mathbf{c}$$

Since $\mathbf{a} \,.\, \mathbf{b} = 0$, we get from the given value of **c**,

$\mathbf{a} \,.\, \mathbf{c} = \alpha\, \mathbf{a} \,.\, \mathrm{a} + \beta\, \mathbf{a} \,.\, \mathbf{b} + v\mathbf{a} \,.\, (\mathbf{a} \times \mathbf{b}) = \alpha$

i.e., $\alpha = \cos\theta$, and similarly $\mathbf{b}.\mathbf{c} = \cos\theta = \beta$

$1 = \mathbf{c} \,.\, \mathbf{c} = 2\alpha^2 + \gamma^2\,|\,\mathbf{a} \times \mathbf{b}\,|^2 = 2\alpha^2 + \gamma^2[|\,\mathbf{a}\,|^2\,|\,\mathbf{b}\,|^2 - (\mathbf{a}.\mathbf{b})^2] = 2\alpha^2 + \gamma^2$

$\Rightarrow \gamma^2 = 1 - 2\alpha^2 = 1 - 2\cos^2\theta = -\cos 2\theta$

$$\Rightarrow \alpha^2 = \beta^2 = \frac{1-\gamma^2}{2} = \frac{1+\cos 2\theta}{2}.$$

18. $|\,\mathbf{a} - \mathbf{b}\,|^2 = \mathbf{a}^2 + \mathbf{b}^2 - 2\mathbf{a}.\mathbf{b} = 4\sin^2\theta$

$\Rightarrow \quad |\,\mathbf{a} - \mathbf{b}\,| = 2\,|\sin\theta\,|.$

$|\,\mathbf{a} - \mathbf{b}| < 1 \Rightarrow |\sin\theta\,| < 1/2$

$$\Rightarrow \theta \in [0, \pi/6] \text{ or } \left(\frac{5\pi}{6}, \pi\right).$$

19. If θ is angle between **a** and **b**, and ϕ the angle between **c** and $\mathbf{a} \times \mathbf{b}$. Then

$|\,(\mathbf{a} \times \mathbf{b}) \,.\, \mathbf{c}\,| = |\,\mathbf{a}\,|\,|\,\mathbf{b}\,|\,|\,\mathbf{c}\,| \sin\theta \cos\phi$

So we must have $\sin\theta\cos\phi = 1 \Rightarrow \sin\theta = 1$, $\cos\phi = 1 \Rightarrow \theta = \pi/2$, $\phi = 0 \Rightarrow$ **a** and **b** are perpendicular so $\mathbf{a} \,.\, \mathbf{b} = 0$. $\phi = 0 \Rightarrow$ **c** must be perpendicular to both **a** and **b**, so that $\mathbf{b} \,.\, \mathbf{c} = 0 = \mathbf{a} \,.\, \mathbf{c}$.

20. Let A, B, C and D be the points with the given position vectors. Then $\mathbf{AB} = -\,2\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$, $\mathbf{AC} = 4\mathbf{i} + 5\mathbf{j} + (\lambda - 10)\mathbf{k}$ and $\mathbf{AD} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

The volume of the tetrahedron is

$$\frac{1}{6}\,|\,(\mathbf{AB}\times\mathbf{AC})\,.\,\mathbf{AD}\,|$$

$$=\frac{1}{6}\left|\begin{vmatrix}-2 & 3 & -3\\ 4 & 5 & \lambda-10\\ 6 & 2 & -3\end{vmatrix}\right|$$

$$=\frac{1}{6}\,|-88+22\lambda\,|=11\text{ (given)}$$

$\Rightarrow \quad \lambda = 1$ or 7

21. $$\begin{vmatrix}x & x+1 & x+2\\ x+3 & x+4 & x+5\\ x+6 & x+7 & x+8\end{vmatrix}=\begin{vmatrix}x & x+1 & x+2\\ 3 & 3 & 3\\ 6 & 6 & 6\end{vmatrix}=0$$

for all x so for $x < 0$.

22. Required vector $= K(2\mathbf{j} - \mathbf{k}) \times (-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ $= K(-4\mathbf{i} + \mathbf{j} + 2\mathbf{k})$. The length of this vector is $\sqrt{21}\ |K|$ so $|K| = 1/\sqrt{3}$. Thus $K = \pm 1/\sqrt{3}$. But it will make obtuse angle with y-axis if $K = -1/\sqrt{3}$. Thus the required vector is $1/\sqrt{3}\,(4\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

23. $|\,\mathbf{p} \times \mathbf{q}\,| = |\,(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})\,| = 2\,|\,\mathbf{a} \times \mathbf{b}\,|$
$4\,|\,\mathbf{a}\,|^2\,|\,\mathbf{b}\,|^2 \cos^2\theta = 4\,(K - |\,\mathbf{a}\,|^2\,|\,\mathbf{b}\,|^2\cos^2\theta)$
$\Rightarrow |\,\mathbf{a}\,|^2\,|\,\mathbf{b}\,|^2 = K \Rightarrow K = 16.$

24. The given conditions mean that **r** is perpendicular to all three vectors **a**, **b** and **c**. This is possible only if they are coplanar which means$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

25. Let A, B and C be the three given vertices and D the fourth vertex, with the position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Since ABCD is a parallelogram, the diagonals AC and BD bisect each other so

$$\frac{(1+7)\mathbf{i}+(1+9)\mathbf{j}+(1+11)\mathbf{k}}{2}$$

$$=\frac{(1+x)\mathbf{i}+(3+y)\mathbf{j}+(5+z)\mathbf{k}}{2}$$

$\Rightarrow x = 7$, $y = 7$ and $z = 7$, so that D is the point $7\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

26. Required volume $= (\mathbf{OA} \times \mathbf{OB})\,.\,\mathbf{OC}$

$$=\begin{vmatrix}2 & -3 & 0\\ 1 & 1 & -1\\ 3 & 0 & -1\end{vmatrix}=4\,.$$

27. If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ then $|\,\mathbf{a} \times \mathbf{i}\,|^2 = a_2^2 + a_3^2$
Therefore $|\,\mathbf{a} \times \mathbf{i}\,|^2 + |\,\mathbf{a} \times \mathbf{j}\,|^2 + |\,\mathbf{a} \times \mathbf{k}\,|^2$
$= 2(a_1^2 + a_2^2 + a_3^2) = 2\mathbf{a}^2.$

28. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a}\,.\,\mathbf{c})\mathbf{b} - (\mathbf{a}\,.\,\mathbf{b})\mathbf{c}$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a}\,.\,\mathbf{c})\mathbf{b} - (\mathbf{b}\,.\,\mathbf{c})\mathbf{a}$, so the given equality implies $(\mathbf{a}\,.\,\mathbf{b})\,\mathbf{c} = (\mathbf{b}\,.\,\mathbf{c})\,\mathbf{a}$ which means that **a** and **c** are collinear.

29. The given expression $= (\mathbf{i}\,.\,\mathbf{i})\mathbf{a} - (\mathbf{i}\,.\,\mathbf{a})\mathbf{i} + (\mathbf{j}\,.\,\mathbf{j})\,\mathbf{a} - (\mathbf{j}\,.\,\mathbf{a})\mathbf{j} + (\mathbf{k}\,.\,\mathbf{k})\mathbf{a} - (\mathbf{k}\,.\,\mathbf{a})\mathbf{k}$
$= 3\mathbf{a} - [(\mathbf{i}\,.\,\mathbf{a})\mathbf{i} + (\mathbf{j}\,.\,\mathbf{a})\mathbf{j} + (\mathbf{k}\,.\,\mathbf{a})\mathbf{k}]$
$= 3\mathbf{a} - \mathbf{a} = 2\mathbf{a}$

30. $[\,\mathbf{a} \times \mathbf{b}\quad \mathbf{b} \times \mathbf{c}\quad \mathbf{c} \times \mathbf{a}]$
$= ((\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c})).\ (\mathbf{c} \times \mathbf{a})$
$= ((\mathbf{a} \times \mathbf{b}\,.\,\mathbf{c})\mathbf{b} - ((\mathbf{a} \times \mathbf{b})\,.\,\mathbf{b})\ \mathbf{c}).\ (\mathbf{c} \times \mathbf{a})$
$= [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\ (\mathbf{b}\,.\,\mathbf{c} \times \mathbf{a}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2.$

31. A vector **c** perpendicular to z-axis is of the form $c_1\,\mathbf{i} + c_2\,\mathbf{j}$. According to given conditions $3c_1 - c_2 = 9$ and $c_1 + 2c_2 = -4$
$\Rightarrow c_2 = 2$ and $c_2 = -3$. Thus required point is $(2, -3, 0)$.

32. Required area $= |\,(\mathbf{a} + 3\mathbf{b}) \times (3\mathbf{a} + \mathbf{b})\,|$
$= |\,9(\mathbf{b} \times \mathbf{a}) + \mathbf{a} \times \mathbf{b}\,| = 8\,|\,\mathbf{a} \times \mathbf{b}\,| =$
$8\,|\,\mathbf{a}\,|\,|\,\mathbf{b}\,| \sin \pi/6 = 4.$

33. $$\begin{vmatrix}x & 5 & 7\\ 1 & 1 & -1\\ 1 & 2 & 2\end{vmatrix}=0 \Rightarrow x = 2.$$

34. $\mathbf{a}\,.\,(\mathbf{b} \times \mathbf{c}) = -\,\mathbf{a}\,.\,(\mathbf{c} \times \mathbf{b})$ so $\mathbf{a}\,.\,(\mathbf{c} \times \mathbf{b}) = -3.$

35. If $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j} + b_3\,\mathbf{k}$, we have the following relations $2b_1 + 2b_2 + b_3 = 14$ and $\mathbf{a} \times \mathbf{b} = (2b_3 - b_2)\mathbf{i} + (b_1 - 2b_3)\mathbf{d} + 2(b_2 - b_1)\mathbf{k}$. So $2b_3 - b_2 = 3$, $1 = b_1 - 2b_3$, $b_2 - b_1 = -4$. Solving $b_1 = 5$, $b_2 = 1$, $b_3 = 2$.

36. Let $\mathbf{AB} = \mathbf{a}$, $\mathbf{BC} = \mathbf{b}$,
then $\quad \mathbf{FC} = 2\mathbf{a}$,
$\mathbf{AD} = 2\mathbf{b}$.
Also, $\quad \mathbf{DC} = \mathbf{AC} - \mathbf{AD}$
$= \mathbf{a} + \mathbf{b} - 2\mathbf{b}$
$= \mathbf{a} - \mathbf{b}$
$\Rightarrow \quad \mathbf{EB} = 2\,\mathbf{DC} = 2(\mathbf{a} - \mathbf{b})$
Now, $\mathbf{AD} + \mathbf{EB} + \mathbf{FC}$
$= 2\mathbf{b} + 2(\mathbf{a} - \mathbf{b}) + 2\mathbf{a} = 4\mathbf{a} = 4\,\mathbf{AB} = 4\,\mathbf{ED}$
$\therefore \quad \lambda = 4$

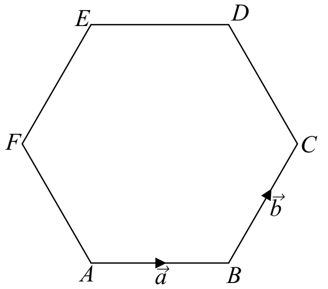


Fig. 22.6

37. Let N_1 be a vector normal to plane determined by vectors **i**, **i** + **k** and N_2 be a vector normal to the plane determined by vectors **i** – **j**, **i** + **k** we have
$\mathbf{N}_1 = \mathbf{i} \times (\mathbf{i} + \mathbf{j}) = \mathbf{k}$

and $\mathbf{N}_2 = (\mathbf{i} - \mathbf{j}) \times (\mathbf{i} + \mathbf{k})$

$= -\mathbf{i} - \mathbf{j} + \mathbf{k}$

Note that **a** is parallel to $\mathbf{N}_1 \times \mathrm{N}_2 = \mathbf{i} - \mathbf{j}$

If θ is the angle between **a** and $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$, then

$$\cos\theta = \pm\frac{1+2}{\sqrt{2}\sqrt{1+4+4}} = \pm\frac{1}{\sqrt{2}}$$

Thus, θ may be taken as $\pi/4$.

38. Let M be the mid point of PR, then position vector of M is

$$\frac{1}{2}(-2\mathbf{i} - \mathbf{j} + 3\mathbf{i} + 3\mathbf{j}) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

Let N be the mid point of QS, then position vector of N is

$$\frac{1}{2}(4\mathbf{i} - 3\mathbf{i} + 2\mathbf{j}) = \frac{1}{2}\mathbf{i} + \mathbf{j}$$

Thus, $PQRS$ is a parallelogram.

Next $\quad PQ = |\mathbf{PQ}| = |4\mathbf{i} - (-2\mathbf{i} - \mathbf{j})|$

$= |6\mathbf{i} + \mathbf{j}| = \sqrt{37}$

and $\quad QR = |\mathbf{QR}| = |3\mathbf{i} + 3\mathbf{j} - 4\mathbf{i}| = |-\mathbf{i} + 3\mathbf{j}|$

$= \sqrt{1+9} = \sqrt{10}$

Thus, $PQRS$ is not a rhombus

Also, $\quad PR = |\mathbf{PR}| = |3\mathbf{i} + 3\mathbf{j} - (-2\mathbf{i} - \mathbf{j})|$

$= |5\mathbf{i} + 4\mathbf{j}| = \sqrt{25+16} = 41$

and $\quad OS = |\mathbf{OS}| = |-3\mathbf{i} + 2\mathbf{j} - (-4\mathbf{i})|$

$= |\mathbf{i} + 2\mathbf{j}| = \sqrt{1+4} = \sqrt{5}$.

$\therefore$ $PQRS$ is not a rectangle.

Thus, $PQRS$ is a parallelogram which is neither a rhombus nor a rectangle.

39. Let θ_1 be the angle between **a** and **b** and θ_2 be the angle between **c** and **d**.

Now,

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta_1\, \mathbf{n}_1 = \sin\theta_1\, \mathbf{n}_1$$

and $\quad \mathbf{c} \times \mathbf{b} = |\mathbf{c}|\,|\mathbf{d}| \sin\theta_2\, \mathbf{n}_2 = \sin\theta_2\, \mathbf{n}_2$

where $\mathbf{n}_1$ is a unit vector parallelogram to the plane of **a** and **b**; and $\mathbf{n}_2$ is a unit vector perpendicular to the plane of **c** and **d**.

As $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 1$, we get

$(\sin\theta_1)(\sin\theta_2)\, \mathbf{n}_1 \cdot \mathbf{n}_2 = 1$

$(\sin\theta_1)(\sin\theta_2)\, |\mathbf{n}_1|\, |\mathbf{n}_2| \cos\phi = 1$

where ϕ is angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

$\Rightarrow \quad (\sin\theta_1)(\sin\theta_2) = \cos\phi = 1$

$\Rightarrow \quad \theta_1 = \pi/2,\ \theta_2 = \pi/2$ and $\phi = 0$

As $\phi = 0$ we get $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel.

Therefore, **a**, **b**, **c** and **d** are coplanar.

Also $\mathbf{a}.\ \mathbf{c} = 1/2 \Rightarrow |\mathbf{a}|\, |\mathbf{c}| \cos\theta = 1/2$,

So the angle between **a** and **c** is $\pi/3$

$\Rightarrow$ the angle between **b** and **d** is $\pi/3$

$\therefore$ **b** and **d** are nonparallel.

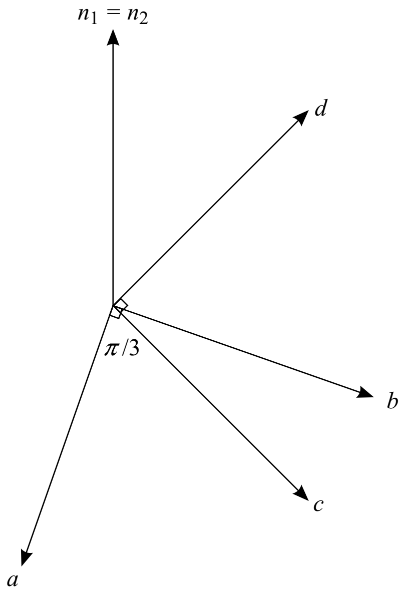


Fig. 22.7

40. If V is volume of the parallelopiped, then

$$V^2 = [\mathbf{abc}]^2 = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & 1/2 & 1/2 \\ 1 & 1 & 1/2 \\ 1 & 1/2 & 1 \end{vmatrix}$$

$[C_1 \to C_1 + C_2 + C_3]$

$$= 2\begin{vmatrix} 1 & 1/2 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{vmatrix} = \frac{1}{2}$$

$\Rightarrow V = 1/\sqrt{2}$

41. $$\begin{vmatrix} a & 2a & -3a \\ 2a+1 & 2a+3 & a+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}$$

$$= a\begin{vmatrix} 1 & 2 & -3 \\ 2a+1 & 2a+3 & 4+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}$$

$$= a\begin{vmatrix} 1 & 0 & 0 \\ 2a+1 & 1-2a & 7a+4 \\ 3a+5 & -5a-5 & 10a+17 \end{vmatrix}$$

$[C_2 \to C_2 - 2C_1,\ C_3 \to C_3 + 3C_1]$

$= a\,(15a^2 + 31a + 37) = 0$ if and only if $a = 0$ as $15a^2 + 31a + 37 > 0$ for any a.

42. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$
$= (\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d}))\mathbf{b} - (\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d}))\, \mathbf{a}$
$= p\mathbf{a} + q\mathbf{b}$
So $p = -(\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})) = \mathbf{b} \cdot (\mathbf{d} \times \mathbf{c}) = [\mathbf{b}\ \mathbf{d}\ \mathbf{c}]$
$= [\mathbf{c}\ \mathbf{b}\ \mathbf{d}]$.

43. $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$. Also

$$\cos\theta = \mathbf{a} \cdot \mathbf{c} = \mathbf{b} \cdot \mathbf{c}$$
$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma(\mathbf{a} \times \mathbf{b}) \text{ so}$$
$$\mathbf{a} \cdot \mathbf{c} = \alpha + \beta(\mathbf{a} \cdot \mathbf{b}) = \alpha$$
$$\mathbf{b} \cdot \mathbf{c} = \alpha(\mathbf{a} \cdot \mathbf{b}) + \beta = \beta$$

Thus $\alpha = \mathbf{a} \cdot \mathbf{c} = \cos\theta = \mathbf{b} \cdot \mathbf{c} = \beta$.

44. $(\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d}) = ((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}) . \mathbf{d}$
$= ((\mathbf{a} . \mathbf{c})\, \mathbf{b} - (\mathbf{b} . \mathbf{c})\, \mathbf{a}) . \mathbf{d}$
$= (\mathbf{a} . \mathbf{c})\, (\mathbf{b} . \mathbf{d}) - (\mathbf{b} . \mathbf{c})\, (\mathbf{a} . \mathbf{d})$

45. $[\mathbf{a} \times \mathbf{b},\ \mathbf{b} \times \mathbf{c},\ \mathbf{c} \times \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$

46. $|\ \mathbf{a} + \mathbf{b}\ |^2 - |\ \mathbf{a} - \mathbf{b}\ |^2 = |\ \mathbf{a}\ |^2 + |\ \mathbf{b}\ |^2 + 2\mathbf{a} . \mathbf{b}$
$- (|\ \mathbf{a}\ |^2 + |\ \mathbf{b}\ |^2 - 2\mathbf{a} . \mathbf{b})$
$= 4\mathbf{a} . \mathbf{b}$

47. $(\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - \mathbf{k})$
$= -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

48. $[\mathbf{a} + \mathbf{b}\ \ \mathbf{b} + \mathbf{c}\ \ \mathbf{c} + \mathbf{a}] = 2[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$.

Level 2

49. Suppose that l, m, n are direction cosine of a line and the cube be a unit cube. The direction ratios of the diagonals will 1,1,1; 1, –1; 1, 1,1; –1,1,1. Hence

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$
$$= \frac{(l+m+n)^2}{1\cdot 3} + \frac{(l-m+n)^2}{1\cdot 3} + \frac{(-l+m+n)^2}{1\cdot 3} + \frac{(l+m-n)^2}{1\cdot 3}$$
$$= \frac{1}{3}[4(l^2 + m^2 + n^2) + 2(lm + mn + ln - lm + ln - mn + lm - ln - mn - lm + mn - ln)]$$
$$= \frac{4}{3}.$$

50. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = ((\mathbf{a} \times \mathbf{b}).\ \mathbf{d})\ \mathbf{c} - ((\mathbf{a} \times \mathbf{b})\cdot\mathbf{c})\ \mathbf{d}$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$

so, $k = -[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{b}\ \mathbf{a}\ \mathbf{c}]$.

51. $$\frac{1}{2} = \cos\frac{\pi}{3} = \frac{\mathbf{c}\cdot\mathbf{d}}{|\mathbf{c}||\mathbf{d}|} = \frac{x + x + 1}{\sqrt{x^2+2}\,\sqrt{x^2+2}} = \frac{2x+1}{x^2+2}$$
$\Rightarrow x^2 + 2 = 4x + 2 \Rightarrow x = 0, 4$.

52. For $t = 3$, the line intersect xy plane at $(-2, 10, 0)$.

53. The vector component of u orthogonal to $\mathbf{a}$ is

$$\mathbf{u} - \frac{\mathbf{u}.\mathbf{a}}{|\mathbf{a}|^2}\ \mathbf{a}$$

$$= (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) - \frac{8+1+6}{\left(\sqrt{16+1+4}\right)^2}\ (4\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$
$$= \frac{1}{7}[(14\mathbf{i} - 7\mathbf{i} + 21\mathbf{k}) - 20\mathbf{i} + 5\mathbf{j} - 10\mathbf{k}]$$
$$= \frac{1}{7}(-6\mathbf{i} - 2\mathbf{j} + 11\mathbf{k})$$
$$= -\frac{1}{7}(6\mathbf{i} - 2\mathbf{j} - 11\mathbf{k}).$$

54. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [a\ b\ d]c - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$

a, **b**, **c**, **d** lie in the same plane so $[\mathbf{a}\ \mathbf{b}\ \mathbf{d}] = 0$ and $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{O}$ so $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0$.

55. $$(\mathbf{a} \times \mathbf{b}).\ (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$$

$= (\mathbf{a}\cdot\mathbf{c})\ (\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})\ (\mathbf{b}\cdot\mathbf{c})$

so $k = -1$.

56. Any point on the given line is of the form $B(3, 7 + t, 1 + t)$. The given line is parallel to the vector $\mathbf{j} + \mathbf{k}$. The direction ratio of AB will be $3 - 5$, $6 + t$, $-2 + t$ where A is $(5, 1, 3)$. If AB is perpendicular to the given line then

$-2.\ 0 + (6 + t).\ 1 + (-2 + t)\ 1 = 0$

$\Rightarrow t = -2$

$$|\mathbf{AB}| = \sqrt{(3-5)^2 + (5-1)^2 + (-1-3)^2}$$
$$= \sqrt{4+16+16} = 6.$$

57. The vector equation of two lines are

$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + t\,(-4\,\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and

$\mathbf{r} = \mathbf{i} + 4\mathbf{j} - \mathbf{k} + \mathbf{s}\,(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ so distance between them is (Ch. 18, P 6, formula 13)

$$\frac{\left|\ \begin{vmatrix} 1 & 4 & -1 \\ -4 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 3 \\ -4 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix}\ \right|}{\left|\ \begin{vmatrix} i & j & k \\ -4 & 1 & 2 \\ 1 & -2 & 1 \end{vmatrix} 1\right.}$$

$$= \frac{|22 - 38|}{|5\mathbf{i} + 6\mathbf{j} + 7\mathbf{k}|}$$

$$= \frac{16}{\sqrt{110}}.$$

58. The vector AC is given by

$\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + t\,(-5\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

Any point M on **AC** is of the form

$(1 - 5t, -2 + 3t, 2 - t)$

d.r. of **BM** are $-5t$, $-6 + 3t$, $2 - t$. Since **BM** is an altitude so

$-5(-5t) + 3(-6 + 3t) - 1(2 - t) = 0$

$\Rightarrow t = \dfrac{4}{7}$

The point M is given by $\left(\dfrac{-13}{7}, \dfrac{-2}{7}, \dfrac{10}{7}\right)$

BM is given by $\dfrac{-20}{7}\mathbf{i} \dfrac{-30}{7}\mathbf{j} + \dfrac{10}{7}\mathbf{k}$.

59. $h = 2h + k$, $3 - 4h + l = k$, $1 + h + k = l$

$\Rightarrow k = -h$ so $3 - 3h + l = 0$, $l = 1$

$\Rightarrow h = 4/3$, $k = -4/3$, $l = 1$.

60. For a unit cube d.r. of diagonal are 1, 1, 1 and 1, -1, 1 so the angle between two diagonals will be $\cos^{-1}\dfrac{1-1+1}{\sqrt{3}\sqrt{3}} = \cos^{-1}\dfrac{1}{3}$.

61. Since $(\mathbf{a} + \mathbf{b}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ so the intersection is $\mathbf{a} + \mathbf{b}$.

62. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\ \mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$

$$= \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \mathbf{c} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \mathbf{d}$$

Also

$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}\ \mathbf{c}\ \mathbf{d}]\mathbf{b} - [\mathbf{b}\ \mathbf{c}\ \mathbf{d}]\mathbf{a}$

$$= \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} \mathbf{b} - \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix} \mathbf{a}$$

Adding, we have $k = 2$.

63. $(\mathbf{b} \times \mathbf{c}).(\mathbf{a} \times \mathbf{d}) + (\mathbf{c} \times \mathbf{a}).(\mathbf{b} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{b}).(\mathbf{c} \times \mathbf{d})$.

$$= \begin{vmatrix} \mathbf{b}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{b}\cdot\mathbf{d} & \mathbf{c}\cdot\mathbf{d} \end{vmatrix} + \begin{vmatrix} \mathbf{c}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{c}\cdot\mathbf{d} & \mathbf{a}\cdot\mathbf{d} \end{vmatrix} + \begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$$

$= (\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\cdot\mathbf{a}) + (\mathbf{c}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d}) - (\mathbf{c}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$

$= 0$.

64. Equating the two equations we get $\lambda = 0$ and $\mu = -1$ so the point of intersection is $\mathbf{b} - 2\mathbf{c}$.

65. $\mathbf{a} \times \mathbf{v} = \mathbf{a} \times \mathbf{b} \Rightarrow \mathbf{a} \times (\mathbf{v} - \mathbf{b}) = 0$

$\Rightarrow \mathbf{v} - \mathbf{b} = t\,\mathbf{a} \Rightarrow \mathbf{v} = \mathbf{b} + t\,\mathbf{a}$

$\mathbf{a}\cdot\mathbf{v} = 0 \Rightarrow \mathbf{a}\cdot\mathbf{b} + t\,\mathbf{a}\cdot\mathbf{a} = 0$

$\Rightarrow t = -\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2} = -\dfrac{2+2+3}{\sqrt{1+4+9}\sqrt{4+1+1}}$

$= -\dfrac{7}{\sqrt{14}\sqrt{7}} = -\dfrac{1}{\sqrt{2}}$.

66. For $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$, we have

$\mathbf{a} \times (\mathbf{i} \times \mathbf{j}) = (\mathbf{a}\cdot\mathbf{i})\mathbf{j} - (\mathbf{a}\cdot\mathbf{j})\mathbf{i}$

$|\mathbf{a} \times (\mathbf{i} \times \mathbf{j})|^2 = (\mathbf{a}\cdot\mathbf{i})^2 + (\mathbf{a}\cdot\mathbf{j})^2 = a_1^2 + a_2^2$

$|\mathbf{a} \times (\mathbf{j} \times \mathbf{k})|^2 = a_2^2 + a_3^2$

$|\mathbf{a} \times (\mathbf{k} \times \mathbf{i})|^2 = a_3^2 + a_1^2$

so the required value $= 2(a_1^2 + a_2^2 + a_3^2) = 2|\mathbf{a}|^2$.

67. The mid point is given by $\dfrac{1}{2}(\mathbf{a} + \mathbf{b})$. The required locus passes through $\dfrac{1}{2}(\mathbf{a} + \mathbf{b})$ so required locus is of the form

$r = \dfrac{1}{2}(\mathbf{a} + \mathbf{b}) + t\,\mathbf{c}$ (i)

where $\mathbf{c}$ is perpendicular to $\mathbf{b} - \mathbf{a}$. Thus

$\mathbf{c}\ .\ (\mathbf{b} - \mathbf{a}) = 0$

Taking dot product in (i) by $\mathbf{a} - \mathbf{b}$, we get $(\mathbf{r} - \dfrac{1}{2}(\mathbf{a} + \mathbf{b})).(\mathbf{a} - \mathbf{b}) = 0$.

68. Since $\mathbf{a}$ is perpendicular to $\mathbf{d}$, so

$x - y + 2z = 0$ (i)

Moreover, $|\mathbf{b}| = |\mathbf{c}|$ so $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$ as $\mathbf{a}$ makes equal angles with $\mathbf{b}$ and $\mathbf{c}$.

Thus $xy - 2yz + 3xz = 2xz + 3xy - yz$

$\Rightarrow xz - 2xy - yz = 0$ (ii)

Also $x^2 + y^2 + z^2 = 12$ (iii)

and $y < 0$.

Substituting the value of y from (i) in (ii) we get

$x^2 + 2xz + z^2 = 0$

So $x = -z$ and $y = z$

Again substituting these values in (iii) we get $z^2 = 4$ i.e. $z = \pm 2$ but $y < 0$ and $y = z$

so $z = -2 = y$ and $x = 2$.

69. $\mathbf{a} \times \mathbf{b} = \mathbf{c} \Rightarrow \mathbf{c}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$.
$\mathbf{b} \times \mathbf{c} = \mathbf{a} \Rightarrow \mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{c}$

Thus $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are orthogonal in pairs. Since

$|\mathbf{c}| = |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|$ and similarly $|\mathbf{a}| = |\mathbf{b}|\,|\mathbf{c}|$ so $|\mathbf{c}| = |\mathbf{c}|\,|\mathbf{b}|^2$

Hence $|\mathbf{b}| = 1$ and $|\mathbf{c}| = |\mathbf{a}|$.

70. Given lines has vector form as

$$\mathbf{r} = -\,2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + t\,(2\mathbf{i} - 4\mathbf{i} + \mathbf{k})$$

$$\mathbf{r} = -\,2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} + t\,(-\,\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

$$\cos\theta = \frac{-2-8+3}{\sqrt{4+16+1}\,\sqrt{1+4+9}} = \frac{-7}{\sqrt{21}\,\sqrt{14}} = -\frac{1}{\sqrt{6}}$$

Acute angle $= \pi - \cos^{-1}\left(-\frac{1}{\sqrt{6}}\right) = \cos^{-1}\frac{1}{\sqrt{6}}$.

71. According to the given conditions,

$c_1^2 + c_3^2 + c_3^2 = 1$, $\mathbf{a}\cdot\mathbf{c} = 0$, $\mathbf{b}\cdot\mathbf{c} = 0$ and

$$\frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{\sum a_i^2}\,\sqrt{\sum b_i^2}}$$

Thus $a_1c_1 + a_2c_2 + a_3c_3 = 0 = b_1c_1 + b_2c_2 + b_3c_3$

and $\frac{\sqrt{3}}{2} = \sqrt{\sum a_i^2}\,\sqrt{\sum b_i^2} = \sum a_i\,b_i$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 = \begin{vmatrix} \sum a_i^2 & \sum a_ib_i & \sum a_ic_i \\ \sum a_ib_i & \sum b_i^2 & \sum b_ic_i \\ \sum a_ic_i & \sum b_ic_i & \sum c_i^2 \end{vmatrix}$$

$$= \begin{vmatrix} \sum a_i^2 & \sum a_ib_i & 0 \\ \sum a_ib_i & \sum b_i^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \left(\sum a_i^2\right)\left(\sum b_i^2\right) - \left(\sum a_ib_i\right)^2$$

$$= \left(\sum a_i^2\right)\left(\sum b_i^2\right) - \frac{3}{4}\left(\sum a_i^2\right)\left(\sum a_i^2\right)$$

$$= \frac{1}{4}\left(\sum a_i^2\right)\left(\sum b_i^2\right).$$

72. Since the rotation of axes does not affect the distance between the origin and the point, we have

$$4p^2 + 1 = (p + 1)^2 + 1$$

$\Rightarrow p + 1 = \pm\, 2p \Rightarrow p = 1$ or $-\,1/3$.

73. $\mathbf{a}.\ ((\mathbf{b} \times \mathbf{c}) \times (\mathbf{a} + (\mathbf{b} \times \mathbf{c})) = \mathbf{a}.\ ((\mathbf{b} \times \mathbf{c}) \times \mathbf{a})$

$= \mathbf{a}.\ ((\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b}) = 0.$

74. $\mathbf{X}\cdot\mathbf{A} = 0$, $\mathbf{X}\cdot\mathbf{B} = \mathbf{X}\cdot\mathbf{C} = 0 \Rightarrow \mathbf{A}, \mathbf{B}, \mathbf{C}$ are perpendicular to $\mathbf{X}$. Thus $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are coplanar $\Rightarrow [\mathbf{A}, \mathbf{B}, \mathbf{C}] = 0$.

75. Let $\mathbf{c} = c_1\,\mathbf{i} + c_2\,\mathbf{j} + c_3\,\mathbf{k}$. $\mathbf{c}\cdot\mathbf{k} = 0 \Rightarrow c_3 = 0$

$$\left.\begin{array}{l} \mathbf{c}\cdot\mathbf{a} = 9 \Rightarrow 3c_1 - 3c_2 = 9 \\ \mathbf{c}\cdot\mathbf{b} = -4 \Rightarrow c_1 + 2c_2 = -4 \end{array}\right] \Rightarrow c_1 = 2, c_2 = -3$$

Thus $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$.

76. A unit vector in XZ plane is of the form $\mathbf{c} = \mathbf{c}_1\,\mathbf{i} + c_3\,\mathbf{k}$ where $c_1^2 + c_3^2 = 1$

Also $\frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \frac{2c_1 - c_3}{\sqrt{9}} \Rightarrow \frac{3}{\sqrt{2}} = 2c_1 - c_3$

$$\frac{1}{2} = \cos\frac{\pi}{3} = \frac{-c_3}{\sqrt{2}} \Rightarrow c_3 = -\frac{1}{\sqrt{2}}$$

so $c_1 = \frac{1}{\sqrt{2}}$. Hence $\mathbf{c} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$.

77. According to the given condition

$$\frac{\mathbf{a}.(\mathbf{a}+\mathbf{b})}{|\mathbf{a}|\,|\mathbf{a}+\mathbf{b}|} = \frac{\mathbf{b}.(\mathbf{a}+\mathbf{b})}{|\mathbf{b}|\,|\mathbf{a}+\mathbf{b}|}$$

$$\Rightarrow \frac{|\mathbf{a}|^2 + \mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2}{|\mathbf{b}|}$$

$$\Rightarrow |\mathbf{a}| + \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} + |\mathbf{b}|$$

$$\Rightarrow |\mathbf{a}| - |\mathbf{b}| = \mathbf{a}\cdot\mathbf{b}\left(\frac{|\mathbf{a}| - |\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}\right)$$

If $|\mathbf{a}| \neq |\mathbf{b}|$ then $|\mathbf{a}|\,|\mathbf{b}| = \mathbf{a}\cdot\mathbf{b}$

$\Rightarrow \theta = 0$ not possible

so $|\mathbf{a}| = |\mathbf{b}|$.

78. Treating A as the origin of reference. The position vector M is given by

$$\frac{5-3}{2}\mathbf{i} + \frac{-2+4}{2}\mathbf{j} + \frac{8}{2}\mathbf{k} = \mathbf{i} + \mathbf{j} + 4\mathbf{k}.$$

The $|\mathbf{AM}| = \sqrt{18}$.

79. Area of the triangle $\mathbf{ABC} = \frac{1}{2}\,|\mathbf{b} \times \mathbf{AC}|$

$= \frac{1}{2}\,|\mathbf{b} \times (m\mathbf{b} + p\mathbf{d})|$

$= \frac{1}{2}\,|\mathrm{p}\ (\mathbf{b} \times \mathbf{d})|$

Similarly area of the triangle ACD

$= \frac{1}{2}\,\mathbf{d} \times (m\mathbf{b} + p\mathbf{d}) = \frac{1}{2}\,|m|\,|\mathbf{b} \times \mathbf{d}|$

Area of the quadrilateral $ABCD$

$= \frac{1}{2}\,(|p| + |m|)\ |\mathbf{b} \times \mathbf{d}|.$

80. $|\mathbf{u} \times \mathbf{v}|^2 = |(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - \mathbf{b})|^2 = |\mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{b}|^2$

$= 4|\mathbf{a} \times \mathbf{b}|^2 = 4|\mathbf{a}|^2\,|\mathbf{b}|^2 \sin^2 \theta$

$= 4k^4\,(1 - \cos^2\theta) = 4k^4\left(1 - \dfrac{(\mathbf{ab})^2}{|\mathbf{a}|^2\,|\mathbf{b}|^2}\right)$

$= 4k^4\left(1 - \dfrac{(\mathbf{a}\cdot\mathbf{b})^2}{k^4}\right)$

$|\mathbf{u} \times \mathbf{v}| = 2(k^4 - (\mathbf{a}\cdot\mathbf{b})^2)^{1/2}$.

Previous Years' AIEEE/JEE Main Questions

1. $(\mathbf{a} \times \mathbf{b})^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 \sin^2(\pi/6)$

$= (4^2)\,(2^2)\,(1/2)^2 = 16$

2. $[\mathbf{a} \times \mathbf{b}\ \ \mathbf{b} \times \mathbf{c}\ \ \mathbf{c} \times \mathbf{a}] = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2 = 4^2 = 16$

3. $|\mathbf{b} + \mathbf{c}|^2 = |-\mathbf{a}|^2$

$\Rightarrow |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2\mathbf{b}.\mathbf{c} = |\mathbf{a}|^2$

$\Rightarrow 25 + 9 + 2|\mathbf{b}|\,|\mathbf{c}| \cos\theta = 49$

$\Rightarrow 2(5)\,(3) \cos\theta = 15 \Rightarrow \cos\theta = 1/2$

$\Rightarrow \theta = 60°$

4. $0 = |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2$

$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{c} + \mathbf{c}.\mathbf{a})$

$\Rightarrow \mathbf{a}.\mathbf{b} + \mathbf{b}.\mathbf{c} + \mathbf{c}.\mathbf{a} = -\dfrac{1}{2}\left(5^2 + 4^2 + 3^2\right) = -25$

5. $\mathbf{c} = \mathbf{a} \times \mathbf{b} = 39\boldsymbol{k}$

$|\mathbf{a}| = \sqrt{34}$, $|\mathbf{b}| = \sqrt{45}$, $|\mathbf{c}| = 39$

$\therefore\ |\mathbf{a}| : |\mathbf{b}| : |\mathbf{c}| = \sqrt{34} : \sqrt{45} : 39$

6. $\mathbf{u} \times \mathbf{v} = -2\boldsymbol{k} \Rightarrow \boldsymbol{n} = \boldsymbol{k}$.

Thus, $\mathbf{w}.\ \boldsymbol{n} = 3 \Rightarrow |\mathbf{w}.\ \boldsymbol{n}| = 3$

7. $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = 7\boldsymbol{i} + 2\boldsymbol{j} - 4\boldsymbol{k}$

$\mathbf{d} = 5\boldsymbol{i} + 4\boldsymbol{j} + \boldsymbol{k} - (\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$

$= 4\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$

Thus, work done $= \mathbf{w} = \mathbf{F}.\mathbf{d} = 28 + 4 + 8 = 40$

8. Let M be mid point of BC,

$\mathbf{AM} = \dfrac{1}{2}(\mathbf{AB} + \mathbf{AC})$

$= 4\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$

$\Rightarrow \text{AM} = |\mathbf{AM}| = \sqrt{33}$

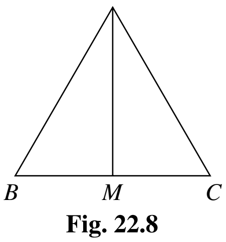


Fig. 22.8

9. As $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are non-coplanar, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$

Now $[\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}\ \ \lambda\mathbf{b} + 4\mathbf{c}\ \ (2\lambda - 1)\mathbf{c}]$

$= (2\lambda - 1)\ [\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}\ \ \lambda\mathbf{b} + 4\mathbf{c}\ \ \mathbf{c}]$

$= (2\lambda - 1)\ [\mathbf{a} + 2\mathbf{b}\ \ \lambda\mathbf{b}\ \ \mathbf{c}]$

$= (2\lambda - 1)\lambda\ [\mathbf{a} + 2\mathbf{b}\ \ \mathbf{b}\ \ \mathbf{c}]$

$= (2\lambda - 1)\lambda\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$

if $\lambda \neq 0,\ 1/2$

10. We are given

$\dfrac{\mathbf{v}.\mathbf{u}}{|\mathbf{u}|} = \dfrac{\mathbf{w}.\mathbf{u}}{|\mathbf{u}|}$

$\Rightarrow \mathbf{v}.\ \mathbf{u} = \mathbf{w}.\ \mathbf{u}$

Also, $\mathbf{v}.\ \mathbf{w} = 0$

Now, $|\mathbf{u} - \mathbf{v} + \mathbf{w}|^2$

$= |\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2\mathbf{u}.\mathbf{v} + 2\mathbf{u}.\mathbf{w} - 2\mathbf{v}.\mathbf{w}$

$= 1 + 4 + 9 = 14$

$\Rightarrow |\mathbf{u} - \mathbf{v} + \mathbf{w}| = \sqrt{14}$

11. $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \dfrac{1}{3}\ |\mathbf{b}|\ |\mathbf{c}|\ \mathbf{a}$

$\Rightarrow (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a} = \dfrac{1}{3}\ |\mathbf{b}|\ |\mathbf{c}|\ \mathbf{a}$

A possible solution is $\mathbf{a}.\mathbf{c} = 0$, $\mathbf{b}.\mathbf{c} = -\dfrac{1}{3}\ |\mathbf{b}|\ |\mathbf{c}|$

$\Rightarrow |\mathbf{b}|\ |\mathbf{c}| \cos\theta = -\dfrac{1}{3}\ |\mathbf{b}|\ |\mathbf{c}| \Rightarrow \cos\theta = -1/3$

$\therefore\ \sin\theta = 2\sqrt{2}/3$

12. Take P as origin

$\therefore\ \mathbf{PC} = \dfrac{1}{2}\ (\mathbf{PA} + \mathbf{PB})$

$\Rightarrow \mathbf{PA} + \mathbf{PB} = 2\mathbf{PC}$

13. Let $\mathbf{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$

$\Rightarrow \mathbf{a} \times \boldsymbol{i} = -a_2\boldsymbol{k} + a_3\boldsymbol{j}$

$\Rightarrow |\mathbf{a} \times \boldsymbol{i}|^2 = a_2^{\,2} + a_3^{\,2}$

Similarly, $|\mathbf{a} \times \mathbf{j}|^2 = a_1^{\,2} + a_3^{\,2}$

and $|\mathbf{a} \times \boldsymbol{k}|^2 = a_1^{\,2} + a_2^{\,2}$

Thus, $|\mathbf{a} \times \boldsymbol{i}|^2 + |\mathbf{a} \times \mathbf{j}|^2 + |\mathbf{a} \times \boldsymbol{k}|^2$

$= 2(a_1^{\,2} + a_2^{\,2} + a_3^{\,2}) = 2|\mathbf{a}|^2$

14. $\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 0 & a & c \\ 1 & 0 & 1 \\ 0 & c & b \end{vmatrix} = 0 \Rightarrow -ab + c^2 = 0$

$\Rightarrow a$, b, c are in G.P.

15. $[\lambda(\mathbf{a}+\mathbf{b})\ \lambda^2\mathbf{b}\ \lambda\mathbf{c}] = [\mathbf{a}\ \mathbf{b}+\mathbf{c}\ \mathbf{b}]$

$\Rightarrow (\lambda)\ (\lambda^2)\ (\lambda)\ [\mathbf{a}+\mathbf{b}\ \mathbf{b}\ \mathbf{c}] = [\mathbf{a}\ \mathbf{c}\ \mathbf{b}]$

$\Rightarrow \lambda^4\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = -\ [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

As $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$, $\lambda^4 = -1$.

Not possible for any real value of λ.

16. $$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix} \quad [C_3 \to C_3 + C_1]$$

$= 1$

$\Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ is independent of x, y.

17. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

$\Rightarrow (\mathbf{a}.\mathbf{c})\ \mathbf{b} - (\mathbf{b}.\mathbf{c})\ \mathbf{a} = (\mathbf{a}.\mathbf{c})\ \mathbf{b} - (\mathbf{a}.\mathbf{b})\ \mathbf{c}$

$\Rightarrow \mathbf{a} = \dfrac{\mathbf{a}.\mathbf{b}}{\mathbf{b}.\mathbf{c}}\mathbf{c}$

$\therefore$ **a** is parallel to **c**

18. $\mathbf{CA} = (a-2)\boldsymbol{i} - 2\boldsymbol{j} + 0\boldsymbol{k}$

$\mathbf{CB} = (a-1)\boldsymbol{i} + 0\boldsymbol{j} + 6\boldsymbol{k}$

As $\angle C = \pi/2$, $\mathbf{CA}.\mathbf{CB} = 0$

$\Rightarrow (a-2)\ (a-1) = 0 \Rightarrow a = 1, 2$

19. As **c** are lies in the plane of **a** and **b**, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x-2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ -1 & -3 & 2 \\ x+1 & x-1 & -1 \end{vmatrix} = 0$$

[using $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$]

$\Rightarrow -(x-1) + 3(x+1) = 0 \Rightarrow x = -2$

20. $|2\mathbf{u}\times 3\mathbf{v}| = 1 \Rightarrow |\mathbf{u}\times\mathbf{v}| = \dfrac{1}{6}$

$\Rightarrow |\mathbf{u}|\ |\mathbf{v}|\sin\theta = \dfrac{1}{6} \Rightarrow \sin\theta = \dfrac{1}{6}$

$\Rightarrow \theta = \sin^{-1}(1/6),\ \pi - \sin^{-1}(1/6)$

21. Unit vector along angle bisector of **b** and **c** is

$$\mathbf{d} = \frac{1}{2}\left(\frac{b}{|b|} + \frac{c}{|c|}\right)$$

$$= \frac{1}{2\sqrt{2}}\ (\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$$

Now, $\mathbf{a} = \lambda\mathbf{d}$

$\Rightarrow \mathrm{a}\boldsymbol{i} + 2\boldsymbol{j} + \beta\boldsymbol{k} = \dfrac{\lambda}{2\sqrt{2}}(\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})$

Equating coefficients of $\boldsymbol{j}$, we get

$\lambda = 2\sqrt{2}$

Thus, $\alpha = 1$, $\beta = 1$

22. $\mathbf{a} = 8\mathbf{b} = -\dfrac{8}{7}(7\mathbf{b}) = -\dfrac{8}{7}\mathbf{c}$

$\Rightarrow$ angle between **a** and **c** is π.

23. $[3\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}]$

$-\ [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}] = 0$

$\Rightarrow 3p^2\ [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] - pq[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] + 2q^2\ [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = 0$

$\Rightarrow (3p^2 - pq + 2q^2)\ [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = 0$

As **u**, **v**, **w** are non-coplanar,

$3p^2 - pq + 2q^2 = 0$

$\Rightarrow p^2 - \dfrac{1}{3}pq + \dfrac{2}{3}q^2 = 0$

$\Rightarrow \left(p - \dfrac{1}{6}q\right)^2 + \dfrac{23}{36}q^2 = 0$

$\Rightarrow p = \dfrac{1}{6}q,\ q = 0.$

$\therefore p = 0,\ q = 0$

That is, there is exactly one value of (p, q).

24. $\mathbf{a}.\mathbf{c} = 0$, $\mathbf{b}.\mathbf{c} = 0$

$\Rightarrow \lambda - 1 + 2\mu = 0,\ 2\lambda + 4 + \mu = 0$

$\Rightarrow \lambda = -3,\ \mu = 2$

25. Let $\mathbf{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 1 & -1 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$= (b_2 + b_3)\boldsymbol{i} - b_1\boldsymbol{j} - b_1\boldsymbol{k}$

Now, $\mathbf{a}\times\mathbf{b} + \mathbf{c} = \mathbf{0}$

$\Rightarrow (b_2 + b_3 + 1)\boldsymbol{i} + (-b_1 - 1)\boldsymbol{j} + (-b_1 - 1)\boldsymbol{k} = \mathbf{0}$

$\Rightarrow b_2 + b_3 + 1 = 0,\ b_1 = -1$

Also, $\mathbf{a}.\mathbf{b} = 3$

$\Rightarrow b_2 - b_3 = 3$

Thus, $\mathbf{b}_2 = 1$, $\mathbf{b}_3 = -2$

Hence, $\mathbf{b} = -\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$

26. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})$

$= (\mathbf{a}.(\mathbf{a} + 2\mathbf{b}))\ \mathbf{b} - (\mathbf{b}.(\mathbf{a} + 2\mathbf{b}))\ \mathbf{a}$

$= (|\mathbf{a}|^2 + 2\mathbf{a}.\mathbf{b})\ \mathbf{b} - (\mathbf{b}.\mathbf{a} + 2|\mathbf{b}|^2)\ \mathbf{a}$

But $\mathbf{a}.\mathbf{b} = 0$, $|\mathbf{a}| = 1$, $|\mathbf{b}| = 1$

Thus, $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b}) = \mathbf{b} - 2\mathbf{a}$

$\Rightarrow (2\mathbf{a} - \mathbf{b}).\ [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})]$

$= -\ |2\mathbf{a} - \mathbf{b}|^2 = -(4|\mathbf{a}|^2 + |\mathbf{b}|^2) = -\ 5.$

27. $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$

$\Rightarrow \mathbf{b} \times (\mathbf{c} - \mathbf{d}) = \mathbf{0}$

$\Rightarrow \mathbf{c} - \mathbf{d} = \alpha\mathbf{b}$ for same $\alpha \in \mathbf{R}$.

$\Rightarrow \mathbf{d} = \mathbf{c} - \alpha\mathbf{b}$

As $\mathbf{a}.\mathbf{d} = 0$, we get $\mathbf{a}.\mathbf{c} - \alpha\ \mathbf{a}.\mathbf{b} = 0$

$\Rightarrow \alpha = \dfrac{\mathbf{a.c}}{\mathbf{a.b}}$

Thus, $\mathbf{d} = \mathbf{c} - \left(\dfrac{\mathbf{a.c}}{\mathbf{a.b}}\right)\mathbf{b}$

28. As $p\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{i} + q\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{i} + \boldsymbol{j} + r\boldsymbol{k}$ are coplanar, so

$$\begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$$

$\Rightarrow pqr + 2 - p - q - r = 0$

$\Rightarrow pqr - p - q - r = -2$

29. $\mathbf{a} + 3\mathbf{b} = \alpha\mathbf{c}$ and $\mathbf{b} + 2\mathbf{c} = \beta\mathbf{a}$

Now, $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (\alpha + 6)\mathbf{c}$

and $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = (1 + 3\beta)\mathbf{a}$

$\therefore\ (\alpha + 6)\ \mathbf{c} = (1 + 3\beta)\ \mathbf{a}$

As $\mathbf{a}$ and $\mathbf{c}$ are non-collinear,

We get $\alpha + 6 = 0$, $1 + 3\beta = 0$.

$\therefore\ \mathbf{a} + 3\mathbf{b} + 6\mathbf{c} = \mathbf{0}$

30. As $\mathbf{c}.\mathbf{d} = 0$

$\Rightarrow (\mathbf{a} + 2\mathbf{b}).(5\mathbf{a} - 4\mathbf{b}) = 0$

$\Rightarrow 5|\mathbf{a}|^2 + 10\mathbf{b}.\mathbf{a} - 4\mathbf{a}.\mathbf{b} - 8|\mathbf{b}|^2 = 0$

$\Rightarrow 5 + 6|\mathbf{a}|\ |\mathbf{b}| \cos\theta - 8 = 0$

$\Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \pi/3$

31. Let $\mathbf{AE} = \alpha\ \mathbf{p}$,

for same $\alpha \in \mathbf{R}$.

$\mathbf{BE} = \mathbf{AE} - \mathbf{AB}$

$\Rightarrow \mathbf{r} = \alpha\ \mathbf{p} - \mathbf{q}$

As, $\mathbf{BE}.\mathbf{AD} = 0$

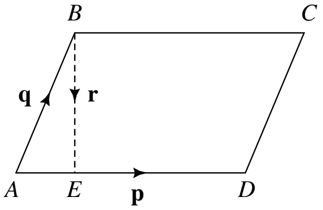


Fig. 22.9

$\Rightarrow 0 = \mathbf{r}.\mathbf{p} = \alpha\mathbf{p}.\mathbf{q} - \mathbf{p}.\mathbf{q}$

$\Rightarrow \alpha = \dfrac{\mathbf{p.q}}{\mathbf{p.p}}$

Thus, $\mathbf{r} = -\mathbf{q} + \dfrac{\mathbf{p.q}}{\mathbf{p.p}}\mathbf{p}$

32. $\mathbf{AD} = \dfrac{1}{2}\ (\mathbf{AB} + \mathbf{AC})$

$= 4\boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k}$

$\Rightarrow |\mathbf{AD}| = \sqrt{33}$

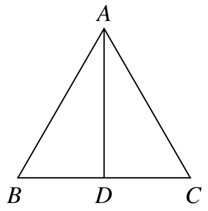


Fig. 22.10

33. As $\mathbf{u}$ and $\mathbf{v}$ are collinear,

$\mathbf{u} = k\mathbf{v}$ for some $k \in \mathbf{R}$.

$\Rightarrow \alpha - 2 = k(2 + 3\alpha)$, $1 = -\ 3k$

$\Rightarrow \alpha - 2 = -\dfrac{1}{3}(2 + 3\alpha) \Rightarrow \alpha = 2/3$

34. Projection of $\mathbf{b} + \lambda\mathbf{c}$ on $\mathbf{a}$

$= \dfrac{(\mathbf{b} + \lambda\mathbf{c}).\mathbf{a}}{|\mathbf{a}|}$

$\Rightarrow \pm\sqrt{\dfrac{2}{3}} = \dfrac{\mathbf{b}.\mathbf{a} + \lambda\mathbf{c}.\mathbf{a}}{\sqrt{6}}$

$\Rightarrow \pm 2 = -\ 1 + \lambda(-1) \Rightarrow \lambda = -3,\ 1$

For $\lambda = 1$, $-\ \mathbf{b} + \lambda\mathbf{c} = 2\boldsymbol{i} + 3\boldsymbol{j} - 3\boldsymbol{k}$

35. $\lambda = 1$ [see Theory]

36. $|2\mathbf{a} - \mathbf{b}|^2 + |2\mathbf{a} + \mathbf{b}|^2$

$= 2(4|\mathbf{a}|^2 + |\mathbf{b}|^2)$

$\Rightarrow 25 + |2\mathbf{a} + \mathbf{b}|^2 = 2(4(2^2) + 3^2)$

$\Rightarrow |2\mathbf{a} + \mathbf{b}| = 5$

37. As $\mathbf{c} \times (\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}) = \mathbf{0}$, is parallel to $\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}$. Let

$\mathbf{c} = \alpha(\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k})$ for same $\alpha \in \mathbf{R}$.

$\Rightarrow 60 = |\mathbf{c}|^2 = \alpha^2(1 + 4 + 25) \Rightarrow \alpha^2 = 2$

Now, $\mathbf{c}.\ (-7\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$

$= \alpha(-7 + 4 + 15)$

$= 12\alpha = 12\sqrt{2}$

38. $|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{y}+\mathbf{z}|^2 + |\mathbf{z}+\mathbf{x}|^2 - |\mathbf{x}+\mathbf{y}+\mathbf{z}|^2$

$= |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}.\ \mathbf{y} + |\mathbf{y}|^2 + |\mathbf{z}|^2 + 2\mathbf{y}.\mathbf{z} + |\mathbf{z}|^2 + |\mathbf{x}|^2 + 2\mathbf{z}.\mathbf{x} - (|\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2 + 2\mathbf{x}.\mathbf{y} + 2\mathbf{x}.\mathbf{z} + 2\mathbf{y}.\mathbf{z})$

$= |\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2 = 3$

Thus, $u = |\mathbf{x}+\mathbf{y}|^2 + |\mathbf{y}+\mathbf{z}|^2 + |\mathbf{z}+\mathbf{x}|^2$

$= 3 + |\mathbf{x}+\mathbf{y}+\mathbf{z}|^2 \ge 3.$

$\therefore$ minimum value u is 3 and its attained when $\mathbf{x}+\mathbf{y}+\mathbf{z} = \mathbf{0}$.

39. $\mathbf{x}\times\mathbf{y} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -6 & -1 \\ 1 & 4 & -3 \end{vmatrix}$

$= 22\boldsymbol{i} + 8\boldsymbol{j} + 18\boldsymbol{k}$

Projection of $\mathbf{x}\times\mathbf{y}$ on $\mathbf{z}$

$$= \frac{|(\mathbf{x}\times\mathbf{y}).\mathbf{z}|}{|\mathbf{z}|} = \frac{|66-32-216|}{\sqrt{9+16+144}} = \frac{182}{13} = 14$$

40. $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \frac{1}{3}|\mathbf{b}|\ |\mathbf{c}|\ \mathbf{a}$

$\Rightarrow (\mathbf{a}.\mathbf{c})\ \mathbf{b} - (\mathbf{b}.\mathbf{c})\ \mathbf{a} = \frac{1}{3}\ |\mathbf{b}|\ |\mathbf{c}|\ \mathbf{a}$

$\Rightarrow |\mathbf{a}|\ |\mathbf{c}| \cos\varphi\mathbf{b} = |\mathbf{b}|\ |\mathbf{c}|\ \left(\cos\theta + \frac{1}{3}\right)\mathbf{a}$

As $\mathbf{a}$ and $\mathbf{b}$ are non-collinear,

$\cos\theta + \frac{1}{3} = 0$ and $\cos\varphi = 0$

$\Rightarrow \sin^2\theta = 1 - 1/9 = 8/9$

$\Rightarrow \sin\theta = \pm 2\sqrt{2}/3.$

Thus, a value of $\sin\theta$ is $2\sqrt{2}/3$

41. $\mathbf{AC} = \mathbf{AB} + \mathbf{BC}$

$\Rightarrow \mathbf{c} = \mathbf{a} + \mathbf{b}$

and $\mathbf{DB} = \mathbf{AB} - \mathbf{AD} = \mathbf{a} - \mathbf{b}$.

Now, $\mathbf{DB}.\mathbf{AB} = (\mathbf{a}-\mathbf{b}).\mathbf{a} = \mathbf{a}^2 - \mathbf{b}.\mathbf{a}$

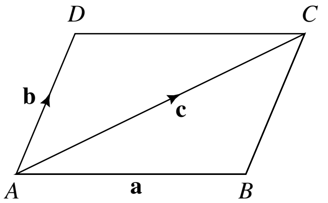


Fig. 22.11

Also

$\mathbf{c}^2 = |\mathbf{c}|^2 = |\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}.\mathbf{b}$

$\Rightarrow \mathbf{a}.\mathbf{b} = \frac{1}{2}\left(c^2 - a^2 - b^2\right)$

Thus,

$\mathbf{DB}.\mathbf{AB} = a^2 - \frac{1}{2}\left(c^2 - a^2 - b^2\right)$

$= \frac{1}{2}\left(3a^2 - b^2 - c^2\right)$

42. $\sqrt{3} = |\mathbf{a}+\mathbf{b}|$

$\Rightarrow 3 = |\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}.\mathbf{b}$

$\Rightarrow \mathbf{a}.\mathbf{b} = 1/2$

Also, $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2\ |\mathbf{b}|^2 - (\mathbf{a}.\mathbf{b})^2$

$= 1 - 1/4 = 3/4$

Now,

$|\mathbf{c}|^2 = |\mathbf{a}|^2 + 9|\mathbf{b}|^2 + 9|\mathbf{a}\times\mathbf{b}|^2 + 4\mathbf{a}.\mathbf{b} + 6\mathbf{a}.(\mathbf{a}\times\mathbf{b}) + 12\ \mathbf{b}.(\mathbf{a}\times\mathbf{b})$

$= 1 + 4 + 9(3/4) + 4(1/2) + 0 + 0$

$\Rightarrow 4|\mathbf{c}|^2 = 55$

$\Rightarrow 2|\mathbf{c}| = \sqrt{55}.$

43. $(\mathbf{a}.\mathbf{c})\ \mathbf{b} - (\mathbf{a}.\mathbf{b})\ \mathbf{c} = \frac{\sqrt{3}}{2}\ (\mathbf{b}+\mathbf{c})$

As $\mathbf{b}$ and $\mathbf{c}$ are not parallel

$\mathbf{a}.\mathbf{c} = \frac{\sqrt{3}}{2}$, $\mathbf{a}.\mathbf{b} = -\frac{\sqrt{3}}{2}$

Now, $\mathbf{a}.\mathbf{b} = -\sqrt{3}/2 \Rightarrow |\mathbf{a}|\ |\mathbf{b}| \cos\theta = -\sqrt{3}/2$

$\Rightarrow \cos\theta = -\sqrt{3}/2 \Rightarrow \theta = 5\pi/6$

44. $\mathbf{AB} = -4\boldsymbol{i} + 2\boldsymbol{j} + (p+1)\boldsymbol{k}$

and $\mathbf{AC} = 2\boldsymbol{i} + (q-1)\boldsymbol{j} - 3\boldsymbol{k}$

As $\angle CAB = \pi/2$, we get

$(-4)\ (2) + 2(q-1) + (p+1)\ (-3) = 0$

$\Rightarrow -3p + 2q - 13 = 0$

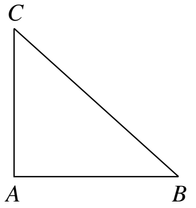


Fig. 22.12

Thus, (p, q) lies on $3x - 2y + 13 = 0$ which makes an acute angle with the x-axis.

45. Let $\mathbf{p}$ be the position vectors of the circumcentre and orthocentre be $\mathbf{p}$ and $\mathbf{h}$.

As G divides HP in the ratio 2:1,

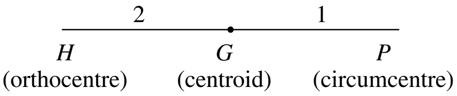


Fig. 22.13

$\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}) = \frac{1}{3}(2\mathbf{p}+\mathbf{h})$

$\Rightarrow \mathbf{h} = \frac{1}{2}\ (\mathbf{a}+\mathbf{b}+\mathbf{c})$

Previous Years' B-Architecture Entrance Examination Questions

1. $0 = |\mathbf{u} + \mathbf{v} + \mathbf{w}|^2$

   $= |\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2 + 2\mathbf{u}.\mathbf{v} + 2\mathbf{v}.\mathbf{w} + 2\mathbf{w}.\mathbf{u}$

   $= 9 + 16 + 25 + 2(\mathbf{u}.\mathbf{v} + \mathbf{v}.\mathbf{w} + \mathbf{w}.\mathbf{u})$

   $\Rightarrow \mathbf{u}.\mathbf{v} + \mathbf{v}.\mathbf{w} + \mathbf{w}.\mathbf{u} = -25$

2. Let $\mathbf{c} = (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} \times \mathbf{b})$

   $= ((\mathbf{a} - \mathbf{b}).\mathbf{b})\mathbf{a} - ((\mathbf{a} - \mathbf{b}).\mathbf{a})\ \mathbf{b}$

   $\Rightarrow (\mathbf{a}.\mathbf{b} - |\mathbf{b}|^2)\mathbf{a} - (|\mathbf{a}|^2 - \mathbf{b}.\mathbf{a})\mathbf{b}$

   $= (\mathbf{a}.\mathbf{b} - |\mathbf{a}|^2)\ (\mathbf{a} - \mathbf{b})\ [\therefore\ |\mathbf{a}| = |\mathbf{b}|]$

   $\Rightarrow$ **c** is parallel to $\mathbf{a} - \mathbf{b}$

3. See solution to Question 14 in Previous Years' AIEEE/JEE Main Questions.

4. $|\mathbf{x} - \mathbf{y}|^2 + |\mathbf{y} - \mathbf{z}|^2 + |\mathbf{z} - \mathbf{x}|^2 = 9$

   $\Rightarrow |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\mathbf{x}.\mathbf{y} + |\mathbf{y}|^2 + |\mathbf{z}|^2 - 2\mathbf{y}.\mathbf{z} + |\mathbf{z}|^2 + |\mathbf{x}|^2 - 2\mathbf{z}.\mathbf{x} = 9$

   $\Rightarrow 2\mathbf{x}.\mathbf{y} + 2\mathbf{y}.\mathbf{z} + 2\mathbf{z}.\mathbf{x} = -3\ [|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = 1]$

   Now, $|\mathbf{x} + \mathbf{y} - \mathbf{z}|^2 - 4\mathbf{x}.\mathbf{y}$

   $= |\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2 + 2\mathbf{x}.\mathbf{y} - 2\mathbf{x}.\mathbf{z} - 2\mathbf{y}.\mathbf{z} - 4\mathbf{x}.\mathbf{y}$

   $= 3 - (-3) = 6$

5. $2\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{c} = \mathbf{0}$

   $\Rightarrow 2(\mathbf{a}.\mathbf{b})\mathbf{a} - 2(\mathbf{a}.\mathbf{a})\mathbf{b} + \mathbf{c} = 0$

   $\Rightarrow \mathbf{c} = 2\mathbf{b} - 2(\mathbf{a}.\mathbf{b})\mathbf{a}$

   $[\because\ |\mathbf{a}| = 1]$

   $\Rightarrow 1 = |\mathbf{c}|^2 = 4|\mathbf{b}|^2 + 4(\mathbf{a}.\mathbf{b})^2\ |\mathbf{a}|^2 - 8(\mathbf{a}.\mathbf{b})\ (\mathbf{b}.\mathbf{a})$

   $\Rightarrow 1 = 4 - 4(\mathbf{a}.\mathbf{b})^2 \Rightarrow (\mathbf{a}.\mathbf{b})^2 = 3/4$

   $\Rightarrow |\mathbf{a}|^2\ |\mathbf{b}|\ \cos^2\theta = 3/4 \Rightarrow \cos\ \theta = \pm\sqrt{3}/2$

   $\therefore$ *a* value of θ is $\pi/6$

6. $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & -1 & 3 \\ 0 & 1 & 2 \end{vmatrix}$

   $= -\ 5\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

   Now, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = \mathbf{a}.(\mathbf{b} \times \mathbf{c})$

   $= |\mathbf{a}|\ |\mathbf{b} \times \mathbf{c}|\ \cos\ \theta$

   where θ = angle between **a** and $\mathbf{b} \times \mathbf{c}$

   $\Rightarrow [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \le (1)\ (29)$

   Max $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 29$ when $\cos\ \theta = 1$.

7. Statement-2 is false as

   $|\boldsymbol{i} - \boldsymbol{j}| = \sqrt{2} = |\boldsymbol{i} + \boldsymbol{j}|$

   Also, $|2\mathbf{a} - \mathbf{b}|^2 + |2\mathbf{a} + \mathbf{b}|^2$

   $= 2[|2\mathbf{a}|^2 + |\mathbf{b}|^2] = 2[4(4) + 9] = 50$

   $\Rightarrow |2\mathbf{a} + \mathbf{b}|^2 = 50 - 5^2 = 25$

   $\Rightarrow |2\mathbf{a} + \mathbf{b}| = 5$

   $\therefore$ Statement-1 is true.

8. $\mathbf{c} = \mathbf{a} \times \mathbf{b}$

   $\Rightarrow |\mathbf{c}|^2 = \mathbf{c}.\mathbf{c} = (\mathbf{a} \times \mathbf{b}).\mathbf{c} = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

   Similarly, $|\mathbf{a}|^2 = |\mathbf{b}|^2 = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

   Thus, $|\mathbf{a}|^2 = |\mathbf{b}|^2 = |\mathbf{c}|^2$

   $\Rightarrow |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$

9. p = Area of quadrilateral

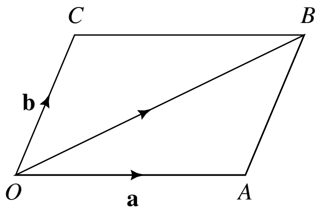
   

   Fig. 22.14

   = area ($\Delta\ OAB$) + area ($\Delta\ OBC$)

   $= \frac{1}{2}|\mathbf{a} \times (2\mathbf{b} + 10\mathbf{a})| + \frac{1}{2}|(2\mathbf{b} + 10\mathbf{a}) \times \mathbf{b}|$

   $= |\mathbf{a} \times \mathbf{b}| + 5|\mathbf{a} \times \mathbf{b}| = 6|\mathbf{a} \times \mathbf{b}| = 6q$

10. Solving two equations, we get

    $\mathbf{a} = \frac{2}{3}\mathbf{e}_1 - \frac{1}{3}\mathbf{e}_2,\ \mathbf{b} = -\frac{1}{3}\mathbf{e}_1 + \frac{2}{3}\mathbf{e}_2$

    $\Rightarrow \mathbf{a} = \frac{1}{3}(1,\ 1,\ 3)$

    $\mathbf{b} = \frac{1}{3}(1,\ 1,\ -3)$

    $\therefore\ \mathbf{a}.\mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|\ \cos\ \theta$

    $\Rightarrow \frac{1}{9}\ (1 + 1 - 9) = \frac{1}{9}\ (11)\ \cos\ \theta$

    $\Rightarrow \theta = \cos^{-1}\ (-7/11)$

11. $2\mathbf{u} + 2\mathbf{v} = -\ 3\mathbf{w}$

    $\Rightarrow 9 = 9|\mathbf{w}|^2 = 4|\mathbf{u}|^2 + 4|\mathbf{v}|^2 + 8\mathbf{u}.\mathbf{v}$

    $\Rightarrow \mathbf{u}.\mathbf{v} = 1/8$

    Now, $|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u}.\mathbf{v}$

    $= 1 + 1 - 1/4 = 7/4$

    $\Rightarrow |\mathbf{u} - \mathbf{v}| = \sqrt{7}/2$

12. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -1 \\ 1 & 0 & 3 \end{vmatrix}$

$= 3\boldsymbol{i} - 7\boldsymbol{j} - \boldsymbol{k}$

$[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u}.(\mathbf{v} \times \mathbf{w})$

$= |\mathbf{u}|\ |\mathbf{v} \times \mathbf{w}| \cos\theta$

where θ = angle between $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$

$\Rightarrow [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = (1)\ \sqrt{59}\cos\theta \le \sqrt{59}$

$\therefore$ Max $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \sqrt{59}$

for $\cos\theta = 1$

13. $\frac{1}{6}\ (\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}) = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$

$= \{(\mathbf{a} \times \mathbf{b}).\mathbf{d}\}\mathbf{c} - \{(\mathbf{a} \times \mathbf{b}).\mathbf{c}\}\mathbf{d}$

$= [\mathbf{a}\ \mathbf{b}\ \mathbf{d}]\mathbf{c} - [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]\mathbf{d}$

As $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = 0$

We have $[\mathbf{a}\ \mathbf{b}\ \mathbf{d}] = (\mathbf{a} \times \mathbf{b}).\mathbf{d}$

$= (\pm\ |\mathbf{a}|\ |\mathbf{b}| \sin\theta\ \mathbf{d}).\mathbf{d}$

$= \pm\frac{1}{2}$

Thus, $\pm\frac{1}{2}\mathbf{d} = \frac{1}{6}\ (\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

$\Rightarrow \mathbf{d} = \pm\frac{1}{3}(\boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k})$

$= \pm\frac{1}{3}\ (-\boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k})$

14. $|\mathbf{z} - \mathbf{x}|^2 = 8 \Rightarrow |\mathbf{z}|^2 - 2\mathbf{z}.\mathbf{x} + |\mathbf{x}|^2 = 8$

$\Rightarrow |\mathbf{z}|^2 - 2|\mathbf{z}| + 9 - 8 = 0 \Rightarrow |\mathbf{z}| = 1.$

Also,

$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix}$

$= 2\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$

$\Rightarrow |\mathbf{x} \times \mathbf{y}| = 3$

Now,

$|(\mathbf{x} \times \mathbf{y}) \times \mathbf{z}|^2 = |\mathbf{x} \times \mathbf{y}|^2\ |\mathbf{z}|^2 \sin^2(30°)$

$= 9(1)\left(\frac{1}{4}\right)$

$\Rightarrow |(\mathbf{x} \times \mathbf{y}) \times \mathbf{z}| = \frac{3}{2}$

15. $\mathbf{AB} = \boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j}) - p(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$and $\mathbf{AC} = -\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j}) - p(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$

As AB is perpendicular to $\mathbf{r} = \boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j})$

$\boldsymbol{k} \cdot (\boldsymbol{i} + \boldsymbol{j}) + \lambda(\boldsymbol{i} + \boldsymbol{j}).(\boldsymbol{i} + \boldsymbol{j}) - p(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}).(\boldsymbol{i} + \boldsymbol{k}) = 0$

$\Rightarrow 0 + 2\lambda - p(2) \Rightarrow \lambda = p$

Similarly $\mu(1 + 1) - p(1 - 1) \Rightarrow \mu = 0$

Thus for any given value of p, $B(p, p, 1)$ lying on the line

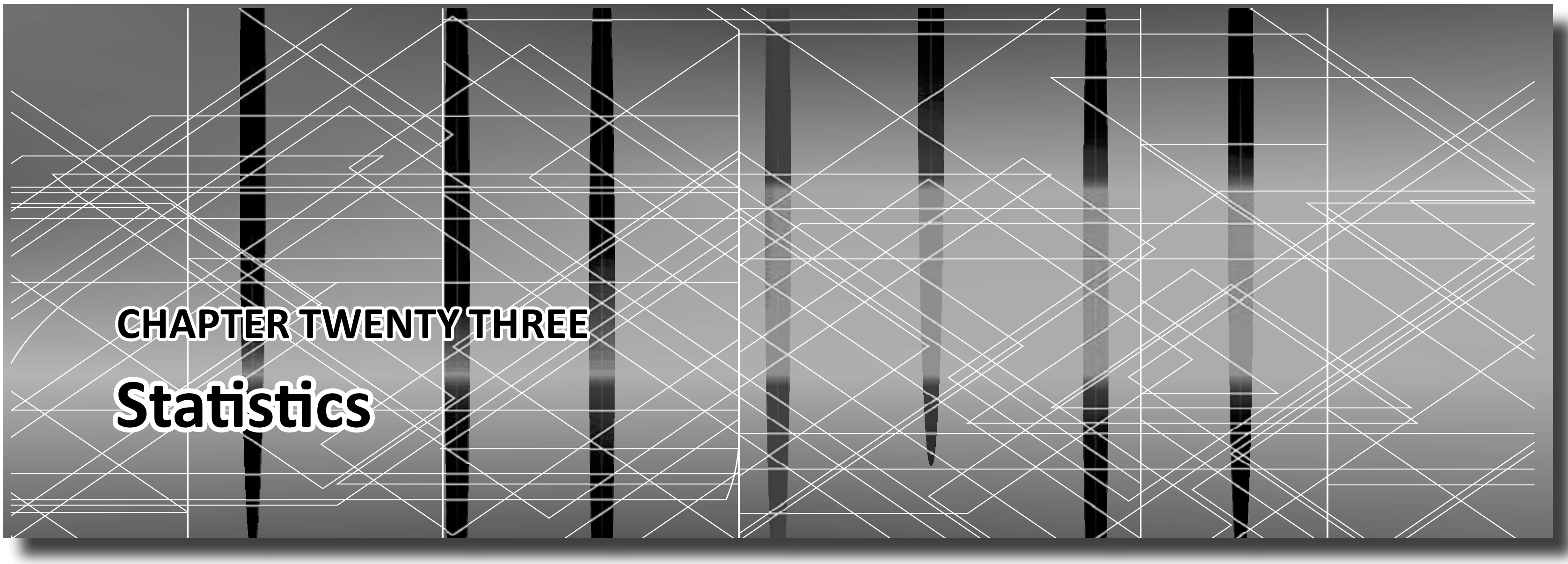
$\mathbf{r} = \boldsymbol{k} + \lambda(\boldsymbol{i} + \boldsymbol{j})$ (1)

is such that AB is perpendicular to (1), and $C(0, 0, -1)$ lying on the line

$\mathbf{r} = -\boldsymbol{k} + \mu(\boldsymbol{i} - \boldsymbol{j})$ (2)

is such that AC is perpendicular to (2).

$\therefore$ all the four answers are correct.



CHAPTER TWENTY THREE

Statistics

MEASURES OF CENTRAL TENDENCY

One of the most important objectives of statistical analysis is to get one single value that describes the characteristic of the entire mass of unwiedely data. Such a value is called the central value or an average.

The following are the important types of averages:

1. **Arithmetic Mean**
2. **Geometric Mean**
3. **Harmonic Mean**
4. **Median**
5. **Mode**

We consider these measures in three cases (i) individual series (i.e. each individual observation is given) (ii) discrete series (i.e. the observations along with number of times a particular observation called the frequency is given (iii) continuous series (i.e. the class intervals along with their frequencies are given)

ARITHMETIC MEAN

1. *Individual Series* If x_1, x_2, …x_n are the values of the variable x, then the arithmetic mean usually denoted by $\bar{x}$ is given by

$$\bar{x} = \frac{x_1 + x_2 \ldots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

2. *Discrete Series* If a variable takes values x_1, x_2 … x_n with corresponding frequencies $f_1, f_2 \ldots f_n$ then the arithmetic mean $\bar{x}$ is given by

$$\bar{x} = \frac{f_1x_1 + f_2x_2 \ldots + f_nx_n}{f_1 + f_2 \ldots + f_n} = \frac{1}{N}\sum_{i=1}^{n} f_ix_i$$

where $N = \sum_{i=1}^{n} f_i$

3. *Continuous Series* In case of a set of data with class intervals, we cannot find the exact value of the mean because we do not know the exact values of the variables. We, therefore, try to obtain an approximate value of the mean. The method of approximation is to replace all the observed values belonging to a class by *mid-value* of that class. If x_1, x_2 … x_n are the mid values of the class intervals having corresponding frequencies f_1, f_2 … f_n then we apply the same formula as in discrete series

i.e. $$\bar{x} = \frac{1}{N}\sum_{i=1}^{n} f_ix_i,\; N = \sum_{i=1}^{n} f_i$$

PROPERTIES OF ARITHMETIC MEAN

1. Sum of all the deviations from arithmetic mean is zero i.e.

   $\sum_{i=1}^{n}(x_i - \bar{x}) = 0$ (in case of individual series)

   $\sum_{i=1}^{n} f_i(x_i - \bar{x}) = 0$ (in case of discrete or continuous series)

2. If each observation is increased or decreased by a given constant, the mean is also increased or decreased. That is, if x_1, x_2 … x_n are given variables and if we take $d_i = x_i - A$ then the mean $\bar{d}$ of the new variables d_i is given by

   $$\bar{d} = \bar{x} - A$$

   or $$\bar{x} = A + \frac{1}{n}\sum d_i$$

   If we have discrete/continuous series then

   $$\bar{x} = A + \frac{1}{N}\sum f_id_i, \quad N = \sum_{i=1}^{n} f_i$$

   This property is also known as effect of *change of origin*. A can be taken to be any number. However, to simplify the calculations, A should be taken as a value which is in the middle of the table.

3. *Step Deviation Method* or change of scale
   If $x_1, x_2 \ldots, x_n$ are mid values of class intervals with corresponding frequencies $f_1, f_{2,} \ldots f_n$, then we may change the scale by taking $d_i = \frac{x_i - A}{h}$, in this case
   $$\bar{x} = A + h \times \left(\frac{1}{N}\sum f_i d_i\right)$$
   A and h can be any numbers but if the lengths of class intervals are equal then h may be taken as width of the class interval.
   In particular if each observation is multiplied or divided by a constant, the mean is also multiplied or divided by the same constant.
4. The sum of the squared deviation of the variate from their mean is minimum i.e. the quantity $\Sigma\,(x_i - A)^2$ or $\Sigma\, f_i\, (x_i – A)^2$ is minimum when $A = \bar{x}$
5. If $\bar{x}_1$ and $\bar{x}_2$ be the means of two related groups having n_1 and n_2 items respectively then the **combined mean** $\bar{x}$ of both the group is given by
   $$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$
   If there are more than two groups then
   $$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_2\bar{x}_3 + \ldots}{n_1 + n_2 + n_3 + \ldots}$$

GEOMETRIC MEAN

In case of individual series $x_1, x_2, \ldots, x_n$

$$\text{G.M} = (x_1 . x_2 \ldots x_n)^{1/n}$$

or $$\text{G} = \text{Antilog}\left(\frac{1}{n}\sum_{i=1}^{n}\log x_i\right)$$

In case of discrete or continuous series

$$\text{G.M.} = \left(x_1^{f_1} x_2^{f_2} \ldots x_n^{f_n}\right)^{1/N}, \text{ where } N = \sum_{i=1}^{n} f_i$$

or $$\text{G} = \text{Antilog}\left(\frac{1}{N}\sum f_i \log x_i\right)$$

HARMONIC MEAN

The harmonic mean is based on the reciprocals of the value of the variable

$$\text{H.M.} = \frac{1}{\frac{1}{n}\left(\frac{1}{x_i} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right)}$$

or $$\frac{1}{H} = \frac{1}{\frac{1}{N}\sum_{i=1}^{n} f_i \frac{1}{x_i}}$$ (in case of individual series)

and $$\frac{1}{H} = \frac{1}{\frac{1}{N}\sum_{i=1}^{n} f_i \frac{1}{x_i}}$$ (in case of discrete series or continuous series)

If $x_1, x_2, \ldots x_n > 0$ then it is known that A.M. ≥ G.M. ≥ H.M.

MEDIAN

It refers to the middle value in a distribution.

In case of *individual series*, in order to find median, arrange the data in ascending or descending order of magnitude. In case of odd number of values Med = size of $\frac{n+1}{2}$th item. In case of even number of values Med = average of $\frac{n}{2}$th and $\frac{n+2}{2}$th observation.

In case of *discrete frequency* distribution the median is obtained by considering the commulative frequency. Find $\frac{N}{2}$, where $N = \sum_{i=1}^{n} f_i$. Find the commulative frequency (c.f.) just more than $N/2$. The corresponding value of x is median.

In case of *continuous distribution*, the class corrsponding to c.f. just more than $N/2$ is called the median class and the median is obtained by

$$\text{Median} = 1 + \frac{h}{f}\left(\frac{N}{2} - C\right)$$

where l = the lower limit of the median class; f = the frequency of the median class; h = the width of the median class; C = the c.f. of the class preceding to the median class and $N = \sum_{i=1}^{n} f_i$.

MODE

The mode is that value in a series of observations which occurs with greatest frequency. In case of *individual series*, for determining the mode, count the number of times the various values repeat themselves and the value occurring the maximum number of times in the model value.

In case of *discrete series*, quite often mode can be determined just by inspection i.e. by looking to that value of the variable around which the items are most heavily concentrated.

In case of *continuous series*,

$$\text{Mode} = l + \frac{f_1 - f_0}{2f_1 - f_2 - f_0} \times h$$

where l = the lower limit of the modal class i.e. the class having maximum frequency; f_1 = frequency of the modal class; f_0 = frequency of the class preceding the modal class; f_2 = frequency of the class succeeding the modal class and h = width of the modal class.

When mode is ill-defined i.e. there are two or more values which occur with equal maximum frequency, the mode can computed by

mode = 3 median – 2 mean.

DISPERSION

Literally, dispersion means 'scatteredness'. Dispersion measures the degree of scatteredness of the variable about a central value. Different measures of dispersion are

1. Range
2. Mean-deviation
3. Quartile deviation
4. Standard deviation

1. Range

The range is the difference between the largest and smallest observations.

2. Mean-deviation

If $x_1, x_2, \dots x_n$ are n observations then mean deviation about a point A is given by

$$\text{M.D.} = \frac{1}{n}\sum |x_i - A|$$

If $A = \bar{x}$, then it is called M.D. from mean.
In case of discrete or continuous series

$$\text{M.D.} = \frac{1}{N}\sum_{i=1}^{n} f_i\,|x_i - A|, \quad N = \sum_{i=1}^{n} f_i$$

M.D. is least when taken from the median i.e. When A is median. M.D. from mean has no effect of change of orgin.

3. Quartile Deviation

Q.D. = $\frac{Q_3 - Q_1}{2}$, where Q_3 and Q_1, are the third quartile and the first quartile. Q_1 and Q_3 can be calculated in a similar manner as median. In fact, quartiles divides the data into four parts. So in case of individual series, arrange the data in ascending or descending order. Q_1 = size of $\frac{n+1}{4}$th and Q_3 = size of $\frac{3\,(n+1)}{4}$th item

In case of discrete frequency distribution, Q_1 is obtained by considering commulative frequency. Find $N/4$, where $N = \Sigma f_i$. Find the commulative frequency (c.f.) just more than $N/4$. The corresponding value of x is Q_1. Similarly for obtaining Q_3, find $3N/4$ and the c.f. just more than $3N/4$. The corresponding value of x is Q_3. In case of continuous distribution

$$Q_i = l + \frac{h}{f}\left(i\frac{N}{4} - C\right), \quad i = 1, 2, 3$$

where l = the lower limit of the class whose c.f. is just more than $iN/4$, f is its frequency and h is its width. C = c.f. of the class preceding to the class whose c.f. is just more than $iN/4$, $i = 1, 2, 3$. Note that $i = 2$ will give us median.

4. Standard Deviation

Variance σ^2 in case of individual series is given by

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2$$

If $x_1, x_2 \dots, x_n$ occur with frequency $f_1, f_2, \dots f_n$ respectively then

$$\sigma^2 \text{ (variance)} = \frac{1}{N}\sum_{i=1}^{n} f_i\,(x_i - \bar{x})^2$$

$$= \frac{1}{N}\sum f_i x_i^2 - \left(\frac{1}{N}\sum f_i x_i\right)^2$$

and standard deviation = $+\sqrt{\text{variance}}$.

There is no effect of change of origin on standard deviation i.e. if $d_i = x_i - A$ then $\sigma_x = \sigma_d$. If $d_i = \frac{x_i - A}{h}$ then

$\sigma_x = |h|\sigma_d$ equivalently

$$\sigma_x^2 = h^2\left[\frac{1}{N}\sum f_i d_i^2 - \left(\frac{1}{N}\sum f_i d_i\right)^2\right]$$

If there are two samples of sizes n_1 and n_2 with $\bar{x}_1$, and $\bar{x}_2$ as their means and σ_1 and σ_2 their standard deviations respectively, then the combined variance is given by

$$\sigma^2 = \frac{n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1 + n_2}$$

where $d_1 = \bar{x}_1 - \bar{x}$ and $d_2 = \bar{x}_2 - \bar{x}$,

$\bar{x}$ being the combined mean. i.e $\bar{x} = \frac{n_1\,\bar{x}_1 + n_2\,\bar{x}_2}{n_1 + n_2}$

COEFFICIENT OF DISPERSION (C.D.)

1. C.D. based on range = $\frac{A - B}{A + B}$, where A and B are the greatest and smallest items in the series.
2. C.D. based on quartile deviation = $\frac{Q_3 - Q_1}{Q_3 + Q_1}$
3. C.D. based on M.D. = $\frac{\text{M.D.}}{\text{Average from which it is calculated}}$
4. C.D. based on S.D. = $\frac{\text{S.D.}}{\bar{x}}$
5. Coefficient of variation = $100 \times \frac{\text{S.D.}}{\bar{x}}$

SOLVED EXAMPLES

Concept-based
Straight Objective Type Questions

Example 1: The mean deviation of an ungrouped data is 50. If each observation is increased by 2%, then the new mean deviation is

(a) 50 (b) 51
(c) 49 (d) 50.5

Ans. (b)

Solution: If $x_1 \ldots x_n$ are the observations then the new observations are $(1.02)\,x_1, (1.02)x_2 \ldots, (1.02)x_n$
Therefore the new mean = $(1.02)\,\bar{x}$

$$\text{New mean deviation} = \frac{1}{n}\sum |(1.02)x_i - (1.02)\,\bar{x}\,|$$
$$= (1.02)\,\frac{1}{n}\sum |x_i - \bar{x}|$$
$$= (1.02) \times 50 = 51$$

Example 2: In a nuclear engineering class there are 22 juniors, 18 seniors and 10 graduate students. If the juniors averaged 71 in the midterm examination, the senior averaged 78 and the graduate students averaged 89, the mean of the centre class is approximately

(a) 77 (b) 80
(c) 74 (d) 81

Ans. (a)

Solution: $\text{Combined mean} = \dfrac{22\times 71 + 18\times 78 + 10\times 89}{22+18+10}$

$$= \frac{1562\times 1404 + 890}{50}$$
$$= \frac{3856}{50} = 77 \text{ (approximately)}$$

Example 3: The mean marks obtained by 300 students in Mathematics are 45. The mean of top 100 Students was 70 and the mean of last 100 was known to be 20. The mean of remaining 100 students is

(a) 40 (b) 50
(c) 45 (d) 43

Ans. (c)

Solution: $45 = \dfrac{70\times 100 + \bar{x}_2 \times 100 + 100\times 20}{300}$

$\Rightarrow \quad 135 = 90 + \bar{x}_2 \Rightarrow x_2 = 45$

Example 4: The mean of variable 1, 2, … n whose corresponding frequencies are 1, 2,… n is given by

(a) $\dfrac{n+1}{2}$ (b) $\dfrac{2n+1}{6}$
(c) $\dfrac{n(n+1)}{2}$ (d) $\dfrac{2n+1}{3}$

Ans. (d)

Solution: $\text{Mean} = \dfrac{1.1 + 2.2 + \ldots. n.n}{1+2\ldots+n}$

$$= \frac{\sum_{K=1}^{n} k^2}{\sum_{K=1}^{n} k} = \frac{n(n+1)(2n+1)}{6} \times \frac{2}{n(n+1)}$$
$$= \frac{2n+1}{3}.$$

Example 5: The mean of 5 observations is 4.4 and the variance is 8.24. If three of the five observations are 1, 2 and 6, the two values are

(a) 4 and 9 (b) 3 and 5
(c) 2 and 6 (d) 4 and 6

Ans. (a)

Solution: $\Sigma x_i = n\,\bar{x} = 5 \times 4.4 = 22$

So $\quad 1 + 2 + 6 + x_4 + x_5 = 22$

$$x_4 + x_5 = 13 \tag{1}$$

$$\sigma^2 = \frac{1}{n}\sum x_1^2 - \left(\frac{1}{n}\sum x_i\right)^2 \Rightarrow \sum x_i^2 = n\left(\sigma^2 + \bar{x}^2\right)$$
$$= 5(8.24 + 19.36)$$
$$= 138.$$

$\Rightarrow \quad 1 + 4 + 36 + x_4^2 + x_5^2 = 138$

$\Rightarrow \quad x_4^2 + x_5^2 = 97$

So $x_4^2 + (13 - x_4)^2 = 97 \Rightarrow x_4^2 + 169 + x_4^2 - 26x_4 = 97$

$\Rightarrow \quad 2x_4^2 - 26x_4 + 72 = 0$

$\Rightarrow \quad x_4^2 - 13x_4 + 36 = 0$

$$x_4 = \frac{13 \pm \sqrt{169-144}}{2} = \frac{13\pm 5}{2} = 9,4$$

So $\quad x_5 = 13 - x_4 = 4, 9$

Hence, the other two nembers are 4 and 9.

Example 6: In a set of 20 observations, each of the observation below the median of all observations is increased by 6 and each of the remaining observation is decreased by 2.Then the mean of the new set of observations

(a) increase by 1 (b) decrease by 1
(c) increase by 4 (d) increase by 2

Ans. (d)

Solution: Arrange all observations in increasing or decreasing order.

Let the observations be $x_1, x_2 \ldots. x_{20}$ so that $\frac{x_{10}+x_1}{2}$ is the median. The new set of observations will be $x_1 + 6, \ldots x_{10} + 6, x_{11} - 2, \ldots, x_{20} - 2$. Their mean is equal to

$$\frac{1}{20}\left(\sum_{i=1}^{20} x_i + 6\times 10 - 2\times 10\right)$$

$$= \frac{1}{20}\sum_{i=1}^{20} x_i + 2$$

LEVEL 1

Straight Objective Type Questions

Example 7: If the mean of a set of observations $x_1, x_2, \ldots, x_{10}$ is 20 then the mean of $x_1 + 4, x_2 + 8, x_3 + 12, \ldots, x_{10} + 40$ is

(a) 34 (b) 42
(c) 38 (d) 40

Ans. (b)

Solution: Required mean $= \frac{1}{10}$ $[(x_1 + x_2 + \ldots x_{10}) + (4 + 8 + \ldots 40)$

$$= \frac{1}{10}(x_1 + \ldots + x_{10}) + \frac{4}{10}[1 + 2 + \ldots + 10]$$

$$= 20 + \frac{4\times 10\times 11}{10\times 2} = 42.$$

Example 8: Let G_1, G_2 be the geometric means of two series $x_1, x_2, \ldots x_n; y_1, y_2 \ldots y_n$. If G is the geometric mean of x_i/y_i, $i = 1, 2, \ldots n$, then G is equal to

(a) $G_1 - G_2$ (b) $\log G_1/\log G_2$
(c) $\log (G_1/G_2)$ (d) none of these

Ans. (d)

Solution: $G_1 = (x_1, x_2 \ldots x_n)^{1/n}$ and $G_2 = (y_1, y_2 \ldots y_n)^{1/n}$

$$G = \left(\frac{x_1}{y_1}\cdot\frac{x_2}{y_2}\cdots\frac{x_n}{y_n}\right)^{1/n} = \frac{(x_1.x_2\ldots x_n)^{1/n}}{(y_1.y_2\ldots y_n)^{1/n}} = \frac{G_1}{G_2}$$

Example 9: The monthly sales for the first 11 months of the year of a certain salesman were Rs. 12,000 but due to his illness during the last month the average sales for the whole year came down to Rs. 11,375. The value of the sale during the last month was

(a) Rs 4,500 (b) Rs 6,000
(c) Rs 10,000 (d) Rs 8,000

Ans. (a)

Solution: Let Rs. x be the sale in the last month. Using the formula of combined mean, we have

$$\frac{12{,}000\times 11 + x}{12} = 11{,}375$$

$$\Rightarrow \quad x = 11{,}375\times 12 - 12{,}000\times 11$$
$$= 1{,}36{,}500 - 1{,}32{,}000 = 4500.$$

Example 10: The A.M. of the observations 1.3.5, 3.5.7, 5.7.9, …, $(2n - 1)(2n + 1)(2n + 3)$ is

(a) $2n^3 + 6n^2 + 7n - 2$ (b) $n^3 + 8n^2 + 7n - 2$
(c) $2n^3 + 5n^2 + 6n - 1$ (d) $2n^3 + 8n^2 + 7n - 2$

Ans. (d)

Solution: A.M. $= \frac{1}{n}$ $[1.3.5 + 3.5.7 + \ldots + (2n - 1)(2n + 1)(2n + 3)]$

$$= \frac{1}{n}\left[\sum_{r=1}^{n}(2r-1)(2r+1)(2r+3)\right]$$

$$= \frac{1}{n}\sum_{r=1}^{n}\left(8r^3 + 12r^2 - 2r - 3\right)$$

$$= \frac{1}{n}\left[8\left(\frac{n(n+1)}{2}\right)^2 + 12\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} - 3n\right]$$

$$= 2n(n + 1)^2 + 2(n + 1)(2n + 1) - (n + 1) - 3$$

$$= 2n^3 + 8n^2 + 7n - 2.$$

Example 11: If $\sum_{i=1}^{18}(x_i - 8) = 9$ and $\sum_{i=1}^{18}(x_i - 8)^2 = 45$ then the standard deviation of $x_1, x_2 \ldots x_{18}$ is

(a) 4/9 (b) 9/4
(c) 3/2 (d) none of these

Ans. (c)

◎ **Solution:** Let $d_i = x_i - 8$ but

$$\sigma_x^2 = \sigma_d^2 = \frac{1}{18}\sum d_i^2 - \left(\frac{1}{18}\sum d_i\right)^2$$

$$= \frac{1}{18}\times 45 - \left(\frac{9}{18}\right)^2 = \frac{5}{2} - \frac{1}{4} = \frac{9}{4}$$

Therefore $\sigma_x = 3/2$.

⦿ **Example 12:** If a variable takes the values 0, 1, 2 … n with frequencies proportional to the binomial coefficients ${}^nC_0, {}^nC_1, \ldots, {}^nC_n$ then the mean of the distribution is

(a) $\frac{n(n+1)}{4}$ (b) $\frac{n}{2}$

(c) $\frac{n(n-1)}{2}$ (d) $\frac{n(n+1)}{2}$

Ans. (b)

◎ **Solution:** $N = \Sigma f_i = k\,[{}^nC_0 + {}^nC_1 + \ldots + {}^nC_n] = k\,(1+1)^n = k2^n$

$$\Sigma f_i x_i = k\,[1.{}^nC_1 + 2\,.\,{}^nC_2 + \ldots + n\,{}^nC_n]$$

$$= kn\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots + 1\right] = kn\,(1+1)^{n-1}$$

$$= kn2^{n-1}$$

Thus $\bar{x} = \frac{1}{2^n}\left(n\,2^{n-1}\right) = \frac{n}{2}$.

⦿ **Example 13:** The mean of two samples of sizes 200 and 300 were found to be 25, 10 respectively. Their standard deviations were 3 and 4 respectively. The variance of combined sample of size 500 is

(a) 64 (b) 65.2

(c) 67.2 (d) 64.2

Ans. (c)

◎ **Solution:** Combined mean $\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$

$$= \frac{200\times 25 + 300\times 10}{500} = 16$$

Let $d_1 = \bar{x}_1 - \bar{x} = 25 - 16 = 9$ and

$d_2 = \bar{x}_2 - \bar{x} = 10 - 16 = -6$

Now we know that

$$\sigma^2 = \frac{n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1 + n_2}$$

$$= \frac{200\,(9+81) + 300(16+36)}{500} = \frac{33600}{500} = 67.2$$

⦿ **Example 14:** The variance of first 20-natural numbers is

(a) 133/4 (b) 379/12

(c) 133/2 (d) 399/4

Ans. (a)

◎ **Solution:** $\sigma^2 = \frac{1}{20}[1^2 + 2^2 + \ldots + 20^2] - \left[\frac{1}{20}(1 + 2 + \ldots + n)\right]^2$

$$= \frac{1}{20}\,\frac{20\times 21\times(2\times 20+1)}{6} - \left[\frac{1}{20}\,\frac{20\times 21}{2}\right]^2$$

$$= \frac{7\times 41}{2} - \frac{441}{4} = \frac{133}{4}.$$

In fact, the variance of first n-natural numbers is $\frac{n^2 - 1}{12}$.

⦿ **Example 15:** If the standard deviation of $x_1, x_2, \ldots, x_n$ is 3.5, then the standard deviation of $-2x, -3, -2x_2, -3, \ldots, -2x_n - 3$ is

(a) – 7 (b) – 4

(c) 7 (d) 1.75

Ans. (c)

◎ **Solution:** We know that if $d_i = \frac{x_i - A}{h}$ then $\sigma_x = |h|\sigma_d$.

In this case $-2x_i - 3 = \frac{x_i + 3/2}{-1/2}$ so $h = -1/2$. Thus $\sigma_d = \frac{1}{|h|}\sigma_x = 2 \times 3.5 = 7$.

⦿ **Example 16:** The geometric mean of $3, 3^2 \ldots, 3^n$ is

(a) $3^{n/2}$ (b) $3^{(n+1)/2}$

(c) $3^{n(n+1)/2}$ (d) none of these

Ans. (b)

◎ **Solution:** G.M. $= (3.3^2.3^3 \ldots 3^n)^{1/n}$

$$= (3^{1+2+\ldots+n})^{1/n} = \left(3^{\frac{n(n+1)}{2}}\right)^{1/n}$$

$$= 3^{(n+1)/2}$$

⦿ **Example 17:** If the median of 21 observations is 40 and if the observations greater than the median are increased by 6 then the median of the new data will be

(a) 40 (b) 46

(c) 46 + 40/21 (d) 46 – 40/21

Ans. (a)

◎ **Solution:** If the observations greater than the median are increased by 6, the median will not be affected.

Example 18: The median from the following data

Wages/week (Rs.)	*No. of workers*	*Wages/week*	*No. of workers*
50-59	15	90–99	45
60-69	40	100–109	40
70-79	50	110–119	15
80-89	60		

is

(a) 83.17 (b) 84.08 (c) 82.17 (d) 85.67

Ans. (b)

Solution:

	Wages/week (Rs.)	No. of workers (f)	c.f.
	50–59	15	15
	60–69	40	55
	70–79	50	105
Median Class	80–89	60	165
	90–99	45	210
	100–109	40	250
	110–119	15	265

$$\text{Median} = \text{size of } \left(\frac{N}{2}\right) \text{ item} = \frac{265}{2} = 132.5\text{th item}$$

Median lies in the class 80–89. But the true limit of the class is 79.5 – 89.5.

Here, $N = 265$, $C = 105$, $f = 60$, $h = 10$ and $l = 79.5$

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - C\right)$$

$$= 79.5 + \frac{10}{60}(132.5 - 105) = 84.08.$$

Example 19: If the standard deviation of numbers 2, 4, 5 and 6 is a constant α, then the standard deviation of the numbers 4, 6, 7 and 8 is

(a) $\alpha + 2$ (b) 2α

(c) α (d) $\sqrt{2\alpha}$

Ans. (c)

Solution: The standard deviation is independent of the change of origin.

Example 20: Assuming the variance of four numbers a, b, c and d as 9, the variance of $5a$, $5b$, $5c$ and $5d$ is

(a) 45 (b) 5/9

(c) 9/5 (d) 225

Ans. (d)

Solution: $\sigma_x^2 = h^2 \sigma_u^2$ if $u = \dfrac{x - A}{h}$

Here $h = 1/5$ so $\sigma_u^2 = 9 \times 25 = 225$.

Example 21: 1, 2, 4, 6, 8, 9 transformed into another series whose mean is 12. Two middle terms of the transformed series are

(a) 7, 8 (b) 10, 14

(c) 9, 15 (d) 11,13

Ans (d)

Solution: Mean of the given series is 5. If we have to transfer the given series into one having 12 as mean, we must take $d = x + 7$. The transformed terms are 8, 9, 11, 13, 15, 16. The two middle terms are 11, 13.

Example 22: If a variable takes the values 0, 1, 2 ----, n with frequencies proportional to the binomial coefficients, $C(n, 0)$, $C(n,1)$ ----, $C(n, n)$ respectively, then the variance of the distribution is:

(a) n (b) $\sqrt{n}/2$

(c) $\dfrac{n}{2}$ (d) $\dfrac{n}{4}$

Ans. (d)

Solution: $\sigma^2 = \dfrac{\sum fx^2}{\sum f} - \left(\dfrac{\sum fx}{\sum f}\right)^2$

We have $\sum f = \sum C(n,k) = 2^n$

Next, $$\sum fx = \sum_{k=0}^{n} k \; C(n,k) = n(2^{n-1})$$

$$\text{and } \sum fx^2 = \sum_{k=0}^{n} k^2 \; C(n,k)$$

$$= \sum_{k=0}^{n} [k(k-1) + k] C(n,k)$$

$$= n(n-1)(2^{n-2}) + n(2^{n-1})$$

Thus, $$\sigma^2 = \frac{n(n-1)}{4} + \frac{n}{2} - \left(\frac{n}{2}\right)^2 = \frac{n}{4}.$$

Example 23: If the mean deviation of the numbers 1, $1 + d$, ..., $1 + 100d$ from their mean is 255, then d is equal to

(a) 10.1 (b) 20.2

(c) 10.0 (d) 20.0

Ans. (a)

Solution: $\bar{x} = \dfrac{1}{101}[1 + (1 + d) + (1 + 2d) + ... + (1 + 100d)]$

$$= \frac{1}{101} \times \frac{101}{2}[1 + (1 + 100d)] = 1 + 50d.$$

∴ Mean deviation from mean

$$= \frac{1}{101}[|1 - (1 + 50d)| + |1 + d -$$

$$(1 + 50d)| + ... + |1 + 100d - (1 + 50d)|]$$

$$= \frac{2|d|}{101}\{1 + 2 + ... + 50\}$$

$$= \frac{2|d|}{101}\frac{50(51)}{2} = \frac{2550}{101}|d|$$

Now, $\frac{2550}{101}|d| = 255 \Rightarrow |d| = 10.1$

Thus, we may take $d = 10.1$

Example 24: For two data sets, each of size 5, the variances are given to be 4 and 5 the corresponding means are given to be 2 and 4 respectively. The variance of the combined data set is

(a) 6 (b) 13/2
(c) 5/2 (d) 11/2

Ans. (d)

Solution: We have

$n_1 = 5, n_2 = 5, \quad \sigma_1{}^2 = 4, \sigma_2{}^2 = 5, \quad \bar{x}_1 = 2, \ \bar{x}_2 = 4$

We have

$$\bar{x} = \frac{n_1\overline{x_1} + n_2\overline{x_2}}{n_1 + n_2} = \frac{(5)(2)+(5)(4)}{5+5} = 3$$

We know that

$$\sigma^2 = \frac{n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1 + n_2}$$

where $d_1 = \bar{x}_1 - \bar{x} = -1$ and $d_2 = \bar{x}_2 - \bar{x} = 1$

Thus, $$\sigma^2 = \frac{5(4+1)+5(5+1)}{5+5} = \frac{11}{2}$$

Example 25: If the mean deviation about the median of the numbers $a, 2a, \cdots, 50a$ is 50 then $|a|$ equal

(a) 5 (b) 2
(c) 3 (d) 4

Ans. (d)

Solution: Median of $a, 2a \cdots, 50a$ is

$$\frac{1}{2}(25a + 26a) = (25.5)\,a.$$

M.D. of given numbers from median

$$= \frac{1}{50}\sum_{k=1}^{50}|ka - (25\cdot 5)a|$$

$$= \frac{|a|}{50}\sum_{k=1}^{50}|k - 25\cdot 5|$$

$$= \frac{|a|}{50}(2(24\cdot 5 + 23\cdot 5 + \ldots + 0\cdot 5))$$

$$= \frac{|a|}{50}(1 + 3 + \ldots + 49)$$

$$= \frac{|a|}{50} \times \frac{25}{2}(1+49) = \frac{25}{2}|a| = 50$$

$\Rightarrow \quad |a| = 4$

Example 26: A scientist is weighing each of 30 fishes. Their mean weight worked out is 30 gm and a standard deviation of 2 gm. Later, it was found that the measuring scale was misaligned and always under reported every fish weight by 2 gm. The correct mean and standard deviation (in gm) of fishes are respectively:

(a) 32, 2 (b) 32, 4
(c) 28, 2 (d) 28, 4

Ans. (a)

Solution: Since mean $(X + b)$ = mean $X + b$ and Var $(X + b)$ = Var X, so we get correct mean as $30 + 2 = 32$ gm and s.d, is 2 gm

Assertion-Reason Type Questions

Example 27: If $\bar{x} = 3$ and $A = \sum_{i=1}^{n}(x_i - 3)^2$,

$B = \sum_{i=1}^{n}(x_i - 4)^2$, $C = \sum_{i=1}^{n}(x_i - 2)^2$

Statement-1: Min $(A, B, C) = C$.
Statement-2: Sum of the squares of deviations is least when taken from mean

Ans. (d)

Solution: Among all the squares of the deviations, the least is when taken from mean. This can be proved by considering, $F(x) = \sum_{i=1}^{n}(x_i - X)^2$, F is least when $X = \bar{x}$.

Example 28: The mean monthly salary paid to all employees of a factory was Rs. 6000. The mean salary paid to male and female employees was Rs. 6200 and Rs. 5200.

Statement-1: The percentage of males is 80.
Statement-2: The exact percentage cannot be determined as the data is insufficient.

Ans. (c)

Solution: Let there be N_1 percentage male and female be N_2 percent so $N_1 + N_2 = 100$

$$6000 = \frac{6200\,N_1 + 5200\,N_2}{N_1 + N_2}$$

$\Rightarrow \quad 6000 = 62\,N_1 + 52\,(100 - N_1)$

$\Rightarrow \quad N_1 = \frac{800}{10} = 80$ and $N_2 = 20.$

Example 29: The arithmetic mean of two observations is 127.5 and geometric mean is 60.

Statement-1: The harmonic mean is approximately 28.

Statement-2: $A.M. \times H.M. = (G.M.)^2$.

Ans. (a)

Solution: $A.M. = \dfrac{a+b}{2} \quad H.M. = \dfrac{2ab}{a+b}$

So $A.M \times H.M. = ab = (G.M)^2$

Thus $\quad H.M. = \dfrac{3600}{127.5} = 28.24$

Example 30: Let $x_1, x_2 \cdots x_n$ be n observations, Let $\bar{x}$ be their arithmetic mean and σ^2 be their variance

Statement 1: Variance of $2x_1, 2x_2 \cdots 2x_n$ is

Statement 2: Arithmetic mean of $2x_1, 2x_2 \cdots 3x_n$ is $4\bar{x}$

Ans. (c)

Solution: If $d = \dfrac{x-A}{h}$ then $\bar{x} = A + h\ \bar{d}$. In this case A = 0, $h = 1/2$ so $\bar{d} = \dfrac{1}{h}\ \bar{x} = 2x$. Also $\sigma_x^2 = h^2\ \sigma_d^2$ so $\sigma_d^2 = \dfrac{1}{h^2}\sigma_x^2 = 4\sigma^2$.

LEVEL 2

Straight Objective Type Questions

Example 31: Given the following frequency distribution with some missing frequencies

Class	Frequency
10–20	180
20–30	--
30–40	34
40–50	180
50–60	136
60–70	--
70–80	50

If the total frequency is 685 and median is 42.6 then the missing frequencies are

(a) 81, 24 (b) 80, 25

(c) 82, 23 (d) 832, 22

Ans. (c)

Solution:

Let the missing frequencies be f_1 and f_2, respectively.

Class	Frequency (f)	Cum. Freq. (c.f.)
10–20	180	180
20–30	f_1	$180 + f_1$
30–40	34	$214 + f_1$
40–50	180	$394 + f_1$
50–60	136	$530 + f_1$
60–70	f_2	$530 + f_1 + f_2$
70–80	50	$580 + f_1 + f_2$
Total	685	

We have $580 + f_1 + f_2 = 685 \quad \Rightarrow \quad f_1 + f_2 = 105$

As median 42.6 lies in the class 40 – 50, median class is 40 – 50. We have that $l = 40, f = 180, C = 214 + f_1$ and $h = 10$

Now, $\quad \text{Median} = l + \dfrac{h}{f}\left(\dfrac{N}{2} - C\right)$

$$\Rightarrow \quad 42.6 = 40 + \frac{10}{180}\left(\frac{685}{2} - 214 - f_1\right)$$

$$\Rightarrow \quad 2.6 \times 18 = 342.5 - 214 - f_1 \quad \Rightarrow \quad f_1 = 81.7$$

As frequency of an item is always a whole number, we take $f_1 = 82$.

$\therefore \quad f_2 = 105 - 82 = 23.$

Example 32: The mean deviation from mean of the observations $a, a + d, a + 2d, \ldots, a + 2nd$ is

(a) $\dfrac{n(n+1)d^2}{3}$ (b) $\dfrac{n(n+1)}{2}d^2$

(c) $a + \dfrac{n(n+1)d^2}{2}$ (d) none of these

Ans. (d)

Solution:
$$\bar{x} = \frac{1}{2n+1}[a + (a+d) + \ldots + (a+2nd)]$$
$$= \frac{1}{2n+1}[(2n+1)a + d(1 + 2 + \ldots + 2n)]$$
$$= a + d\,\frac{2n}{2}\,\frac{(1+2n)}{2n+1} = a + nd$$

M.D. from mean
$$= \frac{1}{2n+1} 2|d|(1+2+\ldots+n) = \frac{n(n+1)|d|}{(2n+1)}$$

Example 33: Sum of absolute deviations about median is

(a) least (b) greatest
(c) zero (d) equal to median

Ans. (a)

Solution: See theory.

Example 34: In any discrete series (when all the values are not same) the relationship between M.D. about mean and S.D. is

(a) M.D. = S.D. (b) M.D. > S.D.
(c) M.D. < S.D. (d) M.D. ≤ S.D.

Ans. (c)

Solution: $\text{M.D.} = \frac{1}{n}\sum_{i=1}^{n}|x_i - \bar{x}|$ and $\text{S.D.} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$

Let $a_i = |x_i - \bar{x}|$ so $(\text{M.D.})^2 = \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)^2$ and $(\text{S.D.})^2 = \frac{1}{n}\sum_{i=1}^{n} a_i^2$. By Cauchy Schwarz inequality

$$\left(\sum_{i=1}^{n} a_i\right)^2 < n\sum_{i=1}^{n} a_i^2$$

$$\Rightarrow \quad \left(\frac{1}{n}\sum a_i\right)^2 < \frac{1}{n}\sum a_i^2 \quad \Rightarrow \quad \text{M.D.} < \text{S.D.}$$

Example 35: The median and standard deviation (S.D.) of a distribution are 20 and 4 respectively. If each item is increased by 2 then

(a) median will increase and S.D. will also increase
(b) median will will go up by 2 but S.D. will remain same
(c) median will increase but S.D. will decrease
(d) none of these

Ans. (b)

Solution: Median will go up by 2 and S.D. will remain the same.

EXERCISE

Concept-based Straight Objective Type Questions

1. The mean of 5 observations is 3 and variance is 2. If three of the five observations are 1, 3, 5 the other two are

(a) 2, 6 (b) 3, 3
(c) 1, 5 (d) 2, 4

2. The mean and standard deviation of a sample of size 20 are found 10 and 2 respectively. If one item 8 is replaced by 12 then

(a) standard deviation increases
(b) standard deviation decreases
(c) standard deviation will remain same
(d) mean increases by 4%

3. The mean weight of 150 students in 60 Kgs. The mean weight of boys is 70 and that of girls in 55 with standard deviations 10 and 15 respectively. The combined variance is

(a) $\frac{700}{3}$ (b) $\frac{500}{3}$
(c) $\frac{400}{7}$ (d) $\frac{300}{7}$

4. If $n = 10$, $\bar{x} = 12$ and $\Sigma x^2 = 1530$, then coefficient of variation is

(a) 16 (b) 9
(c) 25 (d) 36

5. The first of two samples has 100 items with mean 15 and standard deviation 3. If the whole group has 250 items with mean 15.6 and variance 13.44, the standard deviation of the second sample is

(a) 3 (b) 2
(c) 4 (d) 5

6. If the variables with values 10, 20, 30, 40 have frequencies x, $x + 2$, $x - 2$, $x + 4$ and the mean is 2.7 then the value of x is

(a) 2 (b) 4
(c) 6 (d) 5

LEVEL 1

Straight Objective Type Questions

7. The arithmetic mean of the first n natural numbers is
   (a) $(n + 1)/2$ (b) $(n - 1)/2$
   (c) $n/2$ (d) none of these

8. The arithmetic mean of the first n odd natural numbers is
   (a) n (b) $(n + 1)/2$
   (c) $(n - 1)$ (d) none of these

9. The arithmetic mean of the squares of the first n natural numbers is
   (a) $(n + 1)/6$ (b) $(n + 1)(2n + 1)/6$
   (c) $(n^2 - 1)/6$ (d) none of these

10. The arithmetic mean of the series 1, 2, 2^2, …, 2^{n-1} is
    (a) $2^n/n$ (b) $(2^n - 1)/n$
    (c) $(2^{n+1})/n$ (d) none of these

11. The arithmetic mean of the series ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, …, ${}^nC_n$ is
    (a) $2^n/(n + 1)$ (b) $2n/n$
    (c) $2^{n-1}/(n + 1)$ (d) none of these

12. The arithmetic mean of a, $a + d$, $a + 2d$, …, $a + 2nd$ is
    (a) $a + (n - 1)d$ (b) $a + (n + 1)d$
    (c) $a + (n + 2)d$ (d) none of these

13. The standard deviation of the data given by

Variate (x)	0	1	2	3	…	n
Frequency (f)	${}^nC_0$	${}^nC_1$	${}^nC_2$	${}^nC_3$	…	${}^nC_n$

    (a) $\sqrt{(n+1)/2}$ (b) $\sqrt{n/2}$
    (c) $2^n/n$ (d) none of these

14. The mean height of 25 male workers in a factory is 161 cms, and the mean height of 35 female workers in the same factory is 158 cms. The combined mean height of 60 workers in the factory is
    (a) 159.25 (b) 159.5
    (c) 159.75 (d) 158.75

15. The average salary of male employees in a firm was Rs 5200 and that of females was 4200. The mean salary of all the employees was Rs 5000. The percentage of male employees is
    (a) 80 (b) 60
    (c) 40 (d) 20

16. The mean weight of 9 items is 15. If one more item is added to the series the mean becomes 16. The value of 10th item is
    (a) 35 (b) 30
    (c) 25 (d) 20

17. For a certain frequency table which has been partly reproduced here, the arithmetic mean was found to be Rs 28.07.

Income (in Rs):	15	20	25	30	35	40
No. of workers:	8	12	?	16	?	10

    If the total number of workers is 75, then the missing frequencies are
    (a) 14, 15 (b) 15, 14
    (c) 13, 16 (d) 12, 17

18. The value of median for the data

Income (in Rs.)	1000	1100	1200	1300	1400	1500
No. of persons:	14	26	21	18	28	15

    is
    (a) 1300 (b) 1200
    (c) 1250 (d) 1150

19. The median from the following data:

Mid-value	*Frequency*	*Mid-value*	*Frequency*
115	6	165	60
125	25	175	38
135	48	185	22
145	72	195	3
155	116		

    is
    (a) 153.79 (b) 153.91
    (c) 165.18 (d) 165.93

20. In a discrete series, the relation between s.d.s and range r is
    (a) $s \geq r$ (b) $s > r$
    (c) $r = s$ (d) $s \leq r$

21. If A.M., G.M. and H.M. in any series are equal then
    (a) the distribution is symmetric
    (b) all the values are same
    (c) the distribution is unimodal
    (d) none of these

22. For a group containing 100 observation, the mean and standard deviation are 8 and $\sqrt{10.5}$. For 50 observations selected from these 100 observations, the mean and standard deviation are 10 and 2 respectively . The standard deviation of the other half is

(a) 2
(b) 3
(c) $\sqrt{6.5}$
(d) (4)

23. A sample of 90 values has mean 55 and standard deviation 3. A second sample of 110 values has mean 60 and standard deviation 2. The combined variance is equal to (upto two decimal)

(a) 12.43
(b) 11. 24
(c) 10.43
(d) 13.25

Assertion-Reason Type Questions

24. Suppose that the mean and standard deviation of 100 observations are 50 and 4.

Statement-1: Sum of the square of the observations is 2,51,600.

Statement-2: Sum of the observations is 5000.

25. If $\sum x_i = 440$, $\sum (x_i - 44)^2 = 1710$, $N = 10$, then.

Statement-1: *C.V.* is approximately 30.

Statement-2: Variance is 171.

LEVEL 2

Straight Objective Type Questions

26. If the s.d. of a set of observations is 4 and if each observation is divided by 4, the s.d. of the new set of observations will be

(a) 4
(b) 3
(c) 2
(d) 1

27. If s.d. of a set of observations is 8 and if each observation is divided by – 2, the s.d. of the new set of observations will be

(a) – 4
(b) – 8
(c) 8
(d) 4

28. The mean square deviation of a set of n observations $x_1, x_2, \ldots, x_n$ about a point c is defined as

$$\frac{1}{n}\sum_{i=1}^{n}(x_i - c)^2$$

The mean square deviation about –2 and 2 are 18 and 10 respectively, then standard deviation of this set of observation is

(a) 3
(b) 2
(c) 1
(d) none of these

29. If the s.d. of n observations $x_1, x_2, \ldots, x_n$ is 4 and another set of n observations $y_1, y_2, \ldots, y_n$ is 3, the s.d. of n observations $x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n$ is

(a) 1
(b) $2/\sqrt{3}$
(c) 5
(d) none of these

30. If a variable takes values 0, 1, 2 … n with frequencies proportional to q^n, ${}^nC_1\, pq^{n-1}$, ${}^nC_2\, p^2 q^{n-2}$, …, p^n where $p + q = 1$ then the mean is

(a) np
(b) nq
(c) npq
(d) np^2

31. If a variable takes values 0, 1, 2, 3 … with frequencies proportional to $e^{-\lambda}$, $e^{-\lambda}\lambda$, $\dfrac{e^{-\lambda}\lambda^2}{2!}$, $\dfrac{e^{-\lambda}\lambda^3}{3!}$,… then the mean of the distribution is

(a) $e^{-\lambda}$
(b) λ
(c) $e^{-\lambda}\lambda$
(d) $\left(\dfrac{1}{2}\right)e^{-\lambda}\lambda^2$

32. The standard deviation is not affected by the change of

(a) origin
(b) scale
(c) origin and scale
(d) none of these

33. If x_1, x_2, x_3 are three non-zero real numbers such that $(x_1^2 + x_2^2)(x_2^2 + x_3^2) \le (x_1 x_2 + x_2 x_3)^2$ then the G.M. of x_1, x_2, x_3 is

(a) x_1
(b) x_2
(c) x_3
(d) $\dfrac{x_1 x_2 x_3}{3}$

34. The mean and s.d. of a sample of size 10 were found to be 9.5 and 2.5 respectively. Later on, an additional observation 15 was added to the original data. The s.d. of 11 observation is

(a) 2.6 (b) 2.8
(c) 2.86 (d) 3.24

35. For a group of 50 male workers, the mean and s.d. of their daily wages are Rs 630 and Rs 90 respectively. For a group of 40 female workers these Rs 540 and Rs 60 respectively. The s.d. of these 90 workers is

(a) 60 (b) 70
(c) 80 (d) 90

36. The mean marks of 120 students is 20. It was later discovered that two marks were wrongly taken as 50 and 80 instead of 15 and 18. The correct mean of marks is

(a) 19.19 (b) 19.17
(c) 19.21 (d) 19.14

37. Let r be the range and $S^2 = \frac{1}{n-1}\sum(x_i - \overline{x})^2$. If $S^2 \le r^2 k$ then k is equal to

(a) $\frac{1}{n-1}$ (b) $\frac{n}{n-1}$
(c) $\frac{n+1}{2(n-1)}$ (d) none of these

Previous Years' AIEEE/JEE Main Questions

1. In a class of 100 students, there are 70 boys whose average marks in a subject is 75. If the average marks of the complete class is 72, then the average of the girls is

(a) 73 (b) 65
(c) 68 (d) 74 **[2002]**

2. The median of a set of 9 distinct observations is 20.5. If each of the largest 4 observations of the set is increased by 2, then the median of the new set

(a) is decreased by 2
(b) is two times the original median
(c) remains the same as that of the original set
(d) is increased by 2 **[2003]**

3. In an experiment with 15 observations on x, the following results were available

$\Sigma x^2 = 2830$, $\Sigma x = 170$.

One observation, 20, was found to be wrong and was replaced by the correct value 30. Then the corrected variance is

(a) 78 (b) 188.66
(c) 177.33 (d) 833 **[2003]**

4. Consider the following statements:

(1) Mode can be computed from histogram
(2) Median is not independent of change of scale
(3) Variance is independent of change of origin and scale

Which of these is/are correct?

(a) only (1) and (3) (b) only (2)
(c) only (1) (d) (1), (2), (3) **[2004]**

5. In a series of $2n$ observations, half of them equal a and remaining half equal $-a$. If the standard deviation of the observations is 2, then $|a|$ equals

(a) $\frac{1}{n}$ (b) $\sqrt{2}$
(c) 2 (d) $\frac{\sqrt{2}}{n}$ **[2004]**

6. Let $x_1, x_2 \ldots x_n$ be n observations such that $\Sigma x^2 = 400$ and $\Sigma x = 80$. Then a possible value of n among the following is

(a) 15 (b) 18
(c) 9 (d) 12 **[2005]**

7. If in a frequency distribution, the mean and median are 21 and 22 respectively, then the mode is approximately

(a) 25.5 (b) 24.0
(c) 22.0 (d) 20.5 **[2005]**

8. Suppose a population A has 100 observations 101, 102, … 200 and another population B has 100 observations 151, 152, … 250. If V_A and V_B represent the variances of the two populations, respectively, then V_A/V_B is

(a) 2/3 (b) 1
(c) 9/4 (d) 4/9 **[2006]**

9. The average marks of boys in a class is 42 and that of girls is 52. The average marks of boys and girls combined is 50. The percentage of boys in the class is

(a) 40 (b) 20
(c) 80 (d) 60 **[2007]**

10. The mean of the numbers a, b, 8, 5, 10 is 6 and the variance is 6.80. Then which one of the following gives possible values of a and b?

(a) $a = 0, b = 7$ (b) $a = 5, b = 2$
(c) $a = 1, b = 6$ (d) $a = 3, b = 4$ **[2008]**

11. If the mean deviation of the numbers 1, $1 + d$, $1 + 2d$, ..., $1 + 100d$ form their mean is 255 then the d is equal to

(a) 10.1 (b) 20.2
(c) 10.0 (d) 20.0 **[2009]**

12. For two data sets, each of size 5, the variances are given to be 4 and 5 and the corresponding means are given to be 2 and 4 respectively. The variance of combined data set is

(a) 6 (b) 13/2
(c) 5/2 (d) 11/2 **[2010]**

13. If the mean deviation about the median of the numbers a, $2a$, ..., $50a$ is 50 then $|a|$ equals

(a) 5 (b) 2
(c) 3 (d) 4 **[2011]**

14. A scientist is weighing each of 30 fishes. Their mean weight worked out is 30 gm and a standard deviation of 2 gm. Later, it was found that the measuring scale was misaligned and always reported every fish weight by 2 gm. The correct mean and standard deviation (in gm) of fishes are respectively

(a) 32, 2 (b) 32, 4
(c) 28, 2 (d) 28, 4 **[2011]**

15. Let x_1, x_2 ...x_n be n observations, and let $\bar{x}$ be their arithmetic mean and σ^2 their variance.

Statement-1: Variance of $2x_1$, $2x_2$ ... $2x_n$ is $4\sigma^2$

Statement-2: Arithmetic mean of $2x_1$, $2x_2$ ... $2x_4$ is $4\bar{x}$ **[2012]**

16. All the students of a class performed poorly in mathematics. The teacher decided to give marks of 10 to each of the students. Which of the following statistical measures will not change even after the grace marks were given?

(a) mean (b) median
(c) mode (d) variance **[2013]**

17. If the median and the range of four numbers $\{x, y, 2x + y, x - y\}$, where $0 < y < x < 2y$ are 10 and 28 respectively, then the mean of numbers is

(a) 18 (b) 10
(c) 5 (d) 14 **[2013, online]**

18. The mean of a data set consisting of 20 observation is 40. If one observation 53 was wrongly recorded as 33, then the correct mean will be

(a) 41 (b) 49
(c) 40.5 (d) 42.5 **[2013, online]**

19. The variance of first 50 even natural numbers is.

(a) $\frac{833}{4}$ (b) 833
(c) 437 (d) $\frac{437}{4}$ **[2014]**

20. In a set of $2n$ distinct observations, each of the observations below the median of all the observations is increased by 5 and each of the remaining observation is decreased by 3. Then the mean of the new set of observations

(a) increase by 1 (b) decrease by 1
(c) decrease by 2 (d) increase by 2
[2014, online]

21. Let $\bar{x}$ and M.D be the mean and the mean deviation about $\bar{x}$ of n observations x_i, $i = 1, 2 \ldots., n$. If each of the observation is increased by 5, then the new mean and mean deviation about the new mean, respectively are:

(a) $\bar{x}$, M.D (b) $\bar{x}$ + 5, M.D.
(c) $\bar{x}$, M.D. + 5 (d) $\bar{x}$ – 15, m. D+5
[2014, online]

22. Let $\bar{x}$, M and σ^2 be respectively the mean, mode and variance of n observations x_1, $x_2 - x_n$ and $d_i = -x_i - a$, $i = 1, 2 \ldots n$ where a is any number

Statement 1: Variance of d_1, d_2 . . . d_n is σ^2

Statement 2: Mean and Mode of d_1, $d_2 - d_n$ are $-\bar{x} - a$ and $M - a$ respectively.

(a) Statement 1 and Statement 2 are both false.
(b) Statement 1 and Statement 2 are both true.
(c) Statement 1 is true and Statement 2 is false.
(d) Statement 1 is false and Statement 2 is true.
[2014, online]

23. A factory is operating in two shifts, day and night, with 70 and 30 workers respectively. If per day mean wage of the day shift workers is ₹54 and per day mean wage of all the workers is ₹60, then per day mean wage of the night shift (in ₹) is:

(a) 66 (b) 69
(c) 74 (d) 75 **[2015, online]**

24. The mean of the data set comprising of 16 observations is 16. If one of the observations valued 16 is deleted and three new observations valued 3,4 and 5 are added to the data, then the mean of the resultant data is

(a) 16.8 (b) 16.0
(c) 15.8 (d) 14.0 **[2015]**

25. If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true?

(a) $3a^2 - 26a + 55 = 0$ (b) $3a^2 - 32a + 84 = 0$
(c) $3a^2 - 34a + 91 = 0$ (d) $3a^2 - 23a + 44 = 0$
[2016]

26. If the mean deviation of the numbers 1, 1 + d, … 14 + 100d from their means is 255, then a value of d is
(a) 10.1
(b) 5.05
(c) 20.2
(d) 10 **[2016, online]**

27. The mean of 5 observations is 5 and their variance is 124. If three of the observations are 1, 2 and 6; then the mean deviation from the mean of the data is:
(a) 2.5
(b) 2.6
(c) 2.8
(d) (1.6) $(1+3\sqrt{2})$ (modified) **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. The mean deviation of an ungrouped data is 10. If each observation is increased by 4%, the revised mean deviation is
(a) 10.0
(b) 10.4
(c) 10.04
(d) 9.6 **[2006]**

2. There are 30 boys and 20 girls in a class. The mean and variance of their marks in maths are, respectively, 65 and 100. If 5 grace marks are added to the score of each student, then the revised mean and variance will be respectively
(a) 70, 100
(b) 70, variance > 100
(c) 65, variance < 100
(d) 65, 100 **[2007]**

3. The mean and the median of 100 observations have been computed to be 60 and 70 respectively. Later it was discovered that three observations which have been recorded 18, 28 and 98 are actually 80, 26 and 38 respectively. If the mean and median are recalculated with actual observations, then
(a) median will change but mean will not change
(b) neither mean nor median will change
(c) both mean and median will change
(d) mean will change but median will not change **[2009]**

4. Let a distribution be made by combining three distributions, each having mean zero, standard deviations 3, 4 and 5 respectively and frequencies 200, 250 and 300 respectively. Then the variance of the combined distribution is equal to
(a) $\frac{266}{15}$
(b) 17
(c) $\frac{50}{3}$
(d) $\frac{62}{15}$ **[2010]**

5. In a class of 20 students, each student can score either 10 or 0 marks in a certain examination. The maximum possible variance in the marks of the students in the class is
(a) 24
(b) 22
(c) 20
(d) 25 **[2010]**

6. If the mean and standard deviation of 20 observations $x_1, x_2 \dots x_{20}$ are 50 and 10 respectively, then $\sum_{i=1}^{10} x_i^2$ is equal to
(a) 2510
(b) 50200
(c) 52000
(d) 2600 **[2011]**

7. If the mean of a set of observations $x_1, x_2 \; \dots, x_{10}$ is 20 then the mean of $x_1 + 4, x_2 + 8, \dots, x_{10} + 40$ is
(a) 34
(b) 42
(c) 38
(d) 40 **[2012]**

8. Suppose a population A has 100 observation 101, 102 . . . 200 and another population B has 100 observations 151, 152, . . . 250. If V_A and V_B represents the variances of two populations respectively, then the ratio $V_A : V_B$ is
(a) 1 : 1
(b) 2 : 3
(c) 1 : 2
(d) 3 : 2 **[2013]**

9. If the mean and the standard deviations of 10 observations $x_1, x_2 \; \dots x_n$ are 2 and 3 respectively, then the mean of $(x_1 + 1)^2, (x_2 + 1)^2 \dots (x_{10} + 1)^2$ is equal to
(a) 13.5
(b) 14.4
(c) 16.0
(d) 18.0 **[2014]**

Answers

Concept-based

1. (d) **2.** (b) **3.** (a) **4.** (c)
5. (c) **6.** (10)

Level 1

7. (a) **8.** (a) **9.** (b) **10.** (b)
11. (a) **12.** (d) **13.** (d) **14.** (a)
15. (a) **16.** (c) **17.** (b) **18.** (b)
19. (a) **20.** (d) **21.** (b) **22.** (b)
23. (a) **24.** (b) **25.** (a)

Level 2

26. (d) **27.** (d) **28.** (a) **29.** (d)
30. (a) **31.** (b) **32.** (a) **33.** (b)
34. (c) **35.** (d) **36.** (a) **37.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (b) **2.** (c) **3.** (a) **4.** (c)
5. (c) **6.** (b) **7.** (b) **8.** (b)
9. (b) **10.** (d) **11.** (a) **12.** (d)
13. (d) **14.** (a) **15.** (c) **16.** (d)
17. (d) **18.** (a) **19.** (b) **20.** (a)
21. (b) **22.** (b) **23.** (c) **24.** (d)
25. (b) **26.** (a) **27.** (d)

Previous Years' B-Architecture Entrance Examination Questions

1. (a) **2.** (a) **3.** (a) **4.** (a)
5. (d) **6.** (c) **7.** (b) **8.** (a)
9. (d)

Hints and Solutions

Concept-based

1. $\frac{1}{5}\ \Sigma x_i = 3 \Rightarrow 1 + 3 + 5 + x_4 + x_5 = 15$

$$\Rightarrow x_4 + x_5 = 6 \qquad \text{(i)}$$

$$\frac{1}{5}\ \Sigma x_i^2 - \left(\frac{1}{5}\Sigma x_i\right)^2 = 4 \Rightarrow \Sigma x_i^2 = 65$$

So $x_4^2 + x_5^2 = 65 - (1 + 9 + 25) = 30$ (ii)

Solving (i) and (ii) $x_4, x_5 = 2, 4$

2. $\Sigma x_i = 20 \times 10 = 200$ so new sum $= 200 - 8 + 12 = 204$.

New mean = 10.2 so increase of 2%.

Also $\frac{1}{20}\Sigma x_i^2 - 10^2 = 4$. so $\Sigma x_i^2 = 2080$

New sum of squares $= 2080 - 64 + 144$

$= 2160$

New variance $= \frac{1}{20} \times 2160 - \left(\frac{204}{20}\right)^2$

$= 108 - 104.04$

$= 3.96 < 4$

3. $n = n_1 + n_2 = 150$ (n_1 is number of boys, n_2 is number of girls)

$$60 = \frac{n_1\,\bar{x}_1 + n_2\,\bar{x}_2}{n_1 + n_2} \Rightarrow 70n_1 + 55n_2 = 150 \times 60 = 9000$$

$\Rightarrow\ 14n_1 + 11n_2 = 1800$

$\Rightarrow\ 14n_1 + 11(150 - n_1) = 1800$

$\Rightarrow\ n_1 = 50$ so $n_2 = 100$

$d_1 = \bar{x}_1 - x = 70 - 60 = 10$

$d_2 = \bar{x}_2 - x = 55 - 60 = -5$

$$\sigma^2 = \frac{1}{150}\ (n_1\ (\sigma_1^2 + d_1^2) + n_2\ (\sigma_2^2 + d_2^2))$$

$$= \frac{1}{150}\ (50\ (100 + 100) + 100\ (225 + 25))$$

$$= \frac{700}{3}.$$

4. $\sigma^2 = \frac{1}{n}\ \Sigma x^2 - \bar{x}^2 = \frac{1530}{10} - 12^2 = 9$

C.V $= 100 \times \frac{\sigma}{\bar{x}} = 100 \times \frac{3}{12} = 25.$

5. $n_1 = 100$ and $n_1 + n_2 = 250$ so $n_2 = 150$.

Also $\bar{x}_1 = 15$, $\bar{x} = 15.6$

so $15.6 = \bar{x} = \frac{n_1\,\bar{x}_1 + n_2\,\bar{x}_2}{n_1 + n_2} = \frac{1500 + 150\bar{x}_2}{250}$

$\Rightarrow\ \bar{x}_2 = 16$

$d_1 = \bar{x}_1 - \bar{x} = 15 - 15.6 = -0.6$

$d_2 = \bar{x}_2 - \bar{x} = 16 - 15.6 = 0.4$

$(n_1 + n_2)\ \sigma^2 = n_1\ (\sigma_1^2 + d_1^2) + n_2\ (\sigma_2^2 + d_2^2)$

$250 \times 13.44 = 100\ (9 + 0.36) + 150\ (\sigma_2^2 + 0.16)$

$\Rightarrow\ \sigma_2^2 = 16.$

6. $10\left[\frac{x + 2x + 4 + 3x - 6 + 4x + 16}{4x + 4}\right] = 27$

$\Rightarrow 10[10x + 14] = 108\ x + 108$

$\Rightarrow 8\ x = 32 \Rightarrow x = 4.$

Level 1

7. Sum n natural numbers is $\frac{n(n+1)}{2}$ so the mean is $\frac{n+1}{2}$.

8. Sum of first n odd numbers n^2 so the mean is n.

9. $1^2 + 2^2 \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ so the mean $= \frac{(n+1)(2n+1)}{6}$.

10. $1 + 2 + \dots 2^{n-1} = 2^n - 1.$

11. ${}^nC_0 + {}^nC_1 \dots + {}^nC_n = 2^n$

12. $a + (a + d) + \dots + (a + 2nd)$

$= (2n + 1)\ a + d\ n(2n + 1)$

$= (2n + 1)\ (a + dn)$

13. $(S.D.)^2 = \frac{1}{N}\sum fx^2 - \left(\frac{1}{N}\sum fx\right)^2$

$$\frac{1}{N}\sum fx^2 = \frac{1}{2^n}\left[1^2 \,.\, {}^nC_1 + \cdots n^2\ {}^nC_n\right]$$

$$= \frac{n}{2^n}\left[1 + 2(n-1) + \frac{3}{2}(n-1)(n-2) + \cdots + n\right]$$

$$= \frac{n}{2^n}\left[\left\{1 + (n-1) + \frac{(n-1)(n-2)}{21} + \cdots + 1\right\}\right.$$
$$\left. + \{(n-1) + (n-1)(n-2) + \cdots + (n-1)\}\right]$$

$$= \frac{n}{2^n}\left[{}^{n-1}C_0 + {}^{n-1}C_1 \cdots + {}^{n-1}C_{n-1} + (n-1)\right.$$
$$\left.\left[{}^{n-2}C_0 + \cdots + {}^{n-2}C_{n-2}\right]\right]$$

$$= \frac{n(n+1)}{4}$$

$$\sigma^2 = \frac{n(n+1)}{4} - \frac{n^2}{4} = \frac{n}{4}.$$

14. Mean height $= \frac{25 \times 161 + 35 \times 158}{60} = 159.25$

15. $5000 = \frac{m \times 5200 + f \times 4200}{m + f} \Rightarrow m = 4f.$

So percentage of mole $= \frac{m}{m+f} \times 100$

$$= \frac{4}{5} \times 100 = 80\%.$$

16. Sum of 9 items = 15 × 9 = 135. If x is added then

$$\frac{135 + x}{10} = 16$$

$\Rightarrow\ x = 25.$

17. $8 + 12 + f_1 + 16 + f_2 + 10 = 75 \Rightarrow f_1 + f_2 = 29$

$$28.07 = \frac{120 + 240 + 25f_1 + 480 + 35f_2 + 400}{75}$$

$\Rightarrow\ f_1 = 15 \quad f_2 = 14.$

18. Compute *c.f.* find median.

19. Turn the data into class intervals *e.g.* 110 – 120, 120 – 130, …

20. See Theory.

21. See Theory.

22. $n_1 = n_2 = 50$, $\bar{x}_1 = 10$, $\bar{x} = 8$, $n = 100$, $\sigma = \sqrt{10.5}$

We find $\bar{x}_2$ and then σ_2

Using the combined mean formula, we have

$$8 = \frac{50 \times 10 + 50 \times \bar{x}_2}{100}$$

$\Rightarrow \quad \bar{x}_2 = 6$

$d_1 = 10 - 8 = 2$ and $d_2 = 6 - 8 = -2$

$$10.5 = \sigma^2 = \frac{n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1 + n_2}$$

$$= \frac{50(4+4) + 50(\sigma_2^2 + 4)}{100}$$

$\Rightarrow \quad \sigma_2^2 = 9, \sigma_2 = 3.$

23. $n_1 = 90$, $\bar{x}_1 = 55$, $\sigma_1 = 3$, $n_2 = 110$, $\bar{x}_2 = 60$ and $\sigma_2 = 2$

$$\bar{x}^2 = \frac{90 \times 55 + 110 \times 60}{200} = 57.75$$

$d_1 = 55 - 57.75 = -2.75$

$d_2 = 60 - 57.75 = 2.25$

$$\sigma^2 = \frac{90(9 + 7.5625) + 110(4 + 5.0625)}{200}$$

$= 12.43$ (App).

24. $\sum x_i = 100 \times 50 = 5000,$

$$\bar{x}^2 + \sigma^2 = \frac{1}{N}\sum x_i^2 \quad \Rightarrow \sum x_i^2 = N\left(\bar{x}^2 + \sigma^2\right).$$

25. C.V. $= \frac{6}{\bar{x}} \times 100.$

Level 2

26. $\sigma_X = 4$, $d = \frac{X - 0}{4}$ Since $\sigma_X = |4|\sigma_d$

So $\sigma_d = \frac{4}{4} = 1.$

27. $\sigma_X = 8$, $d = \frac{x - 0}{-2}$. Since $\sigma_x = |-2|\ \sigma_d$

So $\sigma_d = \frac{8}{2} = 4.$

28. Given that $18 = \frac{1}{n}\sum (x_i + 2)^2$, $10 = \frac{1}{n}(x_i - 2)^2$

Subtracting, $8 = \frac{1}{n}\left(\sum(x_i + 2)^2 - \sum(x_i - 2)^2\right)$

$$= \frac{8}{n}\sum x_i \Rightarrow \bar{x} = 1$$

Also $18 = \frac{1}{n}\sum\left(x_i^2 + 4 + 4x_i\right)$

$$= \frac{1}{n}\sum x_i^2 + 4 + 4\frac{1}{n}\sum x_i$$

$$= \frac{1}{n}\sum x_i^2 + 8$$

So $10 = \frac{1}{n}\sum x_i^2$

$$\sigma_X^2 = \frac{1}{n}\sum x_i^2 - \left(\frac{1}{n}\sum x_i\right)^2 = 10 - 1 = 9$$

Thus $\sigma_X = 3$.

29. $\sigma_{X-Y}^2 = \frac{1}{n}\Sigma\left(x_i - y_i - (\bar{x} - \bar{y})\right)^2$

$$= \frac{1}{n}\Sigma\left((x_i - \bar{x}) - (y_i - \bar{y})\right)^2$$

$$= \frac{1}{n}\Sigma(x_i - \bar{x})^2 + \frac{1}{n}\Sigma(y_i - \bar{y})^2$$

$$- \frac{2}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

$$= \sigma_X^2 + \sigma_Y^2 - \frac{2}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

$$= 7 - 2\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})\right).$$

30. $\bar{x} = [0\cdot q^n + 1\cdot{}^nC_1\, pq^{n-1} + 2\cdot{}^nC_2\, p^2q^{n-2} + \cdots + n\cdot p^n] \times \frac{K}{K[p+q]^n}$

$$= npq^{n-1} + 2.\frac{n(n-1)}{2}\, p^2q^{n-2} + \cdots + np^n$$

$$= np[q^{n-1} + (n-1)\, pq^{n-2} + \frac{(n-1)(n-2)}{2}\, p^2q^{n-3} + \cdots + p^{n-1}]$$

$= np\,[q + p]^{n-1} = np$ (K is constant of proportionality)

31. $$\bar{x} = \left[0.e^{-\lambda} + e^{-\lambda}\lambda + 2.\frac{e^{-\lambda}\lambda^2}{2!} + 3.\frac{e^{-\lambda}\lambda^3}{3!} + \cdots +\right] \frac{K}{K\left[e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \cdots\right)\right]}$$

$$= \lambda e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \cdots\right]\frac{1}{e^{-\lambda}e^{\lambda}}$$

$$= \lambda e^{-\lambda}\, e^{\lambda} = \lambda.$$

32. If $d_i = x_i - A$ then $\sigma_x = \sigma_d$.

33. $\left(x_1^2 + x_2^2\right)\left(x_2^2 + x_3^2\right) \le (x_1\, x_2 + x_2\, x_3)^2$

$$\Rightarrow x_1^2\, x_2^2 + x_2^4 + x_1^2 x_3^2 + x_2^2 + x_3^2$$

$$\le x_1^2\, x_2^2 + x_2^2\, x_3^2 + 2x_1 x_2^2\, x_2$$

$$\Rightarrow x_2^4 + x_1^2\, x_3^2 - 2x_1\, x_2^2\, x_3 \le 0$$

$$\Rightarrow \left(x_2^2 - x_1\, x_3\right)^2 \le 0 \Rightarrow x_2^2 = x_1\, x_3$$

Hence G.M. of x_1, x_2, x_3 is x_2

34. $\frac{1}{10}\sum x_i = 9.5$ and $\frac{1}{n}\sum x_i^2 - \bar{x}^2 = 6.25$

$\Rightarrow \Sigma x_i = 95$ and $\frac{1}{10}\sum x_i^2 = 96.5$

Corrected Mean $= \frac{95 + 15}{11} = 10$

Corrected variance $= \frac{1}{11}(965 + 15^2) - 10^2$

$$= \frac{1190 - 1100}{11} = \frac{90}{11}$$

$$\sigma = \sqrt{\frac{90}{11}} = \frac{30}{\sqrt{110}} = 2.86.$$

35. $\sigma^2 = \frac{n_1\left(\sigma_1^2 + d_1^2\right) + n_2\left(\sigma_2^2 + d_2^2\right)}{n_1 + n_2}$

$d_1 = \bar{x}_1 - \bar{x}$ and $d_2 = \bar{x}_2 - \bar{x}$

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \frac{1}{90}[50 \times 630 + 40 + 540]$$

$$= \frac{10}{9}[315 + 216] = 590$$

$d_1 = 590 - 630 = -40$

$d_2 = 590 - 540 = 50$

So $\sigma^2 = \frac{50(8100 + 1600) + 40(3600 + 2500)}{90}$

$$= \frac{48500 + 24400}{9} = 8100$$

$\sigma = 90$.

36. $\frac{1}{120}\sum x_1 = 20$ so $\Sigma x_i = 120 \times 120$

Corrected sum of the marks

$= 2400 - 50 - 80 + 15 + 18 = 2303$

Corrected mean $= \frac{2303}{120} = 19.19$.

37. $S^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2 \le \frac{1}{n-1}\sum r^2 = \frac{nr^2}{n-1}$,

so $K = \frac{n}{n-1}$.

Previous Years' AIEEE/JEE Main Questions

1. If the average of girls is x, then

$$72 = \frac{70 \times 75 + 30 \times x}{100}$$

$\Rightarrow x = 65$

2. Median refers to middle value so remain unchanged.

3. Corrected $\Sigma x = 170 - 20 + 30 = 180$

Corrected mean $= \frac{180}{15} = 12$

Corrected $\Sigma x^2 = 2830 - 400 + 900 = 3330$

Corrected variance $= \frac{1}{15} \times 3330 - 12^2 = 78.$

4. See theory.

5. $4 = \frac{1}{2n} \cdot 2n\, a^2 - 0 \Rightarrow a^2 = 4 \Rightarrow |a| = 2$

6. $\sigma^2 = \frac{1}{n} \times 400 - \left(\frac{80}{n}\right)^2 \geq 0 \Rightarrow n \geq 16$

So $n = 15$

7. Mode = 3 Median – 2 mean

$= 3 \times 22 - 2 \times 21$

$= 24$

8. Since variables in B can be obtained by adding 50 to variables in A so $V_A = V_B$ i.e. $V_A/V_B = 1$.

9. Let there be x number of boys and y number of girls.

$$50 = \frac{42 \times x + 52y}{x + y}$$

$\Rightarrow 50x + 50y = 42x + 52y$

$\Rightarrow 8x = 2y \Rightarrow y = 4x$

Percentage of boys $= \frac{x}{x+y} \times 100 = \frac{x}{5x} \times 100 = 20\%$

10. $\frac{a + b + 8 + 5 + 10}{5} = 6 \Rightarrow a + b = 7$

$\frac{1}{5}[a^2 + b^2 + 64 + 25 + 100] - 36 = 6.8$

$\Rightarrow a^2 + b^2 = 25$

So $a^2 + (7 - a)^2 = 25 \Rightarrow a = 4, 3$

$a = 4$, $b = 3$ or $a = 3$, $b = 4$

11. Same as Example 23.

12. Combined mean $= \frac{2 \times 5 + 4 \times 5}{5 + 5} = 3$

$d_1 = \bar{x}_1 - \bar{x} = -1$, $d_2 = \bar{x}_2 - \bar{x} = 1$

$$\sigma^2 \text{ (combined variance)} = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}$$

$= \frac{1}{2}[4 + 1 + 5 + 1] = \frac{11}{2}.$

13. Median $= \frac{25a + 26a}{2} = 25.5a$

$50 = \frac{1}{50}[24.5 + 23.5 + \ldots + .5 + .5 + \ldots + 24.5]\,|a|$

$= \frac{1}{25}[0.5 + 1.5 + .. + 24.5]\,|a|$

$= \frac{1}{25} \times \frac{25}{2}[0.5 + 24.5]\,|a|$

$100 = 25|a| \Rightarrow |a| = 4$

14. Correct mean = 32 and $\sigma = 2$ (see properties of mean and standard deviation).

15. Mean of $2x_1, 2x_2, \ldots, 2x_n$ is equal to $2\bar{x}$.

Variance of $2x, 2x_2 \ldots, 2x_n$ is equal to $\frac{\sigma^2}{(1/2)^2} = 4\sigma^2$.

16. Clearly variance (properties of mean, median, variance).

17. Arranging the four numbers, we have $x - y, y, x, 2x + y$. So the median is $\frac{x+y}{2} = 10 \Rightarrow x + y = 20$ and the range is $2x + y - (x - y) = x + 2y = 28$. Solving $x = 12$, $y = 8$.

The mean of the four numbers $= \frac{1}{4}[4x + y] = \frac{1}{4} \times 56 = 14$.

18. Correct mean $= \frac{1}{20}$ [correct sum of observation]

$= \frac{1}{20}[20 \times 40 - 33 + 53]$

$= 41.$

19. Mean of first 50 even natural numbers

$= \frac{1}{50}[2 + 4 + \ldots + 100]$

$= \frac{1}{50} \times \frac{50}{2} \times 102 = 51$

$\sigma^2 = \frac{1}{50}[2^2 + 4^2 + \ldots + 100^2] - 51^2$

$= \frac{4}{50}[1^2 + 2^2 + \ldots + 50^2] - 51^2$

$= \frac{2}{25}\left[\frac{50 \times 51 \times 101}{6}\right] - 51^2$

$= 34 \times 101 - 51^2 = 17(202 - 153) = 833.$

20. There are exactly n observations below the median and exactly n observations above the median. Thus, exactly n observations are increased by 5 and n observations are decreased by 3.

$\therefore$ Change in mean = $\dfrac{5n+(-3n)}{25}$ = 1

Thus, mean increases by 1.

21. Mean will become $\bar{x}+5$, and there will be no change in M.D. (see theory).

22. Both the statements are true (see theory).

23. Let $\bar{x}$ be the mean wages of workers in the night shift, then

$$60 = \frac{70\times 54+30\bar{x}}{70+30}$$

$$\Rightarrow 600 = 7 \times 54 + 3\bar{x}$$

$$\Rightarrow \bar{x} = 200 - 7 \times 18 = 74.$$

24. Required mean

$$= \frac{16\times 16-16+3+4+5}{16-1+3}$$

$$= \frac{256-16+12}{18} = \frac{252}{18} = 14$$

25. $\bar{x} = \dfrac{1}{4}\ (2 + 3 + a + 11) = 4 + \dfrac{1}{4}\ a$

$$\text{Variance} = \frac{1}{4}\ (2^2 + 3^2 + a^2 + 11^2) - \left(4+\frac{1}{4}a\right)^2$$

$$\left(\frac{7}{2}\right)^2 = \frac{1}{4}\ (134 + a^2) - \frac{1}{4}\left(8+\frac{1}{2}a\right)^2$$

$$\Rightarrow 49 = 134 + a^2 - \left(64+\frac{1}{4}a^2+8a\right)$$

$$\Rightarrow \frac{3}{4}\ a^2 - 8a + 21 = 0$$

$$\Rightarrow 3a^2 - 32a + 84 = 0$$

26. Same as Q. 11 above.

27. Let remaining two observations be a, b, where $a \geq b$.

Then

$$5 = \frac{1}{5}\ (1 + 2 + 6 + a + b) \Rightarrow a + b = 16$$

Also, $124 = \dfrac{1}{5}\ (1^2 + 2^2 + 6^2 + a^2 + b^2) - 5^2$

$$\Rightarrow 745 = 41 + a^2 + b^2$$

$$\Rightarrow a^2 + b^2 = 704$$

Now, $(a - b)^2 = 2(a^2 + b^2) - (a + b)^2$

$= 2(704) - 256 = 1152$

$$\Rightarrow a - b = 24\sqrt{2}$$

Also, $a + b = 16$

$$\Rightarrow a = 12\sqrt{2} + 8,\ b = 8 - 12\sqrt{2}$$

Mean deviation from Mean

$= \dfrac{1}{5}\ [|1 - 5| + |2 - 5| + |6 - 5| +$

$|12\sqrt{2} + 8 - 5| + |8 - 12\sqrt{2} - 5|]$

$= \dfrac{1}{5}\ [4 + 3 + 1 + 12\sqrt{2} + 3 + 12\sqrt{2} - 3]$

$= 1.6 + 4.8\sqrt{2}$

Previous Years' B–Architecture Entrance Examination Questions

1. M.D. remains same with change of origin (see theory).

2. The new mean will be $\bar{x}+5$ = 70 and variance will be same i.e. 100 (see theory).

3. Corrected mean.

$$= \frac{1}{100}\ [60 \times 100 - 18 - 28 - 98 + 80 + 26 + 38]$$

= 60 (same as the original mean)

Median will change (see theory).

4. Combined mean = 0, $d_1 = d_2 = d_3 = 0$

$$\sigma^2 = \frac{n_1(\sigma_1^2+d_1^2)+n_2(\sigma_2^2+d_2^2)+n_3\sigma_3^2}{n_1+n_2+n_3}$$

$$= \frac{200\times 9+250\times 16+300\times 25}{750}$$

$$= \frac{1330}{75} = \frac{266}{15}$$

5. Max difference from mean can be 5 so the maximum possible variance is 25.

6. $100 = \dfrac{1}{20}\ \Sigma x_i^2 - 2500 \Rightarrow \Sigma x_i^2 = 52000$

7. $(x_1 + 4) + (x_2 + 8) + \ldots, x_{10} + 40$

$= (x_1 + x_2 \ldots + x_{10}) + (4 + 8 + \ldots + 40)$

$= (x_1 + x_2 \ldots + x_{10}) + 4(1 + 2 + \ldots + 10)$

$= (x_1 + x_2 \ldots + x_{10}) + \dfrac{4\times 10\times 11}{2}$

$= 10 \times 20 + 220 = 420$

So their mean = $\dfrac{420}{10}$ = 42

8. Same as Q.8 in AIEEE [2006] above.

9. Mean of $(x_1 + 1)^2, \ldots, (x_{10} + 1)^2$

$$= \frac{1}{10}\sum_{j=1}^{10}(x_j+1)^2 = \frac{1}{10}\sum_{j=1}^{10}x_j^2 + \frac{2}{10}\sum_{j=1}^{10}x_j + 1$$

$= \sigma^2 + \bar{x}^2 + 2 \times 2 + 1$

$= 9 + 4 + 4 + 1$

$= 18$

CHAPTER TWENTY FOUR

Probability

DEFINITIONS

1. A *random experiment* is an experiment in which (i) all the outcomes of the experiment are known in advance, and (ii) the exact outcome of any specific performance of the experiment is unpredictable, i.e. not known in advance.
2. A *sample space*, denoted by S, associated with a random experiment is a set of points such that (i) each element of S denotes an outcome of the experiment, and (ii) any performance of the experiment results in an outcome that corresponds to exactly one element of S. In other words, sample space consists of all possible outcomes associated with the random experiment.
   The element of the sample space associated with a random experiment are called the *elementary* or *simple* or *indecomposable* events of that experiment.
3. An *event* (associated with a random experiment) is a subset of the sample space S (associated with that experiment). We say that an event E ($\subseteq S$) has occurred provided the outcome ω of the experiment is an element of E. If the outcome ω of the experiment is such that $\omega \notin E$, we say that the event E has not occurred.
   The empty set $\varnothing$, which is a subset of every set, also represents an event and is said to be an *impossible event*. The set S is itself a subset of S and it represents the *sure event*.
4. A set of events is said to be *mutually exclusive* if the occurrence of one of them precludes the occurrence of any of the remaining events. In set-theoretic notation, events $E_1, E_2, \cdots, E_m$ are mutually exclusive if $E_i \cap E_j = \varnothing$ for $i \neq j$ and $1 \leq i, j \leq m$. See Fig. 24.1

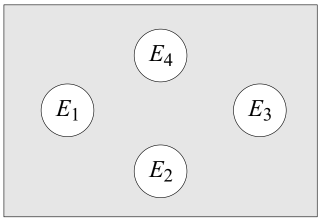


Fig. 24.1

5. A set of events is said to be *equally likely* if, taking into consideration all the relevant factors, there is no reason to expect one of them in preference to the others.
6. A set of events is said to be *exhaustive* if the performance of the experiment always results in the occurrence of at least one of them.

CLASSICAL DEFINITION OF PROBABILITY

If a random experiment results in N mutually exclusive, equally likely and exhaustive outcomes, out of which n are favourable to the occurrence of an event A, then the probability of occurrence of A, usually denoted by $P(A)$, is given by

$$P(A) = \frac{n}{N}$$

Since the number of favourable outcomes can never exceed the total number of outcomes, we have $0 \leq P(A) \leq 1$. Note that $P(\varnothing) = 0$ and $P(S) = 1$.

NOTATIONS

Let A and B be two events. Then

(i) A' or $\bar{A}$ or A^c stands for the non-occurrence or negation of A.
(ii) $A \cup B$ stands for the occurrence of at least one of A and B.
(iii) $A \cap B$ stands for the simultaneous occurrence of A and B.
(iv) $A' \cap B'$ stands for the non-occurrence of both A and B.
(v) $A \subseteq B$ stands for “the occurrence of A implies occurrence of B”.

A FEW THEOREMS ON PROBABILITY

1. If A and B are two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

2. If A is any event, then $P(A') = 1 - P(A)$

3. If A and B are two events, then
$P(A \cap B') = P(A) - P(A \cap B)$

4. If A and B are two events, then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

5. If A and B are two events, then
P(exactly one of A, B occurs)
$= P[(A \cap B') \cup (A' \cap B)]$ [Fig. 24.2]

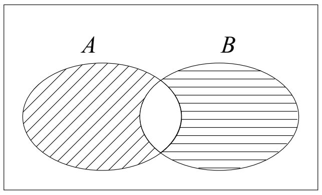


Fig. 24.2

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$
$= P(A) + P(B) - 2P(A \cap B)$
$= P(A \cup B) - P(A \cap B)$

Also, P(exactly one of A, B occurs)
$= P(A \cap B') + P(A' \cap B)$
$= P(B') - P(A' \cap B') + P(A') - P(A' \cap B')$
$= P(A') + P(B') - 2P(A' \cap B')$
$= P(A' \cup B') - P(A' \cap B')$

6. If A and B are two events
$P(A' \cup B') = 1 - P(A \cap B)$ and
$P(A' \cap B') = 1 - P(A \cup B)$

7. If $A_1, A_2, \cdots, A_n$ are n events, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} P(A_1 \cap A_2 \cdots \cap A_n)$$

8. If A, B and C are three events, then

(i) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C) - P(C \cap A) - P(A \cap B) + P(A \cap B \cap C)$

(ii) P (at least two of A, B, C occur)
$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(iii) P (exactly two of A, B, C occur)
$= P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

(iv) P(exactly one of A, B, C occurs)
$= P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

9. If $A_1, A_2, \cdots, A_n$ are n events, then

(i) $P(A_1 \cup A_2 \cup \cdots \cup A_n) \le P(A_1) + P(A_2) + \cdots + P(A_n)$

(ii) $P(A_1 \cap A_2 \cap \ldots \cap A_n) \ge 1 - P(A'_1) - P(A'_2) - \ldots - P(A'_n)$

10. If $A_1, A_2, \cdots, A_n$ are n events, then
$P(A_1 \cap A_2 \cap \ldots \cap A_n) \ge P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1)$

11. If A and B are two events such that $A \subseteq B$, then $P(A) \le P(B)$.

Odds in Favour and against an Event If x is the number of cases favourable to the occurrence of an event A and y that for the event A', then the odds in favour of A are $x : y$ and the odds against A are $y : x$. In this case

$$P(A) = \frac{x}{x+y} \quad \text{and} \quad P(A') = \frac{y}{x+y}$$

CONDITIONAL PROBABILITY

The probability of occurrence of an event A, given that B has already occurred is called the conditional probability of occurrence of A. It is denoted by $P(A \mid B)$.

If the event B has already occurred, then the sample space reduces to B. Now the outcomes favourable to the occurrence of A (given that B has already occurred) are those that are common to both A and B, that is, those which belong to $A \cap B$.

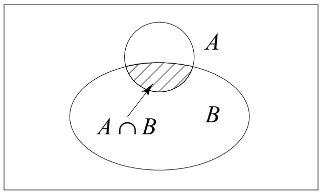


Fig. 24.3

Thus,

$$P(A \mid B) = \frac{N_{A \cap B}}{N_B}$$

where $N_{A \cap B}$ is the number of elements in $A \cap B$ and $N_B \ne 0$ is the number of elements in B and N the total number of elements in S.

$$\Rightarrow \quad P(A \mid B) = \frac{N_{A \cap B}/N}{N_B/N} = \frac{P(A \cap B)}{P(B)}$$

Hence

$$P(A \cap B) = \begin{cases} P(B)\,P(A|B) & \text{if } P(B) \ne 0 \\ P(A)\,P(B|A) & \text{if } P(A) \ne 0 \end{cases}$$

Important Note

The function $P(\mid A)$ is a probability function and hence all theorems on probability apply to this function.
For instance,
$P(B \cup C \mid A) = P(B \mid A) + P(C \mid A) - P(B \cap C \mid A)$.

MULTIPLICATION THEOREM

Let $A_1\ A_2, \cdots, A_n, A_{n+1}$ be $(n+1)$ events such that $P(A_1 \cap A_2 \cap \cdots \cap A_n) > 0$.

Then

$$P\left(\bigcap_{j=1}^{n+1} A_j\right) = P(A_1)\ P(A_2 \mid A_1)\ P(A_3 \mid A_1 \cap A_2) \cdots P(A_{n+1} \mid A_1 \cap \cdots \cap A_n)$$

TOTAL PROBABILITY THEOREM

Let $\{H_i\}$, $i = 1, 2, \cdots, n$, be n events such that $H_i \cap H_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{n} H_i = S$. Suppose that $P(H_i) > 0$ for $1 \le i \le n$. Then for any event A

$$P(A) = \sum_{i=1}^{n} P(H_i)\ P(A \mid H_i)$$

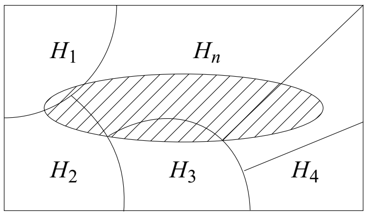


Fig. 24.4

BAYES' RULE

Let S be a sample space and $H_1, \cdots, H_n$ be n mutually exclusive events such that $\bigcup_{j=1}^{n} H_j = S$ and $P(H_j) > 0$ for $j = 1, 2, \cdots, n$. We can think of the H_j's as the 'causes' (or the hypotheses) that lead to the outcome of an experiment. The probabilities $P(H_j), j = 1, 2, \cdots, n$ are called *prior probabilities*. Suppose the experiment results in an outcome of event A, where $P(A) > 0$. What is the probability that the observed event was due to cause H_j? In other words, we seek the conditional probability $P(H_j \mid A)$. These probabilities are frequently called *posterior probabilities*. The information that A has occurred makes us reassess the probability $P(H_j)$ assigned to H_j.

THEOREM (BAYES' RULE)

Let $\{H_j\}$ be mutually exclusive events such that $P(H_j) > 0$ for $j = 1, 2, \cdots, n$ and $S = \bigcup_{j=1}^{n} H_j$. Let A be an event with $P(A) > 0$. Then for $j = 1, 2, \cdots, n$

$$P(H_j \mid A) = \frac{P(H_j)\, P(A \mid H_j)}{\sum_{k=1}^{n} P(H_k)\, P(A \mid H_k)}$$

INDEPENDENT EVENTS

We say that two events A and B are *independent* if the occurrence or non-occurrence of $A(B)$ does not affect the probability of occurrence or non-occurrence of $B(A)$, that is, if $P(B \mid A) = P(B)$ provided $P(A) \neq 0$

$$\Rightarrow \frac{P(B \cap A)}{P(A)} = P(B) \quad \Rightarrow \quad P(A \cap B) = P(A)\ P(B)$$

Thus, two events A and B are independent if and only if $P(A \cap B) = P(A)\ P(B)$.

If two events are not independent, they are said to be *dependent*.

We note that if $P(A) = 0$ then for any event B, $0 \le P(A \cap B) \le P(A) = 0$, so that $P(A \cap B) = P(A)\ P(B)$ and A is independent of any other event.

Difference between Mutually Exclusiveness and Independence

Important Note

We advise the reader to distinguish between independent and mutually exclusive events. Independence is a property of probability, whereas mutual exclusion is a set-theoretic property. If A and B are mutually exclusive events with $P(A) > 0$ and $P(B) > 0$, then $P(A \cap B) = 0 \neq P(A)\ P(B)$, so that A and B cannot be independent. In fact, $P(A \mid B) = 0 = P(B \mid A)$, so that if A occurs, B cannot occur and vice-versa. Thus, A and B are strongly dependent when $A \cap B = \emptyset$.

We also note the following fact; two events A and B are independent if and only if

(i) A and B' are independent, or
(ii) A' and B are independent, or
(iii) A' and B' are independent.

Three events A, B and C are said to be *pairwise independent* if

$$P(A \cap B) = P(A)\ P(B),\ P(B \cap C) = P(B)\ P(C)$$

and $P(C \cap A) = P(C)P(A)$

Three events A, B and C are said to be *mutually independent* if

$$P(A \cap B) = P(A)\ P(B),\ P(B \cap C) = P(B)\ P(C),$$
$$P(C \cap A) = P(C)\ P(A)$$

and $P(A \cap B \cap C) = P(A)\ P(B)\ P(C)$

RANDOM VARIABLES

Let S be a sample space. A *random variable* X is a function from the set S to **R**, the set of real numbers. If X is a random variable defined on the sample space S and r is a real number, then $\{X = r\} = \{w \in S \mid X(w) = r\}$ is an event. If the random variable X takes n distinct values $x_1, x_2, \cdots, x_n$, then

$\{X = x_1\}, \{X = x_2\}, \cdots, \{X = x_n\}$ are mutually exclusive and exhaustive events (Fig. 24.5).

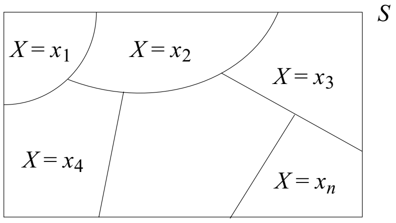


Fig. 24.5

Now, since $(X = x_i)$ is an event, we can talk of $P(X = x_i)$. If $P(X = x_i) = p_i$ $(1 \le i \le n)$, then the system of numbers

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$$

is said to be the *probability distribution* of the random variable X.

Mean and Variance of a Random Variable

The *expectation* (mean) of the random variable X is defined as

$$E(X) = \sum_{i=1}^{n} p_i x_i$$

and the *variance* of X is defined as

$$\text{var}(X) = \sum_{i=1}^{n} p_i (x_i - E(X))^2 = \sum_{i=1}^{n} p_i x_i^2 - (E(X))^2$$

BINOMIAL DISTRIBUTIONS

Consider a random experiment with sample space S and an event E (a subset of S) associated with it. Then the event "not E" may be denoted by E', the complement (in S) of the subset E. Let $P(E) = p$ and $P(E') = q$, so that $p, q > 0$ and $p + q = 1$.

If the experiment results in the event E, we say a success, denoted by S, has occurred. If on the other hand, the event E does not occur (i.e., E' occurs) the experiment is said to have resulted in a failure, denoted by F. The probability of a success is equal to p and that of a failure is $q = 1 - p$.

Suppose the above experiment is carried out n times under identical conditions. If the success (occurrence of E) and failure (non-occurrence of E) are recorded successively as the experiment is repeated, we will get a result of the type

$$S\,F\,F\,S\,S\,F\,F \cdots F\,S\,F$$

There are 2^n such outcomes, and these constitute the sample space of the combined experiment. Since the successive experiments are independent, the probability of the outcome above is

$$p\,q\,q\,p\,p\,q\,q \cdots q\,p\,q = p^r\,q^{n-r}$$

where r is the number of successes in the outcome. If X denotes the random variable "number of successes", then the possible values of X are $0, 1, 2, \cdots, n$.

We shall now calculate the probability of the event $(X = r)$. Out of the 2^n outcomes, r successes and $(n - r)$ failures can occur in ${}^nC_r$ ways. Also, the probability of r successes and $(n - r)$ failures is $p^r\,q^{n-r}$. Thus, $P(X = r) = {}^nC_r\,p^r\,q^{n-r}$. The probability distribution of the random variable X is given by

r	0	1	2	$\cdots$	r	$\cdots$	n
$P(x = r)$	${}^nC_0q^n$	${}^nC_1pq^{n-1}$	${}^nC_2p^2q^{n-2}$	$\cdots$	${}^nC_rp^rq^{n-r}$	$\cdots$	${}^nC_np^n$

Note that the probabilities that the random variable takes the values $0, 1, 2, \cdots, n$ are given by the terms in the binomial expansion of $(q + p)^n$. Because of this, the probability distribution of the random variable X is said to be a *binomial distribution* and X is said to be a *binomial random variable*.

A *binomial distribution* $B(n, p)$ is the probability distribution of a random variable X which takes values $0, 1, 2, \cdots, n$ with probabilities

$${}^nC_0p^0q^n, {}^nC_1pq^{n-1}, \cdots, {}^nC_rp^rq^{n-r}, \cdots, {}^nC_np^nq^0$$

where $p, q > 0$ and $p + q = 1$. The numbers n and p are said to be *parameters* of the binomial distribution $B(n, p)$. We usually write, $X \sim B(n, p)$.

Mode of Binomial Distribution

For the value of r so that $P(X = r)$ is maximum, we have two cases.

Case 1: $(n + 1)p$ is not an integer then, $P(X = r)$ is maximum when $r = m$, where m is the greatest integer $\le (n + 1)p$, that is, $m = [(n + 1)p]$.

Case 2: $(n + 1)p$ is an integer then, $P(X = r)$ is maximum when $r = m - 1$ or m, where

$(n + 1)\,p = m$ and m is an integer.

If $X \sim B(n, p)$, then $E(X) = np$ and $\text{var}(X) = npq$.

SOLVED EXAMPLES

Concept-based Straight Objective Type Questions

Example 1: If 15 boys of different ages are distributed into 3 groups of 4, 5, and 6 boys randomly then the probability that three youngest boys are in different groups is

(a) $\frac{24}{91}$ (b) $\frac{71}{91}$

(c) $\frac{67}{91}$ (d) $\frac{20}{91}$

Ans. (a)

Solution: Total number of ways of distributing 15 boys in three groups is

$$= ({}^{15}C_4)\,({}^{11}C_5)\,({}^{6}C_6) = \frac{15!}{4!\,5!\,6!}$$

The number of ways of distributing 12 remaining boys (excluding three youngest) is $({}^{12}C_3)\,({}^{9}C_4)\,({}^{5}C_5) = \frac{12!}{3!\,4!\,5!}$

and the number of ways of distributing three youngest boys is ${}^3P_3 = 3!$

∴ the required probability

$$= \frac{3!\,(12!)}{3!\,4!\,5!} \times \frac{4!\,5!\,6!}{15!} = \frac{24}{91}$$

Example 2: A bag contains 6 white and 6 black socks. A man randomly takes out two socks from the bag. The probability that socks are the same colour is:

(a) $\frac{2}{11}$ (b) $\frac{5}{11}$

(c) $\frac{7}{11}$ (d) $\frac{4}{11}$

Ans. (b)

Solution: Total number of ways of taking out two socks is ${}^{12}C_2 = 66$.

The number of favourable ways

= Number of ways both the socks are white or both are black

$= {}^6C_2 + {}^6C_2 = 30.$

∴ probability of required event

$$= \frac{30}{66} = \frac{5}{11}$$

Example 3: Two numbers are randomly selected from the first 100 natural numbers. The probability that the product of the numbers is divisible by 7 is

(a) 0 (b) $\frac{1}{14}$

(c) $\frac{4859}{4950}$ (d) $\frac{91}{4950}$

Ans. (c)

Solution: Product of two natural numbers is divisible by 7 if and only if at least one of the number is divisible by 7.

∴ required probability

= 1 – *P*(none of the numbers is divisible by 7)

$$= 1 - \frac{{}^{14}C_2}{{}^{100}C_2} = \frac{4859}{4950}$$

Example 4: Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b\}$. A function $f : A \to B$ is selected randomly. Probability that function is an onto function is

(a) $\frac{1}{8}$ (b) $\frac{5}{8}$

(c) $\frac{7}{8}$ (d) $\frac{3}{8}$

Ans. (c)

Solution: Total number of functions from A to B is $2^4 = 16$. There are exactly two functions which are not onto viz. one in which all the elements of A are mapped to a and other in which all the elements of A are mapped to b. Thus, there are 14 onto function. Therefore, probability of required event is

$$\frac{14}{16} = \frac{7}{8}.$$

Example 5: Sets A, B, C, $A \cap B$, $A \cap C$, $B \cap C$ and $A \cap B \cap C$ have 35, 40, 45, 13, 12, 14 and 5 elements respectively. An element is selected at random from the set $A \cup B \cup C$. The probability that the selected element belongs to only set A is:

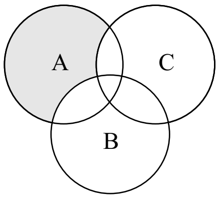
(a) $\frac{13}{86}$ (b) $\frac{35}{86}$

(c) $\frac{5}{86}$ (d) $\frac{15}{86}$

Ans. (d)

Solution: By the principle of inclusion and exclusion,

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(B \cap C)$$
$$- n(A \cap C) - n(A \cap B) + n(A \cap B \cap C)$$
$$= 35 + 40 + 45 - 13 - 12 - 14 + 5 = 86$$



Also,

$$n(A \cap B' \cap C') = n(A) - n[(A \cap B) \cup (A \cap C)]$$
$$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$$
$$= 35 - [13 + 12 - 5] = 15$$

$\therefore$ Probability of required event

$$= \frac{15}{86}$$

Example 6: If $P(A) = 0.4$, and $P(A \cap B) = 0.15$, then $P(A|A' \cup B')$ is equal to

(a) $\frac{1}{17}$ (b) $\frac{2}{17}$

(c) $\frac{5}{17}$ (d) $\frac{9}{17}$

Ans. (c)

Solution: $P(A|A' \cup B') = \dfrac{P(A \cap (A' \cup B'))}{P(A' \cup B')}$

$$= \frac{P((A \cap A') \cup (A \cap B'))}{P(A' \cup B')}$$

$$= \frac{P(\phi \cup (A \cap B'))}{P((A \cap B)')} \text{ [De Margon's Law]}$$

$$= \frac{P(A \cap B')}{1 - P(A \cap B)} = \frac{P(A) - P(A \cap B)}{1 - P(A \cap B)}$$

$$= \frac{0.4 - 0.15}{1 - 0.15} = \frac{0.25}{0.85} = \frac{5}{17}$$

Example 7: A class consists of 80 students, 25 of them are girls. If 10 of the students are rich and 20 of the students are fair complexioned, then the probability of selecting a fair complexioned rich girl from the class (assuming three traits as independent) is

(a) $\frac{1}{10}$ (b) $\frac{1}{32}$

(c) $\frac{5}{512}$ (d) $\frac{7}{512}$

Ans. (c)

Solution: Let G, R and F denote the following events

G: a girl student is selected

R: a rich student is selected

F: a fair complexioned student is selected.

We have

$$P(G) = \frac{25}{80},\ P(R) = \frac{10}{80} \text{ and } P(F) = \frac{20}{80}$$

As G, R and F are independent events,

$$P(G \cap R \cap F) = P(G)\, P(R)\, P(F)$$
$$= \left(\frac{25}{80}\right)\left(\frac{10}{80}\right)\left(\frac{20}{80}\right) = \frac{5}{512}$$

Example 8: Suppose A and B are two mutually exclusive events such that $P(A) = \frac{1}{3}(3p + 1)$ and $P(B) = \frac{1}{4}(1 - p)$, then the set of possible values of p lies in the interval:

(a) $[0, 1]$ (b) $\left[\frac{1}{3}, \frac{2}{9}\right]$

(c) $\left[-\frac{1}{3}, \frac{5}{9}\right]$ (d) $\left[-\frac{7}{9}, \frac{4}{9}\right]$

Ans. (c)

Solution: $\frac{1}{3}(3p + 1) \geq 0$, $\frac{1}{4}(1 - p) \geq 0$

and $\frac{1}{3}(3p + 1) + \frac{1}{4}(1 - p) \leq 1$.

$$\Rightarrow p \geq -\frac{1}{3},\ p \leq 1,\ p \leq \frac{5}{9} \Rightarrow p \in \left[-\frac{1}{3}, \frac{5}{9}\right]$$

Example 9: A number x is chosen at random from the set $\{1, 2, 3, \ldots, 100\}$. Define event : A = the chosen number x satisfies $\dfrac{x - 20}{x - 40} \geq 2$. Then $P(A)$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$

(c) $\frac{1}{8}$ (d) $\frac{1}{10}$

Ans. (b)

Solution: $\dfrac{x - 20}{x - 40} \geq 2 \Leftrightarrow \dfrac{x - 20}{x - 40} - 2 \geq 0$

$$\Leftrightarrow \frac{60 - x}{x - 40} \geq 0 \Leftrightarrow \frac{x - 60}{x - 40} \leq 0$$
$$\Leftrightarrow 41 \leq x \leq 60$$

$\therefore$ A contains 20 elements.

Thus, $P(A) = \dfrac{20}{100} = \dfrac{1}{5}$

Example 10: A number x is chosen at random from the set $S = \{1, 2, 3, \ldots, 100\}$.Then the probability that the expression $\sqrt{\dfrac{x - 15}{(x - 10)(x - 20)}}$ is a positive real number is

(a) 11/25 (b) 7/25

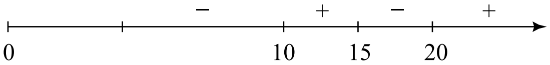
(c) 21/25 (d) 17/25

Ans. (c)

Solution: $\sqrt{\dfrac{x-15}{(x-10)(x-20)}}$ is a positive real number

$$\Leftrightarrow \qquad y = \frac{x-15}{(x-10)(x-20)} > 0$$

Following figure shows sign of y in different intervals:



Thus, $y > 0 \Leftrightarrow 11 \le x \le 14$ or $x \ge 21$

Therefore, there are 84 favourable cases.

Hence, probability of required event is $\dfrac{84}{100} = \dfrac{21}{25}$.

LEVEL 1

Straight Objective Type Questions

Example 11: Suppose A and B are two events such that $0 < P(B) < 1$. If

$$P(A \mid B') = \frac{P(A)}{1-P(B)}$$

then which one of the following is **not** true?

(a) $P(A \cap B) = 0$

(b) $P(A \cap B) = P(A \cup B)$

(c) $P(A \cup B) = P(A) + P(B)$

(d) $P(A \mid B) = 0$

Ans. (b)

Solution: $\dfrac{P(A)}{1-P(B)} = P(A|B') = \dfrac{P(A \cap B')}{P(B')}$

$$\Rightarrow \qquad P(A) = P(A \cap B') = P(A) - P(A \cap B)$$

$$\Rightarrow \qquad P(A \cap B) = 0,$$

Also, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= P(A) + P(B)$$

and $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = 0.$

Example 12: A man is known to speak truth 3 out of 4 times. He takes out a card at random from a well shuffled pack of 52 playing cards, and reports it is a king. The probability that its actually a king is

(a) $\dfrac{1}{4}$ (b) $\dfrac{3}{4}$

(c) $\dfrac{4}{5}$ (d) $\dfrac{1}{5}$

Ans. (d)

Solution: E_1 : card drawn is a king

E_2 : card drawn is not a king

A : man reports it is a king.

We have

$$P(E_1) = \frac{4}{52} = \frac{1}{13},\; P(E_2) = \frac{12}{13}$$

$$P(A|E_1) = \frac{3}{4} \text{ and } P(A \mid E_2) = \frac{1}{4}$$

By the Bayes' rule

$$P(E_1|A) = \frac{P(E_1)P(A \mid E_1)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)}$$

$$= \frac{\left(\frac{1}{13}\right)\left(\frac{3}{4}\right)}{\left(\frac{1}{13}\right)\left(\frac{3}{4}\right) + \left(\frac{12}{13}\right)\left(\frac{1}{4}\right)} = \frac{1}{5}$$

Example 13: Suppose A has 7 fair coins and B has 6 fair coins. If A and B toss these coins simultaneously, then the probability that A and B get equal number of heads is

(a) ${}^{13}C_6\left(\dfrac{1}{2}\right)^{13}$ (b) ${}^{13}C_5\left(\dfrac{1}{2}\right)^{13}$

(c) ${}^{13}C_{12}\left(\dfrac{1}{2}\right)^{13}$ (d) $\left(\dfrac{1}{2}\right)^{13}$

Ans. (a)

Solution: Let X = number of heads obtained by A.

and Y = number of heads obtained by B.

Note that both X and Y follow binomial distribution and $X \sim B$ (7, 1/2) and $Y \sim B$ (6, 1/2)

$$\text{Now, } P(X = Y) = \sum_{k=0}^{6} P(x = k,\, y = k)$$

$$= \sum_{k=0}^{6} \left({}^7C_k\right)\left(\frac{1}{2}\right)^7 \left({}^6C_k\right)\left(\frac{1}{2}\right)^6$$

$$= \frac{1}{2^{13}} \sum_{k=0}^{6} \left({}^7C_k\right)\left({}^6C_{6-k}\right)$$

But $\displaystyle\sum_{k=0}^{6} \left({}^7C_k\right)\left({}^6C_{6-k}\right)$

= Number of ways of choosing 6 persons out of 7 men and 6 women

$= {}^{13}C_6$.

Thus, $P(X = Y) = {}^{13}C_6\left(\dfrac{1}{2^{13}}\right)$

Example 14: A box X contains 1 white ball, 3 red balls and 2 black balls. Another box Y contains 2 white balls, 3 red balls and 4 black balls. If one ball is drawn from each of the two boxes, then the probability the balls are of different colours is

(a) $\frac{19}{54}$ (b) $\frac{35}{54}$

(c) $\frac{17}{54}$ (d) $\frac{37}{54}$

Ans. (b)

Solution:

$$\begin{aligned} P(\text{Balls are of different colours}) &= 1 - P(\text{Balls are of the same colour}) \\ &= 1 - [P(W_1\, W_2) + P(R_1\, R_2) + P(B_1\, B_2)] \\ &= 1 - \left[\left(\frac{1}{6}\right)\left(\frac{2}{9}\right) + \left(\frac{3}{6}\right)\left(\frac{3}{9}\right) + \left(\frac{2}{6}\right)\left(\frac{4}{9}\right)\right] \\ &= 1 - \frac{19}{54} = \frac{35}{54} \end{aligned}$$

Example 15: A sample consists of six points

$S = \{(i, j) |\ i = 1, 2\ , j = 1, 2, 3\}$. Suppose

$P\,((i, j)) = \dfrac{k}{i + j}$. Let $A = \{(i, j) \mid i + j = 4\}$

and $B = \{(i, j) \mid j = 2\}$, then $P(B|A)$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{1}{6}$

Ans. (a)

Solution:

TIP

It is unnecessary to calculate k.

We have

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(2,2)}{P\{(1,3), (2,2)\}}$$

$$= \frac{\frac{k}{4}}{\frac{k}{4} + \frac{k}{4}} = \frac{1}{2}.$$

Example 16: Let A be a set containing n elements. The probability of randomly choosing two subsets P and Q of A so that $P = Q'$, the complement of Q, is.

(a) $\frac{1}{2^n}$ (b) $\frac{1}{2^n - 1}$

(c) $\frac{1}{2^{n-1}}$ (d) $\frac{2}{2^n - 1}$

Ans. (b)

Solution: Number of subsets of A is $2^n = m$ (say). The number of ways of choosing P and Q is ${}^mC_2 = \frac{1}{2} m\,(m - 1)$ $= 2^{n-1}\,(2^n - 1)$.

For $P = Q'$, we just have to choose Q and put the remaining elements of A in P. But the number of ways of choosing P and Q is 2^n. But every selection of P and Q appears twice. Therefore, the number of choosing P and Q is $\frac{1}{2}(2^n) = 2^{n-1}$.

Thus, probability of the required event is

$$\frac{2^{n-1}}{2^{n-1}(2^n - 1)} = \frac{1}{2^n - 1}.$$

Example 17: Four natural numbers are selected at random and are multiplied. The probability that the product is divisible by 5 or 10 is

(a) $\frac{49}{625}$ (b) $\frac{369}{625}$

(c) $\frac{64}{625}$ (d) $\frac{256}{625}$

Ans. (b)

Solution: The product will be divisible by 5 or 10 if at least one of the number has last digit as 0 or 5. Thus, required probability

$= 1 - P$(none of the number has last digit 0 or 5.)

$$= 1 - \left(\frac{8}{10}\right)^4 = 1 - \left(\frac{4}{5}\right)^4 = \frac{369}{625}$$

Example 18: Six boys and six girls sit in a row randomly. The probability that boys and girls sit alternatively is

(a) $\frac{1}{231}$ (b) $\frac{5}{462}$

(c) $\frac{1}{462}$ (d) $\frac{7}{101}$

Ans. (c)

Solution: Total number of ways in which six boys and six girls can sit in a row is ${}^{12}P_{12} = 12!$. The first position can be occupied in two ways either by a boy or by a girl.

BG BG BG BG BG BG

or

GB GB GB GB GB GB

Six boys can take their seats in ${}^6P_6 = 6!$ ways and six girls can take their seats ${}^6P_6 = 6!$ ways. Thus, the number of favourable ways = 2(6!) (6!)

$\therefore$ probability of the required event $= \dfrac{2(6!)\,(6!)}{12!} = \dfrac{1}{462}$.

Example 19: An unbiased die is rolled twice. Let A denote the event that an even number appears on the first throw and B denote the event that an odd number appears on the second throw. Then A and B

(a) are mutually exclusive

(b) are independent and mutually exclusive
(c) are independent
(d) none of these

Ans. (c)

◎ **Solution:** The sample space is given by

$S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$

We have

$A = \{21, 22, 23, 24, 25, 26, 41, 42, 43, 44, 45, 46, 61, 62, 63, 64, 65, 66\}$

$B = \{11, 13, 15, 21, 23, 25, 31, 33, 35, 41, 43, 45, 51, 53, 55, 61, 63, 65\}$

Note that $A \cap B = \{21, 23, 25, 41, 43, 45, 61, 63, 65\}$

As $A \cap B \neq \phi$, A and B cannot be mutually exclusive.

Next, $P(A) = \frac{18}{36} = \frac{1}{2}$, $P(B) = \frac{18}{36} = \frac{1}{2}$,

$$P(A \cap B) = \frac{9}{36} = \frac{1}{4} = P(A)\,P(B)$$

∴ A and B are independent.

⦿ **Example 20:** Two events A and B have probabilities 0.25 and 0.50 respectively. The probability that both A and B occur simultaneously is 0.14. The probability that neither A nor B occurs is

(a) 0.39 (b) 0.25
(c) 0.11 (d) none of these

Ans. (a)

◎ **Solution:** We are given $P(A) = 0.25$, $P(B) = 0.50$, $P(A \cap B) = 0.14$

Now, P (neither A nor B)

$$= P(A' \cap B') = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 1 - [0.25 + 0.50 - 0.14]$$
$$= 1 - 0.61 = 0.39.$$

⦿ **Example 21:** An anti-aircraft gun can take a maximum of four shots at an enemy plane moving away from it. The probabilities of hitting the plane in the first, second, third and fourth shot are 0.4, 0.3, 0.2 and 0.1 respectively. The probability that gun hits the plane is

(a) 0.6976 (b) 0.866
(c) 0.922 (d) 0.934

Ans. (a)

◎ **Solution:** Let E_i denote the event that ith shot hits the plane. Then

$$P(E_1) = 0.4,\ P(E_2) = 0.3,\ P(E_3) = 0.2,\ P(E_4) = 0.1$$

We have

$$P(E_1 \cup E_2 \cup E_3 \cup E_4) = 1 - P(E'_1 \cap E'_2 \cap E'_3 \cap E'_4)$$
$$= 1 - P(E'_1)\,P(E'_2)\,P(E'_3)\,P(E'_4)$$
$$[\because E_1, E_2, E_3, E_4 \text{ are independent}]$$
$$= 1 - (0.6)(0.5)(0.8)(0.9)$$
$$= 1 - 0.3024 = 0.6976.$$

⦿ **Example 22:** A and B are two candidates seeking admission in IIT. The probability that A is selected is 0.5 and the probability that both A and B are selected is at most 0.3. The probability of B getting selected cannot exceed

(a) 0.6 (b) 0.7
(c) 0.8 (d) 0.9

Ans. (c)

◎ **Solution:** We are given $P(A) = 0.5$ and $P(A \cap B) \leq 0.3$

We have $P(A) + P(B) - P(A \cap B) = P(A \cup B) \leq 1$

$\Rightarrow$ $P(A) + P(B) \leq 1 + P(A \cap B) \leq 1 + 0.3 = 1.3$

$\Rightarrow$ $0.5 + P(B) \leq 1.3$

$\Rightarrow$ $P(B) \leq 0.8$

Thus, probability of B getting selected cannot exceed 0.8

⦿ **Example 23:** If letters of the word "ASSASSIN" are written down at random in a row, the probability that no two S's occur together is

(a) $\frac{1}{7}$ (b) $\frac{1}{14}$
(c) $\frac{1}{28}$ (d) $\frac{1}{35}$

Ans. (b)

◎ **Solution:** The number of ways of permuting the letters of the word "ASSASSIN" is

$$\frac{8!}{2!4!} = \frac{8 \times 7 \times 6 \times 5}{2} = 840$$

Let us now find the number of ways in which two S are not together.

We first arrange A, A, I, N. This can be done $\frac{4!}{2!} = 12$ ways.

If in the following figure X's shows one such arrangement, then 4 S's can be arranged at four boxes out of the five boxes.

□×□×□×□×□

Thus, corresponding to each arrangement of A, A, I, N, there are ${}^5C_4 = 5$ ways of arranging S's.

∴ the number of favourable ways = (12) (5) = 60

Thus, probability of the required event is $\frac{60}{840} = \frac{1}{14}$.

⦿ **Example 24:** Fifteen coupons are numbered 1, 2, …, 15 respectively. Three coupons are selected at random without replacement. The probability that maximum number on the selected coupon is 9 is

(a) $\frac{1}{65}$ (b) $\frac{3}{65}$

(c) $\dfrac{1}{13}$ (d) $\dfrac{4}{65}$

Ans. (d)

◎ **Solution:** Total number of ways of selecting 3 coupons out of 15 is ${}^{15}C_3$.

Maximum number on the selected ticket will be 9 if coupon bearing number 9 is selected and two coupons from 1 to 8 are selected. This can be done in $({}^{8}C_2)\,({}^{1}C_1)$ ways.

Thus, probability of the required event is

$$\frac{{}^{8}C_2}{{}^{15}C_3} = \frac{8\times 7}{2}\times\frac{3\times 2}{15\times 14\times 13} = \frac{4}{65}.$$

◉ **Example 25:** A student appears for tests I, II, and III. The student is successful if he passes either in tests I and II or tests I and III. The probabilities of the student passing in tests I, II, III are p, q and 1/2, respectively. If the probability that the student is successful is 1/2, then possible values of p and q are

(a) $p = q = 1$ (b) $p = q = 1/2$

(c) $p = 1, q = 0$ (d) $p = 1, q = 1/2$

Ans. (c)

◎ **Solution:** Let A, B and C be the events that the student is successful in tests I, II and III, respectively. Then

$$\begin{aligned}
&P(\text{the student is successful})\\
&= P[(A\cap B\cap C')\cup(A\cap B'\cap C)\cup(A\cap B\cap C)]\\
&= P(A\cap B\cap C') + P(A\cap B'\cap C) + P(A\cap B\cap C)\\
&= P(A)\,P(B)\,P(C') + P(A)\,P(B')\,P(C)\\
&\quad + P(A)\,P(B)\,P(C)\\
&\qquad\qquad [\because A, B \text{ and } C \text{ are independent}]\\
&= pq\,(1-1/2) + p\,(1-q)\,(1/2) + (pq)\,(1/2)\\
&= \frac{1}{2}\,[pq + p(1-q) + pq] = \frac{1}{2}\,p(1+q)
\end{aligned}$$

$$\therefore\quad \frac{1}{2} = \frac{1}{2}\,p\,(1+q) \quad\Rightarrow\quad p\,(1+q) = 1$$

This equation is satisfied for the pair of values given in (c).

◉ **Example 26:** A bag contains a white and b black balls. Two players, A and B alternately draw a ball from the bag, replacing the ball each time after the draw till one of them draws a white ball and wins the game. A begins the game. If the probability of A winning the game is three times that of B, the ratio $a : b$ is

(a) 1 : 1 (b) 1 : 2

(c) 2 : 1 (d) none of these

Ans. (c)

◎ **Solution:** Let W denote the event of drawing a white ball at any draw and B that for a black ball. Then

$$P(W) = \frac{a}{a+b} \text{ and } P(B) = \frac{b}{a+b}$$

$$\begin{aligned}
P(A \text{ wins the game}) &= P(W \text{ or } BBW \text{ or } BBBBW \text{ or } \cdots)\\
&= P(W) + P(BBW) + P(BBBBW) + \cdots\\
&= P(W) + P(B)\,P(B)\,P(W)\\
&\qquad + P(B)\,P(B)\,P(B)\,P(B)\,P(W) + \cdots\\
&= P(W) + P(W)\cdot P(B)^2 + P(W)\cdot P(B)^4 + \cdots\\
&= \frac{P(W)}{1-P(B)^2} = \frac{a(a+b)}{a^2+2ab} = \frac{a+b}{a+2b}
\end{aligned}$$

Also $P(B \text{ wins the game}) = 1 - \dfrac{a+b}{a+2b} = \dfrac{b}{a+2b}$

According to the given condition,

$$\Rightarrow\quad \frac{a+b}{a+2b} = 3\cdot\frac{b}{a+2b} \quad\Rightarrow\quad a = 2b \Rightarrow a : b = 2 : 1.$$

◉ **Example 27:** A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only. The probability that the determinant chosen is non-zero is

(a) $\dfrac{3}{16}$ (b) $\dfrac{3}{8}$

(c) $\dfrac{1}{4}$ (d) $\dfrac{5}{16}$

Ans. (b)

◎ **Solution:** A determinant of order 2 is of the form

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

It is equal to $ad - bc$. The total number of ways of choosing a, b, c and d is $2\times 2\times 2\times 2 = 16$. Now $\Delta \neq 0$ if and only if either $ad = 1$, $bc = 0$ or $ad = 0$, $bc = 1$. But $ad = 1$, $bc = 0$ if and only if $a = d = 1$ and at least one of b, c is zero. Thus, $ad = 1$, $bc = 0$ in three cases. Similarly, $ad = 0$, $bc = 1$ in three cases. Thus, the probability of the required event is $6/16 = 3/8$.

◉ **Example 28:** A fair coin is tossed 100 times. The probability of getting tails an odd number of times is

(a) 1/2 (b) 1/8

(c) 3/8 (d) none of these

Ans. (a)

◎ **Solution:** Let p = probability of getting a tail in a single trial = 1/2.

n = number of trials = 100

and X = number of tails in 100 trials.

Note that $q = 1/2$

We have

$$\begin{aligned}
P(X = r) &= {}^{100}C_r\, p^r\, q^{\,n-r}\\
&= {}^{100}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{100-r} = {}^{100}C_r \left(\frac{1}{2}\right)^{100}
\end{aligned}$$

Now,

$$P(X = \text{odd}) = P(X = 1) + P(X = 3) + \dots + P(X = 99)$$

$$= {}^{100}C_1 \left(\frac{1}{2}\right)^{100} + {}^{100}C_3 \left(\frac{1}{2}\right)^{100} + \dots$$

$$+ {}^{100}C_{99} \left(\frac{1}{2}\right)^{100}$$

$$= ({}^{100}C_1 + {}^{100}C_3 + \dots + {}^{100}C_{99}) \left(\frac{1}{2}\right)^{100}$$

But $\quad {}^{100}C_1 + {}^{100}C_3 + \dots + {}^{100}C_{99} = 2^{99}$

Thus, $\quad P(X = \text{odd}) = \dfrac{2^{99}}{2^{100}} = \dfrac{1}{2}$.

◉ **Example 29:** The probability that at least one of A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, then $P(A') + P(B')$ is

(a) 0.9 (b) 1.15

(c) 1.1 (d) 1.2

Ans. (c)

◎ **Solution:** We have $P(A \cup B) = 0.6$

and $\quad P(A \cap B) = 0.3$. We know that

$$P(A) + P(B)$$

$$= P(A \cup B) + P(A \cap B) = 0.6 + 0.3 = 0.9$$

$\therefore \quad P(A') + P(B') = 1 - P(A) + 1 - P(B) = 2 - 0.9 = 1.1.$

◉ **Example 30:** An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of the four faces obtained, the probability that the minimum value is not less than 2 and the maximum value is not greater than 5 is

(a) $\dfrac{16}{81}$ (b) $\dfrac{1}{81}$

(c) $\dfrac{80}{81}$ (d) $\dfrac{65}{81}$

Ans. (a)

◎ **Solution:** The probability of getting a number not less than 2 and not greater than 5 in a single throw is 4/6 = 2/3. As die is rolled four times, the probability of required event = $(2/3)^4 = 16/81$.

◉ **Example 31:** Let A and B be two events such that $P(A) = 0.3$ and $P(A \cup B) = 0.8$. If A and B are independent events, then $P(B)$ is

(a) $\dfrac{3}{7}$ (b) $\dfrac{4}{7}$

(c) $\dfrac{5}{7}$ (d) $\dfrac{6}{7}$

Ans. (c)

◎ **Solution:** We have, $0.8 = P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)\,P(B)$$

[$\because$ A and B are independent]

$$= 0.3 + P(B) - (0.3)\,P(B)$$

$\Rightarrow \quad 0.5 = (0.7)\,P(B) \Rightarrow P(B) = 5/7.$

◉ **Example 32:** A pair of unbiased dice is rolled together till a sum of either 5 or 7 is obtained. The probability that 5 comes before 7 is

(a) $\dfrac{2}{5}$ (b) $\dfrac{3}{5}$

(c) $\dfrac{4}{5}$ (d) $\dfrac{1}{5}$

Ans. (a)

◎ **Solution:** Let A denote the event that a sum of 5 occurs, B the event that a sum of 7 occurs and C the event that neither a sum of 5 nor a sum of 7 occurs. We have

$P(A) = \dfrac{4}{36} = \dfrac{1}{9}$, $P(B) = \dfrac{6}{36} = \dfrac{1}{6}$ and $P(C) = \dfrac{26}{36} = \dfrac{13}{18}$. Thus,

$$P(A \text{ occurs before } B) = P\,[A \text{ or } (C \cap A) \text{ or } (C \cap C \cap A) \text{ or } \cdots]$$

$$= P(A) + P(C \cap A) + P(C \cap C \cap A) + \cdots$$

$$= P(A) + P(C)\,P(A) + P(C)^2\,P(A) + \cdots$$

$$= \frac{1}{9} + \left(\frac{13}{18}\right)\frac{1}{9} + \left(\frac{13}{18}\right)^2 \frac{1}{9} + \dots$$

$$= \frac{1/9}{1 - 13/18} = \frac{2}{5} \qquad \text{[sum of an infinite G.P.]}$$

◉ **Example 33:** E and F be two independent events such that $P(E) < P(F)$. The probability that both E and F happen is 1/12 and the probability that neither E nor F happen is 1/2. Then

(a) $P(E) = 1/3$, $P(F) = 1/2$

(b) $P(E) = 1/2$, $P(F) = 2/3$

(c) $P(E) = 2/3$, $P(F) = 3/4$

(d) $P(E) = 1/4$, $P(F) = 1/3$

Ans. (d)

◎ **Solution:** We are given $P(E \cap F) = 1/12$

and $\quad P(E' \cap F') = 1/2$

As E and F are independent, we get $P(E)\,P(F) = 1/12$

and $\quad P(E')\,P(F') = 1/2$

$\Rightarrow \quad (1 - P(E))\,(1 - P(F)) = 1/2$

$\Rightarrow \quad 1 - (P(E) + P(F)) + P(E)\,P(F) = 1/2$

$\Rightarrow \quad P(E) + P(F) = 1 + 1/12 - 1/2 = 7/12$

$\therefore \quad$ Equation whose roots are $P(E)$ and $P(F)$ is

$$x^2 - (P(E) + P(F))\,x + P(E)\,P(F) = 0$$

or $\quad x^2 - \dfrac{7}{12}x + \dfrac{1}{12} \quad \Rightarrow \quad 12x^2 - 7x + 1 = 0$

$$\Rightarrow \qquad (3x - 1)(4x - 1) = 0 \Rightarrow x = 1/3,\ 1/4$$

As $P(E) < P(F)$, we take $P(E) = 1/4$ and $P(F) = 1/3$.

◉ **Example 34:** A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is a six. The probability that it is actually a six is

(a) 3/8 (b) 1/5

(c) 3/4 (d) none of these

Ans. (a)

◎ **Solution:** Let E denote the event that a six occurs and A the event that the man reports that it is a six. We have $P(E) = 1/6$, $P(E') = 5/6$, $P(A \mid E) = 3/4$ and $P(A \mid E') = 1/4$. By Bayes' theorem

$$P(E \mid A) = \frac{P(E)\,P(A \mid E)}{P(E)\,P(A \mid E) + P(E')\,P(A \mid E')}$$

$$= \frac{(1/6)(3/4)}{(1/6)(3/4) + (5/6)(1/4)} = \frac{3}{8}$$

◉ **Example 35:** The event A is independent of itself if and only if $P(A)$ is

(a) 0 or 1 (b) 1/2

(c) 0 (d) 0, 1/2 or 1

Ans. (a)

◎ **Solution:** A is independent of itself if and only if

$$P(A \cap A) = P(A)\,P(A) \quad \Leftrightarrow \quad P(A) = P(A)^2$$
$$\Leftrightarrow P(A) = 0 \text{ or } P(A) = 1.$$

◉ **Example 36:** If three distinct numbers are chosen randomly from the first 100 natural numbers, then the probability that all the three numbers are divisible by both 2 and 3 is

(a) $\frac{4}{57}$ (b) $\frac{7}{99}$

(c) $\frac{4}{1155}$ (d) $\frac{1}{1100}$

Ans. (c)

◎ **Solution:** A number is divisible by both 2 and 3 if and only if it is divisible by 6. In the first 100 natural numbers, there are 16 numbers which are divisible by 6.

Thus, probability of the required event is

$${}^{16}C_3 / {}^{100}C_3 = \frac{4}{1155}$$

◉ **Example 37:** One ticket is selected at random from 100 tickets numbered 00, 01, 02, …, 98, 99. If X and Y denote respectively the sum and the product of the digits on the tickets, then $P(X = 9 \mid Y = 0) =$

(a) $\frac{2}{17}$ (b) $\frac{2}{19}$

(c) $\frac{2}{21}$ (d) $\frac{2}{11}$

Ans. (b)

◎ **Solution:** $(Y = 0)$ is $\{00, 01, \cdots, 09, 10, 20, \cdots, 90\}$. Also, $(X = 9) \cap (Y = 0) = \{09, 90\}$. We have $P(Y = 0) = 19/100$ and $P[(X = 9) \cap (Y = 0)] = 2/100$.

$$P(X = 9 \mid Y = 0) = \frac{P[(X = 9) \cap (Y = 0)]}{P(Y = 0)} = \frac{2}{19}.$$

◉ **Example 38:** If three six-faced fair dice are thrown together, the probability that the sum of the numbers appearing on the dice is 16 is

(a) $\frac{1}{36}$ (b) $\frac{1}{11}$

(c) $\frac{1}{12}$ (d) $\frac{5}{36}$

Ans. (a)

◎ **Solution:** The total number of cases is $6 \times 6 \times 6 = 6^3$.

If the first die shows 6, sum of the faces on the remaining two dice must be 10 which can happen in 3 ways viz. outcome should be (6, 4), (5, 5) or (4, 6).

If the first die shows 5, then other two must show (5, 6) or (6, 5).If the first die shows 4, then other two must show (6, 6).

$\therefore$ number of favourable ways = 3 + 2 + 1 = 6.

Thus, probability of the required event $= \frac{6}{6^3} = \frac{1}{36}$.

◉ **Example 39:** Four person are selected at random from a group of 3 men, 2 women and 4 children. The probability that exactly two of them are children is

(a) $\frac{9}{21}$ (b) $\frac{10}{23}$

(c) $\frac{1}{2}$ (d) $\frac{10}{21}$

Ans. (d)

◎ **Solution:** Total number of ways of selecting 4 persons out of 9 is ${}^9C_4$.

The number of selecting exactly two children is $({}^4C_2)\,({}^5C_2)$.

$\therefore$ probability of the required event

$$= \frac{\left({}^4C_2\right)\left({}^5C_2\right)}{{}^9C_4} = \frac{10}{21}$$

◉ **Example 40:** A die is thrown. Let A be the event that the number obtained is greater than 3. Let B be the event that the number obtained is less than 5. Then $P(A \cup B)$ is

(a) $\frac{2}{5}$ (b) 0

(c) 1 (d) $\frac{3}{5}$

Ans. (c)

◎ **Solution:** $A = \{4, 5, 6\}$ and $B = \{1, 2, 3, 4\}$

then $A \cup B = \{1, 2, 3, 4, 5, 6\}$

$\therefore$ $P(A \cup B) = 1$.

◉ **Example 41:** If A and B are two events. The probability that exactly one of them occurs is equal to

(a) $P(A) + P(B) - 2P(A \cap B)$
(b) $P(A) + P(B) + P(A \cap B)$
(c) $P(A) + P(B)$
(d) $P(A) + P(B) - P(A \cap B)$

Ans. (a)

◎ **Solution:** See Theory.

◉ **Example 42:** If A and B are two events such that $P(A) > 0$ and $P(B) < 1$, then $P(A|\bar{B})$ is equal to

(a) $1 - P(\bar{A}|B)$ (b) $1 - P(A|B)$

(c) $\dfrac{P(\bar{A})}{P(B)}$ (d) $1 - P(\bar{A}|\bar{B})$

Ans. (d)

◎ **Solution:** $P(A|\bar{B}) = \dfrac{P(A \cap \bar{B})}{P(\bar{B})} = \dfrac{P(\bar{B}) - P(\bar{A} \cap \bar{B})}{P(\bar{B})}$

$= 1 - P(\bar{A}|\bar{B})$

◉ **Example 43:** For a biased die, the probabilities for the different faces to turn up are given by the table

Face	1	2	3	4	5	6
Probability	0.1	0.32	0.21	0.15	0.05	0.17

The die is thrown and you are told that either the face 1 or the face 2 has turned up, then the probability that it is face 1, is

(a) $\dfrac{16}{21}$ (b) $\dfrac{1}{10}$

(c) $\dfrac{5}{16}$ (d) $\dfrac{5}{21}$

Ans. (d)

◎ **Solution:** P (face 1 | face 1 or face 2)

$$= \frac{P\text{ (face 1)}}{P\text{ (face 1)} + P\text{(face 2)}}$$

$$= \frac{0.1}{0.1 + 0.32} = \frac{0.1}{0.42} = \frac{5}{21}.$$

◉ **Example 44:** A six faced die is so biased that it is twice likely to show an even number as an odd number when thrown. If the die is thrown twice the probability that sum of the numbers is even is

(a) $\dfrac{1}{3}$ (b) $\dfrac{2}{3}$

(c) $\dfrac{4}{9}$ (d) $\dfrac{5}{9}$

Ans. (d)

◎ **Solution:** Let p denote the probability of showing an odd number, then the probability that die will show an even number is $2p$.

We have $p + 2p = 1 \Rightarrow 3p = 1 \Rightarrow p = 1/3$

Thus, probability of getting an even sum when dice is thrown twice is

$$P\text{ (odd, odd)} + P\text{ (even, even)} = P\text{ (odd)}^2 + P\text{ (even)}^2$$
$$= p^2 + (2p)^2 = 5p^2 = 5/9$$

◉ **Example 45:** While shuffling a pack of playing cards, four are accidently dropped. The probability that the cards that are dropped are one from each suit is

(a) $\dfrac{1}{20825}$ (b) $\dfrac{2197}{20825}$

(c) $\dfrac{1}{256}$ (d) $\dfrac{301}{20825}$

Ans. (b)

◎ **Solution:** Total number of ways of choosing four cards out of 52 is ${}^{52}C_4$. The number of ways of choosing one card from each suit is

$$({}^{13}C_1)\,({}^{13}C_1)\,({}^{13}C_1)\,({}^{13}C_1) = 13^4.$$

Thus, probability of the required event is

$$\frac{13^4}{{}^{52}C_4} = \frac{13 \times 13 \times 13 \times 13 \times 4 \times 3 \times 2 \times 1}{52 \times 51 \times 50 \times 49} = \frac{2197}{20825}.$$

◉ **Example 46:** The probability that a man who is 85 years old will die before attaining the age of 90 is 1/3. Four persons A_1, A_2, A_3 and A_4 are 85 years old. The probability that A_1 will die before attaining the age of 90 and will be the first to die is

(a) $\dfrac{65}{81}$ (b) $\dfrac{13}{81}$

(c) $\dfrac{65}{324}$ (d) $\dfrac{13}{108}$

Ans. (c)

◎ **Solution:** Required probability

= (probability at least one of A_1, A_2, A_3 and A_4 dies) × (probability that A_1 is first to die)

Now, probability that at least one of A_1, A_2, A_3 and A_4 dies

$$= 1 - P\text{(none of them dies)} = 1 - \left(\frac{2}{3}\right)^4 = 1 - \frac{16}{81} = \frac{65}{81}$$

$\therefore$ Required probability $= \dfrac{65}{81} \times \dfrac{1}{4} = \dfrac{65}{324}$.

◉ **Example 47:** Three persons A, B and C are to speak at a function along with 5 others. If the persons speaks in a random order, the probability that A speaks before B and B speaks before C is

(a) $\dfrac{3}{8}$ (b) $\dfrac{1}{12}$

(c) $\frac{1}{8}$ (d) $\frac{1}{6}$

Ans. (d)

◎ **Solution:** The number of ways in which 8 persons can speak is ${}^8P_8 = 8!$.

We can choose 3 places for *A, B, C* (out of 8) in ${}^8C_3$ ways and arrange *A*, *B*, *C* in that order in just one way. We can arrange the remaining 5 persons in 5! ways. Thus, the number of favourable ways is $({}^8C_3)$ (5!).

$\therefore$ probability of the required event $= \frac{\left({}^8C_3\right)(5!)}{8!} = \frac{1}{3!} = \frac{1}{6}$.

◉ **Example 48:** The probability that a man will live for 10 more years is 1/4 and the probability that his wife will live for 10 more years is 1/3. The probability that neither the husband nor the wife will be alive after 10 years is

(a) $\frac{5}{12}$ (b) $\frac{1}{2}$

(c) $\frac{7}{12}$ (d) $\frac{11}{12}$

Ans. (b)

◎ **Solution:** We have $P(H) = 1/4$ and $P(W) = 1/3$.

Now, *P*(neither husband nor wife will be alive)

$$= P(H' \cap W') = P(H')\,P(W')$$
$$= (1 - P(H))\,(1 - P(W))$$
$$= (1 - 1/4)\,(1 - 1/3) = 1/2.$$

◉ **Example 49:** Two numbers *x* and *y* are chosen at random without replacement from the first 30 natural numbers. The probability that $x^2 - y^2$ is divisible by 3 is

(a) $\frac{3}{29}$ (b) $\frac{3}{55}$

(c) $\frac{5}{29}$ (d) $\frac{47}{87}$

Ans. (d)

◎ **Solution:** Total number of ways of choosing three numbers out of 30 is ${}^{30}C_3$.

We rewrite the first 30 natural numbers in three rows as follows:

Row I:	1, 4, 7, 10, 13, 16, 19, 22, 25, 28
Row II:	2, 5, 8, 11, 14, 17, 20, 23, 26, 29
Row III:	3, 6, 9, 12, 15, 18, 21, 24, 27, 30

For $x^2 - y^2$ to be divisible by 3, either both *x* and *y* must be chosen from the same row or exactly one of *x*, *y* from Row I and the other from Row II.

Thus, the number of favourable ways

$$= 3({}^{10}C_2) + 10 \times 10 = 235$$

$\therefore$ probability of the required event

$$= \frac{235}{{}^{30}C_2} = \frac{235 \times 2}{30 \times 29} = \frac{47}{87}.$$

◉ **Example 50:** If *m* is a natural such that $m \le 5$, then the probability that the quadratic equation $x^2 + mx + \frac{1}{2} + \frac{m}{2} = 0$ has real roots is

(a) $\frac{1}{5}$ (b) $\frac{2}{3}$

(c) $\frac{3}{5}$ (d) $\frac{1}{5}$

Ans. (c)

◎ **Solution:** Discriminant *D* of the given quadratic equation is

$$D = m^2 - 4\left(\frac{1}{2} + \frac{m}{2}\right) = m^2 - 2m - 2 = (m - 1)^2 - 3$$

Now, $D \ge 0 \quad \Leftrightarrow \quad (m - 1)^2 \ge 3$

This is possible for $m = 3$, 4 and 5. Also, the total number of ways of choosing *m* is 5.

$\therefore$ probability of the required event = 3/5.

◉ **Example 51:** If two events *A* and *B* such that $P(A') = 0.3$, $P(B) = 0.5$ and $P(A \cap B) = 0.3$, then $P(B|A \cup B')$ is

(a) 0.6 (b) 0.32

(c) 0.31 (d) 0.28

Ans. (a)

◎ **Solution:** We have

$$P(A \cup B') = P(A) + P(B') - P(A \cap B')$$
$$= [1 - P(A')] + [1 - P(B)] - [P(A) - P(A \cap B)]$$
$$= (1 - 0.3) + (1 - 0.5) - (1 - 0.3) = 0.5.$$

Now, $P\,(B|A \cup B')$

$$= \frac{P\,[B \cap (A \cup B')]}{P(A \cup B')} = \frac{P[(B \cap A) \cup (B \cap B')]}{P(A \cup B')}$$
$$= \frac{P\,(A \cap B)}{P\,(A \cup B')} = \frac{0.3}{0.5} = 0.6$$

◉ **Example 52:** Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently equals

(a) $\frac{1}{2}$ (b) $\frac{7}{15}$

(c) $\frac{2}{15}$ (d) $\frac{1}{3}$

Ans. (b)

◎ **Solution:** The number of ways of placing 3 black balls at 10 places is ${}^{10}C_3$. The number of ways in which two black balls are not together is equal to the number of ways of choosing 3 places marked with *X* out of the eight places

X W X W X W X W X W X W X W X

This can be done in $^8C_3$ ways. Thus, probability of the required event is

$$\frac{^8C_3}{^{10}C_3} = \frac{8\times7\times6}{10\times9\times8} = \frac{7}{15}.$$

◉ **Example 53:** If from each of the three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn at random, then the probability that two white and one black ball will be drawn is

(a) $\frac{13}{32}$ (b) $\frac{1}{4}$

(c) $\frac{1}{32}$ (d) $\frac{3}{16}$

Ans. (a)

◎ **Solution:** P (2 white and 1 black)

$= P(W_1\ W_2\ B_3 \text{ or } W_1\ B_2\ W_3 \text{ or } B_1\ W_2\ W_3)$

$= P(W_1)\,P(W_2)\,P(B_3) + P(W_1)\,P(B_2)\,P(W_3) + P(B_1)\,P(W_2)P(W_3)$

$$= \left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{3}{4}\right)+\left(\frac{3}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)\left(\frac{2}{4}\right)\left(\frac{1}{4}\right)$$

$$= \frac{13}{32}$$

◉ **Example 54:** There are four machines and it is known that exactly two of them are faulty. They are tested, one by one, in a random order till both the faulty machines are identified. Then the probability that only two tests are needed is

(a) $\frac{1}{3}$ (b) $\frac{1}{6}$

(c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Ans. (a)

◎ **Solution:** Either both the machines tested should be faulty or both should be good.

$\therefore$ required probability $= \frac{2\times1}{4\times3}\times\frac{2\times1}{4\times3} = \frac{1}{3}.$

◉ **Example 55:** If E and F are two events such that $0 < P(F) < 1$, then

(a) $P(E|\bar{F}) + P\ (\bar{E}\,|\,\bar{F}) = 1$

(b) $P(E|F) + P(E|\bar{F}) = 1$

(c) $P(\bar{E}\,|F) + P(E|\bar{F}) = 1$

(d) none of these

Ans. (a)

◎ **Solution:** As $P(|\bar{F})$ is a probability function, (a) is clearly true.

Alternate Solution

We have $P(E\,|\,\bar{F}) + P(\bar{E}\,|\,\bar{F})$

$$= \frac{P(E\cap\bar{F})}{P(\bar{F})} + \frac{P(\bar{E}\cap\bar{F})}{P(\bar{F})}$$

$$= \frac{P(E\cap\bar{F}) + P(\bar{E}\cap\bar{F})}{P(\bar{F})} = \frac{P(\bar{F})}{P(\bar{F})} = 1$$

◉ **Example 56:** The probability that an event A occurs in a single trial of an experiment is 0.4. Three independent trials of the experiment are performed. The probability that A occurs at least once is

(a) 0.936 (b) 0.784

(c) 0.904 (d) 0.788

Ans. (b)

◎ **Solution:** Let

$p=$ probability that event A occurs in a single trial $= 0.4$

$n =$ number of independent trials = 3

and $X=$ number of times A occurs. Note that $X \sim B\ (n,\ p)$

P (A occurs at least once) $= P\ (X \geq 1)$

$= 1 - P(X = 0) = 1 - q^3$

$= 1 - (0.6)^3 = 1 - 0.216 = 0.784.$

◉ **Example 57:** Fifteen coupons are numbered 1, 2, … , 15 respectively. Seven coupons are selected at random one at a time with replacement. The probability that the largest number on a selected coupon as 9 is

(a) $\left(\frac{9}{15}\right)^6$ (b) $\left(\frac{8}{15}\right)^7$

(c) $\left(\frac{3}{5}\right)^7$ (d) none of these

Ans. (d)

◎ **Solution:** Let p = the probability that a selected coupon bears number ≤ 9.

$\Rightarrow \quad p = \frac{9}{15} = \frac{3}{5}$

n = number of coupons drawn with replacement

and X = the number of coupons bearing number ≤ 9.

Note that $X \sim B\ (n, p)$.

Probability that the largest number on the selected coupons does not exceed 9

= probability that all the coupons bear number ≤ 9

$= P(X = 7) = {}^7C_7\,p^7 = \left(\frac{3}{5}\right)^7.$

Similarly, probability that largest number on the selected coupon is ≤ 8 is $\left(\frac{8}{15}\right)^7$. Hence, probability of the required event $= \left(\frac{3}{5}\right)^7 - \left(\frac{8}{15}\right)^7.$

◉ **Example 58:** Each of two persons A and B toss three fair coins. The probability that both get the same number of heads is

(a) 3/8 (b) 1/9
(c) 5/16 (d) none of these

Ans. (c)

◎ **Solution:** Let X be the number of heads obtained by A and Y be the number of heads obtained by B. Note that both X and Y are binomial variate with parameters $n = 3$ and $p = 1/2$.

Probability that both A and B obtain the same number of heads is

$\therefore$ $P(X = 0)\,P(Y = 0) + P(X = 1)\,P(Y = 1) + P(X = 2)\,P(Y = 2) + P(X = 3)\,P(Y = 3)$

$$= \left[{}^3C_0\left(\frac{1}{2}\right)^3\right]^2 + \left[{}^3C_1\left(\frac{1}{2}\right)^3\right]^2 + \left[{}^3C_2\left(\frac{1}{2}\right)^3\right]^2 + \left[{}^3C_3\left(\frac{1}{2}\right)^3\right]^2$$

$$= \left(\frac{1}{2}\right)^6 [1 + 9 + 9 + 1] = \frac{20}{64} = \frac{5}{16}.$$

◉ **Example 59:** The probability that a student is not a swimmer is 1/5. The probability that out of 5 students exactly 4 are swimmer is

(a) $\left(\frac{4}{5}\right)^3$ (b) $\left(\frac{4}{5}\right)^4$
(c) ${}^5C_4\left(\frac{4}{5}\right)^4$ (d) $\left(\frac{3}{4}\right)^3$

Ans. (b)

◎ **Solution:** Let p = probability that a student selected at random is a swimmer

$$= 1 - 1/5 = 4/5.$$

n = number of students selected = 5

and X = number of swimmers

Then $X \sim B(n, p)$ where $n = 5$ and $p = 4/5$

Probability that exactly 4 students are swimmer is

$$= {}^5C_4\, p^4 q = 5\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right) = \left(\frac{4}{5}\right)^4.$$

◉ **Example 60:** One hundred identical coins, each with probability p of showing up heads, are tossed once. If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of heads showing on 51 coins, then the value of p is

(a) $\frac{1}{2}$ (b) $\frac{49}{101}$
(c) $\frac{50}{101}$ (d) $\frac{51}{101}$

Ans. (d)

◎ **Solution:** Let X be the number of coins showing heads. Then $X \sim B(100, p)$. We have

$$P(X = 51) = P(X = 50)$$

$$\Rightarrow \quad {}^{100}C_{51}\, p^{51} q^{49} = {}^{100}C_{50}\, p^{50} q^{50} \qquad (\text{where } q = 1 - p)$$

$$\Rightarrow \quad \frac{p}{q} = \frac{100!}{50!50!}\,\frac{51!49!}{100!} \Rightarrow \frac{p}{1-p} = \frac{51}{50}$$

$$\Rightarrow \quad 50p = 51 - 51p \Rightarrow 101p = 51$$

$$\Rightarrow \quad p = 51/101.$$

◉ **Example 61:** The least number of times a fair coin must be tossed so that the probability of getting at least one head is at least 0.95

(a) 5 (b) 6
(c) 7 (d) 12

Ans. (a)

◎ **Solution:** Let n be the required number of tosses, and X be the number of heads obtained in n tosses. Then $X \sim B(n, 1/2)$

$$P(\text{at least one head}) = P(X \ge 1) = 1 - P(X = 0)$$

$$= 1 - q^n = 1 - \left(\frac{1}{2}\right)^n$$

Now, $P(X \ge 1) \ge 0.95 \Rightarrow 1 - \left(\frac{1}{2}\right)^n \ge 0.95$

$$\left(\frac{1}{2}\right)^n \le 0.05 \Rightarrow 2^n \ge 20 \Rightarrow n \ge 5$$

$\therefore$ least value of n is 5.

◉ **Example 62:** A box contain N coins, m of which are fair and rest are biased. The probability of getting a head when a fair coin is tossed is 1/2, while it is 2/3 when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. The first time it shows head and the second time it shows tail. The probability that the coin drawn is fair is

(a) $\frac{8m}{8N+m}$ (b) $\frac{m}{8N+m}$
(c) $\frac{9m}{8N+m}$ (d) $\frac{9N}{8N+m}$

Ans. (c)

◎ **Solution:** Let E_1, E_2 and A denote the following events:

E_1 : coin selected is fair

E_2 : coin selected is biased

A : the first toss results in a head and the second toss results in a tail.

$$P(E_1) = \frac{m}{N},\; P(E_2) = \frac{N-m}{N},$$

$$P(A|E_1) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4},\; P(A|E_2) = \frac{2}{3}\times\frac{1}{3} = \frac{2}{9}$$

By Bayes' rule

$$P\,(E_1|A) = \frac{P(E_1)\,P(A\,|\,E_1)}{P(E_1)\,P\,(A\,|\,E_1)+P(E_2)\,P(A\,|\,E_2)}$$

$$= \frac{9m}{8N+m}.$$

Example 63: The probability of India winning a test match against West Indies is 1/2. Assuming independence from match to match, the probability that in a 5 match series India's second win occurs at the third test is

(a) 1/8 (b) 1/4
(c) 1/2 (d) 2/3

Ans. (b)

Solution: Required probability

= P(winning exactly one match in the first two matches) × P(winning third march)

$$= \left[{}^2C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right]\left[\frac{1}{2}\right] = \frac{1}{4}$$

Example 64: Let $0 < P(A) < 1, 0 < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A)\,P(B)$, then

(a) $P(A\,|B) = 0$

(b) $P(B|A) = 0$

(c) $P(A' \cap B') = P(A')\,P(B')$

(d) $P(A|B) + P(B|A) = 1$

Ans. (c)

Solution: We know that

$$P(A \cup B) = P\,(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(A) + P(B) - P(A)\,P(B) = P(A) + P(B) - P\,(A \cap B)$$

$$\Rightarrow \quad P(A)\,P(B) = P(A \cap B)$$

$\Rightarrow$ A and B are independent.

Thus, $P(A' \cap B') = P(A')\,P(B')$.

Example 65: If $A_1, A_2, \ldots, A_n$ are n independent events such that $P(A_i) = \dfrac{1}{i+1}$, $i = 1, 2, \ldots, n$. The probability that none of $A_1, A_2, \ldots A_n$ occurs is

(a) $\dfrac{n}{n+1}$ (b) $\dfrac{1}{n+1}$

(c) $\dfrac{1}{n!}$ (d) $\dfrac{1}{n+2}$

Ans. (b)

Solution: $P(A' \cap A'_2 \ldots \cap A'_n) = P(A'_1)\,P\,(A'_2) \ldots P(A'_n)$

[$\because$ $A_1, A_2, \ldots, A_n$ are independent]

$$= \left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n+1}\right)$$

$$= \frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\ldots\times\frac{n-1}{n}\times\frac{n}{n+1} = \frac{1}{n+1}.$$

Example 66: A four digit number (numbered from 0000 to 9999) is said to be lucky if the sum of its first two digits is equal to the sum of its last two digits. If a four digit number is picked up at random, then the probability that it is lucky is

(a) 0.065 (b) 0.064
(c) 0.066 (d) 0.067

Ans. (d)

Solution: The total number of ways of choosing the ticket is 10000.

Let the four digits number on the ticket be $x_1\,x_2\,x_3\,x_4$. Note that $0 \le x_1 + x_2 \le 18$ and $0 \le x_3 + x_4 \le 18$.

Also, the number of non-negative integral solutions of $x + y = m$ (with $0 \le x, y \le 9$) is $m + 1$ if $0 \le m \le 9$ and is $19 - m$ if $10 \le m \le 18$.

Thus, the number of favourable ways

$$= 1\times1 + 2\times2 + \ldots + 10\times10 + 9\times9 + 8\times8 + \ldots + 1\times1$$

$$= 2\left\{\frac{9\times10\times19}{6}\right\} + 100 = 670$$

$\therefore$ probability of required event $= \dfrac{670}{1000} = 0.067$.

Example 67: Three numbers are chosen at random without replacement from {1, 2, 3, …, 10}. The probability that minimum of the chosen number is 3 or their maximum is 7, is

(a) $\dfrac{11}{30}$ (b) $\dfrac{11}{40}$

(c) $\dfrac{1}{7}$ (d) $\dfrac{1}{8}$

Ans. (b)

Solution: Let A and B denote the following events:

A: minimum of the chosen number is 3

B: maximum of the chosen number is 7

We have

$P(A) = P$ (choosing 3 and two other numbers from 4 to 10)

$$= \frac{{}^7C_2}{{}^{10}C_3} = \frac{7\times6}{2}\times\frac{3\times2}{10\times9\times8} = \frac{7}{40}$$

$P(B) = P$ (choosing 7 and two other numbers from 1 to 6)

$$= \frac{{}^6C_2}{{}^{10}C_3} = \frac{6\times5}{2}\times\frac{3\times2}{10\times9\times8} = \frac{1}{8}$$

$P(A \cap B) = P$ (choosing 3 and 7 and one other number from 4 to 6).

$$= \frac{3}{{}^{10}C_3} = \frac{3\times3\times2}{10\times9\times8} = \frac{1}{40}.$$

Now,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{7}{40} + \frac{1}{8} - \frac{1}{40} = \frac{11}{40}$$

Example 68: One ticket is selected at random from 50 tickets numbered 00, 01, 02, ..., 49. Then the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, equals

(a) $\frac{5}{14}$ (b) $\frac{1}{50}$

(c) $\frac{1}{14}$ (d) $\frac{1}{7}$

Ans. (c)

Solution: Let A and B denote the following events.
A : sum of digits is 8 and B: product of digits is 0.

Then $A \cap B = \{08\}$

and $B = \{00, 01, 02, ..., 09, 10, ... 30, 40\}$

Now, $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{n(A \cap B)}{n(B)} = \frac{1}{14}$

Example 69: An urn contains nine balls of which three are red, four are blue and two are green. Three balls are drawn at random without replacement from the urn. The probability that the three balls have different colours is

(a) $\frac{1}{21}$ (b) $\frac{2}{23}$

(c) $\frac{1}{3}$ (d) $\frac{2}{7}$

Ans. (d)

Solution: Total number of ways of drawing three balls out of 9 is ${}^9C_3 = 84$. The number of ways of drawing the balls so that balls are of different colours is

$$({}^3C_1)\,({}^4C_1)\,({}^2C_1) = (3)\,(4)\,(2) = 24$$

$\therefore$ probability of the required event $= \frac{24}{84} = \frac{2}{7}$

Example 70: Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 the numbers obtained on the die, then the probability that $\omega^{r1} + \omega^{r2} + \omega^{r3} = 0$ is

(a) $\frac{1}{18}$ (b) $\frac{1}{9}$

(c) $\frac{2}{9}$ (d) $\frac{1}{36}$

Ans. (c)

Solution: Total number of ways of throwing a fair die three times is $6 \times 6 \times 6$.

Note that $\omega^{r1} + \omega^{r2} + \omega^{r3} = 0$ if r_1, r_2, r_3 is a permutation of the numbers of the form $\{3k, 3m + 1, 3n + 2\}$ from the set $\{1, 2, 3, 4, 5, 6\}$

Thus, r_1, r_2, r_3 can be chosen in

(2) (2) (2) (3!) = (8) (6) ways.

Hence, probability of the required event $= \frac{(8)(6)}{(6)(6)(6)} = \frac{2}{9}$

Example 71: A signal which can be green or red with probability $\frac{4}{5}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is $\frac{3}{4}$. If the signal received at station B is green, then the probability that the original signal was green is

(a) $\frac{3}{5}$ (b) $\frac{6}{7}$

(c) $\frac{20}{23}$ (d) $\frac{9}{20}$

Ans. (c)

Solution:

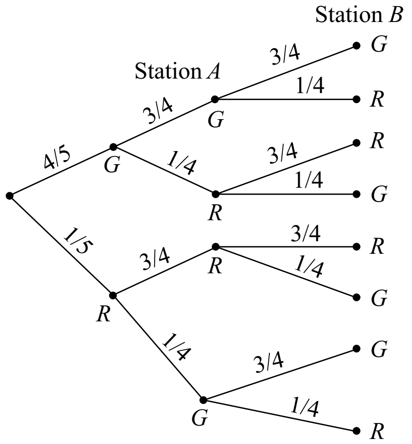


Fig. 24.6

Let G, E_1, E_2, and E denote the following events :

G : Original signal is green

E_1 : A receives the signal correctly

E_2 : B receives the signal correctly

E : B receives the green signal

We have

$$E = G\,E_1\,E_2 \cup G\,E'_1\,E'_2 \cup G'\,E_1\,E'_2 \cup G'\,E'_1\,E_2$$

$$\Rightarrow P(E) = P(G\,E_1\,E_2) + P(G\,E'_1\,E'_2)$$
$$+ P(G'\,E_1\,E'_2) + P(G'\,E'_1\,E'_2)$$
$$= \left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) + \left(\frac{4}{5}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$$

$$+\left(\frac{1}{5}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)+\left(\frac{1}{5}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)$$

$$=\frac{36+4+3+3}{80}=\frac{23}{40}$$

Also, $P(G\cap E) = P(G\,E_1\;E_2) + P(G\,E'_1\;E'_2)$

$$=\frac{40}{80}=\frac{1}{2}$$

$$\therefore \quad P(G \mid E) = \frac{P(G\cap E)}{P(E)}=\frac{1/2}{23/40}=\frac{20}{23}$$

Example 72: Three numbers are chosen at random from the numbers 1, 2, ... 20. The probability that the arithmetic mean of these numbers is 4 is

(a) 7/2280 (b) 3/4

(c) 2/7281 (d) 0

Ans. (a)

Solution: Total number of ways of choosing three numbers out of 20 is ${}^{20}C_3 = 1140$

Let these numbers be x_1, x_2, x_3.

We have $\frac{1}{3}(x_1 + x_2 + x_3) = 4$,

we get $x_1 + x_2 + x_3 = 12$.

The possible choices of (x_1, x_2, x_3) are (1, 2, 9), (1, 3, 8), (1, 4, 7), (1, 5, 6) (2, 3,7), (2, 4, 6), (3, 4, 5)

$\therefore$ probability of the required event $= \frac{7}{1140}$.

Example 73: An unbiased cubical die is thrown 5 times. The probability that the maximum number appearing on the die is 4 is

(a) $7/6^5$ (b) $1023/6^5$

(c) $3781/6^5$ (d) $1781/6^5$

Ans. (b)

Solution: Let X = maximum number appearing on the cubical die.

$$P(X = 4) = P(X \le 4) - P(X \le 3)$$

$$=\left(\frac{4}{6}\right)^5-\left(\frac{3}{6}\right)^5=\left(\frac{2}{3}\right)^5-\left(\frac{1}{2}\right)^5=\frac{2^5}{3^5}-\frac{1}{2^5}=\frac{1023}{6^5}$$

Example 74: Let A and B be two events such that $P(A) = 3/7$, $P(B) = 4/7$ and $P(A \cup B) = 5/7$,

Then $\frac{P(A \mid B)}{P(A' \mid B)}$ equals

(a) 1/3 (b) 2/3

(c) 1/2 (d) 3/8

Ans. (c)

Solution: $P(A \cap B) = P(A) + P(B) - P(A \cup B) = 2/7$

Now, $$\frac{P(A \mid B)}{P(A' \mid B)} = \frac{P(A\cap B)}{P(B)}\frac{P(B)}{P(A'\cap B)}$$

$$=\frac{P(A\cap B)}{P(B)-P(A\cap B)}=\frac{2/7}{4/7-2/7}=\frac{1}{2}$$

Example 75 Suppose A, B, C are three independent events such that $P(A) = P(B) = P(C) = p$, then P (at least two of A, B, C) equals

(a) $2p^2 - 3p^3$ (b) $3p^2 - 2p^3$

(c) $p^2 - p^3$ (d) $2p^2 - p^3$

Ans. (b)

Solution: P(at least two of A, B, C)

$$= P\,[(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C) \cup (A \cap B \cap C)]$$

$$= P(A \cap B \cap C') + P(A \cap B' \cap C) + P(A' \cap B \cap C) + P(A \cap B \cap C)$$

$$= p^2(1 - p) + p^2(1 - p) + p^2(1 - p) + p^3$$

[A, B, C are independent]

$$= 3p^2 - 2p^3$$

Example 76: A and B are two students. Their probabilities of solving a problem correctly are 1/4 and 1/5 respectively. If the probability of their making a common error is 1/40, and they obtain the same answer, then the probability of their answer is correct is

(a) 1/12 (b) 1/20

(c) 10/13 (d) 13/200

Ans. (c)

Solution: Let A, B, C and S denote the following events.

A : A solves the problem correctly;

B : B solves the problem correctly;

C : A and B make common errors;

S : A and B reach the same answer.

We have $S = (A \cap B) \cup (A' \cap B' \cap C)$

$\Rightarrow$ $P(S) = P(A \cap B) + P(A' \cap B' \cap C)$

$$= P(A)\,P(B) + P(A')\,P(B')\,P(C)$$

$$=\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)+\left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\left(\frac{1}{40}\right)=\frac{13}{200}$$

Now P (answer is correct | S)

$$=\frac{P(A\cap B\cap S)}{P(S)}=\frac{P(A\cap B)}{P(S)}$$

$$=\frac{(1/4)(1/5)}{13/200}=\frac{10}{13}$$

Example 77: Suppose A and B are two events such that $P(A) \neq 0$, $P(B) \neq 0$, then

(a) $P(A|B) = P(A)/P(B)$

(b) $P(A|B) = P(A \cap B)/P(B)$

(c) $P(A|B).P(B/A) = 1$

(d) $P(A|B) = P(A)P(B)$

Ans. (b)

Solution: See theory.

Example 78: Fig. 24.7 shows three events A, B and C. Probabilities of different events are shown in the figure. For instance,

$P(A \cap B' \cap C) = 0.18$,

$P(A' \cap B \cap C') = 0.06$ etc.

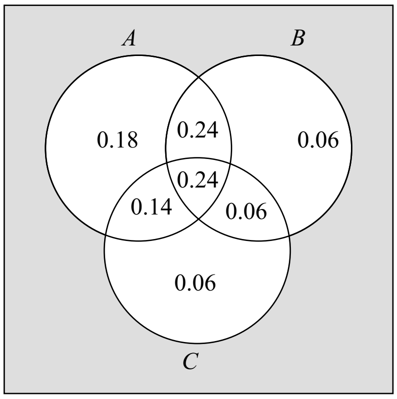


Fig. 24.7

Which of the following is not true ?

(a) A and B are independent
(b) B and C are independent
(c) A and C are independent
(d) A and $B \cap C$ are independent

Ans. (c)

Solution: Note that

$$P(A) = 0.18 + 0.24 + 0.24 + 0.14 = 0.8$$
$$P(B) = 0.06 + 0.06 + 0.24 + 0.24 = 0.6$$
$$P(C) = 0.06 + 0.14 + 0.24 + 0.06 = 0.5$$
$$P(B \cap C) = 0.24 + 0.06 = 0.3 = P(B)\,P(C)$$
$$P(C \cap A) = 0.14 + 0.24 = 0.38 \neq P(C)\,P(A)$$
$$P(A \cap B) = 0.24 + 0.24 = 0.48 = P(A)\,P(B)$$
$$P(A \cap B \cap C) = 0.24 = P(A)\,P(B \cap C)$$

Example 79: A fair die is tossed repeatedly until a six is obtained. Let X denote the number of tosses required. The probability that $X \geq 3$ equals

(a) $\dfrac{125}{216}$ (b) $\dfrac{25}{216}$

(c) $\dfrac{5}{36}$ (d) $\dfrac{25}{36}$

Ans.(d)

Solution: Let $p = \dfrac{1}{6}$, $q = \dfrac{5}{6}$

We have

$$P(X = r) = pq^{r-1} \;\forall\; r \geq 1.$$

Now,

$$P(X \geq 6 \mid X > 3) = \frac{P[(X \geq 6) \cap (X > 3)]}{P(X > 3)} = \frac{P(X \geq 6)}{P(X \geq 4)}$$

But $\quad P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6) + \ldots$

$$= pq^3 + pq^4 + pq^5 + \ldots$$

$$= \frac{pq^3}{1-q} = \frac{pq^3}{p} = q^3$$

and $\quad p(X \geq 6) = q^5$

Thus, $\quad P(X \geq 6 \mid X \geq 3) = \dfrac{q^5}{q^3} = q^2 = \dfrac{25}{36}$.

Example 80: If $P(A) = 0.4$, $P(B') = 0.6$ and $P(A \cap B) = 0.15$ then the value of $P(A/A' \cup B')$ is

(a) $\dfrac{4}{17}$ (b) $\dfrac{5}{17}$

(c) $\dfrac{10}{17}$ (d) $\dfrac{1}{17}$

Ans.(b)

Solution: $P(A/A' \cup B') = \dfrac{P[A \cap (A' \cup B')]}{P(A' \cup B')}$

$$= \frac{P(A \cap B')}{1 - P(A \cap B)}$$

$$= \frac{P(A) - P(A \cap B)}{1 - P(A \cap B)} = \frac{0.4 - 0.15}{1 - 0.15}$$

$$= \frac{0.25}{0.85} = \frac{5}{17}$$

Example 81: Consider 5 independent Bernoulli's trials each with probability of success p. If the probability of at least one failure is greater than or equal to 31/32, then p lies in the interval.

(a) [11/12, 1] (b) [1/2, 3/4]

(c) [3/4, 11/12] (d) [0, 1/2]

Ans.(d)

Solution: Let X = number of failure, then $X \sim B(5, 1 - p)$

P (at least one failure) $= P(X \geq 1)$

$= 1 - p(X = 0)$

$= 1 - {}^5C_0\,[1 - (1-p)]^5$

$= 1\,p^5$

We are given $= 1 - p^5 \geq 31/32 \Rightarrow p^5 \leq 1/32$

$\Rightarrow \quad 0 \leq p \leq 1/2$

Example 82: If C and D are two events such that $C \subseteq D$ and $P(D) \neq 0$, then the correct statement among the following is

(a) $P(C/D) = \dfrac{P(C)}{P(D)}$ (b) $P(C/D) = P(C)$

(c) $P(C/D) \geq P(C)$ (d) $P(C/D) < P(C)$

Ans.(a)

Solution: $P(C/D) = \dfrac{P(C \cap D)}{P(D)} = \dfrac{P(C)}{P(D)} \quad [\because C \subseteq D]$

Example 83: Three numbers are chosen at random without replacement from {1, 2, 3, ... 8}.The probability that their minimum is 3, given that their maximum is 6, is

(a) 1/5 (b) 1/4

(c) 2/5 (d) 3/8

Ans.(a)

Solution: Total number of ways of choosing 3 numbers out of 8 is $^8C_3 = 56$

Let A denote the event that minimum is 3 and B denote the event that maximum number is 6.

Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

But $$P(B) = \frac{^5C_2}{^8C_3} = \frac{10}{56} \text{ and } P(A \cap B) = \frac{^2C_1}{^8C_3} = \frac{2}{56}$$

$$\therefore \quad P(A|B) = \frac{2}{10} = \frac{1}{5}$$

Example 84: Ten unbiased coins are thrown simultaneously. The probability of getting at least seven heads is

(a) $\frac{3}{64}$ (b) $\frac{5}{64}$

(c) $\frac{7}{64}$ (d) $\frac{11}{64}$

Ans. (d)

Solution: Let X = number of heads when 10 unbiased coins are thrown, then $X \sim B$ (10, 1/2)

We have

$$P(X \geq 7) = {}^{10}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 + {}^{10}C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^2$$

$$+ {}^{10}C_9 \left(\frac{1}{2}\right)^9 \left(\frac{1}{2}\right) + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$$

$$= \frac{1}{2^{10}}\left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right]$$

$$= \frac{1}{2^{10}}(176) = \frac{11}{64}$$

Example 85: Out of 13 applicants for a job there are 5 women and 8 men. It is desired to select 2 persons for this job.The probability that at least one of the selected person will be a woman is

(a) 5/13 (b) 10/13

(c) 14/39 (d) 25/39

Ans.(d)

Solution: P (at least one woman)

$= 1 - P$ (no woman)

$$= 1 - {}^8C_2 / {}^{13}C_2 = \frac{25}{39}$$

Example 86: Four persons A, B, C, D are to speak at a function along with 6 others. If they all speak in random order, the probability that A speaks before B, B before C, C before D is

(a) 1/4 (b) 1/6

(c) 1/24 (d) 1/16

Ans.(c)

Solution: 10 persons can speak at the function in 10! ways.

We can choose 4 places out of 10 in $^{10}C_4$ ways and arrange A, B, C, D at the selected places in only one way. The remaining 6 persons can be arranged in 6! ways. Thus, probability of the required event is

$$= \frac{({}^{10}C_4)(6!)}{10!} = \frac{1}{4!} = \frac{1}{24}$$

Example 87: Let A, B, C be pairwise independent events with $P(C) > 0$ and $P(A \cap B \cap C) = 0$. Then $P(A' \cap B'|C)$ is equal to

(a) $P(A) - P(B')$ (b) $P(A') + P(B')$

(c) $P(A') - P(B')$ (d) $P(A') - P(B)$

Ans.(d)

Solution: $P(A' \cap B'|C) = 1 - P(A \cup B|C)$

$= 1 - [P(A|C) + P(B|C) - P(A \cap B|C)]$

$$= 1 - \left[P(A) + P(B) - \frac{P(A \cap B \cap C)}{P(C)}\right]$$

[$\because$ A, B, C are pairwise independent]

$= 1 - P(A) - P(B)$

$= P(A') - P(B)$

Example 88: Four fair dice D_1, D_2, D_3 and D_4, each having six faces numbered 1, 2, 3, 4, 5 and 6 are rolled simultaneously. The probability D_4 shows a number appearing on one of D_1, D_2 and D_3 is

(a) $\frac{91}{216}$ (b) $\frac{108}{216}$

(c) $\frac{125}{216}$ (d) $\frac{127}{216}$

Ans.(a)

Solution: P (D_4 shows number on one of D_1, D_2 and D_3)

$= 1 - P$ (D_4 shows a number different from D_1, D_2 and D_3)

$$= 1 - \frac{{}^6C_1(5^3)}{6^4} = 1 - \frac{125}{216} = \frac{91}{216}$$

Example 89: Let A and B be two events such that $P(A|B) = 1/2$, $P(B|A) = 1/3$, $P(A \cap B) = 1/6$, then

(a) $P(A \cup B) = 1/2$

(b) A and B are independent

(c) $P(A' \cup B) = 1/3$

(d) none of these

Ans.(b)

Solution: $P(A/B) = \frac{1}{2} \Rightarrow \frac{P(A \cap B)}{P(B)} = \frac{1}{2} \Rightarrow P(B) = \frac{1}{3}$

and $P(B/A) = \frac{1}{3} \Rightarrow P(A) = \frac{1}{2}$

As $P(A \cap B) = P(A)\,P(B)$, we get A and B are independent.

Example 90: A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities 1/2, 1/4, and 1/4. For the ship to be operative at least two of its engines must function. Let X denote the event that ship is operational and let X_1, X_2 and X_3 respectively the events that the engines E_1, E_2 and E_3 are functioning. Let

(a) $P(X_1'/X) = 3/8$

(b) $P(X/X_2) = 7/8$

(c) P (Exactly two engines are functioning) = 7/8

(d) $P(X/X_1) = 7/16$

Ans.(d)

Solution: We have

$$X = (X_1X_2X_3') \cup (X_1X_2'X_3) \cup (X_1'X_2X_3) \cup (X_1X_2X_3),$$

and $\quad X_1' \cap X = X_1'X_2X_3$

Now, $\quad P(X_1'/X) = \frac{P(X_1' \cap X)}{P(X)} = \frac{P(X_1'X_2X_3)}{P(X)}$

We have $P(X_1'X_2X_3) = P(X_1')P(X_2)P(X_3)$

$= (1/2)\,(1/4)\,(1/4) = 1/32$

and $\quad P(X) = (1/2)\,(1/4)\,(3/4) + (1/2)\,(3/4)\,(1/4)$

$+ (1/2)\,(1/4)\,(1/4) + (1/2)\,(1/4)\,(1/4) = 1/4$

$\therefore \quad P(X_1'/X) = 1/8$

Next, $\quad X \cap X_2 = X - X_1X_2'X_3$,

$$P(X \cap X_2) = P(X) - P(X_1X_2'X_3) = 5/32$$

$$\therefore \quad P(X/X_2) = \frac{P(X \cap X_2)}{P(X_2)} = \frac{5/32}{1/4} = \frac{5}{8}$$

and P (Exactly two engines are functioning)

$= P(X - X_1X_2X_3) = 7/32$

Next, $\quad X \cap X_1 = X - X_1'X_2X_3$

$\Rightarrow \quad P(X \cap X_1) = 7/32$

$$\therefore \quad P(X/X_1) = \frac{P(X \cap X_1)}{P(X_1)} = \frac{7/32}{1/2} = \frac{7}{16}$$

Assertion-Reason Type Questions

Example 91: Four numbers are chosen at random (without replacement) from the set {1, 2, 3.... 20}.

Statement-1: The probability that the chosen numbers when arranged in some order will form an A.P. is 1/85.

Statement-2: If the four numbers form an A.P., then the set of all possible values of common difference is {± 1, ± 2, ± 3, ± 4, ± 5}.

Ans. (c)

Solution: Statement-2 is false.

For $d = 6$, we have 1, 7, 13, 19 forms an A.P. We, now check the truth of Statement-1.

For $d = 1$, there are 17 A.P.'s. viz. (1, 2, 3, 4), (2, 3, 4, 5), ... (17, 18, 19, 20)

For $d = 2$, there are 14 A.P.'s viz. (1, 3, 5, 7), (2, 4, 6, 8), ... (14, 16, 18, 20)

For $d = 3$, there are 11 A.P.'s

For $d = 4$, there are 8 A.P.'s

For $d = 5$, there are 5 and for $d = 6$, there are 2 A.P.'s.

Thus, there are 17 + 14 + 11 + 8 + 5 + 2 = 57 A.P.'s

Also, total number of ways of choosing 4 numbers out of 20 is ${}^{20}C_4$.

$\therefore \quad$ The required probability $= \frac{57}{{}^{20}C_4} = \frac{1}{85}$.

Example 92: Let A and B be two events such that $P(A \cup B) = P(A \cap B)$.

Statement-1: $P(A \cap B') = P(A' \cap B) = 0$

Statement-2: $P(A) + P(B) = 1$.

Ans. (c)

Solution: $P(A \cup B) = P(A \cap B)$

$\Leftrightarrow \quad P(A) + P(B) - P(A \cap B) = P(A \cap B)$

$\Leftrightarrow \quad [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] = 0 \qquad (1)$

But $P(A), P(B) \geq P(A \cap B)$.

Thus, (1) is possible if and only if

$P(A) - P(A \cap B) = 0$ and $P(B) - P(A \cap B) = 0$

$\Leftrightarrow \quad P(A) = P(B) = P(A \cap B)$

$\therefore \quad P(A) + P(B) = 1$ need not hold.

Next, $P(A \cap B') = P(A) - P(A \cap B) = 0$

Similarly $P(A' \cap B) = P(B) - P(A \cap B) = 0$

Example 93: Statement-1: If $\frac{1}{5}(1 + 5p)$, $\frac{1}{3}(1 + 2p)$, $\frac{1}{3}(1 - p)$ and $\frac{1}{5}(1 - 3p)$ are probabilities of four mutually exclusive events, then p can take infinite number of values.

Statement-2: If A, B, C and D are three mutually exclusive events, then $P(A)$, $P(B)$, $P(C)$, $P(D) \geq 0$ and $P(A) + P(B) + P(C) + P(D) \leq 1$.

Ans. (a)

◎ **Solution:** Statement-2 is true. See theory.

$$\text{Now, } \frac{1}{5}(1+5p),\ \frac{1}{3}(1+2p),$$

$$\frac{1}{3}(1-p),\ \frac{1}{5}(1-3p) \geq 0$$

$$\Rightarrow \quad p \geq -1/5,\ p \geq -1/2,\ p \leq 1,\ p \leq 1/3$$

$$\Rightarrow \quad -1/5 \leq p \leq 1/3 \qquad (1)$$

$$\text{and } \frac{1}{5}(1+5p) + \frac{1}{3}(1+2p) + \frac{1}{3}(1-p) + \frac{1}{5}(1-3p) \leq 1$$

$$\Rightarrow \quad \frac{1}{5}(2+2p) + \frac{1}{3}(2+p) \leq 1$$

$$\Rightarrow \quad 6 + 6p + 10 + 5p \leq 15$$

$$\Rightarrow \quad 11p \leq -1 \quad \Rightarrow p \leq -1/11. \qquad (2)$$

From (1) and (2) we get $-1/5 \leq p \leq -1/11$

$\Rightarrow$ there are infinite values of p.

◉ **Example 94: Statement-1:** A natural number is chosen at random. The probability that sum of the digits of its square is 93 is 0.

Statement-2: A number is divisible by 31 if and only if sum of its digits is divisible by 31.

Ans. (c)

◎ **Solution:** Let $n \in \mathbf{N}$, be such that sum of digits of its square is 93.

Then $3 \mid n^2 \Rightarrow 3 \mid n \Rightarrow 9 \mid n^2$. A contradiction.

Statement-2 is clearly false.

◉ **Example 95:** Let $m \in \mathbf{N}$, and suppose three numbers are chosen at random from the number 1, 2, 3 … m.

Statement-1: If $m = 2n$ for some $n \in \mathbf{N}$, then the chosen numbers are in A.P. with probability $\dfrac{3}{2(2n-1)}$.

Statement-2: If $m = 2n + 1$, then the chosen numbers are in A.P. with probability $\dfrac{3n}{4n^2-1}$

Ans. (b)

◎ **Solution:** We can choose three numbers out of m in ${}^mC_3$ ways. Let numbers be x_1, x_2, x_3. Now, x_1, x_2, x_3 are in A.P. if and only if $x_1 + x_3 = 2x_2$, that is, if and only if other both x_1, x_3 are odd or both x_1, x_3 are even.

If $m = 2n$, x_1 and x_3 can be chosen in ${}^nC_2 + {}^nC_2 = n(n-1)$ ways.

In this case probability of the required event is $\dfrac{n(n-1)}{{}^{2n}C_3}$

$$= \frac{6n(n-1)}{2n(2n-1)(2n-2)} = \frac{3}{2(2n-1)}.$$

If $m = 2n + 1$, x_1 and x_3 can be chosen in ${}^{n+1}C_2 + {}^nC_2 = n^2$ ways.

In this case probability of the required event is

$$\frac{n^2}{{}^{2n+1}C_3} = \frac{3n}{4n^2-1}.$$

◉ **Example 96: Statement-1:** Let A, B, C be three mutually independent events.

Statement-1: A and $B \cup C$ are independent.

Statement-2: A and $B \cap C$ are independent.

Ans. (b)

◎ **Solution:** As A, B, C are mutually independent

$$P(B \cap C) = P(B)\,P(C),\ P(C \cap A) = P(C)\,P(A),$$
$$P(A \cap B) = P(A)\,P(B) \text{ and}$$
$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

$$\begin{aligned}\text{Now,}\quad P[A \cap (B \cup C)] &= P[(A \cap B) \cup (A \cap C)]\\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)\\ &= P(A)\,P(B) + P(A)\,P(C) - P(A)\,P(B)\,P(C)\\ &= P(A)\,[P(B) + P(C) - P(B)\,P(C)]\\ &= P(A)\,[P(B) + P(C) - P(B \cap C)]\\ &= P(A)\,P(B \cup C)\end{aligned}$$

and $P(A \cap B \cap C) = P(A)\,P(B)\,P(C) = P(A)\,P(B \cap C)$

◉ **Example 97:** Let A and B be two events such that $P(A \cup B) \geq \dfrac{3}{4}$ and $\dfrac{1}{8} \leq P(A \cap B) \leq \dfrac{3}{8}$.

Statement-1: $P(A) + P(B) \geq 7/8$

Statement-2: $P(A) + P(B) \leq 11/8$

Ans. (b)

◎ **Solution:** $\dfrac{3}{4} \leq P(A \cup B) \leq 1$

$$\Rightarrow \quad \frac{3}{4} \leq P(A) + P(B) - P(A \cap B) \leq 1$$

$$\Rightarrow \quad \frac{3}{4} + \frac{1}{8} \leq P(A) + P(B) \leq 1 + \frac{3}{8}$$

$$\Rightarrow \quad \frac{7}{8} \leq P(A) + P(B) \leq \frac{11}{8}$$

Thus, both the statements are correct but Statement-2 is not a correct explanation for Statement-1.

Example 98: An urn contains four balls bearing numbers 1, 2, 3 and 123 respectively. A ball is drawn at random from the urn. Let E_i, i =1, 2, 3 denote the event that digit i appears on the ball drawn.

Statement-1: $P(E_1 \cap E_2) = P(E_2 \cap E_3)$

$$= P(E_3 \cap E_1) = \frac{1}{4}$$

Statement-2: $P(E_1) = P(E_2) = P(E_3) = \frac{1}{2}$

Ans. (b)

Solution: We have

$$P(E_1) = \frac{2}{4} = P(E_2) = P(E_3)$$

and $\quad P(E_1 \cap E_2) = \frac{1}{4} \; P(E_2 \cap E_3) = P(E_3 \cap E_1)$

Thus, both the statements are correct but Statement-2 is not a correct explanation of Statement-1.

LEVEL 2

Straight Objective Type Questions

Example 99: If p and q are chosen from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, with replacement, the probability that the roots of the equation $x^2 + 2px + q = 0$ are real is

(a) 0.84 (b) 0.16

(c) 0.62 (d) 0.38

Ans. (a)

Solution: Required probability = 1 – P (the roots of $x^2 + 2px + q = 0$ are non-real)

The roots of $x^2 + 2px + q = 0$ will be non-real if $(2p)^2 - 4q < 0$, i.e. if $p^2 < q$.

We enumerate the possible values of p and q.

When q = 1, there is no value of p

When q = 2, 3, 4, possible value of p is 1

When q = 5, 6, 7, 8, 9 possible values of p are 1 and 2

When q = 10, possible values of p are 1, 2 and 3.

Thus, the number of pairs for which $x^2 + 2px + q = 0$ have non-real roots is

$$0 + 3 \times 1 + 5 \times 2 + 1 \times 3 = 16$$

Also, total number of possible pairs is 10 × 10 = 100.

Thus, probability of the required event $= 1 - \frac{16}{100} = 0.84$.

Example 100: A person writes 4 letters and addresses 4 envelopes. If the letters are placed in the envelopes at random, the probability that not all letters are placed in correct envelopes is

(a) $\frac{1}{24}$ (b) $\frac{11}{24}$

(c) $\frac{5}{8}$ (d) $\frac{23}{24}$

Ans. (d)

Solution: Required probability = 1 – P(all the letters are put in correct envelopes)

The number of ways of putting the letters in the envelopes $= {}^4P_4 = 4!$

The number of ways of putting all the letters in correct envelopes = 1

$\therefore \quad$ Required probability $= 1 - \frac{1}{24} = \frac{23}{24}$.

Example 101: A letter is known to have come from either TATANAGAR or CALCUTTA. On the envelope, just two consecutive letters, TA, are visible. The probability that the letter has come from CALCUTTA is

(a) $\frac{4}{11}$ (b) $\frac{1}{3}$

(c) $\frac{5}{12}$ (d) $\frac{7}{19}$

Ans. (a)

Solution: Let E_1 denote the event that the letter has come from TATANAGAR and E_2 the event that the letter has come from CALCUTTA. Let B denote the event that the two consecutive alphabets visible on the envelope are TA. We have $P(E_1) = 1/2$, $P(E_2) = 1/2$, $P(B|E_1) = 1/7$, $P(B|E_2) = 2/8$. Therefore, by the Bayes' theorem we have

$$P(E_2|B) = \frac{P(E_2)\, P(B \mid E_2)}{P(E_1)\, P(B \mid E_1) + P(E_2)\, P(B \mid E_2)} = \frac{4}{11}.$$

Example 102: A fair coin is tossed n times. If the probability that heads occurs 6 times is equal to the probability that heads occurs 8 times, then value of n is

(a) 24 (b) 48

(c) 14 (d) 16

Ans. (c)

Solution: Let X be the number of times heads occurs. Then X follows a binomial distribution with parameter n and p=1/2. We are given that $P(X = 6) = P(X = 8)$.

$$\Rightarrow \quad {}^nC_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{n-6} = {}^nC_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{n-8}$$

$$\Rightarrow \quad {}^nC_6\left(\frac{1}{2}\right)^n = {}^nC_8\left(\frac{1}{2}\right)^n$$

$\Rightarrow \quad {}^nC_6 = {}^nC_8 = {}^nC_{n-8}$

$\Rightarrow 6 = n-8$ or $n = 14$

◉ **Example 103:** If the standard deviation of the binomial distribution $(q + p)^{16}$ is 2, then mean of the distribution is

(a) 6 (b) 8

(c) 10 (d) 12

Ans. (b)

◎ **Solution:** We have $n = 16$, $npq = 2^2 = 4$

$\therefore \quad p(1 - p) = 1/4 \quad \Rightarrow \quad 4p^2 - 4p + 1 = 0$

$\Rightarrow \quad (2p - 1)^2 = 0 \quad \Rightarrow \quad p = 1/2$

Thus, mean $= np = 16(1/2) = 8$.

◉ **Example 104:** If A and B are two events such that $P(A \cup B) = 3/4$, $P(A \cap B) = 1/4$, $P(A') = 2/3$, then $P(A' \cup B)$ is

(a) $\dfrac{5}{12}$ (b) $\dfrac{3}{8}$

(c) $\dfrac{5}{8}$ (d) $\dfrac{1}{4}$

Ans. (a)

◎ **Solution:** $P(A) + P(B) = P(A \cup B) + P(A \cap B)$

$$= \frac{3}{4} + \frac{1}{4} = 1$$

Also $\quad P(A') = 2/3 \quad \Rightarrow \quad P(A) = 1/3$

$\therefore \quad P(B) = 1 - 1/3 = 2/3$

Now, $\quad P(A' \cap B) = P(B) - P(A \cap B)$

$$= \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

◉ **Example 105:** A die is thrown 5 times. Getting an odd number is consider a success. The variance of the distribution is

(a) $\dfrac{8}{3}$ (b) $\dfrac{3}{8}$

(c) $\dfrac{4}{5}$ (d) $\dfrac{5}{4}$

Ans. (d)

◎ **Solution:** Let X = number of successes.

Then $X \sim B(n, p)$ where $n = 5$, $p = 1/2$

$$\text{Var}(X) = npq = (5)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{5}{4}$$

◉ **Example 106:** The mean and variance of random variable X having a binomial distribution are 4 and 2 respectively, then $P(X = 1)$ is

(a) $\dfrac{1}{16}$ (b) $\dfrac{1}{8}$

(c) $\dfrac{1}{4}$ (d) $\dfrac{1}{32}$

Ans. (d)

◎ **Solution:** We are given

$$np = 4,\ npq = 2 \quad \Rightarrow \quad q = \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow \quad p = \frac{1}{2}$$

Now $\quad n\left(\dfrac{1}{2}\right) = 4 \quad \Rightarrow \quad n = 8$

$$\therefore \quad P(X = 1) = {}^nC_1\, pq^{n-1} = 8pq^7 = 8\left(\frac{1}{2}\right)^8 = \frac{1}{32}$$

◉ **Example 107:** If $(1 + 3p)/3$, $(1 - p)/4$ and $(1 - 2p)/2$ are the probabilities of three mutually exclusive events, then the set of all values of p is

(a) $1/3 \le p \le 1/2$ (b) $1/4 \le p \le 1/3$

(c) $-1 \le p \le 1/5$ (d) $-2 \le p \le 1/3$

Ans. (a)

◎ **Solution:** As $(1 + 3p)/3$, $(1 - p)/4$ and $(1 - 2p)/2$ are the probabilities of three mutually exclusive events

$$\frac{1+3p}{3} \ge 0,\ \frac{1-p}{4} \ge 0,\ \frac{1-2p}{2} \ge 0$$

and $$\frac{1+3p}{3} + \frac{1-p}{4} + \frac{1-2p}{2} \le 1$$

$\Rightarrow \quad p \ge 1/3, \quad p \le 1, \quad p \le 1/2, \quad p \ge 1/3.$

Thus, $\quad 1/3 \le p \le 1/2$.

◉ **Example 108:** A letter is taken at random from the letters of the word 'STATISTICS' and another letter is taken at random from the letters of the word 'ASSISTANT'. The probability that they are the same letter is

(a) 1/45 (b) 13/90

(c) 19/90 (d) 5/18

Ans. (c)

◎ **Solution:** Letters of the word STATISTICS are AIICSSSTTT

Letters of the word ASSISSTANT are AAINSSSTT

Common letters are *A, I, S* and *T*

Probability of choosing *A* is $\dfrac{1}{10} \times \dfrac{2}{9} = \dfrac{2}{90}$

Probability of choosing *I* is $\dfrac{2}{10} \times \dfrac{1}{9} = \dfrac{2}{90}$

Probability of choosing S is $\frac{3}{10} \times \frac{3}{9} = \frac{9}{90}$

Probability of choosing T is $\frac{3}{10} \times \frac{2}{9} = \frac{6}{90}$

$\therefore$ probability of required event $= \frac{2}{90} + \frac{2}{90} + \frac{9}{90} + \frac{6}{90} = \frac{19}{90}$

Example 109: For three events A, B and C, P (exactly one of the events A or B occurs) = P (exactly one of the events B and C occur) = P (exactly one of the events C or A occurs) = p and P (all the three events occur simultaneously) = p^2, where $0 < p < 1/2$. Then the probability of at least one of the three events A, B and C occurring is

(a) $\frac{3p + 2p^2}{2}$ (b) $\frac{p + 3p^2}{4}$

(c) $\frac{p + 3p^2}{2}$ (d) $\frac{3p + 2p^2}{4}$

Ans. (a)

Solution: We know that

P (exactly one of A or B occurs) = $P(A) + P(B) - 2P(A \cap B)$

Therefore, $P(A) + P(B) - 2P(A \cap B) = p$ (1)

Similarly, $P(B) + P(C) - 2P(B \cap C) = p$ (2)

and $P(C) + P(A) - 2P(C \cap A) = p$ (3)

Adding (1), (2) and (3) we get

$$2[P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)] = 3p$$

$$\Rightarrow P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = 3p/2 \quad (4)$$

We are also given that $P(A \cap B \cap C) = p^2$ (5)

Now, P(at least one of A, B and C) = $P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$$= \frac{3p}{2} + p^2 = \frac{3p + 2p^2}{2} \quad \text{[using (4) and (5)]}$$

Example 110: Let A, B, C, be the three mutually independent events. Consider the two statements S_1 and S_2.

S_1 : A and $B \cup C$ are independent

S_2 : A and $B \cap C$ are independent

Then

(a) Both S_1 and S_2 are true

(b) Only S_1 is true

(c) Only S_2 is true

(d) Neither S_1 nor S_2 is true.

Ans. (a)

Solution: We are given that

$P(A \cap B) = P(A)\,P(B)$, $P(B \cap C) = P(B)\,P(C)$,

$P(C \cap A) = P(C)\,P(A)$,

and $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

We have

$$P(A \cap (B \cap C)) = P(A \cap B \cap C) = P(A)\,P(B)\,P(C) = P(A)\,P(B \cap C).$$

$\Rightarrow A$ and $B \cap C$ are independent. Therefore, S_2 is true

Also
$$\begin{aligned} P[(A \cap (B \cup C)] &= P[(A \cap B) \cup (A \cap C)] \\ &= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)] \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \\ &= P(A)\,P(B) + P(A)\,P(C) - P(A)\,P(B)\,P(C) \\ &= P(A)\,[P(B) + P(C) - P(B)\,P(C)] \\ &= P(A)\,[P(B) + P(C) - P(B \cap C)] = P(A)\,P(B \cup C) \end{aligned}$$

$\therefore$ A and $B \cup C$ are independent.

Example 111: Three identical dice are rolled. The probability that the same number appears on each of them is

(a) 1/6 (b) 1/36

(c) 1/18 (d) 3/28

Ans. (b)

Solution: Total number of cases = 6^3.

The number of ways in which the same number appears on each of them is ${}^6C_1 = 6$

Thus, probability of the required event = $\frac{6}{6^3} = \frac{1}{36}$.

Example 112: A bag contains some white and some black balls, all combinations of balls being equally likely. The total number of balls in the bag is 10. If three balls are drawn at random without replacement and all of them are found to be black, the probability that the bag contains 1 white and 9 black balls is

(a) 14/55 (b) 12/55

(c) 2/11 (d) 8/55

Ans. (a)

Solution: Let E_i denote the event that the bag contains i black and $(10 - i)$ white balls $(i = 0, 1, 2, \ldots, 10)$. Let A denote the event that the three balls drawn at random from the bag are black. We have

$$P(E_i) = \frac{1}{11} \quad (i = 0, 1, 2, \ldots, 10)$$

$$P(A|E_i) = 0 \text{ for } i = 0, 1, 2$$

and $$P(A|E_i) = \frac{{}^iC_3}{{}^{10}C_3} \text{ for } i \geq 3$$

Now, by the total probability rule

$$P(A) = \sum_{i=0}^{10} P(Ei)\,P(A\,|\,Ei)$$

$$= \frac{1}{11}\times\frac{1}{^{10}C_3}\;[^3C_3 + {^4C_3} + \ldots + {^{10}C_3}]$$

But $^3C_3 + {^4C_3} + {^5C_3} + \ldots + {^{10}C_3}$

$$= {^4C_4} + {^4C_3} + {^5C_3} + \ldots + {^{10}C_3}$$

$$[^3C_3 = 1 = 4\,C_4]$$

$$= {^5C_4} + {^5C_3} + {^6C_3} + \ldots + {^{10}C_3}$$

$$= {^6C_4} + {^6C_3} + \ldots + {^{10}C_3} = \ldots = {^{11}C_4}$$

Thus, $$P(A) = \frac{^{11}C_4}{11\times{^{10}C_3}} = \frac{1}{4}$$

By the Bayes' rule

$$P\,(E_9|A) = \frac{P\,(E_9)\,P\,(A\,|\,E_9)}{P\,(A)} = \frac{14}{55}.$$

EXERCISES

Concept-based Straight Objective Type Questions

1. If $2n$ boys are randomly divided into 2 teams containing n boys each, the probability, that the 2 tallest boys are in different teams, is

   (a) $\frac{n}{2n-1}$
   (b) $\frac{1}{2}$
   (c) $\frac{2n}{2n+1}$
   (d) $\frac{n^2+1}{n^2+n}$

2. Two events A and B are such that $P(B) = 0.55$ and $P(A \cap B') = 0.15$, then probability that at least one of A, B occurs is

   (a) 0.70
   (b) 0.20
   (c) 0.35
   (d) 0.30

3. Two numbers are randomly selected from the first 100 natural numbers. The probability the product is divisible by 17 is

   (a) $\frac{931}{990}$
   (b) $\frac{97}{990}$
   (c) $\frac{101}{990}$
   (d) $\frac{879}{990}$

4. Out of 30 consecutive numbers, two are chosen at random, then the probability that sum of the numbers is odd is

   (a) $\frac{14}{29}$
   (b) $\frac{15}{29}$
   (c) $\frac{1}{29}$
   (d) $\frac{3}{29}$

5. Suppose A and B are two events such that $P(A) = 0.5$ and $P(B) = 0.8$, then which one of the following is **not** true?

   (a) $P(A \cap B) \leq 0.5$
   (b) $P(A \cap B) \geq 0.3$
   (c) $P(A' \cap B) \leq 0.5$
   (d) $P(A \cap B') \leq 0.1$

6. Let A, B and C be three events and suppose that simultaneous occurrence of A and B implies the occurrence of C, then

   (a) $P(C) \geq P(A) + P(B)$
   (b) $P(C) \geq P(A) + P(B) - 1$
   (c) $P(C) < P(A) + P(B) - P(A \cap B)$
   (d) none of these

7. Let A and B be two events such that $P(A|B) = \frac{1}{2}$, $P(B|A) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{6}$, then which one of the following is **not** true?

   (a) $P(A \cup B) = \frac{2}{3}$
   (b) A and B are independent
   (c) A and B are not independent
   (d) $P(A' \cap B) = \frac{1}{6}$

8. An urn contains four balls bearing numbers 1, 2, 3 and 123 respectively. A ball is drawn at random from the urn. Let E_i, $i = 1, 2, 3$ denote the event that digit i appears on the ball drawn. Which one of the following is **not** true?

   (a) E_1 and E_2 are independent
   (b) E_2 and E_3 are independent

(c) E_3 and E_1 are independent
(d) E_1, E_2, E_3 are independent

9. In a game called "odd man out" each person in a group of $n \geq 2$ persons tosses a coin to determine who will buy refreshment for the entire group. The odd man out is the one whose outcome is different from the rest of the persons. The probability that there is a loser in the game is

(a) $\dfrac{n}{2^{n-1}}$ (b) $\dfrac{n+1}{2^n}$
(c) $\dfrac{n!}{2^n}$ (d) $\dfrac{1}{2^{n-1}}$

10. Consider the circuit

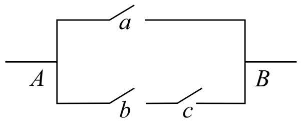


Fig. 24.8

If the probability that a circuit is closed is p and the probability that current flows from A to B is 4/9, then value of p is

(a) $\dfrac{1}{2}$ (b) $\dfrac{1}{9}$
(c) $\dfrac{1}{3}$ (d) $\dfrac{3}{4}$

LEVEL 1

Straight Objective Type Questions

11. A bag contains 100 tickets numbered 1 to 100. If one ticket is picked up at random, then the probability that the number is divisible by 3 or 5 is
(a) 0.47 (b) 0.53
(c) 0.52 (d) 0.61

12. Suppose n students appear in an examination. Let X = the number of students who pass the examination. Suppose $P(X = k) = \lambda k^2$, then value of λ is

(a) $\dfrac{2}{n(n+1)}$ (b) $\dfrac{6}{n(n+1)(2n+1)}$
(c) $\dfrac{4}{n^2(n+1)^2}$ (d) $\dfrac{2n+1}{n^2(n+1)^2}$

13. Suppose two numbers a and b are chosen randomly from the set {1, 2, 3, 4, 5, 6}. The probability that function $f(x) = x^3 + ax^2 + bx$ is strictly increasing function on **R** is:

(a) $\dfrac{1}{15}$ (b) $\dfrac{2}{15}$
(c) $\dfrac{13}{15}$ (d) $\dfrac{14}{15}$

14. Two dice are rolled together and the numbers x, y on them form complex number $z = x + iy$. The probability that $|z| \leq 3$ is

(a) $\dfrac{1}{3}$ (b) $\dfrac{2}{3}$
(c) $\dfrac{1}{12}$ (d) $\dfrac{5}{36}$

15. Suppose A is an event which is independent of itself and B is any other event, then
(a) $P(A \cap B) \leq P(A)\, P(B)$
(b) $P(A \cap B) < P(B)$
(c) $P(A \cap B) = 0$
(d) $P(A' \cup B') = 1$

16. Six boys and six girls sit in a row randomly. The probability that all the six girls sit together is

(a) $\dfrac{1}{132}$ (b) $\dfrac{7}{462}$
(c) $\dfrac{5}{462}$ (d) $\dfrac{1}{924}$

17. If $P(A \cap B) = 1/2$, $P(A' \cap B') = 1/3$, $P(A) = p$ and $P(B) = 2p$ then value of p is

(a) $\dfrac{1}{3}$ (b) $\dfrac{7}{18}$
(c) $\dfrac{4}{9}$ (d) $\dfrac{1}{2}$

18. Out of 20 consecutive integers, two are chosen at random. The probability that their product is odd is

(a) $\dfrac{1}{9}$ (b) $\dfrac{9}{38}$
(c) $\dfrac{2}{19}$ (d) $\dfrac{7}{19}$

19. An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, the number of outcomes that B must have so that A and B are independent, is

(a) 2, 4, or 8 (b) 3, 6 or 9
(c) 4 or 8 (d) 5 or 10

20. A box contains 24 balls of which 12 are white and 12 are black. The balls are drawn at random from the box one at a time with the replacement. The probability that a white ball is drawn for the 4th time on the 7th draw is

(a) $\frac{5}{64}$ (b) $\frac{27}{32}$
(c) $\frac{5}{32}$ (d) $\frac{1}{2}$

21. India plays two matches each with West Indies and Australia. In any match, the probabilities of India getting 0, 1 and 2 points are 0.45, 0.05 and 0.50, respectively. Assuming that the outcomes are independent, the probability of India getting at least 7 points is

(a) 0.875 (b) 0.0875
(c) 0.0625 (d) 0.0250

22. If E and F are two events associated with a random experiment for which $P(E) = 0.6$, $P(E \cup F) = 0.85$ and $P(E \cap F) = 0.42$, then $P(F)$ is

(a) 0.6 (b) 0.67
(c) 0.7 (d) 0.73

23. If E and F are events with $P(E) \leq P(F)$ and $P(E \cap F) > 0$, then

(a) occurrence of $E \Rightarrow$ occurrence of F
(b) occurrence of $F \Rightarrow$ occurrence of E
(c) non-occurrence of $E \Rightarrow$ non-occurrence of F
(d) none of the above implication hold

24. If the integers m and n are chosen at random from 1 and 100, then the probability that a number of the form $7^m + 7^n$ is divisible by 5 equals is

(a) $\frac{1}{4}$ (b) $\frac{1}{7}$
(c) $\frac{1}{8}$ (d) $\frac{1}{49}$

25. The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c respectively. Of these subjects, the student has a 75% chance of passing in at least one, a 50% chance of passing in at least two, and a 40% chance of passing in exactly two. Then

(a) $p + m + c = \frac{19}{20}$ (b) $p + m + c = \frac{27}{20}$
(c) $pmc = \frac{1}{3}$ (d) $pmc = \frac{1}{7}$

26. One ticket is selected at random from 100 tickets numbered 00, 01, 02, ..., 99. Suppose X and Y are the sum and product of the digit found on the ticket then $P(X = 7|Y = 0)$ is given by

(a) $\frac{2}{13}$ (b) $\frac{2}{19}$
(c) $\frac{1}{50}$ (d) $\frac{3}{19}$

27. A mapping is selected at random from the set of all the mappings of the set $A = \{1, 2, \ldots, n\}$ into itself. The probability that the mapping selected is one-to-one is

(a) $\frac{1}{n^n}$ (b) $\frac{1}{n!}$
(c) $\frac{(n-1)!}{n^{n-1}}$ (d) $\frac{n!}{n^{n-1}}$

28. A natural number x is choosen at random from the first 100 natural numbers. The probability that

$$x + \frac{100}{x} > 50$$

is

(a) $\frac{1}{10}$ (b) $\frac{11}{20}$
(c) $\frac{1}{20}$ (d) $\frac{13}{50}$

29. A bag contains $(2n + 1)$ coins. It is known that n of these coins have a head on both sides, whereas the remaining $(n + 1)$ coins are fair. A coin is picked up at random from the bag and tossed. If the probability that the toss results in a head is $\frac{31}{42}$, then n is equal to

(a) 10 (b) 11
(c) 12 (d) 13

30. If X and Y are independent binomial variate $B(5, 1/2)$ and $B(7, 1/2)$, then $P(X + Y = 3)$ is

(a) $\frac{55}{1024}$ (b) $\frac{55}{4098}$
(c) $\frac{55}{2048}$ (d) $\frac{55}{128}$

31. Suppose n (≥ 3) persons are arranged in a row. The probability that two particular persons are not together is

(a) $1 - \frac{2}{n}$ (b) $\frac{2}{n-1}$
(c) $1 - \frac{1}{n}$ (d) $\frac{2}{n-1}$

32. A number x is chosen at random from the set $\{1, 2, 3, ..., 100\}$. Let p = probability that x is divisible by 19 and q = probability that x is divisible by 31, then $p + q$ is equal to

(a) 0.5 (b) 0.6
(c) 0.7 (d) 0.8

33. If the letters of the word PROBABILITY are written down at random in a row, the probability that two Bs are together is

(a) $\frac{2}{11}$ (b) $\frac{10}{11}$
(c) $\frac{3}{11}$ (d) $\frac{6}{11}$

34. If four positive integers are taken at random and multiplied together, then the probability that the last digit is 1, 3, 7 or 9 is

(a) $\frac{1}{8}$ (b) $\frac{2}{7}$
(c) $\frac{1}{625}$ (d) $\frac{16}{625}$

35. Two contestants play a game as follows: each is asked to select a digit from 1 to 9. If the two digits match they both win a prize. The probability that they will win a prize in a single trial is

(a) $\frac{1}{81}$ (b) $\frac{7}{81}$
(c) $\frac{1}{9}$ (d) $\frac{3}{11}$

36. If A and B are two events, then which of the following *does not* represent the probability that exactly one of A, B occurs is

(a) $P(A) + P(B) - P(A \cap B)$
(b) $P(A \cap B') + P(A' \cap B)$
(c) $P(A') + P(B') - 2P(A' \cap B')$
(d) $P(A) + P(B) - 2P(A \cap \text{B})$

37. If A and B are two events, then which one of the following is not always true

(a) $P(A \cap \text{B}) \ge P(A) + P(B) - 1$
(b) $P(A \cap B) \le P(A)$
(c) $P(A' \cap B') \ge P(A') + P(B') - 1$
(d) $P(A \cap B) = P(A)\,P(B)$

38. A sum of money is rounded off to the nearest rupee. The probability that the round off error is at most 10 paise is

(a) $\frac{10}{101}$ (b) $\frac{11}{101}$
(c) $\frac{12}{101}$ (d) $\frac{21}{100}$

39. The probability that an electric bulb will last 150 days or more is 0.7 and that it will last at most 160 days is 0.8. The probability that the bulb will last 150 to 160 days is

(a) 0.5 (b) 0.3
(c) 0.56 (d) 0.28

40. A bag contains three tickets numbered 1, 2 and 3. A ticket is drawn at random and put back in the bag, and this is done four times. The probability that the sum of the numbers drawn is even is

(a) $\frac{40}{81}$ (b) $\frac{41}{81}$
(c) $\frac{14}{27}$ (d) $\frac{13}{81}$

41. Two numbers x and y are selected at random from the set $\{1, 2, 3, \ldots, 3N\}$. The probability that $x^2 - y^2$ is divisible by 3 is

(a) $\frac{(3N-1)}{3N}$ (b) $\frac{(N-1)}{N}$
(c) $\frac{(5N-3)}{(9N-3)}$ (d) $\frac{N}{3N-5}$

42. Three persons A, B and C are to speak at a function along with 7 other persons. The probability that A, B, and C speak together

(a) $\frac{3}{70}$ (b) $\frac{1}{15}$
(c) $\frac{3}{7}$ (d) $\frac{1}{7!}$

43. A positive integer is chosen at random. The probability that the sum of the digits of its square is 33 is

(a) $\frac{1}{33}$ (b) $\frac{2}{33}$
(c) $\frac{1}{11}$ (d) 0

44. A lottery sells n^2 tickets and declares n prizes. If a man purchases n tickets, the probability of his winning exactly one prize is

(a) $\frac{(n^2-n)!}{(n^2)!}$ (b) $\frac{1}{2n}$
(c) $\frac{(n-1)!^2}{(n^2)!}$ (d) none of these

45. Let x be a non-zero real number. A determinant is chosen from the set of all determinants of order 2 with entries x or $-x$ only. The probability that the value of the determinant is non-zero is

(a) $\frac{3}{16}$ (b) $\frac{1}{4}$
(c) $\frac{1}{2}$ (d) $\frac{1}{8}$

46. A die is rolled three times. The probability of getting a number larger than the previous number each time is

(a) $\frac{5}{72}$ (b) $\frac{5}{54}$

(c) $\frac{13}{216}$ (d) $\frac{1}{18}$

47. If $\frac{1+4p}{4}, \frac{1-p}{4}$ and $\frac{1-2p}{2}$ are the probabilities of three mutually exclusive events, then

(a) $\frac{1}{3} \le p \le \frac{1}{2}$ (b) $\frac{1}{2} \le p \le \frac{2}{3}$

(c) $\frac{1}{6} \le p \le \frac{1}{2}$ (d) $0 \le p \le \frac{1}{2}$

48. Two non-negative integers are chosen at random. The probability that the sum of the square is divisible by 5 is

(a) $\frac{9}{16}$ (b) $\frac{9}{25}$

(c) $\frac{9}{17}$ (d) 0

49. A is a set containing n elements. Two subsets P and Q of A are chosen at random. (P and Q may have elements in common). The probability that $P \cap Q = \phi$ is

(a) $\left(\frac{3}{4}\right)^n$ (b) $\frac{1}{4}n$

(c) $\frac{{}^nC_2}{2^n}$ (d) none of these

50. Suppose A is a set containing n elements. Two subsets P and Q of A are chosen at random. If the probability that P is a subset of Q is 243/1024, then n is equal to

(a) 3 (b) 4

(c) 5 (d) 6

51. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with these vertics is equilateral is

(a) $\frac{1}{2}$ (b) $\frac{1}{5}$

(c) $\frac{1}{10}$ (d) $\frac{1}{20}$

52. Each of the n urns contains 4 white and 6 black balls. The (n +1)th urn contains 5 white and 5 black balls. One of the (n +1) urns is chosen at random and two balls are drawn from it without replacement. Both the balls turn out to be black. If the probability that the (n +1)th urn was chosen to draw the balls is 1/16, then value of n is

(a) 10 (b) 11

(c) 12 (d) 13

53. Two numbers x and y are chosen at random from the set {0, 1, 2,…, 10}. The probability that $|x - y| \le 5$, is

(a) $\frac{8}{11}$ (b) $\frac{7}{11}$

(c) $\frac{1}{2}$ (d) $\frac{3}{11}$

54. In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is 1/3 and the probability that he copies the answer is p. The probability that his answer is correct, given that he copied it, is 1/8. If the probability that he knew the answer to question, given that he correctly answered is 8/11, then value of p is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{6}$ (d) $\frac{1}{12}$

55. A fair coin is tossed n times. Let X = the number of times head occurs. If $P(X = 4)$, $P(X = 5)$ and $P(X = 6)$ are in AP, then the value of n is

(a) 7 (b) 10

(c) 12 (d) 24

56. A fair coin is tossed a fixed number of times. If the probability of getting exactly 3 heads equals the probability of getting exactly 5 heads, then probability of getting exactly one head is

(a) $\frac{1}{64}$ (b) $\frac{1}{32}$

(c) $\frac{1}{16}$ (d) $\frac{1}{8}$

57. Let E, F, G be pairwise independent events with $P(G) > 0$ and $P(E \cap F \cap G) = 0$. Then $P(E' \cap F' | G)$ equals

(a) $P(E') + P(F')$

(b) $P(E') - P(F')$

(c) $P(E') - P(F)$

(d) $P(E) - P(F')$

58. If $P(E) = 0.40$, $P(F) = 0.35$, $P(E \cup F) = 0.55$, then $P(F/E)$ is

(a) 0.5 (b) 0.25

(c) 0.6 (d) 0.7

59. If the mean and variance of a binomial distribution are 4 and 3 respectively, then $P(X = 1)$ is

(a) $\left(\frac{3}{4}\right)^{16}$ (b) $4\left(\frac{3}{4}\right)^{15}$

(c) $\frac{1}{4}$ (d) $\frac{1}{16}$

60. A box contains 20 cards of these 10 have letter I printed on them and the remaining 10 have T printed on them. 3 cards are drawn from the box, the probability that we can write IIT with these cards is

(a) $\frac{9}{80}$ (b) $\frac{1}{8}$

(c) $\frac{4}{27}$ (d) $\frac{15}{38}$

61. A and B toss a coin alternatively till one of them gets a head and wins the game. If A begins the game, the probability B wins the game is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{2}{3}$

62. A fair coin is tossed repeatedly. If head and tail appear alternatively on first 7 tosses; then the probability that head appears on the eighth toss is

(a) $\frac{1}{2}$ (b) $\frac{1}{128}$

(c) $\frac{1}{256}$ (d) $\frac{7}{256}$

63. If $P(A) > 0$, then the event A is independent of itself if and only if $P(A)$ is

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) 1 (d) $\frac{1}{4}$

64. If A and B are two events, then $P(A \cup B) = P(A \cap B)$ if and only if

(a) $P(A) + P(B) = 1$ (b) $P(A) = P(B)$

(c) $P(A) + P(B) > 1$ (d) $P(A) + P(B) < 1$

65. A fair die is rolled repeatedly. The probability that 1 occurs for the first time at the even throw is

(a) $\frac{1}{6}$ (b) $\frac{5}{11}$

(c) $\frac{6}{11}$ (d) $\frac{3}{11}$

66. Suppose A and B are two events such that $P(A) > 0$ and $P(B) \neq 1$, then $P(A' \mid B')$ equals

(a) $[1-P(A \cup B)]/P(B')$

(b) $P(A')/P(B')$

(c) $1-P(A|B)$

(d) $1-P(A'|B)$

67. Two unbiased die are rolled together.
Let $A = \{(a, b): a + b = 11\}$ and $B = \{(a, b): a \neq 5\}$, then $P(A|B)$ equals

(a) 1/30 (b) 1/15

(c) 2/15 (d) 5/6

68. A bag contains w white and b black balls. A ball is drawn at random and put back in the bag with k additional balls of the same colour. A ball is again drawn at random from the bag. The probability that it is black is

(a) $(b + k)/(w + b + k)$

(b) $b/(w + b + k)$

(c) $b/(w + b)$

(d) $b/(w + b + 2k)$

69. The probability distribution of a random variable X is given as follows:

x	1	2	3	$2k$	$3k$	$5k$
$P(x = x)$	1/2	1/5	3/25	1/10	1/25	1/25

If $E(X) = 2.3$, then value of k is

(a) 1/2 (b) 5/4

(c) 3 (d) 2

70. Let A and B be two events such that $P(A|B) = P(A'|B') = p$ and $P(B) = 0.05$. The value of p so that $P(B|A) = 0.9$ is

(a) 21/430 (b) 173/430

(c) 171/430 (d) 95/430

71. In the Fig. 24.9 the number represents the probability of the event. Which one of following is not true ?

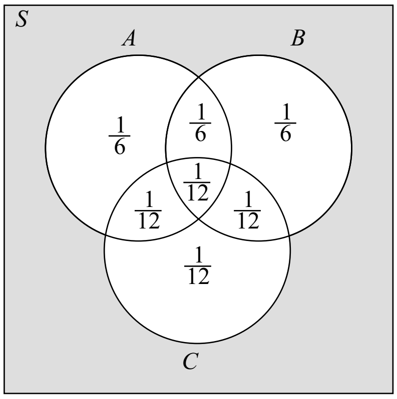


Fig. 24.9

(a) A and B are independent

(b) A and C are independent

(c) B and C are independent

(d) A and $B \cap C$ are dependent.

72. Three numbers are chosen at random from the first 20 numbers 1, 2, 3, … 20. The probability the selected numbers forms a G.P., is

(a) 1/20 (b) 1/30
(c) 1/45 (d) none of these

73. From a bag containing 30 tickets numbered 1, 2, 3, … 30, three tickets are selected at random without replacement. Suppose these tickets are arranged as $x_1 < x_2 < x_3$. The probability that $x_2 = 15$ and $x_3 > 20$, is
(a) 1/29 (b) 1/7
(c) 1/19 (d) 1/11

74. Suppose an unbiased die has n sides numbered $i = 1, 2, \ldots n$. The die is tossed n times (assume independence) and a "match" is defined to be the occurrence of side i on the its roll. The probability of at least one match is
(a) $\left(1-\frac{1}{n}\right)^n$ (b) $\frac{1}{n^n}$
(c) $1-\left(1-\frac{1}{n}\right)^n$ (d) $1-\frac{1}{n^n}$

75. Three numbers are chosen at random (without replacement) from the numbers 1, 2,... 20. The probability that the numbers are not consecutive are
(a) $\frac{3}{190}$ (b) $\frac{7}{190}$
(c) $\frac{171}{190}$ (d) $\frac{187}{190}$

Assertion-Reason Type Questions

76. Let A, B and C be three events such that $P(C) = 0$.
Statement-1: $P(A \cap B \cap C) = 0$
Statement-2: $P(A \cup B \cup C) = P(A \cup B)$

77. Let A, B and C be three events such that A is independent of both B and C.
Statement-1: A is independent of $B \cup C$.
Statement-2: A is independent of $B \cap C$.

78. A set P contains n elements. Two boys Rakshit and Gitansh independently pick up two subsets Q and R of P.
Statement-1: Probability $Q \cap R = \phi$ is $\left(\frac{3}{4}\right)^n$
Statement-2: Probability $Q \cup R = P$ is $\left(\frac{1}{2}\right)^n$.

79. A fair die is thrown twice. Let (a, b) denote the outcome in which the first throw shows a and the second shows b. Let A and B be the following two events.
$A = \{(a, b) \mid a \text{ is odd}\}$, $B = \{(a, b) \mid b \text{ is odd}\}$
Statement-1: If $C = \{(a, b) \mid a + b \text{ is odd}\}$, then $P(A \cap B \cap C) = 1/8$
Statement-2: If $D = \{(a, b) \mid a + b \text{ is even}\}$, then $P(A \cap B \cap D \mid A \cup B) = 1/3$

80. A man P speaks truth with probability p and another man Q speaks truth with probability $2p$.
Statement-1: If P and Q contradict each other with probability 1/2, then there are two values of p.
Statement-2: A quadratic equation with real coefficients has two real roots.

LEVEL 2

Straight Objective Type Questions

81. The probability that an event A occurrs in a single trial of an experiment is 0.6. Three independent trials of the experiment are performed. The probability that the event A occurs at least once is
(a) 0.936 (b) 0.932
(c) 0.948 (d) 0.946

82. The probability of occurrence of an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive, then the probability of occurrence of neither A nor B is
(a) 0.6 (b) 0.5
(c) 0.4 (d) 0.2

83. Suppose $X \sim B(n, p)$ and $P(X = 3) = P(X = 5)$. If $p > 1/2$, then
(a) $n \le 7$ (b) $n > 8$
(c) $n \ge 9$ (d) none of these

84. In a three throw of two dice, the probability throwing doublets not more than twice is
(a) $\frac{1}{6}$ (b) $\frac{5}{72}$
(c) $\frac{215}{216}$ (d) none of these

85. A bag contains 4 brown and 5 white socks. A man pulls two socks at random without replacement. The

probability that the man gets both the socks of the same colour is

(a) $\frac{5}{108}$ (b) $\frac{1}{6}$

(c) $\frac{5}{18}$ (d) $\frac{4}{9}$

86. Given a throw of three dice shows different face, the probability that at least one face shows 6 is

(a) $\frac{1}{2}$ (b) $\frac{1}{6}$

(c) $\frac{1}{3}$ (d) $\frac{5}{19}$

87. The probability that a student is not a car driver is 1/5. The probability that out of 5 students, exactly 4 are car drivers is

(a) $\left(\frac{4}{5}\right)^4$ (b) $({}^5C_4)\left(\frac{4}{5}\right)^4$

(c) $\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right)$ (d) none of these

88. Four persons are selected from a group of 4 men, 2 women and 3 children. The probability that exactly two of them are men is

(a) $\frac{9}{11}$ (b) $\frac{10}{23}$

(c) $\frac{11}{24}$ (d) $\frac{10}{21}$

89. Fifteen persons, among whom are A and B, sit down at random around a round table. The probability that there are exactly 4 persons between A and B is

(a) $\frac{1}{11}$ (b) $\frac{2}{7}$

(c) $\frac{1}{7}$ (d) $\frac{1}{9}$

90. There are 3 works. One is of 3 volumes and one is of 4 volumes and one is of exactly one volume. If the books are placed in a random order on a shelf, the probability that all the volumes of the same work are together is

(a) $\frac{1}{70}$ (b) $\frac{3}{140}$

(c) $\frac{1}{65}$ (d) $\frac{3}{130}$

91. If the mean of a binomial distribution with 9 trials is 6, then its variance is

(a) 2 (b) 3

(c) 4 (d) $\sqrt{2}$

92. For a binomial variable X if $n = 5$ and $P(X = 1) = 8P(X = 3)$, then p is given by

(a) $\frac{4}{5}$ (b) $\frac{1}{5}$

(c) $\frac{1}{3}$ (d) $\frac{2}{3}$

Previous Years' AIEEE/JEE Main Questions

1. Events A, B, C are mutually exclusive events such that $P(A) = \frac{1}{3}(3x + 1)$, $P(B) = \frac{1}{4}(1 - x)$ and $P(C) = \frac{1}{2}(1 - 2x)$. Then set of possible values of x are in the interval

(a) $\left[\frac{1}{3}, \frac{2}{3}\right]$ (b) $\left[\frac{1}{3}, \frac{13}{3}\right]$

(c) [0, 1] (d) $\left[\frac{1}{3}, \frac{1}{2}\right]$ **[2003]**

2. Five horses are in a race. Mr. A selected two of the horses at random and bets on them. The probability than Mr. A selected the wining horse is

(a) $\frac{3}{5}$ (b) $\frac{1}{5}$

(c) $\frac{2}{5}$ (d) $\frac{4}{5}$ **[2003]**

3. The mean and variance of a random variable X having a binomial distribution are 4 and 2 respectively, then $P(X = 1)$ is

(a) $\frac{1}{16}$ (b) $\frac{1}{8}$

(c) $\frac{1}{4}$ (d) $\frac{1}{32}$ **[2003]**

4. A random variable X has the probability distribution:

x:	1	2	3	4	5	6	7	8
$p(x)$:	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events $E = \{X \text{ is prime number}\}$ and $F = \{X < 4\}$, the probability $P(E \cup F)$ is

(a) 0.35 (b) 0.77
(c) 0.87 (d) 0.50 **[2004]**

5. The probability A speaks truth is $\frac{4}{5}$, while the probability for B is $\frac{3}{4}$. The probability that they contradict each other when asked to speak on a fact is

(a) $\frac{7}{20}$ (b) $\frac{1}{5}$
(c) $\frac{3}{20}$ (d) $\frac{4}{5}$ **[2004]**

6. The mean and the variance of a binomial distribution are 4 and 2 respectively. The probability of 2 successes is

(a) $\frac{128}{256}$ (b) $\frac{219}{256}$
(c) $\frac{37}{256}$ (d) $\frac{28}{256}$ **[2004]**

7. If A and B are two events such that $P[(A \cup B)'] = \frac{1}{6}$, $P(A \cup B) = \frac{1}{4}$ and $P(A') = \frac{1}{4}$, then events A and B are

(a) independent but not equally likely
(b) mutually exclusive and independent
(c) equally likely and mutually exclusive
(d) equally likely but not independent **[2005]**

8. Three houses are available in a locality. Three persons apply for the houses. Each applies for one house without consulting others. The probability that all the three apply for the same house.

(a) $\frac{8}{9}$ (b) $\frac{7}{9}$
(c) $\frac{2}{9}$ (d) $\frac{1}{9}$ **[2005]**

9. A pair of fair dice is thrown independently three times. The probability of getting a score of exactly 9 twice is

(a) $\frac{1}{729}$ (b) $\frac{8}{9}$
(c) $\frac{8}{729}$ (d) $\frac{8}{243}$ **[2007]**

10*. Two aeroplanes I and II bomb a target in succession. The probabilities of I and II scoring a hit correctly are 0.3 and 0.2, respectively. The second plane will bomb only if the first misses the target. The probability that the target is hit by the second plane is

(a) 0.06 (b) 0.14
(c) 0.2 (d) 0.7 **[2007]**

11. A die is thrown. Let A be the event that the number obtained is greater than 3. Let B be the event that the number obtained is less than 5. Then $P(A \cup B)$ is

(a) $\frac{3}{5}$ (b) 0
(c) 1 (d) $\frac{2}{5}$ **[2008]**

12. One ticket is selected at random from 50 tickets numbered 00, 01, 02, .., 49. Then the probability that the sum of the digits on the selected ticket is 8, given that product of these digits is zero, equals

(a) $\frac{5}{14}$ (b) $\frac{1}{50}$
(c) $\frac{1}{14}$ (d) $\frac{1}{7}$ **[2009]**

13. In a binomial distribution $B\left(n, p = \frac{1}{4}\right)$, if the probability of at least one success is greater than or equal to $\frac{9}{10}$, then n is greater than

(a) $\frac{9}{\log 4 - \log 3}$ (b) $\frac{4}{\log 4 - \log 3}$
(c) $\frac{1}{\log 4 - \log 3}$ (d) $\frac{1}{\log 4 - \log 3}$ **[2009]**

14. An urn contains nine balls of which three are red, four are blue and two are green. Three balls are drawn at random without replacement from the urn. The probability that three balls have different colours is

(a) $\frac{1}{21}$ (b) $\frac{2}{23}$
(c) $\frac{1}{3}$ (d) $\frac{2}{7}$ **[2010]**

15. Four numbers are chosen at random (without replacement) from the set {1, 2, 3, ..., 20}.

Statement-1: The probability that the chosen numbers when arranged in some order will form an AP is 1/85.

Statement-2: If the four chosen numbers form and AP, then the set of all possible values of common difference is {±1, ±2, ±3, ±4, ±5} **[2010]**

16. Consider 5 independent Bernoulli's trials each with probability of success p. If the probability of at least one failure is greater than or equal to 31/32, then p lies in the interval

(a) [11/12, 1]
(b) [1/2, 3/4]
(c) [3/4, 11/12]
(d) [0, 1/2] **[2011]**

---

* None of the given options is correct.

17. If C and D are two events such that $C \subseteq D$ and $P(D) \neq 0$ then the correct statement among the following is

(a) $P(C|D) = \dfrac{P(C)}{P(D)}$ (b) $P(C|D) = P(C)$

(c) $P(C|D) \geq P(C)$ (d) $P(C|D) < P(C)$ **[2011]**

18. Let A, B, C be pairwise independent events such that $P(C) > 0$ and $P(A \cap B \cap C) = 0$, then $P(A' \cap B' | C)$ is equal to

(a) $P(A) - P(B')$ (b) $P(A') + P(B')$

(c) $P(A') - P(B')$ (d) $P(A') - P(B)$ **[2011]**

19. Three numbers are chosen at random without replacement from $\{1, 2, 3, ..., 8\}$. The probability that their minimum is 3, given that their maximum is 6 is

(a) 1/5 (b) 1/4

(c) 2/5 (d) 3/8 **[2012]**

20. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing is

(a) $\dfrac{10}{3^5}$ (b) $\dfrac{17}{3^5}$

(c) $\dfrac{13}{3^5}$ (d) $\dfrac{11}{3^5}$ **[2013]**

21. Given two independent events, if the probability that exactly one of them occurs is 26/49 and the probability that none of them occurs is 15/49, then the probability of more probable of two events is

(a) $\dfrac{4}{7}$ (b) $\dfrac{6}{7}$

(c) $\dfrac{3}{7}$ (d) $\dfrac{5}{7}$ **[2013, online]**

22. If the events A and B are mutually exclusive events such that

$P(A) = \dfrac{1}{3}(3x + 1)$ and $P(B) = \dfrac{1}{4}(1 - x)$, then the set of possible values of x lies in the interval:

(a) $[0, 1]$ (b) $\left[\dfrac{1}{3}, \dfrac{2}{9}\right]$

(c) $\left[-\dfrac{1}{3}, \dfrac{5}{9}\right]$ (d) $\left[-\dfrac{7}{9}, \dfrac{4}{9}\right]$

[2013, online]

23. The probability of a man hitting a target is $\dfrac{2}{5}$. He fires at the target k times (k, a given number). Then the minimum k, so that the probability of hitting the target at least once is more than $\dfrac{7}{10}$, is

(a) 3 (b) 5

(c) 2 (d) 4 **[2013, online]**

24. A, B, C try to hit a target simultaneously but independently. Their respective probabilities of hitting the target are $\dfrac{3}{4}, \dfrac{1}{2}, \dfrac{5}{8}$. The probability that target is hit by A or B but not by C is

(a) $\dfrac{21}{64}$ (b) $\dfrac{7}{8}$

(c) $\dfrac{7}{32}$ (d) $\dfrac{9}{64}$ **[2013, online]**

25. Let A and B be two events such that $P\left((A \cup B)'\right) = \dfrac{1}{6}$, $P(A \cap B) = \dfrac{1}{4}$ and $P(A') = \dfrac{1}{4}$. Then the events A and B are

(a) equally likely but not independent

(b) independent but not equally likely

(c) independent and equally likely

(d) mutually exclusive and independent **[2014]**

26. If A and B are two events such that $P(A \cup B) = P(A \cap B)$, then the incorrect statement amongst the following is

(a) A and B are equally likely

(b) $P(A \cap B') = 0$

(c) $P(A' \cap B) = 0$

(d) $P(A) + P(B) = 1$. **[2014, online]**

27. If X has a binomial distribution, $B(n, p)$ with parameters n and p such that $P(X = 2) = P(X = 3)$, then $E(X)$, the mean of variable X, is

(a) $2 - p$ (b) $3 - p$

(c) $p/2$ (d) $p/3$ **[2014, online]**

28. A number x is chosen at random from the set $\{1, 2, 3, 4, \ldots, 100\}$. Define the event: A = the chosen number x satisfies $\dfrac{(x-10)(x-50)}{(x-30)} \geq 0$. Then $P(A)$ is

(a) 0.71 (b) 0.70

(c) 0.51 (d) 0.20 **[2014, online]**

29. Let A and E be two events with positive probabilities:

Statement-1: $P(E|A) \geq P(A|E)\, P(E)$

Statement-2: $P(A|E) \geq P(A \cap E)$

(a) Both statements are true

(b) Both statements are false

(c) Statement-1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

[2014, online]

30. A set S contains 7 elements. A non-empty subset A of S and an element x of S are chosen at random. Then the probability that $x \in A$ is

(a) $\frac{1}{2}$ (b) $\frac{64}{127}$

(c) $\frac{63}{128}$ (d) $\frac{31}{128}$ **[2014, online]**

31*. If 12 distinct baalls are to be placed in 3 identical boxes, then the probability that one of the boxes contains exactly 3 balls is

(a) $\frac{55}{3}\left(\frac{2}{3}\right)^{11}$ (b) $55\left(\frac{2}{3}\right)^{10}$

(c) $220\left(\frac{1}{3}\right)^{12}$ (d) $22\left(\frac{1}{3}\right)^{11}$ **[2014]**

32. Let X be a set containing 10 elements and $P(X)$ be its power set. If A and B are picked up at random form $P(X)$, with replacement, then the probability that A and B have equal number of elements, is

(a) $\frac{{}^{20}C_{10}}{2^{10}}$ (b) $\frac{\left(2^{10}-1\right)}{2^{20}}$

(c) $\frac{\left(2^{10}-1\right)}{2^{10}}$ (d) $\frac{{}^{20}C_{10}}{2^{20}}$ **[2015, online]**

33. If the lengths of the sides of a triangle are decided by the three throws of a single fair die, then the probability that the triangle is of maximum area given that it is an isosceles triangle, is

(a) $\frac{1}{26}$ (b) $\frac{1}{27}$

(c) $\frac{1}{21}$ (d) $\frac{1}{15}$ **[2015, online]**

34. If the mean and the variance of a bionomial variate are 2 and 1 respectively, then the probability that X takes a value greater than or equal to one is

(a) $\frac{1}{16}$ (b) $\frac{9}{16}$

(c) $\frac{3}{4}$ (d) $\frac{15}{16}$ **[2015, online]**

35. Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four and E_2 be the event die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements in NOT true?

(a) E_1 and E_2 are independent

(b) E_2 and E_3 are independent

(c) E_1 and E_3 are independent

(d) E_1, E_2 and E_3 are independent **[2016]**

36. If A and B are any two events such that $P(A) = 2/5$ and $P(A \cap B) = 3/20$, then the conditional probability, $P(A \mid A' \cup B')$, where A' denotes the complement of A, is equal to

(a) 11/20 (b) 5/17

(c) 8/17 (d) 1/4 **[2016, online]**

37. An experiment succeeds twice as often as it fails. The probability of at least 5 successes in the six trials of this experiment is

(a) $\frac{496}{729}$ (b) $\frac{192}{729}$

(c) $\frac{240}{729}$ (d) $\frac{256}{729}$ **[2016, online]**

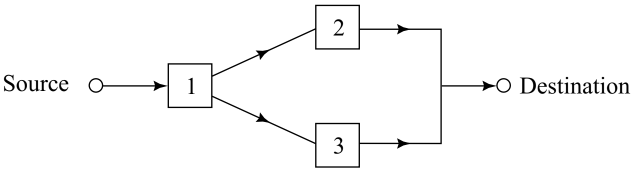
Previous Years' B-Architecture Entrance Examination Questions

1. Two events A and B are such that $P(B) = 0.55$ and $P(AB') = 0.15$. The probability of occurrence of at least one of event is

(a) 0.70 (b) 0.20

(c) 0.35 (d) 0.30 **[2006]**

2. A certain water-supply system consists of a source, three pumping stations, and a destination. Each pumping station has a probability p $(0 < p < 1)$ of being operable at a specified time to and station functions independently of one another. The stations are connected as shown in figure:



The probability that water is available to the destination at time to is

(a) $2p^2$ (b) $p^2(2-p)$

(c) p^3 (d) p^2 **[2007]**

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* Word 'identical' in the question should be replaced by 'distinct'

3. An urn contains four balls bearing numbers 1, 2, 3 and 123 respectively. A ball is drawn at random from the urn. Let E_i, $i = 1, 2, 3$ donote the event that digit i appears on the ball drawn.

   Statement-1: $P(E_1 \cap E_2) = P(E_1 \cap E_3) = P(E_2 \cap E_3) = \frac{1}{4}$

   Statement-2: $P(E_1) = P(E_2) = P(E_3) = \frac{1}{2}$ **[2008]**

4. Three dice, red, blue and green in colour are rolled together. Let B be the event that sum of the numbers shown up is 7. Let A be the event that the red die shows 1. The conditional probability of the event A given B, $P(A|B)$ is

   (a) $\frac{2}{7}$ (b) $\frac{1}{6}$

   (c) $\frac{1}{7}$ (d) $\frac{1}{3}$ **[2009]**

5. Sets A, B, C $A \cap B$, $A \cap C$, $B \cap C$ and $A \cap B \cap C$ have 35, 40, 45, 13, 12, 14 and 5 elements respectively. An element is selected at random from the set $A \cup B \cup C$. The probability that the selected element belongs to only set A is

   (a) $\frac{13}{86}$ (b) $\frac{35}{86}$

   (c) $\frac{5}{86}$ (d) $\frac{15}{86}$ **[2009]**

6. A man is known to speak the truth on an average 3 out of 4 times. He throws a fair die and reports that it is a six. The probability that it is actually a six is

   (a) $\frac{3}{5}$ (b) $\frac{3}{8}$

   (c) $\frac{3}{4}$ (d) $\frac{1}{5}$ **[2010]**

7. If $P(A) = 0.4$, $P(B') = 0.6$, and $P(A \cap B) = 0.15$, then the value of $P(A|\, A' \cup B')$ is

   (a) $\frac{1}{17}$ (b) $\frac{4}{17}$

   (c) $\frac{5}{17}$ (d) $\frac{10}{17}$ **[2011]**

8. A class consists of 80 students, 25 of them are girls. If 10 of the students are rich and 20 of the students are fair complexioned, then the probability of selecting a fair complexioned rich girl from the class (assuming three traits as independent) is

   (a) $\frac{1}{10}$ (b) $\frac{1}{32}$

   (c) $\frac{5}{512}$ (d) $\frac{7}{512}$ **[2012]**

9. Let A and B be two events such that $P(A \cup B) \geq 3/4$ and $1/8 \leq P(A \cap B) \leq 3/8$

   Statement-1: $P(A) + P(B) \geq 7/8$

   Statement-2: $P(A) + P(B) \leq 11/8$ **[2013]**

10. A biased coin with probability p, $0 < p < 1$ of showing head is tossed until a head appears for the first time. If the probability that the number of tosses required is even is 2/5, then p is equal to

    (a) $\frac{2}{3}$ (b) $\frac{1}{2}$

    (c) $\frac{1}{3}$ (d) $\frac{1}{4}$ **[2014]**

11. If p and q are chosen at random from the set {1, 2, 3, ..., 10}, with replacement, then the probability that the roots of equation $x^2 + px + q = 0$ are real, is

    (a) $\frac{31}{50}$ (b) $\frac{9}{25}$

    (c) $\frac{29}{50}$ (d) $\frac{13}{25}$ **[2015]**

12. In a binomial distribution $B\left(n, p = \frac{1}{4}\right)$, if the probability of at least one success is greater than or equal to $\frac{9}{10}$, then n is an integer greater than

    (a) $\frac{9}{\log_{10} 4 - \log_{10} 3}$ (b) $\frac{1 - \log_{10} 9}{\log_{10} 4 - \log_{10} 3}$

    (c) $\frac{1}{\log_{10} 4 - \log_{10} 3}$ (d) $\frac{1}{\log_{10} 4 + \log_{10} 3}$ **[2015]**

13. A box contains 5 black and 4 white balls. A ball is drawn at random and its colour is noted. The ball is then put back in the box along with two additional balls of its opposite colour. If a ball is drawn again from the box, then the probability that the ball drawn now is black, is

    (a) $\frac{5}{11}$ (b) $\frac{53}{99}$

    (c) $\frac{48}{99}$ (d) $\frac{7}{11}$ **[2016]**

Answers

Concept-based

1. (a)	**2.** (a)	**3.** (b)	**4.** (b)
5. (d)	**6.** (b)	**7.** (c)	**8.** (d)
9. (a)	**10.** (c)		

Level 1

11. (a)	**12.** (b)	**13.** (c)	**14.** (c)
15. (a)	**16.** (a)	**17.** (b)	**18.** (b)
19. (d)	**20.** (c)	**21.** (b)	**22.** (b)
23. (d)	**24.** (a)	**25.** (b)	**26.** (b)
27. (c)	**28.** (b)	**29.** (a)	**30.** (a)
31. (a)	**32.** (d)	**33.** (a)	**34.** (d)
35. (c)	**36.** (a)	**37.** (d)	**38.** (d)
39. (a)	**40.** (b)	**41.** (c)	**42.** (b)
43. (d)	**44.** (d)	**45.** (c)	**46.** (b)
47. (d)	**48.** (b)	**49.** (a)	**50.** (c)
51. (c)	**52.** (a)	**53.** (a)	**54.** (b)
55. (a)	**56.** (b)	**57.** (c)	**58.** (a)
59. (b)	**60.** (d)	**61.** (b)	**62.** (a)
63. (c)	**64.** (b)	**65.** (b)	**66.** (a)
67. (a)	**68.** (c)	**69.** (d)	**70.** (c)
71. (d)	**72.** (d)	**73.** (a)	**74.** (c)
75. (d)	**76.** (a)	**77.** (a)	**78.** (c)
79. (d)	**80.** (c)		

Level 2

81. (a)	**82.** (d)	**83.** (a)	**84.** (c)
85. (d)	**86.** (a)	**87.** (a)	**88.** (d)
89. (c)	**90.** (b)	**91.** (a)	**92.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (d)	**2.** (c)	**3.** (d)	**4.** (b)
5. (a)	**6.** (b)	**7.** (a)	**8.** (d)
9. (d)	**10.** None of the answer matches.		
11. (c)	**12.** (c)	**13.** (c)	**14.** (d)
15. (c)	**16.** (d)	**17.** (a)	**18.** (a)
19. (a)	**20.** (d)	**21.** (a)	**22.** (c)
23. (a)	**24.** (a)	**25.** (b)	**26.** (d)
27. (b)	**28.** (a)	**29.** (a)	**30.** (b)
31. (*)	**32.** (a)	**33.** (b)	**34.** (d)
35. (d)	**36.** (b)	**37.** (d)	

Previous Years' B-Architecture Entrance Examination Questions

1. (a)	**2.** (a)	**3.** (b)	**4.** (b)
5. (d)	**6.** (b)	**7.** (c)	**8.** (c)
9. (b)	**10.** (c)		

Hints and Solutions

Concept-based

1. Total number of ways of dividing $2n$ boys is
$({}^{2n}C_n)\ ({}^{n}C_n) = \dfrac{(2n)!}{n!n!}$
Dividing $(2n-2)$ boys (excluding 2 tallest boys) is $\dfrac{(2n-2)!}{((n-1)!)^2}$. Two tallest boys can be divided in ${}^2P_2 = 2!$ ways. Thus, required probability is
$$\frac{(2!)(2n-2)!}{(n-1)!(n-1)!} \times \frac{n!n!}{(2n)!} = \frac{n}{2n-1}$$
2. Use $P(A \cup B) = P(A \cap B') + P(B)$
3. Use $17 | ab \Rightarrow 17 | a$ or $17 | b$.
4. Use : one of them must be even and other must be odd.
5. $P(A \cap B) \le \min\{P(A), P(B)\} = 0.5$,
$P(A \cap B) \ge P(A) + P(B) - 1 = 0.3$,
$P(A' \cap B) = P(B) - P(A \cap B)$
$\le 0.8 - 0.3 = 0.5$
and $P(A \cap B') = P(A) - P(A \cap B) \ge 0.2$
6. $A \cap B \subseteq C \Rightarrow P(C) \ge P(A \cap B) \ge P(A) + P(B) - 1$
7. $P(A)\ P(B|A) = P(A \cap B) \Rightarrow P(A) = 1/2$.
Similarly, $P(B) = 1/3$
Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 2/3$
As $P(A \cap B) = P(A)\ P(B)$, A and B are independent.
$\therefore$ (c) cannot be correct.
Also, $P(A' \cap B) = P(B) - P(A \cap B) = 1/3 - 1/6 = 1/6$
8. $P(E_1) = P(E_2) = P(E_3) = \dfrac{2}{4} = \dfrac{1}{2}$
$P(E_1 \cap E_2) = P(E_2 \cap E_3) = P(E_3 \cap E_1) = \dfrac{1}{4}$
and $P(E_1 \cap E_2 \cap E_3) = \dfrac{1}{4}$
9. Total number of possible outcomes is 2^n. We can choose a loser in n ways and he can lose in 2 ways when he gets a head and rest of them tails or when he gets a tail and rest of them get head. Thus, probability of required event is $2n/2^n = n/2^{n-1}$.
10. Current flows from A to B with a probability of $p + p^2$. Now, $p + p^2 = 4/9$
$\Rightarrow p = 1/3$.

Level 1

11. Let A, B denote the following events.
A : number is divisible by 3
B : number is divisible by 5,
then $A \cap B$: number is divisible by 15.

$$P(A) = \frac{33}{100},\ P(B) = \frac{20}{100},\ P(A \cap B) = \frac{6}{100}$$

Now, use formula for $P(A \cup B)$.

12. Note that X can take values from 0 to n. Now,

$$\sum_{k=0}^{n} P(X = k) = 1$$

$$\Rightarrow \quad \lambda \sum_{k=0}^{n} k^2 = 1 \Rightarrow \frac{6}{n(n+1)(2n+1)} = \lambda$$

13. $f'(x) = 3x^2 + 2ax + b$.

Note that $f'(x) > 0 \Leftrightarrow 4a^2 - 12b < 0$

$\Rightarrow a^2 < 3b$. Possible values of (a, b) are

(1, 2), (1, 3), (2, 3), (1, 4), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5), (1, 6), (2, 6), (3, 6), (4, 6).

Total number of ways of choosing (a, b) is ${}^6C_2 = 15$.

Thus, required probability $= \dfrac{13}{15}$.

14. $|z| \leq 3 \Leftrightarrow |z|^2 \leq 9$

$\Leftrightarrow x^2 + y^2 \leq 9$.

Possible ordered pairs (x, y) are : (1, 1), (1, 2), (2, 1).

Thus, probability of required event is 3/36 = 1/12

15. As A is independent of itself,

$P(A) = P(A \cap A) = P(A)\, P(A)$

$\Rightarrow P(A) = 0$ or $P(A) = 1$.

If $P(A) = 0$, then $\phi \subseteq A \cap B \subseteq A$

$\Rightarrow P(\phi) \leq P(A \cap B) \leq P(A) = 0$

$\Rightarrow P(A \cap B) = 0 \leq P(A)\, P(B)$

If $P(A) = 1$, use $A \cap B \subseteq B$

16. Total number of ways = 12!

The number of ways when all the six girls sit together is (7!) (6!)

$\therefore$ probability of the required event $= \dfrac{7!6!}{12!} = \dfrac{1}{132}$

17. $P(A' \cap B') = P(A') + P(B') - P(A' \cup B')$

$= (1 - p) + (1 - 2p) - \{1 - (A \cap B)\}$

$$\Rightarrow \quad \frac{1}{3} = \frac{3}{2} - 3p$$

$$\Rightarrow \quad 3p = \frac{3}{2} - \frac{1}{3} = \frac{7}{6} \Rightarrow p = \frac{7}{18}$$

18. Use product is odd if and only if both numbers are odd.

19. As A and B are independent

$P\ (A \cap B) = P(A)\ P(B)$

$$\Rightarrow \quad \frac{n_{A \cap B}}{10} = \frac{4}{10} \times \frac{n_B}{10}$$

$\Rightarrow \quad 2n_B = 5n_{A \cap B}$

$\Rightarrow \quad 5/2n_B \Rightarrow 5$

$\Rightarrow \quad n_B = 5$ or 10.

20. Required probability

= (probability of drawing 3 white balls in the first six draws) × (probability of getting a white ball at the 7th draw)

$$= \left[{}^6C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3\right]\left(\frac{1}{2}\right) = \frac{5}{32}$$

21. Probability of getting at least 7 points

= (probability of getting 2 points each from 3 matches)

× (probability of getting 1 point from one match)

+ probability of getting 2 points from each of the four matches

$= {}^4C_3(0.5)^3\ (0.05) + (0.5)^4$

$= (0.5)^3\,(0.2) + (0.5)^4 = (0.125)\ (0.7)$

$= 0.0875$

22. $P(F) = P(E \cup F) + P(E \cap F) - P(E)$

23. See Theory.

24. Last digit of 7^m is 7, 9, 3, 1.

If m is of the form $4r + 1$, then n must be of the form $4s + 3$ etc.

25. $P(M \cup P \cup C) = 0.75$,

P(at least two of M, P, C) = 0.5, and P(exactly two) = 0.4

Now, P(exactly two) = P(at least two) – $P(M \cap P \cap C)$

$\Rightarrow 0.4 = 0.5 - P(M \cap P \cap C)$

$\Rightarrow P(M \cap P \cap C) = 0.1$

As the events are independent

P(exactly two) $= pm(1 - c) + pc(1 - m) + mc(1 - p)$

$$0.4 = pm + pc + mc - 3pmc$$

Also, $P(M \cap P \cap C) = pmc = 0.1$

$\therefore \quad pm + pc + mc = 0.7$

Now, $P(M \cup P \cup C) = p + m + c - (pm + mc + cp) + pmc$

$\Rightarrow \quad 0.75 = p + m + c - 0.7 + 0.1$

$\Rightarrow \quad 1.35 = p + m + c$

26. $\{Y = 0\} = \{00, 01, 02, \ldots, 09, 10, 20, 30 \ldots 90\}$ and $\{(X = 7) \cap (Y = 0)\} = \{07, 70\}$

Thus, $P(X = 7 \mid Y = 0) = \dfrac{2}{19}$

27. Total number of mappings $= n^n$ and number of one-to-one mappings $= n!$

28. Use $x = 1, 2$ or $x \geq 48$ for $x + \dfrac{100}{x} > 50$.

29. E_1 : a biased coin is picked up;

E_2 : a fair coin is picked up.

A : the toss results in a head.

We have

$$P(E_1) = \frac{n}{2n+1},\ P(E_2) = \frac{n+1}{2n+1}$$

$$P(A|E_1) = 1,\ P(A|E_2) = \frac{1}{2}.$$

Now, use the total probability

$$P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2)$$

30. $P(X + Y = 3) = \sum_{k=0}^{3} P(X = k, Y = 3 - k)$

$$= \sum_{k=0}^{3} P(X = k)\,.\,P(Y = 3 - k)$$

[∵ X and Y are independent]

$$= \sum_{k=0}^{3} \left({}^5C_k\right)\left(\frac{1}{2}\right)^5 \left({}^7C_{3-k}\right)\left(\frac{1}{2}\right)^7$$

$$= \left(\frac{1}{2}\right)^{12} \sum_{k=0}^{3} \left({}^5C_k\right)\left({}^7C_{3-k}\right)$$

$$= \left(\frac{1}{2}\right)^{12} \left({}^{12}C_3\right) = \frac{55}{1024}$$

31. Use P(they are not together)

$$= 1 - P \text{ (they are together)}$$

$$= 1 - \frac{(n-1)!(2!)}{n!} = 1 - \frac{2}{n}$$

32. $P = \frac{5}{100},\ q = \frac{3}{100},\ p + q = 0.08$

33. Total number of ways of arranging the letters of the word PROBABILITY is $\frac{11!}{2!\,2!}$ and the number of ways in which two B's are together is $\frac{10!}{2!}$

34. Last digit of the product will be 1, 3, 7 or 9 if the last digit of each of the four numbers is 1, 3, 7 or 9.

35. They can win prize in 9 ways out of 9 × 9 ways.

36. See Theory.

37. See Theory.

38. Sample space is

$$\{-\ 0.50, -\ 0.49, \ldots, 0, 0.01, \ldots, 0{,}49\}$$

and the round off error is at most 10 paise if

$$A = \{-\ 0.10, -\ 0.09, \ldots\ 0.10\}$$

39. Let A : the bulb will last at least 150 days and B : the bulb will last at most 160 days.

Use $P(A) = 0.7$, $P(B) = 0.8$ and $P(A \cup B) = 1$.

40. P(even) = P(all four odd) + P(two evens and two odds) + P(all four even)

$$= {}^4C_0\left(\frac{2}{3}\right)^4 + {}^4C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^2 + {}^4C_4\left(\frac{1}{3}\right)^4 = \frac{41}{81}$$

41. Split the set {1, 2, … 3N} into three sets as follows:

$A = \{1, 4, 7, \ldots\ 3N - 2\}$

$B = \{2, 5, 8, \ldots\ 3N - 1\}$

$C = \{3, 6, \ldots\ 3N\}$

Choose x, y from $A \cup B$ or from C.

∴ required probability

$$= \frac{{}^{2N}C_2 + {}^{N}C_2}{{}^{3N}C_2} = \frac{2N(2N-1) + N(N-1)}{3N(3N-1)}$$

$$= \frac{5N-3}{9N-3}$$

42. The number of ways in which 10 persons can speak is 10!. The number of ways in A, B and C speak together is $\frac{(8!)\,(3!)}{10!} = \frac{1}{15}$

43. Use if $3\,|\,n^2 \Rightarrow 3\,|\,n$ and hence $9\,|\,n^2$

44. Required probability

$$= \frac{\left({}^nC_1\right)\left({}^{n^2-n}C_{n-1}\right)}{{}^{n^2}C_n}$$

45. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Each of the element can take two values, therefore total number of ways = 16.

But $ad - bc \neq 0$ if and only $ad = x^2$ and $bc = -\ x^2$ or $ad = -\ x^2$ and $bc = x^2$

But $ad = x^2$ and $bc = -\ x^2$

⇔ ($a = d = x$ or $a = -\ d = x$) and ($b = x$, $c = -\ x$ or $b = -\ x$, $c = x$)

That is, there are four ways

Similarly, $ad = -\ x^2$ and $bc = x^2$ in four ways.

Thus, required probability = $\frac{4+4}{16} = \frac{1}{2}$

46. Total number of ways = 6 × 6 × 6.

The number of favourable ways = ${}^6C_3$

47. We have $0 \leq \frac{1+4p}{4}, \frac{1-p}{4}, \frac{1-2p}{2} \leq 1$

and $0 \leq \frac{1+4p}{4} + \frac{1-p}{4} + \frac{1-2p}{2} \leq 1$

$\Rightarrow -\frac{1}{4} \leq p \leq \frac{3}{4},\ -\ 3 \leq p \leq 1,\ -\frac{1}{2} \leq p \leq \frac{1}{2}$

and $0 \leq p \leq 4$

$$\Rightarrow \max\left\{-\frac{1}{4}, -3, -\frac{1}{2}, 0\right\} \leq p \leq \min\left\{\frac{3}{4}, 1, \frac{1}{2}, 4\right\}$$

$\Rightarrow \quad 0 \le p \le 1/2.$

48. Let $x = 5m + r$, $y = 5n + s$ where $m, n \in \mathbf{I}$, $0 \le r, s \le 4$

$\Rightarrow \quad x^2 + y^2 = 25(m^2 + n^2) + 10(mr + ns) + r^2 + s^2$

The ordered pair (r, s) can take 5×5 = 25 values.

But $r^2 + s^2$ is divisible by 5 if $(r, s) = (0, 0), (1, 2), (2, 1), (1, 3), (3, 1), (2, 4), (4, 2), (3, 4), (4, 3)$

Thus, probability of the required event = $\dfrac{9}{25}$.

49. Let $A = \{a_1, a_2, \ldots, a_n\}$.

For each a_i, we have four choices:

(i) $a_i \in P, a_i \in Q$

(ii) $a_i \notin P, a_i \in Q$

(iii) $a_i \in P, a_i \notin Q$

(iv) $a_i \notin P, a_i \notin Q$

For $P \cap Q = \phi$, just last three cases are favourable.

50. Let $A = \{x_1, x_2, ..., x_n\}$.

For x_i $(1 \le 1 \le n)$, we have four choice

(i) $x_i \in P, x_i : \in Q$

(ii) $x_i \in P, x_i : \notin Q$

(iii) $x_i \notin P, x_i : \in Q$

(iv) $x_i \notin P, x_i \notin Q$

There are 3 cases out of 4 which are favourable to event $P \subseteq Q$. Thus, probability of required event is $(3/4)_n$.

$\therefore$ $(3/4)^n = 243/1024 = (3/4)5 \Rightarrow n = 5$

51. Total number of cases = ${}^6C_3 = 20$. There are just two favourable cases viz. $\Delta A_1A_3A_5$ or $\Delta A_2\, A_4\, A_6$.

Required probability = $\dfrac{2}{20} = \dfrac{1}{10}$. See Fig. 24.10.

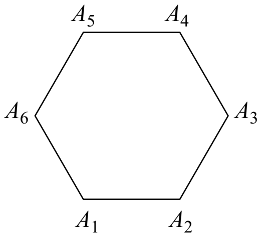


Fig. 24.10

52. E_1 : one of the n urns is chosen

E_2 : $(n + 1)$th urn is chosen

A : both the balls drawn are black.

$$P(E_1) = \frac{n}{n+1}, P(E_2) = \frac{1}{n+1},$$

$$P(A \mid E_1) = \frac{{}^6C_2}{{}^{10}C_2} = \frac{1}{3},$$

$$P(A \mid E_2) = \frac{{}^5C_2}{{}^{10}C_2} = \frac{2}{9}$$

By the Bayes' rule

$$P(E_2 \mid A) = \frac{P(E_2)P(A \mid E_2)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)}$$

Now, use $\quad P(E_2 \mid A) = \dfrac{2}{9} \quad \Rightarrow \quad n = 10.$

53. The number of ways of choosing x and y is ${}^{11}C_2$ = 55. We now count the number of ways in which $|\, x - y \,| > 5$ or $|\, x - y \,| \ge 6$.

For $x = 0$, $y = 6, 7, 8, 9, 10$

For $x = 1$, $y = 7, 8, 9, 10$

For $x = 2$, $y = 8, 9, 10$

For $x = 3$, $y = 9, 10$

For $x = 4$, $y = 10$

$\therefore$ the number of favourable ways for $|\, x - y \,| > 5$ is 15. Thus, probability of the required event 40/55 = 8/11.

54. $P(G) = \dfrac{1}{3}$, $P(C) = p$, $P(K) = \dfrac{2}{3} - p$

$P(R \mid G) = \dfrac{1}{4}$, $P(R \mid C) = \dfrac{1}{8}$, $P(R \mid K) = 1.$

Now, use Bayes' rule.

55. Use $2P(X = 5) = P(X = 4) + P(X = 6)$

56. $P(X = 3) = P(X = 5)$

$$\Rightarrow {}^nC_3\left(\frac{1}{2}\right)^8 = {}^nC_5\left(\frac{1}{2}\right)^8 \quad \Rightarrow \quad n = 8.$$

Thus, $P(X = 1) = {}^8C_1\left(\dfrac{1}{2}\right)^8 = \dfrac{1}{32}$

57. $P(E' \cap F' \mid G) = \dfrac{P(E' \cap F' \cap G)}{P(G)}$

$$= \frac{P(G) - P(G \cap E) - P(G \cap F)}{P(G)}$$

$$= \frac{P(G) - P(G)P(E) - P(G)P(F)}{P(G)}$$

$= 1 - P(E) - P(F) = P(E') - P(F)$

58. $P(E \cap F) = P(E) + P(F) - P(E \cup F) = 0.2.$

Now, $P(F \mid E) = \dfrac{P(F \cap E)}{P(E)} = \dfrac{0.2}{0.4} = 0.5$

59. $np = 4$, $npq = 3 \quad \Rightarrow \quad q = 3/4 \quad \Rightarrow \quad p = 1/4$

Thus, $n = 16$.

$$P(1) = {}^{16}C_1\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{15} = 4\left(\frac{3}{4}\right)^{15}$$

60. $\dfrac{{}^{10}C_2 \times {}^{10}C_1}{{}^{20}C_3}$

61. $P(B$ wins$)$

$= P(TH$ or $T\ T\ T\ H$ or $T\ T\ T\ T\ T\ H$ or $\ldots)$

$$= \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{1/4}{1-1/4} = \frac{1}{3}$$

62. Use result of eighth toss is independent of the first seven tosses.

63. $P(A \cap A) = P(A)\ P(A)$

$\Rightarrow\ P(A) = P(A)^2$

$\Rightarrow\ P(A) = 1$ as $P(A) > 0$.

64. $P(A) + P(B) = P(A \cup B) + P(A \cap B)$

$= 2P(A \cap B)$

$\Rightarrow\ (P(A) - P(A \cap B)) + (P(B) - P(A \cap B)) = 0$

$\Rightarrow\ P(A) = P(B) = P(A \cap B)$

65. Let E : 1 occurs; F : 1 does not occur.

$P(1$ occurs at even throw$)$

$= P(F\ E$ or $F\ F\ F\ E$ or $F\ F\ F\ F\ F\ E$ or $\ldots)$

$$= \frac{5}{36} + \frac{5}{36}\left(\frac{25}{36}\right) + \frac{5}{36}\left(\frac{25}{36}\right)^2 + \cdots = \frac{5/36}{1-25/36} = \frac{5}{11}$$

66. $P(A'|B') = \dfrac{P(A' \cap B')}{P(B')} = \dfrac{1 - P(A \cup B)}{P(B')}$

67. $P(B) = 1 - P(B') = 1 - 6/36 = 5/6$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P\{(6,5)\}}{P(B)} = \frac{1/36}{5/6} = \frac{1}{30}$$

68. Let W and B denote the events of drawing a white and black balls in the first draw. Let A denote the event of drawing a black ball in the second draw.

$$P(W) = \frac{w}{w+b} \text{ and } P(B)\ \frac{b}{w+b}$$

$$P(A|\ W) = \frac{b}{w+b+k} \text{ and } P(A|B) = \frac{b+k}{w+b+k}$$

By the total probability rule,

$$P(A) = P(W)\ P(A|W) + P(B)\ P(A|B)$$

$$= \frac{wb + b(b+k)}{(w+b)(w+b+k)} = \frac{b(w+b+k)}{(w+b)(w+b+k)}$$

$$= \frac{b}{w+b}$$

69. $E(X) = 1\left(\dfrac{1}{2}\right) + 2\left(\dfrac{1}{5}\right) + 3\left(\dfrac{3}{25}\right)$

$$+(2k)\left(\frac{1}{10}\right) + (3k)\left(\frac{1}{25}\right) + (5k)\left(\frac{1}{25}\right)$$

$$= \frac{1}{50}[25 + 20 + 18 + 10k + 6k + 10k]$$

$$\Rightarrow\ 2.3 = \frac{1}{50}(63 + 26k) \ \Rightarrow\ 115 = 63 + 26k$$

$$\Rightarrow\ k = 2$$

70. $p = P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$\Rightarrow\ P(A \cap B) = 0.05p$

Similarly, $P(A' \cap B') = 0.95p$

Next, $0.9 = P(B/A) = \dfrac{P(B \cap A)}{P(A)}$

$\Rightarrow\ P(\mathrm{A}) = \dfrac{0.05p}{0.9} = \dfrac{p}{18}$

We have

$0.95p = 1 - P(A \cup B)$

$\Rightarrow\ P(A) + P(B) - P(A \cap B) = 1 - 0.95p$

$\Rightarrow\ \dfrac{p}{18} + \dfrac{5}{100} - 0.05p = 1 - 0.95p$

$\Rightarrow\ p\left(\dfrac{1}{18} + \dfrac{9}{10}\right) = 0.95$

$\Rightarrow\ p = \dfrac{0.95 \times 90}{86} = \dfrac{171}{430}$

71. We have

$$P(A) = \frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2}$$

$$P(B) = \frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2}$$

$$P(C) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3}$$

$$P(A \cap B) = \frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4} = P(A)\ P(B)$$

$$P(B \cap C) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6} = P(B)\ P(C)$$

$$P(C \cap A) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6} = P(C)\ P(A)$$

$$P(A \cap B \cap C) = \frac{1}{12} = P(A)\ P(B \cap C)$$

72. Three numbers can be chosen out of 20 in ${}^{20}C_3$ ways.

For different values of common ratios, G.P.'s are listed below.

r	*G.P.*	*Number of G.P.'s*
2	(1, 2, 4), (2, 4, 8), (3, 6, 12), (4, 8, 16), (5, 10, 20)	5
3	(1, 3, 9), (2, 6, 18)	2
4	(1, 4, 16)	1
3/2	(4, 6, 9), (8, 12, 18)	2
4/3	(9, 12, 16)	1
	Total	11

$\therefore$ probability of required event = $\dfrac{11}{{}^{20}C_3}$

73. Total number of ways of choosing 3 tickets out of 30 is ${}^{30}C_3$.

    We can choose x_1 is 14 ways, x_3 in 10 ways. Therefore, the probability of the required

    $$\text{events} = \frac{(14)(10)}{{}^{30}C_3} = \frac{14\times10\times3\times2}{30\times29\times28} = \frac{1}{29}$$

74. P(Match does not occur at ith trial)

    $= \dfrac{n-1}{n} = 1 - \dfrac{1}{n}$

    $\therefore$ P(at least one match)

    $$= 1 - P(\text{no match}) = 1-\left(1-\frac{1}{n}\right)^n.$$

75. Required probability = 1 – P (three numbers chosen are consecutive)

    $$= 1-\frac{18}{{}^{20}C_3} = 1-\frac{3}{190} = \frac{187}{190}$$

76. $A \cap B \cap C \subseteq C \Rightarrow P(A \cap B \cap C) \le P(C) = 0$

    $\therefore \quad P(A \cap B \cap C) = 0$

    Similarly, $\quad P(A \cap C) = 0 \quad P(B \cap C) = 0$

    Next, $\quad P(A \cup B \cup C) = P(A \cup B) + P(C) - P[(A \cup B) \cap C]$

    But $\quad P[(A \cup B) \cap C] = P[(A \cap C) \cup (B \cap C)]$
    $= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C) = 0$

    Thus, $\quad P(A \cup B \cup C) = P(A \cup B)$

77. $P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)$
    $= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$
    $= P(A)\,P(B) + P(A)\,P(C) - P(A \cap B \cap C)$

    Now, A and $B \cup C$ are independent

    $\Leftrightarrow P[A \cap (B \cup C)] = P(A)\,P(B \cup C)$

    $\Leftrightarrow \quad P(A)\,P(B) + P(A)\,P(C) - P(A \cap B \cap C)$
    $= P(A)\,[P(B) + P(C) - P(B \cap C)]$

    $\Leftrightarrow \quad P(A \cap B \cap C) = P(A)\,P(B \cap C)$

    $\Leftrightarrow \quad A$ and $B \cap C$ are independent

78. Let $P = \{a_1, a_2, \ldots a_n\}$. For $a_i \in P$ $(1 \le i \le n)$, we have four choices.

    (i) $a_i \in Q$ and $a_i \in R$
    (ii) $a_i \in Q$ and $a_i \notin R$
    (iii) $a_i \notin Q$ and $a_i \in R$
    (iv) $a_i \notin Q$ and $a_i \notin R$

    Thus, total number of choices for picking up Q and R is 4^n.

    For a_i not to belong to $Q \cap R$, we have 3 choices. Therefore, $Q \cap R = \phi$ in 3^n ways.

    Thus, probability $Q \cap R = \phi$ is $(3/4)^n$.

    Next, $a_i \in Q \cup R$ in 3 ways. Therefore, $Q \cup R = P$ in 3^n ways.

    Thus, probability $Q \cup R = P$ is $(3/4)^n$.

79. If a and b are both odd, then $a + b$ is even, therefore, $P(A \cap B \cap C) = 0$.

    We have $P(A) = \dfrac{1}{2}$, $P(B) = \dfrac{1}{2}$, $P(D) = \dfrac{1}{2}$.

    Now, $P[(A \cap B \cap D) \mid A \cup B]$

    $$= \frac{P[(A\cap B\cap D)\cap(A\cup B)]}{P(A\cup B)} = \frac{P(A\cap B)}{P(A\cup B)}$$

    $$= \frac{1/4}{1/2+1/2-1/4} = \frac{1}{3}$$

80. Statement-2 is false.

    P(P and Q contradict each other)

    $= p(1 - 2p) + 2p(1 - p) = 1/2$

    $\Rightarrow 8p^2 - 6p + 1 = 0.$

    $\Rightarrow (2p - 1)(4p - 1) = 0 \quad \Rightarrow \quad p = 1/2, 1/4.$

Level 2

81. Let X = number of times the event occurs, then $X \sim B(n, p)$ where $n = 3$, $p = 0.6$

    P (the event occur at least once)

    $= P(X \ge 1) = 1 - P(X = 0)$

    $= 1 - (0.4)^3 = 0.936$

82. P (neither A nor B)

    $= P(A' \cap B') = 1 - P(A \cup B)$

    $= 1 - [P(A) + P(B)] = 1 - 0.8 = 0.2$

83. $P(X = 3) = P(X = 5)$

    $\Rightarrow {}^nC_3\, p^3\, (1 - p)^{n-3} = {}^nC_5\, p^5\, (1 - p)^{n-5}$

    $$\Rightarrow \frac{(n-3)(n-4)}{(5)(4)} = \frac{(1-p)^2}{p^2} < 1$$

    $\Rightarrow n^2 - 7n - 8 < 0 \Rightarrow (n - 8)(n + 1) < 0$

    $\Rightarrow n - 8 < 0 \Rightarrow n \le 7.$

84. Let X = number of times a doublet is thrown, then $X \sim B(n, p)$ where $n = 3$, $p = 1/6$.

    $\therefore$ P (doublet not more than twice)

    $$= P(X \le 2) = 1 - P(X = 3) = \frac{215}{216}$$

85. Probability of the required event

    $= P$ (both brown or both white)

    $= P$(both brown) + P(both white)

    $$= \frac{{}^4C_2}{{}^9C_2} + \frac{{}^5C_2}{{}^9C_2} = \frac{4}{9}$$

86. Let A denote the event that three dice show different faces and B denote the event that at least one dice shows 6.

We have

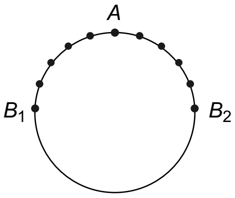
$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{{}^5C_2/6^3}{{}^6C_3/6^3} = \frac{1}{2}$$

87. P (exactly 4 are car drivers)

$$= {}^5C_4\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right) = \left(\frac{4}{5}\right)^4$$

88. Probability of the required event

$$= \frac{\left({}^4C_2\right)\left({}^5C_2\right)}{{}^9C_4} = \frac{10}{21}$$



89. Suppose A occupy any seat at the round table. Then there are 14 seats available for B. If there are four persons between A and B, then B has only two ways to sit, as shown in figure. Thus, probability of the required event
= 2/14 = 1/7.

90. Total number of ways of arranging (3 + 4 + 1) books = 8!
The number of favourable ways
$= ({}^3P_3)\,({}^3P_3)\,({}^4P_4)\,({}^1P_1) = (3!)\,(3!)\,(4!)$
Thus, probability of the required event

$$= \frac{(3!)\,(3!)(4!)}{8!} = \frac{3}{140}$$

91. As $n = 9$ and $np = 6$, we get $p = 2/3$.
Thus, variance $= np\,(1 - p) = 2$

92. $P(X = 1) = 8P(X = 3)$

$$\Rightarrow {}^nC_1\, p\,(1 - p)^{n-1} = 8\,({}^nC_3)\,p^3\,(1 - p)^{n-3}$$

$$\Rightarrow \frac{(1-p)^2}{p^2} = \frac{(8)\,(10)}{5} = 16$$

$$\Rightarrow \frac{1-p}{p} = 4 \Rightarrow p = 1/5$$

Previous Years' AIEEE/JEE Main Questions

1. $\frac{1}{3}(3x+1) \geq 0, \frac{1}{4}(1-x) \geq 0, \frac{1}{2}(1-2x) \geq 0$

and $\frac{1}{3}(3x+1) + \frac{1}{4}(1-x) + \frac{1}{2}(1-2x) \leq 1$

$$\Rightarrow x \geq -\frac{1}{3}, x \leq 1, x \leq \frac{1}{2}$$

and $4(3x + 1) + 3(1 - x) + 6(1 - 2x) \leq 12$
Last equation gives $13 - 3x \leq 12$

$$\Rightarrow x \geq \frac{1}{3}$$

$$\therefore \frac{1}{3} \leq x \leq \frac{1}{2}$$

2. Winning horse can be selected in ${}^5C_1 = 5$ ways. The number of ways in which winning horse is amongst the horses selected by man is 2.

$\therefore$ Required probability $= \frac{2}{5}$

3. $np = 4$, $np(1 - p) = 2$

$$\Rightarrow 1 - p = \frac{2}{4} = \frac{1}{2} \quad \Rightarrow p = \frac{1}{2}$$

Thus. $n = 8$

Now, $P(X = 1) = {}^8C_1\left(\frac{1}{2}\right)^8 = \frac{1}{32}$

4. $P(E) = P(2, 3, 5, 7)$
$= 0.23 + 0.12 + 0.20 + 0.07$
$= 0.62$
$P(F) = P(1, 2, 3) = 0.15 + 0.23 + 0.12$
$= 0.5$
$P(E \cap F) = P(2, 3) = 0.23 + 0.12 = 0.35$
Thus, $P(E \cup F) = P(E) + P(F) - P(E \cap F)$
$= 0.62 + 0.5 - 0.35 = 0.77$

5. Probability they contradict each other
$= P(A'B \text{ or } AB') = P(A'B) + P(AB')$
$= P(A')\,P(B) + P(A)\,P(B')$

$$= \left(1-\frac{4}{5}\right)\left(\frac{3}{4}\right) + \left(\frac{4}{5}\right)\left(1-\frac{3}{4}\right) = \frac{7}{20}$$

6. From solution to Question 3,

$n = 8$, $p = \frac{1}{2}$

Thus, $P(X = 2) = {}^8C_2\left(\frac{1}{2}\right)^8 = \frac{7}{64} = \frac{28}{256}$

7.
$$\frac{1}{6} = P(\overline{A \cup B})$$
$$= 1 - P(A \cup B)$$

$$\Rightarrow \quad P(A \cup B) = \frac{5}{6}$$

$$\Rightarrow \quad P(A) + P(B) - P(A \cap B) = \frac{5}{6}$$

$$\Rightarrow \quad P(A) + P(B) = \frac{5}{6} + \frac{1}{4} = \frac{13}{12}$$

$$\Rightarrow \quad \frac{3}{4} + P(B) = \frac{13}{12}$$

$$\Rightarrow \quad P(B) = \frac{13}{12} - \frac{3}{4} = \frac{1}{3}$$

We have $P(A)\ P(B) = \left(\frac{3}{4}\right)\left(\frac{1}{3}\right)=\frac{1}{4}$

$= P(A \cap B)$

8. The number of ways in which any person can choose a house is 3.

$\therefore$ total number of ways $= 3^3 = 27$

Number of favourable ways = 3

$\therefore$ Required probability $= \frac{3}{27}=\frac{1}{9}$

9. Let S denote the event of getting a score of 9 in a single throw and $F = S'$, then

$$p = P(S)=\frac{4}{36}=\frac{1}{9} \text{ and } q=P(F)=\frac{8}{9}$$

Let X = Number of times a score of 9 occurs.

Then $X \sim B(3, p)$

$\therefore\ P(X = 2) = {}^3C_2p^2q=3\left(\frac{1}{9}\right)^2\left(\frac{8}{9}\right)=\frac{8}{9}$

10. Probability of the required event is

$(0.7)\ (0.2) + (0.7)\ (0.8)\ (0.7)\ (0.2)$
$+ (0.7)\ (0.8)\ (0.7)\ (0.8)\ (0.7)\ (0.2) + ...$

$$= \frac{0.14}{1-0.56}=\frac{0.14}{0.44}=\frac{7}{22}$$

None of the Answer Matches.

11. Let X denote the nummber on the die

$P(A) = P(X > 3) = P(4, 5, 6) = \frac{1}{2}$

$P(B) = P(X < 5) = \frac{4}{6}$

$P(A \cap B) = P(4) = \frac{1}{6}$

$\therefore\ P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{1}{2}+\frac{4}{6}-\frac{1}{6}=1$

12. Let A and B denote the following events

A : sum of digits is 8

and B : product of digits is 0.

Then $A \cap B = \{08\}$

and $B = \{00,01, 02, ..., 09, 10, 20, 30, 40\}$

Now,

$$P(A\,|\,B)=\frac{P(A\cap B)}{P(B)}=\frac{n(A\cap B)}{n(B)}=\frac{1}{14}$$

13. $P(X\geq 1)\geq\frac{9}{10}$

$\Rightarrow\ 1-P(X=0)\geq\frac{9}{10}$

$\Rightarrow\ 1-\left(\frac{3}{4}\right)^n\geq\frac{9}{10}$

$\Rightarrow\ \left(\frac{3}{4}\right)^n\leq\frac{1}{10}$

$\Rightarrow\ \left(\frac{4}{3}\right)^n\geq 10$

$\Rightarrow\ n[\log 4-\log 3]\geq\log_{10}10$

$\Rightarrow\ n\geq\frac{1}{\log 4-\log 3}$

14. Total number of ways of drawing three balls out of 9 is ${}^9C_3$.

The number of ways of drawing the balls so that balls are of different colours is $({}^3C_1)\ ({}^4C_1)\ ({}^2C_1) = (3)\ (4)\ (2)$

$\therefore$ probability of the required event

$$= \frac{(3)(4)(2)}{{}^9C_3}=\frac{(3)(4)(2)(3)(2)}{9\times 8\times 7}=\frac{2}{7}$$

15. Statement-2 is false.

For $d = 6$, we have 1, 7, 13, 19 form an A.P.

Thus, the only choice of answer is (c). However, let us count the number of AP's.

For $d = 1$, there are 17 such AP's, viz.

(1, 2, 3, 4), (2, 3, 4, 5), ..., (17, 18, 19, 20).

For $d = 2$, there are 14 such AP's, viz.

(1, 3, 5, 7), (2, 4, 6, 8), ... (14, 16, 18, 20).

Similarly, for $d = 3$, there are 11 AP's, for $d = 4$, there are 8 AP's.

$d = 5$, there are 5 and for $d = 6$, there are 2 AP's.

Thus, the number of AP's is

$17 + 14 + 11 + 8 + 5 + 2 = (3)\ (19)$

Also, the total number of ways of choosing 4 numbers out of 20 is ${}^{20}C_4$.

Hence, probability of the required event is

$$\frac{(3)(19)}{{}^{20}C_4}=\frac{(3)(19)(4)(3)(2)}{(20)(19)(18)(17)}=\frac{1}{85}$$

$\therefore$ Statement-1 is true.

16. Let X = number of failures, then

$X \sim B(5, 1 - p)$

P(at least one failure) $= P(X \geq 1) = 1 - P(X = 0)$

$= 1 - (1 - (1 - p))^5 = 1 - p^5$

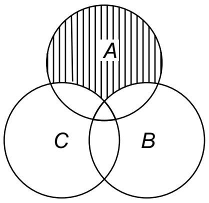
We are given

$1-p^5\geq\frac{31}{32}\ \Rightarrow p^5\leq\frac{1}{32}$

$\Rightarrow\ 0\leq p\leq\frac{1}{2}$

17. $P(C\,|\,D) = \dfrac{P(C \cap D)}{P(D)} = \dfrac{P(C)}{P(D)}$ $\qquad [\because\ C \subseteq D]$

18. We have



$P(A' \cap B' \mid C)$

$= \dfrac{P(A' \cap B' \cap C)}{P(C)}$

$= \dfrac{P[(A \cup B)' \cap C]}{P(C)}$

$= \dfrac{P(C) - P[(A \cap C) \cup (B \cap C)]}{P(C)}$

$= \dfrac{P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}{P(C)}$

$= \dfrac{P(C) - P(A)P(C) - P(B)P(C)}{P(C)}$

[∵ A, B, C are pairwise independent and $P(A \cap B \cap C) = 0$]

$= 1 - P(A) - P(B) = P(A') - P(B)$

19. Total number of ways of choosing 3 numbers out of 8 is ${}^8C_3 = 56$

Let A denote the event that minimum is 3 and B denote the event that maximum number is 6. Then

$P(A\,|\,B) = \dfrac{P(A \cap B)}{P(B)}$

But $P(B) = \dfrac{{}^5C_2}{{}^8C_3} = \dfrac{10}{56}$ and

$P(A \cap B) = \dfrac{{}^2C_1}{{}^8C_3} = \dfrac{2}{56}$

$\therefore\ P(A\,|\,B) = \dfrac{2}{10} = \dfrac{1}{5}$

20. Let X = number of correct answers obtained by just guessing, then $X \sim B(n, p)$ where $n = 5$, $p = 1/3$.

P(getting 4 or more correct answers)

$= P(X \geq 4) = P(X = 4) + P(X = 5)$

$= {}^5C_4\left(\dfrac{1}{3}\right)^4\left(\dfrac{2}{3}\right) + {}^5C_5\left(\dfrac{1}{3}\right)^5 = \dfrac{11}{3^5}$

21. $P((A' \cap B) \cup (A \cap B')) = \dfrac{26}{49}$ $\qquad (1)$

and $P(A' \cap B') = \dfrac{15}{49}$ $\qquad (2)$

Let $P(A) = a$, $P(B) = b$, then (1) and (2) give

$(1-a)b + a(1-b) = \dfrac{26}{49}$

$(1-a)(1-b) = \dfrac{15}{49}$

$\Rightarrow\ a + b - 2ab = \dfrac{26}{49}$

and $1 - (a+b) + ab = \dfrac{15}{49}$

$\Rightarrow\ a + b - ab = \dfrac{34}{49}$

Thus, $a + b = \dfrac{42}{49} = \dfrac{6}{7}$, $ab = \dfrac{8}{49}$

$\Rightarrow\ a = \dfrac{2}{7}$, $b = \dfrac{4}{7}$

$\therefore$ most probable $= \dfrac{4}{7}$

22. $\dfrac{1}{3}(3x+1) \geq 0$, $\dfrac{1}{4}(1-x) \geq 0$

$\Rightarrow\ x \geq -\dfrac{1}{3}$, $x \leq 1$

Also, $P(A) + P(B) \leq 1$

$\Rightarrow 4(3x + 1) + 3(1 - x) \leq 12$

$\Rightarrow\ 9x \leq 5\ \ \Rightarrow x \leq \dfrac{5}{9}$

Thus, $x \in \left[-\dfrac{1}{3}, \dfrac{5}{9}\right]$

23. Let X = number of times the man hits the target

Then $X \sim B(k, p)$ where $p = \dfrac{2}{5}$

Now, $P(X \geq 1) \geq \dfrac{7}{10}$

$\Rightarrow\ 1 - P(X = 0) \geq \dfrac{7}{10}$

$\Rightarrow\ {}^kC_0\left(\dfrac{3}{5}\right)^k \leq \dfrac{3}{10}$

$\Rightarrow 2(3^{k-1}) \leq 5^{k-1}$

Least value of k is 3.

24. $P((A \cup B) \cap C') = P[(A \cap C') \cup (B \cap C')]$

$= P(A \cap C') + P(B \cap C') - P(A \cap B \cap C')$

$= P(A)\,P(C') + P(B)\,P(C') - P(A)\,P(B)\,P(C')$

$= \left(\dfrac{3}{4} + \dfrac{1}{2} - \dfrac{3}{8}\right)\dfrac{3}{8} = \dfrac{21}{64}$

25. $\dfrac{1}{6} = P((A \cup B)')$

$= 1 - P(A \cup B)$

$= 1 - (P(A) + P(B) - P(A \cap B))$

$$= 1-\left(\frac{3}{4}+P(B)-\frac{1}{4}\right)$$

$$= \frac{1}{2}-P(B)$$

$$\Rightarrow\ P(B)=\frac{1}{3}$$

As $P(A\cap B)=\frac{1}{4}=\left(\frac{3}{4}\right)\left(\frac{1}{3}\right)=P(A)P(B)$

we get A and B are independent.

26. $P(A \cup B) = P(A \cap B)$

$\Rightarrow [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)] = 0$

As $P(A \cap B) \le P(A), P(B)$, we get

$P(A) - P(A \cap B) = 0,\ P(B) - P(A \cap B) = 0$

$\Rightarrow P(A) = P(B) = P(A \cap B)$

Thus, A and B are equally likely.

Now, $P(A \cap B') = P(A) - P(A \cap B) = 0$

Smilarly, $P(A' \cap B) = 0$

However, we cannot say $P(A) + P(B) = 1$

27. $P(X = 2) = P(X = 3)$

$\Rightarrow {}^nC_2\, p^2q^{n-2} = {}^nC_3\, p^3q^{n-3}$

$$\Rightarrow\ \frac{p}{q}=\frac{n(n-1)}{2}\cdot\frac{3\times 2}{n(n-1)(n-2)}=\frac{3}{n-2}$$

$\Rightarrow np - 2p = 3q$

Now, $E(X) = np = 2p + 3q = 3(p + q) - p$

$\Rightarrow E(X) = 3 - p.$

28. $\frac{(x-10)(x-50)}{(x-30)}\ge 0$

$\Rightarrow 10 \le x < 30$ or $x \ge 50$

$\therefore$ x can take $(20 + 51)$ values.

Thus, probability of required event is

$$\frac{71}{100}=0.71$$

29. $P(E\,|\,A)=\frac{P(A\cap E)}{P(E)}=\frac{P(E)P(A\,|\,E)}{P(A)}$

$\ge P(E)\ P(A|E)$ as $0 < P(A) \le 1$

Also, $P(A\,|\,E)=\frac{P(A\,|\,E)}{P(E)}\ge P(A\cap E)$

as $0 < P(E) \le 1$

30. Number of ways of choosing A is $27 - 1 = 127$.

Now $x \in A$ if and only if $A = \{x\} \cup B$

where B is a subset of $S - \{x\}$.

The number of ways of choosing B is 64.

$\therefore$ Probability of required event is $\frac{64}{127}$.

31. The question is incorrect

We solve it by taking balls to be distinct.

For each ball we have three choice. Thus, total number of choice is 3^{12}.

We can choose one box three balls out of 12 in ${}^{12}C_3$ ways and distribute the remaining 9 balls in two boxes in 2^9 ways. Thus, number of favourable ways is ${}^{12}C_3\ (2^9)$

$\therefore$ probability of the required event is

$$\frac{{}^{12}C_3(2^9)}{3^{12}}=\frac{12\times 11\times 10}{3\times 2}\cdot\frac{2^9}{3^{12}}=\frac{55}{3}\left(\frac{2}{3}\right)^{11}$$

32. Number of elements in $P(X)$ is 2^{10}. Sets A and B can be chosen in $(2^{10})\ (2^{10}) = 2^{20}$ ways

We now count number of favourable ways.

A and B each can have k $(0 \le k \le 10)$ elements in $({}^{10}C_k)\ ({}^{10}C_k)$ ways.

$\therefore$ Number of favourable ways

$$= \sum_{k=10}^{10}\left({}^{10}C_k\right)^2=\sum_{k=0}^{10}\left({}^{10}C_k\right)\left({}^{10}C_{10-k}\right)$$

= Number of ways of selecting 10 persons from 10 men and 10 women

$= {}^{20}C_{10}$

$\therefore$ probability of required event $= \frac{{}^{20}C_{10}}{2^{20}}$

33*. Possible isosceless triangle are

$(1, 1, 1)$, $(2, 2, a)$ where $a = 1, 2, 3,$

$(3, 3, b)$, where $b = 1, 2, 3, 4, 5$ and

(c, c, d) where $c = 4, 5, 6$ and $d = 1, 2, 3, 4, 5, 6$

Thus, there are 27 isosceles triangle and maximum area occurs when the triangle is $(6, 6, 6)$.

Hence, probability of the required event s 1/27.

34. $np = 2,\ np(1 - p) = 1$

$\Rightarrow 1 - p = 1/2 \Rightarrow p = 1/2$ and $n = 4$

Thus, $P(x \ge 1) = 1 - P\ (X = 0)$

$$= 1-{}^4C_0\left(\frac{1}{2}\right)^4=\frac{15}{16}$$

35. $E_1 = \{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)\}$

$E_2 = \{(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2)\}$

$E_3 = \{(x, y) : 1 \le x, y \le 6,\ x + y \text{ is odd}\}$

We have $E_1 \cap E_2 = \{(4, 2)\}$

$E_2 \cap E_3 = \{(1, 2), (3, 2), (5, 2)\}$

$E_1 \cap E_3 = \{(4, 1), (4, 3), (4, 5)\}$

$E_1 \cap E_2 \cap E_3 = \phi$

We have

$$P(E_1)=\frac{1}{6},\ P(E_2)=\frac{1}{6},\ P(E_3)=\frac{1}{2}$$

$$P(E_1 \cap E_2) = \frac{1}{36}, P(E_2 \cap E_3) = \frac{1}{12}, P(E_3 \cap E_1) = \frac{1}{12}$$

$P(E_1 \cap E_2 \cap E_3) = 0$

As

$P(E_1 \cap E_2) = P(E_1)\, P(E_2)$,

$P(E_2 \cap E_3) = P(E_2)\, P(E_3)$,

$P(E_3 \cap E_1) = P(E_3)\, P(E_1)$

We get (a), (b), (c) are true.

But $P(E_1 \cap E_2 \cap E_3) = 0 \neq P(E_1)\, P(E_2)\, P(E_3)$

$\Rightarrow$ (d) is not true.

36. $P(A) = \frac{2}{5}, P(A \cap B) = \frac{3}{20}$

Now,

$$P(A \mid (A' \cup B')) = \frac{P[A \cap (A' \cup B')]}{P(A' \cup B')}$$

$$= \frac{P(A \cap B')}{1 - P(A \cap B)}$$

$$= \frac{P(A) - P(A \cap B)}{1 - P(A \cap B)}$$

$$= \frac{2/5 - 3/20}{1 - 3/20}$$

$$= \frac{5/20}{17/20} = \frac{5}{17}$$

37. $P(S) = 2P(F)$

Also, $P(S) + P(F) = 1$

$P(F) = 1/3$, $P(S) = 2/3$

Let X = No. of successes, then

$P(X \geq 5) = P(X = 5) + P(X = 6)$

$$= {}^6C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right) + {}^6C_6 \left(\frac{2}{3}\right)^6$$

$$= \left(\frac{2}{3}\right)^5 \left[\frac{6}{3} + \frac{2}{3}\right] = \frac{256}{729}$$

Previous Years' B-Architecture Entrance Examination Questions

1. $P(A \cup B) = P(AB') + P(B)$

$= 0.15 + 0.55 = 0.7$

2. Required probability

= P(1 and 2 are operational) + P(1 and 3 are operational)

$= P(1)\, P(2) + P(1)\, P(3)$

$= (p)\,(p) + (p)\,(p) = 2p^2$

3. $P(E_i) = P$(digit i appears on the ball)

$$= \frac{2}{4} = \frac{1}{2}, i = 1, 2, 3$$

$P(E_i \cap E_j) = P$(digits i and j appear on the ball)

$$= \frac{1}{4} \text{ for } i \neq j$$

Statement 1 and 2 both are true but statement-2 is not a correct explanation for statement-1.

4. Total number of cases = 6^3

Number of favourable cases

= Coefficient of x^7 in $(x + x^2 + \ldots + x^6)^3$

= Coefficient of x^7 in $x^3(1 - x^6)^3\,(1 - x)^{-3}$

= Coefficient of x^4 in $(1 + 3x + {}^4C_2\, x^2 + {}^5C_3\, x^3 + \ldots)$

$= {}^6C_4 = 15$

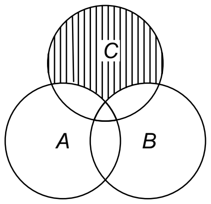
Also, $A \cap B$ = {(1, 1, 5), (1, 2, 4), (1, 3, 3), (1, 4, 2), (1, 5, 1)}

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{5}{15} = \frac{1}{3}$$

5. P (Element belongs to A only)

$= P(A \cap B' \cap C')$

$$= \frac{n(A \cap B' \cap C')}{n(A \cup B \cup C)}$$



We have

$n(A \cup B \cup C) = 35 + 40 + 45 - 13 - 12 - 14 + 5$

$= 86$

and $n(A \cap B' \cap C')$

$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$

$= 35 - 13 - 12 + 5 = 15$

Thus, $P(A \cap B' \cap C') = \frac{15}{86}$

6. E_1 : die shows a six

E_2 : die doesn't show a six

A : man reports it is a six

We have

$$P(E_1) = \frac{1}{6},\ P(E_2) = \frac{5}{6}$$

$$P(A \mid E_1) = \frac{3}{4},\ P(A \mid E_2) = \frac{1}{4}$$

By the Bayes' rule

$$P(E_1 \mid A) = \frac{P(E_1) P(A \mid E_1)}{P(E_1) P(A \mid E_1) + P(E_2) P(A \mid E_2)}$$

$$= \frac{(1/6)(3/4)}{(1/6)(3/4) + (5/6)(1/4)}$$

$= 3/8$

7. $$P(A \mid A' \cup B') = \frac{P(A \cap (A' \cup B'))}{P(A' \cup B')}$$

$$= \frac{P(A \cap B')}{1-(P(A)+P(B)-P(A \cup B))}$$

$$= \frac{P(A)-P(A \cap B)}{1-P(A \cap B)}$$

$$= \frac{0.4-0.15}{1-0.15}$$

$$= \frac{0.25}{0.85}$$

$$= \frac{5}{17}$$

8. $P(F \cap R \cap G) = P(F)\ P(R)\ P(G)$

[$\because$ traits are independent]

$$= \left(\frac{20}{80}\right)\left(\frac{10}{80}\right)\left(\frac{25}{80}\right) = \frac{5}{512}$$

9. $\frac{3}{4} \le P(A \cup B) \le 1$

$\Rightarrow \frac{3}{4} \le P(A)+P(B)-P(A \cap B) \le 1$

$\Rightarrow \frac{3}{4}+\frac{1}{8} \le P(A)+P(B) \le 1+\frac{3}{8}$

$\Rightarrow \frac{7}{8} \le P(A)+P(B) \le \frac{11}{8}$

Thus both the statements are corret but statement-2 is not a correct explanation for statement-2.

10. P(Even number of tosses are required)

$= P(TH$ or $TTTH$ or $TTTTTH$ or ---)

$= P(T)\ P(H) + P(T)^3\ P(H) + P(T)^5\ P(H) +$ ---

$$= \frac{P(T)P(H)}{1-P(T)^2} = \frac{P(T)P(H)}{(1-P(T))(1+P(T))}$$

$$= \frac{P(T)P(H)}{P(H)(1+P(T))} = \frac{1-p}{2-p}$$

Now, $\frac{2}{5} = \frac{1-p}{2-p} \quad \Rightarrow \quad 4 - 2p = 5 - 5p$

$\Rightarrow p = 1/3$

11. P(root are real) = 1 – P(roots are non-real)

Root are non-real if and only if

$p^2 - 4q < 0 \Rightarrow 4q < p^2$

When $q = 1, \quad p = 1$

$q = 2, \quad p = 1, 2$

$q = 3, 4, \quad p = 1, 2, 3$

$q = 5, 6, \quad p = 1, 2, 3, 4$

$q = 7, 8, 9 \quad p = 1, 2, 3, 4, 5$

$q = 10, \quad p = 1, 2, 3, 4, 5, 6$

Thus, (1) is satisfied by

$1 + 2 + 2 \times 3 + 2 \times 4 + 3 \times 5 + 6 = 38$

ordered pairs (p, q)

Also, total number of ways of choosig (p, q) is

$$10 \times 10 = 100$$

$\therefore$ probability of required event

$$= 1 - \frac{38}{100} = \frac{31}{50}.$$

12. $$P\ (x \ge 1) \ge \frac{9}{10}$$

$$\Rightarrow \quad 1 - P(X = 0) \ge \frac{9}{10}$$

$$\Rightarrow \quad P(X = 0) \le \frac{1}{10}$$

$$\Rightarrow \quad \left(\frac{3}{4}\right)^n \le \frac{1}{10}$$

$$\Rightarrow \quad \left(\frac{4}{3}\right)^n \ge 10$$

$$\Rightarrow \quad n \log_{10}(4/3) \ge 1$$

$$\Rightarrow \quad n \ge \frac{1}{\log_{10} 4 - \log_{10} 3}$$

13. Let E_1 : black ball is drawn in the first draw

E_2 : white ball is drawn in the first draw

A : a black ball is drawn in the second draw

$$P(E_1) = \frac{5}{9}, P(E_2) = \frac{4}{9}$$

$$P(A \mid E_1) = \frac{5}{9+2}, P(A \mid E_2) = \frac{5+2}{9+2}$$

By the total probability rule

$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$

$$= \left(\frac{5}{9}\right)\left(\frac{5}{11}\right) + \left(\frac{4}{9}\right)\left(\frac{7}{11}\right) = \frac{53}{99}$$

CHAPTER TWENTY FIVE

Trigonometrical Ratios, Identities and Equations

INTRODUCTION

An angle is the amount of revolution which a line OP revolving about the point O has undergone in passing from its initial position OA into its final position OB.

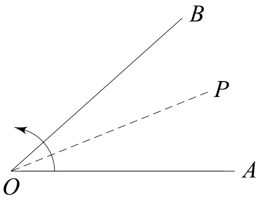


Fig. 25.1

If the rotation is in the clockwise sense, the angle measured is negative and it is positive if the rotation is in the anti-clockwise sense.

The two commonly used systems of measuring an angle are

1. *Sexagesimal system* in which

$$1 \text{ right angle} = 90 \text{ degrees } (90°)$$
$$1 \text{ degree} = 60 \text{ minutes } (60')$$
$$1 \text{ minute} = 60 \text{ seconds } (60'')$$

2. *Circular systems* in which the unit of measurement is the angle subtended at the centre of a circle by an arc whose length is equal to the radius and is called a *radian*.

Relation between degree and radian

$$\pi \text{ radian} = 180 \text{ degrees}$$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$
$$= 57° \; 17' \; 45'' \text{ (approximately)}$$

The *trigonometrical ratios* of an angle are numerical quantities. Each one of them represents the ratio of the length of one side to another of a right angled triangle.

DOMAIN AND RANGE OF TRIGONOMETRICAL FUNCTIONS

	Domain	Range
$\sin A$	$\mathbf{R}$	$[-1, 1]$
$\cos A$	$\mathbf{R}$	$[-1, 1]$
$\tan A$	$\mathbf{R} - \{(2n+1)\,\pi/2, n \in \mathbf{I}\}$	$\mathbf{R} = (-\infty, \infty)$
$\text{cosec } A$	$R - \{n\pi, n \in \mathbf{I}\}$	$(-\infty, -1] \cup [1, \infty)$
$\sec A$	$\mathbf{R} - \{(2n+1)\,\pi/2, n \in \mathbf{I}\}$	$(-\infty, -1] \cup [1, \infty)$
$\cot A$	$\mathbf{R} - \{n\pi, n \in \mathbf{I}\}$	$\mathbf{R} = (-\infty, \infty)$

We find, $|\sin A| \le 1$, $|\cos A| \le 1$

$\sec A \ge 1$ or $\sec A \le -1$

and $\text{cosec } A \ge 1$ or $\text{cosec } A \le -1$

SOME BASIC FORMULAE

1. $\sin^2 A + \cos^2 A = 1$ or $\cos^2 A = 1 - \sin^2 A$ or $\sin^2 A = 1 - \cos^2 A$.
2. $1 + \tan^2 A = \sec^2 A$ or $\sec^2 A - \tan^2 A = 1$.
3. $1 + \cot^2 A = \text{cosec}^2 A$ or $\text{cosec}^2 A - \cot^2 A = 1$.
4. $\sin A \text{ cosec } A = \tan A \cot A = \cos A \sec A = 1$.

A system of rectangular coordinate axes divides a plane into four quadrants. An angle θ lies in one and only one of these quadrants. The values of the trigonometric ratios in the various quadrants are shown in Fig. 25.2.

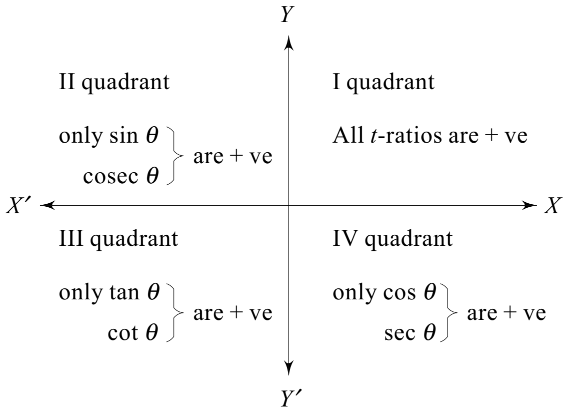


Fig. 25.2

Sine, cosine and tangent of some angles less than 90°

	0°	**15°**	**18°**	**30°**
sin	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$
cos	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}}{2}$
tan	0	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\frac{1}{\sqrt{3}}$
	36°	**45°**	**60°**	**90°**
sin	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	$\sqrt{5-2\sqrt{5}}$	1	$\sqrt{3}$	not defined

ALLIED OR RELATED ANGLES

The angles $\frac{n\pi}{2} + \theta$ and $\frac{n\pi}{2} - \theta$, where n is any integer, are known as *allied* or *related angles*. The trigonometric functions of these angles can be expressed as trigonometric functions of θ, with either a plus or a minus sign. The following working rules can be used in determining these functions:

1. Assuming that $0 < \theta < 90°$, note the quadrant in which the given angle lies. The result has a plus or minus sign according as the given function is positive or negative in that quadrant.
2. If n is even, the result contains the same trigonometric function as the given function. But if n is odd, the result contains the corresponding *co-function*, i.e., sine becomes cosine, tangent becomes cotangent, secant becomes cosecant and *vice versa*.

Illustration 1

(a) To determine $\sin(540° - \theta)$, we note that $540° - \theta = 6 \times 90° - \theta$ is a second quadrant angle if $0° < \theta < 90°$. In this quadrant, sine is positive and, since the given angle contains an even multiple of $\pi/2$, the sine function is retained. Hence, $\sin(540° - \theta) = \sin\theta$.

(b) To determine $\cos(630° - \theta)$, we note $630° - \theta = 7 \times 90° - \theta$ is a third quadrant angle if $0° < \theta < 90°$. In this quadrant, cosine is negative and, since the given angle contains an odd multiple of $\pi/2$, cosine is replaced by sine. Hence, $\cos(630° - \theta) = -\sin\theta$.

COMPOUND ANGLES

An angle made up of the algebraic sum of two or more angles is called a compound angle.

Some formulae and results regarding compound angles:

1. $\sin(A + B) = \sin A \cos B + \cos A \sin B$.
2. $\sin(A - B) = \sin A \cos B - \cos A \sin B$.
3. $\cos(A + B) = \cos A \cos B - \sin A \sin B$.
4. $\cos(A - B) = \cos A \cos B + \sin A \sin B$.
5. $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.
6. $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.
7. $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$ $= \cos^2 B - \cos^2 A$.
8. $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$ $= \cos^2 B - \sin^2 A$.
9. $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1 + \tan^2 A}$.
10. $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$ $= 2\cos^2 A - 1 = \frac{1 - \tan^2 A}{1 + \tan^2 A}$.
11. $1 + \cos 2A = 2\cos^2 A$, $\quad 1 - \cos 2A = 2\sin^2 A$.
12. $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$.
13. (a) $\frac{1 - \cos\theta}{\sin\theta} = \tan\frac{\theta}{2}$ (b) $\frac{1 + \cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$
    (c) $\frac{1 - \cos\theta}{1 + \cos\theta} = \tan^2\frac{\theta}{2}$.

14. (a) $\sin 3x = 3 \sin x - 4 \sin^3 x$
    (b) $\cos 3x = 4 \cos^3 x - 3 \cos x$
    (c) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$.
15. $\tan (A + B + C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

16. (a) $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$
    (b) $\sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$
    (c) $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$
    (d) $\cos C - \cos D = 2 \sin \dfrac{C+D}{2} \sin \dfrac{D-C}{2}$.

(*Note*)

17. (a) $\tan A + \tan B = \dfrac{\sin (A+B)}{\cos A \cos B}$
    (b) $\tan A - \tan B = \dfrac{\sin (A-B)}{\cos A \cos B}$.
18. (a) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$
    (b) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
    (c) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
    (d) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$

(*Note*)

19. $\sin nA = \cos^n A\, ({}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A - \ldots)$
20. $\cos nA = \cos^n A\, (1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A \ldots)$
21. $\tan nA = \dfrac{{}^nC_1 \tan A - {}^nC_3 \tan^3 A + {}^nC_5 \tan^5 A \ldots}{1 - {}^nC_2 \tan^2 A + {}^nC_4 \tan^4 A \ldots}$
22. $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + \ldots + \sin (\alpha + (n - 1)\beta)$

$$= \frac{\sin(\alpha + (n-1)\beta/2)}{\sin(\beta/2)} \sin (n\beta/2)$$

23. $\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + \ldots + \cos (\alpha + (n - 1)\beta)$

$$= \frac{\cos(\alpha + (n-1)\beta/2)}{\sin(\beta/2)} \sin (n\beta/2)$$

24. $\tan (A_1 + A_2 + \ldots + A_n) = \dfrac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}$

where $S_1 = \tan A_1 + \tan A_2 + \ldots + \tan A_n$
(sum of the tangents taken one at a time)
$S_2 = \tan A_1 \tan A_2 + \tan A_2 \tan A_3 + \ldots$
(sum of the tangents taken two at a time, there are ${}^nC_2$ such terms)
$S_3 = \tan A_1 \tan A_2 \tan A_3 + \tan A_2 \tan A_3 \tan A_4 + \ldots$
(sum of the tangents taken three at a time, there are ${}^nC_3$ such terms and so on.

SOME IMPORTANT RESULTS

1. $\sin \theta \sin (60° - \theta) \sin (60° + \theta) = (1/4) \sin 3\theta$.
2. $\cos \theta \cos (60° - \theta) \cos (60° + \theta) = (1/4) \cos 3\theta$.
3. $\tan \theta \tan (60° - \theta) \tan (60° + \theta) = \tan 3\theta$
4. $\sin 9° = (1/4)\left[\sqrt{3+\sqrt{5}} - \sqrt{5-\sqrt{5}}\right] = \cos 81°$
5. $\cos 9° = (1/4)\left[\sqrt{3+\sqrt{5}} + \sqrt{5-\sqrt{5}}\right] = \sin 81°$
6. $\cos 36° - \cos 72° = 1/2$
7. $\cos 36° \cos 72° = 1/4$
8. $\sin 22\frac{1}{2}° = \left(\frac{1}{2}\right)\left[\sqrt{2-\sqrt{2}}\right]$
9. $\cos 22\frac{1}{2}° = \left(\frac{1}{2}\right)\left[\sqrt{2+\sqrt{2}}\right]$
10. $\tan 22\frac{1}{2}° = \sqrt{2} - 1$
11. $\cot 22\frac{1}{2}° = \sqrt{2} + 1$
12. $-\sqrt{a^2 + b^2} \le a \sin x + b \cos x \le \sqrt{a^2 + b^2}$ for all $x \in \mathbf{R}$.

IDENTITIES

A trigonometric equation is an identity if it is true for all values of the angle or angles involved. A given identity may be established by (i) reducing either side to the other one, or (ii) reducing each side to the same expression, or (iii) any convenient modifications of the methods given in (i) and (ii).

CONDITIONAL IDENTITIES

When the angles A, B and C satisfy a given relation, many interesting identities can be established connecting the trigonometric functions of these angles. In proving these identities, we require the properties of complementary and supplementary angles. For example, if $A + B + C = \pi$, then

1. $\sin (B + C) = \sin A$, $\cos B = -\cos (C + A)$
2. $\cos (A + B) = -\cos C$, $\sin C = \sin (A + B)$
3. $\tan (C + A) = -\tan B$, $\cot A = -\cot (B + C)$.
4. $\cos \dfrac{A+B}{2} = \sin \dfrac{C}{2}$, $\cos \dfrac{C}{2} = \sin \dfrac{A+B}{2}$.
5. $\sin \dfrac{C+A}{2} = \cos \dfrac{B}{2}$, $\sin \dfrac{A}{2} = \cos \dfrac{B+C}{2}$.

6. $\tan\dfrac{B+C}{2} = \cot\dfrac{A}{2}$, $\tan\dfrac{B}{2} = \cot\dfrac{C+A}{2}$.

Some Important Identities If $A + B + C = \pi$, then

1. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.
2. $\cot B \cot C + \cot C \cot A + \cot A \cot B = 1$.
3. $\tan\dfrac{B}{2}\tan\dfrac{C}{2} + \tan\dfrac{C}{2}\tan\dfrac{A}{2} + \tan\dfrac{A}{2}\tan\dfrac{B}{2} = 1$.
4. $\cot\dfrac{A}{2} + \cot\dfrac{B}{2} + \cot\dfrac{C}{2} = \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$.
5. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.
6. $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C$.
7. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$.
8. $\sin A + \sin B + \sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$.
9. $\cos A + \cos B + \cos C = 1 + 4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$.

GENERAL SOLUTIONS OF TRIGONOMETRICAL EQUATIONS

The following formulae are used in solving trigonometrical equations:

1. If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n \alpha$ $(n \in \mathbf{I})$
2. If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$ $(n \in \mathbf{I})$
3. If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$ $(n \in \mathbf{I})$
4. If $\sin\theta = \sin\alpha$, $\cos\theta = \cos\alpha$, then $\theta = 2n\pi + \alpha$ $(n \in \mathbf{I})$

Illustration 2

If $\sin\theta = 1/2$, then $\sin\theta = \sin(\pi/6)$ so that $\theta = n\pi + (-1)^n \pi/6$ $(n \in \mathbf{I})$

In solving trigonometrical equations, the general values of the angle should be given, unless the solution is required in any specified interval or range.

Principal Value

Since all the trigonometrical ratios are periodic functions, an equation of the form $\sin\theta = k$, $\cos\theta = k$ or $\tan\theta = k$ etc. can have an infinite number of angles satisfying it. The set of all such values is called general value of θ.
Numerically least value in this set is called **Principal Value.**

Illustration 3

If $\sin\theta = 1/\sqrt{2}$, the general value of θ is $n\pi + (-1)^n \pi/4$ and the principal value is $\pi/4$.

Method for Finding Principal Value

(i) First note the quadrant in which the angle lies.
(ii) For 1st and 2nd quadrants consider anticlockwise direction and for 3rd and 4th clockwise direction.
(iii) Find the angles in the Ist rotation.
(iv) Select the numerically least angle among these two values. The angle thus found will be the principal value.

Illustration 4

If $\tan\theta = -1$, θ lies in 2nd or 4th quadrant. For 2nd quadrant we will select anticlockwise and for 4th quadrant we will select clockwise direction. So we get two values $3\pi/4$ and $-\pi/4$, of which $-\pi/4$ is numerically least angle. Hence principal value is $-\pi/4$.

Illustration 5

For $\sin\theta = -1/2$, the principal value of θ is $-\pi/6$ and for $\cos\theta = 1/2$, the principal value of θ is $\pi/3$. [Note $\cos\theta$ is positive in Ist and 4th quadrant, so we get two values $-\pi/3$ and $\pi/3$ of θ satisfying the relation. In such case. We select the angle with positive sign as principal value].

SOLVED EXAMPLES

Concept-based Straight Objective Type Questions

Example 1: If $0 < \theta < \pi/2$, and $f(\theta) = (1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta)$, then $f(\pi/4)$ is equal to

(a) 0 (b) 1
(c) 2 (d) $2 + \sqrt{2}$

Ans. (c)

Solution:
$$f(\theta) = \left(1 + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1 + \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}\right)$$
$$= \left(\frac{\sin\theta + \cos\theta - 1}{\sin\theta}\right)\left(\frac{\sin\theta + \cos\theta + 1}{\cos\theta}\right)$$

$$= \frac{(\sin\theta+\cos\theta)^2-1}{\sin\theta\cos\theta} = \frac{1+2\sin\theta+\cos\theta-1}{\sin\theta\cos\theta} = 2$$

$\therefore \quad f(\pi/4) = 2$

◉ **Example 2:** If $2\cos\theta = \sqrt{x}+\dfrac{1}{\sqrt{x}}$ for some $x > 0$, then which one of the following is **not** equal to $\cos(5\theta)$

(a) $\frac{1}{2}\left(x^{5/2}+\frac{1}{x^{5/2}}\right)$ (b) $\frac{1}{2}\left(x^{3/2}+\frac{1}{x^{3/2}}\right)$

(c) 1 (d) – 1

Ans. (d)

◎ **Solution:** For $x > 0$, $\sqrt{x}+\dfrac{1}{\sqrt{x}} \geq 2$ and

$$\sqrt{x}+\frac{1}{\sqrt{x}} = 2 \Leftrightarrow \sqrt{x} = 1 \text{ or } x = 1.$$

We have $2\cos\theta \leq 2$ and $\sqrt{x}+\dfrac{1}{\sqrt{x}} \geq 2$.

Thus, $2\cos\theta = \sqrt{x}+\dfrac{1}{\sqrt{x}} \Leftrightarrow x = 1, \cos\theta = 1$

$\therefore \quad \theta = 2n\pi$, where $n \in \mathbf{Z}$.

$\Rightarrow \quad \cos(5\theta) = \cos(10n\pi) = 1 \neq -1$

Also, for $x = 1$, $\frac{1}{2}\left(x^{5/2}+\frac{1}{x^{5/2}}\right) = \frac{1}{2}\left(x^{3/2}+\frac{1}{x^{3/2}}\right) = 1$

◉ **Example 3:** For $0 < \alpha, \beta < \pi/2$, let $E = (\tan\alpha+\tan\beta)(1-\cot\alpha\cot\beta)+(\cot\alpha+\cot\beta)(1-\tan\alpha\tan\beta)$ then E is

(a) independent of α only

(b) independent of β only

(c) independent of α and β both

(d) equal to $2(\tan\alpha+\tan\beta)$

Ans. (c)

◎ **Solution:** $E = (\tan\alpha+\tan\beta)\left(1-\dfrac{1}{\tan\alpha\tan\beta}\right)$

$$+\frac{(\tan\alpha+\tan\beta)}{\tan\alpha\tan\beta}(1-\tan\alpha\tan\beta)$$

$$= (\tan\alpha+\tan\beta)(1-\tan\alpha\tan\beta)p$$

where

$$p = -\frac{1}{\tan\alpha\tan\beta}+\frac{1}{\tan\alpha\tan\beta} = 0$$

$\therefore \quad E = 0$

◉ **Example 4:** If $\cot A = \tan((n-1)A)$, where $n \in \mathbf{N}$, then a value of A can be

(a) $\pi/2n$ (b) π/n

(c) $2\pi/n$ (d) $\pi/4n$

Ans. (a)

◎ **Solution:** $\tan(\pi/2 - A) = \tan((n-1)A)$

One possible solution is

$$\pi/2 - A = (n-1)A \Rightarrow A = \pi/2n$$

◉ **Example 5:** $\frac{1}{2}(\sqrt{3}\cos 23° - \sin 23°)$ is not equal to

(a) cos 53° (b) sin 53°

(c) sin 37° (d) sin 143°

Ans. (b)

◎ **Solution:** $\frac{1}{2}(\sqrt{3}\cos 23° - \sin 23°)$

$$= \cos 30°\cos 23° - \sin 30°\sin 23°$$
$$= \cos(30°+23°) = \cos 53°$$
$$= \cos(90°-37°) = \sin 37° = \sin(180°-37°)$$
$$= \sin(143°)$$

◉ **Example 6:** Suppose $\alpha, \beta \in \mathbf{R}$, and let $E = \sin^2(\alpha+\beta) + \sin^2\alpha + \sin^2\beta - 2\cos\alpha\cos\beta\cos(\alpha+\beta)$ then E is equal to:

(a) 0 (b) 1

(c) 2 (d) $2\cos\alpha\cos\beta$

Ans. (c)

◎ **Solution:**

$$E = 1+\sin^2\alpha+\sin^2\beta-\cos^2(\alpha+\beta)$$
$$-2\cos\alpha\cos\beta\cos(\alpha+\beta)$$
$$= 1+\sin^2\alpha+\sin^2\beta-\cos(\alpha+\beta)[\cos(\alpha+\beta)-2\cos\alpha\cos\beta]$$
$$= 1+\sin^2\alpha+\sin^2\beta+\cos(\alpha+\beta)\cos(\alpha-\beta)$$
$$= 1+\sin^2\alpha+\sin^2\beta+\cos^2\alpha-\sin^2\beta$$
$$= 2$$

◉ **Example 7:** If α, β, γ are three distinct real numbers such that $0 < \alpha, \beta, \gamma < \pi/2$, then $\tan(\alpha-\beta)+\tan(\beta-\gamma)+\tan(\gamma-\alpha)$ is equal to:

(a) 0 (b) – 1

(c) 1 (d) none of these

Ans. (d)

◎ **Solution:** Let $A = \alpha-\beta$, $B = \beta-\gamma$, $C = \gamma-\alpha$, then $A+B+C = 0$.

We have $\tan(A+B) = \tan(-C)$

$$\Rightarrow \quad \frac{\tan A+\tan B}{1-\tan A\tan B} = -\tan C$$

$$\Rightarrow \quad \tan A+\tan B+\tan C = \tan A\tan B\tan C$$

◉ **Example 8:** If $\tan A - \tan B = \dfrac{1}{2}$ and $\cot A - \cot B = \dfrac{1}{3}$, then $\cot(A-B)$ is equal to:

(a) 1 (b) –1

(c) 5 (d) –5

Ans. (b)

◎ **Solution:** $\dfrac{1}{2} = \dfrac{1}{\cot A}-\dfrac{1}{\cot B} = -\dfrac{\cot A-\cot B}{\cot A\cot B}$

$\Rightarrow \cot A\cot B = -2/3.$

Now,

$$\cot(A-B) = \frac{\cot B-\cot A}{1+\cot A\cot B} = \frac{-1/3}{1-2/3} = -1$$

◉ **Example 9:** If $\tan\beta = \dfrac{3\sin\alpha\cos\alpha}{1-3\sin^2\alpha}$, then $\dfrac{\tan(\alpha-\beta)}{\tan\alpha}$ is equal to:

(a) 3 (b) 2
(c) – 2 (d) none of these

Ans. (c)

◎ **Solution:** $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{3\sin\alpha\cos\alpha}{1-3\sin^2\alpha}}{1+\dfrac{\sin\alpha}{\cos\alpha}\cdot\dfrac{3\sin\alpha\cos\alpha}{1-3\sin^2\alpha}}$$

$$= -2\tan\alpha$$

◉ **Example 10:** Let $f(\theta) = \tan\left(\dfrac{\pi}{3}+\theta\right)\tan\left(\dfrac{\pi}{3}-\theta\right)$, $-\dfrac{\pi}{3} < \theta < \dfrac{\pi}{3}$, then $f\left(\dfrac{\pi}{12}\right)$ is equal to:

(a) $2+\sqrt{3}$ (b) $2-\sqrt{3}$
(c) $\sqrt{3}+1$ (d) $\sqrt{3}-1$

Ans. (a)

◎ **Solution:** $f(\theta) = \dfrac{\sin\left(\dfrac{\pi}{3}+\theta\right)\sin\left(\dfrac{\pi}{3}-\theta\right)}{\cos\left(\dfrac{\pi}{3}+\theta\right)\cos\left(\dfrac{\pi}{3}-\theta\right)}$

$$= \frac{\sin^2\left(\dfrac{\pi}{3}\right)-\sin^2\theta}{\cos^2\left(\dfrac{\pi}{3}\right)-\sin^2\theta}$$

$$= \frac{3-4\sin^2\theta}{1-4\sin^2\theta} = \frac{2\cos(2\theta)+1}{2\cos(2\theta)-1}$$

$$\therefore \quad f\left(\frac{\pi}{12}\right) = \frac{2\cos\left(\dfrac{\pi}{6}\right)+1}{2\cos\left(\dfrac{\pi}{6}\right)-1} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= 2+\sqrt{3}$$

◉ **Example 11:** If $\operatorname{cosec}\theta = \dfrac{p+q}{p-q}$ where $p > q > 0$, then $\left|\cot\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)\right|$ is equal to:

(a) $\sqrt{\dfrac{p}{q}}$ (b) $\sqrt{\dfrac{q}{p}}$
(c) $\sqrt{pq}$ (d) pq

Ans. (a)

◎ **Solution:** $\sin\theta = \dfrac{p-q}{p+q} \Rightarrow \cos\left(\dfrac{\pi}{2}+\theta\right) = \dfrac{p-q}{p+q}$

$$\Rightarrow \quad \frac{1-\tan^2\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)}{1+\tan^2\left(\dfrac{\theta}{4}+\dfrac{\theta}{2}\right)} = \frac{1-q/p}{1+q/p}$$

$$\Rightarrow \quad \tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{q}{p}$$

$$\Rightarrow \quad \left|\cot\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right| = \sqrt{\frac{p}{q}}$$

◉ **Example 12:** If $2\cos\theta + \sin\theta = 1$, $(\theta \neq (4k+1)\pi/2$, $k \in \mathbf{I})$ then $7\cos\theta + 6\sin\theta$ is equal to:

(a) 1/2 (b) 2
(c) 11/2 (d) 46/5

Ans. (b)

◎ **Solution:** $2\cos\theta + \sin\theta = 1 \Rightarrow 4\cos^2\theta = (1-\sin\theta)^2$

$\Rightarrow \quad 4(1-\sin^2\theta) = (1-\sin\theta)^2$

$\Rightarrow \quad 4(1+\sin\theta) = 1-\sin\theta$

$\Rightarrow \quad \sin\theta = -3/5$

As $2\cos\theta + \sin\theta = 1$, we get $\cos\theta = 4/5$.

Thus, $7\cos\theta + 6\sin\theta = 7\left(\dfrac{4}{5}\right)+6\left(\dfrac{-3}{5}\right) = 2$

◉ **Example 13:** The number of solutions of the equation $\tan x + \sec x = 2\cos x$, $x \in [0, 2\pi]$ is

(a) 1 (b) 2
(c) 3 (d) 0

Ans.(b)

◎ **Solution:** $\tan x + \sec x = 2\cos x$

$\Rightarrow \quad 1+\sin x = 2\cos^2 x = 2(1-\sin^2 x)$

$\Rightarrow \quad (1+\sin x)(1-2+2\sin x) = 0$

$\Rightarrow \quad \sin x = -1$ or $\sin x = 1/2$

But $\sin x = -1 \Rightarrow \cos x = 0$. This is not possible.

Thus, $\sin x = 1/2 \Rightarrow x = \pi/6, 5\pi/6, \in [0, 2\pi]$

◉ **Example 14:** Let $A = \sin^4 x + \cos^4 x$, $x \in \mathbf{R}$, then

(a) $\dfrac{1}{2} \le A \le 1$ (b) $\dfrac{1}{\sqrt{2}} \le A \le 1$
(c) $\dfrac{1}{4} \le A \le 1$ (d) $\dfrac{1}{2\sqrt{2}} \le A \le 1$

Ans. (a)

◎ **Solution:** $A = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$$= 1-\frac{1}{2}\sin^2 2x$$

Now, $\quad 0 \le \sin^2 2x \le 1$

$$\Rightarrow \quad \frac{1}{2} = 1-\frac{1}{2} \le 1-\frac{1}{2}\sin^2 2x = A \le 1$$

◉ **Example 15:** If $0 < x < \pi$, and $\sin x + \cos x = 1/3$, then a possible value of $\tan x$ is

(a) $\frac{1}{8}\left(-9+\sqrt{17}\right)$ (b) $\frac{1}{8}\left(9-\sqrt{17}\right)$

(c) $-\frac{1}{5}\left(9+\sqrt{5}\right)$ (d) $\frac{1}{5}\left(9-\sqrt{5}\right)$

Ans. (a)

◎ **Solution:** $3(\tan x + 1) = \sec x$

$\Rightarrow \quad 9\,(1 + \tan^2 x + 2\tan x) = 1 + \tan^2 x$

$\Rightarrow \quad 4\tan^2 x + 9\tan x + 4 = 0$

$$\Rightarrow \quad \tan x = \frac{-9 \pm \sqrt{81-64}}{8} = \frac{-9 \pm \sqrt{17}}{8}$$

◉ **Example 16:** Let $f(\theta) = (1 + \cos\theta)\,(1 + \cos 3\theta)\,(1 + \cos 5\theta)\,(1 + \cos 7\theta)$ then $f\left(-\frac{\pi}{8}\right)$ is equal to:

(a) – 1/8 (b) 1/4

(c) – 1/4 (d) 1/8

Ans. (d)

◎ **Solution:** As $f(-\theta) = f(\theta)$, we evaluate $f(\pi/8)$.

For $\theta = \pi/8$, $3\theta = \pi/2 - \pi/8$, $5\theta = \pi/2 + \pi/8$ and $7\theta = \pi - \pi/8$, so that

$$f\left(\frac{\pi}{8}\right) = \left(1+\cos\frac{\pi}{8}\right)\left(1+\sin\frac{\pi}{8}\right)\left(1-\sin\frac{\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$$

$$= \cos^2\left(\frac{\pi}{8}\right)\sin^2\left(\frac{\pi}{8}\right)$$

$$= \frac{1}{4}\left[2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)\right]^2$$

$$= \frac{1}{4}\sin^2\left(\frac{\pi}{4}\right) = \frac{1}{8}$$

◉ **Example 17:** If $\theta = \pi/8$, and $f(\theta) = \cos^6\theta + \cos^6 3\theta + \cos^6 5\theta + \cos^6 7\theta$ then $f(\theta)$ is equal to:

(a) 3/4 (b) 1

(c) 5/4 (d) 3/2

Ans. (c)

◎ **Solution:** As in example 16, using

$3\theta = \pi/2 - \pi/8$, $5\theta = \pi/2 + \pi/8$, $7\theta = \pi - \pi/8$, we get

$$f(\theta) = 2(\cos^6\theta + \sin^6\theta)$$

$$= 2[(\cos^2\theta + \sin^2\theta)^3 - 3\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)]$$

$$= 2\left[1 - \frac{3}{4}(2\cos\theta\sin\theta)^2\right] = 2\left[1-\frac{3}{4}\left(\frac{1}{\sqrt{2}}\right)^2\right]$$

$$= 5/4$$

◉ **Example 18:** If α, β, γ are three real numbers, then

$\cos^2(\beta-\gamma) + \cos^2(\gamma-\alpha) + \cos^2(\alpha-\beta)$
$- 2\cos(\beta-\gamma)\cos(\gamma-\alpha)\cos(\alpha-\beta)$ is equal to

(a) 0 (b) 1

(c) –1 (d) none of these

Ans. (b)

◎ **Solution:** Put $A = \beta - \gamma$, $B = \gamma - \alpha$, $C = \alpha - \beta$, so that $A + B + C = 0$. Now

$$\cos^2 A + \cos^2 B + \cos^2 C - 2\cos A\cos B\cos C$$

$$= 1 + \cos^2 A - \sin^2 B + \cos C\,[\cos C - 2\cos A\cos B]$$

$$= 1 + \cos(A+B)\cos(A-B) + \cos C\,[\cos(-A-B) - 2\cos A\cos B]$$

$$= 1 + \cos(-C)\cos(A-B) - \cos C\,[\sin A\sin B + \cos A\cos B]$$

$$= 1 + \cos C\cos(A-B) - \cos C\cos(A-B)$$

$$= 1$$

◉ **Example 19:** Number of values of $x \in [0, 2\pi]$ and satisfying the equation

$\cos x\cos 2x\cos 3x = 1/4$ is:

(a) 6 (b) 8

(c) 20 (d) 24

Ans. (b)

◎ **Solution:** $2\,(2\cos 3x\cos x)\cos 2x = 1$

$\Rightarrow \quad 2\,(\cos 4x + \cos 2x)\cos 2x = 1$

$\Rightarrow \quad 2\cos 4x\cos 2x = 1 - 2\cos^2 2x = -\cos 4x$

$\Rightarrow \quad \cos 4x\,(2\cos 2x + 1) = 0$

$\Rightarrow \quad \cos 4x = 0$ or $\cos 2x = -1/2$

Now $\cos 4x = 0 \Rightarrow 4x = (2n+1)\pi/2$, $n \in \mathbf{I}$

$\Rightarrow \quad x = (2n+1)\,\pi/8$, $n \in \mathbf{I}$.

and $0 \le (2n+1)\pi/4 \le 2\pi \Rightarrow 0 \le 2n+1 \le 8$

$\Rightarrow \quad n = 0, 1, 2, 3$.

Also, $\cos(2x) = -1/2 = \cos(2\pi/3)$

$\Rightarrow \quad 2x = 2n\pi \pm 2\pi/3$, $n \in \mathbf{I}$

$\Rightarrow \quad x = n\pi \pm \pi/3$, $n \in \mathbf{I}$.

As $0 \le x \le 2\pi$, $x = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$.

Thus, there are 8 values of x.

◉ **Example 20:** Number of values of $x \in [0, 4\pi]$ and satisfying $\sqrt{2}\sec x + \tan x = 1$ is

(a) 0 (b) 1

(c) 2 (d) 4

Ans. (c)

◎ **Solution:** We can write the given equation as

$$\sqrt{2} = \cos x - \sin x$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x = 1$$

$$\Rightarrow \quad \cos\left(x + \frac{\pi}{4}\right) = 1$$

$$\Rightarrow \quad x + \frac{\pi}{4} = 2n\pi,\ n \in \mathbf{I}.$$

$$\Rightarrow \quad x = \frac{7\pi}{4}, \frac{15\pi}{4}$$

LEVEL 1

Straight Objective Type Questions

Example 21: $\cos(540° - \theta) - \sin(630° - \theta)$ is equal to

(a) 0　(b) $2\cos\theta$

(c) $2\sin\theta$　(d) $\sin\theta - \cos\theta$

Ans. (a)

Solution: $\cos(540° - \theta) = \cos(6(\pi/2) - \theta) = -\cos\theta$

$\sin(630° - \theta) = \sin(7(\pi/2) - \theta) = -\cos\theta$

Example 22: $2\sec^2\alpha - \sec^4\alpha - 2\operatorname{cosec}^2\alpha + \operatorname{cosec}^4\alpha = 15/4$ if $\tan\alpha$ is equal to

(a) $1/\sqrt{2}$　(b) $1/2$

(c) $1/2\sqrt{2}$　(d) $1/4$

Ans. (a)

Solution: L.H.S. $= 2(1 + \tan^2\alpha - 1 - \cot^2\alpha) - [(1 + \tan^2\alpha)^2 - (1 + \cot^2\alpha)^2]$

$= \cot^4\alpha - \tan^4\alpha = 15/4$

$\Rightarrow \quad \tan^4\alpha = 1/4 \Rightarrow \tan\alpha = \pm 1/\sqrt{2}$.

Example 23: If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15°, then the values of $2 + q - p$ is

(a) 1　(b) 2

(c) 3　(d) 0

Ans. (c)

Solution: $\tan 30° + \tan 15° = -p$

and $\quad \tan 30° \tan 15° = q$

Now $\quad 1 = \tan 45° = \dfrac{\tan 30° + \tan 15°}{1 - \tan 30° \tan 15°}$

$\Rightarrow \quad 1 = \dfrac{-p}{1-q} \Rightarrow 1 - q + p = 0$

$\Rightarrow \quad 2 + q - p = 2 + 1 = 3.$

Example 24: If $\tan 25° = x$, then $\dfrac{\tan 155° - \tan 115°}{1 + \tan 155° \tan 115°}$ is equal to

(a) $\dfrac{1-x^2}{2x}$　(b) $\dfrac{1+x^2}{2x}$

(c) $\dfrac{1+x^2}{1-x^2}$　(d) $\dfrac{1-x^2}{1+x^2}$

Ans. (a)

Solution: $\dfrac{\tan 155° - \tan 115°}{1 + \tan 155° \tan 115°}$

$$= \frac{\tan(180° - 25°) - \tan(90° + 25°)}{1 + \tan(180° - 25°)\tan(90° + 25°)}$$

$$= \frac{-\tan 25° + \cot 25°}{1 + \tan 25° \cot 25°} = \frac{1}{2}\left(-x + \frac{1}{x}\right) = \frac{1-x^2}{2x}.$$

Example 25: The value of the determinant

$$\begin{vmatrix} \sin^2 13° & \sin^2 77° & \tan 135° \\ \sin^2 77° & \tan 135° & \sin^2 13° \\ \tan 135° & \sin^2 13° & \sin^2 77° \end{vmatrix} \text{ is equal to}$$

(a) – 1　(b) 0

(c) 1　(d) 2

Ans. (b)

Solution: The given determinant is equal to

$$\begin{vmatrix} \sin^2 13° & \cos^2 13° & -1 \\ \cos^2 13° & -1 & \sin^2 13° \\ -1 & \sin^2 13° & \cos^2 13° \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \cos^2 13° & -1 \\ 0 & -1 & \sin^2 13° \\ 0 & \sin^2 13° & \cos^2 13° \end{vmatrix} = 0$$

(Applying $C_1 \to C_1 + C_2 + C_3$)

Example 26: If $A = 130°$ and $x = \sin A + \cos A$, then

(a) $x > 0$　(b) $x < 0$

(c) $x = 0$　(d) $x \geq 0$

Ans. (a)

Solution: $x = \sin 130° + \cos 130° = \sin 50° - \sin 40° > 0$

($\because$ $\sin x$ is increasing for $0 < x < \pi/2$)

Example 27: If $\tan^2 36° + k(\sin 18° + \cos 36°) = 5$, then the value of k is

(a) 2　(b) $2\sqrt{5}$

(c) 4　(d) $4\sqrt{5}$

Ans. (c)

Solution: From the tables, we have

$$\text{L.H.S.} = 5 - 2\sqrt{5} + k\left(\frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}+1}{4}\right)$$

$$= 5 - 2\sqrt{5} + \frac{\sqrt{5}}{2}k = 5 \text{ (given)} \quad \Rightarrow \quad k = 4$$

Example 28: If $\frac{\sin x}{a} = \frac{\cos x}{b} = \frac{\tan x}{c} = k$, then $bc + \frac{1}{ck} + \frac{ak}{1+bk}$ is equal to

(a) $k\left(a+\frac{1}{a}\right)$ (b) $\frac{1}{k}\left(a+\frac{1}{a}\right)$

(c) $\frac{1}{k^2}$ (d) $\frac{a}{k}$

Ans. (b)

Solution: The given expression is equal to

$$\frac{\cos x \cdot \tan x}{k^2} + \frac{1}{\tan x} + \frac{\sin x}{1+\cos x}$$

$$= \frac{\sin x}{k^2} + \frac{\cos x(1+\cos x)+\sin^2 x}{\sin x(1+\cos x)}$$

$$= \frac{a}{k} + \frac{1}{\sin x} = \frac{a}{k} + \frac{1}{ak} = \frac{1}{k}\left(a+\frac{1}{a}\right).$$

Example 29: $\sin^2 \alpha + \cos^2 (\alpha + \beta) + 2 \sin \alpha \sin \beta \cos (\alpha + \beta)$ is independent of

(a) α (b) β

(c) both α and β (d) none

Ans. (a)

Solution: The given expression is equal to

$$\sin^2 \alpha + \cos (\alpha + \beta) [\cos (\alpha + \beta) + 2 \sin \alpha \sin \beta]$$

$$= \sin^2 \alpha + \cos (\alpha + \beta) [\cos \alpha \cos \beta + \sin \alpha \sin \beta]$$

$$= \sin^2 \alpha + \cos (\alpha + \beta) \cos (\alpha - \beta)$$

$$= \sin^2 \alpha + \cos^2 \alpha - \sin^2 \beta = 1 - \sin^2 \beta = \cos^2 \beta$$

which is independent of α only.

Example 30: tan 203° + tan 22° + tan 203° tan 22° =

(a) – 1 (b) 0

(c) 1 (d) 2

Ans. (c)

Solution: $\tan (203° + 22°) = \frac{\tan 203° + \tan 22°}{1 - \tan 203° \tan 22°}$

$\Rightarrow$ $1 = \tan (180° + 45°) = \frac{\tan 203° + \tan 22°}{1 - \tan 203° \tan 22°}$

$\Rightarrow$ tan 203° + tan 22° + tan 203° tan 22° = 1.

Example 31: If sin 32° = k and $\cos x = 1 - 2k^2$; α, β are the values of x between 0° and 360° with $\alpha < \beta$, then

(a) $\alpha + \beta = 180°$ (b) $\beta - \alpha = 200°$

(c) $\beta = 4\alpha + 40°$ (d) $\beta = 5\alpha - 20°$

Ans. (c)

Solution: $\cos x = 1 - 2k^2 = 1 - 2 \sin^2 32° = \cos 64°$

$\Rightarrow$ $x = 64°$ or $296°$

$\therefore$ $\alpha = 64°$ and $\beta = 296°$

which satisfy (c).

Example 32: The minimum value of $27 \tan^2 \theta + 3 \cot^2 \theta$ is

(a) 9 (b) 18

(c) 27 (d) 30

Ans. (b)

Solution: A.M. ≥ G.M.

$$\Rightarrow \quad \frac{27\tan^2 \theta + 3\cot^2 \theta}{2} \geq \sqrt{27\tan^2 \theta \cdot 3\cot^2 \theta}$$

$$\Rightarrow \quad 27 \tan^2 \theta + 3 \cot^2 \theta \geq 18.$$

Example 33: If $3 \sin \beta = \sin (2\alpha + \beta)$, then $\tan (\alpha + \beta) - 2 \tan \alpha$ is

(a) independent of α

(b) independent of β

(c) independent of both α and β

(d) independent of none of them

Ans. (c)

Solution: $\sin (2\alpha + \beta) = 3 \sin \beta$

$$\Rightarrow \quad \frac{\sin(2\alpha+\beta)+\sin\beta}{\sin(2\alpha+\beta)-\sin\beta} = \frac{3+1}{3-1}$$

$$\Rightarrow \quad \frac{2\sin(\alpha+\beta)\cos\alpha}{2\cos(\alpha+\beta)\sin\alpha} = 2 \Rightarrow \tan(\alpha+\beta) - 2\tan \alpha = 0$$

Example 34: If $A = \sin^2 \theta + \cos^4 \theta$, then for all values of θ

(a) $1 \leq A \leq 2$ (b) $3/4 \leq A \leq 1$

(c) $13/16 \leq A \leq 1$ (d) $3/4 \leq A \leq 13/16$

Ans. (b)

Solution: $A = \sin^2 \theta + (1 - \sin^2 \theta)^2$

$= 1 + \sin^2 \theta \ (\sin^2 \theta - 1)$

$= 1 - \sin^2 \theta \cos^2 \theta \leq 1$

Also $A = 1 - (1/4) \sin^2 2\theta \geq 1 - (1/4) = (3/4)$. Hence $3/4 \leq A \leq 1$

Example 35: If $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$ where $a, b > 0$. Then the difference between the maximum and minimum values of u^2 is given by

(a) $(a + b)^2$ (b) $2\sqrt{a^2 + b^2}$

(c) $2(a^2 + b^2)$ (d) $(a - b)^2$

Ans. (d)

Solution: We have

$$u = \sqrt{\frac{a^2}{2}(1+\cos 2\theta) + \frac{b^2}{2}(1-\cos 2\theta)}$$

$$+ \sqrt{\frac{a^2}{2}(1-\cos 2\theta) + \frac{b^2}{2}(1+\cos 2\theta)}$$

$$= \sqrt{\frac{a^2+b^2}{2} + \frac{a^2-b^2}{2}\cos 2\theta}\ .$$

$$+\sqrt{\frac{a^2+b^2}{2}-\frac{a^2-b^2}{2}\cos 2\theta}$$

Squaring we get

$$u^2 = a^2+b^2+2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2-\left(\frac{a^2-b^2}{2}\right)^2\cos^2 2\theta}$$

Thus max $(u^2) = a^2+b^2+a^2+b^2 = 2(a^2+b^2)$

and min $(u^2) = a^2+b^2+2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2-\left(\frac{a^2-b^2}{2}\right)^2}$

$$= a^2+b^2+2ab = (a+b)^2$$

So the required difference $= 2(a^2+b^2)-(a+b)^2 = (a-b)^2$.

Example 36: $\begin{vmatrix} \sin^2 x & \cos^2 x & 1 \\ \cos^2 x & \sin^2 x & 1 \\ -10 & 12 & 2 \end{vmatrix}$ is equal to

(a) 0
(b) $12\cos^2 x - 10\sin^2 x$
(c) $12\sin^2 x - 10\cos^2 x - 2$
(d) $10\sin^2 x$

Ans. (a)

Solution: Apply $C_1 \to C_1 + C_2 - C_3$, then the given determinant is equal to

$$\begin{vmatrix} 0 & \cos^2 x & 1 \\ 0 & \sin^2 x & 1 \\ 0 & 12 & 2 \end{vmatrix} = 0.$$

Example 37: If $\frac{\cos A}{3} = \frac{\cos B}{4} = \frac{1}{5}, -\pi/2 < A, B < 0,$ then value of $2\sin A + 4\sin B$ is equal to

(a) 4 (b) 2
(c) – 4 (d) 0

Ans. (c)

Solution: $\cos A = 3/5 \;\Rightarrow\; \sin A = -(4/5)$

$\cos B = (4/5) \;\Rightarrow\; \sin B = -3/5.$

So that $2\sin A + 4\sin B$

$= -(8/5)-(12/5) = -4.$

Example 38: If $\sin A = 3\sin(A+2B)$, angle B is acute and A is obtuse; then

(a) $\tan B = 1/\sqrt{2}$ (b) $\tan B > 1/\sqrt{2}$
(c) $\tan B < 1/\sqrt{2}$ (d) $0 < \tan B < 1/\sqrt{2}$

Ans. (d)

Solution: $\frac{\sin(A+2B)}{\sin A} = \frac{1}{3},$

By componendo and dividendo, $\tan(A+B) = -2\tan B$

$\Rightarrow \quad \frac{\tan A + \tan B}{1-\tan A\tan B} = -2\tan B$

$\Rightarrow \quad \tan A = \frac{3\tan B}{2\tan^2 B - 1} < 0$ (as A is obtuse)

$\Rightarrow \quad 2\tan^2 B - 1 < 0$ (as $\tan B > 0$)

$\Rightarrow \quad 0 < \tan B < 1/\sqrt{2}$

Example 39: The value of $\cos^2\left(\frac{3\pi}{5}\right)+\cos^2\left(\frac{4\pi}{5}\right)$ is equal to

(a) 3/4 (b) 5/4
(c) 5/2 (d) 4/5

Ans. (a)

Solution: $\cos^2\left(\frac{3\pi}{5}\right)+\cos^2\left(\frac{4\pi}{5}\right)$

$= \cos^2(108°)+\cos^2(144°)$

$= (\cos(90°+18°))^2+(\cos(180°-36°))^2$

$= \sin^2 18° + \cos^2 36°$

$= \left(\frac{\sqrt{5}-1}{4}\right)^2+\left(\frac{\sqrt{5}+1}{4}\right)^2 = \frac{2\times 6}{16} = \frac{3}{4}.$

Example 40: The value of

$\cos 12° + \cos 84° + \cos 156° + \cos 132°$ is

(a) 1/8 (b) – 1/2
(c) 1 (d) 1/2

Ans. (b)

Solution: $(\cos 12° + \cos 132°) + (\cos 84° + \cos 156°)$

$= 2\cos\frac{12°+132°}{2}\cos\frac{12°-132°}{2} +$

$2\cos\frac{84°+156°}{2}\cos\frac{84°-156°}{2}.$

$= 2\cos 72°\cos 60° + 2\cos 120°\cos 36°$

$= 2\times\frac{\sqrt{5}-1}{4}\times\frac{1}{2}+2\times\left(-\frac{1}{2}\right)\times\frac{\sqrt{5}+1}{4} = -\frac{1}{2}.$

Example 41: Let $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$, then Δ lies in the interval.

(a) [2, 4] (b) [3, 4]
(c) [1, 4] (d) none of these

Ans. (a)

Solution: $\Delta = \begin{vmatrix} 0 & 0 & 2 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$ $[R_1 \to R_1 + R_3]$

$= 2(1+\sin^2\theta)$

$\Rightarrow \quad 2 \le \Delta \le 4$ as $0 \le \sin^2\theta \le 1$

Example 42: If $\sin(120° - \alpha) = \sin(120° - \beta)$ and $0 < \alpha, \beta < \pi$, then all values of α, β are given by

(a) $\alpha+\beta = \pi/3$ (b) $\alpha = \beta$ or $\alpha+\beta = \pi/3$
(c) $\alpha = \beta$ (d) $\alpha+\beta = 0$

Ans. (b)

Solution: $\sin(120° - \alpha) = \sin(120° - \beta)$

$\Rightarrow$ either $120° - \alpha = 120° - \beta \Rightarrow \alpha = \beta$

or $120° - \alpha = 180° - (120° - \beta)$

$\Rightarrow$ $\alpha + \beta = 60° = \pi/3$

Example 43: If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\sin^n\theta + \operatorname{cosec}^n\theta$ is equal to

(a) 2 (b) 2^n
(c) 4^n (d) none of these

Ans. (a)

Solution: We can write $\sin^2\theta + 1 = 2\sin\theta$

$\Rightarrow$ $\sin^2\theta - 2\sin\theta + 1 = 0$ $\Rightarrow$ $(\sin\theta - 1)^2 = 0$

$\Rightarrow$ $\sin\theta = 1$ $\Rightarrow$ $\operatorname{cosec}\theta = 1$

and thus $\sin^n\theta + \operatorname{cosec}^n\theta = 2$.

Example 44: If $a = \cos\phi\cos\psi + \sin\phi\sin\psi\cos\delta$

$b = \cos\phi\sin\psi - \sin\phi\cos\psi\cos\delta$

and $c = \sin\phi\sin\delta$. Then $a^2 + b^2 + c^2 =$

(a) -1 (b) 0
(c) 1 (d) none of these

Ans. (c)

Solution: $a^2 + b^2 + c^2$

$$= \cos^2\phi\cos^2\psi + \sin^2\phi\sin^2\psi\cos^2\delta$$
$$+ \cos^2\phi\sin^2\psi + \sin^2\phi\cos^2\psi\cos^2\delta + \sin^2\phi\sin^2\delta$$
$$= \cos^2\phi + \sin^2\phi\cos^2\delta + \sin^2\phi\sin^2\delta$$
$$= \cos^2\phi + \sin^2\phi = 1.$$

Example 45: In a triangle ABC, BP is drawn perpendicular to BC to meet CA in P, such that $CA = AP$, then $\dfrac{BP}{AB} =$

(a) $2\sin A$ (b) $2\sin B$
(c) $2\sin C$ (d) none of these

Ans. (c)

Solution: We have $\dfrac{BP}{AB} = \dfrac{BP}{AC}$

(AB, AC are the radii of the circle on CP as diametre)

$$= 2\frac{BP}{CP} = 2\sin C.$$

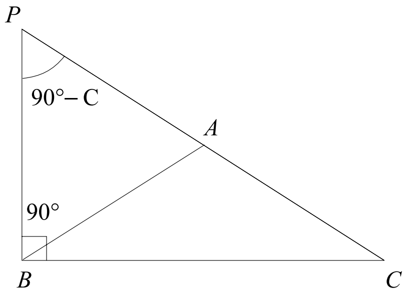


Fig. 25.3

Example 46: If $\cos\theta = \cos\alpha\cos\beta$, then $\tan\dfrac{\theta+\alpha}{2}\tan\dfrac{\theta-\alpha}{2}$ is equal to

(a) $\tan^2(\alpha/2)$ (b) $\tan^2(\beta/2)$
(c) $\tan^2(\theta/2)$ (d) $\cot^2(\beta/2)$

Ans. (b)

Solution: $\tan\dfrac{\theta+\alpha}{2}\tan\dfrac{\theta-\alpha}{2}$

$$= \frac{\tan^2(\theta/2) - \tan^2(\alpha/2)}{1 - \tan^2(\theta/2)\tan^2(\alpha/2)}$$

$$= \frac{\dfrac{1-\cos\theta}{1+\cos\theta} - \dfrac{1-\cos\alpha}{1+\cos\alpha}}{1 - \dfrac{1-\cos\theta}{1+\cos\theta}\cdot\dfrac{1-\cos\alpha}{1+\cos\alpha}}$$

$$= \frac{2(\cos\alpha - \cos\theta)}{2(\cos\alpha + \cos\theta)} = \frac{\cos\alpha(1-\cos\beta)}{\cos\alpha(1+\cos\beta)} = \tan^2\frac{\beta}{2}.$$

Example 47: The equation $a\sin x + b\cos x = c$, where $|c| > \sqrt{a^2+b^2}$ has

(a) a unique solution
(b) infinite number of solutions
(c) no solution
(d) none of these

Ans. (c)

Solution: Let $a = r\cos\alpha$, $b = r\sin\alpha$ so that $r = \sqrt{a^2+b^2}$.

The given equation can be written as

$$r\sin(x+\alpha) = c \Rightarrow \sin(x+\alpha) = \frac{c}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad |\sin(x+\alpha)| = \frac{|c|}{\sqrt{a^2+b^2}} > 1 \text{ as } |c| > \sqrt{a^2+b^2}$$

which is not possible for any value of x.

Example 48: If $\cot\alpha$ equals the integral solution of the inequality $4x^2 - 16x + 15 < 0$ and $\sin\beta$ equals to the slope of the bisector of the first quadrant, then $\sin(\alpha+\beta)\sin(\alpha-\beta)$ is equal to

(a) $-3/5$ (b) $-4/5$
(c) $2/\sqrt{5}$ (d) 3

Ans. (b)

Solution: We have $4x^2 - 16x + 15 < 0$

$\Rightarrow$ $3/2 < x < 5/2$

$\Rightarrow \cot\alpha = 2$, the integral solution of the given inequality and $\sin\beta = \tan 45° = 1$

$\therefore$ $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$

$$= \frac{1}{1+\cot^2\alpha} - 1 = \frac{1}{1+4} - 1 = -\frac{4}{5}.$$

Example 49: The value of $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{7\pi}{7}$ is

(a) 1 (b) – 1

(c) 1/2 (d) – 3/2

Ans. (d)

Solution: $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{7\pi}{7}$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[2\sin\frac{\pi}{7}\cos\frac{2\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{4\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{6\pi}{7}\right] - 1$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left[\sin\frac{3\pi}{7} - \sin\frac{\pi}{7} + \sin\frac{5\pi}{7} - \sin\frac{3\pi}{7} + \sin\frac{7\pi}{7} - \sin\frac{5\pi}{7}\right] - 1$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\left(-\sin\frac{\pi}{7}\right) - 1 = -\frac{1}{2} - 1 = -\frac{3}{2}.$$

Example 50: The greatest value of $f(x) = 2\sin x + \sin 2x$ on $[0, 3\pi/2]$, is given by

(a) 9/2 (b) 5/2

(c) $3\sqrt{3}/2$ (d) 3/2

Ans. (c)

Solution: $f'(x) = 2\cos x + 2\cos 2x$

and $f''(x) = -2\sin x - 4\sin 2x$

For extreme value $f'(x) = 0$

$\Rightarrow$ $\cos x + 2\cos^2 x - 1 = 0$

$\Rightarrow$ $\cos x = -1$ or $1/2$

$\Rightarrow$ $x = \pi$ or $\pi/3$ as $x \in [0, 3\pi/2]$

Now $f(\pi) = 0$ and $f(\pi/3) = \frac{2\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

Also $f(0) = 0$ and $f(3\pi/2) = -2$ so the greatest value of $f(x)$ is $3\sqrt{3}/2$

Example 51: If $x = a\cos^3\theta\sin^2\theta$, $y = a\sin^3\theta\cos^2\theta$ and $\frac{(x^2+y^2)^p}{(xy)^q}$ ($p, q, \in \mathbf{N}$) is independent of θ, then

(a) $4p = 5q$ (b) $4q = 5p$

(c) $p + q = 9$ (d) $pq = 20$

Ans. (a)

Solution: $$\frac{(x^2+y^2)^p}{(xy)^q} = \frac{\left[a\sin^2\theta\cos^2\theta\right]^{2p}}{\left[a^2\sin^5\theta\cos^5\theta\right]^q}$$

$$= \frac{a^{2p}(\sin\theta\cos\theta)^{4p}}{a^{2q}(\sin\theta\cos\theta)^{5q}}$$

which is independent of θ if $4p = 5q$.

Example 52: If $\cos\alpha + \cos\beta = a$, $\sin\alpha + \sin\beta = b$ and $\alpha - \beta = 2\theta$, then $\frac{\cos 3\theta}{\cos\theta} =$

(a) $a^2 + b^2 - 2$ (b) $a^2 + b^2 - 3$

(c) $3 - a^2 - b^2$ (d) $(a^2 + b^2)/4$

Ans. (b)

Solution: From the given relations we have

$$a^2 + b^2 = \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta + \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta$$

$$= 2 + 2\cos(\alpha - \beta)$$

$$= 2 + 2\cos 2\theta = 4\cos^2\theta$$

Now $$\frac{\cos 3\theta}{\cos\theta} = \frac{4\cos^3\theta - 3\cos\theta}{\cos\theta}$$

$$= 4\cos^2\theta - 3 = a^2 + b^2 - 3.$$

Example 53: If $\cos A = 3/4$, then the value of $16\cos^2(A/2) - 32\sin(A/2)\sin(5A/2)$ is

(a) – 4 (b) – 3

(c) 3 (d) 4

Ans. (c)

Solution: The given expression is equal to

$$8(1 + \cos A) - 16(\cos 2A - \cos 3A)$$

$$= 8(1 + \cos A) - 16\left[2\cos^2 A - 1 - \cos A(4\cos^2 A - 3)\right]$$

$$= 8\left(1 + \frac{3}{4}\right) - 16\left[2 \times \frac{9}{16} - 1 - \frac{3}{4}\left(4 \times \frac{9}{16} - 3\right)\right]$$

$$= 14 - (18 - 16 - 27 + 36) = 3.$$

Example 54: If $D = \begin{vmatrix} 1 & \cos\theta & 1 \\ -\sin\theta & 1 & -\cos\theta \\ -1 & \sin\theta & 1 \end{vmatrix}$ then D lies in the interval

(a) [0, 4] (b) [2, 4]

(c) $[2-\sqrt{2}, 2+\sqrt{2}]$ (d) [–2, 2]

Ans. (c)

Solution: Expanding D, we get

$$D = 1 + \sin\theta\cos\theta - \cos\theta(-\sin\theta - \cos\theta) + (-\sin^2\theta + 1)$$

$$= 2 + \sin 2\theta + \cos 2\theta = 2 + \sqrt{2}\cos(2\theta - \pi/4)$$

As $-1 \le \cos(2\theta - \pi/4) \le 1$, $2 - \sqrt{2} \le D \le 2 + \sqrt{2}$

Example 55: A value of θ lying between $\theta = 0$ and $\theta = \pi/2$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0 \text{ is}$$

(a) $3\pi/24$ (b) $5\pi/24$

(c) $11\pi/24$ (d) $\pi/24$

Ans. (c)

Solution: Applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$ to the given determinant we get

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

$\Rightarrow$ $1 + 4\sin 4\theta + \cos^2\theta + \sin^2\theta = 0$

$\Rightarrow$ $\sin 4\theta = -1/2 \Rightarrow 4\theta = \pi + \pi/6$ or $2\pi - \pi/6$ $[\because 0 < 4\theta < 2\pi]$

$\Rightarrow$ $\theta = 7\pi/24$ or $11\pi/24$.

Example 56: Let $\theta \in (\pi/4, \pi/2)$, which of the following statements is true?

(a) $(\cos\theta)^{\cos\theta} < (\sin\theta)^{\cos\theta} < (\cos\theta)^{\sin\theta}$

(b) $(\cos\theta)^{\sin\theta} < (\cos\theta)^{\cos\theta} < (\sin\theta)^{\cos\theta}$

(c) $(\sin\theta)^{\cos\theta} < (\cos\theta)^{\cos\theta} < (\cos\theta)^{\sin\theta}$

(d) $(\cos\theta)^{\cos\theta} < (\cos\theta)^{\sin\theta} < (\sin\theta)^{\cos\theta}$

Ans. (b)

Solution: For $\theta \in (\pi/4, \pi/2)$, $0 < \cos\theta < 1/\sqrt{2} < \sin\theta < 1$

$\Rightarrow$ $(\cos\theta)^{\cos\theta} < (\sin\theta)^{\cos\theta}$ and $(\cos\theta)^{\cos\theta} > (\cos\theta)^{\sin\theta}$

(as $0 < \cos\theta < 1$ and $\cos\theta < \sin\theta$)

Showing that (b) is correct.

Example 57: If $a\cos A - b\sin A = c$, then $a\sin A + b\cos A$ is equal to

(a) $\pm\sqrt{a^2+b^2-c^2}$ (b) $\pm\sqrt{b^2+c^2-a^2}$

(c) $\pm\sqrt{c^2+a^2-b^2}$ (d) $\pm\sqrt{a^2+b^2+c^2}$

Ans. (a)

Solution: Put $a\sin A + b\cos A = x$, then

$$c^2 + x^2 = (a\cos A - b\sin A)^2 + (a\sin A + b\cos A)^2$$
$$= a^2 + b^2$$

$\Rightarrow$ $x = \pm\sqrt{a^2+b^2-c^2}$

Example 58: The general solution of the trigonometrical equation $\sin x + \cos x = 1$ is

(a) $x = 2n\pi, n \in \mathbf{I}$

(b) $x = 2n\pi + \pi/2, n \in \mathbf{I}$

(c) $x = n\pi + (-1)^n \pi/4 - \pi/4, n \in \mathbf{I}$

(d) none of these

Ans. (c)

Solution: $\sin x + \cos x = 1 \Rightarrow \sin(x + \pi/4) = 1/\sqrt{2}$ $= \sin\pi/4$

$\Rightarrow$ $x + \pi/4 = n\pi + (-1)^n \pi/4$

$\Rightarrow$ $x = n\pi + (-1)^n \pi/4 - \pi/4$

which includes $x = 2n\pi$ and $x = 2n\pi + \pi/2$.

Example 59: The values of x between 0 and 2π which satisfy the equation $\sin x\sqrt{8\cos^2 x} = 1$ are in A.P. The common difference of the A.P. is

(a) $\pi/8$ (b) $\pi/4$

(c) $3\pi/8$ (d) $5\pi/8$

Ans. (b)

Solution: From the given equation we have $2\sin x\,|\cos x| = 1/\sqrt{2} \Rightarrow \sin 2x = 1/\sqrt{2}$ if $\cos x > 0$ and $\sin 2x = -1/2$ if $\cos x < 0$

$\therefore$ when $\cos x > 0$, $\sin 2x = 1/\sqrt{2} \Rightarrow x = \pi/8, 3\pi/8$

when $\cos x < 0$, $\sin 2x = -1/\sqrt{2} \Rightarrow x = 5\pi/8, 7\pi/8$. So the required values of x are

$$\pi/8, 3\pi/8, 5\pi/8, 7\pi/8$$

which form an A.P. with common difference $\pi/4$.

Example 60: $15[\tan 2\theta + \sin 2\theta] + 8 = 0$ if

(a) $\tan\theta = 1/2$ (b) $\sin\theta = 1/4$

(c) $\tan\theta = 2$ (d) $\cos\theta = 1/5$

Ans. (c)

Solution: We have $\tan 2\theta + \sin 2\theta = -8/15$

$$\Rightarrow \quad \frac{2\tan\theta}{1-\tan^2\theta} + \frac{2\tan\theta}{1+\tan^2\theta} = \frac{-8}{15}$$

$\Rightarrow$ $15 \times 4\tan\theta + 8(1 - \tan^4\theta) = 0$

$\Rightarrow$ $8\tan^4\theta + 60\tan\theta - 8 = 0$

which is satisfied if $\tan\theta = 2$

Example 61: The numbers of solutions of the pair of equations

$$2\sin^2\theta - \cos 2\theta = 0$$
$$2\cos^2\theta - 3\sin\theta = 0$$

in the interval $[0, 2\pi]$ is

(a) zero (b) one

(c) two (d) four

Ans. (c)

Solution: $2\sin^2\theta - \cos 2\theta = 0$

$\Rightarrow$ $1 - \cos 2\theta - \cos 2\theta = 0 \Rightarrow \cos 2\theta = 1/2$

$\Rightarrow \quad 2\cos^2\theta - 1 = 1/2 \Rightarrow 2\cos^2\theta = 3/2$

So that from $2\cos^2\theta - 3\sin\theta = 0$, we have

$\sin\theta = 1/2 \Rightarrow \theta = \pi/6, 5\pi/6$ as $\theta \in [0, 2\pi]$.

Example 62: The solution set of the equation

$$\tan(\pi\tan x) = \cot(\pi\cot x) \text{ is}$$

(a) $\{0\}$ (b) $\{\pi/4\}$

(c) ϕ (d) none of these

Ans. (c)

Solution: $\tan(\pi\tan x) = \tan(\pi/2 - \pi\cot x)$

$\Rightarrow \quad \pi\tan x = \pi/2 - \pi\cot x \Rightarrow \tan x + \cot x = 1/2$

$\Rightarrow \quad 2\tan^2 x - \tan x + 2 = 0$

$$\Rightarrow \quad \tan x = \frac{1 \pm \sqrt{1-16}}{4}$$

which does not give real values of tan x.

Example 63: The least positive root of the function $\sin x - \pi/2 + 1 = 0$ lies in the interval

(a) $[0, \pi/2]$ (b) $[\pi/2, \pi]$

(c) $[\pi/2, 3\pi/2]$ (d) none of these

Ans. (a)

Solution: $\sin x = \dfrac{\pi}{2} - 1 = \dfrac{22}{14} - 1 = \dfrac{8}{14}$

$= \dfrac{4}{7}$ so the least positive root lies in $[0, \pi/2]$

Example 64: The number of solutions of the equation

$$\cos(\pi\sqrt{x-4})\cos(\pi\sqrt{x}) = 1 \text{ is}$$

(a) 1 (b) 2

(c) more than two (d) 0

Ans. (a)

Solution: Given equation is possible if $\cos(\pi\sqrt{x-4}) = 1$ and $\cos(\pi\sqrt{x}) = 1$.

Since $\quad x - 4 \geq 0 \Rightarrow x \geq 4$ and $x \geq 0$

So $\quad x = 4$ is the only solution.

Example 65: The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is

(a) 2 (b) 4

(c) 6 (d) 1

Ans. (b)

Solution: We have $(\sin x + 3)(2\sin x - 1) = 0$

$\Rightarrow \quad \sin x = -3$ which is not possible

or $\quad \sin x = 1/2 \Rightarrow x = \pi/6, 5\pi/6, 13\pi/6, 17\pi/6$

as $x \in [0, 3\pi]$.

Example 66: If A, B, C are the angles of a triangle and $\cos B + \cos C = 4\sin^2(A/2)$, then $\tan B/2 \tan C/2$ is equal to

(a) 1/2 (b) 1/3

(c) 2/3 (d) 3/2

Ans. (b)

Solution: $\cos B + \cos C = 4\sin^2(A/2)$

$$\Rightarrow \quad 2\cos\frac{B+C}{2}\cos\frac{B-C}{2} = 4\sin\frac{A}{2}\sin\left(\frac{\pi}{2} - \frac{B+C}{2}\right)$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{B-C}{2} = 4\sin\frac{A}{2}\cos\frac{B+C}{2}$$

$$\Rightarrow \quad \cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2} = 2\left(\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$\Rightarrow \quad 3\sin\frac{B}{2}\sin\frac{C}{2} = \cos\frac{B}{2}\cos\frac{C}{2}$$

$$\Rightarrow \quad \tan\frac{B}{2}\tan\frac{C}{2} = \frac{1}{3}.$$

Example 67: If $15\sin^4 x + 10\cos^4 x = 6$, Then $\tan^2 x =$

(a) 1/5 (b) 2/5

(c) 2/3 (d) 1/3

Ans. (c)

Solution: $15\sin^4 x + 10\cos^4 x = 6(\sin^2 x + \cos^2 x)^2$

$\Rightarrow \quad 9\sin^4 x + 4\cos^4 x - 12\sin^2 x\cos^2 x = 0$

$\Rightarrow \quad (3\sin^2 x - 2\cos^2 x)^2 = 0$

$\Rightarrow \quad \tan^2 x = 2/3.$

Example 68: Sum of the root of the equation $2\sin^2\theta + \sin^2 2\theta = 2$; $0 \leq \theta \leq \pi/2$ is

(a) $\pi/2$ (b) $3\pi/4$

(c) $7\pi/2$ (d) $5\pi/12$

Ans. (b)

Solution: $4\sin^2\theta\cos^2\theta = 2(1 - \sin^2\theta)$

$\Rightarrow \quad (2\sin^2\theta - 1)\cos^2\theta = 0$

$\Rightarrow \quad \sin^2\theta = 1/2 \quad \text{or} \quad \cos^2\theta = 0$

$\Rightarrow \quad \theta = \pi/4 \quad \text{or} \quad \theta = \pi/2$

Example 69: If $\tan x/2 = \operatorname{cosec} x - \sin x$, then $\sec^2(x/2) =$

(a) $\sqrt{5} + 1$ (b) $\sqrt{5} - 1$

(c) $\sqrt{5} - 2$ (d) $\sqrt{5} + 2$.

Ans. (b)

Solution: $\tan(x/2) = \dfrac{1 + \tan^2(x/2)}{2\tan(x/2)} - \dfrac{2\tan(x/2)}{1 + \tan^2(x/2)}$

$\Rightarrow \; 2\tan^2(x/2)(1 + \tan^2(x/2) = [1 + \tan^2(x/2)]^2 - 4\tan^2(x/2)$

$\Rightarrow \; 2\tan^4(x/2) + 2\tan^2(x/2) = 1 + \tan^4(x/2) - 2\tan^2(x/2)$

$\Rightarrow \; \tan^4(x/2) + 4\tan^2(x/2) - 1 = 0$

$\Rightarrow \; \tan^2(x/2) = \sqrt{5} - 2 \quad \Rightarrow \; \sec^2(x/2) = \sqrt{5} - 1$

Example 70: Let A and B denote the statements

$A : \cos\alpha + \cos\beta + \cos\gamma = 0$

$B : \sin\alpha + \sin\beta + \sin\gamma = 0$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -3/2$

Then

(a) both A and B are true
(b) both A and B are false
(c) A is true B is false
(d) A is false B is true.

Ans. (a)

Solution: $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -3/2$

$$\Rightarrow \quad 2[\cos\beta\cos\gamma + \cos\gamma\cos\alpha + \cos\alpha\cos\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha + \sin\alpha\sin\beta] + \left(\sin^2\alpha + \cos^2\alpha\right) + \left(\sin^2\beta + \cos^2\beta\right) + \left(\sin^2\gamma + \cos^2\gamma\right) = 0$$

$$\Rightarrow \quad (\sin\alpha + \sin\beta + \sin\gamma)^2 + (\cos\alpha + \cos\beta + \cos\gamma)^2 = 0$$

$$\Rightarrow \quad \cos\alpha + \cos\beta + \cos\gamma = 0$$

and $\quad \sin\alpha + \sin\beta + \sin\gamma = 0$

so both A and B are true.

Example 71: $\cos^2 u + \cos^2(u + x) - 2\cos u \cos x \cos(u + x) = 1/2$ if

(a) $x = \pi/4$ (b) $u = \pi/4$
(c) $x = \pi/2$ (d) $u = \pi/2$

Ans. (a)

Solution: L.H.S $= \cos^2 u + \cos^2(u + x) - [\cos(u + x) + \cos(u - x)]\cos(u + x)$

$$= \cos^2 u - \cos(u - x)\cos(u + x)$$

$$= \cos^2 u - (\cos^2 u - \sin^2 x) = \sin^2 x.$$

so $\quad \sin^2 x = 1/2 \quad \Rightarrow \quad \sin x = \pm 1/\sqrt{2}$

Example 72: For a regular polygon, let r, R be the radii of the inscribed and circumscribed circles. There is no regular polygon with

(a) $\dfrac{r}{R} = \dfrac{2}{3}$ (b) $\dfrac{r}{R} = \dfrac{\sqrt{3}}{2}$
(c) $\dfrac{r}{R} = \dfrac{1}{2}$ (d) $\dfrac{r}{R} = \dfrac{1}{\sqrt{2}}$.

Ans. (a)

Solution: We have $\dfrac{r}{R} = \cos\dfrac{\pi}{n}$

When $\dfrac{r}{R} = \dfrac{\sqrt{3}}{2}$, $\cos\dfrac{\pi}{n} = \cos\dfrac{\pi}{6}$. $\Rightarrow n = 6$

when $\dfrac{r}{R} = \dfrac{1}{2}$, $\cos\dfrac{\pi}{n} = \cos\dfrac{\pi}{3} \quad \Rightarrow \quad n = 3$

and when $\dfrac{r}{R} = \dfrac{1}{\sqrt{2}}$, $\cos\dfrac{\pi}{n} = \cos\dfrac{\pi}{4} \quad \Rightarrow \quad n = 4$

But when $\dfrac{r}{R} = \dfrac{2}{3}$, $\cos\dfrac{\pi}{n} = \dfrac{2}{3}$

Which does not give a positive integral value of n.

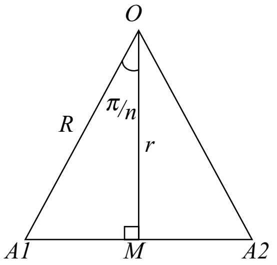


Fig. 25.4

Example 73: If $\cos(\alpha + \beta) = 4/5$ and $\sin(\alpha - \beta) = 5/13$ where $0 \le \alpha, \beta \le \pi/4$, then $\tan 2\alpha =$

(a) 19/12 (b) 20/7
(c) 25/16 (d) 56/33

Ans. (d)

Solution: $0 \le \alpha, \beta \le \pi/4$

$\Rightarrow \quad 0 \le \alpha + \beta \le \pi/2 \quad \Rightarrow \quad -\pi/4 \le \alpha - \beta \le \pi/4$

Now $\quad \cos(\alpha + \beta) = 4/5 \quad \Rightarrow \quad \tan(\alpha + \beta) = 3/4$

and $\quad \sin(\alpha - \beta) = 5/13 \quad \Rightarrow \quad \tan(\alpha - \beta) = 5/12$

we have $\quad \tan 2\alpha = \tan[(\alpha + \beta) + (\alpha - \beta)]$

$$= \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)}$$

$$= \frac{(3/4) + (5/12)}{1 - (3/4)(5/12)} = \frac{14/12}{33/48} = \frac{56}{33}.$$

Example 74: The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are

(a) $\dfrac{\pi}{4}, \dfrac{5\pi}{12}, \dfrac{\pi}{12}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}, \dfrac{8\pi}{9}$

(b) $\dfrac{2\pi}{9}, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}, \dfrac{35\pi}{36}$

(c) $\dfrac{2\pi}{9}, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}, \dfrac{8\pi}{9}$

(d) $\dfrac{2\pi}{9}, \dfrac{\pi}{4}, \dfrac{4\pi}{9}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \dfrac{8\pi}{9}$

Ans. (d)

Solution: $\sin\theta + \sin 4\theta + \sin 7\theta = 0$

$\Rightarrow \quad 2\sin 4\theta \cos 3\theta + \sin 4\theta = 0$

$\Rightarrow \quad \sin 4\theta(2\cos 3\theta + 1) = 0$

$\Rightarrow \quad \sin 4\theta = 0 \quad$ or $\quad \cos 3\theta = -1/2 = \cos(2\pi/3)$

$$\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4} \text{ or } 3\theta = 2\pi/3, \quad 2\pi \pm 2\pi/3$$

$$\Rightarrow \quad \theta = \frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{4\pi}{9}, \frac{3\pi}{4}, \frac{8\pi}{9}.$$

Example 75: In a ΔPQR, if $3 \sin P + 4 \cos Q = 6$ and $4 \sin Q + 3 \cos P = 1$, Then the angle R is equal to

(a) $\pi/4$ (b) $3\pi/4$

(c) $5\pi/6$ (d) $\pi/6$.

Ans. (d)

Solution: Squaring and adding the given relations we get

$$16 + 9 + 24 \sin (P + Q) = 37$$

$\Rightarrow \quad \sin (P + Q) = 1/2 \Rightarrow \sin R = 1/2$

$\Rightarrow \quad R = \pi/6 \quad \text{or} \quad 5\pi/6$

If R = $5\pi/6$, then $P < \pi/6 \quad \Rightarrow \quad 3\sin P < 3/2$.

$\Rightarrow 3 \sin P + 4 \cos Q < 3/2 + 4 < 6$

So $R \neq 5\pi/6$

Assertion-Reason Type Questions

Example 76: Statement-1:

$1/2 \le \sin^2 \theta + 3 \sin \theta \cos \theta + 5\cos^2 \theta \le 11/2$

Statement-2: $-\sqrt{a^2 + b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2 + b^2}$.

Ans. (a)

Solution: Take $a = r \cos\alpha$ and $b = r \sin\alpha$

$\Rightarrow \quad r = \sqrt{a^2 + b^2}$ and $a \sin\theta + b \cos\theta = r \sin(\theta + \alpha)$

Now $\quad - r \le r \sin (\theta + \alpha) \le r$.

$\Rightarrow \quad -\sqrt{a^2 + b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2 + b^2}$

$\Rightarrow$ Statement-2 is true.

Now $\sin^2\theta + 3 \sin\theta \cos\theta + 5 \cos^2\theta$

$$= 1 + (3/2)\sin 2\theta + 4\cos^2 \theta$$

$$= 1 + (3/2)\sin 2\theta + 2(1 + \cos 2\theta)$$

$$= 3 + (3/2)\sin 2\theta + 2 \cos 2\theta = A(\text{say})$$

From statement-2

$$3 - \sqrt{(9/4) + 4} \le A \le 3 + \sqrt{9/4 + 4}$$

$$\Rightarrow \quad 1/2 \le A \le 11/2$$

and the statement-1 is also true.

Example 77: Statement-1: The number of values of θ in the interval $(-\pi/2, \pi/2)$ such that $\theta \neq n\pi/5$ for $n = 0, \pm 1, \pm 2$ and $\tan \theta = \cot 5\theta$ is 6.

Statement-2: No value of θ in statement-1 satisfies $\sin 2\theta = \cos 4\theta$.

Ans. (c)

Solution: $\tan \theta = \cot 5\theta$

$\Rightarrow \quad 2 \sin\theta \sin 5\theta = 2 \cos\theta \cos 5\theta$

$\Rightarrow \quad \cos 4\theta - \cos 6\theta = \cos 6\theta + \cos 4\theta$

$\Rightarrow \quad \cos 6\theta = 0$

$$\Rightarrow \quad 6\theta = \pm\frac{5\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{\pi}{2}$$

as $-3\pi < 6\theta < 3\pi$

Gives 6 values of θ, $\pm\frac{5\pi}{12}, \pm\frac{3\pi}{12}, \pm\frac{\pi}{12}$

out of which $\frac{5\pi}{12}, \frac{-\pi}{4}$ and $\frac{\pi}{12}$

satisfy $\sin 2\theta = \cos 4\theta$.

so statement -1: is true and statement-2 is false.

Example 78: Statement-1: $\sin 52° + \sin 78° + \sin 50°$

$$= 4 \cos 26° \cos 39° \cos 25°$$

Statement-2: If $A + B + C = \pi$, then

$\sin A + \sin B + \sin C = 4 \cos (A/2) \cos (B/2) \cos (C/2)$

Ans. (a)

Solution: Statement-2 is True from conditional identities

$\Rightarrow$ statement-1, is also True.

Example 79: Statement-1: If A, B, C are the angles of a triangle such that angle A is obtuse, then $\tan B \tan C < 1$

Statement-2: In a triangle ABC

$$\tan A = \frac{\tan B + \tan C}{1 - \tan B \tan C}$$

Ans. (c)

Solution: Statement-2 is false because

$A = \tan (\pi - (B + C))$

$$\tan A = -\tan (B + C) = \frac{\tan B + \tan C}{\tan B \tan C - 1}.$$

In statement-1 If A is obtuse, $\tan A < 0 \quad \Rightarrow \quad \tan B \tan C < 1$ and the statement-1 is True.

Example 80: Statement-1:

$$\frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)} = \tan A$$

Statement-2: $\sin(A+B) + \sin(A-B) = \sin A$
and $\cos(A+B) + \cos(A-B) = \cos A$

Ans. (c)

Solution: L.H.S in statement-1

$$= \frac{\sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B}{\cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B}$$

$$= \frac{2\sin A\cos B}{2\cos A\cos B} = \tan A$$

$\Rightarrow$ statement-1 is True and statement-2 is false.

Example 81: Statement-1: If $2\sin^2((\pi/2)\cos^2 x) = 1 - \cos(\pi\sin 2x)$, $x \neq (2n+1)\pi/2$, n is a integer, then $\sin 2x + \cos 2x$ is equal to 1/5.

Statement-2: $\sin 2x + \cos 2x = \dfrac{2-(\tan x - 1)^2}{1+\tan^2 x}$.

Ans. (d)

Solution: $\sin 2x + \cos 2x =$

$$\frac{2\tan x + 1 - \tan^2 x}{1+\tan^2 x} = \frac{2-(\tan x-1)^2}{1+\tan^2 x}$$

$\Rightarrow$ statement-2 is True.

In statement-1, $2\sin^2\dfrac{\pi\cos^2 x}{2} = 2\sin^2\dfrac{\pi\sin 2x}{2}$

$\Rightarrow$ $\cos^2 x = \sin 2x$

$\Rightarrow$ $\cos x(\cos x - 2\sin x) = 0 \Rightarrow \tan x = 1/2$ as $\cos x \neq 0$.

From statement-2, $\sin 2x + \cos 2x = 7/5$

$\Rightarrow$ statement-1 is False.

Example 82: Statement-1: The system of linear equations

$x + (\sin\alpha)y + (\cos\alpha)z = 0$, $x + (\cos\alpha)y + (\sin\alpha)z = 0$ and $x - (\sin\alpha)y + (\cos\alpha)z = 0$ has a non-trivial solution for only one value of α lying between 0 and π

Statement-2: $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ has only one solution lying between 0 and $\pi/2$

Ans. (b)

Solution: In statement-1, equations have a non-trivial solution if

$$\begin{vmatrix} 1 & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ 1 & -\sin\alpha & \cos\alpha \end{vmatrix} = 0$$

$\Rightarrow$ $2\sin\alpha(\cos\alpha - \sin\alpha) = 0$ $\Rightarrow$ $\tan\alpha = 1$
$[\because \sin\alpha \neq 0$ as $0 < \alpha < \pi]$

$\Rightarrow \alpha = \pi/4$ is the only solution $\Rightarrow$ statement-1 is True

In statements, $(\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$

$\Rightarrow$ $(\sin x + 2\cos x)(\cos x - \sin x)^2 = 0$

$\Rightarrow$ $\tan x = -2$ or $\tan x = 1$ which gives only one values of x. i.e $x = \pi/4$ as $0 < x < \pi/2$.

$\Rightarrow$ Statement-2 is also True but does not lead to statement-1.

LEVEL 2

Straight Objective Type Questions

Example 83: If f : R → S is given by $f(x) = \sin x - \sqrt{3}\cos x + 1$ is on to, then the interval of S is

(a) [0, 1] (b) [– 1, 1]
(c) [0, 3] (d) [– 1, 3]

Ans. (d)

Solution: We have

$$-\sqrt{a^2+b^2} \leq a\sin x + b\cos x \leq \sqrt{a^2+b^2}$$

So $-\sqrt{1+3} \leq \sin x - \sqrt{3}\cos x \leq \sqrt{1+3}$

$\Rightarrow$ $-2 \leq \sin x - \sqrt{3}\cos x \leq 2$

$\Rightarrow$ $-2 + 1 \leq \sin x - \sqrt{3}\cos x + 1 \leq 2 + 1$

$\Rightarrow$ $-1 \leq f(x) \leq 3$.

Example 84: If $0 < x < \pi$, and $\cos x + \sin x = 1/2$, then $\tan x$ is

(a) $(1+\sqrt{7})/4$ (b) $(1-\sqrt{7})/4$
(c) $(4-\sqrt{7})/3$ (d) $-(4+\sqrt{7})/3$

Ans. (d)

Solution: Clearly $x \neq \pi/2$ so $\cos x \neq 0$ and we have $2(1 + \tan x) = \sec x$.

Squaring,

$$4(1+\tan x)^2 = 1+\tan^2 x$$

$$\Rightarrow \qquad 3\tan^2 x + 8\tan x + 3 = 0$$

$$\Rightarrow \qquad \tan x = \frac{-8 \pm \sqrt{64-36}}{6} = \frac{-4 \pm \sqrt{7}}{3}$$

(Take the negative sign)

Example 85: If cos A = 3/4 then value of 32 sin (A/2) sin (5 A/2) is equal to

(a) $\sqrt{11}$	(b) $-\sqrt{11}$
(c) 11	(d) – 11

Ans. (c)

Solution: 32 sin (A/2) sin (5A/2)

$$= 16\,[\cos 2A - \cos 3A]$$

$$= 16\,[2\cos^2 A - 1 - 4\cos^3 A + 3\cos A]$$

$$= 16\left[2\times\frac{9}{16} - 1 - 4\times\frac{27}{64} + 3\times\frac{3}{4}\right]$$

$$= 18 - 16 - 27 + 36 = 11.$$

Example 86: $\dfrac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ}$ is equal to

(a) tan 55°	(b) cot 55°
(c) – tan 35°	(d) – cot 35°

Ans. (a)

Solution: $\dfrac{\cos 10^\circ + \sin 10^\circ}{\cos 10^\circ - \sin 10^\circ} = \dfrac{1+\tan 10^\circ}{1-\tan 10^\circ}$

$$= \tan(45^\circ + 10^\circ) = \tan 55^\circ.$$

Example 87: If $\tan x = \dfrac{b}{a}$ then $\sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{a-b}{a+b}}$ is equal to

(a) $\dfrac{2\sin x}{\sqrt{\sin 2x}}$	(b) $\dfrac{2\cos x}{\sqrt{\cos 2x}}$
(c) $\dfrac{2\cos x}{\sqrt{\sin 2x}}$	(d) $\dfrac{2\sin x}{\sqrt{\cos 2x}}$

Ans. (b)

Solution: $\sqrt{\dfrac{a+b}{a-b}} + \sqrt{\dfrac{a-b}{a+b}}$

$$= \frac{a+b+a-b}{\sqrt{a^2-b^2}} = \frac{2a}{a\sqrt{1-\dfrac{b^2}{a^2}}}$$

$$= \frac{2}{\sqrt{1-\tan^2 x}} = \frac{2\cos x}{\sqrt{\cos 2x}}.$$

Example 88: If $\dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} = x$, then $\dfrac{\cos\alpha}{1+\sin\alpha}$ is equal to

(a) 1/*x*	(b) *x*
(c) 1 + *x*	(d) 1 – *x*

Ans. (d)

Solution: $x = \dfrac{2\sin\alpha}{1+\cos\alpha+\sin\alpha} \times \dfrac{1+\sin\alpha-\cos\alpha}{1+\sin\alpha-\cos\alpha}$

$$= \frac{2\sin\alpha(1+\sin\alpha-\cos\alpha)}{(1+\sin\alpha)^2 - (1-\sin^2\alpha)}$$

$$= \frac{2\sin\alpha\,(1+\sin\alpha-\cos\alpha)}{(1+\sin\alpha)\,2\sin\alpha}$$

$$= 1 - \frac{\cos\alpha}{1+\sin\alpha}$$

$$\Rightarrow \qquad \frac{\cos\alpha}{1+\sin\alpha} = 1 - x$$

Example 89: If sin *x* + cos *y* = *a* and cos *x* + sin *y* = *b*, then $\tan\dfrac{x-y}{2}$ is equal to

(a) *a* + *b*	(b) *a* – *b*
(c) $\dfrac{a+b}{a-b}$	(d) $\dfrac{a-b}{a+b}$

Ans. (d)

Solution: From the given relations we have

$\sin x + \sin\left((\pi/2) - y\right) = a$ and $\cos x + \cos\left((\pi/2) - y\right) = b$

$$\Rightarrow \quad 2\sin\frac{x+(\pi/2)-y}{2}\cos\frac{x-(\pi/2)+y}{2} = a$$

$$\text{and} \quad 2\cos\frac{x+(\pi/2)-y}{2}\cos\frac{x-(\pi/2)+y}{2} = b$$

Dividing we get,

$$\tan\left(\frac{\pi}{4} + \frac{x-y}{2}\right) = \frac{a}{b} \quad \Rightarrow \quad \frac{1+\tan\dfrac{x-y}{2}}{1-\tan\dfrac{x-y}{2}} = \frac{a}{b}$$

$$\text{or} \qquad \tan\frac{x-y}{2} = \frac{a-b}{a+b}.$$

Example 90: $\dfrac{\sin 3\alpha}{\cos 2\alpha} < 0$ if α lies in

(a) (13π/48, 14π/48)	(b) (14π/48, 18π/48)
(c) (18π/48, 23π/48)	(d) any of these intervals

Ans. (a)

Solution: $\dfrac{\sin 3\alpha}{\cos 2\alpha} < 0$ if $\sin 3\alpha > 0$ and $\cos 2\alpha < 0$

or $\sin 3\alpha < 0$ and $\cos 2\alpha > 0$

i.e. if $3\alpha \in (0, \pi)$ and $2\alpha \in (\pi/2, 3\pi/2)$

or $3\alpha \in (\pi, 2\pi)$ and $2\alpha \in (-\pi/2, \pi/2)$

i.e. if $\alpha \in (0, \pi/3)$ and $\alpha \in (\pi/4, 3\pi/4)$
or $\alpha \in (\pi/3, 2\pi/3)$ and $\alpha \in (-\pi/4, \pi/4)$
i.e. if $\alpha \in (\pi/4, \pi/3)$
since $(13\pi/48, 14\pi/48) \subset (\pi/4, \pi/3)$, (a) is correct

◉ **Example 91:** If $\cos\alpha + \cos\beta = a$, $\sin\alpha + \sin\beta = b$ and θ is the arithmetic mean between α and β then $\sin 2\theta + \cos 2\theta$ is equal to

(a) $(a+b)^2/(a^2+b^2)$ (b) $(a-b)^2/(a^2+b^2)$
(c) $(a^2-b^2)/(a^2+b^2)$ (d) none of these

Ans. (d)

◎ **Solution:** From the given relations we have

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = a \text{ and } 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = b$$

By dividing we get $\tan\dfrac{\alpha+\beta}{2} = \dfrac{b}{a} \Rightarrow \tan\theta = \dfrac{b}{a}$

so that $\cos 2\theta = \dfrac{1-b^2/a^2}{1+b^2/a^2} = \dfrac{a^2-b^2}{a^2+b^2}$ and $\sin 2\theta = \dfrac{2ab}{a^2+b^2}$.

$$\therefore \sin 2\theta + \cos 2\theta = \frac{a^2-b^2+2ab}{a^2+b^2}$$

◉ **Example 92:** If $\dfrac{1}{\cos\alpha\cos\beta} + \tan\alpha\tan\beta = \tan\gamma$, $0 < \alpha, \beta < \pi$ then $1 - \tan^2\gamma < 0$ for

(a) all values of α and β
(b) no values of α and β
(c) finite number of values of α and β
(d) infinite number of values of α and β

Ans. (a)

◎ **Solution:** We have $1 - \tan^2\gamma =$

$$\frac{\cos^2\alpha\cos^2\beta - (1+\sin\alpha\sin\beta)^2}{\cos^2\alpha\cos^2\beta}$$

$$= \frac{(1-\sin^2\alpha)(1-\sin^2\beta) - (1+2\sin\alpha\sin\beta+\sin^2\alpha\sin^2\beta)}{\cos^2\alpha\cos^2\beta}$$

$$= \frac{-(\sin\alpha+\sin\beta)^2}{\cos^2\alpha\cos^2\beta} < 0 \quad (\sin\alpha + \sin\beta \neq 0, \text{ as } 0 < \alpha, \beta < \pi)$$

◉ **Example 93:** If $x + y = z$, then $\cos^2 x + \cos^2 y + \cos^2 z - 2\cos x\cos y\cos z$ is equal to

(a) $\cos^2 z$ (b) $\sin^2 z$
(c) 0 (d) 1

Ans. (d)

◎ **Solution:** The given expression can be written as

$$\cos^2 x + \cos^2 y + \cos^2 z - \cos z\,[\cos(x+y) + \cos(x-y)]$$
$$= \cos^2 x + \cos^2 y + \cos^2 z - \cos^2 z - \cos(x+y)\cos(x-y)$$
$$= \cos^2 x + \cos^2 y - (1/2)[\cos 2x + \cos 2y]$$
$$= (1/2)\,[2\cos^2 x + 2\cos^2 y - \cos 2x - \cos 2y]$$
$$= (1/2)\,[2\cos^2 x + 2\cos^2 y - 2\cos^2 x + 1 - 2\cos^2 y + 1] = 1$$

◉ **Example 94:** If $\sin 2\theta = k$, then the value of $\dfrac{\tan^3\theta}{1+\tan^2\theta} + \dfrac{\cot^3\theta}{1+\cot^2\theta}$ is equal to

(a) $\dfrac{1-k^2}{k}$ (b) $\dfrac{2-k^2}{k}$
(c) $k^2 + 1$ (d) $2 - k^2$

Ans. (b)

◎ **Solution:** We have $\dfrac{\tan^3\theta}{1+\tan^2\theta} + \dfrac{\cot^3\theta}{1+\cot^2\theta}$

$$= \frac{\sin^3\theta\cdot\cos^2\theta}{\cos^3\theta} + \frac{\cos^3\theta}{\sin^3\theta}\cdot\sin^2\theta = \frac{\sin^3\theta}{\cos\theta} + \frac{\cos^3\theta}{\sin\theta}$$

$$= \frac{\sin^4\theta + \cos^4\theta}{\sin\theta\cos\theta} = \frac{(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{1-k^2/2}{k/2} = \frac{2-k^2}{k}, \qquad \left[\because \sin\theta\cos\theta = \frac{k}{2}\right].$$

◉ **Example 95:** If $\sin^2 A = x$, then $\sin A \sin 2A \sin 3A \sin 4A$ is a polynomial in x, the sum of whose coefficients is

(a) 0 (b) 40
(c) 168 (d) 336

Ans. (a)

◎ **Solution:** We have $\sin A \sin 2A \sin 3A \sin 4A$

$$= \sin A\,(2\sin A\cos A)\,(3\sin A - 4\sin^3 A) \times 2\sin 2A\cos 2A$$
$$= 2\sin^2 A\cos A \times \sin A\,(3 - 4\sin^2 A) \times 2 \times 2\sin A\cos A\,(1 - 2\sin^2 A)$$
$$= 8\sin^4 A\cos^2 A\,(3 - 4\sin^2 A)\,(1 - 2\sin^2 A)$$
$$= 8x^2(1-x)(3-4x)(1-2x)$$
$$= 24x^2 - 104x^3 + 144x^4 - 64x^5.$$

The required sum $= 24 - 104 + 144 - 64 = 0$.

◉ **Example 96:** If $\dfrac{\sin A}{\sin B} = \dfrac{\sqrt{3}}{2}$ and $\dfrac{\cos A}{\cos B} = \dfrac{\sqrt{5}}{2}$, $0 < A, B < \pi/2$, then

$\tan A + \tan B$ is equal to

(a) $\sqrt{3}/\sqrt{5}$ (b) $\sqrt{5}/\sqrt{3}$
(c) 1 (d) $(\sqrt{3}+\sqrt{5})/\sqrt{5}$

Ans. (d)

◎ **Solution:** From the given relation we have

$$\Rightarrow \quad \frac{\tan A}{\sqrt{3}} = \frac{\tan B}{\sqrt{5}} = k \text{ (say), (clearly } k > 0)$$

Also $2 \sin A = \sqrt{3} \sin B$.

$$\Rightarrow \frac{2\tan A}{\sqrt{1+\tan^2 A}} = \frac{\sqrt{3}\tan B}{\sqrt{1+\tan^2 B}} \Rightarrow \frac{2\sqrt{3}k}{\sqrt{1+3k^2}} = \frac{\sqrt{3}\times\sqrt{5}k}{\sqrt{1+5k^2}}$$

$\Rightarrow 4(1 + 5k^2) = 5(1 + 3k^2)$

$\Rightarrow k^2 = 1/5 \Rightarrow k = 1/\sqrt{5}$

so that $\tan A = \dfrac{\sqrt{3}}{\sqrt{5}}$, $\tan B = 1 \Rightarrow \tan A + \tan B = \dfrac{\sqrt{3}+\sqrt{5}}{\sqrt{5}}$.

◉ **Example 97:** If α, β, γ, δ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity k, then the values of $4 \sin \dfrac{\alpha}{2} + 3 \sin \dfrac{\beta}{2} + 2 \sin \dfrac{\gamma}{2} + \sin \dfrac{\delta}{2}$ is equal to

(a) $2\sqrt{1-k}$ (b) $2\sqrt{1+k}$

(c) $2\sqrt{k}$ (d) none of these

Ans. (b)

◎ **Solution:** $\alpha < \beta < \gamma < \delta$ and

$\sin \alpha = \sin \beta = \sin \gamma = \sin \delta = k$

$\Rightarrow \quad \beta = \pi - \alpha, \gamma = 2\pi + \alpha, \delta = 3\pi - \alpha.$

so that the given expression is equal to

$$4 \sin \frac{\alpha}{2} + 3 \sin \frac{\pi-\alpha}{2} + 2 \sin \frac{2\pi+\alpha}{2} + \sin \frac{3\pi-\alpha}{2}$$

$$= 4 \sin \frac{\alpha}{2} + 3 \cos \frac{\alpha}{2} - 2 \sin \frac{\alpha}{2} - \cos \frac{\alpha}{2}$$

$$= 2\left(\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2}\right)$$

$$= 2\sqrt{1 + 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}} = 2\sqrt{1+k}.$$

◉ **Example 98:** If $(a - b) \sin (\theta + \phi) = (a + b) \sin (\theta - \phi)$ and $a \tan (\theta/2) - b \tan (\phi/2) = c$, then the value of $\sin \phi$ is equal to

(a) $2ab/(a^2 - b^2 - c^2)$

(b) $2bc/(a^2 - b^2 - c^2)$

(c) $2bc/(a^2 - b^2 + c^2)$

(d) $2ab/(a^2 - b^2 + c^2)$

Ans. (b)

◎ **Solution:** From the first relation we have

$a [\sin (\theta + \phi) - \sin (\theta - \phi)] = b[\sin (\theta - \phi) + \sin (\theta + \phi)]$

$\Rightarrow \quad 2a \sin \phi \cos \theta = 2b \sin \theta \cos \phi$

$\Rightarrow \quad a \tan \phi = b \tan \theta$

$$\Rightarrow \quad \frac{2a\tan(\phi/2)}{1-\tan^2(\phi/2)} = \frac{2b\tan(\theta/2)}{1-\tan^2(\theta/2)}$$

From the second relation we have

$$\tan (\theta/2) = (b \tan (\phi/2) + c)/a$$

$$\text{so that} \quad \frac{a\tan(\phi/2)}{1-\tan^2(\phi/2)} = \frac{b(b\tan(\phi/2)+c)/a}{1-(b\tan(\phi/2)+c)^2/a^2}$$

$\Rightarrow \quad \tan (\phi/2) (a^2 - b^2 - c^2) = bc (1 + \tan^2 (\phi/2))$

$$\Rightarrow \quad \sin\phi = \frac{2\tan(\phi/2)}{1+\tan^2(\phi/2)} = \frac{2bc}{a^2-b^2-c^2}$$

◉ **Example 99:** If $\dfrac{\cos x - \cos\alpha}{\cos x - \cos\beta} = \dfrac{\sin^2\alpha\cos\beta}{\sin^2\beta\cos\alpha}$ then $\cos x$ is equal to

(a) $\dfrac{\cos\alpha-\cos\beta}{1+\cos\alpha\cos\beta}$ (b) $\dfrac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$

(c) $\dfrac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}$ (d) none of these

Ans. (c)

◎ **Solution:** The given relation can be written as

$$\cos x \,(\sin^2 \beta \cos \alpha - \sin^2 \alpha \cos \beta) = \cos^2 \alpha \sin^2 \beta - \sin^2 \alpha \cos^2 \beta$$

$$\Rightarrow \quad \cos x = \frac{\cos^2\alpha\left(1-\cos^2\beta\right)-\left(1-\cos^2\alpha\right)\cos^2\beta}{\cos\alpha\left(1-\cos^2\beta\right)-\cos\beta\left(1-\cos^2\alpha\right)}$$

$$= \frac{\cos^2\alpha-\cos^2\beta}{(\cos\alpha-\cos\beta)(1+\cos\alpha\cos\beta)}$$

$$= \frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}.$$

◉ **Example 100:** If $0 < \alpha, \beta < \pi$ and $\cos \alpha + \cos \beta - \cos (\alpha + \beta) = 3/2$ then $\sin \alpha + \cos \beta$ is equal to

(a) 0 (b) 1

(c) $\left(\sqrt{3}+1\right)/2$ (d) $\sqrt{3}$

Ans. (c)

◎ **Solution:** From the given equation we have

$$2 \cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2} - 2 \cos^2 \frac{\alpha+\beta}{2} + 1 = \frac{3}{2}$$

$$\Rightarrow 4 \cos^2 \frac{\alpha+\beta}{2} - 4 \cos \frac{\alpha-\beta}{2} \cos \frac{\alpha+\beta}{2} + 1 = 0$$

$$\Rightarrow \quad \left(2\cos\frac{\alpha+\beta}{2} - \cos\frac{\alpha-\beta}{2}\right)^2 = \cos^2\frac{\alpha-\beta}{2} - 1 \qquad \text{(i)}$$

so the only possibility is $\cos^2 \dfrac{\alpha-\beta}{2} - 1 = 0$

since $\cos^2\left(\dfrac{\alpha-\beta}{2}\right) \le 1$

As $0 < \alpha, \beta < \pi$, we have $\alpha = \beta$

From (i) we get $\cos\alpha = 1/2 = \cos\beta$

and $\sin\beta = \sin\alpha = \dfrac{\sqrt{3}}{2}$. so that $\sin\alpha + \cos\beta = \dfrac{\sqrt{3}+1}{2}$

Example 101: If $\sin\alpha, \sin\beta, \sin\gamma$ are in A.P.and $\cos\alpha$, $\cos\beta, \cos\gamma$ are in G.P. then $\dfrac{\cos^2\alpha + \cos^2\gamma - 4\cos\alpha\cos\gamma}{1-\sin\alpha\sin\gamma} =$

(a) -2 (b) -1

(c) 0 (d) 2

Ans. (a)

Solution: From the given conditions we have

$$2\sin\beta = \sin\alpha + \sin\gamma \quad \text{(i)}$$

$$\cos^2\beta = \cos\alpha\cos\gamma \quad \text{(ii)}$$

Squaring (i), $4\sin^2\beta = \sin^2\alpha + \sin^2\gamma + 2\sin\alpha\sin\gamma$

Using (ii), $4(1 - \cos\alpha\cos\gamma) = 1 - \cos^2\alpha + 1 - \cos^2\gamma + 2\sin\alpha\sin\gamma$

$\Rightarrow \cos^2\alpha + \cos^2\gamma - 4\cos\alpha\cos\gamma = 2(\sin\alpha\sin\gamma - 1)$

$$\Rightarrow \frac{\cos^2\alpha + \cos^2\gamma - 4\cos\alpha\cos\gamma}{1-\sin\alpha\sin\gamma} = -2.$$

Example 102: If $x = \dfrac{\sin^3 p}{\cos^2 p}$, $y = \dfrac{\cos^3 p}{\sin^2 p}$; and $\sin p + \cos p = 1/2$, then $x + y =$

(a) 75/18 (b) 44/9

(c) 79/18 (d) 48/9

Ans. (c)

Solution: $x + y = \dfrac{\sin^5 p + \cos^5 p}{\sin^2 p\cos^2 p}$

$$= \frac{\sin^3 p\left(1-\cos^2 p\right) + \cos^3 p\left(1-\sin^2 p\right)}{\sin^2 p\cos^2 p}$$

$$= \frac{\sin^3 p + \cos^3 p - \sin^2 p\cos^2 p(\sin p + \cos p)}{\sin^2 p\cos^2 p}$$

$$= \frac{(\sin p + \cos p)^3 - 3\sin p\cos p(\sin p + \cos p) - \sin^2 p\cos^2 p(\sin p + \cos p)}{\sin^2 p\cos^2 p}$$

Now $2\sin p\cos p = (\sin p + \cos p)^2 - 1 = -3/4$

$$\therefore \quad x + y = \frac{\left(\frac{1}{2}\right)^3 - 3\left(-\frac{3}{8}\right)\left(\frac{1}{2}\right) - \left(-\frac{3}{8}\right)^2\left(\frac{1}{2}\right)}{\left(-\frac{3}{8}\right)^2} = \frac{79}{18}.$$

Example 103: $\sin x + 2\sin 2x = 3 + \sin 3x$, $0 \le x \le 2\pi$ has

(a) 2 solutions in I quadrant

(b) one solution in II quadrant

(c) no solution in any quadrant

(d) one solution in each quadrant

Ans. (c)

Solution: From the given relation we have

$$\sin x - \sin 3x + 2\sin 2x = 3$$

$\Rightarrow 2\sin x\cos 2x - 2\sin 2x + 3 = 0$

$\Rightarrow (\sin x + \cos 2x)^2 + (\sin 2x - 1)^2 + 3$
$= \sin^2 x + \cos^2 2x + \sin^2 2x + 1$

$\Rightarrow (\sin x + \cos 2x)^2 + (\sin 2x - 1)^2 + \cos^2 x = 0$

which is possible only if

$$\sin x + \cos 2x = 0, \ \sin 2x = 1 \text{ and } \cos x = 0$$

which is not possible for any value of x.

Example 104: The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$ where x is a variable, has real roots if p lies in the interval

(a) $(0, 2\pi)$ (b) $(-\pi, 0)$

(c) $(-\pi/2, \pi/2)$ (d) $(0, \pi)$

Ans. (d)

Solution: The equation has real roots if

$\cos^2 p - 4(\cos p - 1)\sin p \ge 0$

or $\cos^2 p - 4\cos p\sin p + 4\sin p \ge 0$

or $(\cos p - 2\sin p)^2 - 4\sin^2 p + 4\sin p \ge 0$

or $(\cos p - 2\sin p)^2 + 4\sin p(1 - \sin p) \ge 0$

since $(\cos p - 2\sin p)^2 \ge 0$, $1 - \sin p \ge 0$ for all values of p, and for $p \in (0, \pi)$, $\sin p \ge 0$ so that the discriminant is non-negative.

Example 105: The general solution of the equation

$$\frac{1 - \sin x + \dots + (-1)^n\sin^n x + \dots}{1 + \sin x + \dots + \sin^n x + \dots} = \frac{1-\cos 2x}{1+\cos 2x}, \ x \ne (2n+1)\pi/2, \ n \in \mathbf{I} \text{ is}$$

(a) $(-1)^n(\pi/3) + n\pi$

(b) $(-1)^n(\pi/6) + n\pi$

(c) $(-1)^{n+1}(\pi/6) + n\pi$

(d) $(-1)^{n-1}(\pi/3) + n\pi$, $(n \in \mathbf{I})$

Ans. (b)

Solution: The equation

$$\frac{1 - \sin x + \dots + (-1)^n\sin^n x + \dots}{1 + \sin x + \dots + \sin^n x + \dots} = \frac{1-\cos 2x}{1+\cos 2x}$$

$$\Rightarrow \qquad \frac{1}{1+\sin x}\times\frac{1-\sin x}{1} = \frac{2\sin^2 x}{2\cos^2 x}$$

as $-1 < \sin x < 1$

$$\Rightarrow \qquad 1 - \sin x = \frac{\sin^2 x\,(1+\sin x)}{1-\sin^2 x}$$

$$\Rightarrow \qquad (1 - \sin x)^2 = \sin^2 x \quad \Rightarrow \quad 1 - 2\sin x = 0$$

$$\Rightarrow \qquad \sin x = 1/2 = \sin(\pi/6)$$

$$\Rightarrow \qquad x = n\pi + (-1)^n\,\pi/6.$$

Example 106: If $\sin^4 x + \cos^4 y + 2 = 4\sin x\cos y$, $0 \le x, y \le \pi/2$ then $\sin x + \cos y =$

(a) -2 (b) 0

(c) 2 (d) none of these

Ans. (c)

Solution: The given equation can be written as

$$\sin^4 x + \cos^4 y + 2 - 4\sin x\cos y = 0$$

$$\Rightarrow \quad (\sin^2 x - 1)^2 + (\cos^2 y - 1)^2 + 2\sin^2 x + 2\cos^2 y - 4\sin x\cos y = 0$$

$$\Rightarrow \quad (\sin^2 x - 1)^2 + (\cos^2 y - 1)^2 + 2(\sin x - \cos y)^2 = 0$$

which is true if $\sin^2 x = 1$, $\cos^2 y = 1$ and $\sin x = \cos y$, so $\sin x + \cos y = 2$ as $0 \le x, y \le \pi/2$.

Example 107: A solution (x, y) of the system of equation $x - y = 1/3$ and $\cos^2(\pi x) - \sin^2(\pi y) = 1/2$ is given by

(a) (2/3, 1/3) (b) (7/6, 1/6)

(c) (13/6, 11/6) (d) (1/6, 5/6)

Ans. (c)

Solution: $\cos^2(\pi x) - \sin^2(\pi y) = 1/2$

$$\Rightarrow \quad \cos\pi(x+y)\cos\pi(x-y) = 1/2$$

$$\Rightarrow \quad \cos\pi(x+y)\cos(\pi/3) = 1/2 \qquad [\because x - y = 1/3]$$

$$\Rightarrow \quad \cos\pi(x+y) = 1$$

$$\Rightarrow \quad \pi(x+y) = 2n\pi \Rightarrow x + y = 2n$$

Now $x + y = 2n$ and $x - y = 1/3$

$$\Rightarrow \quad x = n + 1/6,\ y = n - 1/6, \qquad (n \in I)$$

$\therefore (x, y) = \left(n+\frac{1}{6}, n-\frac{1}{6}\right)$ which is satisfied by (*c*) for $n = 2$

Example 108: $\cos(x - y) - 2\sin x + 2\sin y = 3$ if

(a) $\sin x = \sin y$

(b) $x + y = 2n\pi$, $x - y = (2k - 1)\pi/2$

(c) $x = 2k\pi - \pi/2$, $y = 2n\pi + \pi/2$

(d) $\cos(x - y) = -1$ $(n, k \in \mathbf{I})$

Ans. (c)

Solution: $\cos(x - y) - 2\sin x + 2\sin y = 3$

$$\Rightarrow \qquad 1 - 2\sin^2\frac{x-y}{2} - 4\sin\frac{x-y}{2}\cos\frac{x+y}{2} - 3 = 0$$

$$\Rightarrow \qquad \sin^2\frac{x-y}{2} + 2\sin\frac{x-y}{2}\cos\frac{x+y}{2} + 1 = 0$$

$$\Rightarrow \qquad \sin\frac{x-y}{2} = \frac{-2\cos\frac{x+y}{2} \pm \sqrt{4\cos^2\frac{x+y}{2} - 4}}{2}$$

For real values of $\sin\frac{x-y}{2}$, we have $\cos^2\frac{x+y}{2} = 1$

$$\Rightarrow \qquad \sin^2\frac{x+y}{2} = 0 \text{ or } \sin\frac{x+y}{2} = 0$$

$$\Rightarrow \qquad x + y = 2n\pi$$

and then $\sin\frac{x-y}{2} = \pm 1 \quad \Rightarrow \quad \frac{x-y}{2} = (2k-1)\frac{\pi}{2}$

$$\Rightarrow \qquad x - y = (2k - 1)\pi$$

so (b) is not correct.

Also if $\sin x = \sin y$ then the given equation becomes $\cos(x - y) = 3$ which is not correct; (a) is not correct.

Next if $x = 2k\pi - \pi/2$ and $y = 2n\pi + \pi/2$.

Then $\cos(x - y) - 2\sin x + 2\sin y = -1 + 2 + 2 = 3$, so (c) is correct.

Finally if $\cos(x - y) = -1$, the given equation becomes $\sin x - \sin y = -2$ which is not true for any real values of x and y so (d) is not correct.

Example 109: If $6\cos 2\theta + 2\cos^2(\theta/2) + 2\sin^2\theta = 0$, $-\pi < \theta < \pi$, then θ is equal to

(a) $\pi/3$ (b) $\pi/3, \cos^{-1}(3/5)$

(c) $\cos^{-1}(3/5)$ (d) $\pi/3, \pi - \cos^{-1}(3/5)$

Ans. (d)

Solution: The given equation can be written as

$$6(2\cos^2\theta - 1) + (1 + \cos\theta) + 2(1 - \cos^2\theta) = 0$$

$$\Rightarrow \qquad 10\cos^2\theta + \cos\theta - 3 = 0$$

$$\Rightarrow \qquad (5\cos\theta + 3)(2\cos\theta - 1) = 0$$

$$\Rightarrow \qquad \cos\theta = 1/2 \text{ or } \cos\theta = -3/5$$

$$\Rightarrow \quad \theta = \pi/3 \text{ or } \theta = \pi - \cos^{-1}(3/5) \text{ as } -\pi < \theta < \pi$$

$$\Rightarrow \quad \theta = \pi/3,\ \pi - \cos^{-1}(3/5).$$

Example 110: The number of integral values of a for which the equation $\cos 2x + a\sin x = 2a - 7$ possesses solution is

(a) 2 (b) 3

(c) 4 (d) 5

Ans. (d)

Solution: The given equation can be written as

$$1 - 2\sin^2 x + a\sin x = 2a - 7$$

$$\Rightarrow \qquad 2\sin^2 x - a\sin x + 2a - 8 = 0$$

$$\Rightarrow \qquad \sin x = \frac{a \pm \sqrt{a^2 - 8(2a-8)}}{4} = \frac{a \pm (a-8)}{4}$$

$\Rightarrow \quad \sin x = \dfrac{a-4}{2}$ which is possible if $-1 \le \dfrac{a-4}{2} \le 1$ or $2 \le a \le 6$.

so the required values of a are 2, 3, 4, 5, 6 and hence the required number is 5.

Example 111: The least difference between the roots of the equation

$4 \cos x\,(2 - 3 \sin^2 x) + (\cos 2x + 1) = 0 \ (0 \le x \le \pi/2)$ is

(a) $\pi/6$ (b) $\pi/4$

(c) $\pi/3$ (d) $\pi/2$

Ans. (a)

Solution: The given equation can be written as

$$4 \cos x\,(3 \cos^2 x - 1) + 2 \cos^2 x = 0$$

$$\Rightarrow \quad 2 \cos x\,(6 \cos^2 x + \cos x - 2) = 0$$

$$\Rightarrow \quad 2 \cos x\,(3 \cos x + 2)\,(2 \cos x - 1) = 0$$

$\Rightarrow$ either $\cos x = 0$ which gives $x = \pi/2$

or $\cos x = -2/3$, which gives no value of x for $0 \le x \le \pi/2$

or $\cos x = 1/2$, which gives $x = \pi/3$.

so the required difference $= \pi/2 - \pi/3 = \pi/6$

Example 112: The solution of $|\cos x| = \cos x - 2 \sin x$ is

(a) $x = n\pi$

(b) $x = n\pi + \pi/4$

(c) $x = n\pi + (-1)^n (\pi/4)\ (n \in \mathbf{I})$

(d) $x = (2n + 1)\pi + \pi/4$

Ans. (d)

Solution: $|\cos x| = \cos x - 2 \sin x$

$\Rightarrow \cos x = \cos x - 2 \sin x$ if $\cos x \ge 0$

$\Rightarrow \sin x = 0 \Rightarrow x = 2n\pi$ (as $\cos x \ge 0$) $n \in \mathbf{I}$ and $x \ne n\pi$

(as $x = \pi \Rightarrow \cos x < 0$, for $n = 1$)

Next $|\cos x| = \cos x - 2 \sin x$

$\Rightarrow \quad -\cos x = \cos x - 2 \sin x$ if $\cos x < 0$

$\Rightarrow \cos x - \sin x = 0 \Rightarrow \tan x = 1$

Now $\cos x < 0$ and $\tan x = 1 \Rightarrow \tan x = \tan (5\pi/4)$

$\Rightarrow \quad x = 2n\pi + (5\pi/4) = (2n + 1)\pi + \pi/4$

and $\quad x \ne 2n\pi + \pi/4$ (as $x = \pi/4 \Rightarrow \cos x > 0$, for $n = 0$).

Example 113: If $0 \le a, b \le 3$ and the equation $x^2 + 4 + 3\cos (ax + b) = 2x$ has at least one solution, then the value of $a + b$ is

(a) $\pi/4$ (b) $\pi/2$

(c) π (d) 2π

Ans. (c)

Solution: We have $x^2 - 2x + 4 = -3\cos (ax + b)$

$$\Rightarrow \quad (x-1)^2 + 3 = -3 \cos (ax + b) \qquad (1)$$

As $(x - 1) \ge 0$ and $-1 \le \cos (ax + b) \le 1$

Equation (1) s possible if $\cos (ax + b) = -1$

and $x - 1 = 0$

$\Rightarrow \quad a + b = \pi, 3\pi, 5\pi, \ldots\ldots$

Since $a + b \le 6$ and $3\pi > 6$, $a + b = \pi$

Example 114: If $0 < \theta < \pi/2$, number of solutions of

$$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right) \operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2} \quad \text{is}$$

(a) 0 (b) 1

(c) 2 (d) 4

Ans. (c)

Solution: We can write

$$\frac{1}{\sqrt{2}} \sum_{m=1}^{6} \frac{1}{\sin\left(\theta + \dfrac{(m-1)\pi}{4}\right) \sin\left(\theta + \dfrac{m\pi}{4}\right)} = 4$$

$$\Rightarrow \quad \sum_{m=1}^{6} \frac{\sin\left[\left(\theta + \dfrac{m\pi}{4}\right) - \left(\theta + \dfrac{(m-1)\pi}{4}\right)\right]}{\sin\left(\theta + \dfrac{(m-1)\pi}{4}\right) \sin\left(\theta + \dfrac{m\pi}{4}\right)} = 4$$

$$\Rightarrow \quad \sum_{m=1}^{6} \left[\cot\left(\theta + \frac{(m-1)\pi}{4}\right) - \cot\left(\theta + \frac{m\pi}{4}\right)\right] = 4$$

$$\Rightarrow \quad \cot\theta - \cot\left(\theta + \frac{6\pi}{4}\right) = 4$$

$$\Rightarrow \quad \cot\theta + \tan\theta = 4$$

$$\Rightarrow \quad \sin 2\theta = 1/2$$

$$\Rightarrow \quad 2\theta = \pi/6,\ 5\pi/6 \Rightarrow \theta = \pi/12,\ 5\pi/12$$

EXERCISE

Concept-based Straight Objective Type Questions

1. Let $f(\theta) = \dfrac{\cot\theta}{1 - \tan\theta} + \dfrac{\tan\theta}{1 - \cot\theta}, \pi < \theta < \dfrac{3\pi}{4}$, then $f\left(\dfrac{9\pi}{8}\right)$ is equal to:

(a) $1 + \sqrt{2}$ (b) $1 + 2\sqrt{2}$

(c) $\sqrt{2} - 1$ (d) $2\sqrt{2} - 1$

2. Let $f(\theta) = \dfrac{\tan^3\theta}{1+\tan^2\theta} - \dfrac{\cot^3\theta}{1+\cot^2\theta}, 0 < \theta < \dfrac{\pi}{4}$.
Then $f(\theta)$ is equal to
(a) $\tan\theta + \cot\theta$ (b) $2\sin(2\theta)$
(c) $-2\cot(2\theta)$ (d) 0

3. If $0 < \theta < \dfrac{\pi}{2}$, then $\dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$ is equal to:
(a) $1 + \sin\theta + \cos\theta$ (b) $\dfrac{1+\sin\theta}{\cos\theta}$
(c) $\dfrac{1-\cos\theta}{\sin\theta}$ (d) $\tan\theta - \sec\theta$

4. tan 15°+ tan 75° is equal to:
(a) 1 (b) 2
(c) 3 (d) 4

5. $\sin^2\left(\dfrac{\pi}{8}+\dfrac{A}{2}\right) - \sin^2\left(\dfrac{\pi}{8}-\dfrac{A}{2}\right)$ is equal to:
(a) $\left(1/\sqrt{2}\right)\sin A$ (b) $\left(1/\sqrt{2}\right)\cos A$
(c) $\sqrt{2}\sin A$ (d) $\sqrt{2}\cos A$

6. $\dfrac{\cos 11^\circ - \sin 11^\circ}{\cos 11^\circ + \sin 11^\circ}$ is equal to:
(a) cot 56° (b) tan 11°
(c) tan 56° (d) cot 11°

7. Suppose $\alpha, \beta > 0$ and $\alpha + 2\beta = \pi/2$, then $\tan(\alpha+\beta) - 2\tan\alpha - \tan\beta$ is equal to:
(a) 0 (b) $\tan\beta$
(c) $\cot\beta$ (d) $\tan\alpha - \cot\beta$

8. If $\sin\theta + \operatorname{cosec}\theta = 2$, then $\cos^{2015}\theta + \operatorname{cosec}^{2015}\theta$ is equal to:
(a) – 1 (b) 0
(c) 1 (d) 2015

9. If $(1 - \sin A)(1 - \sin B)(1 - \sin C) = (1 + \sin A)(1 + \sin B)(1 + \sin C)$, then each side is equal to:
(a) $\pm\cos A\cos B\cos C$ (b) 0
(c) $\pm\sin A\sin B\sin C$ (d) 1

10. $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13} + \cos\dfrac{3\pi}{13} + \cos\dfrac{5\pi}{13}$ is equal to:
(a) – 1 (b) 0
(c) 1 (d) $\sqrt{3}/2$

11. If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$, then $\tan A\tan B$ is equal to:
(a) $\tan\left(\dfrac{A+B}{2}\right)$ (b) $\cot\left(\dfrac{A+B}{2}\right)$
(c) $\cot\left(\dfrac{A-B}{2}\right)$ (d) $\tan\left(\dfrac{A-B}{2}\right)$

12. If $0 < \theta < \pi/8$, then $\sqrt{2+\sqrt{2+2\cos(4\theta)}}$ is equal to:
(a) $2\cos\theta$ (b) $-2\cos\theta$
(c) $2\sin\theta$ (d) $-2\sin\theta$

13. Let $f(\theta) = \cos\theta\cos 2\theta\cos 4\theta\cos 7\theta$, then $f(\pi/15)$ is equal to:
(a) 1/4 (b) 1/8
(c) 1/16 (d) 1/32

14. Let $f(\theta) = \sin\theta\sin 3\theta\sin 5\theta$, then $f(\pi/14)$ is equal to:
(a) 1/8 (b) 1/4
(c) 1/7 (d) 1/14

15. If $A + B + C = \pi/2$, then
$\sin^2 A + \sin^2 B + \sin^2 C + 2\sin A\sin B\sin C$
is equal to
(a) 0 (b) 1
(c) – 1 (d) none of these

16. Number of values of x lying in the interval $[0, 4\pi]$ and satisfying the equation $\tan 5x = \cot 3x = 0$ is:
(a) 2 (b) 4
(c) 6 (d) 8

17. Number of values of x lying in the interval $[0, 5\pi]$ and satisfying the equation $\sin x = \tan x$ is:
(a) 5 (b) 6
(c) 9 (d) 12

18. Number of values of $x \in [0, 4\pi]$ and satisfying the equation $\sin x + \cos x = 3/2$ is:
(a) 0 (b) 1
(c) 2 (d) 4

19. Number of values of $x \in [0, 5\pi]$ and satisfying $\sin^2 x = \cos^2 x$ is:
(a) 5 (b) 10
(c) 20 (d) 40

20. For $0 < \theta < \pi/4$, $\dfrac{\sec(8\theta)-1}{\sec(4\theta)-1}\cdot\dfrac{\tan(2\theta)}{\tan(8\theta)}$ is equal to
(a) – 1 (b) 0
(c) 1 (d) none of these

LEVEL 1

Straight Objective Type Questions

21. If 2 tan α + cot β = tan β, then the value of tan ($\beta - \alpha$) is

(a) tan α (b) cot α
(c) tan β (d) cot β

22. If cos $(x - y) = a$ cos $(x + y)$, then cot x cot y is equal to

(a) $\dfrac{a-1}{a+1}$ (b) $\dfrac{a+1}{a-1}$
(c) $a - 1$ (d) $a + 1$

23. $\sin^2 A + \sin^2 (A - B) + 2 \sin A \cos B \sin (B - A)$ is equal to

(a) $\sin^2 A$ (b) $\sin^2 B$
(c) $\cos^2 A$ (d) $\cos^2 B$

24. If $\dfrac{3 \sin 2\theta}{5 + 4\cos 2\theta} = 1$, then the value of $\tan\theta$ is equal to

(a) 1 (b) 1/3
(c) 3 (d) none of these

25. If $x = a \sec^3 \theta \tan \theta$, $y = b \tan^3 \theta \sec \theta$, then $\sin^2 \theta$ is equal to

(a) $\dfrac{x}{a} - \dfrac{y}{b}$ (b) $\dfrac{x}{a} + \dfrac{y}{b}$
(c) $\dfrac{xy}{ab}$ (d) $\dfrac{ay}{bx}$

26. cot θ – cot 3θ is equal to

(a) 2 sin θ sin 3θ (b) 2 cos θ cos 3θ
(c) 2 cos θ cosec 3θ (d) 2 sin θ cosec 3θ

27. If $0 < x, y < 2\pi$, the number of solutions of the system of equations

sin x sin y = 3/4 and cos x cos y = 1/4 is

(a) 0 (b) 1
(c) 2 (d) infinite

28. If A and B be acute positive angles satisfying

$3 \sin^2 A + 2 \sin^2 B = 1$, $3 \sin 2A - 2 \sin 2B = 0$

then

(a) $B = \pi/4 - A/2$ (b) $A = \pi/4 - 2B$
(c) $B = \pi/2 - A/4$ (d) $A = \pi/4 - B/2$

29. If tan α, tan β, tan γ are the roots of the equation $x^3 - px^2 - r = 0$, then the value of $(1 + \tan^2\alpha)$ $(1 + \tan^2\beta)$ $(1 + \tan^2\gamma)$ is equal to

(a) $(p - r)^2$ (b) $1 + (p - r)^2$
(c) $1 - (p - r)^2$ (d) none of these

30. If A, B, C are the angles of a triangle such that angle A is obtuse then

(a) tan A tan B < 1 (b) tan B tan C < 1
(c) tan C tan A < 1 (d) tan A tan B tan C < 1

31. If tan (θ/2) = cosec θ – sin θ, then $\cos^2 (\theta/2)$ is equal to

(a) sin 18° (b) cos 36°
(c) sin 36° (d) cos 18°

32. If $a \sin^2\theta + b \cos^2\theta = a \cos^2\phi + b \sin^2\phi = 1$ and $a \tan \theta = b \tan \phi$ $(a \neq b)$ then

(a) $a + b = 2ab$ (b) $a - b = 2ab$
(c) $a - b + 2ab = 0$ (d) $a + b + 2ab = 0$

33. The equation $\tan^2 x + \cot^2 x = 4a \operatorname{cosec}^2 2x$ has a real solution if

(a) $0 < a \le 1$ (b) $1/2 \le a \le 1$
(c) $1/4 \le a \le 1/2$ (d) $-1 \le a \le 1$

34. For $n \in \mathbf{I}$, the line $x = n\pi + \pi/2$ does not intersect the graph of

(a) cot $(x + \pi)$ (b) cos $(x - \pi)$
(c) sin x (d) tan x

35. If $\dfrac{\tan x}{2} = \dfrac{\tan y}{3} = \dfrac{\tan z}{5}$ and $x + y + z = \pi$, then the value of

$\tan^2 x + \tan^2 y + \tan^2 z$ is

(a) 38/3 (b) 38
(c) 114 (d) none of these

36. If the angles A, B, C of a triangle are in A.P. such that sin $(2A + B)$ = 1/2 then sin $(B + 2C)$ =

(a) – 1/2 (b) 1/2
(c) $\sqrt{3}/2$ (d) $1/\sqrt{2}$

37. If tan θ + tan ϕ = a, cot θ + cot ϕ = b, $\theta - \phi = \alpha$ $(\neq 0)$ then

(a) $ab < 4$ (b) $ab = 4$
(c) $ab > 4$ (d) $ab = 0$

38. If $\dfrac{x}{y} = \dfrac{\cos A}{\cos B}$ then $\dfrac{x \tan A + y \tan B}{x + y}$ =

(a) $\dfrac{\sin A + \cos B}{\cos A + \sin B}$ (b) $\dfrac{\sin A + \sin B}{\cos A \cos B}$
(c) $\tan \dfrac{A+B}{2}$ (d) $\cot \dfrac{A-B}{2}$

39. If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ then $a\theta =$

(a) $x + y - a$ (b) $\sqrt{x^2 + y^2 - a^2}$

(c) $\sqrt{x^2 - y^2 + a^2}$ (d) $x - y + a$

40. $(1 + \cos(\pi/8))(1 + \cos(3\pi/8))(1 + \cos(5\pi/8))(1 + \cos(7\pi/8)) =$

(a) 1/2 (b) $\cos(\pi/8)$

(c) 1/8 (d) $(1 + \sqrt{2})/2\sqrt{2}$

41. If $2\cos x + 2\cos 3x = \cos y$, $2\sin x + 2\sin 3x = \sin y$ then the value of $\cos 2x$ is

(a) – 7/8 (b) 1/8

(c) – 1/8 (d) 7/8

42. $\frac{\cos A}{3} = \frac{\cos B}{4} = \frac{1}{5}$, $-\frac{\pi}{2} < A < 0$, $0 < B < \frac{\pi}{2}$, then $3\sin A + 4\sin B =$

(a) 0 (b) – 1

(c) 24/5 (d) 1

43. The value of $\log_3 \tan 1° + \log_3 \tan 2° + \ldots + \log_3 \tan 89°$ is

(a) 3 (b) 1

(c) 2 (d) 0

44. If $\frac{\tan^3((\pi/2)-\theta)}{\sec^2\theta} \cdot \frac{\cot^2\theta}{\sec((\pi/2)-\theta)} \cdot \frac{\sin((\pi/2)-\theta)}{\sin^4\theta}$ $= \cot^n\theta$ then $n =$

(a) 2 (b) 4

(c) 6 (d) 8

45. The number of solutions of

$\sin\theta + 2\sin 2\theta + 3\sin 3\theta + 4\sin 4\theta = 10$, $0 < \theta < \pi$ is

(a) 0 (b) 1

(c) 2 (d) 4

46. cos 11° – cos 2° is

(a) a positive integer

(b) a negative integer

(c) a positive rational number

(d) a negative rational number

47. If $\sin A$, $\cos A$ and $\tan A$ are in G.P., then $\cot^6 A - \cot^2 A =$

(a) – 1 (b) 0

(c) 1 (d) none of these

48. If $\cos x - \sin x = 1/2$, then $\tan 2x =$

(a) $\sqrt{7}/3$ (b) $\sqrt{7}/4$

(c) $3/\sqrt{7}$ (d) $2/\sqrt{7}$

49. Which of the following gives the least value of A

(a) $\cos 2A = \sin 3A$ (b) $\cos 3A = \sin 7A$

(c) $\tan A = \cot 3A$ (d) $\cot A = \tan 2A$

50. If $n \in \mathbf{N}$ and $\sin\left(\frac{\pi}{2n}\right) + \cos\left(\frac{\pi}{2n}\right) = \frac{\sqrt{n}}{2}$, then a possible value of n is

(a) 4 (b) 6

(c) 8 (d) 12

51. If $4n\alpha = \pi$, then the value of $\tan\alpha \tan 2\alpha \tan 3\alpha \ldots \tan(2n-1)\alpha$ is

(a) – 1 (b) 0

(c) 1 (d) none of these

52. If $x > 0$ and $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = 0$ then

(a) $x < \sqrt{2}$ (b) $x = \sqrt{2}$

(c) $x > \sqrt{2}$ (d) none of these

53. If $2\tan(\pi/3)\cos(2\pi x) = \sqrt{3}$, the general solution of the equation is

(a) $2n\pi \pm \pi/3$ (b) $n \pm 1/3$

(c) $n \pm 1/6$ (d) $n \pm 1/2$ $(n \in \mathbf{I})$

54. $2\cos^2 x + 4\cos x = 3\sin^2 x$ if

(a) $\cos x = \frac{-2+\sqrt{14}}{5}$ (b) $\cos x = \frac{-2+\sqrt{19}}{5}$

(c) $\sin x = \frac{-2+\sqrt{14}}{5}$ (d) $\sin x = \frac{-2+\sqrt{19}}{5}$

55. $6\tan^2 x - 2\cos^2 x = \cos 2x$ if

(a) $\cos 2x = -1$ (b) $\cos 2x = 1$

(c) $\cos 3x = -1/2$ (d) $\cos 2x = 1/2$

56. The greatest value of $\cos\theta$ for which $\cos 5\theta = 0$ is

(a) 0 (b) $(1 + \sqrt{5})/4$

(c) $\sqrt{\frac{5+\sqrt{5}}{8}}$ (d) $\sqrt{\frac{\sqrt{5}+1}{4}}$

57. If $\tan p\theta = \tan q\theta$, then the values of θ form an A.P. with common difference

(a) $\pi/(p+q)$ (b) π/p

(c) π/q (d) $\pi/(p-q)$

58. The equation $\int_0^x (t^2 - 8t + 13)\,dt = x\sin(a/x)$ has a solution if $\sin(a/x) =$

(a) 0 (b) 1

(c) 3 (d) 6

59. The smallest positive root of the equation $\sqrt{\sin(1-x)} = \sqrt{\cos x}$ is

(a) $1/2 + \pi/4$ (b) $1/2 + 3\pi/4$

(c) $1/2 + 5\pi/4$ (d) $1/2 + 7\pi/4$

60. The sum of the roots of the equation $4\cos^3 x - 4\cos^2 x - \cos(\pi + x) - 1 = 0$ in the interval [0, 315] is $p\pi$, where p is equal to
    (a) 2500 (b) 2550
    (c) 2600 (d) 2651

61. A solution (x, y) of $x^2 + 2x \sin xy + 1 = 0$ is
    (a) (1, 0) (b) $(1, 7\pi/2)$
    (c) $(-1, -7\pi/2)$ (d) $(-1, 0)$

62. $e^{\sin x} - e^{-\sin x} = 4$ for
    (a) all real values of x (b) some $x \in [0, \pi/2]$
    (c) some $x \in (-\pi/2, \pi/2)$ (d) no real value of x

63. The number of values of x satisfying the condition $\sin x + \sin 5x = \sin 3x$ in the interval $[0, \pi]$ is
    (a) 6 (b) 2
    (c) 10 (d) 0

64. $\sin x$, $\sin 2x$, $\sin 3x$ are in A.P. if
    (a) $x = 165°$ (b) $x = 195°$
    (c) $x = 15°$ (d) $x = 2n\pi \quad n \in \boldsymbol{I}$.

65. $2\sin^2((\pi/2)\cos^2 x) = 1 - \cos(\pi \sin 2x)$ if
    (a) $x = (2n + 1)\pi/2$ (b) $x = n\pi/2$
    (c) $x = 2n\pi$ (d) none of these, $n \in \boldsymbol{I}$

66. Number of solutions of the equation $4\sin^2 x + \tan^2 x + \cot^2 x + \operatorname{cosec}^2 x = 6$ in $[0, \pi]$ is
    (a) 0 (b) 2
    (c) 4 (d) 8.

67. Sum of integral values of n such that $\sin x(2\sin x + \cos x) = n$, has at least one real solution is
    (a) 1 (b) 2
    (c) 3 (d) 0.

68. If $0 < x < \pi/2$ and $\sin(2\sin x) = \cos(2\cos x)$ then $\tan x + \cot x = \dfrac{a}{\pi^C - b}$ where $a + b + c =$
    (a) 32 (b) 16
    (c) 50 (d) 64

69. If $15\sin^4\alpha + 10\cos^4\alpha = 6$ then the value of $8\operatorname{cosec}^6\alpha - 27\sec^6\alpha$ is
    (a) 0 (b) 1
    (c) –1 (d) 5

70. If $\displaystyle\sum_{r=1}^{\infty} \tan^{-1}\frac{1}{2r^2} = t$, then $\tan t$ is equal to
    (a) 0 (b) 1
    (c) $\sqrt{3}$ (d) $1/\sqrt{3}$.

71. If $\theta \in [-2011\pi, 2011\pi]$, then the number of solutions of the equation.
    $[\sec\theta - 1] + \{\tan\theta \tan\frac{\theta}{2}\}$
    $= [\{\sec\theta - 1 + \tan\theta \tan\frac{\theta}{2}\}]$
    (where {·} denotes the fractional part function and [·] denotes the greatest integer function) is
    (a) 0 (b) 2010
    (c) 2011 (d) 2012.

72. If $x = \sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} + \sin\dfrac{8\pi}{7}$
    and $y = \cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{8\pi}{7}$, then $x^2 + y^2$ is
    (a) 1 (b) 2
    (c) 3 (d) 4.

73. Number of solutions of the equation
    $\sin\left(\dfrac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}x + 4$ is
    (a) 0 (b) 1
    (c) 2 (d) 4

Assertion-Reason Type Questions

74. **Statement-1:** $\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}\cos\dfrac{4\pi}{7} = -\dfrac{1}{8}$
    Statement-2: $\cos\theta\cos 2\theta\cos 4\theta \ldots \cos 2^{n-1}\theta = -\dfrac{1}{2^n}$ if $\theta = \dfrac{\pi}{2^n - 1}$.

75. **Statement-1:** Let $f(x) = \dfrac{\sin 2x}{1 + \cos 2x}$ and $g(x) = \dfrac{1 + \sin x - \cos x}{1 + \sin x + \cos x}$
    If $\beta, \gamma, \in (0, \pi)$ such that
    $$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + \beta + \gamma) = 0$$
    and $\quad \sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + \beta + \gamma) = 0$
    Then $\quad f(\beta) + g(\gamma) = 0$
    Statement-2: For all values of x, $f(x) + g(x) = 0$, $f(x)$ and $g(x)$ are functions defined in statement-1.

76. **Statement-1:** If x and y are real number such that $x^2 + y^2 = 27$, then the maximum possible value of $x - y$ is $3\sqrt{6}$.

Statement-2: $-1 \le \cos(\theta + \pi/4) \le 1$

77. **Statement-1:** If $p = 7 + \tan\alpha\tan\beta$, $q = 5 + \tan\beta\tan\gamma$ and $r = 3 + \tan\gamma\tan\alpha$, then the maximum value of $\sqrt{p} + \sqrt{q} + \sqrt{r}$ is $4\sqrt{3}$; $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = \pi/2$
Statement-2: If $\alpha + \beta + \gamma = \pi/2$, $\alpha, \beta, \gamma > 0$, then $\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1$.

78. **Statement-1:** If $x\sin^3\theta + y\cos^3\theta = \sin\theta\cos\theta$. and $x\sin\theta = y\cos\theta$, then $x^2 + y^2 = 1$
Statement-2: $4\sin^3\theta = 3\sin\theta - \sin 3\theta$
$4\cos^3\theta = 3\cos\theta + \cos 3\theta$

79. **Statement-1:** $2\sin^2 x + 3\sin x - 2 > 0$ and $x^2 - x - 2 < 0$, if $-1 < x < 2$.
Statement-2: $x^2 - x - 2 < 0$ if $-1 < x < 2$

80. **Statement-1:** $\cos^7 x + \sin^4 x = 1$ has only two non-zero solution in the interval $-\pi < x < \pi$.
Statement-2: $\cos^5 x + \cos^2 x - 2 = 0$ is possible only when $\cos x = 1$

LEVEL 2

Straight Objective Type Questions

81. The acute angle of a rhombus whose side is a mean proportional between its diagonals is
(a) 15° (b) 20°
(c) 30° (d) 80°

82. Given the height h and the angle bisector l drawn from the vertex of the right angle of a triangle, then cosine of an acute angle of the triangle is given by
(a) $\dfrac{h + \sqrt{l^2 - h^2}}{\sqrt{2}\,h}$ (b) $\dfrac{h - \sqrt{l^2 - h^2}}{\sqrt{2}\,h}$
(c) $\dfrac{h}{l}$ (d) $\dfrac{h - \sqrt{l^2 - h^2}}{\sqrt{2}\,l}$

83. If $2\sin^2(x + \pi/4) + \sqrt{3}\cos 2x > 0$, then
(a) $\cos(2x - \pi/6) > -1/2$ (b) $\sin(2x - \pi/6) < -1/2$
(c) $\sin(2x - \pi/6) > -1/2$ (d) $\cos(2x - \pi/6) < -1/2$

84. $x = \sum_{n=0}^{\infty}\cos^{2n}\theta,\ y = \sum_{n=0}^{\infty}\sin^{2n}\theta,\ z = \sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\theta,$
$|\cos\theta| < 1$, $|\sin\theta| < 1$ then $x + y + z$ is equal to
(a) xy (b) yz
(c) zx (d) xyz

85. The least positive value of x satisfying
$$\frac{\sin^2 2x + 4\sin^4 x - 4\sin^2 x\cos^2 x}{4 - \sin^2 2x - 4\sin^2 x} = \frac{1}{9} \text{ is}$$
(a) $\pi/3$ (b) $\pi/6$
(c) $2\pi/3$ (d) $5\pi/6$

86. $\cos 7.5° =$
(a) $\sqrt{\dfrac{2 + \sqrt{2} + \sqrt{6}}{8}}$ (b) $\sqrt{\dfrac{4 + \sqrt{2} + \sqrt{6}}{8}}$
(c) $\dfrac{2\sqrt{2} + \sqrt{3} + 1}{2\sqrt{2}}$ (d) $\sqrt{\dfrac{4 + \sqrt{2} + \sqrt{6}}{4}}$

87. If $x = X\cos\theta - Y\sin\theta$, $y = X\sin\theta + Y\cos\theta$; and $ax^2 + 2bxy + cy^2 = AX^2 + 2HXY + BY^2$, then
(a) $H = 0$ If $\theta = 0$ (b) $H = b$ if $\theta = \pi/2$
(c) $A + B = a + c$ (d) $H = c - a$ if $\theta = \pi/4$

88. If $\sin 5\theta = a\sin^5\theta + b\sin^3\theta + c\sin\theta + d$, then
(a) $a + b + c = 0$ (b) $a + b + c + d = 0$
(c) $5a + 3b - 4c = 0$ (d) $a - 3c + d = 0$

89. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then the value of $\cot(A - B)$ is
(a) $\dfrac{x - y}{xy}$ (b) $\dfrac{1}{x^2} + \dfrac{1}{y^2}$
(c) $\dfrac{x + y}{xy}$ (d) xy

90. $\dfrac{\cos 3\theta}{\cos^3\theta} + \dfrac{\sin 3\theta}{\sin^3\theta}$ is equal to
(a) $3\cos 2\theta\operatorname{cosec} 2\theta$ (b) $3\cot 2\theta\operatorname{cosec} 2\theta$
(c) $12\cot 2\theta\operatorname{cosec} 2\theta$ (d) $12\tan 2\theta\sec 2\theta$

91. $(x\tan\alpha + y\cot\alpha)(x\cot\alpha + y\tan\alpha) - 4xy\cot^2 2\alpha =$
(a) $x^2 + y^2$ (b) $4xy$
(c) $(x + y)^2$ (d) none of these

92. If $\tan A$, $\tan B$, $\tan C$ satisfy the equation $3\tan^3\theta - 4\tan^2\theta + 3\tan\theta + 1 = 0$, then $A + B + C =$
(a) 0 (b) $\pi/2$
(c) $3\pi/4$ (d) 2π

93. If $x \sin \theta + y \sin 2\theta + z \sin 3\theta = \sin 4\theta$, $(\theta \neq n\pi)$ then $8 \cos^3 \theta - 4z \cos^2 \theta - 2(y + 2) \cos \theta$ is equal to
(a) $x - y$ (b) $x - z$
(c) $y - z$ (d) none of these

94. The number of values of $\sin x$ satisfying $\sin 5x = 5 \sin x$ is
(a) 0 (b) 1
(c) 2 (d) 3

95. If $\sin \alpha$, $\sin \beta$ are the roots of the equation $a \sin^2 \theta + b \sin \theta + c = 0$ and $\sin \alpha + 2 \sin \beta = 1$ then $a^2 + 2b^2 + 3ab + ac =$
(a) -1 (b) 0
(c) 1 (d) $a + b + c$

96. If $\sin (\theta/2) = a$, $\cos (\theta/2) = b$, then $(1 + \sin \theta)(3 \sin \theta + 4 \cos \theta + 5) =$
(a) $(a + b)^2 (a + 3b)^2$ (b) $(a + b)^2 (3a + b)^2$
(c) $(a - b)^2 (a - 3b)^2$ (d) $(a - b)^2 (3a - b)^2$

97. If $\sin A = \sin B$ and $\cos A = \cos B$; $A \neq B$, then
(a) $\tan \dfrac{A - B}{2} = 0$ (b) $\cos (A + B) = 1$
(c) $\tan \dfrac{A + B}{2} = 0$ (d) $\sin (A - B) = 1/2$

98. $\cos 22° + \cos 78° + \cos 80° =$
(a) $4 \sin 11° \sin 39° \sin 40°$
(b) $1 + 4 \cos 11° \cos 39° \cos 40°$
(c) $1 + 4 \sin 11° \sin 39° \sin 40°$
(d) $4 \cos 11° \cos 39° \cos 40°$

99. $\tan x + \dfrac{1}{2} \tan \dfrac{x}{2} + \dfrac{1}{2^2} \tan \dfrac{x}{2^2} + \ldots + \dfrac{1}{2^{n-1}} \tan \dfrac{x}{2^{n-1}}$ is equal to
(a) $\dfrac{1}{2^n} \cot \dfrac{x}{2^n} - 2 \cot 2x$
(b) $\dfrac{1}{2^{n-1}} \cot \left(\dfrac{x}{2^{n-1}}\right) - 2 \cot 2x$
(c) $\tan \dfrac{2^n - 1}{2^{n-1}} x$
(d) $2 \cot 2x - \dfrac{1}{2^{n-1}} \cot \dfrac{x}{2^{n-1}}$

100. The value of $\dfrac{3 + \cot 76° \cot 16°}{\cot 76° + \cot 16°}$ is
(a) $\cot 44°$ (b) $\cot 46°$
(c) $\tan 2°$ (d) $\cot 92°$

101. If $x \cos \theta = y \cos (\theta + 2\pi/3) = z \cos (\theta + 4\pi/3)$ then $xy + yz + zx =$
(a) $\cos^2 \theta$ (b) $\sin^2 \theta$
(c) 1 (d) 0

102. If $A > 0$, $B > 0$ and $A + B = \pi/3$ then the maximum value of $\tan A \tan B$ is
(a) $1/\sqrt{3}$ (b) $1/3$
(c) $\sqrt{3}$ (d) 3

103. If $\tan \theta$, $2 \tan \theta + 2$, $3 \tan \theta + 3$ are in G.P., then the value of $\dfrac{7 - 5 \cot \theta}{9 - 4\sqrt{\sec^2 \theta - 1}}$ is
(a) $12/5$ (b) $-33/28$
(c) $33/100$ (d) $12/13$

104. If $\sin \theta + \cos \theta = a$ and $\cos \theta - \sin \theta = b$, then $\sin \theta (\sin \theta - \cos \theta) + \sin^2 \theta (\sin^2 \theta - \cos^2 \theta) + \sin^3 \theta (\sin^3 \theta - \cos^3 \theta) + \ldots$ is equal to
(a) $\dfrac{1 - ab}{1 + ab}$ (b) $\dfrac{1 - a^2}{3 - a^2}$
(c) $\dfrac{1 - ab}{1 + ab} + \dfrac{1 - a^2}{3 - a^2}$ (d) $\dfrac{1 + ab}{1 - ab} - \dfrac{a^2 - 1}{3 - a^2}$

105. If $\left(1 + \sqrt{1 + x}\right) \tan \alpha = \left(1 - \sqrt{1 - x}\right)$ then $x =$
(a) $\sin \alpha$ (b) $\sin 2\alpha$
(c) $\sin 4\alpha$ (d) $\cos 4\alpha$

106. If $f(\theta) = \sin \theta (\sin \theta + \sin 3\theta)$, then $f(\theta)$
(a) ≥ 0 only when $\theta \geq 0$ (b) ≤ 0 for all real θ
(c) ≥ 0 for all real θ (d) ≤ 0 only when $\theta \leq 0$

107. In a right angled triangle, the hypotenuse is $2\sqrt{2}$ times the length of the perpendicular drawn from the opposite vertex on its hypotenuse then the other two angles are
(a) $\pi/3, \pi/6$ (b) $\pi/4, \pi/4$
(c) $\pi/8, 3\pi/8$ (d) $\pi/12, 5\pi/12$

108. $\sqrt{\cos 2x} + \sqrt{1 + \sin 2x} = 2\sqrt{\sin x + \cos x}$ if
(a) $\sin x + \cos x = 1$ (b) $x = 2n\pi$
(c) $x = n\pi + \pi/4$ (d) $\sin x - \cos x = 0$

109. $\sin x + 2 \sin 2x = 3 + \sin 3x$
(a) if $\sin x + \cos 2x = 0$
(b) if $\sin 2x - 1 = 0$
(c) if $\cos x = 0$
(d) for no real value of x

110. The number of pairs (x, y) satisfying the equation $\sin x + \sin y = \sin (x + y)$ and $|x| + |y| = 1$ is
(a) 2 (b) 4
(c) 6 (d) infinite

Previous Years' AIEEE/JEE Main Questions

1. The numbers of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$ is
   (a) 2 (b) 3
   (c) 0 (d) 1 **[2003]**

2. Let α, β be such that $\pi < \alpha - \beta < 3\pi$. If $\sin\alpha + \sin\beta = -21/65$ and $\cos\alpha + \cos\beta = -27/65$ then the values of $\cos\dfrac{\alpha-\beta}{2}$ is
   (a) $6/65$ (b) $3/\sqrt{130}$
   (c) $-3/\sqrt{130}$ (d) $-6/65$ **[2004]**

3. If $u = \sqrt{a^2\cos^2\theta + b^2\sin^2\theta} + \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$, then the difference between the maximum and minimum values of u^2 is given by
   (a) $(a+b)^2$ (b) $2\sqrt{a^2+b^2}$
   (c) $2(a^2+b^2)$ (d) $(a-b)^2$ **[2004]**

4. If $f : R \to S$ is defined by $f(x) = \sin x - \sqrt{3}\cos x + 1$ is onto, then the interval of S is
   (a) $[0, 1]$ (b) $[-1, 1]$
   (c) $[0, 3]$ (d) $[-1, 3]$ **[2004]**

5. The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is
   (a) 2 (b) 4
   (c) 6 (d) 1 **[2006]**

6. If the roots of the quadratic equation $x^2 + px + q = 0$ are tan 30° and tan 15°, then the value of $2 + q - p$ is
   (a) 1 (b) 2
   (c) 3 (d) 0. **[2006]**

7. If $0 < x < \pi$, and $\cos x + \sin x = 1/2$, then $\tan x$ is
   (a) $(1+\sqrt{7})/4$ (b) $(1-\sqrt{7})/4$
   (c) $(4-\sqrt{7})/3$ (d) $-(4+\sqrt{7})/3$ **[2006]**

8. Let A and B denote the statements
   $A : \cos\alpha + \cos\beta + \cos\gamma = 0$
   $B : \sin\alpha + \sin\beta + \sin\gamma = 0$
   If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -3/2$ then
   (a) both A and B are true
   (b) both A and B are false
   (c) A is ture and B is false
   (d) A is false and B is true. **[2009]**

9. For a regular polygon, let r, R be the radii of the inscribed and the circumscribed circles. A **false** statement among the following is
   (a) There is a regular polygon with $r/R = 3/3$
   (b) There is a regular polygon with $r/R = \sqrt{3}/2$
   (c) There is a regular polygon with $r/R = 1/2$
   (d) There is a regular polygon with $r/R = 1/\sqrt{2}$ **[2010]**

10. Let $\cos(\alpha + \beta) = 4/5$ and let $\sin(\alpha - \beta) = 5/13$ where $0 \le \alpha, \beta \le \pi/4$. Then $\tan 2\alpha =$
    (a) 19/12 (b) 20/7
    (c) 25/16 (d) 56/33 **[2010]**

11. If $A = \sin^2 x + \cos^4 x$, then for all real x
    (a) $3/4 \le A \le 13/16$ (b) $3/4 \le A \le 1$
    (c) $13/16 \le A \le 1$ (d) $1 \le A \le 2$ **[2011]**

12. The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are:
    (a) $\dfrac{\pi}{4}, \dfrac{5\pi}{12}, \dfrac{\pi}{12}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}, \dfrac{8\pi}{9}$
    (b) $\dfrac{2\pi}{9}, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}, \dfrac{35\pi}{36}$
    (c) $\dfrac{2\pi}{9}, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{4}, \dfrac{8\pi}{9}$
    (d) $\dfrac{2\pi}{9}, \dfrac{\pi}{4}, \dfrac{4\pi}{9}, \dfrac{\pi}{2}, \dfrac{3\pi}{4}, \dfrac{8\pi}{9}$ **[2011]**

13. In a ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to
    (a) $\pi/4$ (b) $3\pi/4$
    (c) $5\pi/6$ (d) $\pi/6$ **[2012]**

14. $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\underline{|ADB} = \theta$, $BC = p$ and $CD = q$, then AB is equal to
    (a) $\dfrac{p^2 + q^2\cos\theta}{p\cos\theta + q\sin\theta}$ (b) $\dfrac{p^2+q^2}{p^2\cos\theta + q^2\sin\theta}$
    (c) $\dfrac{(p^2+q^2)\cos\theta}{(p\cos\theta + q\sin\theta)^2}$ (d) $\dfrac{(p^2+q^2)\sin\theta}{p\cos\theta + q\sin\theta}$ **[2013]**

15. The expression $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}$ can be written as
(a) $\sec A \operatorname{cosec} A + 1$ (b) $\tan A + \cot A$
(c) $\sec A + \operatorname{cosec} A$ (d) $\sin A \cos A + 1$ **[2013]**

16. **Statement-1:** The number of common solutions of the trigonometric equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is two.

Statement-2: The number of solutions of the equation, $2\cos^2\theta - 3\sin\theta = 0$ in the interval $[0, \pi]$ is two **[2013, online]**

17. The number of solutions of the equation $\sin 2x - 2\cos x + 4\sin x = 4$ in the interval $[0, 5\pi]$ is:
(a) 3 (b) 4
(c) 5 (d) 6 **[2013, online]**

18. Let $f_k(x) = (1/k)\left(\sin^k x + \cos^k x\right)$ where $x \in \mathbf{R}$ and $k \geq 1$. Then $f_4(x) - f_6(x)$ equals:
(a) 1/6 (b) 1/3
(c) 1/4 (d) 1/12 **[2014]**

19. The number of values of α in $[0, 2\pi]$ for which $2\sin^3\alpha - 7\sin^2\alpha + 7\sin\alpha = 2$, is:
(a) 6 (b) 4
(c) 3 (d) 1 **[2014, online]**

20. If $\operatorname{cosec}\theta = \dfrac{p+q}{p-q}$ $(p \neq q \neq 0)$, then $\left|\cot\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right)\right|$ is equal to:
(a) $\sqrt{\dfrac{p}{q}}$ (b) $\sqrt{\dfrac{q}{p}}$
(c) $\sqrt{pq}$ (d) pq **[2014, online]**

21. If $2\cos\theta + \sin\theta = 1$, $(\theta \neq \pi/2)$, then $7\cos\theta + 6\sin\theta$ is equal to:
(a) 1/2 (b) 2
(c) 11/2 (d) 46/5 **[2014, online]**

22. If $f(\theta) = \begin{vmatrix} 1 & \cos\theta & 1 \\ -\sin\theta & 1 & -\cos\theta \\ -1 & \sin\theta & 1 \end{vmatrix}$ and A and B are respectively the maximum and the minimum values of $f(\theta)$, then (A, B) is equal to
(a) $(3, -1)$ (b) $\left(4, 2-\sqrt{2}\right)$
(c) $\left(2+\sqrt{2}, 2-\sqrt{2}\right)$ (d) $\left(2+\sqrt{2}, -1\right)$
[2014, online]

23. If $\cos\alpha + \cos\beta = \dfrac{3}{2}$ and $\sin\alpha + \sin\beta = \dfrac{1}{2}$ and θ is the arithmetic mean of α and β, then $\sin 2\theta$ is equal to:
(a) $\dfrac{3}{5}$ (b) $\dfrac{4}{5}$
(c) $\dfrac{7}{5}$ (d) $\dfrac{8}{5}$ **[2015, online]**

24. In a ΔABC, $\dfrac{a}{b} = 2+\sqrt{3}$ and $\underline{|c} = 60°$. Then the ordered pair $(\underline{|A}, \underline{|B})$ is equal to:
(a) (15°, 105°) (b) (105°, 15°)
(c) (45°, 75°) (d) (75°, 45°)
[2015, online]

25. If $0 \leq x < 2\pi$, then the number of real values of x which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is:
(a) 3 (b) 5
(c) 7 (d) 9 **[2016]**

26. If m and M are the minimum and the maximum values of $4+\dfrac{1}{2}\sin^2 2x - 2\cos^4 x$, $x \in \mathbf{R}$, then $M - m$ is equal to:
(a) $\dfrac{9}{4}$ (b) $\dfrac{15}{4}$
(c) $\dfrac{7}{4}$ (d) $\dfrac{1}{4}$ **[2016, online]**

27. The number of $x \in [0, 2\pi]$ for which $\left|\sqrt{2\sin^4 x + 18\cos^2 x} - \sqrt{2\cos^4 x + 18\sin^2 x}\right| = 1$ is:
(a) 2 (b) 6
(c) 4 (d) 8 **[2016, online]**

28. The number of distinct real roots of the equation,
$$\Delta = \begin{vmatrix} \cos x & \sin x & \sin x \\ \sin x & \cos x & \sin x \\ \sin x & \sin x & \cos x \end{vmatrix} = 0$$
in the interval $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ is:
(a) 1 (b) 4
(c) 2 (d) 3 **[2016, online]**

29. If $A > 0$, $B > 0$ and $A+B = \dfrac{\pi}{6}$, then the minimum value of $\tan A + \tan B$ is:
(a) $\sqrt{3}-\sqrt{2}$ (b) ü – $\sqrt{\ }$
(c) $\dfrac{2}{\sqrt{3}}$ (d) $2-\sqrt{3}$ **[2016, online]**

30. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then:
(a) $P \subset Q$ and $Q - P \neq \phi$
(b) $Q \not\subset P$
(c) $P = Q$ (d) $P \not\subset Q$ **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. The number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, 2\pi]$ is
(a) 1 (b) 2
(c) 3 (d) 0 **[2006]**

2. For $\theta \neq 0$, if $\cos\theta + \sec\theta = 2$, then $\cos^n\theta + \sec^n\theta$ equals
(a) 2^n (b) $(-2)^n$
(c) 2^{n+1} (d) 2 **[2006]**

3. If for real p the equation $\sin^6 x + \cos^6 x = p$ always has a real solutions, then p lies in the interval:
(a) $]1, 2]$ (b) $\left[\frac{1}{4}, 1\right[$
(c) $\left[\frac{1}{4}, 1\right]$ (d) $[1, 2]$ **[2007]**

4. The value of $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$ is
(a) 3/2 (b) 1/2
(c) 3/4 (d) 1/4 **[2008]**

5. The value of $\sin\frac{\pi}{14}\cdot\sin\frac{3\pi}{14}\cdot\sin\frac{5\pi}{14}$ is
(a) 1/4 (b) 1/8
(c) 1/2 (d) 1/16 **[2008]**

6. Let $0 < \alpha, \beta \leq \pi/4$. If $\cos(\alpha + \beta) = 4/5$ and $\sin(\alpha - \beta) = 5/13$ then the value of $\tan 2\alpha$ is:
(a) 19/12 (b) 56/33
(c) 20/7 (d) 25/16 **[2009]**

7. The sides of a triangle are $\sin\beta, \cos\beta$ and $\sqrt{1 + a\sin 2\beta}$, for some $a > 0$, $0 < \beta < \frac{\pi}{2}$. If the greatest angle of this triangle is 120°, then a equals
(a) 1 (b) 3/2
(c) 2/3 (d) 1/2 **[2010]**

8. If $\cos 25° + \sin 25° = K$, then $\cos 50°$ equals
(a) $-K\sqrt{2-K^2}$ (b) $K\sqrt{1-K^2}$
(c) $-K\sqrt{1-K^2}$ (d) $K\sqrt{2-K^2}$ **[2010]**

9. $\tan 9° - \tan 27° - \tan 63° + \tan 81°$ is equal to:
(a) 4 (b) 0
(c) 1 (d) -1 **[2011, 12]**

10. If a and b are such that $a\sin\theta = b\cos\theta$ for $0 \leq \theta < \frac{\pi}{4}$, then $\sqrt{\frac{a-b}{a+b}} + \sqrt{\frac{a+b}{a-b}}$ equals
(a) $\frac{2}{\sqrt{\cos 2\theta}}$ (b) $2\cos\theta$
(c) $\frac{2\cos\theta}{\sqrt{\cos 2\theta}}$ (d) $\frac{2\sin\theta}{\sqrt{\cos 2\theta}}$

11. The value of $\tan 9° - \tan 27° - \tan 63° + \tan 81°$ is
(a) 4 (b) 3
(c) 2 (d) 0 **[2012]**

12. In a ΔABC, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, then $\sin^2 A + \sin^2 B + \sin^2 C$ is:
(a) 9/4 (b) 5/4
(c) 2 (d) $3\sqrt{3}/2$ **[2014]**

13. Let $\frac{3\pi}{4} < \theta < \pi$ and $\sqrt{2\cot\theta + \frac{1}{\sin^2\theta}} = K - \cot\theta$, then K equals
(a) -1 (b) 0
(c) 1/2 (d) 1 **[2014]**

14. The maximum value of $f(x) = 2\sin x + \sin 2x$, in the interval $\left[0, \frac{3\pi}{2}\right]$ is:
(a) $\sqrt{2}+1$ (b) $2\sqrt{3}$
(c) $\frac{3\sqrt{3}}{2}$ (d) $\sqrt{3}$ **[2015]**

15. For all values of $\theta \in \left(0, \frac{\pi}{2}\right)$, the determinant of the matrix $\Delta = \begin{bmatrix} -2 & \tan\theta + \sec^2\theta & 3 \\ -\sin\theta & \cos\theta & \sin\theta \\ -3 & -4 & 3 \end{bmatrix}$ always lies in the interval:
(a) $[3, 5]$ (b) $(4, 6)$
(c) $\left(\frac{5}{2}, \frac{19}{4}\right)$ (d) $\left[\frac{7}{2}, \frac{21}{4}\right]$ **[2016]**

Answers

Concept-based

1. (b) **2.** (c) **3.** (b) **4.** (d)
5. (a) **6.** (a) **7.** (a) **8.** (c)
9. (a) **10.** (b) **11.** (b) **12.** (a)

13. (c) **14.** (a) **15.** (b) **16.** (d)
17. (b) **18.** (a) **19.** (b) **20.** (c)

Level 1

21. (d) **22.** (b) **23.** (b) **24.** (c)
25. (d) **26.** (c) **27.** (c) **28.** (a)
29. (b) **30.** (b) **31.** (b) **32.** (a)
33. (b) **34.** (d) **35.** (a) **36.** (a)
37. (c) **38.** (c) **39.** (b) **40.** (c)
41. (a) **42.** (a) **43.** (d) **44.** (d)
45. (a) **46.** (d) **47.** (c) **48.** (c)
49. (b) **50.** (b) **51.** (c) **52.** (d)
53. (c) **54.** (b) **55.** (d) **56.** (c)
57. (d) **58.** (b) **59.** (d) **60.** (b)
61. (b) **62.** (d) **63.** (a) **64.** (d)
65. (a) **66.** (b) **67.** (c) **68.** (c)
69. (a) **70.** (b) **71.** (c) **72.** (b)
73. (c) **74.** (a) **75.** (c) **76.** (a)
77. (c) **78.** (b) **79.** (d) **80.** (a)

Level 2

81. (c) **82.** (d) **83.** (a) **84.** (d)
85. (b) **86.** (b) **87.** (c) **88.** (c)
89. (c) **90.** (c) **91.** (c) **92.** (b)
93. (b) **94.** (b) **95.** (b) **96.** (a)
97. (a) **98.** (c) **99.** (b) **100.** (a)
101. (d) **102.** (b) **103.** (c) **104.** (c)
105. (c) **106.** (c) **107.** (c) **108.** (b)
109. (d) **110.** (c)

Previous Years' AIEEE/JEE Main Questions

1. (a) **2.** (c) **3.** (d) **4.** (d)
5. (b) **6.** (c) **7.** (d). **8.** (d)
9. (a) **10.** (d) **11.** (b) **12.** (d)
13. (d) **14.** (d) **15.** (a) **16.** (b)
17. (a) **18.** (a) **19.** (c) **20.** (a)
21. (b) **22.** (c) **23.** (a) **24.** (b)
25. (c) **26.** (a) **27.** (d) **28.** (c)
29. (b) **30.** (c)

Previous Years' B-Architecture Entrance Examination Questions

1. (b) **2.** (d) **3.** (c) **4.** (a)
5. (b) **6.** (b) **7.** (d) **8.** (d)
9. (a) **10.** (c) **11.** (a) **12.** (a)
13. (a) **14.** (c) **15.** (a)

Hints and Solutions

Concept-based

1. $$f(\theta) = \frac{1}{\tan\theta}\frac{1}{(1-\tan\theta)} + \frac{\tan\theta\cdot\tan\theta}{\tan\theta - 1}$$
$$= \frac{1-\tan^3\theta}{\tan\theta(1-\tan\theta)} = \frac{1+\tan\theta+\tan^2\theta}{\tan\theta}$$
$$= 1 + \frac{2}{\sin 2\theta}$$
$$\therefore f\left(\frac{9\pi}{8}\right) = 1 + \frac{2}{\sin\left(\frac{9\pi}{4}\right)} = 1 + 2\sqrt{2}$$

2. $$f(\theta) = \frac{\sin^3\theta}{\cos\theta} - \frac{\cos^3\theta}{\sin\theta}$$
$$= \frac{\sin^4\theta - \cos^4\theta}{\cos\theta\sin\theta} = \frac{-(\cos^2\theta - \sin^2\theta)(1)}{\cos\theta\sin\theta}$$
$$= -2\cot(2\theta)$$

3. $$\frac{\tan\theta + \sec\theta - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1}$$
$$= \frac{(\sec\theta + \tan\theta)(1 - \sec\theta + \tan\theta)}{\tan\theta - \sec\theta + 1}$$
$$= \frac{1+\sin\theta}{\cos\theta}$$

4. $$\tan 15° + \tan 75° = \tan 15° + \cot 15°$$
$$= \frac{\tan^2 15° + 1}{\tan 15°}$$
$$= 2\left(\frac{1}{\sin 30°}\right) = 4$$

5. Use : $\sin^2 A - \sin^2 B = \sin(A - B)\sin(A + B)$

6. $$\frac{\cos 11° - \sin 11°}{\cos 11° + \sin 11°} = \frac{\cos 11° + \cos(90° + 11°)}{\cos 11° - \cos(90° + 11°)}$$
$$= \frac{2\cos 56°\cos 45°}{2\sin 56°\sin 45°} = \cot 56°$$

7. $\tan(\alpha + \beta) = \tan(\pi/2 - \beta)$
$$\Rightarrow \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \cot\beta$$
$\Rightarrow \tan\alpha + \tan\beta = \cot\beta - \tan\alpha$
$\Rightarrow 2\tan\alpha + \tan\beta = \cot\beta = \tan(\alpha + \beta)$

8. $\sin\theta + \operatorname{cosec}\theta = 2 \Rightarrow \sin\theta + \dfrac{1}{\sin\theta} = 2$

$\Rightarrow \sin\theta = 1 = \operatorname{cosec}\theta$

$\therefore \cos\theta = 0$. Thus, $\cos^{2015}\theta + \operatorname{cosec}^{2015}\theta = 1$

9. Multiply both the sides by $(1 - \sin A)(1 - \sin B)(1 - \sin C)$ to obtain $(1 - \sin A)^2(1 - \sin B)^2(1 - \sin C)^2$ $= \cos^2 A \cos^2 B \cos^2 C$ $\Rightarrow (1 - \sin A)(1 - \sin B)(1 - \sin C)$ $= \pm \cos A \cos B \cos C$

10. $$2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13}$$

$$= 2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + 2\cos\frac{\pi}{13}\cos\frac{4\pi}{13}$$

$$= 2\cos\frac{\pi}{13}\left[\cos\left(\frac{9\pi}{13}\right) + \cos\left(\frac{4\pi}{13}\right)\right]$$

$$= 2\cos\left(\frac{\pi}{13}\right)\left[2\cos\frac{\pi}{2}\cos\frac{5\pi}{26}\right] = 0$$

11. $\sec A - \sec B = \operatorname{cosec} B - \operatorname{cosec} A$

$$\Rightarrow \frac{\cos B - \cos A}{\cos A \cos B} = \frac{\sin A - \sin B}{\sin A \sin B}$$

$$\Rightarrow \frac{\sin A \sin B}{\cos A \cos B} = \frac{\sin A - \sin B}{\cos B - \cos A}$$

$$\Rightarrow \tan A \tan B = \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

$$= \cot\left(\frac{A+B}{2}\right)$$

12. Use $1 + \cos 2A = 2\cos^2 A$.

13. $$f(\theta) = \frac{1}{2\sin\theta}(2\sin\theta\cos\theta)\cos 2\theta\cos 4\theta\cos 7\theta$$

$$= \frac{1}{2\times 2\sin\theta}(2\sin 2\theta\cos 2\theta)\cos 4\theta\cos 7\theta$$

$$= \frac{1}{2\times 4\sin\theta}(2\sin 4\theta\cos 4\theta)\cos 7\theta$$

$$= \frac{1}{2\times 8\sin\theta}[2\sin 8\theta\cos 7\theta]$$

$$= \frac{1}{16\sin\theta}[\sin(15\theta) + \sin\theta]$$

When $\theta = \dfrac{\pi}{15}$, $f(\theta) = \dfrac{1}{16}$.

14. As $\dfrac{\pi}{14} = \dfrac{\pi}{2} - \dfrac{3\pi}{7}$, $\dfrac{3\pi}{14} = \dfrac{\pi}{2} - \dfrac{2\pi}{7}$

and $\dfrac{5\pi}{14} = \dfrac{\pi}{2} - \dfrac{\pi}{7}$, we can write

$$f\left(\frac{\pi}{14}\right) = \cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}.$$

$$\Rightarrow 2\sin\frac{\pi}{7}f\left(\frac{\pi}{14}\right) = \sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$$

$$\Rightarrow 4\sin\frac{\pi}{7}f\left(\frac{\pi}{14}\right) = \sin\frac{4\pi}{7}\cos\frac{3\pi}{7}$$

$$\Rightarrow 8\sin\left(\frac{\pi}{7}\right)f\left(\frac{\pi}{14}\right) = \sin\left(\frac{7\pi}{7}\right) + \sin\left(\frac{\pi}{7}\right) = \sin\frac{\pi}{7}$$

$$\Rightarrow f\left(\frac{\pi}{14}\right) = \frac{1}{8}$$

15. $\sin^2 A + \sin^2 B + \sin^2 C + 2\sin A\sin B\sin C$

$= 1 - (\cos^2 A - \sin^2 B) + \sin C\,[\sin C + 2\sin A\sin B]$

$= 1 - \cos(A + B)\cos(A - B) + \sin C\,[\cos(A + B) + 2\sin A\sin B]$

$[\because C = \pi/2 - A - B]$

$= 1 - \cos(\pi/2 - C)\cos(A - B) + \sin C\cos(A - B)$

$= 1$

16. $\tan 5x = -\cot 3x = \tan(\pi/2 + 3x)$

$\Rightarrow 5x = n\pi + \dfrac{\pi}{2} + 3x$, $n \in \mathbf{I}$

$\Rightarrow 2x = (2n + 1)\dfrac{\pi}{2}$, $n \in \mathbf{I}$

$\Rightarrow x = (2n + 1)\dfrac{\pi}{4}$, $n \in \mathbf{I}$.

Now, $x \in [0, 4\pi]$

$\Rightarrow 0 \le (2n + 1)\dfrac{\pi}{4} \le 4\pi$

$\Rightarrow -\dfrac{1}{2} \le n \le \dfrac{15}{2} \Rightarrow 0 \le n \le 7$

$\therefore$ There are 8 values of x.

17. $\sin x = \tan x \Rightarrow \sin x(\cos x - 1) = 0$

$\Rightarrow \sin x = 0$ or $\cos x = 1$.

However, when $\cos x = 1$, then $\sin x = 0$

$\therefore \sin x = 0 \Rightarrow x = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi$

18. $\sin x + \cos x = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right) \le \sqrt{2} < \dfrac{3}{2}$

Thus, no values of x.

19. $\tan^2 x = 1 \Rightarrow \tan x = \pm 1$

$\tan x = 1 \Rightarrow x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}, \dfrac{9\pi}{4}, \dfrac{13\pi}{4}, \dfrac{17\pi}{4}$

and $\tan x = -1 \Rightarrow x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \dfrac{11\pi}{4}, \dfrac{15\pi}{4}, \dfrac{19\pi}{4}$

Thus, there are 10 values of x.

20. $$\frac{\sec(8\theta) - 1}{\sec(4\theta) - 1}\cdot\frac{\tan(2\theta)}{\tan(8\theta)}$$

$$= \frac{1-\cos(8\theta)}{1-\cos(4\theta)}\cdot\frac{\cos(4\theta)}{\cos(8\theta)}\cdot\frac{\sin(2\theta)}{\cos(2\theta)}\cdot\frac{\cos(8\theta)}{\sin(8\theta)}$$

$$= \frac{2\sin^2(4\theta)}{2\sin^2(2\theta)}\,\frac{\cos(4\theta)\sin(2\theta)}{\cos(2\theta)\sin(8\theta)} = 1$$

Level 1

21. $\tan(\beta - \alpha) = \dfrac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha} = \dfrac{\tan\alpha + \cot\beta}{1+\tan\beta\tan\alpha}$

$= \dfrac{\tan\alpha\tan\beta + 1}{\tan\beta\,(1+\tan\alpha\tan\beta)}$

$= \cot\beta.$

22. $\dfrac{\cos(x-y)}{\cos(x+y)} = a$

$\Rightarrow \dfrac{\cos(x-y)+\cos(x+y)}{\cos(x-y)-\cos(x+y)} = \dfrac{a+1}{a-1}$

$\Rightarrow \cot x\cot y = \dfrac{a+1}{a-1}$

23. $\sin^2 A + \sin^2(A-B) + [\sin(A+B) + \sin(A-B)]\sin(B-A)$

$= \sin^2 A + \sin^2(A-B) + \sin(A+B)\sin(B-A) - \sin^2(A-B)$

$= \sin^2 A + \sin^2 B - \sin^2 A = \sin^2 B.$

24. $\dfrac{6\sin\theta\cos\theta}{8\cos^2\theta+1} = \dfrac{6\tan\theta}{9+\tan^2\theta} = 1$

$\Rightarrow \tan^2\theta - 6\tan\theta + 9 = 0$

$\Rightarrow \tan\theta = 3$

25. $\dfrac{x}{a} = \dfrac{\sin\theta}{\cos^4\theta},\ \dfrac{y}{b} = \dfrac{\sin^3\theta}{\cos^4\theta} \Rightarrow \sin^2\theta = \dfrac{y}{b}\times\dfrac{a}{x} = \dfrac{ay}{bx}$

26. $\cot\theta - \cot 3\theta = \dfrac{\cos\theta\sin 3\theta - \sin\theta\cos 3\theta}{\sin\theta\sin 3\theta}$

$= \dfrac{\sin 2\theta}{\sin\theta\sin 3\theta}$

$= 2\cos\theta\,\mathrm{cosec}\,3\theta.$

27. $\cos(x-y) = 1 \Rightarrow x - y = 0$ (Note $x - y \neq 2\pi$)

$\Rightarrow x = y.$

$\cos(x+y) = -1/2 \Rightarrow x + y = 120°$ or $240° \Rightarrow x = y = 60°$

or $x = y = 120°.$

28. $\sin 2B = (3/2)\sin 2A,\ \cos 2B = 3\sin^2 A.$

$\Rightarrow \tan 2B = \cot A \Rightarrow \tan A\tan 2B = 1$

$\Rightarrow A + 2B = \pi/2$

$\Rightarrow B = \pi/4 - A/2$

29. $x^3 - px^2 - r = (x - \tan\alpha)(x - \tan\beta)(x - \tan\gamma)$

$\Rightarrow i^3 - pi^2 - r = (i - \tan\alpha)(i - \tan\beta)(i - \tan\gamma)$

and $-i^3 - pi^2 - r = (-i - \tan\alpha)(-i - \tan\beta)(-i - \tan\gamma)$

Multiplying we get

$(p - r)^2 - (i)^2 = -[((i)^2 - \tan^2\alpha)(i^2 - \tan^2\beta)(i^2 - \tan^2\gamma)$

$\Rightarrow (p - r)^2 + 1 = (1 + \tan^2\alpha)(1 + \tan^2\beta)(1 + \tan^2\gamma)$

30. $A > \pi/2 \quad B + C < \pi/2 \Rightarrow \tan(B + C) > 0$

$\Rightarrow \dfrac{\tan B + \tan C}{1 - \tan B\tan C} > 0$

$\Rightarrow \tan B\tan C < 1$ (as $\tan B$, $\tan C > 0$)

31. $\dfrac{\sin(\theta/2)}{\cos(\theta/2)} = \dfrac{1-\sin^2\theta}{\sin\theta} \Rightarrow 2\sin^2(\theta/2) = \cos^2\theta$

$\Rightarrow \cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \cos\theta = \dfrac{-1\pm\sqrt{1+4}}{2} = \dfrac{-1\pm\sqrt{5}}{2}$

But $\cos\theta \neq \dfrac{-1-\sqrt{5}}{2}$ so $\cos\theta = \dfrac{\sqrt{5}-1}{2}$

$\Rightarrow 2\cos^2(\theta/2) = \dfrac{\sqrt{5}-1}{2} + 1 = \dfrac{\sqrt{5}+1}{2}$

$\Rightarrow \cos^2(\theta/2) = \dfrac{\sqrt{5}+1}{4} = \cos 36°.$

32. $a\tan^2\theta + b = 1 + \tan^2\theta \Rightarrow (a - 1)\tan^2\theta = 1 - b$

and $(b - 1)\tan^2\phi = 1 - a \Rightarrow \dfrac{\tan^2\theta}{\tan^2\phi} = \dfrac{(1-b)^2}{(1-a)^2} = \dfrac{b^2}{a^2}$

$\Rightarrow (a - b)(a + b - 2ab) = 0$

$\Rightarrow a + b = 2ab$ [as $a \neq b$]

33. $a = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$\Rightarrow \sin^2 2x = -2(a - 1) \Rightarrow 1 - \cos 4x = -4(a - 1)$

$\Rightarrow \cos 4x = 4a - 3$ so $-1 \le 4a - 3 \le 1$

$\Rightarrow 1/2 \le a \le 1$

34. For $x = n\pi + \pi/2$, $\tan x$ is infinite and all other graphs have finite values.

35. $x + y + z = \pi \Rightarrow \tan x + \tan y + \tan z = \tan x\tan y\tan z$

$\Rightarrow 2k + 3k + 5k = 2k \times 3k \times 5k \Rightarrow k^2 = 1/3.$

$\tan^2 x + \tan^2 y + \tan^2 z = (1/3)(4 + 9 + 25) = 38/3.$

36. $B = 60°$, $\sin(2A + B) = 1/2 \Rightarrow 2A + B = 150°$

$\Rightarrow A = 45°,\ C = 75°$

so that $B + 2C = 210° = 180° + 30°$

$\Rightarrow \sin(B + 2C) = -\sin 30° = -1/2.$

37. $\dfrac{a}{b} = \tan\theta\tan\phi,\ \tan^2\alpha = \tan^2(\theta - \varphi)$

$= \left(\dfrac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}\right)^2$

$$= \frac{a^2 - 4(a/b)}{(1+(a/b))^2} = \frac{ab\,(ab-4)}{(a+b)^2}$$

Since $\tan^2 a > 0$, $ab > 4$.

38. $\dfrac{x}{\cos A} = \dfrac{y}{\cos B}$ so $\dfrac{x \tan A + y \tan B}{x+y}$

$$= \frac{\cos A \tan A + \cos B \tan B}{\cos A + \cos B}$$

$$= \frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}}{2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}}$$

$$= \tan \frac{A+B}{2}.$$

39. $x^2 + y^2 = a^2[\cos^2\theta + \sin^2\theta + \theta^2 \sin^2\theta + \theta^2 \cos^2\theta]$
$= a^2(1 + \theta^2)$

$\Rightarrow \quad a\theta = \sqrt{x^2 + y^2 - a^2}$

40. We can write $(1 + \cos(\pi/8))\,(1 + \cos(3\pi/8))$
$(1 - \cos(3\pi/8))\,(1 - \cos(\pi/8))$
$= (1 - \cos^2(\pi/8))\,(1 - \cos^2(3\pi/8))$
$= \sin^2 \pi/8 \sin^2 3\pi/8$
$= \sin^2 \pi/8 \cos^2 \pi/8 = (1/4) \sin^2 \pi/4) = 1/8.$

41. $\cos y = 2 \times 2 \cos 2x \cos x$
$\sin y = 2 \times 2 \sin 2x \cos x$
$\Rightarrow 1 = \cos^2 y + \sin^2 y = 16 \cos^2 x$
So $\cos 2x = 2 \cos^2 x - 1 = -7/8$.

42. $\cos A = 3/5 \Rightarrow \sin A = -4/5$, $\cos B = 4/5$
$\Rightarrow \sin B = 3/5$.

43. $\log_3 \tan 1° \tan 2° \ldots \tan 44° \tan 45° \cot 44° \ldots \cot 1° = \log_3 1 = 0$

44. $\cot^n \theta = \dfrac{\cos^5\theta}{\sin^3\theta} \times \dfrac{\cos^2\theta}{\sin\theta} \times \dfrac{\cos\theta}{\sin^4\theta} = \cot^8 \theta$

$\Rightarrow n = 8$

45. Given equation is possible if $\sin\theta = \sin 2\theta = \sin 3\theta = \sin 4\theta = 1$ for some θ, $0 < \theta < \pi$ which is not possible.

46. $\cos 11° < \cos 2° \Rightarrow \cos 11° - \cos 2° < 0$, $\cos 11°$ and $\cos 2°$ are rational numbers. $\cos 11° - \cos 2°$ is a negative rational number.

47. $\cos^2 A = \sin A \tan A = \sin^2 A \,.\, \sec A$
$\Rightarrow \cot^2 A = \sec \mathrm{A}$.

$\therefore \quad \cot^6 \mathrm{A} - \cot^2 \mathrm{A} = \cot^2 \mathrm{A}(\cot^4 \mathrm{A} - 1)$
$= \cot^2 \mathrm{A}(\sec^2 \mathrm{A} - 1)$
$= \cot^2 A \times \tan^2 A = 1$

48. $(\cos x - \sin x)^2 = 1/4 \Rightarrow 1 - \sin 2x = 1/4 \Rightarrow \sin 2x = 3/4 \Rightarrow \cos 2x = \sqrt{7}/4 = \tan 2x = 3/\sqrt{7}$.

49. (a) $\cos 2A = \sin 3A = \cos(\pi/2 - 3A)$
$\Rightarrow 5A = \pi/2 = A = \pi/10$
(b) $\Rightarrow 10A = \pi/2 \Rightarrow A = \pi/20$,
(c) $\Rightarrow A = \pi/8$, (d) $\Rightarrow A = \pi/6$
Least value of A is $\pi/20$ given by (b).

50. $\dfrac{n}{4} = \sin^2\left(\dfrac{\pi}{2n}\right) + \cos^2\left(\dfrac{\pi}{2n}\right) + \sin\left(\dfrac{2\pi}{2n}\right)$

$\Rightarrow \quad \sin\left(\dfrac{\pi}{n}\right) = \dfrac{n-4}{4}$

For $n \in \mathbf{N}$, $\sin\left(\dfrac{\pi}{n}\right) > 0$

$\therefore \quad \dfrac{n-4}{4} > 0$ or $n > 4$

For $n > 4$, $\dfrac{\pi}{n} < \dfrac{\pi}{2} \Rightarrow \sin\left(\dfrac{\pi}{n}\right) < 1$

$\Rightarrow \quad 0 < \dfrac{1}{4}(n-4) < 1 \Rightarrow 4 < n < 8$

Thus, a possible value of $n = 6$.
In fact, for $n = 6$

$$\sin\left(\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)$$

$$= \sqrt{2} \sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right) = \sqrt{2} \sin\left(\frac{\pi}{3}\right)$$

$$= \sqrt{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}}{4} = \frac{\sqrt{n}}{4}$$

51. $\tan(2n-1)\alpha = \tan(\pi/2 - \alpha) = \cot \alpha$.
$\tan(2n-2)\alpha = \cot 2\alpha$ and so on. So the required value is 1.

52. $x(x^2 - 1) + \sin\theta(x \sin\theta - \cos\theta) + \cos\theta(\sin\theta - x\cos\theta) = 0$
$\Rightarrow \quad x^3 - x + x(\sin^2\theta - \cos^2\theta) = 0$
$\Rightarrow \quad x^2 - (1 + \cos 2\theta) = 0$ as $x \neq 0$
$\Rightarrow \quad x^2 = 2\cos^2\theta \leq 2$

53. $2 \times \sqrt{3} \cos(2\pi x) = \sqrt{3}$
$\Rightarrow \cos(2\pi x) = 1/2 = \cos(2n\pi \pm \pi/3)$
$\Rightarrow x = n \pm 1/6$.

54. $2\cos^2 x + 4\cos x = 3(1 - \cos^2 x)$
$\Rightarrow 5\cos^2 x + 4\cos x - 3 = 0$

$\Rightarrow \cos x = \dfrac{-4 \pm \sqrt{16+60}}{10} = \dfrac{-2 \pm \sqrt{19}}{5}$

But $\cos x \neq \dfrac{-2-\sqrt{19}}{5}$

55. $6\sin^2 x - 2\cos^4 x = \cos^2 x(2\cos^2 x - 1)$
$\Rightarrow 4\cos^4 x + 5\cos^2 x - 6 = 0$
$\Rightarrow \cos^2 x = 3/4 = \cos 2x = 1/2$

56. $\cos 5\theta = 0 \Rightarrow 5\theta = 2n\pi \pm \pi/2$

As $\cos\theta$ decreases in $[0, \pi/2]$, it is greatest when θ is least,

i.e. $\theta = \pi/10 = 18°$ and $\cos\theta = \cos 18° = \sqrt{\dfrac{5+\sqrt{5}}{8}}$

57. $p\theta = n\pi + q\theta \Rightarrow (p - q)\theta = n\pi \Rightarrow \theta = \dfrac{n\pi}{p-q}$.

So the values of θ form an A.P. with common difference $\pi/(p - q)$.

58. $\dfrac{x^3}{3} - \dfrac{8x^2}{2} + 13x = x \sin(a/x)$

$\Rightarrow x^2 - 12x + 3(13 - \sin(a/x)) = 0$

which has a solution if

$144 - 12(13 - \sin(a/x)) \geq 0$

or $\sin(a/x) \geq 1$ but $\sin(a/x) \leq 1$

so $\sin(a/x) = 1$.

59. $\sin(1 - x) \geq 0$, $\cos x \geq 0$ and $\sin(1 - x) = \cos x$

$\Rightarrow \quad 1 - x = n\pi + (-1)^n (\pi/2 - x)$ where $n \in \boldsymbol{I}$.

For $n = 2m$ we get no values of x, $m \in \boldsymbol{I}$.

$\therefore \quad 1 - x = (2m + 1)\pi - \pi/2 + x$

$\Rightarrow \quad x = \dfrac{1}{2} - \dfrac{4m+1}{4}\pi$

For $m \geq 0$, $x < 0$

For $m = -1$, $x = \dfrac{1}{2} + \dfrac{3\pi}{4}$

and $\sin(1-x) = \sin\left(\dfrac{1}{2} - \dfrac{3\pi}{4}\right) = -\sin\left(\dfrac{1}{2} + \dfrac{\pi}{4}\right) < 0$

so $x \neq \dfrac{1}{2} + \dfrac{3\pi}{4}$.

For $m = -2$, $x = \dfrac{1}{2} + \dfrac{7\pi}{4}$ and

$\sin(1 - x) = \sin\left(\dfrac{1}{2} + \dfrac{\pi}{4} - 2\pi\right) > 0$ and

$\cos x = \cos\left(\dfrac{1}{2} + 2\pi - \dfrac{\pi}{4}\right) > 0.$

Hence $x = \dfrac{1}{2} + \dfrac{7\pi}{4}$ is the smallest positive root of the equation.

60. $4\cos^3 x - 4\cos^2 x + \cos x - 1 = 0$

$\Rightarrow \quad (4\cos^2 x + 1)(\cos x - 1) = 0$

$\Rightarrow \cos x = 1 \Rightarrow x = 2n\pi$

$100\pi < 315 < 101\pi$

Required sum $= 2\pi + 4\pi + \dots 100\pi$

$= 2(1 + 2 + \dots 50)\pi$

$= \dfrac{2 \times 50 \times 51}{2}\pi = 2550\pi$

61. When $x = 1$, $2 + 2\sin y = 0 \Rightarrow \sin y = -1$

and $\sin\dfrac{7\pi}{2} = \sin\left(3\pi + \dfrac{\pi}{2}\right) = -1$

So $(1, 7\pi/2)$ is a solution.

For $x = -1$, $\sin y = 1$ which is not satisfied by $y = 7\pi/2$ or 0.

62. $e^{\sin x} = 4 + \dfrac{1}{e^{\sin x}}$ (1)

$-1 \leq \sin x \leq 1$ and $e > 1$

$e^{-1} \leq e^{\sin x} \leq e < 3.$

$\Rightarrow \quad e^{\sin x} < 3$, also $e^{\sin x} > 0$.

$\therefore \quad \dfrac{1}{e^{\sin x}} + 4 > 4.$

So two sides of (1) cannot be equal for any real value of x.

63. $2\sin 3x \cos 2x = \sin 3x$

$\Rightarrow \sin 3x (2\cos 2x - 1) = 0$

$\Rightarrow \sin 3x = 0$ or $2\cos 2x = 1$,

$\Rightarrow x = 0, \pi/3, 2\pi/3, \pi$

or $x = \pi/6, 5\pi/6$ as $x \in [0, \pi]$.

64. $\sin x$, $\sin 2x$, $\sin 3x$ are in A.P.

If $2\sin 2x = \sin x + \sin 3x = 2\sin 2x \cos x$

or if $2\sin 2x(1 - \cos x) = 0$

or if either $\sin 2x = 0$ or $\cos x = 1$

which is satisfied by $x = 2n\pi$ for which $\cos x = 1$.

65. $2\sin^2((\pi/2)\cos^2 x) = 2\sin^2((\pi/2)\sin 2x)$

or if $(\pi/2)\cos^2 x = \pm(\pi/2)\sin 2x$

or if either $\cos x = 0$ or $\cos x = \pm 2\sin x$.

and $\cos x = 0$ if $x = (2n + 1)\pi/2$.

66. $\tan x = t \quad \Rightarrow \quad \dfrac{4t^2}{1+t^2} + t^2 + \dfrac{2}{t^2} = 5$

$\Rightarrow t^6 - 3t^2 + 2 = 0 \quad \Rightarrow \quad (t^2 - 1)(t^4 + t^2 - 2) = 0$

$\Rightarrow t^2 = 1, \quad t^2 = -2 \quad$ so $\quad t = \pm 1, x = \pi/4, \quad 3\pi/4$

67. $2\sin^2 x + \dfrac{2\sin x \cos x}{2} = n$

$\Rightarrow \sin 2x + 2(1 - \cos 2x) = 2n$

$\Rightarrow \sin 2x - 2\cos 2x = 2(n - 1)$

$\Rightarrow \quad -\sqrt{5} \leq 2n - 2 \leq \sqrt{5}$

$\Rightarrow \quad 1 - \dfrac{\sqrt{5}}{2} \leq n \leq 1 + \dfrac{\sqrt{5}}{2}$

Integral values of n are 0, 1, 2.

68. $\sin(2\sin x) = \sin(\pi/2 - 2\cos x)$

$\Rightarrow \quad 2\sin x = \pi/2 - 2\cos x$

$\Rightarrow \quad \sin x + \cos x = \pi/4$

$\Rightarrow \quad 1 + \sin 2x = \pi^2/16$

$$\tan x + \cot x = \frac{2}{\sin 2x} = \frac{32}{\pi^2 - 16}$$

$$a = 32, b = 16, c = 2$$

69. $15\tan^4\alpha + 10 = 6(1 + \tan^2\alpha)^2$

$\Rightarrow \quad 9\tan^4\alpha - 12\tan^2\alpha + 4 = 0$

$\Rightarrow \quad \tan^2\alpha = 2/3.$

so $8\operatorname{cosec}^6\alpha - 27\sec^6\alpha$

$= \quad 8(1 + 3/2)^3 - 27(1 + 2/3)^3 = 0$

70. $\displaystyle\sum_{r=1}^{\infty} \tan^{-1}\frac{1}{2r^2} = \sum_{1}^{\infty}\left[\tan^{-1}(2r+1) - \tan^{-1}(2r-1)\right]$

$= \tan^{-1}3 - \tan^{-1}1 + \tan^{-1}5 - \tan^{-1}3 + \ldots$

$= \pi/2 - \pi/4 = \pi/4$

71. $[x] + \{x\} = x$ and $[\{x\}] = 0.$

$$\sec\theta - 1 = \frac{1-\cos\theta}{\cos\theta} = \frac{2\sin^2(\theta/2)\times\sin\theta}{\cos\theta\times 2\sin(\theta/2)\cos(\theta/2)}$$

$= \tan\theta\tan(\theta/2)$

Given equation reduces to $\cos\theta = 1$

$\Rightarrow \quad \theta = 2n\pi, \quad n \in I$

Number of solutions in $[-2011\pi, 2011\pi]$ is 2011.

72. $x^2 + y^2 = 3 + 2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right)$

$= 3 + 2(-1/2) = 2$

(Ref. Ex 49,)

73. $\sin\frac{\pi x}{2\sqrt{3}} = \left(x - \sqrt{3}\right)^2 + 1 \quad \Rightarrow \quad x = \sqrt{3}.$

74. $\cos\theta\cos 2\theta \ldots \cos 2^{n-1}\theta$

$= \dfrac{1}{2\sin\theta}\left[2\sin\theta\cos\theta\cos 2\theta\ldots\cos 2^{n-1}\theta\right]$

$= \dfrac{1}{2^n\sin\theta}\left(\sin 2^n\theta\right) = \dfrac{1}{2^n\sin\theta}\sin(\pi+\theta)$

$= \dfrac{1}{2^n\sin\theta}(-\sin\theta) = -\dfrac{1}{2^n}.$

So statement-2 is true which implies statement-1 is also true.

75. $f(x) = \tan x$, $g(x) = \tan(x/2)$

$\Rightarrow f(x) + g(x) \neq 0$ for all x.

so statement-2 is False.

we have $[\cos\alpha + \cos(\alpha + \beta)]^2 +$

$[\sin\alpha + \sin(\alpha + \beta)]^2 = 1$

$\Rightarrow \quad 2 + 2\cos(\alpha - \alpha - \beta) = 1$

$\Rightarrow \quad \cos\beta = -1/2 \quad \Rightarrow \quad \beta = 2\pi/3.$

Similarly $\gamma = 2\pi/3.$

$\Rightarrow \quad f(\beta) + g(\gamma) = \tan(2\pi/3) + \tan(\pi/3) = 0$

$\Rightarrow$ statement-1 is true.

76. Statement-2 is True. In statement-1, $x = 3\sqrt{3}\cos\theta$, $y = 3\sqrt{3}\sin\theta$ then $x - y = 3\sqrt{3}(\cos\theta - \sin\theta) = 3\sqrt{3}\times\sqrt{2}\cos(\theta + \pi/4) \leq 3\sqrt{6}$ follows from statement-2

77. If $\alpha + \beta + \gamma = \pi/2$ $\tan(\alpha + \beta) = \tan(\pi/2 - \gamma)$ $= \dfrac{1}{\tan\gamma}.$

$\Rightarrow \quad \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \dfrac{1}{\tan\gamma}$

$\Rightarrow \quad \tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = 1$

$\Rightarrow$ statement-2 is False.

In statement-1, $\left(\sqrt{p} + \sqrt{q} + \sqrt{r}\right)^2$

$= p + q + r + 2\left(\sqrt{pq} + \sqrt{qr} + \sqrt{rp}\right)$

$\leq p + q + r + 2(p + q + r)$

$= 3(p + q + r) = 48$

78. Statement-2 is True.

Statement-1 $\Rightarrow y\cos\theta \,.\, \sin^2\theta + y\cos^3\theta$

$= \sin\theta\cos\theta \Rightarrow y\cos\theta = \sin\theta\cos\theta$

$\Rightarrow \quad y = \sin\theta \quad \Rightarrow \quad x = \cos\theta$

$\Rightarrow x^2 + y^2 = 1 \Rightarrow$ statement-1 is also true but does not follow from statement-1.

79. $x^2 - x - 2 = (x - 2)(x + 1) < 0$

$\Rightarrow -1 < x < 2 \Rightarrow$ Statement-2 is True

$2\sin^2 x + 3\sin x - 2$

$= (2\sin x - 1)(\sin x + 2) > 0 \Rightarrow \sin x > 1/2$

which is true if $\pi/6 < x \leq \pi/2$

i.e. $\dfrac{11}{21} < x < \dfrac{11}{7}$, so statement-1 is false.

80. Statement-2 is True as $-1 \leq \cos x \leq 1$.

In statement-1, $\cos^7 x + \sin^4 x = 1$

$\Rightarrow \cos^7 x + (1 - \cos^2 x)^2 = 1$

$\Rightarrow \cos^7 x + \cos^4 x - 2\cos^2 x = 0$

$\Rightarrow \cos^2 x(\cos^5 x + \cos^2 x - 2) = 0$

$\Rightarrow$ Either $\cos x = 0$ or $\cos^5 x + \cos^2 x - 2 = 0$

$\Rightarrow \cos x = 1$ from statement-2

$\Rightarrow x = -\pi/2, \pi/2$ as $\cos x = 1 \Rightarrow x = 0$

Level 2

81. Let the length of the diagonals of the rhombus be a and b and x be its side then

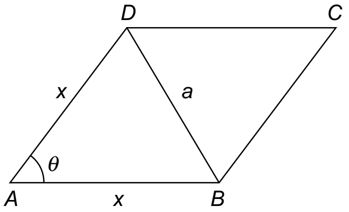


Fig. 25.5

$$\frac{x}{a} = \frac{b}{x} \Rightarrow x^2 = ab$$

Now $\cos\theta = \dfrac{x^2 + x^2 - a^2}{2x\cdot x} = \dfrac{2ab - a^2}{2ab} = 1 - \dfrac{a}{2b}$

Similarly $\cos(180° - \theta) = 1 - \dfrac{b}{2a}$

$$\Rightarrow \frac{a}{2b} = 1 - \cos\theta,\ \frac{b}{2a} = 1 + \cos\theta$$

$$\Rightarrow 1 - \cos^2\theta = \frac{1}{4} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = 30°$$

82. Let $\underline{|ACB} = \theta$

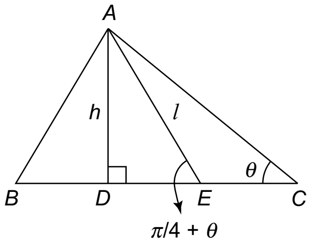


Fig. 25.6

$$\underline{|BAE} = \underline{|EAC} = \pi/4$$

$$\Rightarrow \underline{|AED} = \pi/4 + \theta\ l\ \underline{|EAD} = \pi/4 - \theta$$

From Δ ADE

$$\frac{h}{l} = \cos(\pi/4 - \theta) = \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)$$

$$\frac{\sqrt{l^2 - h^2}}{l} = \sin(\pi/4 - \theta) = \frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)$$

$$\Rightarrow \frac{h - \sqrt{l^2 - h^2}}{l} = \frac{1}{\sqrt{2}} \times 2\sin\theta = \sqrt{2}\sin C$$

$$\Rightarrow \frac{h - \sqrt{l^2 - h^2}}{\sqrt{2}l} = \cos B.$$

83. $2\sin^2(x + \pi/4) + \sqrt{3}\cos 2x > 0$

$$\Rightarrow 1 - \cos(2x + \pi/2) + \sqrt{3}\cos 2x > 0$$

$$\Rightarrow \frac{1}{2}\sin 2x + \frac{\sqrt{3}}{2}\cos 2x > -\frac{1}{2}$$

$$\Rightarrow \cos(2x - \pi/6) > -\frac{1}{2}$$

84. $x = \dfrac{1}{1 - \cos^2 x} = \dfrac{1}{\sin^2 x},\ y = \dfrac{1}{\cos^2 x}$

and $z = \dfrac{1}{1 - \sin^2 x\cos^2 x} = \dfrac{1}{1 - \frac{1}{x}\cdot\frac{1}{y}} = \dfrac{xy}{xy - 1}$

$$\Rightarrow z(xy - 1) = xy = x + y$$

$$\Rightarrow x + y + z = xyz.$$

85. $\dfrac{4\sin^4 x}{4\cos^2 x - 4\sin^2 x\cos^2 x} = \dfrac{1}{9}$

$$\Rightarrow \frac{\sin^4 x}{\cos^4 x} = \frac{1}{9} \Rightarrow \tan^4 x = \frac{1}{9} \Rightarrow \tan x = \frac{1}{\sqrt{3}}$$

$$\Rightarrow x = \pi/6$$

86. Let $\theta = 7.5° \Rightarrow 2\theta = 15° = 45° - 30°$

$$\Rightarrow \cos 2\theta = \frac{1}{2\sqrt{2}}\left(\sqrt{3} + 1\right)$$

$$\Rightarrow 2\cos^2\theta = \frac{1}{2\sqrt{2}}\left(\sqrt{3} + 1\right) + 1$$

$$\Rightarrow \cos^2\theta = \frac{\sqrt{3} + 1 + 2\sqrt{2}}{2 \times 2\sqrt{2}} = \frac{\sqrt{6} + \sqrt{2} + 4}{8}$$

87. We have

$A = a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta$

$B = a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta$

$H = (c - a)\sin 2\theta + 2b\cos 2\theta$

So $A + B = a + c$

88. $\text{Sin}5\theta = \sin(2\theta + 3\theta)$

$= \sin 2\theta\cos 3\theta + \cos 2\theta\sin 3\theta$

$= 2\sin\theta\cos\theta \times \cos\theta\,(4\cos^2\theta - 3)$
$\quad + (1 - 2\sin^2\theta)(3\sin\theta - 4\sin^3\theta)$

$= 2\sin\theta\,(1 - \sin^2\theta)(1 - 4\sin^2\theta)$
$\quad + (1 - 2\sin^2\theta)(3\sin\theta - 4\sin^3\theta)$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

So $a = 16,\ b = -20,\ c = 5,\ d = 0$

$\Rightarrow 5a + 3b - 4c = 0$

89. $\cot(A - B) = \dfrac{\cot A\cot B + 1}{\cot B - \cot A} = \dfrac{y/x + 1}{y} = \dfrac{x + y}{xy}$

90. $\dfrac{\cos 3\theta}{\cos^3\theta} + \dfrac{\sin 3\theta}{\sin^3\theta}$

$$= \frac{4\cos^3\theta - 3\cos\theta}{\cos^3\theta} + \frac{3\sin\theta - 4\sin^3\theta}{\sin^3\theta}$$

$$= 4 \times 3\ \frac{\left(\cos^2\theta - \sin^2\theta\right)}{4\sin^2\theta\cos^2\theta} = \frac{12\cos 2\theta}{\sin 2\theta} \times \frac{1}{\sin 2\theta}$$

91. The given expression is equal to
$x^2 + y^2 + xy\ [(\tan^2\alpha + \cot^2\alpha) - 4\cot^2 2\alpha]$
$= x^2 + y^2 + xy$
$$\left[\frac{1-2\sin^2\alpha\cos^2\alpha}{\sin^2\alpha\cos^2\alpha} - 4\frac{\left(\cos^2\alpha - \sin^2\alpha\right)^2}{4\sin^2\alpha\cos^2\alpha}\right]$$
$= x^2 + y^2 + xy$
$$\left[\frac{1}{\sin^2\alpha\cos^2\alpha} - 2 - \frac{1-4\sin^2\alpha\cos^2\alpha}{\sin^2\alpha\cos^2\alpha}\right]$$
$= x^2 + y^2 + 2xy = (x + y)^2$

92. $S_1 = \tan A + \tan B + \tan C = 4/3$
$S_2 = \Sigma\tan A\tan B = 1,$
$S_3 = \tan A\tan B\tan C = -1/3$
$$\tan(A + B + C) = \frac{S_1 - S_3}{1 - S_2} \to \infty$$
$\Rightarrow A + B + C = \pi/2$

93. We have $\sin\theta\ [x + 2y\cos\theta + z(3 - 4 + 4\cos^2\theta)]$
$= 4\cos\theta(2\cos^2\theta - 1)\sin\theta$
$\Rightarrow 8\cos^3\theta - 4z\cos^2\theta - 2(y + 2)\cos\theta = x - z$
$(\because \sin\theta \neq 0)$

94. We have $16\sin^5 x - 20\sin^3 x + 5\sin x = 5\sin x$ (see Ex 68)
$\Rightarrow \sin^3 x(16\sin^2 x - 20) = 0 \Rightarrow \sin x = 0$, only one value

95. $\sin\alpha + \sin\beta = -b/a$, $\sin\alpha\sin\beta = c/a$,
$\sin\alpha + 2\sin\beta = 1$
$\Rightarrow \sin\alpha = \dfrac{c}{a+b}$ which satisfies the given equation
(Note $c \neq 0$) and the required value is 0.

96. $(1 + \sin\theta)(3\sin\theta + 4\cos\theta + 5)$
$= (\cos(\theta/2) + \sin(\theta/2))^2\ [6\sin(\theta/2)\cos(\theta/2)$
$+ 4(2\cos^2\theta/2 - 1) + 5]$
$= (a + b)^2\ [6ab + 8b^2 + a^2 + b^2]$
$= (a + b)^2(a + 3b)^2$

97. $\sin A = \sin B$ and $\cos A = \cos B$
$\Rightarrow A = 2n\pi + B$ where n is an integer, $n \neq 0$
$$\Rightarrow \frac{A-B}{2} = n\pi \Rightarrow \tan\frac{A-B}{2} = 0$$

98. Apply the identity, if $A + B + C = 180°$
then $\cos A + \cos B + \cos C$
$= 1 + 4\sin(A/2)\sin(B/2)\sin(C/2)$

99. We can write $\tan x = \dfrac{\tan^2 x - 1 + 1}{\tan x}$
$= \cot x - 2\cot 2x$
so $\tan x + \dfrac{1}{2}\tan\dfrac{x}{2} + \dfrac{1}{2^2}\tan\dfrac{x}{2^2} + \cdots$
$+ \dfrac{1}{2^{n-1}}\tan\dfrac{x}{2^{n-1}}$
$= \cot x - 2\cot 2x + \dfrac{1}{2}\left(\cot\dfrac{x}{2} - 2\cot x\right)$
$+ \dfrac{1}{2^2}\left(\cot\dfrac{x}{2^2} - 2\cot\dfrac{x}{2}\right)$
$+ \cdots + \dfrac{1}{2^{n-1}}\left(\cot\dfrac{x}{2^{n-1}} - 2\cot\dfrac{x}{2^{n-2}}\right)$
$$= \frac{1}{2^{n-1}}\cot\left(\frac{x}{2^{n-1}}\right) - 2\cot 2x$$

Note Remember $\tan x + 2\cot 2x = \cot x$

100. We can write $\dfrac{4 + \cot 76°\cot 16° - 1}{\cot 76° + \cot 16°}$
$$= \frac{4\sin 76°\sin 16°}{\sin 92°} + \cot 92°$$
$$= \frac{2(\cos 60° - \cos 92°) + \cos 92°}{\sin 92°} = \frac{1 - \cos 92°}{\sin 92°}$$
$= \cot 46° = \tan 44°$

101. $K\left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right) = \cos\theta + \cos(\theta + 2\pi/3)$
$+ \cos(\theta + 4\pi/3)$
$= \cos\theta + 2\cos(\theta + \pi)\cos\pi/3$
$= \cos\theta - \cos\theta = 0$
$\Rightarrow \quad xy + yz + zx = 0$

102. Let $\tan A\tan B = x$
so $\tan(A + B) = \tan(\pi/3) = \sqrt{3}$
$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A\tan B} = \sqrt{3}$$
$\Rightarrow \sqrt{3}\ (1 - x) = \tan A + \tan B \leq 2\sqrt{\tan A\tan B} = 2\sqrt{x}$
$\Rightarrow 3(1 - x)^2 \leq 4x$
$\Rightarrow 3x^2 - 10x + 3 \leq 0$
$\Rightarrow (3x - 1)(x - 3) \leq 0$
$\Rightarrow \dfrac{1}{3} \leq x \leq 3$

103. $\tan\theta(3\tan\theta + 3) = (2 + 2\tan\theta)^2$
$\Rightarrow (1 + \tan\theta)(4 + \tan\theta) = 0$
$\Rightarrow \tan\theta = -4$ as $\tan\theta \neq -1$
$$\text{So } \frac{7 - 5\cot\theta}{9 - 4\sqrt{\sec^2\theta - 1}} = \frac{7 + 5/4}{9 + 16} = \frac{33}{100}$$

104. We have
$\sin^2\theta + \sin^4\theta + \sin^6\theta + \cdots$
$-(\sin\theta\cos\theta + \sin^2\theta\cos^2\theta + \sin^3\theta + \cos^3\theta + \cdots)$

$$= \frac{\sin^2\theta}{1-\sin^2\theta} - \frac{\sin\theta\cos\theta}{1-\sin\theta\cos\theta}$$

$$= \frac{1-\cos 2\theta}{1+\cos 2\theta} - \frac{\sin 2\theta}{(2-\sin 2\theta)}$$

$$= \frac{1-ab}{1+ab} + \frac{1-a^2}{3-a^2}$$

105. $\left(1+\sqrt{1+x}\right)\ \tan\alpha = 1-\sqrt{1-x}$

$$\Rightarrow \frac{1+\sqrt{1+x}}{1-\sqrt{1-x}} = \frac{\cos\alpha}{\sin\alpha} = \frac{2\cos\alpha\,(\cos\alpha+\sin\alpha)}{2\sin\alpha\,(\cos\alpha+\sin\alpha)}$$

$$= \frac{2\cos^2\alpha + 2\sin\alpha\cos\alpha}{2\sin\alpha\cos\alpha + 2\sin^2\alpha}$$

$$= \frac{1+\cos 2\alpha+\sin 2\alpha}{\sin 2\alpha + 1 - \cos 2\alpha}$$

$$= \frac{1+\sqrt{(\cos 2\alpha+\sin 2\alpha)^2}}{1-\sqrt{(\cos 2\alpha-\sin 2\alpha)^2}}$$

$$= \frac{1+\sqrt{(1+\sin 4\alpha)}}{1-\sqrt{1-\sin 4\alpha}}$$

106. $f(\theta) = \sin\theta\ (\sin\theta + 3\sin\theta - 4\sin^3\theta)$

$= 4\sin^2\theta\ (1-\sin^2\theta) = \sin^2 2\theta \geq 0$ for all real θ.

107. $l \cot B + l \cot C = h = 2\sqrt{2}\, l$

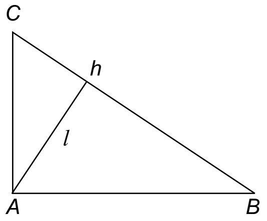


Fig. 25.7

$\Rightarrow \cot B + \cot C = 2\sqrt{2}$

$B + C = \pi/2 \Rightarrow \cot B \cot C = 1$

$\Rightarrow \cot B = \sqrt{2}-1,\ \cot C = \sqrt{2}+1$

The required angles are $\pi/8$ and $3\pi/8$

108. $\sqrt{\cos^2 x - \sin^2 x} + \cos x + \sin x = 2\sqrt{\sin x + \cos x}$

$\Rightarrow \sqrt{\cos x + \sin x}\left[\sqrt{\cos x - \sin x} + \sqrt{\cos x + \sin x} - 2\right] = 0$

$\Rightarrow$ either $\sin x + \cos x = 0$

or $\sqrt{\cos x - \sin x} + \sqrt{\cos x + \sin x} = 2$

$\Rightarrow 2\cos x + 2\sqrt{\cos 2x} = 4$

$\Rightarrow 2\cos^2 x - 1 = (2-\cos x)^2 = 4 - 4\cos x + \cos^2 x$

$\Rightarrow \cos^2 x + 4\cos x - 5 = 0$

$\Rightarrow \cos x = 1 \Rightarrow x = 2n\pi$

Note The answer could be obtained by trial method, taking $x = 2n\pi$

109. $\sin x + 2\sin 2x - \sin 3x = 3$

$\Rightarrow \sin 3x \geq 0 \Rightarrow 0 \leq 3x \leq \pi$

$\Rightarrow 0 \leq x \leq \pi/3 \Rightarrow 0 \leq \sin x \leq \sqrt{3}/2$

and $0 \leq 2x \leq 2\pi/3 \Rightarrow 0 \leq \sin 2x \leq 1$

$-1 \leq -\sin 3x \leq 0$

So $-1 \leq \sin x + \sin 2x - \sin 3x \leq \sqrt{3}/2 + 1 < 3$

So the given equation is not satisfied for any real value of x

110. $\sin x + \sin y = \sin(x+y)$

$$\Rightarrow 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} = 2\sin\frac{x+y}{2}\cos\frac{x+y}{2}$$

$$\Rightarrow \sin\frac{x+y}{2}\left(\cos\frac{x+y}{2} - \cos\frac{x-y}{2}\right) = 0$$

$$\Rightarrow \text{either } \sin\frac{x+y}{2} = 0 \text{ or } \cos\frac{x+y}{2} = \cos\frac{x-y}{2}$$

Now $\sin\dfrac{x+y}{2} = 0$ and $|x| + |y| = 1$

$(x, y) = \left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$ or $\left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$

and $\cos\dfrac{x+y}{2} = \cos\dfrac{x-y}{2}$

$$\Rightarrow \sin\frac{x}{2}\ \sin\frac{y}{2} = 0$$

$$\Rightarrow \sin\frac{x}{2} = 0 \text{ or } \sin\frac{y}{2} = 0$$

$\Rightarrow (x, y) = (1, 0), (-1, 0), (0, 1), (0, -1)$

So the required number of pairs is 6.

Previous Years' AIEEE/JEE Main Questions

1. $\tan x + \sec x = 2\cos x$

$\Rightarrow\ 1 + \sin x = 2\cos^2 x = 2(1-\sin^2 x)$

$\Rightarrow\ (1+\sin x)(1 - 2 + 2\sin x) = 0$

$\Rightarrow\ \sin x = -1$ or $\sin x = \dfrac{1}{2}$

But $\sin x = -1 \Rightarrow \cos x = 0$. This is not possible

Thus $\sin x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6} \in [0, 2\pi]$

2. $\sin\alpha + \sin\beta = \dfrac{-21}{65}, \cos\alpha + \cos\beta = \dfrac{-27}{65}$

Squaring and adding we get

$$2 + 2\cos(\alpha - \beta) = \frac{441+729}{(65)^2} = \frac{1170}{(65)^2} = \frac{18}{65}$$

$$\Rightarrow \quad 2\times 2\cos^2\frac{\alpha-\beta}{2} = \frac{18}{65}$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2} = \pm\frac{3}{\sqrt{130}}$$

$$\pi < \alpha - \beta < 3\pi$$

$$\Rightarrow \quad \frac{\pi}{2} < \frac{\alpha-\beta}{2} < \frac{3\pi}{2}$$

$$\Rightarrow \quad \cos\frac{\alpha-\beta}{2} = \frac{-3}{\sqrt{130}}$$

3. We have

$$u = \sqrt{\frac{a^2}{2}(1+\cos 2\theta) + \frac{b^2}{2}(1-\cos 2\theta)} + \sqrt{\frac{a^2}{2}(1-\cos 2\theta) + \frac{b^2}{2}(1+\cos 2\theta)}$$

$$= \sqrt{\frac{a^2+b^2}{2} + \frac{a^2-b^2}{2}\cos 2\theta} + \sqrt{\frac{a^2+b^2}{2} - \frac{a^2-b^2}{2}\cos 2\theta}$$

Squaring we get

$$u^2 = a^2 + b^2 + 2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2 - \left(\frac{a^2-b^2}{2}\right)^2\cos^2 2\theta}$$

Thus max $(u^2) = a^2 + b^2 + a^2 + b^2 = 2(a^2 + b^2)$

and min $(u^2) = a^2 + b^2 + 2\sqrt{\left(\frac{a^2+b^2}{2}\right)^2 - \left(\frac{a^2-b^2}{2}\right)^2}$

$$= a^2 + b^2 + 2ab = (a+b)^2$$

So the required difference $= 2(a^2 + b^2) - (a+b)^2 = (a-b)^2$.

4. $f(x) = \sin x - \sqrt{3}\cos x + 1$

$$= 2\left[\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right] + 1$$

$$= 2\sin(x - \pi/3) + 1$$

$$-2 \le 2\sin(x - \pi/3) \le 2$$

$$\Rightarrow \quad -2 + 1 \le 2\sin(x - \pi/3) + 1 \le 2 + 1$$

$$\Rightarrow \quad -1 \le f(x) \le 3$$

$$\Rightarrow \quad S = [-1, 3]$$

5. $2\sin^2 x + 5\sin x - 3 = 0$

$$\Rightarrow \quad (2\sin x - 1)(\sin x + 3) = 0$$

$$\Rightarrow \quad \sin x = 1/2 \qquad [\because \sin x \ne -3]$$

$$\Rightarrow \quad x = \pi/6,\ 5\pi/6,\ 13\pi/6,\ 17\pi/6.$$

6. Let $\alpha = \tan 30°$, $\beta = \tan 15°$, then $\alpha = \frac{1}{\sqrt{3}}$ and $\beta = 2 - \sqrt{3}$.

$$\therefore \quad -p = \frac{1}{\sqrt{3}} + (2 - \sqrt{3})$$

$$\text{and} \quad q = \frac{1}{\sqrt{3}}(2 - \sqrt{3}) = \frac{2}{\sqrt{3}} - 1$$

$$\text{Thus} \quad q - p = \frac{3}{\sqrt{3}} + 2 - \sqrt{3} - 1 = 1$$

$$\therefore \quad 2 + q - p = 3$$

7. $2\sin x\cos x = (\sin x + \cos x)^2 - 1$

$$= 1/4 - 1 = -3/4$$

As $0 < x < \pi$, $\sin x > 0$.

Thus, $\cos x < 0$

$$\Rightarrow \quad \pi/2 < x < \pi$$

Also, $(\sin x - \cos x)^2$

$$= \sin^2 x + \cos^2 x - 2\sin x\cos x$$

$$= 1 + 3/4 = 7/4$$

$$\Rightarrow \quad \sin x - \cos x = \frac{\sqrt{7}}{2}$$

Since $\sin x + \cos x = 1/2$

We get

$$2\sin x = (\sqrt{7} + 1)/2$$

$$\text{and} \quad 2\cos x = (1 - \sqrt{7})/2$$

$$\therefore \quad \tan x = -\frac{\sqrt{7}+1}{\sqrt{7}-1} = -\frac{(\sqrt{7}+1)^2}{7-1}$$

$$= -\frac{1}{3}(4 + \sqrt{7})$$

8. $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -3/2$

$\Rightarrow 2[\cos\beta\cos\gamma + \cos\gamma\cos\alpha + \cos\alpha\cos\beta + \sin\beta\sin\gamma + \sin\gamma\sin\alpha + \sin\alpha\sin\beta] + (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + (\cos^2\gamma + \sin^2\gamma) = 0$

$\Rightarrow (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$

$\Rightarrow \cos\alpha + \cos\beta + \cos\gamma = 0$

and $\sin\alpha + \sin\beta + \sin\gamma = 0$

Thus, both A and B are true.

9. We have

$$\frac{r}{R} = \cos\left(\frac{\pi}{n}\right)$$

When

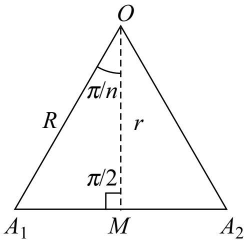


Fig. 25.8

$$\frac{r}{R} = \frac{\sqrt{3}}{2}, \text{ we get}$$

$$\cos\left(\frac{\pi}{n}\right) = \cos\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \qquad n = 6$$

when $\frac{r}{R} = \frac{3}{3} = \cos\frac{\pi}{n}$

there is no value of n

10. $0 \le \alpha, \beta \le \pi/4$

$\Rightarrow 0 \le \alpha + \beta \le \pi/2$ and $-\pi/4 \le \alpha - \beta \le \pi/4$

Now, $\cos(\alpha + \beta) = 4/5$

$\Rightarrow \quad \tan(\alpha + \beta) = 3/4$

and $\sin(\alpha - \beta) = 5/13 \quad \Rightarrow \quad \tan(\alpha - \beta) = \pm 5/12$

We have

$$\tan(2\alpha) = \tan[(\alpha + \beta) + (\alpha - \beta)]$$
$$= \frac{\tan(\alpha+\beta) + \tan(\alpha-\beta)}{1 - \tan(\alpha+\beta)\tan(\alpha-\beta)}$$
$$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \left(\frac{3}{4}\right)\left(\frac{5}{12}\right)} = \frac{\frac{14}{12}}{\frac{33}{48}} = \frac{56}{33}$$

Note $\tan(\alpha - \beta) = -\frac{5}{12}$ does not satisty any of the choices.

11. $A = \sin^2 x + \cos^4 x \le \sin^2 x + \cos^2 x = 1$

$[\because 0 \le \cos^2 x \le 1 \Rightarrow \cos^4 x \le \cos^2 x]$

Also, $A = 1 - \cos^2 x + \cos^4 x$

$= (\cos^2 x - 1/2)^2 + 3/4 \ge 3/4$

Thus, $\frac{3}{4} \le A \le 1$

12. We have

$\sin\theta + \sin(4\theta) + \sin(7\theta) = 0$

$\Rightarrow \quad 2\sin(4\theta)\cos(3\theta) + \sin(4\theta) = 0$

$\Rightarrow \quad \sin(4\theta)[2\cos(3\theta) + 1] = 0$

$\Rightarrow \quad \sin(4\theta) = 0$ or $\cos(3\theta) = -1/2 = \cos(2\pi/3)$

$\Rightarrow \quad \theta = \pi/4, \pi/2, 3\pi/4$ or $3\theta = 2\pi/3, 2\pi \pm 2\pi/3$

$$\Rightarrow \quad \theta = \frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{4\pi}{9}, \frac{3\pi}{4}, \frac{8\pi}{9}$$

13. We have

$(3\sin P + 4\cos Q)^2 + (3\cos P + 4\sin Q)^2 = 6^2 + 1$

$\Rightarrow \quad 9(\sin^2 P + \cos^2 P) + 16(\cos^2 Q + \sin^2 Q)$

$+ 24(\sin P\cos Q + \cos P\sin Q) = 37$

$\Rightarrow \quad \sin(P + Q) = 1/2 \quad \Rightarrow \quad \sin(\pi - R) = 1/2$

$\Rightarrow \quad \sin R = 1/2 \quad \Rightarrow \quad R = \pi/6$ or $5\pi/6$

$R = 5\pi/6 \quad \Rightarrow \quad P < \pi/6 \quad \Rightarrow \quad 3\sin P < 3/2$

$\Rightarrow \quad 3\sin P + 4\cos Q < 6$, so $R = \pi/6$.

14. Let $\angle BDC = \phi$, then $\angle DBA = \phi$. We have

$$\sin\phi = \frac{p}{BD} = \frac{p}{\sqrt{p^2 + q^2}}$$

and $\cos\phi = \frac{q}{\sqrt{p^2 + q^2}}$

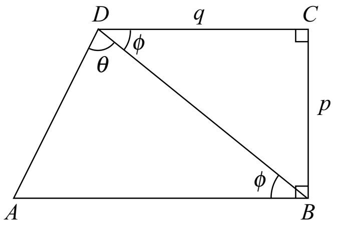


Fig. 25.9

By the law of sines

$$\frac{\sin(\pi - \theta - \phi)}{BD} = \frac{\sin\theta}{AB}$$

$$AB = \frac{\sin\theta}{\sin(\theta + \varphi)} BD$$
$$= \frac{\sqrt{p^2 + q^2}\sin\theta}{\sin\theta\cos\varphi + \cos\theta\sin\varphi}$$
$$= \frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}$$

15. $$\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$$
$$= \frac{\tan^2 A}{\tan A - 1} - \frac{1}{\tan A(\tan A - 1)}$$
$$= \frac{\tan^3 A - 1}{\tan A(\tan A - 1)} = \frac{\tan^2 A + \tan A + 1}{\tan A}$$
$$= \frac{\sec^2 A}{\tan A} + 1 = \sec A \operatorname{cosec} A + 1$$

16. $2\cos^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad 2\sin^2\theta + 3\sin\theta - 2 = 0$

$\Rightarrow \quad \sin\theta = \frac{1}{2} \quad \Rightarrow \quad \frac{\pi}{6}, \frac{5\pi}{6}$

$\Rightarrow$ Statement-2 is True.

Next, $2\sin^2\theta - \cos 2\theta = 0$

$\Rightarrow \quad 4\sin^2\theta = 1 \quad \Rightarrow \quad \sin\theta = \pm\frac{1}{2}$

Which gives four values of θ in $[0, 2\pi]$

$2\cos^2\theta - 3\sin\theta = 0$ gives two values of θ in $[0, 2\pi]$

So, Statement-1 is also true but does not follow from Statement-2.

17. $\sin 2x - 2\cos x + 4\sin x = 4$

$\Rightarrow \quad (2\cos x + 4)(\sin x - 1) = 0$

$$\Rightarrow \sin x = 1 \Rightarrow x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$$

18. $f_k(x) = \frac{1}{k}\left(\sin^k x + \cos^k x\right)$

$$\Rightarrow f_4(x) - f_6(x) = \frac{1}{4}\left(\sin^4 x + \cos^4 x\right) - \frac{1}{6}\left(\sin^6 x + \cos^6 x\right)$$

$$= \frac{1}{4}\left(1 - 2\sin^2 x\cos^2 x\right) - \frac{1}{6}\left(1 - 3\sin^2 x\cos^2 x\right)$$

$$= \frac{1}{12}$$

19. $2\sin^3\alpha - 7\sin^2\alpha + 7\sin\alpha = 2$

$\Rightarrow 2(\sin^3\alpha - 1) - 7\sin\alpha(\sin\alpha - 1) = 0$

$\Rightarrow (\sin\alpha - 1)[2(\sin^2\alpha + \sin\alpha + 1) - 7\sin\alpha] = 0$

$\Rightarrow (\sin\alpha - 1)[2\sin^2\alpha - 5\sin\alpha + 2] = 0$

$\Rightarrow (\sin\alpha - 1)(2\sin\alpha - 1)(\sin\alpha - 2) = 0$

$\Rightarrow \alpha = \pi/6, 5\pi/6, \pi/2 \in [0, 2\pi]$

$\therefore$ There are 3 solutions.

20. $\cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{\cot(\theta/2) - 1}{\cot(\theta/2) + 1}$

$$= \frac{\cos(\theta/2) - \sin(\theta/2)}{\cos(\theta/2) + \sin(\theta/2)}$$

$$= \frac{\cos^2(\theta/2) - \sin^2(\theta/2)}{1 + 2\sin(\theta/2)\cos(\theta/2)} = \frac{\cos\theta}{1 + \sin\theta}$$

$$\Rightarrow \cot^2\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \frac{1 - \sin\theta}{1 + \sin\theta} = \frac{\operatorname{cosec}\theta - 1}{\operatorname{cosec}\theta + 1} = \frac{q}{p}$$

$$\left|\cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right| = \sqrt{\frac{q}{p}}$$

21. $2\cos\theta + \sin\theta = 1$ (1)

$\Rightarrow 4\cos^2\theta = (1 - \sin\theta)^2$

$\Rightarrow 4(1 - \sin^2\theta) = (1 - \sin\theta)^2$

$\Rightarrow 4(1 + \sin\theta) = 1 - \sin\theta$ $(\because \theta \neq \pi/2)$

$\Rightarrow \sin\theta = -\frac{3}{5}$ (2)

From (1) and (2) $\cos\theta = 4/5$

Thus, $7\cos\theta + 6\sin\theta$

$$= 7\left(\frac{4}{5}\right) + 6\left(-\frac{3}{5}\right) = 2$$

22. $f(\theta) = 1 + \cos^2\theta - \sin^2\theta + 1 + \sin 2\theta$

$$= 2 + \cos(2\theta) + \sin(2\theta)$$

$$= 2 + \sqrt{2}\sin\left(2\theta + \frac{\pi}{4}\right)$$

$A = \operatorname{Max}(f(\theta)) = 2 + \sqrt{2}$

$B = \operatorname{Min}(f(\theta)) = 2 - \sqrt{2}$

$\therefore (A, B) = \left(2 + \sqrt{2}, 2 - \sqrt{2}\right)$

23. $\cos\alpha + \cos\beta = 3/2$

$$\Rightarrow 2\cos\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2} = \frac{3}{2} \quad (1)$$

and $\sin\alpha + \sin\beta = \frac{1}{2}$

$$\Rightarrow 2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2} = \frac{1}{2} \quad (2)$$

From (1) and (2), we get

$\tan\theta = \frac{1}{3}$, where $\theta = \frac{1}{2}(\alpha + \beta)$

$$\sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{2\left(\frac{1}{3}\right)}{1 + \left(\frac{1}{3}\right)^2} = \frac{2}{3} \times \frac{9}{10} = \frac{3}{5}$$

24. $2 + \sqrt{3} = \frac{a}{b} = \frac{\sin A}{\sin B}$

$$\Rightarrow 2 + \sqrt{3} = \frac{\sin(120° - B)}{\sin B}$$

$\Rightarrow (2 + \sqrt{3})\sin B = \sin 120°\cos B - \cos 120°\sin B$

$\Rightarrow (2 + \sqrt{3} + \cos(180° - 60°))\sin B$

$= \sin(180° - 60°)\cos B$

$$\Rightarrow \left(2 + \sqrt{3} - \frac{1}{2}\right)\sin B = \frac{\sqrt{3}}{2}\cos B$$

$$\Rightarrow \tan B = \frac{\sqrt{3}}{3 + 2\sqrt{3}} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$$

$\Rightarrow B = 15°$ [use $\tan 15° = \tan(60° - 45°)$]

$\therefore A = 105°$

$\Rightarrow (A, B) = (105°, 15°)$

25. $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$

$\Rightarrow (\cos 4x + \cos x) + (\cos 3x + \cos 2x) = 0$

$$\Rightarrow 2\cos\frac{5x}{2}\cos\frac{3x}{2} + 2\cos\frac{5x}{2}\cos\frac{x}{2} = 0$$

$$\Rightarrow 2\cos\frac{5x}{2}\left[\cos\frac{3x}{2} + \cos\frac{x}{2}\right] = 0$$

$$\Rightarrow 2\cos\frac{5x}{2}\left[2\cos x \cdot \cos\frac{x}{2}\right] = 0$$

$$\Rightarrow \cos\frac{x}{2}=0, \cos x = 0 \text{ or } \cos\frac{5x}{2}=0$$

$$\Rightarrow \frac{x}{2}=\frac{\pi}{2}; x=\frac{\pi}{2},\frac{3\pi}{2}; \frac{5x}{2}=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\frac{9\pi}{2}$$

$$\Rightarrow x=\pi,\frac{\pi}{2},\frac{3\pi}{2},\frac{\pi}{5},\frac{3\pi}{5},\frac{7\pi}{5},\frac{9\pi}{5}$$

26. Let $E = 4+\frac{1}{2}\sin^2 2x - 2\cos^4 x$

$$= 4+\frac{1}{2}\sin^2 2x-\frac{1}{2}(2\cos^2 x)^2$$

$$= 4+\frac{1}{2}\left(1-\cos^2 2x\right)-\frac{1}{2}(1+\cos 2x)^2$$

$$= 4+\frac{1}{2}(1+\cos 2x)[1-\cos 2x-1-\cos 2x]$$

$$= 4-\frac{1}{2}(1+\cos 2x)(2\cos 2x)$$

$$= 4 - \cos 2x - \cos^2 2x$$

$$= \frac{17}{4}-\left(\frac{1}{2}+\cos 2x\right)^2$$

Note that

$m = \frac{17}{4}-\frac{9}{4}=2$ when $\cos 2x = 1$

and $M=\frac{17}{4}$ when $\cos 2x = -1/2 \Rightarrow M - m = \frac{9}{4}$

27. Let $a=\sqrt{2\sin^4 x+18\cos^2 x}$, $b=\sqrt{2\cos^4 x+18\sin^2 x}$

Now, $2a^2 = (2\sin^2 x)^2 + 18(2\cos^2 x)$

$= (1 - \cos 2x)^2 + 18(1 + \cos 2x)$

$= 19 + 16 \cos 2x + \cos^2 2x$

and $2b^2 = (2\cos^2 x)^2 + 18(2\sin^2 x)$

$= (1 + \cos 2x)^2 + 18(1 - \cos 2x)$

$= 19 - 16\cos 2x + \cos^2 2x$

We have $|a - b| = 1$

and $2(a^2 - b^2) = 32\cos 2x$

If $a - b = 1$, $2(a - b)(a + b) = 32 \cos 2x$

$\Rightarrow a + b = 16 \cos 2x$

$\therefore 2a = 1 + 16 \cos 2x$

$\Rightarrow 4a^2 = (1 + 16 \cos 2x)^2$

$= 1 + 32 \cos 2x + 256 \cos^2 2x$

$\Rightarrow 38 + 32 \cos 2x + 2 \cos^2 2x$

$= 1 + 32 \cos 2x + 256 \cos^2 2x$

$$\Rightarrow 37 = 254\cos^2 2x \Rightarrow \cos 2x = \pm\sqrt{\frac{37}{254}}$$

which gives us 8 solutions in $[0, 2\pi]$.

When $a - b = -1$, we get the same 8 solutions in $[0, 2\pi]$

28. Using $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = (2\sin x+\cos x)\begin{vmatrix}1 & \sin x & \sin x\\ 1 & \cos x & \sin x\\ 1 & \sin x & \cos x\end{vmatrix}$$

Using $C_2 \to C_2 - (\sin x)C_1$

and $C_3 \to C_3 - (\sin x)C_1$,

we get

$$\Delta = (2\sin x+\cos x)\begin{vmatrix}\text{ü} & & \\ 1 & \cos x-\sin x & 0\\ 1 & 0 & \cos x-\sin x\end{vmatrix}$$

$= (2\sin x + \cos x)(\cos x - \sin x)^2$

$\therefore \Delta = 0 \Rightarrow \tan x = -\frac{1}{2}, 1$

As $-\pi/4 \le x \le \pi/4$, $-1 \le \tan x \le 1$

and $\tan x$ is one-to-one in the interval $[-\pi/4, \pi/4]$. Thus, there are two values of x.

29. Let $A = \frac{\pi}{12}+\theta$, $B = \frac{\pi}{12}-\theta$

where $-\pi/12 < \theta < \pi/12$

We have

$S = \tan A + \tan B$

$$= \frac{\sin(A+B)}{\cos A\cos B}=\frac{2\sin(\pi/6)}{\cos(A+B)+\cos(A-B)}$$

$$= \frac{2(1/2)}{\left(\sqrt{3}/2\right)+\cos(2\theta)}$$

$$= \frac{2}{\sqrt{3}+\cos(2\theta)}$$

As $-\pi/6 < 2\theta < \pi/6$

$\Rightarrow \sqrt{3}/2 < \cos(2\theta) \le 1$

$\Rightarrow \sqrt{3} < 2\cos(2\theta) \le 2$

$\Rightarrow \sqrt{3}+\sqrt{3} < \sqrt{3}+2\cos(2\theta) \le \sqrt{3}+2$

$$\Rightarrow \frac{1}{\sqrt{3}+2\cos(2\theta)} \ge \frac{1}{2+\sqrt{3}} = 2-\sqrt{3}$$

30. $\theta \in P$

$\Leftrightarrow \sin\theta-\cos\theta=\sqrt{2}\cos\theta$

$\Leftrightarrow \sin\theta=(\sqrt{2}+1)\cos\theta$

$$\Leftrightarrow \sin\theta=\frac{1}{\sqrt{2}-1}\cos\theta$$

$\Leftrightarrow (\sqrt{2}-1)\sin\theta=\cos\theta$

$\Leftrightarrow\ \sqrt{2}\sin\theta = \sin\theta + \cos\theta$

$\Leftrightarrow\ \theta \in Q$

$\therefore\ P = Q$

Previous Years' B-Architecture Entrance Examination Questions

1. See Question No. 1 in AIEEE/JEE Main Questions

2. $\cos\theta + \sec\theta = 2 \ \Rightarrow\ \cos^2\theta - 2\cos\theta + 1 = 0$

   $\Rightarrow\ \cos\theta = \sec\theta = 1 \ \Rightarrow\ \cos^n\theta + \sec^n\theta = 2$

3. $(\sin^2 x + \cos^2 x)^3 = \sin^6 x + \cos^6 x + 3\sin^2 x\cos^2 x(\sin^2 x + \cos^2 x)$

$$\Rightarrow\quad 1 = p + \frac{3}{4}\sin^2 2x$$

$$\Rightarrow\quad \sin^2 2x = \frac{4}{3}(1-p)$$

$$\Rightarrow\quad 0 \le \frac{4}{3}(1-p) \le 1$$

$$\Rightarrow\quad \frac{1}{4} \le p \le 1 \ \Rightarrow p \in \left[\frac{1}{4}, 1\right]$$

4. $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$

$$= 2\left(\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8}\right)$$

$$= \frac{1}{2}\left[\left(1+\cos\frac{\pi}{4}\right)^2 + \left(1+\cos\frac{3\pi}{4}\right)^2\right]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{\sqrt{2}}\right)^2 + \left(1-\frac{1}{\sqrt{2}}\right)^2\right] = \frac{1}{2}\times 2\left(1+\frac{1}{2}\right) = \frac{3}{2}$$

5. $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$

$$= \sin\left(\frac{\pi}{2}-\frac{3\pi}{7}\right)\sin\left(\frac{\pi}{2}-\frac{2\pi}{7}\right)\sin\left(\frac{\pi}{2}-\frac{\pi}{7}\right)$$

$$= \cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\frac{\pi}{7}$$

$$= \frac{1}{2\sin\frac{\pi}{7}}\cdot\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$$

$$= \frac{1}{4\sin\frac{\pi}{7}}\sin\frac{4\pi}{7}\cos\frac{3\pi}{7}$$

$$= \frac{1}{8\sin\frac{\pi}{7}}\left[\sin\frac{7\pi}{7} + \sin\frac{\pi}{7}\right] = \frac{1}{8}$$

6. $0 \le \alpha, \beta \le \pi/4$

   $\Rightarrow 0 \le \alpha + \beta \le \pi/2 \ \Rightarrow\ -\pi/4 \le \alpha - \beta \le \pi/4$

   Now $\cos(\alpha + \beta) = 4/5 \ \Rightarrow\ \tan(\alpha + \beta) = 3/4$

   and $\sin(\alpha - \beta) = 5/13 \ \Rightarrow\ \tan(\alpha - \beta) = 5/12$

   we have $\tan 2\alpha = \tan[(\alpha + \beta) + (\alpha - \beta)]$

$$= \frac{\tan(\alpha+\beta) + \tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$= \frac{(3/4)+(5/12)}{1-(3/4)(5/12)} = \frac{14/12}{33/48} = \frac{56}{33}$$

   Same as Q10 AIEEE/JEE Main

7. $1 + a\sin 2\beta = \sin^2\beta + \cos^2\beta - 2\sin\beta\cos\beta\cos A$

   $\Rightarrow \cos A = -a$

   $\Rightarrow A$ is obtuse.

   $\Rightarrow A = 120°$

   $\Rightarrow\ a = -\cos 120° = \frac{1}{2}$

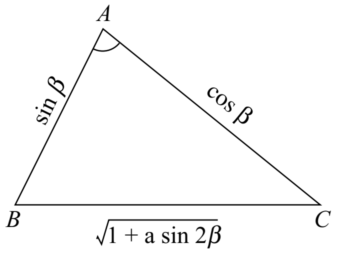


Fig. 25.10

8. $\cos 25° + \sin 25° = k$

   Squaring we get

   $1 + \sin 50° = k^2$

$$\Rightarrow\quad \cos 50° = \sqrt{1-\left(k^2-1\right)^2} = k\sqrt{2-k^2}$$

9. $\tan 9° - \tan 27° - \tan 63° + \tan 81°$

   $= \tan 9° + \tan(90° - 9) - (\tan 27° + \tan(90° - 27°))$

   $= \tan 9° + \cot 9° - (\tan 27° + \cot 27°)$

$$= \frac{2}{\sin 18°} - \frac{2}{\sin 54°} = \frac{2\times 4}{\sqrt{5}-1} - \frac{2\times 4}{\sqrt{5}+1} = 4$$

10. $\sqrt{\frac{a+b}{a-b}} + \sqrt{\frac{a-b}{a+b}}$

$$= \frac{a+b+a-b}{\sqrt{a^2-b^2}} = \frac{2a}{a\sqrt{1-\frac{b^2}{a^2}}}$$

$$= \frac{2}{\sqrt{1-\tan^2\theta}} = \frac{2\cos\theta}{\sqrt{\cos 2\theta}}$$

11. Same as Question No. 9.

12. $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$

   $\Rightarrow a^2 + b^2 + c^2 - (ab + bc + ca) = 0$

   $\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

   $\Rightarrow a = b = c \ \Rightarrow\ A = B = C = 60°$

$$\Rightarrow \sin^2 A + \sin^2 B + \sin^2 C = 3\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4}$$

13. L.H.S. $= \dfrac{\sqrt{2\cos\theta\sin\theta + 1}}{\sin\theta} = \dfrac{\pm(\cos\theta + \sin\theta)}{\sin\theta}$

$= \pm\,[\cot\,\theta + 1] = k - \cot\,\theta$

$\Rightarrow k = -1$

14. $f(x) = 2\sin x + \sin 2x = 2\sin x + 2\sin x\cos x$

$= 2\sin x\,(1 + \cos x)$

$f'(x) = 2\cos x\,(1 + \cos x) - 2\sin x\sin x$

$= 2\,[\cos x + \cos^2 x - (1 - \cos^2 x)]$

$= 2\,[2\cos^2 x + \cos x - 1]$

$= 2\,(\cos x + 1)\,(2\cos x - 1)$

Now, $f'(x) = 0 \;\Rightarrow\; x = \pi/3,\ \pi \quad (\because\ 0 \le x \le 3\pi/2)$

We have $f(0) = 0$, of $\left(\dfrac{3\pi}{2}\right) = -2$

$$f\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} = \frac{3}{2}\sqrt{3},$$

and $f(\pi) = 0,\ f\left(\dfrac{3\pi}{2}\right) = -2$

Thus, maximum value is $\dfrac{3}{2}\sqrt{3}$.

15. Using $C_1 \to C_1 + C_3$, we get

$$\Delta = \begin{vmatrix} 1 & \tan\theta + \sec^2\theta & 3 \\ 0 & \cos\theta & \sin\theta \\ 0 & -4 & 3 \end{vmatrix}$$

$= 3\cos\,\theta + 4\sin\,\theta$

$$\frac{d\Delta}{d\theta} = -3\sin\theta + 4\cos\theta$$

$$\frac{d\Delta}{d\theta} = 0 \Rightarrow \frac{\sin\theta}{4} = \frac{\cos\theta}{3} = \frac{1}{5}$$

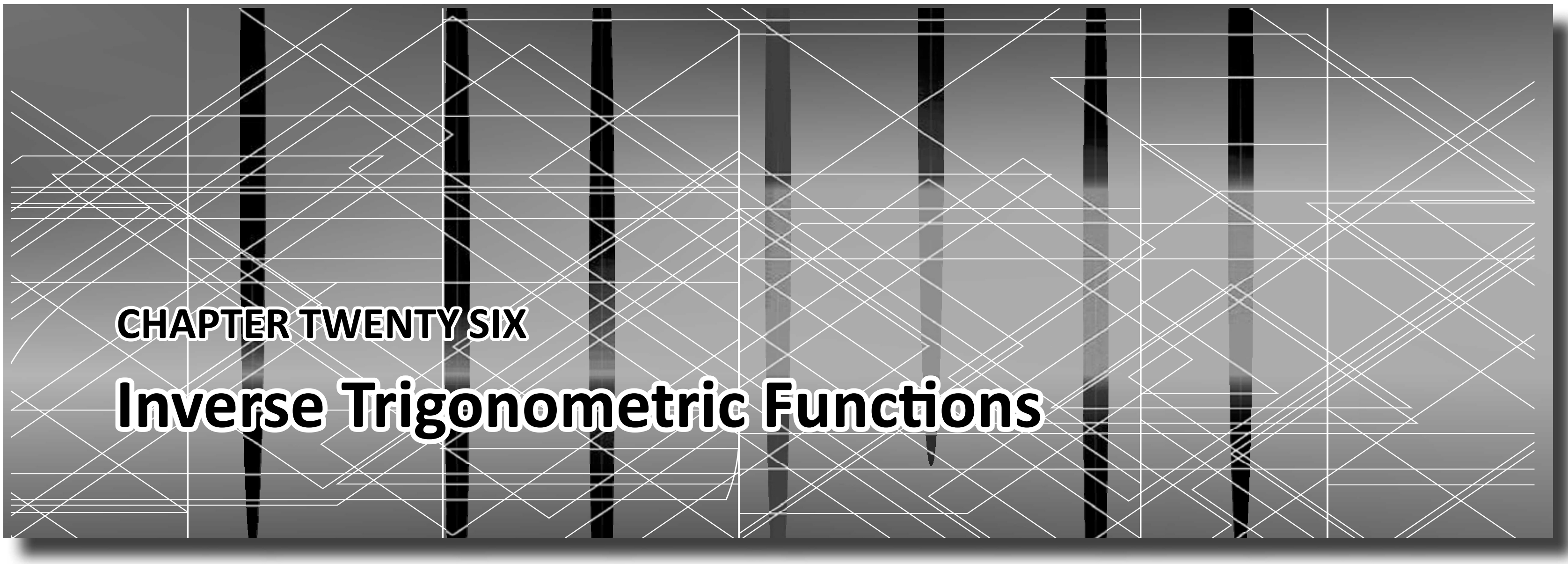
$\Rightarrow \sin\,\theta = 4/5,\ \cos\,\theta = 3/5.$

Max $\Delta = \max\,\{\Delta(0),\ \Delta(\sin^{-1}(4/5)),\ \Delta\left(\dfrac{\pi}{2}\right)\}$

$= \max\left\{3, \dfrac{24}{5}, 4\right\} = \dfrac{24}{5}$

min $\Delta = 3$

$\therefore\ \Delta \in [3, 5]$



CHAPTER TWENTY SIX

Inverse Trigonometric Functions

INVERSE TRIGONOMETRIC FUNCTIONS AND FORMULAE

Function	Domain	Range
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\pi/2 < y < \pi/2$
$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \sec^{-1} x$	$-\infty < x \le -1$	$0 \le y \le \pi$, $y \ne \pi/2$
	or $1 \le x < \infty$	
$y = \text{cosec}^{-1} x$	$-\infty < x \le -1$	$-\pi/2 \le y \le \pi/2, y \ne 0$
	or $1 \le x < \infty$	

(a) (i) $\sin(\sin^{-1} x) = x$ if $-1 \le x \le 1$ and $\sin^{-1}(\sin\theta) = \theta$ if $-\pi/2 \le \theta \le \pi/2$

(ii) $\cos(\cos^{-1} x) = x$ if $-1 \le x \le 1$ and $\cos^{-1}(\cos\theta) = \theta$ if $0 \le \theta \le \pi$

Illustration 1

Find the value of $\sin^{-1}(\sin 12)$ and $\cos^{-1}(\cos 12)$

Solution: $4\pi - \dfrac{\pi}{2} < 12 < 4\pi$

$$\Rightarrow \quad -\frac{\pi}{2} < 12 - 4\pi < 0 < \frac{\pi}{2}$$

So $\sin^{-1}(\sin 12) = \sin^{-1}(\sin(12 - 4\pi))$
$= 12 - 4\pi$

Also $0 < 4\pi - 12 < \pi$

So $\cos^{-1}(\cos 12) = \cos^{-1}(\cos(4\pi - 12)) = 4\pi - 12$

(iii) $\tan(\tan^{-1} x) = x$ if $-\infty < x < \infty$ and $\tan^{-1}(\tan\theta) = \theta$ if $-\pi/2 < \theta < \pi/2$

(iv) $\cot(\cot^{-1} x) = x$ if $-\infty < x < \infty$ and $\cot^{-1}(\cot\theta) = \theta$ if $0 < \theta < \pi$

(v) $\sec(\sec^{-1} x) = x$ if $|x| \ge 1$ and $\sec^{-1}(\sec\theta) = \theta$ if $0 \le \theta \le \pi$, $\theta \ne \pi/2$

(vi) $\text{cosec}(\text{cosec}^{-1} x) = x$ if $|x| \ge 1$ and $\text{cosec}^{1}(\text{cosec}\,\theta) = \theta$ if $-\pi/2 \le \theta \le \pi/2$, $\theta \ne 0$.

(b) (i) $\sin^{-1} x + \cos^{-1} x = \pi/2, \quad -1 \le x \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \pi/2, \quad -\infty < x < \infty$

(iii) $\sec^{-1} x + \text{cosec}^{-1} x = \pi/2, -\infty < x \le -1, \quad 1 \le x < \infty.$

(c) (i) $\sin^{-1} x = \text{cosec}^{-1}(1/x), \quad -1 \le x \le 1, x \ne 0$

(ii) $\cos^{-1} x = \sec^{-1}(1/x), \quad -1 \le x \le 1, x \ne 0$

(iii) $\tan^{-1} x = \cot^{-1}(1/x)$, if $x > 0$ and $\tan^{-1} x = \cot^{-1}(1/x) - \pi$ if $x < 0$.

(d) (i) $\sin^{-1}(-x) = -\sin^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1} x \; (-1 \le x \le 1)$

(ii) $\tan^{-1}(-x) = -\tan^{-1} x$, $\cot^{-1}(-x) = \pi - \cot^{-1} x \; (-\infty < x < \infty)$

(iii) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x \; (|x| \ge 1)$

(e) (i) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ if $-1 \le x, y \le 1, x^2 + y^2 \le 1$ or if $xy < 0, x^2 + y^2 > 1$

$= \pi - \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$ if $0 < x, y \le 1, x^2 + y^2 > 1$

$= -\sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] - \pi$ if $-1 \le x, y < 0, x^2 + y^2 > 1.$

Illustration 2

Find the value of

$$\sin^{-1}\frac{2}{3} + \sin^{-1}\frac{3}{4} + \sin^{-1}\frac{2\sqrt{7} + 3\sqrt{5}}{12}$$

Solution: $0 < \frac{2}{3}, \frac{3}{4} < 1$ and $\left(\frac{2}{3}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{145}{144} > 1$

So, $\sin^{-1}\frac{2}{3} + \sin^{-1}\frac{3}{4} + \sin^{-1}\frac{2\sqrt{7}+3\sqrt{5}}{12}$

$= \pi - \sin^{-1}\left[\frac{2}{3}\sqrt{1-\frac{9}{16}} + \frac{3}{4}\sqrt{1-\frac{4}{9}}\right] + \sin^{-1}\frac{2\sqrt{7}+3\sqrt{5}}{12}$

$= \pi - \sin^{-1}\frac{2\sqrt{7}+3\sqrt{5}}{12} + \sin^{-1}\frac{2\sqrt{7}+3\sqrt{5}}{12} = \pi$

(ii) $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$ if $-1 \le x, y \le 1, x^2 + y^2 \le 1$ or if $xy > 0$, and $x^2 + y^2 > 1$

$= \pi - \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$ if $0 < x \le 1, -1 \le y < 0, x^2 + y^2 > 1$

$= -\sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right] - \pi$ if $-1 \le x < 0, 0 < y \le 1, x^2 + y^2 > 1$

(iii) $\cos^{-1} \mathrm{x} + \cos^{-1} y$

$= \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$ if $-1 \le x, y \le 1, x + y \ge 0$

$= 2\pi - \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$ if $-1 \le x, y < 0, x + y \le 0$

(iv) $\cos^{-1} x - \cos^{-1} y$

$= \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$ if $-1 \le x, y \le 1, x \le y$

$= -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$ if $-1 \le y < 0, 0 < x \le 1, x \ge y$

(v) $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy < 1$

$= \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy > 1, x > 0, y > 0$

$= \tan^{-1}\left(\frac{x+y}{1-xy}\right) - \pi$ if $xy > 1, x < 0, y < 0$

Illustration 3

Find the value of

$\tan^{-1}(1/2) - \tan^{-1}(-3) - \tan^{-1}(-7)$

Solution: $-3 < 0, -7 < 0, (-3)(-7) > 1$

So $\tan^{-1}(1/2) - \tan^{-1}(-3) - \tan^{-1}(-7)$

$= \tan^{-1}(1/2) - \left[\tan^{-1}\frac{-3-7}{1-(-3)(-7)}\right] + \pi = \pi$

(vi) $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ if $xy > -1$

$$= \tan^{-1}\left(\frac{x-y}{1+xy}\right) - \pi \qquad \text{if } x < 0,\ y > 0,\ xy < -1$$

$$= \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right) \qquad \text{if } x > 0,\ y < 0,\ xy < -1$$

(vii) $$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1} x \qquad \text{if } x \le 1$$

$$= \tan^{-1} x - \frac{3\pi}{4} \qquad \text{if } x > 1$$

(viii) $$\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{\pi}{4} - \tan^{-1} x \qquad \text{if } x \ge -1$$

$$= \tan^{-1} x - \frac{\pi}{4} \qquad \text{if } x < -1$$

(f) (i) $$2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \qquad \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$$

$$= \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right) \qquad \text{if } \frac{1}{\sqrt{2}} < x \le 1$$

$$= \sin^{-1}\left(2x\sqrt{1-x^2}\right) - \pi \qquad \text{if } -1 \le x < -\frac{1}{\sqrt{2}}$$

(ii) $$2\cos^{-1} x = \cos^{-1}(2x^2 - 1) \qquad \text{if } 0 \le x \le 1$$

$$= \pi - \cos^{-1}(2x^2 - 1) \qquad \text{if } -1 \le x \le 0$$

Illustration 4

Find the value of

$$\cos\left(2\cos^{-1}\left(-\frac{3}{4}\right)\right)$$

Solution: $\cos\left(2\cos^{-1}\left(-\frac{3}{4}\right)\right)$

$$= \cos\left(\pi - \cos^{-1}\left(2 \times \frac{9}{16} - 1\right)\right)$$

$$= \cos\left(\pi - \cos^{-1}\left(\frac{1}{8}\right)\right)$$

$$= -\cos\left(\cos^{-1}\left(\frac{1}{8}\right)\right) = -\frac{1}{8}.$$

(iii) $$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \qquad \text{if } -1 < x < 1$$

$$= \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) \qquad \text{if } x > 1$$

$$= \tan^{-1}\left(\frac{2x}{1-x^2}\right) - \pi \qquad \text{if } x < -1$$

(iv) $$2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \qquad \text{if } -1 \le x \le 1$$

$$= \pi + \sin^{-1}\left(\frac{2x}{1+x^2}\right) \qquad \text{if } x > 1$$

$$= \sin^{-1}\left(\frac{2x}{1+x^2}\right) - \pi \qquad \text{if } x < -1$$

(v) $$2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \qquad \text{if } 0 \le x < \infty$$

$$= -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \qquad \text{if } -\infty < x \le 0$$

(g) (i) $$3\sin^{-1}x = \sin^{-1}(3x - 4x^3) \qquad \text{if } -\frac{1}{2} \le x \le \frac{1}{2}$$

(ii) $$3\cos^{-1}x = \cos^{-1}(4x^3 - 3x) \qquad \text{if } \frac{1}{2} \le x \le 1$$

(iii) $$3\tan^{-1}x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right) \qquad \text{if } -\frac{1}{\sqrt{3}} \le x \le \frac{1}{\sqrt{3}}$$

Illustration 5

Find the values of

$$\sin\left(3\sin^{-1}\left(\frac{1}{3}\right)\right), \cos\left(3\cos^{-1}\left(\frac{2}{3}\right)\right), \tan\left(3\tan^{-1}\left(\frac{1}{2}\right)\right)$$

Solution:

$$\sin\left(3\sin^{-1}\left(\frac{1}{3}\right)\right) = \sin\left(\sin^{-1}\left(3\times\frac{1}{3} - 4\times\frac{1}{27}\right)\right) = \frac{23}{27}$$

$$\cos\left(3\cos^{-1}\left(\frac{2}{3}\right)\right) = \cos\left(\cos^{-1}\left(4\times\frac{8}{27} - 3\times\frac{2}{3}\right)\right)$$

$$= \cos\ \cos^{-1}\left(-\frac{22}{27}\right)$$

$$= \cos\left(\pi - \cos^{-1}\left(\frac{22}{27}\right)\right)$$

$$= -\frac{22}{27}.$$

$$\tan\left(3\tan^{-1}\left(\frac{1}{2}\right)\right) = \tan\left(\tan^{-1}\left(\frac{3\times\frac{1}{2} - \frac{1}{8}}{1 - 3\times\frac{1}{4}}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{11}{2}\right)\right) = \frac{11}{2}$$

If an expression contains

$$\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$$

simplify them by applying the following substitution.

Expression	**Substitution**	
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$(0 < \theta < \pi)$
$\sqrt{a^2 - x^2}$ and $\sqrt{a^2 + x^2}$ (Both)	$x^2 = a^2 \cos \theta$	$\left(-\frac{\pi}{4} < \theta < \frac{\pi}{4}\right)$

SOLVED EXAMPLES

Concept-based

Straight Objective Type Questions

Example 1: If $\cos^{-1}x + \cos^{-1}(2/3) = (\pi/2)$, the value of x is

(a) $1/3$ (b) $\sqrt{5}/3$

(c) $\sqrt{5}/2$ (d) $1/2$

Ans. (b)

Solution: $\cos^{-1} x = \pi/2 - \cos^{-1} 2/3$

$$\Rightarrow \quad x = \cos\left(\frac{\pi}{2} - \cos^{-1}\frac{2}{3}\right) = \sin \cos^{-1}\left(\frac{2}{3}\right)$$

$$= \sqrt{1 - \frac{4}{9}} = \frac{\sqrt{5}}{3}$$

Example 2: The value of

$\cos\left(3\cos^{-1}\frac{4}{5}\right) + \sin\left(3\sin^{-1}\frac{4}{5}\right)$ is equal to

(a) -1 (b) 0

(c) $4/5$ (d) 1

Ans. (b)

Solution: $\cos\left[\cos^{-1}\left(4 \times (4/5)^3 - 3 \times 4/5\right)\right] +$ $\sin\left[\sin^{-1}\left(3 \times 4/5 - 4 \times (4/5)^3\right)\right]$

$$= 4 \times (4/5)^3 - 3 \times (4/5) + 3 \times (4/5) - 4 \times (4/5)^3 = 0$$

Example 3: A value of x for which $\sin(\cot^{-1}(1 + x)) = \cos(\tan^{-1} x)$ is

(a) $-(1/2)$ (b) 1

(c) 0 (d) $1/2$

Ans. (a)

Solution: $\sin(\cot^{-1}(1 + x)) = \sin(\pi/2 + \tan^{-1} x)$

$$\Rightarrow \quad \cot^{-1}(1 + x) = \pi/2 + \tan^{-1} x$$

$$\Rightarrow \quad 1 + x = \cot(\pi/2 + \tan^{-1} x) = -\tan(\tan^{-1} x) = -x$$

$$\Rightarrow \quad 2x = -1 \Rightarrow x = -(1/2)$$

Example 4: The number of solutions of the equation $\sin^{-1} x = 2\tan^{-1} x$ (in principal values) is

(a) 1 (b) 4

(c) 2 (d) 3

Ans. (d)

Solution: Let $x = \tan \theta$, then $\sin^{-1} x = 2\tan^{-1} x = 2\theta$

$$\Rightarrow \quad x = \sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta} = \frac{2x}{1 + x^2}.$$

$$\Rightarrow \quad x(x^2 - 1) = 0 \Rightarrow x = 0, \pm 1$$

Example 5: The principal value of $\tan^{-1}(\cot 43\pi/4)$ is

(a) $-3\pi/4$ (b) $3\pi/4$
(c) $\pi/4$ (d) $-\pi/4$

Ans. (d)

Solution: $\tan^{-1}\cot(11\pi - \pi/4)$
$= \tan^{-1}\cot(-\pi/4) = \tan^{-1}(-1) = -(\pi/4)$.

Example 6: $2\cot^{-1}(7) + \cos^{-1}(3/5)$, in principal value is equal to

(a) $\cos^{-1}\left(\dfrac{44}{125}\right)$ (b) $\cot^{-1}\left(\dfrac{44}{117}\right)$
(c) $\tan^{-1}\left(\dfrac{41}{117}\right)$ (d) $\operatorname{cosec}^{-1}\left(\dfrac{117}{125}\right)$

Ans. (b)

Solution: $2\cot^{-1}(7) + \cos^{-1}(3/5)$

$$= 2\tan^{-1}(1/7) + \tan^{-1}(4/3)$$

$$= \tan^{-1}\frac{2(1/7)}{1-(1/7)^2} + \tan^{-1}(4/3)$$

$$= \tan^{-1}(7/24) + \tan^{-1}(4/3).$$

$$= \tan^{-1}\frac{(7/24)+(4/3)}{1-(7/24)\times(4/3)} = \tan^{-1}\frac{117}{44} = \cot^{-1}\frac{44}{117}$$

Example 7: Number of solutions of the equation $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = 2\pi^3$ is

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (a)

Solution: $\sin^{-1}x \le \pi/2$, $\cos^{-1}x \le \pi$

$$\Rightarrow \quad (\sin^{-1}x)^3 + (\cos^{-1}x)^3 \le \frac{9\pi^3}{8}$$

So the given equation has no solution.

Example 8: If $\cot(\cot^{-1}x - \tan^{-1}x) = 24/7$, then the values of x are

(a) $-(3/4), (4/3)$ (b) $(3/4), (4/3)$
(c) $-(3/4), (4/3)$ (d) $(3/4), (4/3)$

Ans. (b)

Solution: $\cot(\cot^{-1}x - \tan^{-1}x) = 24/7$

$$\Rightarrow \quad \cot(\pi/2 - 2\tan^{-1}x) = 24/7$$

$$\Rightarrow \quad \tan(2\tan^{-1}x) = 24/7$$

$$\Rightarrow \quad \frac{2x}{1-x^2} = \frac{24}{7}$$

$$\Rightarrow \quad 12x^2 + 7x - 12 = 0$$

$$\Rightarrow \quad (3x+4)(4x-3) = 0$$

$$\Rightarrow \quad x = -(4/3), (3/4).$$

Example 9: $\cos(2\cos^{-1}x) + \cos(2\sin^{-1}x) = 0$ for

(a) all real values of x (b) two values of x
(c) no value of x (d) $-1 \le x \le 1$

Ans. (d)

Solution: $\cos(2\cos^{-1}x) + \cos(2\sin^{-1}x)$
$= 2x^2 - 1 + 1 - 2x^2 = 0$

for all x for which $\cos^{-1}x$ and $\sin^{-1}x$ are defined ie $-1 \le x \le 1$.

Example 10: The value of $\tan\left[\sin^{-1}(7/\sqrt{50}) + \cos^{-1}(1/\sqrt{50}) + \cot^{-1}(1/7)\right]$ is equal to

(a) 7 (b) $\sqrt{50}$
(c) 161/73 (d) 73/161

Ans. (c)

Solution: Let $\sin^{-1}(7/\sqrt{50}) = \theta \Rightarrow (7/\sqrt{50}) = \sin\theta$

$\Rightarrow \cos\theta = (1/\sqrt{50})$

$$\Rightarrow \quad \tan\theta = 7 \Rightarrow \cot\theta = 1/7.$$

So $\tan\left[\sin^{-1}(7/\sqrt{50}) + \cos^{-1}(1/\sqrt{50}) + \cot^{-1}(1/7)\right]$

$$= \tan(3\theta)$$

$$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$= \frac{3\times 7 - 7^3}{1 - 7^2} = \frac{7\times 46}{146} = \frac{161}{73}.$$

LEVEL 1

Straight Objective Type Questions

Example 11: If $x = 1/5$, the value of $\cos(\cos^{-1}x + 2\sin^{-1}x)$ is

(a) $-\sqrt{24/25}$ (b) $\sqrt{24/25}$
(c) $-1/5$ (d) $1/5$

Ans. (c)

Solution: The given expression is equal to

$$\cos(\cos^{-1}x + \sin^{-1}x + \sin^{-1}x) = \cos(\pi/2 + \sin^{-1}x)$$

$$= -\sin(\sin^{-1}x) = -x = -1/5.$$

Example 12: $\tan^{-1}(1/11) + \tan^{-1}(2/12)$ is equal to

(a) $\tan^{-1}(33/132)$　　(b) $\tan^{-1}(1/2)$
(c) $\tan^{-1}(132/33)$　　(d) none of these

Ans. (d)

Solution: $\tan^{-1}(1/11) + \tan^{-1}(1/6)$

$$= \tan^{-1}\frac{\frac{1}{11}+\frac{1}{6}}{1-\frac{1}{11}\times\frac{1}{6}} = \tan^{-1}\left(\frac{17}{65}\right)$$

Example 13: The value of $\sin^{-1}(\sin 10)$ is

(a) 10　　(b) $3\pi - 10$
(c) $10 - 3\pi$　　(d) none of these

Ans. (b)

Solution: $y = \sin^{-1}(\sin 10)$

$$\Rightarrow \quad \sin y = \sin 10 = \sin(3\pi + (10 - 3\pi)) \quad (\because 3\pi < 10 < 3\pi + \pi/2)$$
$$= -\sin(10 - 3\pi)$$
$$= \sin(3\pi - 10)$$
$$\Rightarrow \quad y = 3\pi - 10$$

Example 14: If $\sin^{-1} x + \sin^{-1} y = 2\pi/3$, then $\cos^{-1} x + \cos^{-1} y$ is equal to

(a) $2\pi/3$　　(b) $\pi/3$
(c) $\pi/6$　　(d) π

Ans. (b)

Solution: Let $\cos^{-1} x + \cos^{-1} y = \theta$

then $\quad \sin^{-1} x + \cos^{-1} x + \sin^{-1} y + \cos^{-1} y = 2\pi/3 + \theta$

$$\Rightarrow \quad \frac{\pi}{2} + \frac{\pi}{2} = \frac{2\pi}{3} + \theta \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

Example 15: $\sin^{-1}(1 - x) - 2\sin^{-1} x = \pi/2$, then x is equal to

(a) 0, 1/2　　(b) 1, 1/2
(c) 0　　(d) 1/2

Ans. (c)

Solution: $\sin^{-1}(1 - x) = \pi/2 + 2\sin^{-1} x$

$$\Rightarrow \quad 1 - x = \sin\left(\pi/2 + 2\sin^{-1} x\right) = \cos(2\sin^{-1} x)$$
$$= 2(1 - x^2) - 1$$
$$\Rightarrow \quad 2x^2 - x = 0 \quad \Rightarrow \quad x = 0 \text{ or } 1/2$$

But $x = 1/2$ does not satisfy the given equation.
Hence $x = 0$.

Example 16: If x, y, z are in G.P., $\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are in A.P., then

(a) $x = y = z$ or $y \neq 1$
(b) $x = 1/z$
(c) $x = y = z$, but their common value is not necessary 0
(d) $x + z = y$

Ans. (c)

Solution: x, y, z are in G.P $\Rightarrow y^2 = xz$

$\tan^{-1} x, \tan^{-1} y, \tan^{-1} z$ are in A.P.

$$\Rightarrow \quad 2\tan^{-1} y = \tan^{-1} x + \tan^{-1} z$$
$$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$
$$\Rightarrow \quad 2y = x + z \text{ as } y^2 = xz.$$

$\Rightarrow$ x, y, z are in A.P.

So x, y, z are in A.P. and are in G.P.

$\Rightarrow$ $x = y = z$ for any real value of x.

Example 17: If $\cos^{-1} x + \cos^{-1}(y/2) = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to

(a) $4\sin^2\alpha$　　(b) $-4\sin^2\alpha$
(c) $2\sin 2\alpha$　　(d) 4

Ans. (a)

Solution: $\cos\alpha = x(y/2) + \sqrt{(1-x^2)}\,\sqrt{1-(y^2/4)}$

$$\Rightarrow \quad (2\cos\alpha - xy)^2 = (1 - x^2)(4 - y^2)$$
$$\Rightarrow \quad 4\cos^2\alpha - 4xy\cos\alpha = 4 - 4x^2 - y^2$$
$$\Rightarrow \quad 4x^2 - 4xy\cos\alpha + y^2 = 4(1 - \cos^2\alpha) = 4\sin^2\alpha.$$

Example 18: If $\sin^{-1}(x/5) + \operatorname{cosec}^{-1}(5/4) = \pi/2$ then a values of x is

(a) 1　　(b) 3
(c) 4　　(d) 5

Ans. (b)

Solution: $\sin^{-1}(x/5) + \operatorname{cosec}^{-1}(5/4) = \pi/2$

$$\Rightarrow \quad \sin^{-1}(x/5) + \sin^{-1}(4/5) = \pi/2$$
$$\Rightarrow \quad \sin^{-1}(x/5) + \cos^{-1}\sqrt{1 - 16/25} = \pi/2$$
$$\Rightarrow \quad \sin^{-1}(x/5) + \cos^{-1}(3/5) = \pi/2$$
$$\Rightarrow \quad x = 3 \text{ as } \sin^{-1} x + \cos^{-1} x = \pi/2 \ \forall\ x.$$

Example 19: $\tan^{-1}(1/4) + \tan^{-1}(2/9)$ is equal to

(a) $(1/2)\cos^{-1}(3/5)$　　(b) $\sin^{-1}(4/5)$
(c) $(1/2)\tan^{-1}(3/5)$　　(d) $\tan^{-1}(8/9)$

Ans. (a)

Solution: $\tan^{-1}(1/4) + \tan^{-1}(2/9)$

$$= \tan^{-1}\frac{1/4 + 2/9}{1 - (1/4)(2/9)}$$
$$= \tan^{-1}(17/34) = \tan^{-1}(1/2)$$

If $\theta = \tan^{-1}(1/2) \quad \Rightarrow \quad \tan\theta = (1/2)$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} = \frac{4}{5},\ \cos 2\theta = \frac{3}{5}$$

$\Rightarrow \qquad \theta = \frac{1}{2}\cos^{-1}\frac{3}{5}$

Example 20: $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$ for

(a) $|x| \geq 1$ (b) $x \geq 0$
(c) $|x| \leq 1$ (d) all $x \in \mathbb{R}$.

Ans. (c)

Solution: $-\frac{\pi}{2} \leq \sin^{-1}\left(\frac{2x}{1+x^2}\right) \leq \frac{\pi}{2}$

$\Rightarrow \qquad -(\pi/2) \leq 2\tan^{-1}x \leq (\pi/2)$

$\Rightarrow \qquad -(\pi/4) \leq \tan^{-1}x \leq (\pi/4)$

$\Rightarrow \qquad \tan(-\pi/4) \leq x \leq \tan(\pi/4)$

$\Rightarrow \qquad -1 \leq x \leq 1 \quad \Rightarrow \quad |x| \leq 1$

Example 21: $\cot^{-1}[(\cos\alpha)^{1/2}] - \tan^{-1}[(\cos\alpha)^{1/2}] = x$ then $\sin x =$

(a) $\tan^2(\alpha/2)$ (b) $\cot^2(\alpha/2)$
(c) $\tan\alpha$ (d) $\cot\alpha$

Ans. (a)

Solution: $x = (\pi/2) - 2\tan^{-1}[(\cos\alpha)^{1/2}]$

$\Rightarrow \qquad (\pi/2) - x = 2\tan^{-1}[(\cos\alpha)^{1/2}]$

$\Rightarrow \qquad \tan(\pi/2 - x) = \frac{2(\cos\alpha)^{1/2}}{1-\cos\alpha}$

$\Rightarrow \qquad \cot x = \frac{2(\cos\alpha)^{1/2}}{1-\cos\alpha}$

$\Rightarrow \qquad \operatorname{cosec}^2 x = 1 + \frac{4\cos\alpha}{(1-\cos\alpha)^2} = \left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)^2$

$\Rightarrow \qquad \sin x = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{2\sin^2(\alpha/2)}{2\cos^2(\alpha/2)}$

$\qquad\qquad = \tan^2(\alpha/2)$

Example 22: The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for

(a) all real values of a (b) $|a| \leq \frac{1}{\sqrt{2}}$

(c) $|a| \geq \frac{1}{\sqrt{2}}$ (d) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$

Ans. (b)

Solution: $\sin^{-1}x = \sin^{-1}2a\sqrt{1-a^2}$ if $|a| \leq 1/\sqrt{2}$

$\Rightarrow \quad x = 2a\sqrt{1-a^2}$ which is possible

if $\quad x^2 = 4a^2(1 - a^2) \leq 1 \qquad [\because -1 \leq \sin^{-1}x \leq 1]$

or if $4a^4 - 4a^2 + 1 \geq 0$ if $(2a^2 - 1)^2 \geq 0$

which is true, so $|a| \leq 1/\sqrt{2}$

Example 23: If $0 \leq x \leq 1$ and $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x$, then

(a) $\theta \leq \pi/2$ (b) $\theta \geq \pi/4$
(c) $\theta = \pi/4$ (d) $\pi/4 \leq \theta \leq \pi/2$

Ans. (d)

Solution: $\theta = \sin^{-1}x + \cos^{-1}x - \tan^{-1}x = \pi/2 - \tan^{-1}x$

Since $0 \leq x \leq 1 \quad \Rightarrow \quad 0 \leq \tan^{-1}x \leq \pi/4 \quad \Rightarrow \quad \pi/4 \leq \theta \leq \pi/2$.

Example 24: If $x > 0$, $y > 0$ and $x > y$, then $\tan^{-1}(x/y) + \tan^{-1}[(x + y)/(x - y)]$ is equal to

(a) $-\pi/4$ (b) $\pi/4$
(c) $3\pi/4$ (d) none of these

Ans. (c)

Solution: Since $\frac{x}{y}.\frac{x+y}{x-y} > 1$. The given expression is equal to

$$\pi + \tan^{-1}\left[\frac{\frac{x}{y}+\frac{x+y}{x-y}}{1-\frac{x}{y}\times\frac{x+y}{x-y}}\right]$$

$$= \pi + \tan^{-1}\frac{x^2+y^2}{-(x^2+y^2)} = \pi + \tan^{-1}(-1) = 3\pi/4.$$

Example 25: The principal value of $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}\cos(7\pi/6)$ is

(a) $5\pi/6$ (b) $\pi/2$
(c) $3\pi/2$ (d) none of these

Ans. (b)

Solution: $\sin^{-1}(-\sqrt{3}/2) = -\sin^{-1}(\sqrt{3}/2) = -\pi/3$

and $\qquad \cos^{-1}(\cos(7\pi/6) = \cos^{-1}\cos(2\pi - 5\pi/6)$

$\qquad\qquad = \cos^{-1}\cos(5\pi/6) = 5\pi/6$

Hence $\sin^{-1}(-\sqrt{3}/2) + \cos^{-1}(\cos 7\pi/6)$

$= -(\pi/3) + (5\pi/6) = (\pi/2)$.

Example 26: The value of $\cos^{-1}(-1/2) - 2\sin^{-1}(1/2) + 3\cos^{-1}(-1/\sqrt{2}) - 4\tan^{-1}(-1)$ is equal to

(a) $7\pi/4$ (b) $11\pi/4$
(c) $\pi/12$ (d) $25\pi/12$

Ans. (d)

Solution: The given expression is equal to

$$(2\pi/3) - 2\times(\pi/6) + 3\times(3\pi/4) - 4\,(-\pi/4) = 25\pi/12.$$

Example 27: If $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x} = 4$ then $x =$

(a) tan 2 (b) tan 4
(c) tan (1/4) (d) tan 8

Ans. (d)

Solution: Taking $x = \tan\theta$, $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$

$$= \tan^{-1}\frac{\sec\theta - 1}{\tan\theta}$$

$$= \tan^{-1}\frac{1-\cos\theta}{\sin\theta} = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= (1/2)\theta = (1/2)\tan^{-1}(x)$$

so that according to the given condition,

$$(1/2)\tan^{-1} x = 4 \quad\Rightarrow\quad \tan^{-1} x = 8 \text{ or } x = \tan 8.$$

Example 28: $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$ is equal to

(a) 1 (b) 5
(c) 10 (d) 15

Ans. (d)

Solution: The given expression is equal to

$$1 + \tan^2(\tan^{-1} 2) + 1 + \cot^2(\cot^{-1} 3)$$
$$= 1 + 4 + 1 + 9 = 15.$$

Example 29: The equation $2\cos^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$ is valid for all values of x satisfying

(a) $-1 \le x \le 1$ (b) $0 \le x \le 1$
(c) $0 \le x \le 1/\sqrt{2}$ (d) $1/\sqrt{2} \le x \le 1$

Ans. (d)

Solution: If we denote $\cos^{-1}x$ by y, then

$$\text{since } 0 \le \cos^{-1} x \le \pi \;\Rightarrow\; 0 \le 2y \le 2\pi \qquad (1)$$

Also since $-\pi/2 \le \sin^{-1}(2x\sqrt{1-x^2}) \le \pi/2$

$\Rightarrow -\pi/2 \le \sin^{-1}\sin(2y) \le \pi/2$

$\Rightarrow \quad -\pi/2 \le 2y \le \pi/2 \qquad (2)$

From (1) and (2) we find $0 \le 2y \le \pi/2$

$\Rightarrow \quad 0 \le y \le \pi/4 \;\Rightarrow\; 0 \le \cos^{-1} x \le \pi/4$

which holds if $1/\sqrt{2} \le x \le 1$.

Example 30: If $\tan^{-1}\dfrac{1}{1+2} + \tan^{-1}\dfrac{1}{1+(2)(3)} + \tan^{-1}\dfrac{1}{1+(3)(4)} + \ldots \tan^{-1}\dfrac{1}{1+n(n+1)} = \tan^{-1}\theta$, then $\theta =$

(a) $\dfrac{n}{n+1}$ (b) $\dfrac{n+1}{n+2}$
(c) $\dfrac{n}{n+2}$ (d) $\dfrac{n-1}{n+2}$

Ans. (c)

Solution: $\tan^{-1}\dfrac{1}{1+n(n+1)} = \tan^{-1}\dfrac{n+1-n}{1+n(n+1)}$

$$= \tan^{-1}(n+1) - \tan^{-1}(n)$$

so that L.H.S. of the given equation is

$\tan^{-1} 2 - \tan^{-1} 1 + \tan^{-1} 3 - \tan^{-1} 2 + \cdots + \tan^{-1}(n+1) - \tan^{-1} n.$

$$= \tan^{-1}(n+1) - \tan^{-1} 1$$

$$= \tan^{-1}\frac{n+1-1}{1+(n+1)} = \tan^{-1}\frac{n}{n+2}$$

so that $\quad \tan^{-1}\dfrac{n}{n+2} = \tan^{-1}\theta \;\Rightarrow\; \theta = \dfrac{n}{n+2}.$

Example 31: A value of x satisfying $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}(3/5)$ is

(a) 0 (b) 2
(c) 4 (d) 8

Ans. (c)

Solution: $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}(3/5)$

$\Rightarrow \quad \tan^{-1}\dfrac{(x+3)-(x-3)}{1+(x+3)(x-3)} = \tan^{-1}(3/4)$

$\Rightarrow \quad \dfrac{6}{1+x^2-9} = \dfrac{3}{4} \quad \Rightarrow x^2 = 16$

$\Rightarrow \quad x = \pm 4.$

Example 32: If $x = \operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1} a)))))$ $a \in [0, 1]$, then

(a) $x^2 - a^2 = 3$ (b) $x^2 + a^2 = 3$
(c) $x^2 - a^2 = 2$ (d) $x^2 + a^2 = 2$

Ans. (b)

Solution $\quad x = \operatorname{cosec}\left(\tan^{-1}\left(\cos\left(\cot^{-1}\left(\dfrac{1}{\sqrt{1-a^2}}\right)\right)\right)\right)$

$$= \operatorname{cosec}(\tan^{-1}(\cos(\sec^{-1}(\sqrt{2-a^2}))))$$

$$= \operatorname{cosec}\left(\tan^{-1}\frac{1}{\sqrt{2-a^2}}\right)$$

$$= \operatorname{cosec}(\cot^{-1}\sqrt{2-a^2}) = \sqrt{3-a^2}$$

$\Rightarrow \quad x^2 + a^2 = 3$

Example 33: If $\sin^{-1}\left(\sin\frac{33\pi}{7}\right) + \cos^{-1}\left(\cos\frac{46\pi}{7}\right)$ $+ \tan^{-1}\left(-\tan\frac{13\pi}{8}\right) + \cot^{-1}\left(\cot\frac{-19\pi}{8}\right) = \frac{a\pi}{b}$

where a and b are in their lowest form, then $(a + b)$ is equal to

(a) 17 (b) 20

(c) 23 (d) none of these

Ans. (b)

Solution: We have L.H.S $= \sin^{-1}(\sin(5\pi - 2\pi/7)) + \cos^{-1}(\cos(7\pi - 3\pi/7)) + \tan^{-1}(-\tan(2\pi - 3\pi/8)) + \cot^{-1}(-\cot(3\pi - 5\pi/8))$

$$= \sin^{-1}(\sin(2\pi/7)) + \cos^{-1}(-\cos(3\pi/7)) + \tan^{-1}(\tan(3\pi/8)) + \cot^{-1}(\cot(5\pi/8))$$

$$= \frac{2\pi}{7} + \pi - \frac{3\pi}{7} + \frac{3\pi}{8} + \frac{5\pi}{8}$$

$$= \frac{13\pi}{7} = \frac{a\pi}{b} \quad \Rightarrow a + b = 20$$

Example 34: If $0 < x < 1$, the number of solutions of the equation $\tan^{-1}(x - 1) + \tan^{-1}(x) + \tan^{-1}(x + 1) = \tan^{-1}(3x)$ is

(a) 0 (b) 1

(c) 2 (d) 3

Ans. (b)

Solution: $\tan^{-1}(x - 1) + \tan^{-1}(x + 1) = \tan^{-1}3x - \tan^{-1}x$

$$\Rightarrow \quad \frac{(x-1)+(x+1)}{1-(x+1)(x-1)} = \frac{3x-x}{1+3x^2}$$

$$\Rightarrow \quad 1 + 3x^2 = 2 - x^2, \quad x \neq 0$$

$\Rightarrow$ $x = \pm 1/2$, so $x = 1/2$ as $0 < x < 1$.

Example 35: $\tan^{-1}(\tan 4) - \tan^{-1}(\tan(-6)) + \cos^{-1}(\cos 10)$ is equal to

(a) 0 (b) π

(c) $-\pi$ (d) 5π

Ans. (b)

Solution: $\pi < 4 < 3\pi/2 \quad \Rightarrow 0 < 4 - \pi < \pi/2$

$\Rightarrow \tan^{-1}(\tan 4) = 4 - \pi \quad \because \tan(4 - \pi) = \tan 4$

$2\pi - 6 \in (-\pi/2, \pi/2)$, $\tan(2\pi - 6) = -\tan 6$

$\Rightarrow \tan^{-1}(\tan(-6)) = \tan^{-1}(-\tan 6) = 2\pi - 6$ and

$4\pi - 10 \in (0, \pi)$, $\cos(4\pi - 10) = \cos 10$

so the given expression is equal to

$4 - \pi - (2\pi - 6) + 4\pi - 10 = \pi$

Example 36: If $x < -1/\sqrt{3}$, then the value of

$3\tan^{-1}x - \tan^{-1}\frac{3x - x^3}{1 - 3x^2}$ equals

(a) $-\pi$ (b) π

(c) $-\pi/2$ (d) 0

Ans. (a)

Solution: Let $x = \tan\theta$, then $x < -1/\sqrt{3}$

$$\Rightarrow \quad \tan\theta < -1/\sqrt{3} \quad \Rightarrow -\pi/2 < \theta < -\pi/6$$

$$\Rightarrow \quad -3\pi/2 < 3\theta < -\pi/2$$

$$\Rightarrow \quad -\pi/2 < \pi + 3\theta < \pi/2$$

so $\quad 3\tan^{-1}x - \tan^{-1}\frac{3x - x^3}{1 - 3x^2}$

$$= 3\tan^{-1}(\tan\theta) - \tan^{-1}(\tan 3\theta)$$

$$= 3\tan^{-1}(\tan\theta) - \tan^{-1}(\tan(\pi + 3\theta))$$

$$= 3\theta - (\pi + 3\theta) = -\pi$$

Example 37: $\sin^{-1}(4/5) + \sin^{-1}(5/13) + \sin^{-1}(16/65)$ is equal to

(a) 0 (b) $\pi/2$

(c) π (d) $\sin^{-1}(63/65)$

Ans. (b)

Solution

$$\sin^{-1}\left[(4/5)\sqrt{1-(5/13)^2} + \frac{5}{13}\sqrt{1-(4/5)^2}\right] + \sin^{-1}(16/65)$$

$$= \sin^{-1}[(4/5)\times(12/13) + (5/13)\times(3/5)] + \sin^{-1}(16/65)$$

$$= \sin^{-1}(63/65) + \sin^{-1}(16/65)$$

$$= \cos^{-1}\sqrt{1-(63/65)^2} + \sin^{-1}(16/65)$$

$$= \cos^{-1}(16/65) + \sin^{-1}(16/65) = \pi/2$$

Example 38: If $\tan^{-1}(1/4) + \tan^{-1}(2/9) = (1/2)\cos^{-1}x$

then x is equal to

(a) 1/2 (b) 2/5

(c) 3/5 (d) none of these

Ans. (c)

Solution: $(1/2)\cos^{-1}x$

$$= \tan^{-1}\frac{(1/4)+(2/9)}{1-(1/4)(2/9)} = \tan^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\times 2\tan^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\cos^{-1}\left[\frac{1-(1/4)}{1+(1/4)}\right]$$

$$\left(\text{using } 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \text{ for } x \geq 0\right)$$

$$= \frac{1}{2}\cos^{-1}\frac{3}{5}.$$

Assertion-Reason Type Questions

◉ **Example 39: Statement-1:** The sum of

$$\cot^{-1}\left(\frac{7}{4}\right)+\cot^{-1}\left(\frac{19}{4}\right)+....+\cot^{-1}\frac{4r^2+3}{4}+.... \text{ upto}$$

infinity is $\tan^{-1}2$.

Statement-2: $\sum_{r=1}^{n}\tan^{-1}\frac{x_r-x_{r-1}}{1+x_{r-1}x_r}=\tan^{-1}x_n-\tan^{-1}x_0$

Ans. (a)

◎ **Solution:** L.H.S in Statement-2 is

$$\sum_{r=1}^{n}\left(\tan^{-1}x_r-\tan^{-1}x_{r-1}\right)=\tan^{-1}x_n-\tan^{-1}x_0$$

⇒ Statement-2 is true.

In Statement-1, the given expression is equal to

$$\sum_{r=1}^{\infty}\tan^{-1}\frac{4}{4r^2+3}=\sum_{r=1}^{\infty}\tan^{-1}\frac{1}{1+(r^2-1/4)}$$

$$=\sum_{r=1}^{\infty}\tan^{-1}\frac{(r+1/2)-(r-1/2)}{1+(r+1/2)(r-1/2)}$$

$$=\lim_{n\to\infty}\left[\tan^{-1}(n+1/2)-\tan^{-1}(1/2)\right]$$

(using statement–2)

$$=\pi/2-\tan^{-1}(1/2)=\cot^{-1}(1/2)=\tan^{-1}2.$$

So Statement-1 is also true.

◉ **Example 40: Statement-1:** The equation $(\sin^{-1}x)^3+(\cos^{-1}x)^3=a\pi^3$ has a solution if $\frac{1}{32}\le a\le\frac{7}{8}$.

Statement-2: $0\le(\sin^{-1}x-\pi/4)^2\le 9\pi^2/16$.

Ans. (a)

◎ **Solution:** $-\pi/2\le\sin^{-1}x\le\pi/2$

⇒ $-3\pi/4\le\sin^{-1}x-\pi/4\le\pi/4$

⇒ $(\sin^{-1}x-\pi/4)^2\le 9\pi^2/16$

⇒ Statement-2 is true.

$(\sin^{-1}x)^3+(\cos^{-1}x)^3=a\pi^3$

⇒ $(\sin^{-1}x+\cos^{-1}x)\,[(\sin^{-1}x+\cos^{-1}x)^2-3\sin^{-1}x\cos^{-1}x]=a\pi^3$

⇒ $\pi^2/4-3\sin^{-1}x\cos^{-1}x=2a\pi^2$

⇒ $\sin^{-1}x(\pi/2-\sin^{-1}x)=\pi^2(1-8a)/12$

⇒ $(\sin^{-1}x)^2-(\pi/2)\sin^{-1}x+(\pi^2/12)(1-8a)=0$

⇒ $(\sin^{-1}x-\pi/4)^2=(\pi^2/48)(32a-1)$

using Statement-2,

$0\le(\pi^2/48)(32a-1)\le(9\pi^2/16)$

⇒ $0\le 32a-1\le 27$ ⇒ $1/32\le a\le 7/8$.

⇒ Statement-1 is also true.

◉ **Example 41: Statement-1:** If $1/2\le x\le 1$, then

$$\cos^{-1}x+\cos^{-1}\left[\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right]=\pi/3$$

Statement-2: $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x$ if $x\in\left[-1/\sqrt{2},1/\sqrt{2}\right]$

Ans. (b)

◎ **Solution:** In Statement-1, put $x=\cos\theta$ then $0\le\theta\le\pi/3$

$$\text{L.H.S }=\cos^{-1}(\cos\theta)+\cos^{-1}\left[\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right]$$

$=\theta+\pi/3-\theta=\pi/3$ ⇒ Statement-1 is true

In Statement-2, put $x=\sin\theta$, then $-\pi/4\le\theta\le\pi/4$

L.H.S. $=\sin^{-1}(2\sin\theta\cos\theta)=2\theta=2\sin^{-1}x$.

⇒ Statement-2 is also true but does not lead to Statement-1

◉ **Example 42: Statement-1:** $\operatorname{cosec}^{-1}(3/2)+\cos^{-1}(2/3)-2\cot^{-1}(1/7)-\cot^{-1}7$ is equal to $\cot^{-1}7$.

Statement-2: $\sin^{-1}x+\cos^{-1}x=\pi/2$, $\tan^{-1}x+\cot^{-1}x=\pi/2$

$\operatorname{cosec}^{-1}(x)=\sin^{-1}(1/x)$,

$\cot^{-1}x=\tan^{-1}(1/x)$, $-1\le x\le 1$, $x\ne 0$

Ans. (d)

◎ **Solution:** Statement-2 is true, using in Statement-1

L.H.S $=\sin^{-1}(2/3)+\cos^{-1}(2/3)-(\tan^{-1}7+\cot^{-1}7)-\cot^{-1}(1/7)$

$=\pi/2-\pi/2-\tan^{-1}7=-\tan^{-1}7$.

⇒ Statement-1 is false.

◉ **Example 43: Statement-1:** $\cos^{-1}x=2\sin^{-1}\sqrt{\frac{1-x}{2}}=2\cos^{-1}\sqrt{\frac{1+x}{2}}$

Statement-2: $1+\cos\theta=2\cos^2(\theta/2)$, $1-\cos\theta=2\sin^2(\theta/2)$

Ans. (a)

◎ **Solution:** Statement-2 is true. In Statement-1, put $x=\cos\theta$

then $\cos^{-1}(\cos\theta)=2\sin^{-1}\sqrt{\frac{1-\cos\theta}{2}}=2\cos^{-1}\sqrt{\frac{1+\cos\theta}{2}}$.

Using Statement-2, $\theta=2\times\sin^{-1}(\sin(\theta/2))$

$=2\cos^{-1}(\cos(\theta/2))$

which is true $\Rightarrow$ Statement-1 is also true.

Example 44: Statement-1: If $x = 1/5\sqrt{2}$, then

$\{x \cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 = 51/50.$

Statement-2: $\tan\left[\cos^{-1}\dfrac{1}{5\sqrt{2}} - \sin^{-1}\dfrac{4}{\sqrt{17}}\right] = \dfrac{29}{3}$.

Ans. (c)

Solution: Put $x = \cot y$ in Statement-1,

then L.H.S $= (\cot y \cos y + \sin y)^2 = \left[\dfrac{\cos^2 y + \sin^2 y}{\sin y}\right]^2$

$= 1 + \cot^2 y = 1 + x^2 = 1 + 1/50 = 51/50$

$\Rightarrow$ Statement-1 is true.

In Statement-2, L.H.S $= \tan[\tan^{-1} 7 - \tan^{-1} 4]$

$[\because \cos\theta = 1/5\sqrt{2} \quad \Rightarrow \quad \tan\theta = 7, \sin\theta = 4/\sqrt{17}$

$\Rightarrow \quad \tan\theta = 4]$

$= \dfrac{7-4}{1+7\times 4} = \dfrac{3}{29}$ Statement-2 is false.

Example 45: Statement-1: If $\sin^{-1} x + \cos^{-1}(1 - x) = \sin^{-1}(-x)$, then x satisfies the equation $2x^2 - 3x = 0$

Statement-2: $\sin^{-1}(x) + \sin^{-1}(-x) = 0$

Ans. (a)

Solution: Statement-2 is true as $\sin^{-1}(-x) = -\sin^{-1} x$

In Statement-1,

$\cos^{-1}(1 - x) = \sin^{-1}(-x) - \sin^{-1}(x) = -2\sin^{-1}x$ (using Statement-2)

$\Rightarrow \quad 1 - x = \cos(-2\sin^{-1}x) = \cos(2\sin^{-1}x)$

$= 1 - 2\sin^2(\sin^{-1} x)$

$\Rightarrow \quad 1 - x = 1 - 2x^2 \quad \Rightarrow \quad x = 0$ or $x = 1/2$

But $x = 1/2$ does not satisfy the equtaion.

so $x = 0$, which satisfies $2x^2 - 3x = 0$

$\Rightarrow$ Statement-1 is true.

LEVEL 2

Straight Objective Type Questions

Example 46: If $\operatorname{cosec}^{-1} x = 2\cot^{-1} 7 + \cos^{-1}(3/5)$ then the value of x is

(a) 44/117 (b) 125/117
(c) 24/7 (d) 5/3

Ans. (b)

Solution: $2\cot^{-1} 7 + \cos^{-1}(3/5) = \cot^{-1}\dfrac{7^2 - 1}{2\times 7} + \cot^{-1}\dfrac{3}{4}$

$[\because$ If $\theta = \cos^{-1}(3/5)$, $\cos\theta = 3/5$, $\cot\theta = 3/4]$

$= \cot^{-1}(24/7) + \cot^{-1}(3/4)$

$$= \cot^{-1}\left[\frac{\frac{24}{7}\times\frac{3}{4} - 1}{\frac{24}{7} + \frac{3}{4}}\right].$$

$$= \cot^{-1}\frac{44}{117} = \operatorname{cosec}^{-1}\frac{125}{117}$$

Example 47: $\theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}((1/3)\tan\theta)$ if $\tan\theta$ is equal to

(a) – 2 (b) – 1
(c) 2/3 (d) 2

Ans. (a)

Solution: $\theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}((1/3)\tan\theta)$

$$\Rightarrow \quad \tan\theta = \frac{2\tan^2\theta - (1/3)\tan\theta}{1 + (2/3)\tan^3\theta}$$

$$\Rightarrow \quad \tan\theta\left[\frac{2\tan\theta - (1/3)}{1 + (2/3)\tan^3\theta} - 1\right] = 0$$

which is true if $\tan\theta = 0$

or $\quad \dfrac{2\tan\theta - (1/3)}{1 + (2/3)\tan^3\theta} = 1 \Rightarrow \tan^3\theta - 3\tan\theta + 2 = 0$

$\Rightarrow \quad (\tan\theta - 1)^2(\tan\theta + 2) = 0$ which holds if $\tan\theta = 1$ or $\tan\theta = -2$.

Example 48: If $u = \cot^{-1}\sqrt{\tan\alpha} - \tan^{-1}\sqrt{\tan\alpha}$, then $\tan\left(\dfrac{\pi}{4} - \dfrac{u}{2}\right) =$

(a) $\sqrt{\tan\alpha}$ (b) $\sqrt{\cot\alpha}$
(c) $\tan\alpha$ (d) $\cot\alpha$

Ans. (a)

Solution: Let $\sqrt{\tan\alpha} = \tan x$, then $u = \cot^{-1}(\tan x) - \tan^{-1}(\tan x)$

$$= (\pi/2) - x - x = (\pi/2) - 2x$$

$$\Rightarrow \quad 2x = (\pi/2) - u \quad \Rightarrow \quad x = (\pi/4) - (u/2)$$

$$\Rightarrow \quad \tan x = \tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$$

$$\Rightarrow \quad \sqrt{\tan\alpha} = \tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$$

Example 49: If $\tan^{-1} y = 4\tan^{-1} x$, then $1/y$ is zero for

(a) $x = 1 \pm \sqrt{2}$ (b) $x = \sqrt{2} \pm \sqrt{3}$

(c) $3 \pm 2\sqrt{2}$ (d) all values of x

Ans. (a)

Solution: If we put $x = \tan\theta$, the given equality becomes $\tan^{-1} y = 4\theta$.

$$\Rightarrow \quad y = \tan 4\theta = \frac{2\tan 2\theta}{1-\tan^2 2\theta} = \frac{2\left[\dfrac{2\tan\theta}{1-\tan^2\theta}\right]}{1-\left(\dfrac{2\tan\theta}{1-\tan^2\theta}\right)^2}$$

$$= \frac{2\times 2x\left(1-x^2\right)}{\left(1-x^2\right)^2-4x^2} = \frac{4x\left(1-x^2\right)}{1-6x^2+x^4}$$

so that $1/y$ is zero if $x^4 - 6x^2 + 1 = 0$

$$\Rightarrow \quad x^2 = \frac{6\pm\sqrt{36-4}}{2} = 3 \pm 2\sqrt{2} = (1 \pm \sqrt{2})^2$$

Example 50: $3\cos^{-1} x - \pi x - \pi/2 = 0$ has

(a) one solution
(b) one and only one solution
(c) no solution
(d) more than one solution

Ans. (b)

Solution: $x = 1/2$ is clearly a solution of the given equation which can be obtained by trial and error method. The given equation can be written as

$$3\cos^{-1} x = \pi x + \pi/2 \qquad (1)$$

since the L.H.S. of (1) is a decreasing function and R.H.S. of (1) is an increasing function of x. The equation (1) has either no solution or only one solution. So $x = 1/2$ is one and only one solution of the given equation.

Example 51: If $\cos^{-1} x = \tan^{-1} x$, then $\sin(\cos^{-1} x) =$

(a) x (b) x^2
(c) $1/x$ (d) $1/x^2$

Ans. (b)

Solution: $\cos^{-1} x = \tan^{-1} x = \theta$ (say)

$$\Rightarrow \quad x = \cos\theta = \tan\theta$$

$$\Rightarrow \quad \cos^2\theta = \sin\theta \Rightarrow \sin^2\theta + \sin\theta - 1 = 0$$

$$\Rightarrow \quad \sin\theta = \frac{-1\pm\sqrt{1+4}}{2} \quad \Rightarrow \quad \sin\theta = \frac{\sqrt{5}-1}{2}$$

So $\quad x^2 = \cos^2\theta = \dfrac{\sqrt{5}-1}{2}$

and $\quad \sin(\cos^{-1} x) = \sin\theta = \dfrac{\sqrt{5}-1}{2} = x^2.$

Example 52: $x = n\pi - \tan^{-1} 3$ is a solution of the equation $12\tan 2x + \dfrac{\sqrt{10}}{\cos x} + 1 = 0$ if

(a) n is any integer (b) n is an even integer
(c) n is a positive integer (d) n is an odd integer

Ans. (d)

Solution: $x = n\pi - \tan^{-1} 3 \quad \Rightarrow \quad \tan^{-1} 3 = n\pi - x$

$$\Rightarrow \quad \tan(n\pi - x) = 3 \quad \Rightarrow \quad -\tan x = 3$$

$$\Rightarrow \quad \tan 2x = \frac{2\tan x}{1-\tan^2 x} = \frac{3}{4}$$

and $\quad \cos x = \pm\dfrac{1}{\sqrt{1+\tan^2 x}} = \pm\dfrac{1}{\sqrt{10}}$

on substituting these value in the given equation we find only $\cos x = -1/\sqrt{10}$ satisfies the equation.

So that the given equation holds for values of x for which $\tan x = -3$ and $\cos x = -1/\sqrt{10}$. Which is possible if x lies in the second quadrant only and so n must be an odd integer.

Example 53: If $\dfrac{1}{2}\sin^{-1}\left[\dfrac{3\sin 2\theta}{5+4\cos 2\theta}\right] = \tan^{-1} x$, then $x =$

(a) $\tan 3\theta$ (b) $3\tan\theta$
(c) $(1/3)\tan\theta$ (d) $3\cot\theta$

Ans. (c)

Solution: $\dfrac{3\sin 2\theta}{5+4\cos 2\theta} = \dfrac{6\tan\theta}{9+\tan^2\theta} = \dfrac{2\tan\phi}{1+\tan^2\phi} = \sin 2\phi$

where $\tan\theta = 3\tan\phi$

$$\Rightarrow \quad \left(\frac{1}{2}\right)\sin^{-1}\left[\frac{3\sin 2\theta}{5+4\cos 2\theta}\right] = \phi = \tan^{-1}\left[\left(\frac{1}{3}\right)\tan\theta\right]$$

so that $x = (1/3)\tan\theta$.

Example 54: If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = 3\pi/2$, then the value of

$$x^{100} + y^{100} + z^{100} - \frac{9}{x^{101} + y^{101} + z^{101}} \text{ is}$$

(a) -1 (b) 0
(c) 1 (d) 3

Ans. (b)

Solution: Since $|\sin^{-1} x| \le \pi/2$, from the given relation we have $\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \pi/2$

$\Rightarrow \quad x = y = z = \sin(\pi/2) = 1$

So that the required value is 0.

Example 55: The number of real solutions of

$$\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \pi/2 \text{ is}$$

(a) 0 (b) 1
(c) 2 (d) infinite

Ans. (c)

Solution: $\cos^{-1}\dfrac{1}{\sqrt{1+(x^2+x)}} = \pi/2 - \sin^{-1}\sqrt{x^2+x+1}$

$$= \cos^{-1}\sqrt{x^2+x+1}$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+(x^2+x)}} = \sqrt{x^2+x+1}$$

$$\Rightarrow \quad (x^2+x+1) = 1$$

$$\Rightarrow \quad x^2 + x = 0 \Rightarrow x = -1, 0.$$

EXERCISES

Concept-based Straight Objective Type Questions

1. Value of $\sin(\sin^{-1}(3/5) + \cos^{-1}(4/5))$ is equal to
(a) 12/25 (b) 24/25
(c) 6/25 (d) 7/25

2. $2\tan^{-1} 7 - \sin^{-1}\left(14/\sqrt{50}\right)$ is equal to
(a) 0 (b) $\pi/2$
(c) π (d) 2π

3. $2\sin^{-1} x + 3\cos^{-1} y = \alpha$, $3\cos^{-1} x + 2\sin^{-1} y = \beta$. If $\alpha + \beta = \dfrac{7\pi}{3}$, then $xy - \sqrt{(1-x^2)(1-y^2)}$ is equal to
(a) 1 (b) 1/2
(c) $\sqrt{3}/2$ (d) $1/\sqrt{2}$

4. $2\sin^{-1}(-1/2) + 3\cos^{-1}\left(-\sqrt{3}/2\right) + 4\sin^{-1}\left(1/\sqrt{2}\right) + 5\tan^{-1}(-1)$ is equal to (considering the principal values)
(a) $15\pi/6$ (b) $27\pi/12$
(c) $23\pi/12$ (d) $13\pi/6$

5. If $x + y + z = xyz$ and $x, y, z, > 0$, then the value of $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$ is equal to
(a) $\pi/2$ (b) $\pi/4$
(c) π (d) $-\pi/4$

6. If $\tan(\cos^{-1} x) = \sin \cot^{-1}(1/2)$, then the value of x is
(a) $\sqrt{5}$ (b) $-\sqrt{5}$
(c) $\sqrt{5}/3$ (d) $-\sqrt{5}/3$

7. The value of $\sin^{-1}(\cos(\cos^{-1}(\cos x) + \sin^{-1}(\sin x)))$ where $x \in \left(\dfrac{\pi}{2}, \pi\right)$ is equal to
(a) $-\pi/2$ (b) $\pi/2$
(c) $-\pi$ (d) π

8. If $\tan(x + y) = 33$ and $x = \tan^{-1} 3$, the value of y is
(a) 0.3 (b) $\tan^{-1}(0.3)$
(c) $\tan^{-1}(1.3)$ (d) $\tan^{-1}(1/3)$

9. If $\sin^{-1}(x-1) + \cos^{-1}(x-3) + \tan^{-1}\dfrac{x}{2-x^2} = \cos^{-1} k + \pi$, then the value of k is
(a) 1/2 (b) $1/\sqrt{2}$
(c) $\sqrt{3}/2$ (d) 1

10. $\sin\left[2\sin^{-1}(3/5) + 2\sin^{-1}(4/5)\right]$ is equal to
(a) 24/125 (b) 336/625
(c) 7/125 (d) 168/625

LEVEL 1

Straight Objective Type Questions

11. If $x = 2/3$, then $\sin^2(\tan^{-1} x) + \cos^2(\sin^{-1} x)$ is equal to
(a) 99/101 (b) 107/117
(c) 101/117 (d) none of these

12. A root of the equation
$17x^2 + 17x\tan[2\tan^{-1}(1/5) - \pi/4] - 10 = 0$ is
(a) 10/17 (b) -1
(c) $-7/17$ (d) 1

13. $\cos^{-1}\sqrt{\dfrac{a-x}{a-b}} = \sin^{-1}\sqrt{\dfrac{x-b}{a-b}}$ is possible if

(a) $a > x > b$
(b) $a < b$ and x takes any value
(c) $a > b$ and x takes any value
(d) $a = x = b$

14. The value of $\cos^{-1}(\cos(-17\pi/5))$, is equal to

(a) $-17\pi/5$ (b) $3\pi/5$
(c) $2\pi/5$ (d) none of these

15. The value of $(1/2)\cos^{-1}(3/5)$ is

(a) $\sin^{-1}(1/2)$ (b) $\cos^{-1}(1/2)$
(c) $\cot^{-1}(1/2)$ (d) $\tan^{-1}(1/2)$

16. The value of $\sin(\cot^{-1}(\cos(\tan^{-1}x)))$ is

(a) $\sqrt{\dfrac{x^2+2}{x^2+1}}$ (b) $\sqrt{\dfrac{x^2+1}{x^2+2}}$

(c) $\dfrac{x}{\sqrt{x^2+2}}$ (d) $\dfrac{1}{\sqrt{x^2+2}}$

17. If $\tan^{-1}(a/x) + \tan^{-1}(b/x) = \pi/2$, then x is equal to

(a) ab (b) a/b

(c) b/a (d) $\sqrt{ab}$

18. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$, where $-1 \le x, y \le 1$ and $x + y \ge 0$, then $x^2 + y^2 + z^2 + 2xyz$ is equal to

(a) 0 (b) 1
(c) $(x + y + z)^2$ (d) $xy + yz + zx$

19. If $\cos^{-1}(x/2) + \cos^{-1}(y/3) = \theta$, then $9x^2 - 12xy\cos\theta + 4y^2$ is equal to

(a) 0 (b) 36
(c) $36\sin^2\theta$ (d) $36\cos^2\theta$

20. $\tan^{-1}\sqrt{\dfrac{a(a+b+c)}{bc}} + \tan^{-1}\sqrt{\dfrac{b(a+b+c)}{ca}} + \tan^{-1}\sqrt{\dfrac{c(a+b+c)}{ab}}$ is equal to

(a) $\pi/4$ (b) $\pi/2$
(c) π (d) 0

21. If $a_1, a_2, a_3, \ldots, a_n$ is an A.P. with common ratio d, then

$$\tan\left[\tan^{-1}\frac{d}{1+a_1a_2} + \tan^{-1}\frac{d}{1+a_2a_3} + \cdots + \tan^{-1}\frac{d}{1+a_{n-1}a_n}\right] =$$

(a) $\dfrac{(n-1)d}{a_1+a_n}$ (b) $\dfrac{(n-1)d}{1+a_1a_n}$

(c) $\dfrac{nd}{1+a_1a_n}$ (d) $\dfrac{a_n-a_1}{a_n+a_1}$

22. Two angles of a triangle are $\sin^{-1}(1/\sqrt{5})$ and $\sin^{-1}(1/\sqrt{10})$ then the third angle is

(a) $\pi/4$ (b) $3\pi/4$
(c) $\pi/6$ (d) $\pi/3$

23. If $x + 1/x = 5/2$, then the principal value of $\sin^{-1}x$ is

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) $5\pi/6$

24. The number of positive integral pairs (a, b) satisfying the equation $\tan^{-1}a + \tan^{-1}b = \tan^{-1}7$ is

(a) 0 (b) 2
(c) 4 (d) infinite

25. The value of

$$\sin^{-1}\left\{\cot\left[\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{12}}{4} + \sec^{-1}\sqrt{2}\right]\right\}$$

is equal to

(a) 0 (b) $\pi/12$
(c) $\pi/6$ (d) $\pi/4$

26. The sum of the infinite series

$\cot^{-1}2 + \cot^{-1}8 + \cot^{-1}18 + \cot^{-1}32 + \ldots$ is equal to

(a) $\pi/4$ (b) $\pi/2$
(c) $3\pi/4$ (d) none of these

27. The inequality $\sin^{-1}(\sin 5) > x^2 - 4x$ holds if

(a) $x = 2 - \sqrt{9-2\pi}$

(b) $x = 2 + \sqrt{9-2\pi}$

(c) $x \in \left(2-\sqrt{9-2\pi}, 2+\sqrt{9-2\pi}\right)$

(d) $x > 2 + \sqrt{9-2\pi}$

28. $\sin^{-1}x > \cos^{-1}x$ holds for

(a) all values of x (b) $x \in (0, 1/\sqrt{2})$

(c) $x \in (1/\sqrt{2}, 1)$ (d) $x = 1.75$

29. $2\sin^{-1}x = \cos^{-1}(1 - 2x^2)$ is true for

(a) all values of x (b) $-1 \le x \le 0$
(c) $0 \le x \le 1$ (d) no value of x

30. If $\cos^{-1}x + \cos^{-1}y = \pi/2$ and $\tan^{-1}x - \tan^{-1}y = 0$ then $x^2 + xy + y^2$ is equal to

(a) 0 (b) $1/\sqrt{2}$
(c) 3/2 (d) 1/8

31. $x = \tan^{-1}3 + \tan^{-1}2$

$y = \tan^{-1}3 - \tan^{-1}2$

If a tan x + b tan y = 0, then $a + b$ is equal to ka where k is equal to

(a) –1　　(b) 1/7
(c) 1　　(d) 8

32. If $x > 1$ and $2\tan^{-1}x - \sin^{-1}(7/25) = \pi$ then the value of x is

(a) 7　　(b) 2
(c) 5　　(d) 25

33. Number of values of x satisfying $\sin^{-1}(1 - x) - 2\sin^{-1}x = \pi/2$ is

(a) 0　　(b) 1
(c) 2　　(d) infinite

34. $\sin(3\sin^{-1}(0.4))$ is equal to

(a) 8.624　　(b) 0.944
(c) 0.256　　(d) 1.2

35. If $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\theta\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\theta\right)$ is equal to $\frac{14}{\pi}$, then θ is equal to

(a) $\pi/2$　　(b) π
(c) $\pi/7$　　(d) none of these

36. $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$, $0 < x < \pi/2$ is equal to

(a) $x/2$　　(b) x
(c) $\pi/4 + x/2$　　(d) $\pi/2$

37. If $x < 0$, then $\tan^{-1}x + \tan^{-1}(1/x)$ equals

(a) $\pi/4$　　(b) $\pi/3$
(c) $-\pi/2$　　(d) π

38. If $\sin^{-1}x + \sin^{-1}2x = \pi/3$, then x =

(a) $\sqrt{(3/7)}$　　(b) $(1/2)\sqrt{(3/7)}$
(c) $-\sqrt{(3/7)}$　　(d) $-(1/2)\sqrt{(3/7)}$

Assertion-Reason Type Questions

39. **Statement-1:** $\tan^{-1}\left\{x+\sqrt{1+x^2}\right\}=\frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$

Statement-2: $\sin^2\left[2\tan^{-1}\sqrt{\frac{1+x}{1-x}}\right]=1-x^2$

40. **Statement-1:** $\tan^{-1}\frac{2}{5}+\tan^{-1}\frac{3}{7}=\frac{\pi}{4}$

Statement-2: $\tan^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y-x}{y+x}\right)=\frac{\pi}{4}$

$(x, y > 0)$

41. **Statement-1:** If $x^2 - px + q = 0$, where p is twice the tangent of the arithmetic mean of $\sin^{-1}x$ and $\cos^{-1}x$, q is the geometric mean of $\tan^{-1}x$ and $\cot^{-1}x$, then $x = 1$

Statement-2: $\tan(\sin^{-1}x + \cos^{-1}x) = 1$

42. **Statement-1:** If α, β are the roots of the equation $18(\tan^{-1}x)^2 - 9\pi\tan^{-1}x + \pi^2 = 0$, then $\alpha + \beta = 4/\sqrt{3}$.

Statement-2: $\sec^2(\cos^{-1}(1/4)) + \text{cosec}^2(\sin^{-1}(1/5)) = 41$.

43. **Statement-1:** $\sin^{-1}\tan(\tan^{-1}x + \tan^{-1}(1 - x)) = \pi/2$ has no non-zero integral solution.

Statement 2 : The number of positive integral solutions of the equation $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1-y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$ is 2.

44. **Statement-1:** $\cos\frac{1}{2}\cos^{-1}\left(\frac{1}{8}\right)=\frac{3}{4}$.

Statement-2: If $\cos\theta = x$, then $\cos(\theta/2) = \sqrt{\frac{1+x}{2}}$

45. **Statement-1:** $\sin\cot^{-1}\tan\cos^{-1}\left(3/\sqrt{13}\right) = 3/\sqrt{13}$

Statement-2: If $\alpha = \cos^{-1}(4/5)$, then $\cos^{-1}(44/125) = \pi - 3\alpha$.

LEVEL 2

Straight Objective Type Questions

46. If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = 5\pi^2/8$, then x is equal to

(a) – 1　　(b) 0
(c) 1　　(d) none of these

47. $\sum_{m=1}^{n}\tan^{-1}\frac{2m}{m^4+m^2+2}=$

(a) $\tan^{-1}(n^2 + n + 1)$ (b) $\tan^{-1}(n^2 - n + 1)$

(c) $\tan^{-1}\dfrac{n^2 + n}{n^2 + n + 2}$ (d) none of these

48. If $\cos^{-1}(x/a) + \cos^{-1}(y/b) = \alpha$. Then $x^2/a^2 + y^2/b^2$ is equal to
(a) $(2xy/ab)\cos\alpha + \sin^2\alpha$
(b) $(2xy/ab)\sin\alpha + \cos^2\alpha$
(c) $(2xy/ab)\cos^2\alpha + \sin\alpha$
(d) $(2xy/ab)\sin^2\alpha + \cos\alpha$

49. If $\alpha = 2\tan^{-1}\left(2\sqrt{2} - 1\right)$ and $\beta = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$ then
(a) $\alpha < \beta$ (b) $\alpha = \beta$
(c) $\alpha > \beta$ (d) none of these

50. If $y = \tan^{-1}\dfrac{1-x}{1+x}$, $0 \le x \le 1$, then
(a) $0 \le y \le \pi$ (b) $0 \le y \le \pi/4$
(c) $-\pi/4 \le y \le \pi/4$ (d) $\pi/4 \le y \le \pi/2$

51. $\dfrac{\alpha^3}{2}\operatorname{cosec}^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\alpha}{\beta}\right) + \dfrac{\beta^3}{2}\sec^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\beta}{\alpha}\right)$ is equal to
(a) $(\alpha + \beta)(\alpha^2 + \beta^2)$ (b) $(\alpha - \beta)(\alpha^2 + \beta^2)$
(c) $(\alpha - \beta)(\alpha^2 - \beta^2)$ (d) $(\alpha + \beta)(\alpha^2 - \beta^2)$

52. The value of $\sin^{-1} x - \sin^{-1}\left(\dfrac{x}{2} - \dfrac{\sqrt{3-3x^2}}{2}\right)$; $0 \le x \le 1/2$ is equal to
(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) 0

53. The equation $\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}x = -\pi/2$ has
(a) only integral solutions
(b) two integral solutions
(c) no integral solution
(d) two real solutions

54. If $2\tan^{-1}\left[\sqrt{\dfrac{a-b}{a+b}}\cdot\tan\left(\dfrac{\theta}{2}\right)\right] = \cos^{-1}\left[\dfrac{2a+3b}{3a+2b}\right]$, then $\cos\theta$ is equal to
(a) 1/2 (b) 1/3
(c) 2/3 (d) $2a/3b$

55. $\tan^{-1}\left(\dfrac{x\cos\theta}{1 - x\sin\theta}\right) - \cot^{-1}\left(\dfrac{\cos\theta}{x - \sin\theta}\right)$ is equal to
(a) θ (b) $\pi/2 - \theta$
(c) $\theta/2$ (d) $\pi/4 - \theta/2$

Previous Years' AIEEE/JEE Main Questions

1. $\cot^{-1}[(\cos\alpha)^{1/2}] - \tan^{-1}[(\cos\alpha)^{1/2}] = x$, then $\sin x =$
(a) $\tan^2(\alpha/2)$ (b) $\cot^2(\alpha/2)$
(c) $\tan\alpha$ (d) $\cot(\alpha/2)$ **[2002]**

2. The trigonometric equation $\sin^{-1} x = 2\sin^{-1} a$ has a solution for
(a) all real values of a
(b) $|a| \le 1/\sqrt{2}$
(c) $|a| \ge 1/\sqrt{2}$
(d) $1/2\,|a| < 1/\sqrt{2}$ **[2003]**

3. If $\cos^{-1} x - \cos^{-1}(y/2) = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2$ is equal to
(a) $4\sin^2\alpha$ (b) $-4\sin^2\alpha$
(c) $2\sin 2\alpha$ (d) 4 **[2005]**

4. If $\sin^{-1}(x/5) + \operatorname{cosec}^{-1}(5/4) = \pi/2$ then a value of x is
(a) 1 (b) 3
(c) 4 (d) 6 **[2007]**

5. The value of $\cot(\operatorname{cosec}^{-1}(5/3) + \tan^{-1}(2/3))$ is
(a) 5/17 (b) 6/17
(c) 3/17 (d) 4/17 **[2008]**

6. If x, y, z are in A.P and $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ are also in A.P, then
(a) $2x = 3y = 6z$ (b) $6x = 3y = 2z$
(c) $6x = 4y = 3z$ (d) $x = y = z$ **[2013]**

7. A value of x for which $\sin(\cot^{-1}(1 + x)) = \cos(\tan^{-1}x)$, is:
(a) $-1/2$ (b) 1
(c) 0 (d) 1/2. **[2013, online]**

8. The number of solutions of the equation $\sin^{-1} x = 2\tan^{-1} x$ (in principal values) is:
   (a) 1 (b) 4
   (c) 2 (d) 3 **[2013, online]**

9. If $S = \tan^{-1}\frac{1}{x^2+x+1} + \tan^{-1}\frac{1}{x^2+3x+3} + \dots + \tan^{-1}\frac{1}{1+(x+19)(x+20)}$, then $\tan S$ is equal to
   (a) $\frac{20}{401+20x}$ (b) $\frac{x}{x^2+20x+1}$
   (c) $\frac{20}{x^2+20x+1}$ (d) $\frac{x}{401+20x}$
   [2013, online]

10. **Statement I:** The equation $(\sin^{-1} x)^3 + (\cos^{-1} x)^3 - a\pi^3 = 0$ has a solution for all $a \geq \frac{1}{32}$. For any $x \in \mathbf{R}$.
    Statement II: $\sin^{-1}x + \cos^{-1}x = \pi/2$ and $0 \leq (\sin^{-1}x - \pi/4)^2 \leq 9\pi^2/16$
    (a) Both statements I and II are true
    (b) Both statements I and II are false
    (c) Statement I is true and statement II is false
    (d) Statement I is false and statement II is true
    [2014, online]

11. The principal value of $\tan^{-1}\left(\cot\frac{43\pi}{4}\right)$ is:
    (a) $-(3\pi/4)$ (b) $3\pi/4$
    (c) $\pi/4$ (d) $-(\pi/4)$ **[2014, online]**

12. Let $\tan^{-1}y = \tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ where $|x| < \frac{1}{\sqrt{3}}$, then a value of y is:
    (a) $\frac{3x-x^3}{1-3x^2}$ (b) $\frac{3x+x^3}{1-3x^2}$
    (c) $\frac{3x-x^3}{1+3x^2}$ (d) $\frac{3x+x^3}{1+3x^2}$ **[2015]**

13. If $f(x) = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, $x > 1$, then $f(5)$ is equal to:
    (a) $\frac{\pi}{2}$ (b) π
    (c) $4\tan^{-1}(5)$ (d) $\tan^{-1}\left(\frac{65}{156}\right)$
    [2015, online]

Previous Years' B-Architecture Entrance Examination Questions

1. $2\cot^{-1}(7) + \cos^{-1}(3/5)$, in principal value, is equal to
   (a) $\cos^{-1}\left(\frac{44}{125}\right)$ (b) $\cot^{-1}\left(\frac{44}{117}\right)$
   (c) $\tan^{-1}\left(\frac{41}{117}\right)$ (d) $\operatorname{cosec}^{-1}\left(\frac{117}{125}\right)$ **[2013]**

2. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is:
   (a) $\frac{20}{19}$ (b) $\frac{19}{21}$
   (c) $\frac{21}{19}$ (d) $\frac{19}{20}$ **[2016]**

Answers

Concept-based

1. (b) **2.** (c) **3.** (b) **4.** (c)
5. (c) **6.** (c) **7.** (a) **8.** (a)
9. (b) **10.** (b)

Level 1

11. (c) **12.** (d) **13.** (a) **14.** (b)
15. (d) **16.** (b) **17.** (d) **18.** (b)
19. (c) **20.** (d) **21.** (b) **22.** (b)
23. (a) **24.** (b) **25.** (a) **26.** (a)
27. (c) **28.** (c) **29.** (c) **30.** (c)
31. (d) **32.** (a) **33.** (b) **34.** (b)
35. (c) **36.** (a) **37.** (c) **38.** (b)
39. (b) **40.** (a) **41.** (c) **42.** (b)
43. (d) **44.** (a) **45.** (b)

Level 2

46. (a) **47.** (c) **48.** (a) **49.** (c)
50. (b) **51.** (a) **52.** (c) **53.** (c)
54. (c) **55.** (a)

Previous Years' AIEEE/JEE Main Questions

1. (a) **2.** (b) **3.** (a) **4.** (b)
5. (b) **6.** (d) **7.** (a) **8.** (d)

9. (c) **10.** (b) **11.** (d) **12.** (c)
13. (b)

Previous Years' B-Architecture Entrance Examination Questions

1. (b) **2.** (c)

Hints and Solutions

Concept-based

1. $\sin(\sin^{-1}(3/5) + \cos^{-1}(4/5)) = \sin(2\sin^{-1}(3/5)) = 2 \times (3/5) \times (4/5) = 24/25$

2. $2\tan^{-1} 7 = \pi + \sin^{-1} \dfrac{2\times 7}{\sqrt{1+7^2}}$

   (use $2\tan^{-1} x = \pi + \sin^{-1} \dfrac{2x}{\sqrt{1+x^2}}$ if $x > 1$)

3. $\dfrac{7\pi}{3} = \alpha + \beta = 2 \times \dfrac{\pi}{2} + \cos^{-1} x + 2 \times \dfrac{\pi}{2} + \cos^{-1} y$

   $\Rightarrow \cos^{-1} x + \cos^{-1} y = \dfrac{7\pi}{3} - 2\pi = \dfrac{\pi}{3}$

   $\Rightarrow \cos(\cos^{-1} x + \cos^{-1} y) = \cos(\pi/3) = 1/2$

   $\Rightarrow xy - \sqrt{(1-x^2)(1-y^2)} = 1/2.$

4. $2\left(-\dfrac{\pi}{6}\right) + 3\left(\dfrac{5\pi}{6}\right) + 4\left(\dfrac{\pi}{4}\right) + 5\left(-\dfrac{\pi}{4}\right)$

   $= \dfrac{23\pi}{12}$

5. $xy = 1 + \dfrac{x}{z} + \dfrac{y}{z} > 1$

   So $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$

   $= \pi + \tan^{-1}\left[\dfrac{x+y}{1-xy}\right] + \tan^{-1} z$

   $= \pi + \tan^{-1}\left[\dfrac{xyz - z}{1-xy}\right] + \tan^{-1} z$

   $= \pi + \tan^{-1}(-z) + \tan^{-1} z = \pi$

6. $\tan(\cos^{-1} x) = \sin\left(\cot^{-1} \dfrac{1}{2}\right)$

   $\Rightarrow \tan\left(\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)\right) = \sin\left(\sin^{-1}\left(\dfrac{2}{\sqrt{5}}\right)\right)$

   $\Rightarrow \dfrac{1-x^2}{x^2} = \dfrac{4}{5} \Rightarrow x = \dfrac{\sqrt{5}}{3}$

   [**Note:** $x = -\dfrac{\sqrt{5}}{3}$ makes L.H.S < 0 but R.H.S > 0]

7. $\sin^{-1}(\cos(x + \pi - x)) = \sin^{-1}(\cos \pi) = \sin^{-1}(-1) = -(\pi/2)$

8. $x = \tan^{-1}(3) \Rightarrow \tan x = 3$

   So $\tan(x + y) = 33$

   $\Rightarrow \dfrac{\tan x + \tan y}{1 - \tan x \tan y} = 33$

   $\Rightarrow \dfrac{3 + \tan y}{1 - 3\tan y} = 33 \Rightarrow \tan y = \dfrac{30}{100} = 0.3$

9. $\sin^{-1}(x - 1)$ is defined for $-1 \le x - 1 \le 1 \Rightarrow 0 \le x \le 2$

   $\cos^{-1}(x - 3)$ is defined for $-1 \le x - 3 \le 1 \Rightarrow 2 \le x \le 4$.

   so we get $x = 2$ and

   $\cos^{-1} k + \pi = \sin^{-1}(2 - 1) + \cos^{-1}(2 - 3) + \tan^{-1}(-1)$

   $= \dfrac{\pi}{2} + \pi - \dfrac{\pi}{4}$

   $\Rightarrow \cos^{-1} k = \pi/4 \Rightarrow k = 1/\sqrt{2}$

10. $\sin[2\sin^{-1}(3/5) + 2\sin^{-1}(4/5)]$

    $= \sin\left[\sin^{-1} 2 \times \dfrac{3}{5} \times \dfrac{4}{5} + \sin^{-1} 2 \times \dfrac{4}{5} \times \dfrac{3}{5}\right]$

    $= \sin\left[2\sin^{-1} \dfrac{24}{25}\right] = \sin\left[\sin^{-1} 2 \times \dfrac{24}{25} \times \dfrac{7}{25}\right]$

    $= \dfrac{336}{625}.$

Level 1

11. $\tan^{-1} x = \sin^{-1} \dfrac{x}{\sqrt{1+x^2}}$, $\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$

    So the required value is

    $\dfrac{x^2}{1+x^2} + 1 - x^2 = \dfrac{4/9}{1+4/9} + 1 - \dfrac{4}{9} = \dfrac{101}{117}.$

12. $\tan[2(\tan^{-1}(1/5) - \pi/4]$

    $$= \frac{\tan\left(2\tan^{-1}\left(\frac{1}{5}\right)\right) - 1}{1 + \tan\left(2\tan^{-1}\left(\frac{1}{5}\right)\right)} = \frac{\frac{2\times(1/5)}{1-(1/25)} - 1}{1 + \frac{10}{24}} = -\frac{7}{17}.$$

13. $a \ne b$, $\dfrac{a-x}{a-b} \le 1$, $\dfrac{x-b}{a-b} \le 1$

    $\Rightarrow b \le x,\ x \le a \Rightarrow a > x > b.$

14. $\cos^{-1}(\cos(-17\pi/5))$

    $= \cos^{-1}\left(\cos\left(\dfrac{17\pi}{5}\right)\right) = \cos^{-1}\left(\cos\left(4\pi - \dfrac{3\pi}{5}\right)\right)$

    $= \cos^{-1}\left(\cos\left(\dfrac{3\pi}{5}\right)\right) = \dfrac{3\pi}{5}$

15. $2\theta = \cos^{-1}(3/5) \;\Rightarrow\; \cos 2\theta = 3/5$

$$\Rightarrow \quad \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{1}{4}.$$

16. $x = \tan\theta \Rightarrow \sin(\cot^{-1}(\cos(\tan^{-1} x)))$

$$= \sin(\cot^{-1}(\cos\theta)) = \sin\left(\cot^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)\right)$$

$$= \sin\left(\tan^{-1}\left(\sqrt{1+x^2}\right)\right)$$

$$= \sin\left(\sin^{-1}\frac{\sqrt{1+x^2}}{\sqrt{1+1+x^2}}\right) = \sqrt{\frac{x^2+1}{x^2+2}}.$$

17. $\tan^{-1}(b/x) = \cot^{-1}(a/x) \;\Rightarrow\; x^2 = ab.$

18. $\cos^{-1}\left[xy - \sqrt{1-x^2}\,\sqrt{1-y^2}\right] = \pi - \cos^{-1} z =$ $\cos^{-1}(-z)$

$\Rightarrow \sqrt{1-x^2}\,\sqrt{1-y^2} = xy + z$

$\Rightarrow (1-x^2)(1-y^2) = (xy+z)^2$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

19. $$\cos\theta = \frac{x}{2}\times\frac{y}{3} - \sqrt{1-\frac{x^2}{2^2}}\,\sqrt{1-\frac{y^2}{3^2}}$$

$\Rightarrow (4-x^2)(9-y^2) = (xy - 6\cos\theta)^2$

$\Rightarrow 36 - 9x^2 - 4y^2 = 36\cos^2\theta - 12\,xy\cos\theta$

$\Rightarrow 9x^2 - 12xy\cos\theta + 4y^2 = 36(1-\cos^2\theta)$
$= 36\sin^2\theta$

20. Use $\tan(A_1 + A_2 + A_3) = \dfrac{S_1 - S_3}{1 - S_2}$.

If the given expression is θ, then $\tan\theta = \dfrac{S_1 - S_3}{1 - S_2}$

where $S_1 = \dfrac{\sqrt{a+b+c}}{\sqrt{abc}}\,(a+b+c)$ and

$S_3 = \dfrac{(a+b+c)^{3/2}}{\sqrt{abc}}$.

$\Rightarrow S_1 - S_3 = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = \tan^{-1} 0 = 0$

21. $$\tan\left[\tan^{-1}\frac{a_2-a_1}{1+a_1a_2} + \tan^{-1}\frac{a_3-a_2}{1+a_2a_3} + \cdots + \tan^{-1}\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right]$$

$= \tan[\tan^{-1} a_2 - \tan^{-1} a_1 + \tan^{-1} a_3 - \tan^{-1} a_2 + \ldots + \tan^{-1} a_n - \tan^{-1} a_{n-1}]$

$$= \tan[\tan^{-1} a_n - \tan^{-1} a_1] = \frac{a_n - a_1}{1+a_1a_n} = \frac{(n-1)d}{1+a_1a_n}.$$

22. Let $A = \sin^{-1}\left(1/\sqrt{5}\right)$, $B = \sin^{-1}\left(1/\sqrt{10}\right)$

then $$\sin(A+B) = \frac{1}{\sqrt{5}}\sqrt{1-\frac{1}{10}} + \frac{1}{\sqrt{10}}\sqrt{1-\frac{1}{5}}$$

$$= \frac{3}{\sqrt{5}}\sqrt{1-\frac{1}{10}} + \frac{1}{\sqrt{10}}\sqrt{1-\frac{1}{5}}$$

$\Rightarrow \quad A + B = \pi/4 \quad\Rightarrow\quad C = 3\pi/4$

23. Let $y = \sin^{-1} x \;\Rightarrow\; \sin y + \dfrac{1}{\sin y} = \dfrac{5}{2}$

$\Rightarrow 2\sin^2 y - 5\sin y + 2 = 0$

$\Rightarrow \sin y = (1/2) \;\Rightarrow\; y = \sin^{-1}(1/2) = \pi/6$

24. $\pi + \tan^{-1}\dfrac{a+b}{1-ab} = \tan^{-1} 7$ as $a > 0$, $b > 0$, $ab > 1$

$\Rightarrow \dfrac{a+b}{1-ab} = 7 \;\Rightarrow\; a = \dfrac{7-b}{1+7b}$.

which does not hold for any positive integral pairs (a, b).

25. $\alpha = \sin^{-1}\left[\cot\left(\theta + \dfrac{\pi}{6} + \dfrac{\pi}{4}\right)\right]$ where $\sin^2\theta = \dfrac{2-\sqrt{3}}{4}$.

$$= \sin^{-1}\left[\frac{\cot\theta\cot(\pi/4+\pi/6) - 1}{\cot\theta + \cot(\pi/4+\pi/6)}\right]$$

Now,

$\cot\theta = \sqrt{\dfrac{4}{2-\sqrt{3}} - 1} = \sqrt{\dfrac{2+\sqrt{3}}{2-\sqrt{3}}} = 2+\sqrt{3}$ and

$$\cot = \left(\frac{\pi}{4}+\frac{\pi}{6}\right) = \frac{\cot\frac{\pi}{6} - 1}{1+\cot\frac{\pi}{6}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} =$$

$$\frac{\left(\sqrt{3}-1\right)^2}{2} = 2 - \sqrt{3}$$

So $\alpha = \sin^{-1} 0 = 0$.

26. $u_n = \cot^{-1}(2n^2) = \tan^{-1}\dfrac{1}{2n^2}$

$$= \tan^{-1}\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}$$

$= \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

$\Rightarrow S_n = \tan^{-1}(2n+1) - \tan^{-1} 1$

$$\lim_{n\to\infty} S_n = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

27. $3\pi/2 < 5 < 2\pi$ so $\sin^{-1}(\sin 5) = \sin^{-1}\sin(5 - 2\pi)$ $= 5 - 2\pi$

$\Rightarrow -(2\pi - 5) > x^2 - 4x \quad \Rightarrow \quad x^2 - 4x + (2\pi - 5) < 0$

$$\Rightarrow \left[x - \frac{4-\sqrt{16-4(2\pi-5)}}{2}\right] \times \left[x - \frac{4+\sqrt{16-4(2\pi-5)}}{2}\right] < 0$$

$\Rightarrow x \in \left(2-\sqrt{9-2\pi},\, 2+\sqrt{9-2\pi}\right)$

28. $x \in \left(1/\sqrt{2}, 1\right) \quad \Rightarrow \quad \sin^{-1}(x) \in (\pi/4, \pi/2)$,

$\cos^{-1} x \in (0, \pi/4)$

so $\sin^{-1} x > \cos^{-1} x$ if $x \in \left(1/\sqrt{2}, 1\right)$

29. Let $x = \sin\theta$, then $2\sin^{-1} x = 2\theta$

$-\pi/2 \le \theta \le \pi/2$

Also $\cos^{-1}(1 - 2x^2) = \cos^{-1}(1 - 2\sin^2\theta) = 2\theta$.

$\Rightarrow \quad 0 \le 2\theta \le \pi \quad \Rightarrow \quad 0 \le \theta \in \pi/2$

$\therefore \quad 2\sin^{-1} x = \cos^{-1}(1 - 2x^2)$ holds for $0 \le \theta \le \pi/2$

$\Rightarrow \quad 0 \le x \le 1$

30. $\tan^{-1} x - \tan^{-1} y = 0 \quad \Rightarrow \quad x = y$.

$\cos^{-1} x + \cos^{-1} y = \pi/2 \quad \Rightarrow \quad 2\cos^{-1} x = \pi/2 \quad \Rightarrow \quad \cos^{-1} x = \pi/4$

$\Rightarrow \quad x = \cos\pi/4 = 1/\sqrt{2} \quad \Rightarrow \quad x^2 = 1/2$

and $x^2 + xy + y^2 = 3x^2 = 3/2$.

31. $\tan x = -1$, $\tan y = 1/7$

32. If $x > 1$, $2\tan^{-1}x = \pi + \sin^{-1}\dfrac{2x}{1+x^2}$

33. $1 - x = \sin(\pi/2 + 2\sin^{-1}x)$
$= \cos(2\sin^{-1}x) = \cos(\cos^{-1}(1 - 2x^2))$
$= 1 - 2x^2 \quad \Rightarrow x = 0$ or $1/2$

But $x = 1/2$ does not satisfy the equation

34. Apply $3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

35. $\tan\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right) + \tan\left(\dfrac{\pi}{4}-\dfrac{\alpha}{2}\right) \qquad [\theta = \cos\alpha]$

$$= \frac{(1+\tan(\alpha/2))^2 + (1-\tan(\alpha/2))^2}{1-\tan^2(\alpha/2)}$$

$$= \frac{2}{\cos\alpha} = \frac{2}{\theta} = \frac{14}{\pi} \quad \Rightarrow \quad \theta = \pi/7.$$

36. $1 \pm \sin x = (\cos x/2 \pm \sin x/2)^2$

37. If $x < 0$, $\tan^{-1}(1/x) = \cot^{-1}x - \pi$ and $\tan^{-1}x + \cot^{-1}x = \pi/2$

38. $\sin^{-1}2x = \pi/3 - \sin^{-1}x$

$\Rightarrow \quad 2x = \sin(\pi/3 - \sin^{-1}x) = \dfrac{\sqrt{3}}{2}\sqrt{1-x^2} - \dfrac{1}{2}x$

$\Rightarrow \quad (5x)^2 = 3(1 - x^2) \quad \Rightarrow \quad 28x^2 = 3$

$\Rightarrow \quad x = \pm\dfrac{1}{2}\cdot\sqrt{\dfrac{3}{7}}$.

$x \ne -\dfrac{1}{2}\cdot\sqrt{\dfrac{3}{7}}$ as the sum of two negative angles can not be positive.

39. Put $x = \cot\theta$, then L.H.S

$$= \tan^{-1}\left[\cot\theta + \sqrt{1+\cot^2\theta}\right] = \tan^{-1}\frac{1+\cos\theta}{\sin\theta}$$

$$= \tan^{-1}(\cot(\theta/2)) = \tan^{-1}(\tan(\pi/2 - \theta/2)) = \pi/2 - \theta/2.$$

$\Rightarrow$ Statement-1 is true,

for Statement-2 L.H.S $= \sin^2 2\theta$, $\theta = \tan^{-1}\sqrt{\dfrac{1+x}{1-x}}$

$$= \left(\frac{2\tan\theta}{1+\tan^2\theta}\right)^2 = \frac{4\tan^2\theta}{\left(1+\tan^2\theta\right)^2}$$

$$= \frac{4\left(\dfrac{1+x}{1-x}\right)}{\left[1+\left(\dfrac{1+x}{1-x}\right)\right]^2} = 1 - x^2$$

$\Rightarrow$ Statement-2 is also true.

40. In Statement-2, L.H.S $= \tan^{-1}\dfrac{\dfrac{x}{y}+\dfrac{y-x}{y+x}}{1-\dfrac{x(y-x)}{y(y+x)}}$

$$= \tan^{-1}\frac{x^2+y^2}{x^2+y^2} = \tan^{-1}1 = \pi/4$$

$\Rightarrow$ Statement-2 is true and which shows that Statement–1 is true by taking $x = 2$, $y = 5$.

41. $p = 2\tan\dfrac{\sin^{-1}x + \cos^{-1}x}{2} = 2\tan\dfrac{\pi}{4} = 2$,

$q = \sqrt{\tan^{-1}x\cot^{-1}x} = 1$

so $x^2 - 2x + 1 = 0 \quad \Rightarrow \quad x = 1$, the Statement-1 is true.

Statement-2 is false.

42. In Statement-1 $\tan^{-1}x = \pi/6$ or $\pi/3$

$\Rightarrow \quad x = 1/\sqrt{3}$ or $\sqrt{3}$.

$\Rightarrow \quad \alpha + \beta = 4/\sqrt{3} \quad \Rightarrow$ Statement-1 is true.

In Statement-2, L.H.S = $\sec^2(\sec^{-1} 4) + \operatorname{cosec}^2(\operatorname{cosec}^{-1} 5)$

= 16 + 25 = 41 and the Statement-2 is true but does not lead to Statement-1

43. In Statement-1 $\dfrac{x+1-x}{1-x(1-x)} = \sin(\pi/2) = 1$

$\Rightarrow\ 1 - x(1 - x) = 1$

$\Rightarrow\ x = 1$ is a non-zero solution $\Rightarrow$ Statement-1 is false

In Statement-2:

$$\tan^{-1} x + \cos^{-1}\frac{y}{\sqrt{1-y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$$

$\Rightarrow\ \tan^{-1} x + \tan^{-1}(1/y) = \tan^{-1}(3)$

$\Rightarrow\ \tan^{-1}(1/y) = \tan^{-1} 3 - \tan^{-1} x$

$\Rightarrow\ y = \dfrac{1+3x}{3-x}$.

As x, y are positive integers

$x = 1, 2 \Rightarrow y = 2, 7$ and the solutions are (1, 2), (2, 7)

$\Rightarrow$ Statement-2 is true.

44. Statement-2 is true, Taking $x = 1/8$, Statement-1 is true.

45. Put $\cos\theta = 3/\sqrt{3}$ in Statement-1,

L.H.S = $\sin(\cot^{-1}(\tan\theta)) = \sin\left(\dfrac{\pi}{2} - \theta\right)$

$= \cos\theta = 3/\sqrt{13}$

$\Rightarrow$ Statement-1 is true.

In Statement-2, $\cos\alpha = 4/5$

$\Rightarrow \cos(\pi - 3\alpha) = -\cos 3\alpha$

$= 3\cos\alpha - 4\cos^3\alpha = 44/125$

$\Rightarrow$ Statement-2 is also true but does not lead to Statement-1.

Level 2

46. We can write $(\tan^{-1}x + \cot^{-1}x)^2 - 2\tan^{-1}x\cot^{-1}x = 5\pi^2/8$

$\Rightarrow \left(\dfrac{\pi}{2}\right)^2 - 2\tan^{-1}x(\pi/2 - \tan^{-1}x) = 5\pi^2/8$

$\Rightarrow (\tan^{-1}x)^2 - (\pi/2)\tan^{-1}x - 3\pi^2/16 = 0$

$\Rightarrow \tan^{-1}x = 3\pi/4$ or $-\pi/4 \Rightarrow x = -1$

47. $\displaystyle\sum_{m=1}^{n}\tan^{-1}\frac{2m}{m^4+m^2+2}$

$$= \sum_{m=1}^{n}\tan^{-1}\frac{1+m+m^2-(1-m+m^2)}{1+(1+m+m^2)(1-m+m^2)}$$

$$= \sum_{m=1}^{n}\left[\tan^{-1}(1+m+m^2) - \tan^{-1}(1-m+m^2)\right]$$

$$= \sum_{m=1}^{n}\left[\tan^{-1}(1+m+m^2) - \tan^{-1}(1+(m-1)+(m-1)^2)\right]$$

$= \tan^{-1}(1 + n + n^2) - \tan^{-1} 1$

$= \tan^{-1}\dfrac{n+n^2}{1+1+n+n^2} = \tan^{-1}\dfrac{n+n^2}{2+n+n^2}$

48. We have $\cos\alpha = \dfrac{xy}{ab} - \sqrt{\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{b^2}\right)}$

$\Rightarrow \left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{b^2}\right) = \left(\dfrac{xy}{ab} - \cos\alpha\right)^2$

$\Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 - \cos^2\alpha + \dfrac{2xy}{ab}\cos\alpha\cdot$

49. $\beta = 3\sin^{-1}(1/3) + \sin^{-1}(3/5)$

$= \sin^{-1}(3 \times (1/3) - 4(1/3)^3) + \sin^{-1}(3/5)$

$= \sin^{-1}(23/27) + \sin^{-1}(3/5)$

$< \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) + \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \dfrac{\pi}{3} + \dfrac{\pi}{3} = \dfrac{2\pi}{3}$

$\left[\because \dfrac{23}{27} = 0.85,\ \dfrac{3}{5} = 0.6 \text{ and } \dfrac{\sqrt{3}}{2} = 1.7\right.$

$\Rightarrow \beta < 2\pi/3$

$\alpha = 2\tan^{-1}\left(2\sqrt{2} - 1\right) > 2\tan^{-1}\sqrt{3} = 2\pi/3$

$[\because 2\sqrt{2} - 1 = 1.8,\ \sqrt{3} = 1.7]$

$\Rightarrow \alpha > 2\pi/3$

Hence $\alpha > \beta$

50. Let $x = \tan\theta$, then $0 \le x \le 1 \Rightarrow 0 \le \theta \le \pi/4$

$y = \tan^{-1}\dfrac{1-\tan\theta}{1+\tan\theta} = \pi/4 - \theta$

$\Rightarrow \theta = \pi/4 - y$ so $0 \le y \le \pi/4$

51. $\dfrac{\alpha^3}{2}\operatorname{cosec}^2(\theta/2) + \dfrac{\beta^3}{2}\sec^2(\pi/4 - \theta/2)$

(Taking $\dfrac{\alpha}{\beta} = \tan\theta \Rightarrow \sin\theta = \dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}$,

$\cos\theta = \dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}$)

$= \dfrac{\alpha^3}{2}[1 + \cot^2(\theta/2)] + \dfrac{\beta^3}{2}[1 + \tan^2(\pi/4 - \theta/2)]$

$$= \frac{\alpha^3+\beta^3}{2} + \frac{\alpha^3}{2}\frac{1+\cos\theta}{1-\cos\theta} + \frac{\beta^3}{2}\left[\frac{1-\tan\theta/2}{1+\tan\theta/2}\right]^2$$

$$= \frac{\alpha^3+\beta^3}{2} + \frac{\alpha^3}{2}\left[\frac{1+\cos\theta}{1-\cos\theta}\right] + \frac{\beta^3}{2}\left[\frac{1-\sin\theta}{1+\sin\theta}\right]$$

$$= \frac{\alpha^3+\beta^3}{2} + \frac{\alpha^3}{2}\frac{\sqrt{\alpha^2+\beta^2}+\beta}{\sqrt{\alpha^2+\beta^2}-\beta} + \frac{\beta^3}{2}\frac{\sqrt{\alpha^2+\beta^2}-\alpha}{\sqrt{\alpha^2+\beta^2}+\alpha}$$

$$= \frac{\alpha^3+\beta^3}{2} + \frac{\alpha^3}{2}\frac{\alpha^2+\beta^2+\beta^2+2\beta\sqrt{\alpha^2+\beta^2}}{\alpha^2+\beta^2-\beta^2}$$

$$+ \frac{\beta^3}{2}\frac{\alpha^2+\beta^2+\alpha^2-2\alpha\sqrt{\alpha^2+\beta^2}}{\alpha^2+\beta^2-\alpha^2}$$

$$= \frac{\alpha^3+\beta^3}{2} + \frac{\alpha}{2}\left[\alpha^2+2\beta^2+2\beta\sqrt{\alpha^2+\beta^2}\right] +$$

$$\frac{\beta}{2}\left(2\alpha^2+\beta^2-2\alpha\sqrt{\alpha^2+\beta^2}\right)$$

$= \alpha^3 + \beta^3 + \alpha\beta^2 + \alpha^2\beta$

$= (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2 + \alpha\beta)$

$= (\alpha + \beta)(\alpha^2 + \beta^2)$

52. Put $x = \sin\theta$, then $0 \le x \le 1/2 \Rightarrow 0 \le \theta \le \pi/6$.

So, the given expression is equal to

$\sin^{-1}(\sin\theta) - \sin^{-1}\left[\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta\right]$

$= \theta - \sin^{-1}(\sin(\theta - \pi/3))$

$= \theta - (\theta - \pi/3) = \pi/3$

53. $\sin^{-1} 6x = -\pi/2 - \sin^{-1} 6\sqrt{3}x$

$\Rightarrow 6x = -\sin(\pi/2 + \sin^{-1} 6\sqrt{3}x)$

$= -\cos(\sin^{-1} 6\sqrt{3}x)$

$= \sqrt{1-108x^2}$

$\Rightarrow 36x^2 = 1 - 108x^2 \Rightarrow 144x^2 = 1 \Rightarrow x = \pm 1/12$.

But $x = 1/12$ does not satisfy the equation.

So the given equation has only one non-integral solution

54. Let $\frac{2a+3b}{3a+2b} = \cos\alpha$

$$\Rightarrow \frac{a}{2\cos\alpha - 3} = \frac{b}{2-3\cos\alpha}$$

$$\Rightarrow \sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{5(1-\cos\alpha)}{1+\cos\alpha}} = \sqrt{5}\tan(\alpha/2)$$

So $2\tan^{-1}[\sqrt{5}\tan(\alpha/2)\tan(\theta/2)] = \alpha$.

$\Rightarrow \sqrt{5}\tan(\alpha/2)\tan(\theta/2) = \tan(\alpha/2)$

$$\Rightarrow \tan(\theta/2) = 1/\sqrt{5} \Rightarrow \cos\theta = \frac{1-(1/5)}{1+(1/5)} = \frac{2}{3}$$

55. $\tan^{-1}\left(\frac{x\cos\theta}{1-x\sin\theta}\right) - \cot^{-1}\left(\frac{\cos\theta}{x-\sin\theta}\right)$

$$= \tan^{-1}\frac{x\cos\theta}{1-x\sin\theta} - \tan^{-1}\frac{x-\sin\theta}{\cos\theta}$$

$$= \tan^{-1}\frac{\dfrac{x\cos\theta}{1-x\sin\theta} - \dfrac{x-\sin\theta}{\cos\theta}}{1+\dfrac{x\cos\theta}{1-x\sin\theta}\times\dfrac{x-\sin\theta}{\cos\theta}}$$

$$= \tan^{-1}\frac{\sin\theta(x^2-2x\sin\theta+1)}{\cos\theta(x^2-2x\sin\theta+1)}$$

$= \tan^{-1}(\tan\theta) = \theta$

Previous Years' AIEEE/JEE Main Questions

1. $x = (\pi/2) - 2\tan^{-1}[(\cos\alpha)^{1/2}]$

$\Rightarrow (\pi/2) - x = 2\tan^{-1}[(\cos\alpha)^{1/2}]$

$$\Rightarrow \tan(\pi/2 - x) = \frac{2(\cos\alpha)^{1/2}}{1-\cos\alpha}$$

$$\Rightarrow \cot x = \frac{2(\cos\alpha)^{1/2}}{1-\cos\alpha}$$

$$\Rightarrow \operatorname{cosec}^2 x = 1 + \frac{4\cos\alpha}{(1-\cos\alpha)^2} = \left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)^2$$

$$\Rightarrow \sin x = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{2\sin^2(\alpha/2)}{2\cos^2(\alpha/2)}$$

$= \tan^2(\alpha/2)$

2. $\sin^{-1} x = \sin^{-1} 2a\sqrt{1-a^2}$ if $|a| \le 1/\sqrt{2}$

$\Rightarrow x = 2a\sqrt{1-a^2}$ which is possible

if $x^2 = 4a^2(1 - a^2) \le 1$ $[\because -1 \le \sin^{-1} x \le 1]]$

or if $4a^4 - 4a^2 + 1 \ge 0$ if $(2a^2 - 1)^2 \ge 0$

which is true, so $|a| \le 1/\sqrt{2}$

3. $\cos\alpha = x\left(\frac{y}{2}\right) + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}$

$\Rightarrow (2\cos\alpha - xy)^2 = (1 - x^2)(4 - y^2)$

$\Rightarrow 4\cos^2\alpha - 4xy\cos\alpha = 4 - 4x^2 - y^2$

$\Rightarrow 4x^2 - 4xy\cos\alpha + y^2 = 4(1 - \cos^2\alpha) = 4\sin^2\alpha$

4. $\sin^{-1}\left(\frac{x}{5}\right) + \operatorname{cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{x}{5}\right) + \cos^{-1}\frac{3}{5} = \frac{\pi}{2}$$

$$\Rightarrow \quad x = 3 \text{ as } \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

5. $\operatorname{cosec}^{-1}\left(\frac{5}{3}\right) = \cot^{-1}\left(\frac{4}{3}\right)$

so $\cot\left(\cot^{-1}\left(\frac{4}{3}\right) + \cot^{-1}\left(\frac{3}{2}\right)\right) = \dfrac{\frac{4}{3}\times\frac{3}{2}-1}{\frac{4}{3}+\frac{3}{2}} = \dfrac{6}{17}$

6. We have $2y = x + z$ and $2\tan^{-1}y = \tan^{-1}x + \tan^{-1}z$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$$

$$\Rightarrow \quad \frac{2y}{1-y^2} = \frac{x+z}{1+xz} \Rightarrow y^2 = xz$$

$$\Rightarrow \quad 4xz = (2y)^2 = (x+z)^2$$

$$\Rightarrow \quad (x-z)^2 = 0 \quad \Rightarrow x = z.$$

Thus, $z = y = x$

7. We can write $\cot^{-1}(1 + x) = \sin^{-1}(\cos(\tan^{-1} x)$

$$= \sin^{-1}\left(\sin\left(\frac{\pi}{2} + \tan^{-1} x\right)\right)$$

$$= \frac{\pi}{2} + \tan^{-1} x$$

$$\Rightarrow \quad 1 + x = \cot\left(\frac{\pi}{2} + \tan^{-1} x\right) = -x$$

$$\Rightarrow \quad 2x = -1 \quad \Rightarrow \quad x = \frac{-1}{2}$$

8. Let $x = \tan\theta$.

Then we have $\tan\theta = \sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \quad \sin\theta(1 - 2\cos^2\theta) = 0$

$\Rightarrow \quad \sin\theta = 0$ or $\cos\theta = \pm\frac{1}{\sqrt{2}}$ which gives 3 values of θ.

9. We can write

$S = \tan^{-1}(x + 1) - \tan^{-1}x + \tan^{-1}(x + 2)$

$\quad - \tan^{-1}(x + 1) + \cdots \tan^{-1}(x + 20) - \tan^{-1}(x + 19)$

$= \tan^{-1}(x + 20) - \tan^{-1}x$

$$\Rightarrow \quad \tan S = \frac{x+20-x}{1+(x+20)x} = \frac{20}{x^2+20x+1}.$$

10. We have $\sin^{-1}x \le \frac{\pi}{2}$, $\cos^{-1} x \le \pi$

$$\Rightarrow (\sin^{-1} x)^3 + (\cos^{-1} x)^3 \le \frac{9}{8}\pi^3$$

Thus, $(\sin^{-1}x)^3 + (\cos^{-1} x)^3 = 2\pi^3$ has no solution.

Also, $\sin^{-1}x + \cos^{-1} x = \frac{\pi}{2}$ for $-1 \le x \le 1$, not for each real x.

$\therefore$ Both the Statements are false.

11. $\tan^{-1}\left(\cot\left(\frac{43}{4}\pi\right)\right) = \tan^{-1}(\cot(11\pi - \pi/4))$

$= \tan^{-1}(-\cot(\pi/4)) = \tan^{-1}(-1)$

$= -\pi/4$

12. As $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1} x$ for $|x| < 1$ we get

$\tan^{-1} y = \tan^{-1} x + 2\tan^{-1} x = 3\tan^{-1} x$

$$= \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\therefore \ y = \frac{3x-x^3}{1-3x^2}$$

13. As $x > 1$, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}(x)$

$\therefore \ f(x) = 2\tan^{-1}(x) + \pi - 2\tan^{-1}(x)$

$\Rightarrow \pi \ \forall \ x > 1$

$\Rightarrow f(5) = \pi$

Previous Years' B-Architecture Entrance Examination Questions

1. $2\cot^{-1}(7) + \cos^{-1}\left(\frac{3}{5}\right)$

$$= 2\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}\frac{\frac{2}{7}}{1-\frac{1}{49}} + \tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}\frac{7}{24} + \tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}\frac{\frac{7}{24}+\frac{4}{3}}{1-\frac{7\times4}{24\times3}} = \tan^{-1}\frac{117}{44} = \cot^{-1}\frac{44}{117}$$

2. $1+\sum_{p=1}^{n}(2p) = 1+\frac{2n(n+1)}{2} = 1+n^2+n$

$$\therefore\ \cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)=\cot^{-1}(1+n(n+1))$$

$$= \tan^{-1}\left(\frac{n+1-n}{n+(n+1)n}\right)$$

$$= \tan^{-1}(n+1)-\tan^{-1}(n)$$

$$\Rightarrow S=\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)=\tan^{-1}(20)-\frac{\pi}{4}$$

Thus, $\cot(S)=\cot\left(\tan^{-1}(20)-\frac{\pi}{4}\right)$

$$=\frac{\cot(\tan^{-1}(20))\cot\left(\frac{\pi}{4}\right)+1}{\cot\left(\frac{\pi}{4}\right)-\cot(\tan^{-1}(20))}$$

$$=\frac{\frac{1}{20}+1}{1-\frac{1}{20}}=\frac{21}{19}$$

CHAPTER TWENTY SEVEN

Heights and Distances

ANGLES OF ELEVATION AND DEPRESSION

Let *OP* be a horizontal line in the vertical plane in which an object *R* is given and let *OR* be joined.

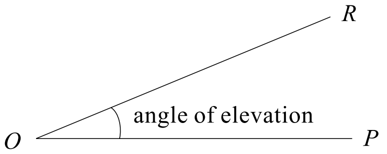


Fig. 27(a)

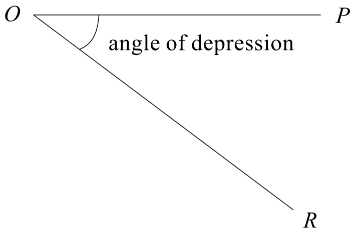


Fig. 27(b)

In Fig. 27(a), where the object *R* is above the horizontal line *OP*, the angle *POR* is called the *angle of elevation* of the object *R* as seen from the point *O*. In Fig. 27(b), where the object *R* is below the horizontal line *OP*, the angle *POR* is called the *angle of depression* of the object *R* as seen from the point *O*.

Remark

Unless stated to the contrary, it is assumed that the height of the observer is neglected, and that the angles of elevation are measured from the ground.

The following results regarding the sides and angles of a triangle are useful in solving the problems on heights and distances.

Illustration 1

Solution:

$$\frac{OP}{OA} = \tan 30° = 1/\sqrt{3}$$

$$\Rightarrow OP = (1/\sqrt{3})\,OA = 10/\sqrt{3}$$

So height of the tower is $10\sqrt{3}$

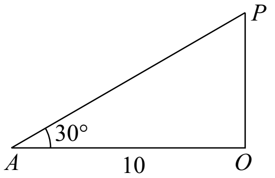


Fig. 27.1

Illustration 2

A car is moving towards a pole of height *a*. The angle of depression of the car from the top of the pole at a time is observed to be α. After 10 minutes it is observed to be *B*. Find the distance travelled by the car in 10 minutes.

Solution: Let *OP* be the pole of height *a*. *A* and *B* be the two positions of the car then

$\underline{|OAP} = \alpha, \underline{|OBP} = \beta$

$OA = a \cot \alpha,\ OB = a \cot \beta$

$AB = OA - OB = a(\cot \alpha - \cot \beta)$

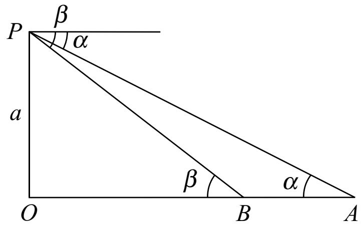


Fig. 27.2

If the angle of elevation of the top *P* of a tower *OP* at a point A on the ground is 30° and *OA* = 10 m. Find the height of the tower.

In a triangle *ABC*, the angles are denoted by the capital letters *A*, *B*, *C* and the lengths of the sides opposite these angles are denoted by *a*, *b*, *c*, respectively (Fig. 27.3).

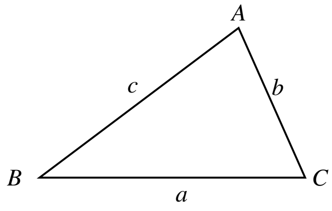


Fig. 27.3

1. *The law of sines* In any triangle ABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the radius of the circumcircle of the triangle ABC. R is also known as the *circumradius* of the triangle.

2. *The law of cosines* In any triangle ABC

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

or $a^2 = b^2 + c^2 - 2bc \cos A$

Similar formulae for $\cos B$, $\cos C$ exist.

3. *Projection rule* In any triangle ABC

$a = b \cos C + c \cos B$, $b = c \cos A + a \cos C$ and $C = a \cos B + b \cos A$.

4. *Area of the triangle* If Δ represents the area of the triangle ABC, then

$\Delta = (1/2)bc \sin A = (1/2)ca \sin B = (1/2)ab \sin C$.

$$= \frac{abc}{4R} = rs = \sqrt{s(s-a)(s-b)(s-c)}$$

where $2s = a + b + c$, R is the radius of the circum circle of triangle ABC and r is the radius of the circle inscribed in the triangle ABC.

SOLVED EXAMPLES

Concept-based Straight Objective Type Questions

Example 1: The angle of elevation of the top of two poles at a point on the line joining the foot of the towers on the ground is 45°. If the distance between the towers is 1 m., the difference between the heights of the tower is

(a) 2m (b) 1m

(c) 1/2 m (d) 3/2 m

Ans. (b)

Solution: Let PQ and LM be the two towers

$$\underline{|PAQ} = \underline{|LAM} = 45°$$

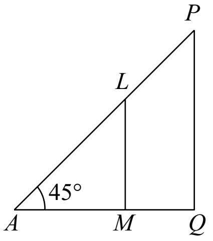


Fig. 27.4

$\Rightarrow$ $AQ = PQ$ and $AM = LM$.

$\Rightarrow$ $PQ - LM = AQ - AM = MQ = 1$m.

Hence the required difference is 1m.

Example 2: Apoorv is standing in the centre of a rectangular park and observes that the angle of elevation of the top of a lamp post at a corner of the park in 60°. He then moves diagonally towards the opposite corner of the park and observes that the angle of elevation is now β, then the value of β is

(a) 45° (b) 30°

(c) $\tan^{-1}(\sqrt{3}/2)$ (d) $\tan^{-1}(2/\sqrt{3})$

Ans. (c)

Solution: Let P and Q be the opposite corners of the park with centre O. If h is the height of the lampost at P

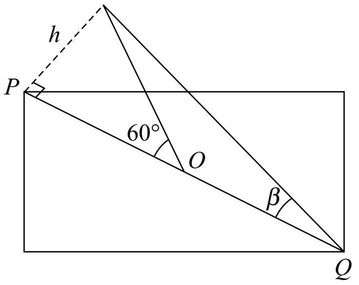


Fig. 27.5

then $OP = h \cot 60° = h/\sqrt{3}$.

$PQ = h \cot \beta = 2\,OP = 2h/\sqrt{3}$.

$\Rightarrow \cot \beta = (2/\sqrt{3}) \Rightarrow \beta = \tan^{-1}(\sqrt{3}/2)$.

Example 3: A bird is sitting on the top of a vertical pole 20m high and its elevation from a point O on the ground is 45°. It flies off horizontally straight away from the point O. After one second, the elevation of the bird from O is reduced to 30°. Then the speed (m/s) of the bird is

(a) $40\left(\sqrt{2}-1\right)$ (b) $40\left(\sqrt{3}-\sqrt{2}\right)$

(c) $20\sqrt{2}$ (d) $20\left(\sqrt{3}-1\right)$

Ans. (d)

Solution: Let PQ be the pole of height 20 m $\angle POQ = 45^\circ$

Let P' be the position of the bird after one second

$\angle P'OQ' = 30^\circ$.

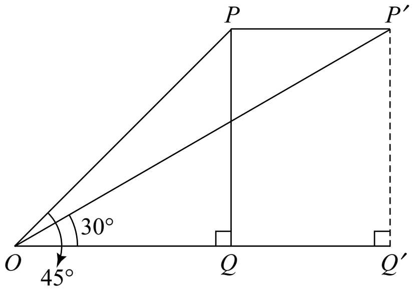


Fig. 27.6

$$PP' = QQ' = PQ' - PQ = 20\cot 30^\circ - 20\cot 45^\circ$$
$$= 20\left(\sqrt{3}-1\right)$$

Example 4: Two ships A and B are sailing straight away from the foot of a tower OP along routes such that $\angle AOB$ is always 120°. At a certain instance, the angles of depression of the ships A and B from the top P of the towers are 60° and 30° respectively. The distance between the ships when the height of the tower is 15m is

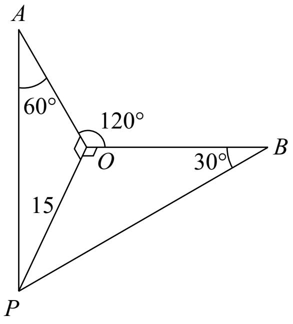


Fig. 27.7

(a) $5\sqrt{39}$ m (b) $5\sqrt{30}$ m

(c) $5\sqrt{21}$ m (d) $5\sqrt{3}$ m

Ans. (a)

Solution: $OP = 15$m

$\angle PAO = 60^\circ$

$\angle PBO = 30^\circ$

$\angle AOB = 120^\circ$

$$OA = OP\cot 60^\circ = 15/\sqrt{3}$$
$$OB = OP\cot 30^\circ = 15\sqrt{3}$$

From $\Delta\, AOB$

$$(AB)^2 = (OA)^2 + (OB)^2 - 2OA \times OB \times \cos 120^\circ$$
$$= \left(15^2\right)\left[\frac{1}{3} + 3 - 2\left(-\frac{1}{2}\right)\right]$$
$$= 15^2 \times \frac{13}{3} = \frac{15\times15\times13}{3} = 5\sqrt{39}$$

Example 5: *A* vertical pole stands at a point A on the boundary of a circular park of radius 2 km. and subtends an angle 60° at another point B on the boundary. If the chord AB subtends the same angle 60° at the centre of the park, the height of the pole is

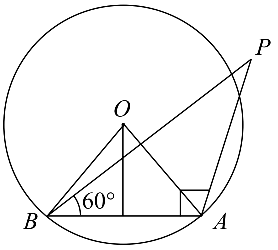


Fig. 27.8

(a) $2\sqrt{3}$ km (b) $\sqrt{3}$ km

(c) $2/\sqrt{3}$ km (d) 1 km.

Ans. (a)

Solution: Let AP be the pole, $\angle PBA = 60^\circ$

$\Rightarrow AB = AP\cot 60^\circ = \dfrac{h}{\sqrt{3}}$, h being the height of the pole.

Let O be the centre of the park.

Then $\angle AOB = 60^\circ \Rightarrow \Delta OAB$ is equilateral and hence $AB = OB = 2$km.

$$\Rightarrow \frac{h}{\sqrt{3}} = 2 \Rightarrow h = 2\sqrt{3}\text{ km}.$$

LEVEL 1

Straight Objective Type Questions

Example 6: A flagstaff stands in the centre of a rectangular field whose diagonal is 1200 m, and subtends angles 15° and 45° at the mid points of the sides of the field. The height of the flagstaff is

(a) 200 m (b) $300\sqrt{2+\sqrt{3}}$ m

(c) $300\sqrt{2-\sqrt{3}}$ m (d) 400 m

Ans. (c)

◎ **Solution:** Let OP be the flagstaff of height h standing at the centre O of the rectangular field $ABCD$ subtending angles 15° and 45° at E and F the mid points of the sides AD and DC of the field.(Fig. 27.9)

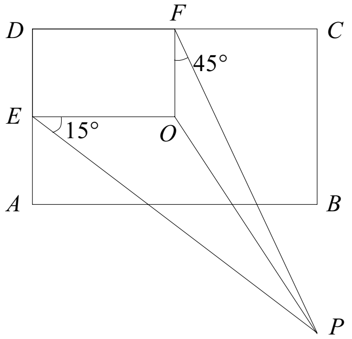


Fig. 27.9

then $OE = h \cot 15°$

$= h(2 + \sqrt{3})$

and $OF = h.\ \cot 45° = h.$

$\Rightarrow$ $EF = h\sqrt{1+\left(2+\sqrt{3}\right)^2}$

$= 2h\sqrt{2+\sqrt{3}}$

$\Rightarrow$ $1200 = AC = 2EF = 4h\sqrt{2+\sqrt{3}}$

$\Rightarrow$ $h = \dfrac{300}{\sqrt{2+\sqrt{3}}} = 300\sqrt{2-\sqrt{3}}.$

◉ **Example 7:** Two flagstaffs stand on a horizontal plane. A and B are two points on the line joining their feet and between them. The angles of elevation of the tops of the flagstaffs as seen from A are 30° and 60° and as seen from B are 60° and 45°. If AB is 30 m, the distance between the flagstaffs in metres is

(a) $30 + 15\sqrt{3}$ (b) $45 + 15\sqrt{3}$

(c) $60 - 15\sqrt{3}$ (d) $60 + 15\sqrt{3}$

Ans. (d)

◎ **Solution:** Let x and y be the heights of the flagstaffs at P and Q respectively.

Then $AP = x \cot 60° = x/\sqrt{3}$,

$AQ = y \cot 30° = y\sqrt{3}$

$BP = x \cot 45° = x,$

$BQ = y \cot 60° = y/\sqrt{3}$ (Fig. 27.10)

$\Rightarrow$ $BP - AP = x - x/\sqrt{3} = AB$

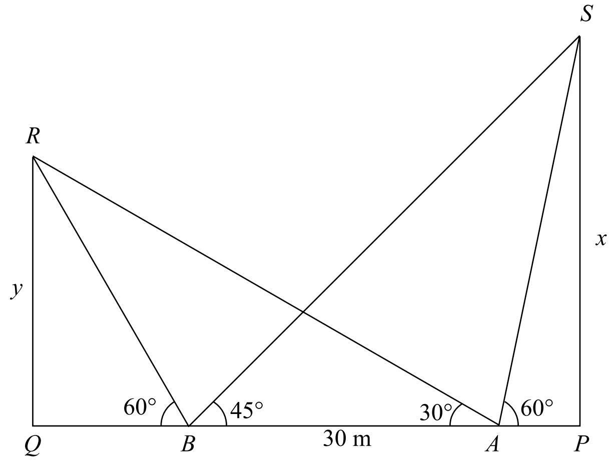


Fig. 27.10

$\Rightarrow$ $30\sqrt{3} = \left(\sqrt{3}-1\right)x$

$\Rightarrow$ $x = 15\left(3+\sqrt{3}\right)$

Similarly $30 = y\left(\sqrt{3} - 1/\sqrt{3}\right)$

$\Rightarrow$ $y = 15\sqrt{3}.$

so that $PQ = BP + BQ = x + y/\sqrt{3}$

$= 15\left(3+\sqrt{3}\right) + 15$

$= (60 + 15\sqrt{3})$ m.

◉ **Example 8:** In a cubical hall $A\ B\ C\ D\ P\ Q\ R\ S$ with each side 10 m, G is the centre of the wall $B\ C\ R\ Q$ and T is the mid point of the side AB. The angle of elevation of G at the point T is

(a) $\sin^{-1}(1/\sqrt{3})$ (b) $\cos^{-1}(1/\sqrt{3})$

(c) $\tan^{-1}(1/\sqrt{3})$ (d) $\cot^{-1}(1/\sqrt{3})$

Ans. (a)

◎ **Solution:** Let H be the mid point of BC since $\angle TBH = 90°$, $TH^2 = BT^2 + BH^2 = 5^2 + 5^2 = 50$

Also since $\angle THG = 90°$, $TG^2 = TH^2 + GH^2 = 50 + 25 = 75$

Let θ be the required angle of elevation of G at T. (Fig. 27.11)

then $\sin\theta = \dfrac{GH}{TG} = \dfrac{5}{5\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

$\Rightarrow$ $\theta = \sin^{-1}(1/\sqrt{3})$

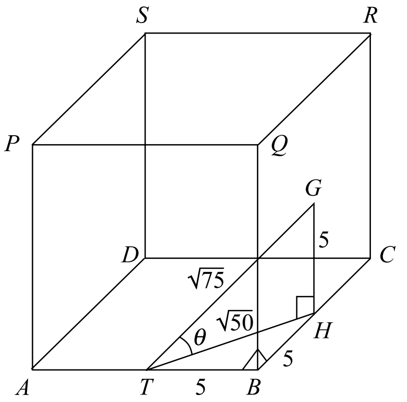


Fig. 27.11

Example 9: Two vertical poles 20 m and 80 m high stand apart on a horizontal plane. The height of the point of intersection of the lines joining the top of each pole to the foot of the other is

(a) 15 m (b) 16 m
(c) 18 m (d) 50 m

Ans. (b)

Solution: Let PQ and RS be the poles of height 20 m and 80 m subtending angles α and β at R and P respectively. Let h be the height of the point T, the intersection of QR and PS. (Fig. 27.12)

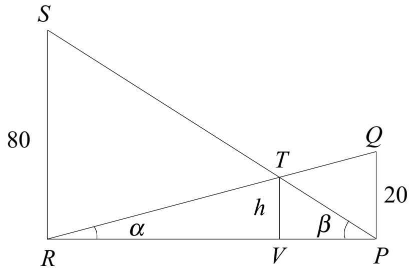


Fig. 27.12

Then $PR = h \cot \alpha + h \cot \beta = 20 \cot \alpha = 80 \cot \beta$

$$\Rightarrow \quad \cot \alpha = 4 \cot \beta \text{ or } \frac{\cot \alpha}{\cot \beta} = 4$$

Again $h \cot \alpha + h \cot \beta = 20 \cot \alpha$

$$\Rightarrow \quad (h - 20) \cot \alpha = - h \cot \beta$$

$$\Rightarrow \quad \frac{\cot \alpha}{\cot \beta} = \frac{h}{20 - h} = 4$$

$$\Rightarrow \quad h = 80 - 4h \quad \Rightarrow \quad h = 16 \text{ m.}$$

Example 10: A man from the top of a 100 metres high tower observes a car moving towards the tower at an angle of depression of 30°. After some time, the angle of depression becomes 60°. The distance (in metres) travelled by the car during this time is

(a) $100\sqrt{3}$ (b) $200/\sqrt{3}$
(c) $100/\sqrt{3}$ (d) $200\sqrt{3}$

Ans. (b)

Solution: Let OP be the tower and C_1, C_2 be the positions of the car (Fig. 27.13)

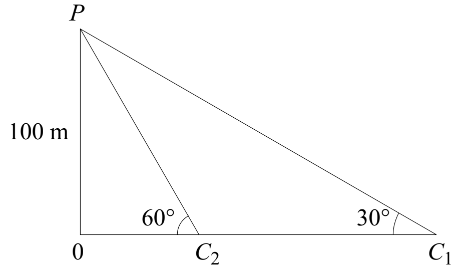


Fig. 27.13

We have $\dfrac{OC_1}{OP} = \cot 30° = \sqrt{3}$

$\Rightarrow \quad OC_1 = 100\sqrt{3}$

Similarly $\quad OC_2 = 100/\sqrt{3}$

Now $\quad C_1 C_2 = OC_1 - OC_2$

$= 100(\sqrt{3} - 1/\sqrt{3})$

$= 200/\sqrt{3}$.

Example 11: A pole stands vertically, inside a triangular park ABC. If the angle of elevation of the top of the pole from each corner of the park is same, then in ΔABC, the foot of the pole is at the

(a) centroid (b) circumcentre
(c) incentre (d) orthocentre

Ans. (b)

Solution: Let OP be the pole with O as the foot, and the angle of elevation of the top P of the pole at each of the corner A, B, C be α. (Fig. 27.14)

then $\quad \dfrac{OP}{OA} = \tan \alpha$

$\Rightarrow \quad OA = OP \cot \alpha$

Similarly $\quad OB = OC = OP \cot \alpha$

Since $OA = OB = OC$, O is the circumcentre of the triangle ABC.

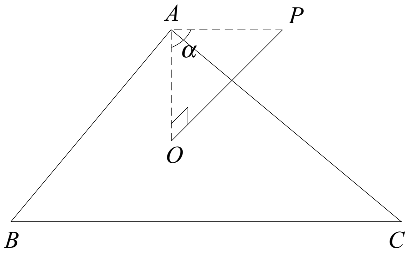


Fig. 27.14

Example 12: A man observes that the angle of elevation of the top of a tower from a point P on the ground is θ.

He moves a certain distance towards the foot of the tower and finds that the angle of elevation of the top has doubled. He further moves a distance 3/4 of the previous and finds that the angle of elevation is three times that at *P*. The angle θ is given by

(a) $\sin\theta = \sqrt{5/12}$ (b) $\cos\theta = \sqrt{5/12}$

(c) $\sin\theta = 3/4$ (d) $\cos\theta = 3/8$

Ans. (a)

Solution: Let *AB* be the tower,

$$\angle APB = \theta,\ \angle AQB = 2\theta \text{ and } \angle ARB = 3\theta$$

Then $QR = (3/4)\ PQ$ and $\angle PBQ = \angle QBR = \theta$

$\Rightarrow$ *BQ* is the bisector of $\angle PBR$ (Fig. 27.15)

$$\Rightarrow \frac{PB}{BR} = \frac{PQ}{QR} \Rightarrow \frac{AB\operatorname{cosec}\theta}{AB\operatorname{cosec}3\theta} = \frac{4}{3} \Rightarrow \frac{\sin 3\theta}{\sin\theta} = \frac{4}{3}$$

$$\Rightarrow 3 - 4\sin^2\theta = 4/3 \Rightarrow 12\sin^2\theta = 5 \Rightarrow \sin\theta = \sqrt{5/12}\,.$$

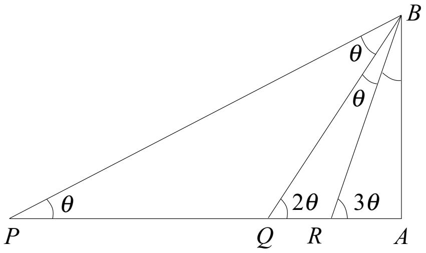


Fig. 27.15

Example 13: A vertical pole subtends an angle $\tan^{-1}(1/2)$ at a point *P* on the ground. The angle subtended by the upper half of the pole at the point *P* is

(a) $\tan^{-1}(1/4)$ (b) $\tan^{-1}(2/9)$

(c) $\tan^{-1}(1/8)$ (d) $\tan^{-1}(2/3)$

Ans. (b)

Solution: Let the pole *AB* subtend angle θ at *P* and the upper half *BC* of the pole subtend angle α at *P* (Fig. 27.16)

then $\tan\theta = \dfrac{1}{2} = \dfrac{AB}{AP}$

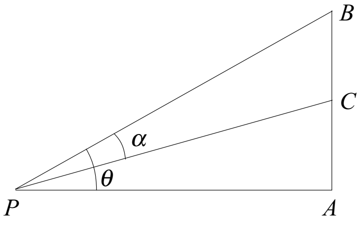


Fig. 27.16

$$\tan(\theta - \alpha) = \frac{AC}{AP} = \frac{(1/2)AB}{AP} = \frac{1}{4}.$$

Now $\tan\alpha = \tan(\theta - (\theta - \alpha))$

$$= \frac{\tan\theta - \tan(\theta - \alpha)}{1 + \tan\theta\tan(\theta - \alpha)} = \frac{\frac{1}{2} - \frac{1}{4}}{1 + \frac{1}{2} \times \frac{1}{4}} = \frac{2}{9}$$

$\Rightarrow \alpha = \tan^{-1}(2/9)$.

Example 14: An aeroplane flying at a height of 3000 m above the ground passes vertically above another plane at an instant when the angles of elevation of the two planes from the same point on the ground are 60° and 45° respectively. The height of the lower plane from the ground is

(a) $1000\sqrt{3}$ m (b) $1000/\sqrt{3}$ m

(c) 500 m (d) $1500(\sqrt{3} + 1)$ m

Ans. (a)

Solution: Let *Q* and *P* be the position of the two planes and *A* be the point on the ground such that $\angle QAO = 60°$, $\angle PAO = 45°$ (Fig. 27.17) and *OP*, the height of lower plane be *h*

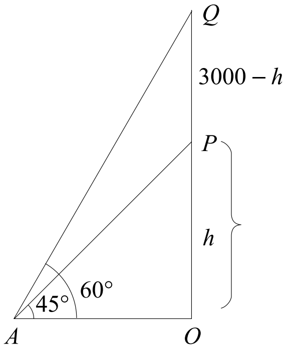


Fig. 27.17

Also $OQ = 3000$ m

Then $OA = OP = h$

and $OQ = h\tan 60° = 3000$

$$\Rightarrow h = 3000\cot 60° = \frac{3000}{\sqrt{3}} = 1000\sqrt{3}\ \text{m}$$

Example 15: A pole of height *h* stands at one corner of a park in the shape of an equilateral triangle. If α is the angle which the pole subtends at the midpoint of the opposite side, the length of each side of the park is

(a) $(\sqrt{3}/2)\ h\cot\alpha$ (b) $(2/\sqrt{3})\ h\cot\alpha$

(c) $(\sqrt{3}/2)\ h\tan\alpha$ (d) $(2/\sqrt{3})\ h\tan\alpha$

Ans. (b)

Solution: Let *ABC* be the triangular park, *AP* the pole at *A*, and *D* the midpoint of *BC*. Let each side of the equilateral triangle *ABC* be *a*. Then (Fig. 27.18)

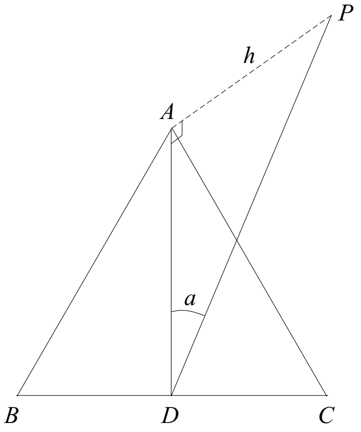


Fig. 27.18

$$AD^2 = AB^2 - BD^2 = a^2 - a^2/4 = 3a^2/4$$

$$\Rightarrow \qquad AD = a\sqrt{3}/2$$

and, since $AP = h$ and $\angle ADP = \alpha$,

we have $AD = h \cot \alpha$.

Therefore, $a\sqrt{3}/2 = h \cot \alpha$

$$\Rightarrow \qquad a = \left(2/\sqrt{3}\right) h \cot \alpha$$

◉ **Example 16:** If each side of length a of an equilateral triangle subtends an angle of 60° at the top of a tower h metre high situated at the centre of the triangle, then

(a) $3a^2 = 2h^2$	(b) $2a^2 = 3h^2$
(c) $a^2 = 3h^2$	(d) $3a^2 = h^2$

Ans. (b)

◎ **Solution:** Let O be the centre of the equilateral triangle ABC and OP be the tower of height h. (Fig. 27.19)
Then each of the triangles PAB, PBC and PCA are equilateral and thus

$$PA = PB = AB = a.$$

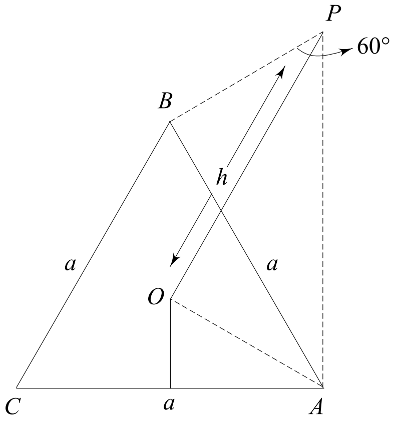


Fig. 27.19

In triangle ABC, OA is the bisector of the angle A,

So $$\frac{OA}{a/2} = \sec 30°$$

$$\Rightarrow \qquad OA = \frac{a}{2} \cdot \frac{2}{\sqrt{3}} = \frac{a}{\sqrt{3}}.$$

Now from right angled triangle POA

$$PA^2 = OP^2 + OA^2$$

$$\Rightarrow \qquad a^2 = h^2 + \frac{a^2}{3} \Rightarrow \frac{2a^2}{3} = h^2$$

$$\Rightarrow \qquad 2a^2 = 3h^2.$$

◉ **Example 17:** A pole 50 m high stands on a building 250 m high. To an observer at a height of 300 m, the building and the pole subtend equal angles. The horizontal distance of the observer from the pole is

(a) 25 m	(b) 50 m
(c) $25\sqrt{6}$ m	(d) $25\sqrt{3}$ m

Ans. (c)

◎ **Solution:** Let PQ be the pole on the building QR and O be the observer.

Then $PQ = 50$, $QR = 250$ (Fig. 27.20)

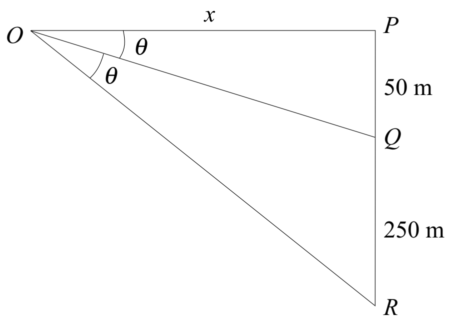


Fig. 27.20

$\Rightarrow$ $PR = 300$ so the observer is at the same height as the top P of the pole. Let $OP = x$. Then from right angled triangles OPQ and OPR,

$$\tan \theta = \frac{50}{x} \text{ and } \tan 2\theta = \frac{300}{x}.$$

so that $$\frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{300}{x}$$

$$\Rightarrow \quad \frac{2 \times (50/x)}{1 - (50/x)^2} = \frac{300}{x} \Rightarrow 3\left\{1 - \left(\frac{50}{x}\right)^2\right\} = 1$$

$$\Rightarrow \quad \left(\frac{50}{x}\right)^2 = \frac{2}{3} \Rightarrow x = \frac{50\sqrt{3}}{\sqrt{2}} = 25\sqrt{6}.$$

◉ **Example 18:** Two vertical poles of height a and b subtend the same angle 45° at a point on the line joining their feet, the square of the distance between their tops is

(a) $(1/2)\,(a^2 + b^2)$ (b) $a^2 + b^2$

(c) $2\,(a^2 + b^2)$ (d) $(a + b)^2$

Ans. (c)

◎ **Solution:** Let OP and QR be the poles of height a and b respectively subtending angle 45° at S, a point between O and Q. (Fig. 27.21)

then $OS = a$, $QS = b$

$$PS^2 = 2a^2 \text{ and } RS^2 = 2b^2$$

Since $\angle PSR = 180° - (45° + 45°) = 90°$

$$PR^2 = PS^2 + RS^2 = 2(a^2 + b^2)$$

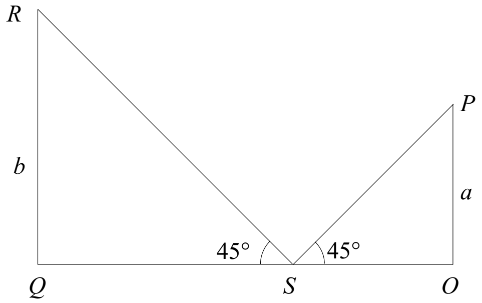


Fig. 27.21

◉ **Example 19:** A monument $ABCD$ stands at A on a level ground. At a point P on the ground the portions AB, AC, AD subtend angles α, β, γ respectively. If $AB = a$, $AC = b$, $AD = c$, $AP = x$ and $\alpha + \beta + \gamma = 180°$ then x^2 is equal to

(a) $\dfrac{a}{a+b+c}$ (b) $\dfrac{b}{a+b+c}$

(c) $\dfrac{c}{a+b+c}$ (d) $\dfrac{abc}{a+b+c}$

Ans. (d)

◎ **Solution:** We have

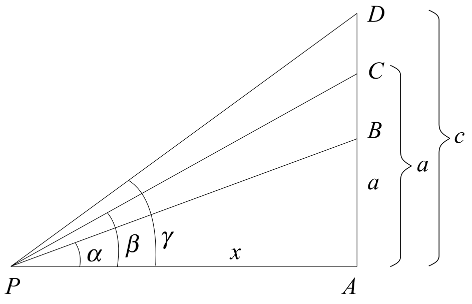


Fig. 27.22

$$a = AB = AP \tan \alpha = x \tan \alpha \text{ (Fig. 27.22)}$$
$$b = AC = x \tan \beta$$

and $c = AD = x \tan \gamma$

so that $a + b + c = x(\tan \alpha + \tan \beta + \tan \gamma)$

$= x \tan \alpha \tan \beta \tan \gamma$ $[\because \alpha + \beta + \gamma = 180°]$

and $abc = x^3 \tan \alpha \tan \beta \tan \gamma$

$$\Rightarrow \quad x^2 = \frac{abc}{a+b+c}$$

◉ **Example 20:** A vertical tower CP subtends the same angle θ, at point B on the horizontal plane through C, the foot of the tower, and at point A in the vertical plane. If the triangle ABC is equilateral with length of each side equal to 4 m, the height of the tower is

(a) $8\sqrt{3}$ m (b) $4\sqrt{3}/3$ m

(c) $4\sqrt{3}$ m (d) $8/\sqrt{3}$ m

Ans. (b)

◎ **Solution:** Since $\angle CBP = \angle CAP = \theta$ (angles in the same segment are equal) $PCBA$ is a cyclic quadrilateral. (Fig. 27.23)

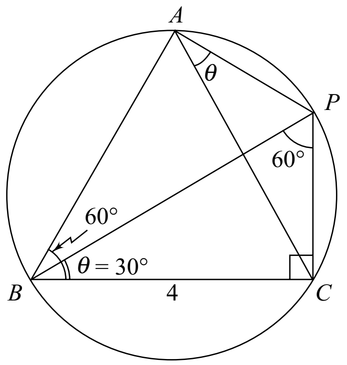


Fig. 27.23

$\Rightarrow \quad \angle BAP + \angle BCP = 180°$

$\Rightarrow \quad \angle BAP = 90°$ $[\because \angle BCP = 90°]$

$\Rightarrow \quad \angle PBC = \theta = \angle BAP - \angle BAC$

$= 90° - 60° = 30°$

If h is the height of the tower CP, then $\dfrac{CP}{CB} = \tan \theta = \tan A$

$$\Rightarrow \quad \frac{h}{4} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad h = \frac{4}{\sqrt{3}} \text{ m}$$

◉ **Example 21:** A man on the ground observes that the angle of elevation of the top of a tower is 68° 11′, and a flagstaff 24 m high on the summit of the tower subtends an angle of 2° 10′ at the observer's eye. If tan 70° 21′ = 2.8 and cot 68° 11′ = 0.4, the height of the tower is

(a) 120 m (b) 168 m

(c) 200 m (d) 300 m

Ans. (c)

Solution: Let PQ be the tower of height h whose elevation at A is $68° \, 11'$ and QR be the flagstaff of height 24 m subtending an angle of $2° \, 10'$ at A. (Fig. 27.24)

Then $\quad h \cot 68° \, 11' = AP$

$$= (h + 24) \cot (68° \, 11' + 2° \, 10')$$

$$\Rightarrow \quad h \cot 68°11' = \frac{(h+24)}{\tan 70° \, 21'}$$

$$\Rightarrow \quad h\,(0\cdot 4)\,(2\cdot 8) = h + 24$$

$$\Rightarrow \quad h = \frac{24}{1\cdot 12 - 1} = \frac{24}{\cdot 12}$$

$$= 200 \text{ m}$$

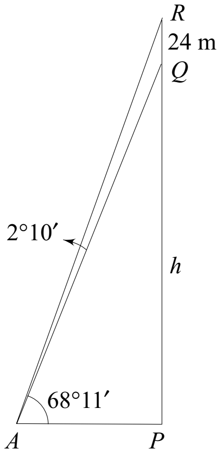


Fig. 27.24

Example 22: A statue, standing on the top of a pillar 25 m high, subtends an angle whose tangent is 0.125 at a point 60 m from the foot of the pillar. The best approximation for the height of the statue is

(a) 9.28 m	(b) 9.29 m
(c) 9.30 m	(d) 10 m

Ans. (b)

Solution: Let PQ be the statue of height x on the top of the pillar OP, 25 m high, subtending an angle α at the point A, 60 m from O, the foot of the tower and $\angle OAP = \theta$. (Fig. 27.25)

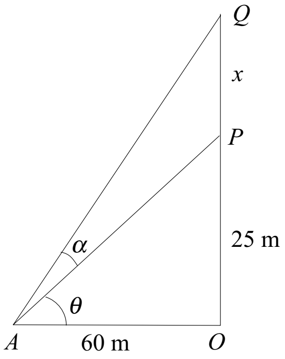


Fig. 27.25

Then $\qquad \tan \alpha = \cdot 125$ (given)

and $\qquad \tan \theta = \dfrac{OP}{OA} = \dfrac{25}{60} = \dfrac{5}{12}$

then $\quad (x + 25) \cot (\alpha + \theta) = 60$

$\Rightarrow \quad x + 25 = 60 \tan (\alpha + \theta)$

$$= 60 \left[\frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta} \right]$$

$$\Rightarrow \qquad x = 60 \left[\frac{(1/8)+(5/12)}{1-(1/8)\times(5/12)} \right] - 25$$

$$= 60 \times (52/91) - 25 = 60 \times (4/7) - 25$$

$$= 65/7$$

$$= 9.2857 = 9.29 \text{ (approximately).}$$

Example 23: A tower BCD surmounted by a spire DE stands on a horizontal plane. At the extremity A of a horizontal line BA it is found that BC and DE subtend equal angles. If $BC = 3$ m, $CD = 28$ m and $DE = 5$ m, then BA is equal to

(a) $\sqrt{18 \times 93}$	(b) $\sqrt{36 \times 93}$
(c) $\sqrt{34 \times 93}$	(d) $\sqrt{34 \times 36}$

Ans. (a)

Solution: Let $\angle BAC = \angle DAE = \theta$

$\angle DAB = \alpha$, $AB = x$. (Fig. 27.26)

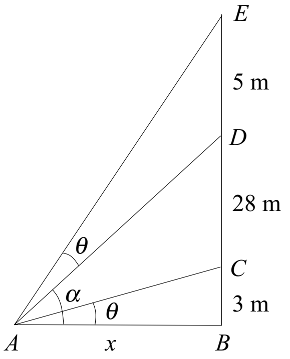


Fig. 27.26

Now $BC = 3$ m, $CD = 28$ m, $DE = 5$ m.

$\therefore \qquad \tan (\alpha + \theta) = 36/x$

$\qquad \tan (\alpha) = 31/x$, $\tan \theta = 3/x$

so, $\tan (\alpha + \theta) = \dfrac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta}$

$$\Rightarrow \quad \frac{36}{x} = \frac{\frac{31}{x} + \frac{3}{x}}{1 - \frac{31}{x} \times \frac{3}{x}}$$

$\Rightarrow \quad 36(x^2 - 93) = 34\,x^2 \Rightarrow 2x^2 = 36 \times 93$

$\Rightarrow \quad x^2 = 18 \times 93$

$\Rightarrow \quad x = \sqrt{18 \times 93}$.

Example 24: A lamp post standing at a point A on a circular path of radius r subtends an angle α at some point B on the path, and AB subtends an angle of 45° at any other point on the path, then height of the lampost is

(a) $\sqrt{2}\ r\cot\alpha$ (b) $(r/\sqrt{2})\ \tan\alpha$

(c) $\sqrt{2}\ r\tan\alpha$ (d) $(r/\sqrt{2})\ \cot\alpha$

Ans. (c)

◎ **Solution:** Let AP be the lampost of height h at a point A on a circular path of radius r and centre C. Let B be the point on this path such that $\angle PBA = \alpha$. $\Rightarrow$ $AB = h\cot\alpha$ (Fig. 27.27)

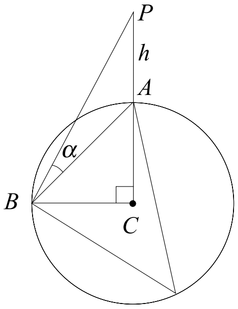


Fig. 27.27

Since AB subtends an angle 45° at another point of the path, it subtends an angle of 90° at the centre C so that $\angle BCA = 90°$

Also $CA = CB = r \Rightarrow AB = \sqrt{2}\ r$

and then $h\cot\alpha = \sqrt{2}\ r$

$\Rightarrow \quad h = \sqrt{2}\ r\tan\alpha.$

Example 25: A tree is broken by wind, its upper part touches the ground at a point 10 m from the foot of the tree and makes an angle 45° with the ground. The entire length of the tree was

(a) 15m (b) 20m

(c) $10(\sqrt{2}+1)$m (d) $5(\sqrt{3}+2)$ m

Ans. (c)

◎ **Solution:** Let OPQ be the tree whose upper part PQ makes an angle 45° with the ground.

Then the length of the tree (Fig. 27.28)

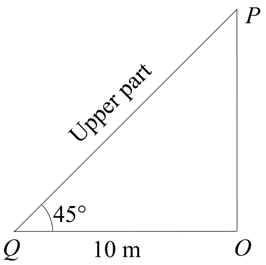


Fig. 27.28

$= OP + PQ = 10\tan 45° + 10\sec 45°$

$= 10 + 10\sqrt{2} = 10(\sqrt{2}+1)$ m

Example 26: From the top of a cliff x m high, the angle of depression of the foot of a tower is found to be double the angle of elevation of the tower. If the height of the tower is h, the angle of elevation is

(a) $\sin^{-1}\sqrt{x/(2-h)}$ (b) $\tan^{-1}\sqrt{3-2h/x}$

(c) $\sin^{-1}\sqrt{2h/x}$ (d) $\cos^{-1}\sqrt{2h/x}$

Ans. (b)

◎ **Solution:** Let OQ be the tower of height h with O as its foot. AB be the cliff of height x. (Fig. 27.29)

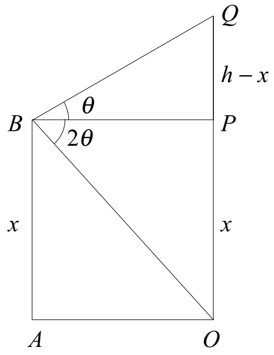


Fig. 27.29

$PQ = h - x,\ \angle QBP = \theta$ and $\angle PBO = 2\theta$

Then $(h-x)\cot\theta = x\cot 2\theta$

$\Rightarrow (h-x)\tan 2\theta = x\tan\theta$

$$\Rightarrow \frac{h-x}{x} = \frac{\tan\theta}{\tan 2\theta}$$

$$\Rightarrow \frac{h-x}{x} + 1 = \frac{1-\tan^2\theta}{2} + 1$$

$$\Rightarrow \frac{2h}{x} = 3 - \tan^2\theta$$

$\Rightarrow \theta = \tan^{-1}\sqrt{3-2h/x}$.

Example 27: A river flows due North, and a tower stands on its left bank. From a point A upstream and on the same bank as the tower, the elevation of the tower is 60°, and from a point B just opposite A on the other bank the elevation is 45°. If the tower is 360 m high, the breadth of the river is

(a) $120\sqrt{6}$ m (b) $240/\sqrt{3}$ m

(c) $240\sqrt{3}$ m (d) $240\sqrt{6}$ m

Ans. (a)

◎ **Solution:** Let PQ be the tower of height 360 m standing on the left bank of the river subtending an angle 60° at A on the same bank as the tower. (Fig. 27.30) Then PA = 360 cot 60°

Let B be the point opposite to A on the other bank where the elevation of the tower is 45°. So that $PB = 360 \cot 45° = 360$.

If x be the width of the river, then $AB = x$ and from the right angled triangle PAB,

$$(PA)^2 + (AB)^2 = (PB)^2$$

$$\Rightarrow \quad \left(360 \times \frac{1}{\sqrt{3}}\right)^2 + x^2 = (360)^2$$

$$\Rightarrow \quad (360)^2 \left(1 - \frac{1}{3}\right) = x^2$$

$$\Rightarrow \quad x = 360 \times \frac{\sqrt{2}}{\sqrt{3}} = 120\sqrt{6} \text{ m.}$$

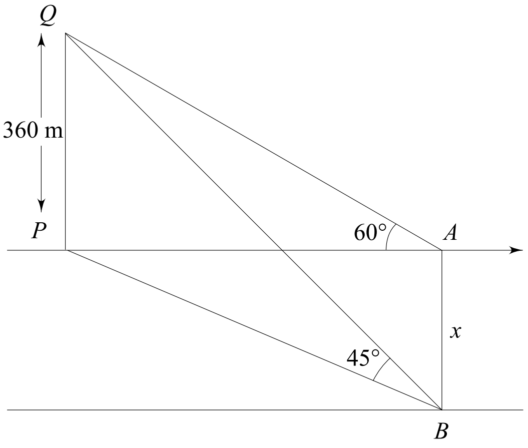


Fig. 27.30

◉ **Example 28:** The angle of elevation of a cloud at a height h above the level of water in a lake is α, and the angle of depression of its image in the lake is β. The height of the cloud above the surface of the lake is

(a) $\dfrac{h(\cot\alpha + \cot\beta)}{\cot\beta - \cot\alpha}$ (b) $\dfrac{h(\tan\alpha - \tan\beta)}{\tan\alpha + \tan\beta}$

(c) $\dfrac{h\sin(\alpha+\beta)}{\sin(\beta-\alpha)}$ (d) $\dfrac{h\sin(\alpha-\beta)}{\sin(\alpha+\beta)}$

Ans. (c)

◎ **Solution:** Let the height of the cloud above the water level be x. Let A be the point at a height h above the water level from which the cloud's angles of depression and elevation are measured. Let L and L' be the locations of the cloud and its image, respectively. Further, as shown in Fig. 27.31, let P be the point directly below the cloud, midway between L and L', and Q the point directly above P at a height h. Then $\angle LAQ = \alpha$, $\angle QAL' = \beta$, $PQ = h$, $QL = x - h$, $PL' = x$, and we have $(x - h)\cot\alpha = (x + h)\cot\beta = AQ$, i.e.,

$$\frac{x+h}{x-h} = \frac{\tan\beta}{\tan\alpha} \Rightarrow \frac{2x}{2h} = \frac{\tan\beta + \tan\alpha}{\tan\beta - \tan\alpha}$$

$$\Rightarrow \quad x = h\left(\frac{\tan\beta + \tan\alpha}{\tan\beta - \tan\alpha}\right) = h\left(\frac{\cot\alpha + \cot\beta}{\cot\alpha - \cot\beta}\right)$$

$$= h\left[\frac{\sin(\alpha+\beta)}{\sin(\beta-\alpha)}\right],$$

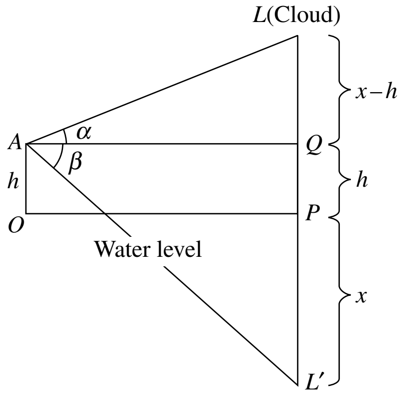


Fig. 27.31

◉ **Example 29:** A balloon of radius r subtends an angle α at the eye of an observer and the elevation of the centre of the balloon from the eye is β, the height h of the centre of the balloon is given by

(a) $\dfrac{r\sin\beta}{\sin\alpha}$ (b) $r \sin b \sin a$

(c) $\dfrac{r\sin\beta}{\sin(\alpha/2)}$ (d) $\dfrac{r\sin\alpha}{\sin(\beta/2)}$

Ans. (c)

◎ **Solution:** Let O be the centre of the balloon of radius r which subtends an angle α at E, the eye of the observer. If EA and EB are the tangents to the balloon, then $\angle OEA = \angle OEB = \alpha/2$ (Fig. 27.32)

Let $OL = h$ be the height of the centre of the balloon, then from ΔOLE

$$h = OE \sin\beta$$

$$= r \operatorname{cosec}(\alpha/2)\sin\beta$$

$$= r\sin\beta/\sin(\alpha/2).$$

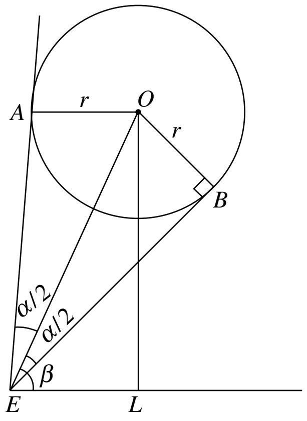


Fig. 27.32

Example 30: Two poles of height a and b stand at the centres of two circular plots which touch each other externally at a point and the two poles subtend angles of 30° and 60° respectively at this point, then distance between the centres of these poles is

(a) $a + b$ (b) $(3a + b)/\sqrt{3}$

(c) $(a + 3b)/\sqrt{3}$ (d) $a\sqrt{3} + b$

Ans. (b)

Solution: Let A and B be the centres of the two circles where the poles of height a and b respectively stand making angles 30° and 60° respectively at the point O where these circle touch each other externally. (Fig. 27.33)

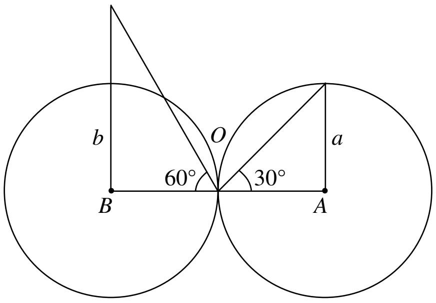


Fig. 27.33

sum of the radii of the two circles

$$= a\cot 30° + b\cot 60°$$

$$= a\sqrt{3} + \frac{b}{\sqrt{3}} = \frac{3a+b}{\sqrt{3}}$$

= distance between the centres of the two circles.

Example 31: A tower is standing in the centre of an elliptic field. If Adya observes that the angle of elevation of the top of the tower at an extremity of the major axis of the field is α, at its focus is β and an extremity of the minor axis is γ, then

(a) $\cot^2\alpha = \cot^2\beta - \cot^2\gamma$
(b) $\cot^2\beta = \cot^2\gamma - \cot^2\alpha$
(c) $\cot^2\gamma = \cot^2\alpha - \cot^2\beta$
(d) none of these

Ans. (c)

Solution: Let the equation of the elliptic field be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

OP be the tower of height h at O, the centre of the field, Let A & B be extremities of major and minor axis respectively and S be a focus, then

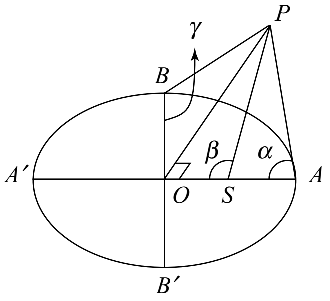


Fig. 27.34

$$OA = h\cot\alpha = a$$

$$OS = h\cot\beta = ae$$

$$OB = h\cot\gamma = b$$

Since $b^2 = a^2(1 - e^2)$

$\Rightarrow$ $\cot^2\gamma = \cot^2\alpha - \cot^2\beta$

Example 32: A tower of height h stands at a point O on the ground. Two poles of height a and b stand at the points A and B respectively such that O lies on the line joining A and B. If the angle of elevation of the top of the tower at the foot of one pole is same as at the top of the other pole, then h is equal to

(a) $\frac{a+b}{ab}$ (b) $\frac{ab}{a+b}$

(c) $a + b$ (d) $\frac{a+b}{|a-b|}$

Ans. (b)

Solution: Let L and M be the tops of the poles at A and B respectively. Angle of elevation of P at A and M be α, and at B and L be β.

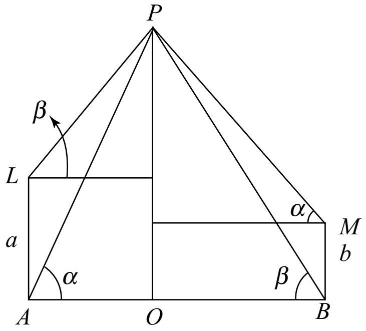


Fig. 27.35

then $OA = h\cot\alpha = (h - a)\cot\beta$

and $OB = h\cot\beta = (h - b)\cot\alpha$

$$\Rightarrow \frac{h-a}{h} = \frac{h}{h-b} \Rightarrow h = \frac{ab}{a+b}$$

Example 33: Rajat observes that the angle of elevation of the first floor of a building at a point A on the ground is 30°. He moves $\sqrt{3}$ units towards the building to the point B and finds that the angle of elevation of the second floor of the building is 60°. If each floor has the same height, height of the 7th floor from the ground in units is

(a) 5 (b) 7

(c) 21 (d) 35

Ans. (c)

Solution: Let the height of each floor be x units

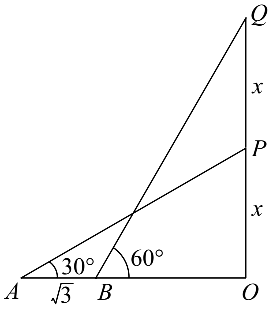


Fig. 27.36

$$\begin{aligned}\sqrt{3} &= AB = OA - OB\\ &= x\cot 30° - 2x\cot 60°\\ &= x\sqrt{3} - \frac{2x}{\sqrt{3}}\end{aligned}$$

$\Rightarrow$ $x = 3$ and hence the required height is 21 units.

Example 34: A pole of height h stands in the centre of a circular platform in the centre of a circular field. Another pole of equal height is at a point on the boundary of the field. The angles of elevation of the top of the first pole from the bottom and top of the second pole are respectively α and β. Height of the platform from the ground is

(a) $\dfrac{h\cot\alpha}{\cot\beta - \cot\alpha}$ (b) $\dfrac{h\cot\beta}{\cot\alpha - \cot\beta}$

(c) $\cot\alpha - \cot\beta$ (d) none of these

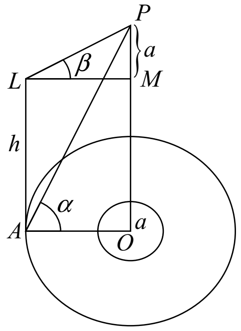


Fig. 27.37

Ans. (a)

Solution: Let a be the height of the platform then height of the first pole from the ground is $h + a$ and from the top of the second pole is a, so

$$(h + a)\cot\alpha = a\cot\beta$$

$$\Rightarrow \qquad a = \frac{h\cot\alpha}{\cot\beta - \cot\alpha}$$

Example 35: The tangents of the angles subtended by a tower at four points A, B, C and D on the ground are in H.P. If O be the foot of the tower on the ground, then

(a) $OA + OC = OB + OD$

(b) $OA + OB = OC + OD$

(c) $OA + OD = OB + OC$

(d) $AB + CD = BC + CD$

Ans. (c)

Solution: If $\alpha, \beta, \gamma, \delta$ be the four angles such that $\tan\alpha$, $\tan\beta$, $\tan\gamma$, $\tan\delta$ are in H.P, then $\cot\alpha$, $\cot\beta$, $\cot\gamma$, $\cot\delta$ are in A.P

$\Rightarrow$ $h\cot\alpha$, $h\cot\beta$, $h\cot\gamma$, $h\cot\delta$ are in A.P.

$\Rightarrow$ OA, OB, OC, OD are in A.P

$\Rightarrow$ $OA + OD = OB + OC$

Assertion-Reason Type Questions

Example 36: Statement-1: A pole standing in the centre of a rectangular field of area 2500 sq. units subtends angle α and β respectively at the mid-points of two adjacent sides of the field such that $\alpha + \beta = \pi/2$, the height of the pole is 25 sq. units.

Statement-2: Area of a rectangle is equal to the product of the length of the adjacent sides.

Ans. (a)

Solution: Statement-2 is True. In statement-1 if the height of the pole is h, then the length of the adjacent sides of the field are $2h\cot\alpha$ and $2h\cot\beta$ and the area is $4h^2\cot\alpha\cot\beta = 4h^2$ as $\alpha + \beta = \pi/2 \Rightarrow \cot\alpha\cot\beta = 1$. So $4h^2 = 2500 \Rightarrow h = 25$ sq. units and the statement-1 is True using statement-2

Example 37: Statement-1: Apoorv, standing on the ground wants to observe the angle α of elevation of the top of a tower in front of him. He walks half the distance towards the foot of the tower and finds the angle of elevation is $\pi/4$. He observes $\alpha = \tan^{-1}(1/2)$

Statement-2: If the angles of elevation of the top of a tower at three distinct points on the ground is α, the points lie on a circle with centre at the foot of the tower.

Ans. (b)

Solution: Statement-2 is True, because if h is the height of the tower, the points are at a distance of $h \cot \alpha$ from the foot of the tower and hence lie on a circle. In statement-1, if h is the height of the tower then $h \cot \alpha = 2h \cot \pi/4 = 2h \quad \Rightarrow \quad \alpha = \tan^{-1} 1/2$

$\Rightarrow$ Statement-1 is true but does not follow from statement-2

Example 38: A and B are two points in a line on the horizontal plane through the foot O of a tower lying on opposite sides of the tower.

Statement-1: If the angles of elevation of the top of the tower at A and B are α and 2α respectively and the distance between the points is twice the height of the tower, then $\tan^2 \alpha + 4 \tan \alpha = 3$.

Statement-2: If $OB = 2(OA)$, α, β are respectively the angles of elevation of the top of the tower at A and B, then $\beta = 2\alpha$.

Ans. (c)

Solution: If h is the height of the tower, in statement-1 $OA = h \cot \alpha$, $OB = h \cot 2\alpha$ so $h(\cot \alpha + \cot 2\alpha) = 2h$.

$$\Rightarrow \quad \frac{1}{\tan\alpha} + \frac{1}{\tan 2\alpha} = 2$$

$$\Rightarrow \quad \frac{1}{\tan\alpha} + \frac{1-\tan^2\alpha}{2\tan\alpha} = 2$$

$\Rightarrow \quad \tan^2 \alpha + 4 \tan \alpha = 3$

$\Rightarrow$ statement-1 is True.

In statement 2, $h \cot \beta = 2h \cot \alpha$

$\Rightarrow \quad \cot \beta = 2\cot \alpha \Rightarrow \beta = 2\alpha$ and the statement is false.

Example 39: Mansi observes that the angle of elevation of a vertical pole of height h at two points A and B on the horizontal plane through the foot O of the pole is $\pi/3$. $AB = a$.

Statement-1: If AB subtends an angle $\pi/2$ at the foot of the tower, then $2h = 3a$.

Statement-2: If AB subtends an angle $\pi/3$ at the foot of the tower then $h = a\sqrt{3}$

Ans. (d)

Solution: $OA = OB = h \cot \pi/3 = h/\sqrt{3}$.

In statement-1 AOB is a right angled triangle.

so $(AB)^2 = (OA)^2 + (OB)^2 \Rightarrow a^2 = 2h^2/3 \Rightarrow 3a^2 = 2h^2$ and the statement is false.

In statement-2 AOB is an equilateral triangle.

so $AB = OA \Rightarrow a = h/\sqrt{3}$

$\Rightarrow \quad h = a\sqrt{3}$ and the statement is True.

Example 40: ABC is an equilateral triangle on the horizontal ground with length of each side equal to a.

Statement-1: If a tower standing at the centre O of the triangle makes an angle α at each corner such that $\alpha = \tan^{-1} 9$, then height of the tower is $3\sqrt{3}$.

Statement-2: If a tower of height $2a$ standing at one corner of the triangle makes an α at any other corner, then $a = \tan^{-1} 2$.

Ans. (b)

Solution: Since ABC is an equilateral triangle. $OA = OB = OC = a/\sqrt{3}$. In statement-1 if the height of the tower is h, $h \cot \alpha = a/\sqrt{3}$

$$\Rightarrow \quad h = \frac{a}{\sqrt{3}} \times 9 = 3\sqrt{3}\,a.$$

$\Rightarrow$ statement-1 is True.

In statement-2, $2a \cot \alpha = a$

$\Rightarrow \quad \tan \alpha = 2 \quad \Rightarrow \quad \alpha = \tan^{-1} 2$

so the statement-2 is True but does not lead to statement-1.

LEVEL 2

Straight Objective Type Questions

Example 41: A and B are two points 30 m apart in a line on the horizontal plane through the foot of a tower lying on opposite sides of the tower. If the distances of the top of the tower from A and B are 20 m and 15 m respectively, the angle of elevation of the top of the tower at A is

(a) $\cos^{-1}(43/48)$
(b) $\sin^{-1}(43/48)$
(c) $\cos^{-1}(29/36)$
(d) $\sin^{-1}(29/36)$

Ans. (a)

◎ **Solution**: Let *OP* be the tower, then $AB = 30$, $AP = 20$ and $BP = 15$. (Fig. 27.38)

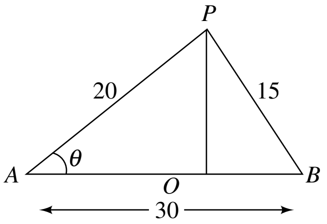


Fig. 27.38

If θ be the elevation at *A* of the top *P* of the tower, then from triangle *ABP*

$$\cos\theta = \frac{20^2 + 30^2 - 15^2}{2 \times 20 \times 30} = \frac{1075}{1200} = \frac{43}{48}$$

$\Rightarrow \quad \theta = \cos^{-1}(43/48)$.

◉ **Example 42:** The angle of elevation of the top *C* of a vertical tower *CD* of height *h* from a point *A* in the horizontal plane is 45° and from a point *B* at a distance *a* from *A* on the line making an angle 30° with *AD*, it is 60°, then

(a) $a = h(\sqrt{3} + 1)$ (b) $h = a(\sqrt{3} + 1)$

(c) $a = h(\sqrt{3} - 1)$ (d) $h = a(\sqrt{3} - 1)$

Ans. (c)

◎ **Solution:** We have

$\angle CAD = 45°$, $\angle BAD = 30°$, $\angle CBH = 60°$ (Fig. 27.39)

$\Rightarrow \angle ACD = 45°$, $\angle BCH = 30°$ so that $\angle ACB = 15°$

and $\angle CAB = 45° - 30° = 15°$

$\Rightarrow \quad \angle ABC = 150°$

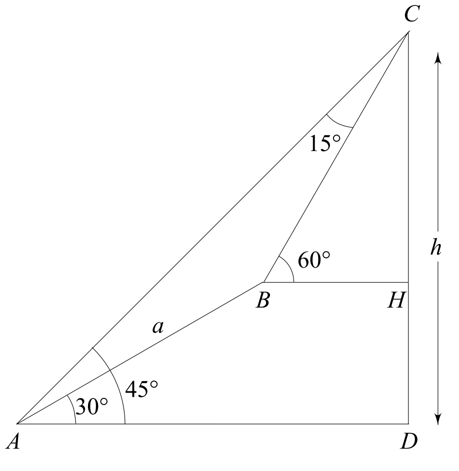


Fig. 27.39

From $\Delta\, ADC$, $AC^2 = h^2 + h^2 = 2h^2$ and from $\Delta\, ABC$

$$\frac{AB}{\sin 15^\circ} = \frac{AC}{\sin 150^\circ} \Rightarrow \frac{a}{\sin 15^\circ} = \frac{\sqrt{2}\,h}{\sin 150^\circ}$$

$$\Rightarrow \quad a = \frac{\sqrt{2}\,h \cdot \sin 15^\circ}{\sin 30^\circ} \quad \Rightarrow a = h(\sqrt{3} - 1).$$

◉ **Example 43:** The angles of elevation of a vertical tower standing inside a triangular field at the vertices of the field are each equal to θ. If the length of the sides of the field are 30 m, 50 m and 70 m, the height of the tower is

(a) $(70\sqrt{3})\tan\theta$ m (b) $\left(70/\sqrt{3}\right)\tan\theta$ m

(c) $\left(50/\sqrt{3}\right)\tan\theta$ m (d) $(75\sqrt{3})\tan\theta$ m

Ans. (b)

◎ **Solution:** Let *OP* be the tower of height *h* at the point *O* in the triangular field *ABC* with sides $BC = 30$, $CA = 50$ and $AB = 70$.

Then $OA = OB = OC = h\cot\theta$

$\Rightarrow$ *O* is the circumcentre of the triangle *ABC*

$\Rightarrow$ $h\cot\theta = R$ (the radius of the circumcircle)

$\Rightarrow$ $h = R\tan\theta$

We know that $R = abc/4\Delta$

where $$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{75 \times 45 \times 25 \times 5} = 25 \times 5 \times 3\sqrt{3}$$

Thus, $$h = \frac{30 \times 50 \times 70}{4 \times 25 \times 5 \times 3\sqrt{3}}\tan\theta$$

$$= \frac{70}{\sqrt{3}}\tan\theta \text{ m.}$$

◉ **Example 44:** In a triangular plot *ABC* with $BC = 7$ m, $CA = 8$ m and $AB = 9$ m. A lamp post is situated at the middle point *E* of the side *AC* and subtends an angle $\tan^{-1} 3$ at the point *B*, the height of the lamp post is

(a) 21 m (b) 24 m

(c) 27 m (d) cannot be determined

Ans. (a)

◎ **Solution:** In triangle *ABC*

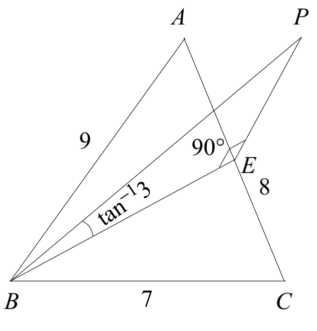


Fig. 27.40

$a = BC = 7$, $b = CA = 8$ and $c = AB = 9$

so that $$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49 + 64 - 81}{2 \times 7 \times 8} = \frac{2}{7}$$

In triangle BEC, $BC = 7$, $CE = 4$ and $\cos C = \frac{2}{7}$

so that $BE^2 = 4^2 + 7^2 - 2 \times 4 \times 7 \times \frac{2}{7} = 49$

$\Rightarrow \quad BE = 7.$

If h is the height of the lamp post EP at E

then $\frac{PE}{BE} = \frac{h}{7} = \tan(\tan^{-1} 3) = 3 \quad \Rightarrow h = 21$ m.

Example 45: Two objects at the points P and Q subtend an angle of 30° at a point A. Lengths $AR = 20$ m and $AS = 10$ m are measured from A at right angles to AP and AQ respectively. If PQ subtends equal angles of 30°, at R and S, then length of PQ is

(a) $\sqrt{300 - 200\sqrt{3}}$ (b) $\sqrt{500 - 200\sqrt{3}}$

(c) $\sqrt{500\sqrt{3} - 200}$ (d) $\sqrt{300}$

Ans. (b)

Solution: Since PQ subtends the same angle of 30° at each of the points A, R and S, the points A, P, Q, R and S lie on a circle. (Fig. 27.41)

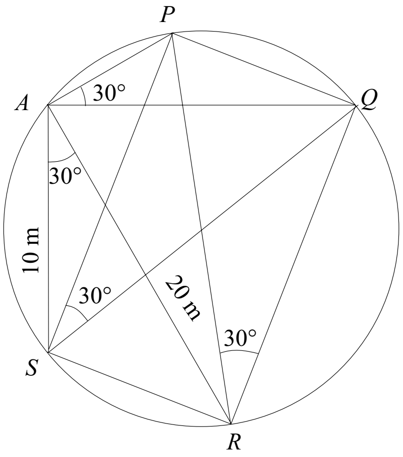


Fig. 27.41

Also $AR \perp AP$ and $AS \perp AQ$

$$\Rightarrow \quad \angle RAS = 90° - \angle RAQ = \angle PAQ = 30°$$

As it is given

$$AR = 20 \text{ m}, AS = 10 \text{ m}.$$

Then PQ and RS are chords of the same circle making an angle of 30° at a point A on the circumference of the circle and hence are equal in length.

Now from $\Delta\, SAR$

$$RS^2 = 20^2 + 10^2 - 2 \times 20 \times 10 \cos 30°$$

$$\Rightarrow \quad PQ^2 = 500 - 200\sqrt{3}$$

$$\Rightarrow \quad PQ = \sqrt{500 - 200\sqrt{3}}.$$

Example 46: From a ship at sea it is observed that the angle subtended by feet A and B of two light houses, at the ship is 30°. The ship sails 4 km towards A and this angle is then 48°, the distance of B from the ship at the second observation is

(a) 6.460 km (b) 6.472 km

(c) 6.476 km (d) 6.478 km

Ans. (b)

Solution: If S_1 and S_2 are the two positions of the ship then $S_1S_2 = 4$.

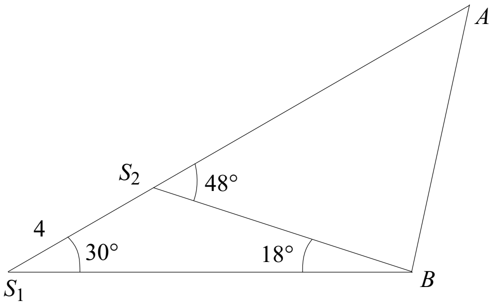


Fig. 27.42

$$\angle AS_1B = 30° \text{ and } \angle AS_2B = 48°$$

$$\Rightarrow \quad \angle S_1BS_2 = 18° \text{ (Fig. 27.42)}$$

Now from ΔS_1BS_2

$$\frac{S_1\, S_2}{\sin 18°} = \frac{S_2\, B}{\sin 30°}$$

$$\Rightarrow S_2B = \frac{4 \times (1/2)}{\sin 18°} = 2(\sqrt{5} + 1) = 6.472.$$

Example 47: From a point on the horizontal plane, the elevation of the top of a hill is 45°. After walking 500 m towards its summit up a slope inclined at an angle of 15° to the horizon the elevation is 75°, the height of the hill is

(a) $500\sqrt{6}$ m (b) $500\sqrt{3}$ m

(c) $250\sqrt{6}$ m (d) $250\sqrt{3}$ m

Ans. (c)

Solution: Let P be the top of the hill CP, A and B be the points where the angles of elevation of the top P of the hill are 45° and 75° respectively. Let BE be the line parallel to AC through B. (Fig. 27.43)

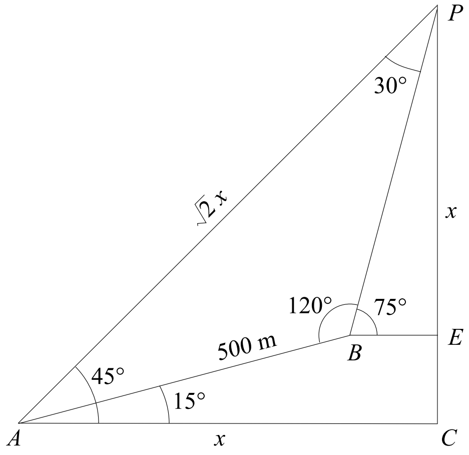


Fig. 27.43

Then $\angle PAC = 45°$ and $\angle BAC = 15°$

$\Rightarrow \quad CA = CP = x$ (say)

$\Rightarrow \quad PA = \sqrt{2}\,x$

$\angle PBE = 75° \;\Rightarrow\; \angle BPE = 15°$

$\Rightarrow \quad \angle APB = 30° = \angle BAP$

and $\quad \angle ABP = 120° \quad \therefore$ From $\Delta\, ABP$

$$\frac{\sqrt{2}x}{\sin 120^\circ} = \frac{500}{\sin 30} \quad \Rightarrow \quad x = 250\sqrt{6} \text{ m.}$$

Example 48: The elevation of a steeple at a place due south of it is 45° and at a place *B* due west of *A* the elevation is 15°. If $AB = 2a$, the height of the steeple is

(a) $a\dfrac{(\sqrt{3}-1)}{\sqrt{2}}$ (b) $a\dfrac{(\sqrt{3}+1)}{\sqrt{2}}$

(c) $a\left[3^{\frac{1}{4}} - 3^{-\frac{1}{4}}\right]$ (d) $a\left[3^{\frac{1}{4}} + 3^{-\frac{1}{4}}\right]$

Ans. (c)

Solution: Let *OP* be the steeple of height *h*. *A* be due south of *OP* and $\angle OAP = 45°$ and *B* be due west of *A* and $\angle OBP = 15°$ then $OA = h \cot 45° = h$ and $OB = h \cot 15°$ and $AB = 2a$ (given) so that from right angled ΔOAB, $OB^2 = OA^2 + AB^2$

$\Rightarrow \quad h^2 \cot^2 15 = h^2 + 4a^2$ (Fig. 27.44)

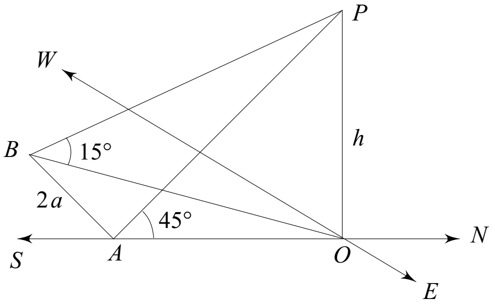


Fig. 27.44

$$\Rightarrow \quad h^2\left[\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2 - 1\right] = 4a^2$$

$$\Rightarrow \quad h^2 \times 4\sqrt{3} = 4a^2\left(\sqrt{3}-1\right)^2$$

$$\Rightarrow h = \frac{\sqrt{3}-1}{\left(\sqrt{3}\right)^{1/2}}a = [3^{\frac{1}{4}} - 3^{-\frac{1}{4}}]\,a.$$

Example 49: The top of a pole, placed against a wall at an angle α with the horizon, just touches the coping, and when its foot is moved *a* m, away from the wall and its angle of inclination is β, it rests on the sill of a window; the vertical distance of the sill from the coping is

(a) $a \sin((\alpha+\beta)/2)$ (b) $a \cos((\alpha+\beta)/2)$

(c) $a \cot((\alpha+\beta)/2)$ (d) $a \tan((\alpha+\beta)/2)$

Ans. (c)

Solution: Let *L* be the coping and *M* the sill of the window and *A* and *B* be the two positions of the pole such that $\angle LAX = \alpha$ and $\angle MBX = \beta$. $LM = x$, *X* being the foot of the vertical wall. (Fig. 27.45)

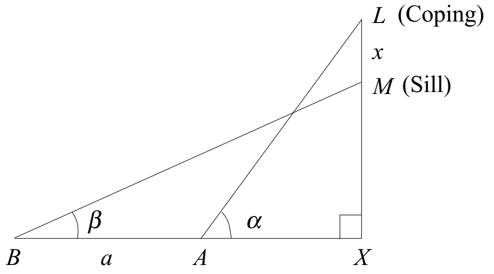


Fig. 27.45

Let $\quad AL = BM = l$ and $AB = a$ (given)

then $\quad x = LX - MX = l(\sin\alpha - \sin\beta)$

$a = BA = BX - AX = l(\cos\beta - \cos\alpha)$.

$$\Rightarrow \quad \frac{x}{a} = \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \frac{2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}}{2\sin\cdot\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}}$$

$$\Rightarrow \quad x = a\cot\frac{\alpha+\beta}{2}.$$

Example 50: *OAB* is a triangle in the horizontal plane through the foot *P* of the tower at the middle point of the side *OB* of the triangle. If $OA = 2$ m, $OB = 6$ m, $AB = 5$ m and $\angle AOB$ is equal to the angle subtended by the tower at A then the height of the tower is

(a) $\sqrt{\dfrac{11\times 39}{25\times 3}}$ (b) $\sqrt{\dfrac{11\times 39}{25\times 2}}$

(c) $\sqrt{\dfrac{11\times 25}{39\times 2}}$ (d) none of these

Ans. (b)

◎ **Solution:** Let PQ be the tower of height h at the middle point P of the side OB of the triangle OAB, where (Fig. 27.46)

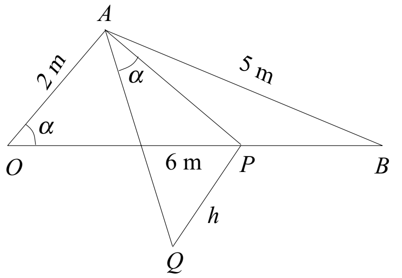


Fig. 27.46

$OA = 2$, $OB = 6$, $AB = 5$ and $\angle AOB = \angle PAQ = \alpha$,

then $AP = h \cot \alpha$, $OP = 3$

From triangle OAB, $\cos \alpha = \dfrac{2^2 + 6^2 - 5^2}{2 \times 2 \times 6} = \dfrac{5}{8}$

and from Δ OAP

$$\cos \alpha = \frac{2^2 + 3^2 - h^2 \cot^2 \alpha}{2 \times 2 \times 3}$$

$$\Rightarrow \quad \frac{5}{8} = \frac{13 - h^2 \times \dfrac{25}{39}}{12}$$

$$\Rightarrow \quad h^2 = \frac{11 \times 39}{25 \times 2}$$

◉ **Example 51:** If two vertical towers PQ and RS of lengths a and b ($a > b$) respectively subtend the same angle α at a point A on the line joining their feet P and R in the horizontal plane and angles β and γ at another point B on this line nearer the towers on the same side of the towers as A, then $\dfrac{\sin(\beta - \gamma)}{\sin(\beta - \alpha)}$ is equal to

(a) $\dfrac{b \sin \alpha}{(a - b) \sin \gamma}$ (b) $\dfrac{(b - a) \sin \gamma}{b \sin \alpha}$

(c) $\dfrac{\sin \gamma}{\sin \alpha}$ (d) $\dfrac{(b - a) \sin \alpha}{b \sin \gamma}$

Ans. (b)

◎ **Solution:** We have

$$\angle PAQ = \alpha = \angle RAS$$

$$\angle PBQ = \beta, \angle RBS = \gamma$$

$$\Rightarrow \quad \angle QBS = \gamma - \beta$$

and $\angle BQS = 90° - \alpha - (90° - \beta) = \beta - \alpha$.

$\therefore$ From Δ SBQ, we have

$$\frac{\sin(\gamma - \beta)}{\sin(\beta - \alpha)} = \frac{SQ}{SB} = \frac{a - b}{\sin \alpha} \times \frac{\sin \gamma}{b}$$

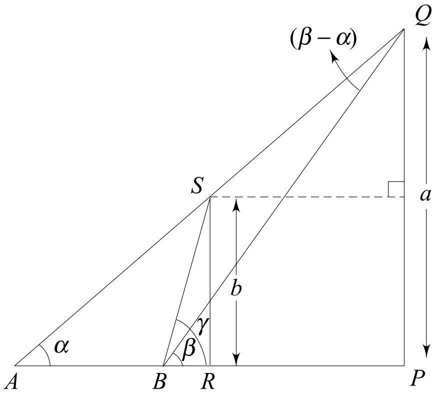


Fig. 27.47

◉ **Example 52:** A person standing on the ground observes the angle of elevation of the top of a tower to be 30°. On walking a distance a in a certain direction, he finds the elevation of the top to be the same as before. He then walks a distance $5a/3$ at right angles to his former direction, and finds that the elevation of the top has doubled. The height of the tower is

(a) a (b) $\sqrt{85/48}\, a$

(c) $\sqrt{6/5}\, a$ (d) $\sqrt{48/85}\, a$

Ans. (b)

◎ **Solution:** Let PQ be the tower of height h and A, B, C be the three positions of the person from initial to final then (Fig. 27.48)

$$\angle PAQ = \angle PBQ = 30° \text{ and } \angle PCQ = 60°$$

$$\Rightarrow \quad AQ = BQ = h \cot 30° = h\sqrt{3}$$

$$\text{and} \quad CQ = h \cot 60° = h/\sqrt{3}.$$

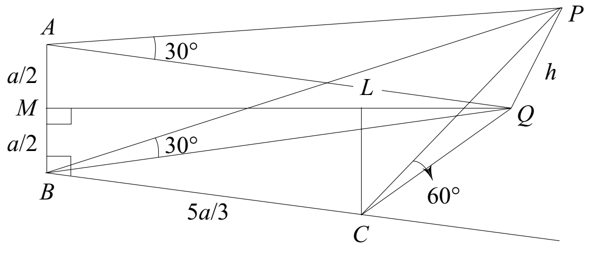


Fig. 27.48

Let QM be perpendicular to AB and CL be perpendicular to QM

Now $AB = a$ and $BC = 5a/3$ and $\angle ABC = 90°$

$\Rightarrow \quad ML = BC = 5a/3$ and $CL = BM = a/2$

From right angled triangle BMQ,

$$QM^2 = BQ^2 - BM^2 = 3h^2 - (a/2)^2 = 3h^2 - (a^2/4)$$

and from right angled triangle CLQ.

$$QL^2 = QC^2 - CL^2 = \frac{h^2}{3} - \frac{a^2}{4}$$

Since $QM = QL + LM$

$$\Rightarrow \quad 3h^2 - \frac{a^2}{4} = \frac{h^2}{3} - \frac{a^2}{4} + \frac{25a^2}{9} + \frac{10a}{3}\sqrt{\frac{h^2}{3} - \frac{a^2}{4}}$$

$$\Rightarrow \quad \frac{8h^2}{3} - \frac{25a^2}{9} = \frac{10a}{3}\sqrt{\frac{h^2}{3} - \frac{a^2}{4}}$$

$$\Rightarrow \quad 64h^4 - \frac{400a^2h^2}{3} + \frac{625a^4}{9} = \frac{100a^2h^2}{3} - 25a^4$$

$$\Rightarrow \quad 576h^4 - 1500a^2h^2 + 850a^4 = 0$$

$$\Rightarrow \quad 288h^4 - 750a^2h^2 + 425a^4 = 0$$

$$\Rightarrow \quad (48h^2 - 85a^2)(6h^2 - 5a^2) = 0$$

$$\Rightarrow \quad h = \sqrt{85/48}\,a \text{ or } \sqrt{5/6}\cdot a$$

Example 53: A tower PQ subtends an angle α at a point A on the same level as the foot Q of the tower. It also subtends the same angle α at a point B where AB subtends the angle α with AP then

(a) $AB = BQ$ (b) $BQ = 2AQ$

(c) $\frac{AB}{BQ} = (1/2)\sin\alpha$ (d) $\frac{AB}{BQ} = (1/2)\operatorname{cosec}\alpha$

Ans. (d)

Solution: Since $\angle PAQ = \angle PBQ = \alpha$ (angles in the same segment are equal) $PQAB$ is a cyclic quadrilateral (Fig. 27.49)

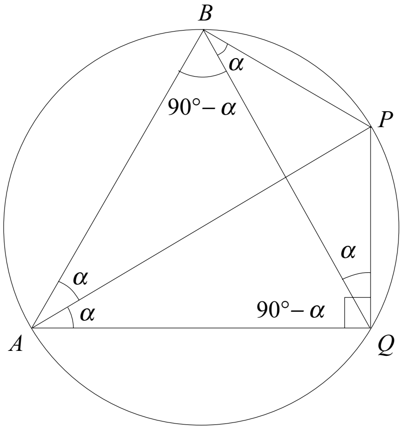


Fig. 27.49

Also since $\angle PAB = \alpha$ given, $\angle PQB = \alpha$ (angles in the same segment.)

$\Rightarrow$ Now $\angle PBQ = \alpha = \angle PQB$

$\Rightarrow$ $\angle ABQ = \angle AQB = 90° - \alpha$

$\Rightarrow$ ABQ is an isosceles triangle.

$\Rightarrow$ $AB = AQ$, and

$$\frac{AB}{\sin(90° - \alpha)} = \frac{BQ}{\sin 2\alpha} \quad \Rightarrow \quad 2AB\sin\alpha = BQ.$$

$$\Rightarrow \quad \frac{AB}{BQ} = \frac{1}{2\sin\alpha} = \frac{1}{2}\operatorname{cosec}\alpha.$$

Example 54: The angle of elevation of the top of a tree at a point B due south of it is 60° and at a point C due north of it is 30°. D is a point due north of C where the angle of elevation is 15°, then given $\sqrt{3} = 1\frac{8}{11}$ and $BC \times CD = 2^3 \times 3^2 \times 19 \times 11$, the height of the tree is

(a) 33 (b) 38

(c) 57 (d) 88

Ans. (c)

Solution: Let A be the top of the tree $OA = h$ (Fig. 27.50) B, C, D be the three point of observation such then $\angle ABO = 60°$, $\angle ACO = 30°$ and $\angle ADO = 15°$

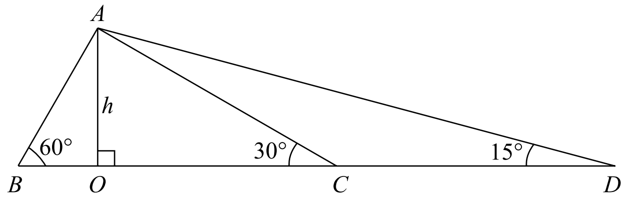


Fig. 27.50

Then $BC = BO + OC = h(\cot 60° + \cot 30)$

$$= h\left(\sqrt{3} + 1/\sqrt{3}\right) = \left(4/\sqrt{3}\right)h$$

$$CD = h\cot 15° - h\cot 30°$$

$$= h\left(2 + \sqrt{3} - \sqrt{3}\right) = 2h$$

So that $2^3 \times 3^2 \times 19 \times 11$

$$= \frac{4}{\sqrt{3}} \times 2h^2 = \frac{4 \times 2 \times 11}{19}h^2 \quad \left(\because \sqrt{3} = \frac{19}{11}\right)$$

$$\Rightarrow \quad h^2 = 3^2 \times 19^2 \Rightarrow h = 57$$

Example 55: n poles standing at equal distances on a straight road subtend the same angle α at a point O on the road. If the height of the largest pole is h and the distance of the foot of the smallest pole from O is a, the distance between two consecutive poles is

(a) $\dfrac{h\sin\alpha - a\cos\alpha}{(n-1)\sin\alpha}$ (b) $\dfrac{h\cos\alpha - a\cos\alpha}{(n-1)\cos\alpha}$

(c) $\dfrac{h\cos\alpha - a\sin\alpha}{(n-1)\sin\alpha}$ (d) $\dfrac{h\sin\alpha - a\cos\alpha}{(n-1)\cos\alpha}$

Ans. (c)

◎ **Solution:** Let $A_1, A_2 \ldots A_n$ be the feet of the n poles subtending angle α at O, such that $OA_1 = a$, if d be the distance between two consecutive poles

$$OA_2 = a + d$$
$$OA_3 = a + 2d$$
$$\vdots$$
$$OA_n = a + (n-1)d$$

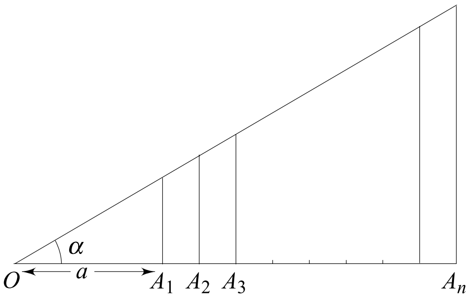


Fig. 27.51

Since the height of the pole at A_n is h;

$$OA_n = h\cot\alpha.$$

$$\Rightarrow \quad a + (n-1)d = h\cot\alpha$$

$$\Rightarrow \quad d = \frac{h\cot\alpha - a}{n-1} = \frac{h\cos\alpha - a\sin\alpha}{(n-1)\sin\alpha}.$$

EXERCISE

Concept-based Straight Objective Type Questions

1. A pole is standing at a point O between two milestones at A and B such that the angles of elevation of the top of the pole at A and B are respectively α and β. If the distance between the milestones is half the height of the pole then
   (a) $2\sin(\alpha + \beta) = \sin\alpha\sin\beta$
   (b) $2\sin(\alpha + \beta) = \cos\alpha\cos\beta$
   (c) $\sin(\alpha + \beta) = 2\sin\alpha\sin\beta$
   (d) $\sin(\alpha + \beta) = 2\cos\alpha\cos\beta$

2. OP is a tower of height 20 m and AB is a pole of height 5 m. The angle of elevation of the top P of the tower from the top B of the pole is 45°. Both pole and tower stand on the same ground. The angle of elevation of the top P of the tower from the base A of the pole is
   (a) $\cos^{-1}\dfrac{3}{5}$ (b) $\sin^{-1}\dfrac{3}{5}$
   (c) $\tan^{-1}\dfrac{3}{4}$ (d) $\cot^{-1}\dfrac{4}{3}$

3. A circular path is 50 m. wide. The angle of elevation of the top of a pole at the centre of the circular park at a point on the inner circle is α and at a point on the outer circle is 45°. The height of the pole is
   (a) $\dfrac{50\cos\alpha}{\cos\alpha - \sin\alpha}$ (b) $\dfrac{50\sin\alpha}{\sin\alpha - \cos\alpha}$
   (c) $\dfrac{50}{\cos\alpha - \sin\alpha}$ (d) $\dfrac{50}{\cos\alpha + \sin\alpha}$

4. A pole 10 m high stands on a tower 30 m. high. The angle of elevation of the top of the pole at a point A on the ground in 45° and the pole subtends an angle α at the same point A then α is equal to
   (a) $\cot^{-1}(1/7)$ (b) $\cot^{-1} 7$
   (c) $\cos^{-1}(1/7)$ (d) $\sin^{-1}(1/7)$

   with each side equal to 100 m.

5. $ABCD$ is a square field with each side equal to 100 m. Two poles of equal heights stand at E, the mid point of DC and at the corner B of the field, subtending respectively angles α and 30° at the corner A of the field. The value of α satisfies
   (a) $\cos 2\alpha = \dfrac{11}{19}$ (b) $\sin 2\alpha = \dfrac{15}{19}$
   (c) $\tan 2\alpha = \dfrac{4}{19}$ (d) $\tan 2\alpha = \dfrac{19}{15}$

LEVEL 1

Straight Objective Type Questions

6. The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 100 m from its base is 45°. If the angle of elevation of the top of the pillar after completion at the same point is 60°, then the height to be increased for the completion of the pillar in metres is
   (a) $50\sqrt{3}$ (b) $100\sqrt{2}$
   (c) $100\sqrt{3}$ (d) $100(\sqrt{3}-1)$

7. The angles of elevation of the top of a tower at the top and the foot of a pole 10 m high are 30° and 60° respectively. The height of the tower is
   (a) 15 m (b) 20 m
   (c) $10\sqrt{3}$ m (d) $25\sqrt{3}$ m

8. A tower subtends an angle α at a point A on the ground, and the angle of depression of its foot from a point B just above A and at distance b from A, is β. The height of the tower is
   (a) $b \tan\alpha \tan\beta$ (b) $b \tan\alpha \cot\beta$
   (c) $b \cot\alpha \cot\beta$ (d) $b \cot\alpha \tan\beta$

9. A person walking along a straight road observes that at two points 1 km apart, the angles of elevation of a pole in front of him are 30° and 75°. The height of the pole is
   (a) $250(\sqrt{3}+1)$ m (b) $250(\sqrt{3}-1)$ m
   (c) $500(\sqrt{2}+1)$ m (d) $500(\sqrt{2}-1)$ m

10. If a flagstaff subtends the same angle at the points A, B, C and D on the horizontal plane through its foot, then $ABCD$ is a
    (a) square (b) cyclic quadrilateral
    (c) rectangle (d) none of these

11. From a point on the ground 100 m away from the base of a building, the angle of elevation of the top of the building is 60°. Which of the following is the best approximation for the height of the building?
    (a) 172 m (b) 173 m
    (c) 174 m (d) 175 m

12. From the top of a tower 100 m height, the angles of depression of two objects 200 m apart on the horizontal plane and in a line passing through the foot of the tower and on the same side of the tower are $45° - A$ and $45° + A$, then angle A is equal to
    (a) 15° (b) 22.5°
    (c) 30° (d) 35°

13. An observer finds that the angular elevation of a tower is A, on advancing 3 m towards the tower the elevation is 45° and on advancing 2 m nearer, the elevation is $90° - A$, the height of the tower is
    (a) 1m (b) 5m
    (c) 6m (d) 8m

14. ABC is a triangular park with all sides equal. If a pillar at A subtends an angle of 45° at C, the angle of elevation of the pillar at D, the middle point of BC is
    (a) $\tan^{-1}\left(\sqrt{3}/2\right)$ (b) $\tan^{-1}\left(2/\sqrt{3}\right)$
    (c) $\cot^{-1}\sqrt{3}$ (d) $\tan^{-1}\sqrt{3}$

15. A kite is flying with the string inclined at 75° to the horizon. If the length of the string is 25 m, the height of the kite is
    (a) $(25/2)(\sqrt{3}-1)^2$ (b) $(25/4)(\sqrt{3}+1)\sqrt{2}$
    (c) $(25/2)(\sqrt{3}+1)^2$ (d) $(25/2)(\sqrt{6}+\sqrt{2})$

16. AB is a vertical pole. The end A is on the level ground, C is the middle point of AB. P is a point on the level ground. The portion BC subtends an angle β at P. If $AP = n\,AB$, then $\tan\beta =$
    (a) $\dfrac{n}{2n^2+1}$ (b) $\dfrac{n}{n^2-1}$
    (c) $\dfrac{n}{n^2+1}$ (d) none of these

17. A man in a boat rowing away from a cliff 150 metres high observes that it takes 2 minutes to change the angle of elevation of the top of the cliff from 60° to 45°. The speed of the boat is
    (a) $(1/2)(9-3\sqrt{3})$ km/h
    (b) $(1/2)(9+3\sqrt{3})$ km/h
    (c) $(1/2)(9\sqrt{3})$ km/h
    (d) none of these

18. A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°, when he retires 40 metres from the banks he finds the angle to 30°. The breadth of the river is
    (a) 40 m (b) 60 m
    (c) 20 m (d) 30 m

19. The elevation of the top of a mountain at each of the three angular points A, B and C of a plane horizontal triangle is α, if $BC = a$ the height of the mountain is

(a) $(a/2)$ cosec A tan α (b) $(a/2)$ sec A tan α

(c) $(a/2)$ cosec α cot A (d) $(a/2)$ sec α tan A

20. The angles of elevation of the top of a tower standing on a horizontal plane, from two points on a line passing through its foot at distances a and b, respectively, are complementary angles. If the line joining the two points subtends an angle θ at the top of the tower, then if $a > b$, $\sin\theta =$

(a) $\dfrac{a-b}{a+b}$ (b) $\dfrac{a+b}{a-b}$

(c) $\dfrac{2\sqrt{ab}}{a+b}$ (d) $\dfrac{2\sqrt{ab}}{a-b}$

21. The upper three-quarters of a vertical pole subtends an angle $\tan^{-1}(3/5)$ at a point in the horizontal plane through its foot and distant 40 m from it. The height of the pole is

(a) 80 m (b) 100 m

(c) 160 m (d) 200 m

22. PQ is a vertical tower and A, B, C are three points on a horizontal line through Q, the foot of the tower and on the same side of the tower. If the angles of elevation of the top of the tower from A, B and C are α, β, γ respectively, then $AB/BC =$

(a) $\dfrac{\cot\alpha - \cot\gamma}{\cot\beta - \cot\gamma}$ (b) $\dfrac{\cot\alpha - \cot\beta}{\cot\beta - \cot\gamma}$

(c) $\dfrac{\cot\alpha - \cot\beta}{\cot\alpha - \cot\gamma}$ (d) $\dfrac{\cot\alpha - \cot\gamma}{\cot\alpha - \cot\beta}$

23. $ABCD$ is a rectangular park with $AB = a$. A tower standing at C makes angles α and β at A and B respectively, the height of the tower is

(a) $\dfrac{a}{\sqrt{\cot^2\alpha + \cot^2\beta}}$ (b) $\dfrac{a}{\sqrt{\cot^2\alpha - \cot^2\beta}}$

(c) $\dfrac{a\tan\alpha\tan\beta}{\sqrt{\tan^2\beta + \tan^2\alpha}}$ (d) $\dfrac{a\cot\alpha\cot\beta}{\sqrt{\cot^2\alpha - \cot^2\beta}}$

24. Two circular path of radii a and b intersect at a point O and AB is a line through O meeting the circles at A and B respectively. Chords OA and OB subtend equal angles of 60° at their respective centres. A vertical pole at O subtends angles α and β respectively at A and B then height of the pole is

(a) $a\cot\alpha$ (b) $b\cot\beta$

(c) $\dfrac{a+b}{\cot\alpha + \cot\beta}$ (d) none of these

25. Three poles of height a, b, c stand on the same side of a road and subtend an angle of 45° at a point on the line joining their feet. The pole of height a subtends an angle α at the foot of the pole of height b which subtends an angle β at the foot of the pole with height c, if $a > b > c$, then $\cot\alpha - \cot\beta =$

(a) $\dfrac{ac-b^2}{ab}$ (b) $\dfrac{bc-a^2}{ab}$

(c) $\dfrac{ab-c^2}{bc}$ (d) $\dfrac{ac-b^2}{bc}$

26. An aeroplane flying horizontally 1 km above the ground is observed at an elevation of 60°. If after 10 seconds, the elevation is observed to be 30°, then the uniform speed of the aeroplane per hour is

(a) 120 km (b) 240 km

(c) $240\sqrt{3}$ km (d) $240/\sqrt{3}$ km

27. If a flagstaff 6 metres high placed on the top of a tower throws a shadow of $2\sqrt{3}$ metres along the ground then the angle (in degrees) that the sun makes with the ground is

(a) 15° (b) 30°

(c) 60° (d) $\tan^{-1} 2\sqrt{3}$

28. Three poles whose feet lie on a circle subtend angle α, β, γ respectively at the centre of the circle. If the height of the poles are in A.P. then $\cot\alpha$, $\cot\beta$, $\cot\gamma$ are in

(a) A.P. (b) G.P.

(c) H.P. (d) none of these

29. A, B, C are three points on a vertical pole whose distances from the foot of the pole are in A.P. and whose angles of elevation at a point on the ground are α, β and γ respectively. If $\alpha + \beta + \gamma = \pi$, then $\tan\alpha\tan\gamma$ is equal to

(a) 3 (b) 2

(c) 1 (d) -1

30. A ladder rests against a wall at an angle of 35°. Its foot is pulled away through a distance a, so that it slides a distance b down the wall, finally making an angle of 25° with the horizontal, then $a/b =$

(a) 1 (b) $1/\sqrt{3}$

(c) $\sqrt{3}$ (d) $\sqrt{3}/2$

Assertion-Reason Type Questions

31. **Statement-1:** Three poles of height a, b, c stand at the points A, B, C respectively and subtend the same angle α at a point O on the horizontal line through the feet of the poles. If a, b, c are in A.P., then $AB = BC$.

    Statement-2: O is the centre of a circular field and A is any point on its boundary. Two poles standing at A and O subtend the same angle α at a point B on the other end of the diameter through A. Height of the pole at A is twice the height of the plot at O.

32. **Statement-1:** Rajat observes that the angle of elevation of the top P of a tower OP at a point A on the ground is α. He travels a distance a in the direction AP and reaches the point B. He then travels a horizontal distance a towards the tower and reaches the point C, where the angle of elevation of the top of the tower is $\pi/4$, the height of the tower is $\dfrac{a(\cos\alpha + 1 - \sin\alpha)}{\cot\alpha - 1}$

    Statement-2: On the top of building a pole of height equal to 1/3 of the height of the building is placed so that the angles of elevations of the top of the pole and the top of the building at a point on the ground are α and β respectively then $\alpha = (3/4)\beta$.

33. **Statement-1:** ABC is a triangular field with $AC = b$ and $AB = c$. A pole standing at a point D on BC subtends angles α at B and β at C.

    If $\angle BAD = \angle DAC$ then $b \cot \alpha = c \cot \beta$.

    Statement-2: Bisector of an angle of a triangle divides the opposite side in the ratio of the side containing the angle.

34. **Statement-1:** The angle of elevation of the top P of a tower OP at a point A on the ground is α, the angle of elevation of the mid-point Q of the tower at the mid-point B of OA is also α.

    Statement-2: The line joining the mid-points of two sides of a triangle is parallel to the third side.

35. **Statement-1:** A tower standing at the centre of a square field subtends an angle α at each corner. If the height of the tower is twice the length of a side of the square, then $\alpha = \tan^{-1} 2$.

    Statement-2: A, B, C are three points on the horizontal line through the foot of a tower and the angles of elevation of the top of the tower at these points are 30°, 45° and 60° respectively, $\dfrac{AB}{BC} = \sqrt{3}$.

LEVEL 2

Straight Objective Type Questions

36. The angle of elevation of the top Q of a tower PQ at a point A on the horizontal plane through P the foot of the tower is α. At a point B on AQ at a vertical height of a, the angle of elevation of the middle point R of the tower PQ is β, then the height of the tower is

    (a) $\dfrac{2a(\tan\alpha - \tan\beta)}{\tan\alpha - 2\tan\beta}$ (b) $\dfrac{2a(\tan\alpha - 2\tan\beta)}{\tan\alpha - \tan\beta}$

    (c) $\dfrac{2a(\tan\alpha\tan\beta - 1)}{2\tan\alpha\cot\beta - 1}$ (d) $\dfrac{2a(\tan\alpha\cot\beta - 1)}{2\tan\alpha\cot\beta - 1}$

37. A lamppost stands in the centre of a circular garden and makes angle α at points A and B on the boundary where AB subtends an angle 2β at the foot of the lamppost. If γ is the angle which the lamppost subtends at C, the middle point of the line joining A and B, then $\tan\gamma =$

    (a) $\tan\alpha\tan\beta$ (b) $\sec\alpha\tan\beta$
    (c) $\tan\alpha\sec\beta$ (d) none of these

38. From a point on the ground, if the angles of elevation of a bird flying at constant speed in a horizontal direction, measured at equal intervals of time are α, β, γ and δ, then

    (a) $\cot^2\beta - \cot^2\gamma = 3(\cot^2\alpha - \cot^2\delta)$
    (b) $\cot^2\beta - \cot^2\delta = 3(\cot^2\alpha - \cot^2\gamma)$
    (c) $\cot^2\gamma - \cot^2\delta = 3(\cot^2\alpha - \cot^2\beta)$
    (d) $\cot^2\alpha - \cot^2\delta = 3(\cot^2\beta - \cot^2\gamma)$

39. A vertical tower standing at O has marks P, Q, R, S at heights of 1m, 2m, 3m and 4m from the foot O and A is a point on the horizontal plane through O. If PQ and RS subtend angles α and β respectively at A where OA = 2m then $\cos(\alpha + \beta) =$

(a) $5/\sqrt{26}$ (b) $24/\sqrt{650}$
(c) $23/\sqrt{650}$ (d) $1/\sqrt{26}$

40. *ABCD* is a rectangular field with $AB = a$ and $BC = b$. A lamp post of height h at A subtends an angle α at P, the middle point of CD and another lamp post of equal height at D subtends an angle β at Q, the middle point of BC. If PQ subtends an angle θ at A, then $\cot^2 \alpha \cot^2 \beta \cos^2 \theta = k^2$, where $k =$
(a) $(a^2 + b^2)/2h^2$ (b) $(a^2 - b^2)/2h^2$
(c) $2h^2/(a^2 + b^2)$ (d) $2(a^2 + b^2)h^2$

41. A vertical tower OP of height h subtends angle α, β, γ respectively at the points A, B, C on the horizontal plane through the foot O of the tower. A is due west of the tower. B is due east of A and on the same side of the tower as A. C is due south of B, then $AC =$
(a) $h(\cot \alpha - \cot \beta)$
(b) $h\sqrt{\cot^2 \gamma - \cot^2 \beta}$
(c) $h\sqrt{\cot^2 \alpha + \cot^2 \gamma - 2\cot \alpha \cot \beta}$
(d) $h\sqrt{\cot^2 \gamma + \cot^2 \beta - 2\cot \alpha \cot \beta}$

42. PQ and RS are two vertical towers of the same height where S is on the ground Q is above the ground. The line joining the top P and the foot S of the two towers meets the horizontal line through Q at a point A where the angles of elevation of the tops P and R of the two towers are α and β respectively. If $AS = a$, the height of the towers is
(a) $\dfrac{a \sin(\beta + \alpha)}{\cos \beta}$ (b) $\dfrac{a \cos(\beta + \alpha)}{\cos \beta}$
(c) $\dfrac{a \sin(\beta + \alpha)}{\sin \beta}$ (d) $\dfrac{a \cos(\beta + \alpha)}{\sin \beta}$

43. From the top of a building of height h, a tower standing on the ground is observed to make an angle θ. If the horizontal distance between the building and the tower is h, the height of the tower is
(a) $\dfrac{2h \cos \theta}{\sin \theta + \cos \theta}$ (b) $\dfrac{2h}{1 + \cot \theta}$
(c) $\dfrac{2h}{1 + \tan \theta}$ (d) $\dfrac{2h}{\sin \theta + \cos \theta}$

44. A tower stands at the foot of a hill whose inclination to the horizon is 9°; at a point 40 m up the hill the tower subtends an angle of 54°. The height of the tower is
(a) 17.56 m (b) 45.76 m
(c) 54.76 m (d) none of these

45. The angle of elevation of a stationary cloud from a point 2500 metres above a lake is 15° and the angle of depression of its reflection in the lake is 45°. The height of the cloud above the lake level is
(a) $2500/\sqrt{3}$ m (b) 2500 m
(c) $2500\sqrt{3}$ m (d) $5000\sqrt{3}$ m

46. A tower PQ stands at a point P with in the triangular park ABC such that the sides a, b, c of the triangle subtend equal angles at P, the foot of the tower and the tower subtends angles α, β, γ at A, B, C respectively, then $a^2(\cot \beta - \cot \gamma) + b^2(\cot \gamma - \cot \alpha) + c^2(\cot \alpha - \cot \beta)$ is equal to
(a) -1 (b) 0
(c) 1 (d) $a + b + c$

47. A spherical balloon subtends an angle 2α at a man's eye and the elevation of its centre is β. If θ is the elevation of the highest point of the balloon at A then $\tan \theta$ is equal to
(a) $\dfrac{\sin \alpha + \cos \beta}{\sin \beta}$ (b) $\dfrac{\sin \alpha + \sin \beta}{\cos \beta}$
(c) $\dfrac{\sin \alpha + \cos \beta}{\sin \alpha}$ (d) $\dfrac{\sin \alpha + \sin \beta}{\cos \alpha}$

48. A person stands at a point A due south of a tower and observes that its elevation is 60°. He then walks westwards towards B, where the elevation is 45°. At a point C on AB produced, he finds it to be 30°. Then AB/BC is equal to
(a) 1/2 (b) 1
(c) 2 (d) 5/2

49. A pole stands at a point A on the boundary of a circular park of radius a and subtends an angle α at another point B on the boundary. If the chord AB subtends an angle α at the centre of the path, the height of the pole is
(a) $2a \cos(\alpha/2) \tan \alpha$ (b) $2a \sin(\alpha/2) \cot \alpha$
(c) $2a \sin(\alpha/2) \tan \alpha$ (d) $2a \cos(\alpha/2) \cot \alpha$

50. A, B, C are three points on a horizontal line through the base O of a pillar OP, such that OA, OB, OC are in A.P. If α, β, γ the angles of elevation of the top of the pillar at A, B, C respectively are also in A.P. then $\sin \alpha$, $\sin \beta$, $\sin \gamma$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) none of these

Previous Years' AIEEE/JEE Main Questions

1. The upper 3/4th portion of a vertical pole subtends an angle $\tan^{-1}$ (3/5) at a point in the horizontal plane through its foot and at a distance 40 m from the foot. A possible height of the vertical pole is

(a) 40 m (b) 60 m
(c) 80 m (d) 20 m **[2003]**

2. A person standing on the bank of a river observe that the angle of elevation of the top of a tree on the opposite bank of the river is 60º and when he retires 40 m away from the tree the angle of elevation becomes 30º. The breadth of the river is

(a) 40 m (b) 30 m
(c) 20 m (d) 60 m **[2004]**

3. A tower stands at the centre of a circular park. A and B are two points on the boundary of the park such that AB (= *a*) subtends an angle of 60º at the foot of the tower and the angle of elevation of the top of the tower from A or B is 30º. The height of the tower is

(a) $2a/\sqrt{3}$ (b) $2a\sqrt{3}$
(c) $a/\sqrt{3}$ (d) $a\sqrt{3}$ **[2007]**

4. AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is 60°. He moves away from the pole along the line BC to a point D such that CD = 7m. From D the angle of elevation of the point A is 45°. Then the height of the pole is

(a) $\frac{7\sqrt{3}}{2}.\frac{1}{\sqrt{3}+1}m.$ (b) $\frac{7\sqrt{3}}{2}.\frac{1}{\sqrt{3}-1}m.$
(c) $\frac{7\sqrt{3}}{2}.\left(\sqrt{3}+1\right)m.$ (d) $\frac{7\sqrt{3}}{2}.\left(\sqrt{3}-1\right)m.$

[2008]

5. A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point *O* on the ground is 45°. It flies off horizontally straight away from the point *O*. After one second, the elevation of the bird from *O* is reduced to 30°. Then the speed (in m/s) of the bird is

(a) $40\left(\sqrt{2}-1\right)$ (b) $40\left(\sqrt{3}-\sqrt{2}\right)$
(c) $20\sqrt{2}$ (d) $20\left(\sqrt{3}-1\right)$ **[2014]**

6. The angle of elevation of the top of a vertical tower from a point *P* on the horizontal ground was observed to be α. After moving a distance 2 meters from *P* towards the foot of the tower, the angle of elevation changes to β. Then the height (in meters) of the tower is:

(a) $\frac{2\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$ (b) $\frac{\sin\alpha\sin\beta}{\cos(\beta-\alpha)}$
(c) $\frac{2\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$ (d) $\frac{\cos(\beta-\alpha)}{\sin\alpha\sin\beta}$

[2014, online]

7. If the angles of elevation of the top of a tower from three collinear points *A*, *B* and *C*, on a line leading to the foot of the tower, art 30°, 45° and 60° respectively, then the ration *AB*: *BC*, is:

(a) $\sqrt{3}:1$ (b) $\sqrt{3}:\sqrt{2}$
(c) $1:\sqrt{3}$ (d) 2 : 3 **[2015]**

8. Let 10 vertical poles standing at equal distances on a straight line, subtend the same angle of elevation α at a point *O* on this line and all the poles are on the same side of *O*. If the height of the longest pole is '*h*' and the distance of the foot of the smallest pole from *O* is '*a*' then the distance between two consecutive poles, is:

(a) $\frac{h\sin\alpha+a\cos\alpha}{9\sin\alpha}$ (b) $\frac{h\cos\alpha-a\sin\alpha}{9\cos\alpha}$
(c) $\frac{h\cos\alpha-a\sin\alpha}{9\sin\alpha}$ (d) $\frac{h\sin\alpha+a\cos\alpha}{9\cos\alpha}$

[2014, online]

9. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point *A* on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 minutes from *A* in the same direction, at a point *B*, he observes that the angle of elevation of the top of the pillar is 60°. Then the time taken (in minutes) by him, from *B* to reach the pillar is

(a) 6 (b) 10
(c) 20 (d) 5 **[2016]**

10. The angle of elevation of the top of a vertical tower from a point *A*, due east of it is 45°. The angle of elevation of the top of the same tower from a point *B*, due south of *A* is 30°. If the distance between *A* and *B* is $54\sqrt{2}$ m, then the height of the tower (in meters) is;

(a) 108 (b) $36\sqrt{3}$
(c) $54\sqrt{3}$ (d) 54 **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. A vertical pole stands at a point A on the boundary of a circular park of radius a and subtends an angle α at another point β on the boundary. If the chord AB subtends an angle α at the centre of the park, the height of the pole is

   (a) $2a \sin \frac{\alpha}{2} \tan \alpha$ (b) $2a \cos \frac{\alpha}{2} \tan \alpha$

   (c) $2a \sin \frac{\alpha}{2} \cot \alpha$ (d) $2a \cos \frac{\alpha}{2} \cot \alpha$. **[2014]**

2. Two vehicles C_1 and C_2 start from a point P and travel east of P at the speeds 20 km/hr and 60 km/hr respectively. If an observer, one kilometre north of P, is able to see both the vehicles at the same time, then the maximum angle of sight between the observer's view of C_1 and C_2, is:

   (a) $\frac{\pi}{3}$ (b) $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

   (c) $\frac{\pi}{6}$ (d) $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ **[2015]**

Answers

Concept-based

1. (a) **2.** (a) **3.** (b) **4.** (b)
5. (a)

Level 1

6. (d) **7.** (a) **8.** (b) **9.** (a)
10. (b) **11.** (b) **12.** (b) **13.** (c)
14. (b) **15.** (b) **16.** (a) **17.** (a)
18. (c) **19.** (a) **20.** (a) **21.** (c)
22. (b) **23.** (b) **24.** (c) **25.** (a)
26. (c) **27.** (c) **28.** (c) **29.** (a)
30. (b) **31.** (b) **32.** (c) **33.** (a)
34. (a) **35.** (d)

Level 2

36. (a) **37.** (c) **38.** (d) **39.** (c)
40. (a) **41.** (c) **42.** (a) **43.** (b)
44. (b) **45.** (c) **46.** (b) **47.** (b)
48. (b) **49.** (c) **50.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (a) **2.** (c) **3.** (c) **4.** (c)
5. (d) **6.** (a) **7.** (a) **8.** (c)
9. (d) **10.** (d)

Previous Years' B-Architecture Entrance Examination Questions

1. (a) **2.** (c)

Hints and Solutions

Concept-based

1. $h \cot \alpha + h \cot \beta = \frac{h}{2}$

$$\Rightarrow 2 \sin (\alpha + \beta) = \sin \alpha \sin \beta$$

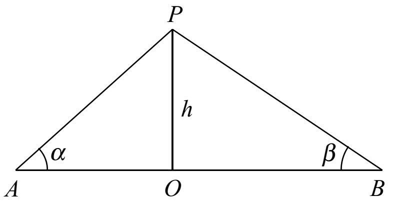


Fig. 27.52

2. $OA = BL = 15 \cot 45° = 15$

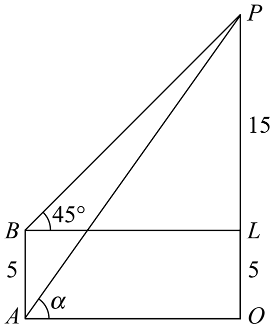


Fig. 27.53

$$\tan \alpha = \frac{OP}{OA} = \frac{20}{15} = \frac{4}{3}$$

$$\Rightarrow \cos \alpha = \frac{3}{5}.$$

3. $OA = h \cot \alpha$

$OB = h \cot 45° = h$

$50 = OB - OA$

$= h\,[1 - \cot \alpha]$

$$\Rightarrow h = \frac{50}{1-\cot\alpha} = \frac{50 \sin\alpha}{\sin\alpha - \cos\alpha}$$

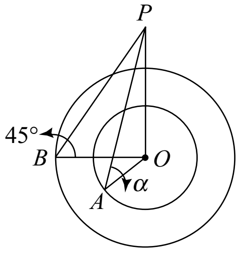


Fig. 27.54

4. $OA = 40 \cot - 45° \ \alpha = 40$

$= 30 \cot (45° - \alpha)$

$\Rightarrow 4 \tan (45° - \alpha) = 3$

$\Rightarrow 4\,(1 - \tan \alpha) = 3\,(1 + \tan \alpha)$

$\Rightarrow \tan \alpha = 1/7 \Rightarrow \cot \alpha = 7$

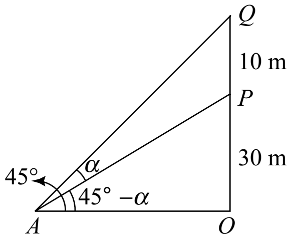


Fig. 27.55

5. Let the height of the poles be h.

then $h \cot \alpha = AE = \dfrac{100\sqrt{5}}{2}$

$h \cot 30° = 100$

$h = \dfrac{100}{\sqrt{3}}$.

$\cot \alpha = \dfrac{\sqrt{15}}{2}$

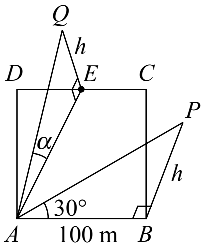


Fig. 27.56

$\cos^2\alpha = \dfrac{15}{19}$, $\sin^2 \alpha = \dfrac{4}{19}$

$\cos 2\alpha = \dfrac{11}{19}$.

Level 1

6. $PQ = OQ - OP = 100 \tan 60° - 100 \tan 45°$

$= 100(\sqrt{3} - 1)$

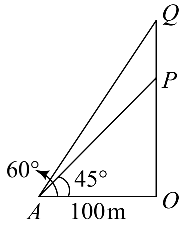


Fig. 27.57

7. $(h - 10) \cot 30° = h \cot 60°$

$$\Rightarrow h = \frac{10 \cot 30°}{\cot 30° - \cot 60°}$$

$$= \frac{10\sqrt{3}}{\sqrt{3} - (1/\sqrt{3})} = 15$$

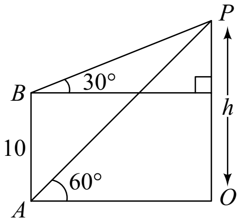


Fig. 27.58

8. $h \cot \alpha = b \cot \beta$

$\Rightarrow \quad h = b \tan \alpha \cot \beta$

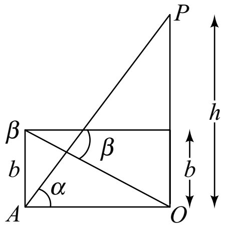


Fig. 27.59

9. $h \cot 30° - h \cot 75° = 1000$ m.

$$\Rightarrow \quad h = \frac{1000}{\cot 30° - \cot 75°}$$

$$= \frac{1000}{\sqrt{3} - \dfrac{\sqrt{3}-1}{\sqrt{3}+1}} = 250(\sqrt{3}+1)\text{m}$$

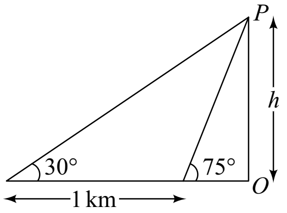


Fig. 27.60

10. A, B, C, D lie on a circle with centre at O, the foot of the flagstaff as $OA = OB = OC = OD = h \cot \alpha$, h being the height of the flagstaff and α the angle given.

11. Height = 100 tan 60º = 100 × $\sqrt{3}$ = 100 × 1.73 = 173 m.

12. $100(\cot(45^\circ - A) - \cot(45^\circ + A)) = 200$

$$\Rightarrow \frac{1+\tan A}{1-\tan A} - \frac{1-\tan A}{1+\tan A} = 2$$

$$\Rightarrow 4 \tan A = 2(1 - \tan^2 A)$$

$$\Rightarrow \frac{2 \tan A}{1-\tan^2 A} = 1 = \tan 45^\circ$$

$$\Rightarrow \tan 2A = 45^\circ \Rightarrow A = 22\frac{1}{2}{}^\circ.$$

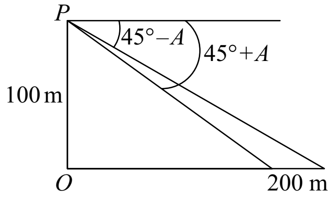


Fig. 27.61

13. $h \cot A = OM + ML = h + 3$

$h \cot(90^\circ - \text{A}) = h - 2$

$h^2 = (h + 3)(h - 2)$

$\Rightarrow \quad h = 6$ m

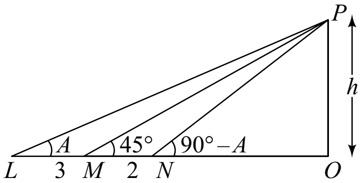


Fig. 27.62

14. $AC = h \cot 45^\circ = h$

$$AD = \frac{\sqrt{3}\,h}{2} = h \cot \theta$$

$$\Rightarrow \theta = \tan^{-1}(2/\sqrt{3})$$

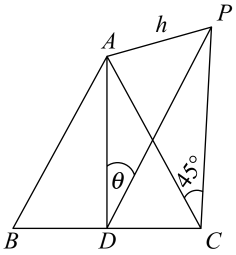


Fig. 27.63

15. $\sin 75^\circ = h/25$

$$\Rightarrow \quad h = 25 \times \frac{\sqrt{3}+1}{2\sqrt{2}} = (25/4)\ (\sqrt{3}+1)\sqrt{2}.$$

16. $AC = AP \tan \alpha$

$\Rightarrow (1/2)\ AB = n\ AB \tan \alpha$

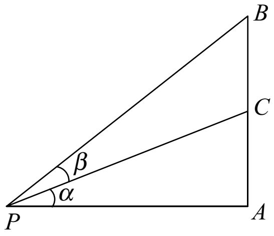


Fig. 27.64

$$\Rightarrow \quad \tan \alpha = \frac{1}{2n}$$

$$\tan(\alpha + \beta) = \frac{AB}{AP} = \frac{1}{n}.$$

$$\Rightarrow \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{1}{n}.$$

$$\Rightarrow n\left(\frac{1}{2n} + \tan \beta\right) = 1 - \frac{1}{2n} \tan \beta$$

$$\Rightarrow \quad \tan \beta = \frac{n}{2n^2+1}.$$

17. $\text{AB} = (\cot 45^\circ - \cot 60^\circ)\ 150$

$$= 150\left(1 - \frac{1}{\sqrt{3}}\right) = 50(3-\sqrt{3})$$

$$\text{speed} = \frac{50(3-\sqrt{3})}{2}\ \text{m/min.}$$

$$= 25(3-\sqrt{3}) \times \frac{60}{1000}\ \text{km/hr}$$

$$= \frac{9-3\sqrt{3}}{2}\ \text{km/h.}$$

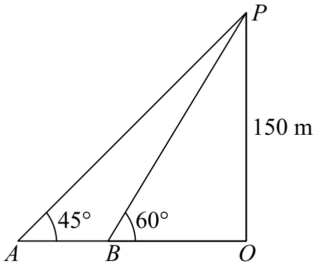


Fig. 27.65

18. $OP = x \tan 60° = (40 + x) \tan 30°$

$$\Rightarrow \qquad x = \frac{40 \tan 30°}{\tan 60° - \tan 30°}$$

$$= \frac{40}{\sqrt{3}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)} = 20 \text{ m}$$

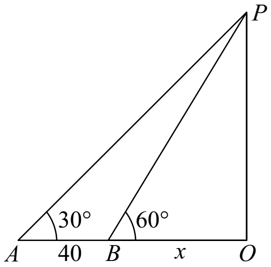


Fig. 27.66

19. The base of the mountain is at the circumcentre of the circle passing through A, B and C if O is the centre and R the radius of the circle, then $R = h \cot \alpha$.

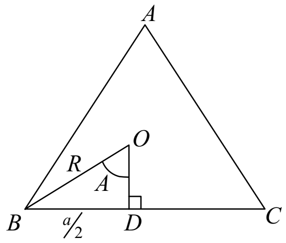


Fig. 27.67

$$\sin A = \frac{a}{2R} = \frac{a}{2h \cot \alpha}$$

$$\Rightarrow h = \frac{a}{2} \tan \alpha \cdot \operatorname{cosec} A$$

20. $\tan \alpha = h/a$, $\cot \alpha = h/b$.

$$\Rightarrow \qquad h = \sqrt{ab}.$$

$\tan (\alpha + \theta) = a/h$.

$$\Rightarrow \frac{\tan \alpha + \tan \theta}{1 - \tan \alpha \tan \theta} = \frac{a}{h}.$$

$$\Rightarrow h \tan \alpha + h \tan \theta = a - a \tan \alpha \tan \theta$$

$$\Rightarrow \tan \theta = \frac{a - b}{h + a \tan \alpha} = \frac{a - b}{2\sqrt{ab}}.$$

$$\Rightarrow \sin \theta = \frac{a - b}{a + b}$$

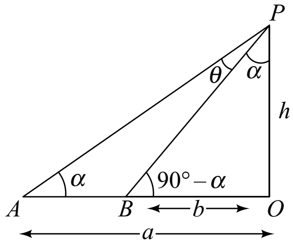


Fig. 27.68

21. $\tan\theta = \frac{h}{160}$, $\tan (\alpha + \theta) = \frac{h}{40}$.

$$\Rightarrow \frac{\tan \alpha + \frac{h}{160}}{1 - \frac{h}{160} \cdot \tan \alpha} = \frac{h}{40}$$

$$\Rightarrow \frac{\frac{3}{5} + \frac{h}{160}}{1 - \frac{h}{160} \times \frac{3}{5}} = \frac{h}{40}$$

$\Rightarrow 3h^2 - 600h + 19200 = 0$

$\Rightarrow h = 40$ or 160.

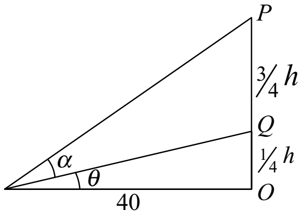


Fig. 27.69

22. $AB = h[\cot \alpha - \cot \beta]$

$BC = h[\cot \beta - \cot \gamma]$

$$\Rightarrow \frac{AB}{BC} = \frac{\cot \alpha - \cot \beta}{\cot \beta - \cot \gamma}.$$

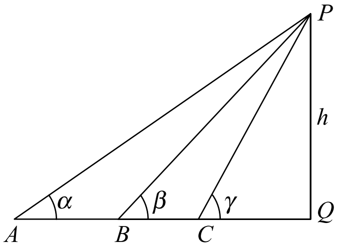


Fig. 27.70

23. $BC = h \cot \beta$

$AC = h \cot \alpha.$

$(AC)^2 = a^2 + (BC)^2$

$\Rightarrow h^2 \cot^2 \alpha = a^2 + h^2 \cot^2 \beta$

$$\Rightarrow \qquad h = \frac{a}{\sqrt{\cot^2 \alpha - \cot^2 \beta}}.$$

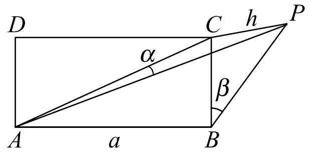


Fig. 27.71

24. $OA = a$, $OB = b$.

$h \cot \alpha = a$, $h \cot \beta = b$

$$\Rightarrow \qquad h = \frac{a+b}{\cot \alpha + \cot \beta}.$$

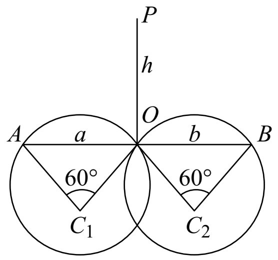


Fig. 27.72

25. $PQ = a - b \quad \Rightarrow \quad QL = a - b$

$\Rightarrow \qquad OA = a - b$

$\Rightarrow \qquad a \cot \alpha = a - b$

$$\Rightarrow \qquad \cot \alpha = \frac{a-b}{a}$$

Similarly $\cot \beta = \dfrac{b-c}{b}$.

So that $\cot \alpha - \cot \beta = \dfrac{c}{b} - \dfrac{b}{a} = \dfrac{ac - b^2}{ab}$

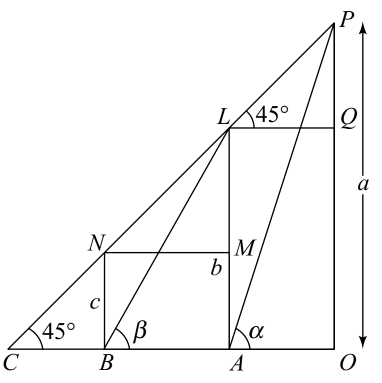


Fig. 27.73

26. Distance travelled in

10 sec $= AB = PQ.$

$= OQ - OP$

$= (\cot 30° - \cot 60°)$ km.

in 1 hour $= \dfrac{\sqrt{3} - 1/\sqrt{3}}{10} \times 60 \times 60$ km.

$$= \frac{2 \times 60 \times 60\sqrt{3}}{30} = 240\sqrt{3} \text{ km.}$$

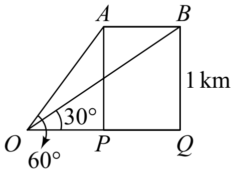


Fig. 27.74

27. AB is the shadow of the tower PQ (= 6 m) on the ground.

So $2\sqrt{3} = (x + 6) \cot \theta - x \cot \theta$

$$\Rightarrow \cot \theta = \frac{2\sqrt{3}}{6} = \frac{1}{\sqrt{3}}$$

$\Rightarrow \qquad \theta = 60°.$

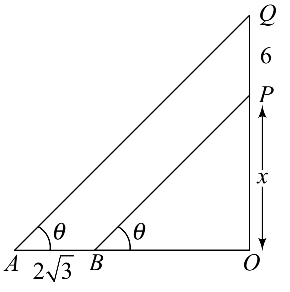


Fig. 27.75

28. If r is the radius of the circle, then the heights of the poles are $r \tan \alpha$, $r \tan \beta$, $r \tan \gamma$. Since they are in A.P. $\cot \alpha$, $\cot \beta$, $\cot \gamma$ are in H.P.

29. If a is the distance of the point from the foot O, of the poles, then the distances of the points A, B, C from O are $a \tan \alpha$, $a \tan \beta$, $a \tan \gamma$, since they are in A.P; $\tan \alpha$, $\tan \beta$, $\tan \gamma$ are in A.P.

So $2 \tan \beta = \tan \alpha + \tan \gamma$.

Also $\alpha + \beta + \gamma = \pi \quad \Rightarrow \quad \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$

$\Rightarrow \tan \alpha \tan \gamma = 3.$

30. AP and BQ be the two positions of the ladder of length l.

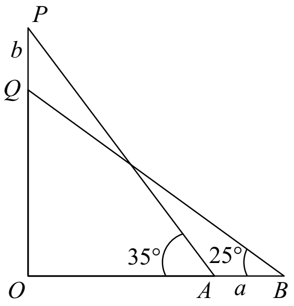


Fig. 27.76

$$\frac{a}{b} = \frac{l\,(\cos 25° - \cos 35°)}{l\,(\sin 35° - \sin 25°)}$$

$$= \frac{2 \sin 30° \sin 5°}{2 \cos 30° \sin 5°}$$

$$= \tan 30° = 1/\sqrt{3}\,.$$

31. $AB = OA - OB = (a - b) \cot \alpha$

$BC = OB - OC = (b - c) \cot \alpha$

$\Rightarrow AB = BC$

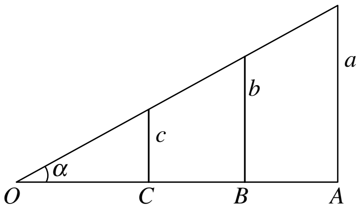


Fig. 27.77

[$\because$ a, b, c are in A.P. $\Rightarrow$ $b - a = c - b$]

$\Rightarrow$ Statement-1 is True.

In statement-2 if r is the radius of the circular field, then $OB = r$, $AB = 2r$

Height of the pole at $A = 2r \tan \alpha = 2 \times$ height of the pole at O.

$\Rightarrow$ Statement-2 is also correct but does not lead to statement-1.

32. Let h be the height of the tower, $BD \perp OA$; $BE \perp OP$ then (Fig. 27.78)

$OA = AD + BC + CE$

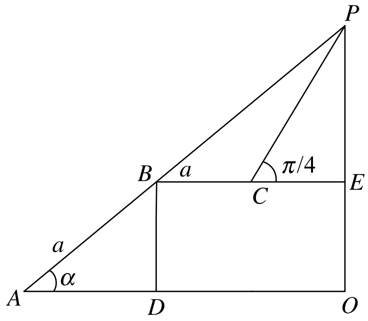


Fig. 27.78

$$h \cot \alpha = a \cos \alpha + a + h - a \sin \alpha.$$

$$h\,(\cot \alpha - 1) = a(\cos \alpha + 1 - \sin \alpha)$$

$\Rightarrow$ statement-1 is True.

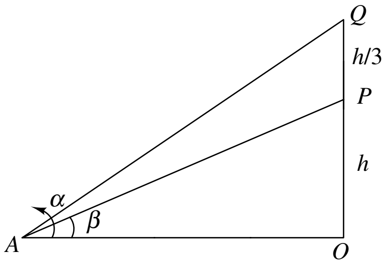


Fig. 27.79

In statement-2 (Fig. 27.79)

$$OA = h \cot \beta = (h + h/3) \cot \alpha.$$

$\Rightarrow$ $\cot \alpha = (3/4) \cot \beta$

$\nRightarrow$ $\alpha = (3/4)\,\beta$

$\Rightarrow$ statement-2 is False.

33. $BD = h \cot \alpha$, $CD = h \cot \beta$.

Statement-2 is True, so using it $\dfrac{BD}{CD} = \dfrac{c}{b} = \dfrac{h \cot \alpha}{h \cot \beta}$.

$\Rightarrow$ $b \cot \alpha = c \cot \beta$ $\Rightarrow$ statement-1 is True.

34. Statement-2 is True. Using it $BQ \parallel AP$

$\Rightarrow$ $\angle QBO = \angle PAO = \alpha$ and the statement-1 is also True.

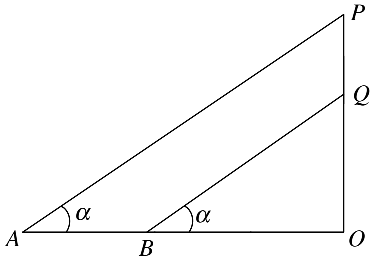


Fig. 27.80

35. Let h be the height of the tower in statement-1 and x be the side of the square, then diagonal of the square is $2h \cot \alpha$

$\Rightarrow$ $(2h \cot \alpha)^2 = 2x^2$

$\Rightarrow$ $4h^2 \cot^2 \alpha = 2x^2$

$\Rightarrow$ $16 \cot^2 \alpha = 2$ [$\because$ $h = 2x$]

$\Rightarrow$ $\tan \alpha = 2\sqrt{2}$

$\Rightarrow$ statement-1 is false.

In statement-2

$$\frac{AB}{BC} = \frac{\cot 30° - \cot 45°}{\cot 45° - \cot 60°} = \frac{\sqrt{3} - 1}{1 - \left(1/\sqrt{3}\right)} = \sqrt{3}$$

$\Rightarrow$ statement-2 is True.

Level 2

36. Let the height of the tower be h

$\angle PAQ = \alpha$, $\angle RBN = \beta$, $BM = a$

$\Rightarrow RN = (h/2) - a$

$AP = h \cot \alpha = AM + MP = a \cot\alpha + (h/2 - a) \cot\beta$

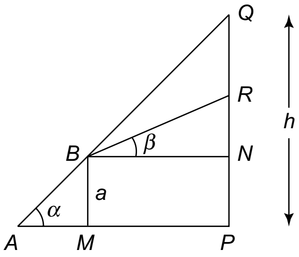


Fig. 27.81

$\Rightarrow h (\cot\alpha - (h/2) \cot\beta) = a (\cot\alpha - \cot\beta)$

$$\Rightarrow h = 2a\frac{(\tan\beta - \tan\alpha)}{2\tan\beta - \tan\alpha}$$

37. If h is the height of the tower OP at the centre O of the circular garden.

Then $\angle AOB = 2\beta \Rightarrow \angle AOC = \beta$

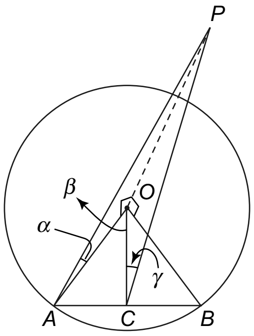


Fig. 27.82

$OA = OB = h \cot \alpha$

$OC = h \cot \gamma$.

$$\Rightarrow \frac{OA}{OC} = \sec\beta \Rightarrow \tan\gamma = \tan\alpha \sec\beta.$$

38. Let P, Q, R, S be the positions of the bird when it makes angles

α, β, γ, δ respectively at a point, O, on the ground.

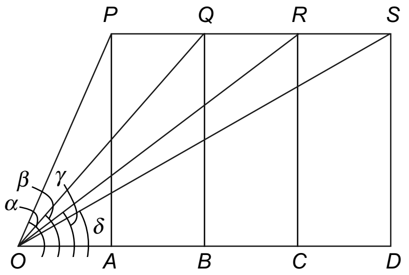


Fig. 27.83

$\angle POA = \alpha$, $\angle QOB = \beta$, $\angle ROC = \gamma$, $\angle SOD = \delta$

$\Rightarrow OA = h \cot\alpha$, $OB = h \cot\beta$, $OC = h \cot \gamma$

and $OD = h \cot\delta$, $AB = BC = CD$.

$AD = h (\cot \delta - \cot\alpha) = 3\, BC = 3h (\cot\gamma - \cot \beta)$

$\Rightarrow \cot\alpha - \cot\delta = 3 (\cot\beta - \cot\gamma)$

Also $\cot\alpha + \cot\delta = \cot\beta + \cot\gamma$

So $\cot^2\alpha - \cot^2\delta = 3 (\cot^2\beta - \cot^2\gamma)$

39. $(AP)^2 = 4 + 1 = 5$

$(AQ)^2 = 4 + 4 = 8$

$(AR)^2 = 4 + 9 = 13$

$(AS)^2 = 4 + 4 = 16$

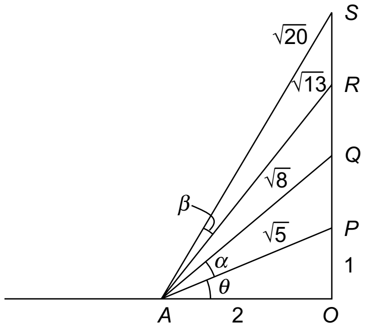


Fig. 27.84

$$\cos\alpha = \frac{8+5-1}{2\sqrt{8\times5}} = \frac{3}{\sqrt{10}} \Rightarrow \sin\alpha = \frac{1}{\sqrt{10}}$$

$$\cos\beta = \frac{20+13-1}{2\sqrt{20\times13}} = \frac{8}{\sqrt{65}} \Rightarrow \sin\beta = \frac{1}{\sqrt{65}}$$

$$\cos(\alpha+\beta) = \frac{3\times8}{\sqrt{650}} - \frac{1\times1}{\sqrt{650}} = \frac{23}{\sqrt{650}}$$

40. Let the height of the lamppost be h.

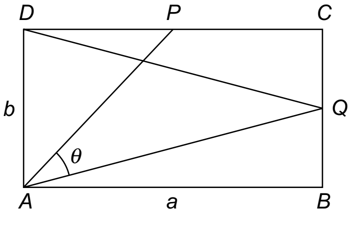


Fig. 27.85

$$AP = h \cot\alpha = \sqrt{b^2 + a^2/4}$$

$$AQ = DQ = h\cot\beta = \sqrt{a^2 + b^2/4}$$

$$\cos\theta = \frac{(AP)^2 + (AQ)^2 - (a^2/4 + b^2/4)}{2\,AP \cdot AQ \cdot}$$

$$= \frac{a^2 + b^2}{2\,h\cot\alpha \times h\cot\beta}$$

$$\Rightarrow k = \cot\alpha\,\cot\beta\,\cos\theta = \frac{a^2 + b^2}{2h^2}$$

41. $AB = h\,(\cot\alpha - \cot\beta)$

$OB = h\cot\beta,\ OC = h\cot\gamma$

From right angled triangle OBC

$$(BC)^2 = h^2\,[\cot^2\gamma - \cot^2\beta]$$

From right angled triangle ABC

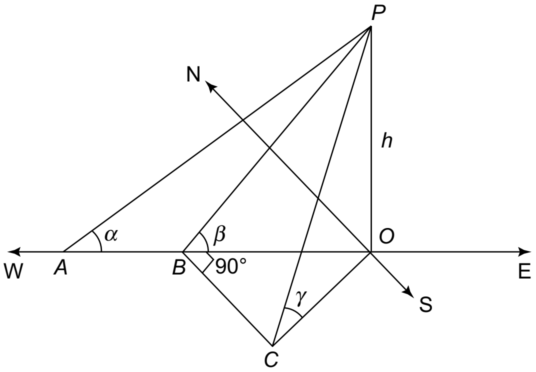


Fig. 27.86

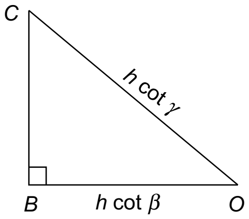


Fig. 27.87

$$(AC)^2 = h^2\,[(\cot\alpha - \cot\beta)^2 + \cot^2\gamma - \cot^2\beta]$$

$$= h^2\,[\cot^2\alpha + \cot^2\gamma - 2\cot\alpha\cot\beta]$$

42. $h = PQ = RS$

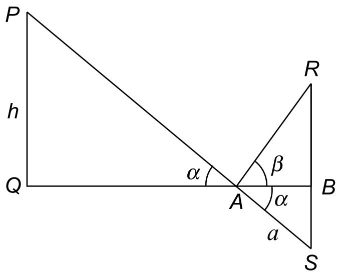


Fig. 27.88

$= RB + BS$

$= AB\,(\tan\alpha + \tan\beta)$

$= a\cos\alpha\,(\tan\alpha + \tan\beta)$

$$= a\,\frac{\sin(\alpha + \beta)}{\cos\beta}$$

43. $\angle PBO = \theta,\ \angle QBO = 45°$

$\Rightarrow \angle PBQ = \theta - 45°$

$PQ = h\tan(\theta - 45°)$

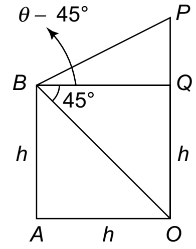


Fig. 27.89

Height of the tower is $PQ + OQ$.

$$= h\left[\frac{\tan\theta - 1}{\tan\theta + 1} + 1\right] = \frac{2h}{1 + \cot\theta}$$

44. $\angle ROX = 9° = \angle ORQ$.

$\angle ORP = 54° \Rightarrow \angle QRP = 45°$

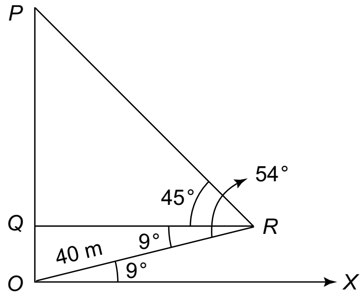


Fig. 27.90

$\Rightarrow \angle QPR = 45° \Rightarrow PQ = QR$

$OR = 40$ m

$\Rightarrow QR = 40\cos 9°,\ OQ = 40\sin 9°$

Height of the tower $OP = OQ + QP$

$= 40\,(\sin 9° + \cos 9°)$

$$= 40\sqrt{1 + \sin 18°} = 40\sqrt{1 + \frac{\sqrt{5} - 1}{4}}$$

$$= 20\sqrt{3 + \sqrt{5}} = 45.76 \text{ m.}$$

45. $\angle QPC = 15°,\ \angle QPR = 45°,\ CQ = x$

$PQ = QR = 2500 + 2500 + x$

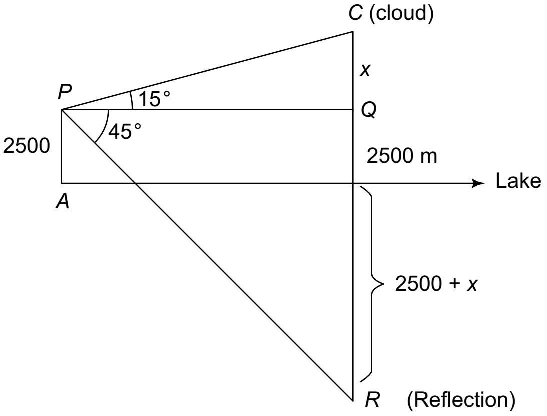


Fig. 27.91

Also $PQ = x \cot 15°$

So $x = \dfrac{5000}{\cot 15° - 1} = 2500\left(\sqrt{3} - 1\right)$

Height of the cloud above the lake

$= x + 2500 = 2500\sqrt{3}$ m.

46. Let h be the height of the tower at P, then

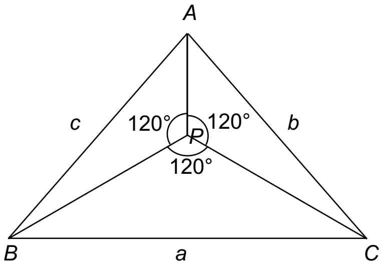


Fig. 27.92

$PA = h \cot\alpha$, $PB = h \cot\beta$, $PC = h \cot \gamma$.
From triangle PBC

$$\cos 120° = \frac{h^2 \cot^2 \gamma + h^2 \cot^2 \beta - a^2}{2\, h^2 \cot\beta \cot\gamma}$$

$\Rightarrow a^2 = h^2\,[\cot^2 \beta + \cot^2 \gamma + \cot \beta \cot \gamma]$
$\Rightarrow a^2\,(\cot \beta - \cot \gamma) = h^2\,[\cot^3 \beta - \cot^3 \gamma]$
Similarly for $b^2\,(\cot \gamma - \cot \alpha)$ and
$c^2(\cot \alpha - \cot \beta)$

47. Let r be the radius of the sphere with centre O

$\underline{|OPA} = \alpha$, $\underline{|OPQ} = \beta$, $\underline{|RPQ} = \theta$

$OP = r \operatorname{cosec} \alpha$ (From $\Delta\, OAP$)
$OQ = OP \sin \beta = r \operatorname{cosec} \alpha \sin \beta$
$PQ = OP \cos \beta = r \operatorname{cosec} \alpha \cos \beta$

$$\tan \theta = \frac{QR}{PQ} = \frac{r + r \operatorname{cosec} \alpha \sin\beta}{r \operatorname{cosec} \alpha \cos\beta} = \frac{\sin\alpha + \sin\beta}{\cos\beta}$$

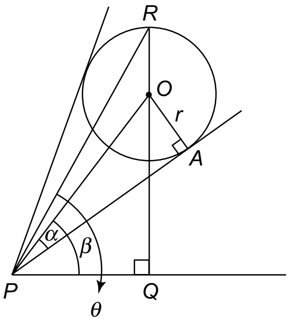


Fig. 27.93

48. Let OP be the tower of height h.

$\underline{|OAP} = 60°$, $\underline{|OBP} = 45°$, $\underline{|OCP} = 30°$,

$\underline{|OAC} = 90°$

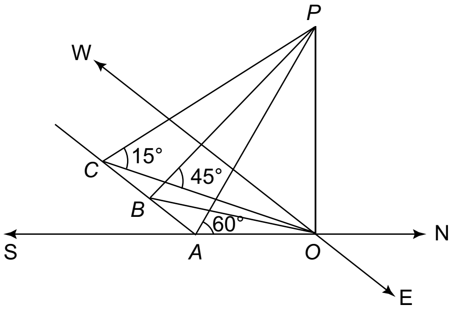


Fig. 27.94

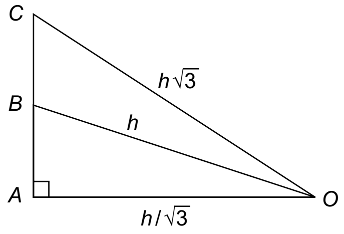


Fig. 27.95

$OA = h \cot 60° = h/\sqrt{3}$

$OB = h$, $OC = h \cot 30° = h/\sqrt{3}$

$(AB)^2 = h^2\,(1 - 1/3) = (2/3)\,h^2$

$(AC)^2 = h^2\,(3 - 1/3) = (8/3)\,h^2$

$BC = AC - AB = \left(2\sqrt{2/3} - \sqrt{2/3}\right)h = \sqrt{2/3}\,h = AB$

Hence $AB/BC = 1$

49. Let AP be the pole of height h.

$\underline{|PBA} = \alpha \Rightarrow AB = h \cot a$

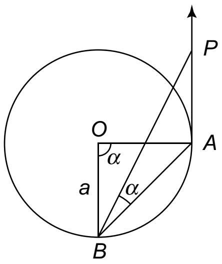


Fig. 27.96

$OA = OB = a$

From triangle OAB

$$\cos\alpha = \frac{a^2 + a^2 - h^2\cot^2\alpha}{2a^2}$$

$$\Rightarrow h^2\cot^2\alpha = 2a^2(1 - \cos\alpha) = 4a^2\sin^2(\alpha/2)$$

$$\Rightarrow h = 2a\sin(\alpha/2)\tan\alpha$$

50. Let OP be the pillar of height h

$\lfloor\underline{OAP} = \alpha$, $\lfloor\underline{OBP} = \beta$, $\lfloor\underline{OCP} = \gamma$

$2h\cot\beta = h(\cot\alpha + \cot\gamma)$

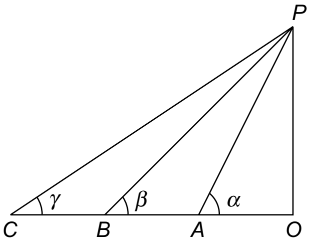


Fig. 27.97

$$\Rightarrow 2\frac{\cos\beta}{\sin\beta} = \frac{\sin(\alpha+\gamma)}{\sin\alpha\sin\gamma}$$

$$= \frac{\sin 2\beta}{\sin\alpha\sin\gamma}$$

$$= \frac{2\sin\beta\cos\beta}{\sin\alpha\sin\gamma}$$

$\Rightarrow \sin\alpha\sin\gamma = \sin^2\beta \Rightarrow \sin\alpha, \sin\beta, \sin\gamma$ are in G.P.

Previous Year's AIEEE/JEE Main Questions

1. $\tan\theta = \dfrac{h}{160}$, $\tan(\alpha + \theta) = \dfrac{h}{40}$.

$$\Rightarrow \frac{\tan\alpha + \dfrac{h}{160}}{1 - \dfrac{h}{160}\cdot\tan\alpha} = \frac{h}{40}$$

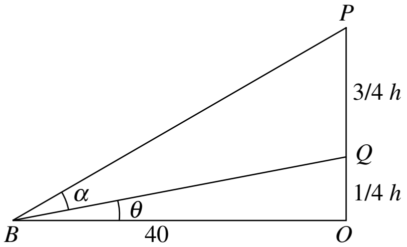


Fig. 27.98

$$\Rightarrow \frac{\dfrac{3}{5} + \dfrac{h}{160}}{1 - \dfrac{h}{160}\times\dfrac{3}{5}} = \frac{h}{40}$$

$$\Rightarrow 3h^2 - 600h + 19200 = 0$$

$$\Rightarrow h = 40 \quad \text{or} \quad 160.$$

2. We have

$$\frac{h}{b} = \tan 60°$$

$$\Rightarrow h = \sqrt{3}\,b$$

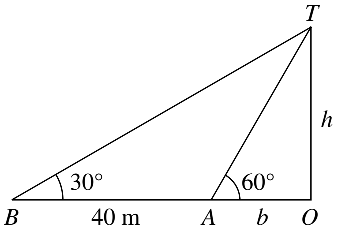


Fig. 27.99

and $\dfrac{h}{40+b} = \tan 30° = \dfrac{1}{\sqrt{3}}$

$$\Rightarrow (\sqrt{3}b)\sqrt{3} = 40 + b \Rightarrow 2b = 40 \Rightarrow b = 20$$

3. $OA = OB = AB = a$

$$\Rightarrow \frac{h}{a} = \tan 30°$$

$$\Rightarrow h = \frac{a}{\sqrt{3}}$$

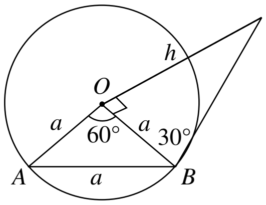


Fig. 27.100

4. $h - h\cot 60° = 7$

$$\Rightarrow h = \frac{7}{1 - \dfrac{1}{\sqrt{3}}}$$

$$= \frac{7\sqrt{3}}{2}\left(\sqrt{3} + 1\right)$$

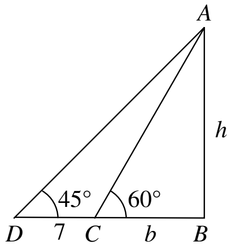


Fig. 27.101

5. $RS = PQ = 20$

$\Rightarrow OS = 20$

Let $SP = d = 20\cot 30° - 20$

$= 20\left(\sqrt{3} - 1\right)$

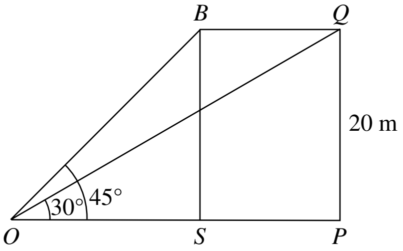


Fig. 27.102

Required speed is

$20\left(\sqrt{3}-1\right)$ m/s

6. $h \cot \alpha - h \cot \beta = 2$

$$\Rightarrow h = \frac{2}{\cot\alpha - \cot\beta}$$

$$= \frac{2\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$$

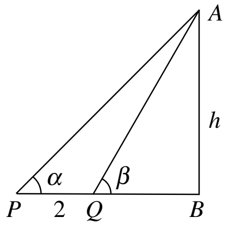


Fig. 27.103

7. $\dfrac{PQ}{CQ} = \tan 60°$

$\Rightarrow$ CQ = h cot 60° = $hl\sqrt{3}$

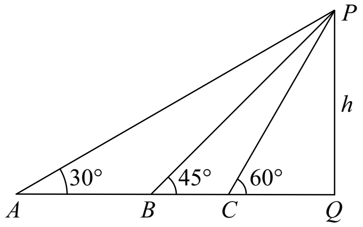


Fig. 27.104

Similarly

$BQ = h$ and $AQ = \sqrt{3}\, h$

Now,

$$\frac{AB}{BC} = \frac{AQ-BQ}{BQ-CQ} = \frac{\sqrt{3}h - h}{h - h/\sqrt{3}} = \frac{\sqrt{3}}{1}$$

8. Let $OA_1 = a$ and $A_1A_2 = A_2A_3 = \ldots\ d.$

$A_1B_1 = C$

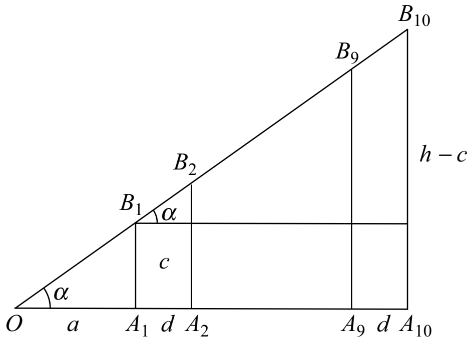


Fig. 27.105

Then $OA_{10} = a + 9d$

Now,

$$\tan\alpha = \frac{c}{a}$$

$$\text{and } \tan\alpha = \frac{h-c}{9d} = \frac{h - a\tan\alpha}{9d}$$

$9d \tan\alpha = h - a\tan\alpha$

$$\Rightarrow\quad d = \frac{h\cos\alpha - a\sin\alpha}{9\sin\alpha}$$

9. Refer figure

$$\frac{h}{a+b} = \tan 30°$$

$$\text{and}\quad \frac{h}{b} = \tan 60°$$

$$\Rightarrow\quad a+b = \sqrt{3}h,\ b = \frac{h}{\sqrt{3}}$$

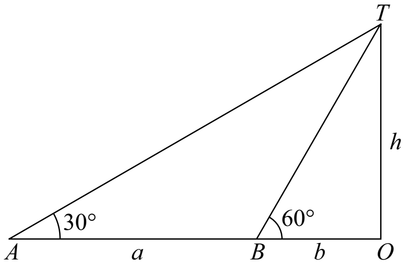


Fig. 27.106

$$\Rightarrow\quad a = \left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)h = \frac{2}{\sqrt{3}}h$$

Time taken to cover $a = \dfrac{2}{\sqrt{3}}h$ units is 10 minutes

$\Rightarrow$ time taken to cover $b = \dfrac{1}{\sqrt{3}}h$ units is 5 minutes.

10. Let $OT = h$

In right triangle

ΔAOT,

$$\frac{OT}{OB} = \tan 45° = 1$$

$\Rightarrow$ $OA = OT = h$

In right triangle, ΔBOT

$$\frac{OT}{OB} = \tan 30° = \frac{1}{\sqrt{3}}$$

$$\Rightarrow\quad OB = \sqrt{3}OT = \sqrt{3}h$$

Also, $OB^2 = OA^2 + AB^2$

$$\Rightarrow\quad 3h^2 = h^2 + (54\sqrt{2})^2$$

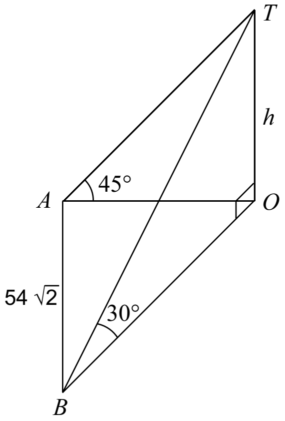


Fig. 27.107

$$\Rightarrow \quad 2h^2 = (54\sqrt{2})^2$$

$$\Rightarrow \quad h = 54$$

Previous Years' B-Architecture Entrance Examination Questions

1. $AP = h$

$\underline{|ABP} = \alpha, \underline{|AOB} = \alpha$

$$\Rightarrow \quad \underline{|BOQ} = \frac{\alpha}{2}. \; OB = a$$

$AB = h \cot \alpha$

Also $AB = 2BQ$

$$\Rightarrow \quad h \cot \alpha = 2a \sin \frac{\alpha}{2}$$

$$\Rightarrow \quad h = 2a \sin \frac{\alpha}{2} \tan \alpha$$

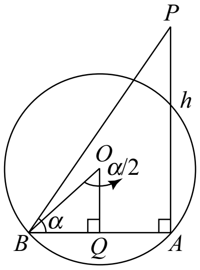


Fig. 27.108

2. At time t, $PC_1 = 20t$

and $PC_2 = 60t$

As $OP = 1$,

$$\tan \theta_1 = \frac{PC_1}{OP} = 20t, \; \tan \theta_2 = \frac{PC_2}{OP} = 60t$$

Now

$$\tan \theta = \tan(\theta_2 - \theta_1) = \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1}$$

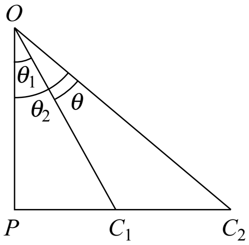


Fig. 27.109

$$= \frac{40t}{1 + 1200t^2}$$

$$\sec^2 \theta \frac{d\theta}{dt} = 40 \frac{(1 + 1200t^2)(1) - t(2400t)}{(1 + 1200t^2)^2}$$

$$= 40 \frac{1 - 1200t^2}{(1 + 1200t^2)^2}$$

We have $\dfrac{d\theta}{dt} = 0 \Rightarrow t = \dfrac{1}{20\sqrt{3}}$

and $\dfrac{d\theta}{dt} > 0$ for $0 < t < \dfrac{1}{20\sqrt{3}}$

and $\dfrac{d\theta}{dt} < 0$ for $t < \dfrac{1}{20\sqrt{3}}$

$\therefore$ θ is maximum when

$$t = \frac{1}{20\sqrt{3}} \Rightarrow \tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$$

CHAPTER TWENTY EIGHT

Mathematical Reasoning

SOME DEFINITIONS

1. A *statement* or proposition is a declarative sentence which is either true or false but not both simultaneously.
2. A paradox is a sentence which is both true and false simultaneously.
3. A statement is said to be simple if it cannot be broken further into simple statements otherwise statement is said to be compound.
   Truth value of a compound statement is completely determined by its constituent statements.
4. Table for basic logical connectives.

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow p$	$\sim p$
T	T	T	T	T	T	F
T	F	F	T	F	F	F
F	T	F	T	T	F	T
F	F	F	F	T	T	T

5. Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$
6. Converse of the conditional statement $p \rightarrow q$ is $q \rightarrow p$
7. Inverse of the conditional statement $p \rightarrow q$ is $\sim p \rightarrow \sim q$.
8. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

TIP

Negation of an if statement does not start with the word if.

Some Definitions

1. A statement is said to be a *tautology* if it is always true.
2. A statement is a *contradiction* if it is always false.
3. Dual of a compound statement not involving logical connective other than $\wedge$ and $\vee$ is obtained by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$ respectively, and each occurance of T by F and F by T.

Some Logical Equivalences

Given any statement variables p, q and r, a tautology t and a contradiction c, the following logical equivalances hold:

1. Commutative laws:
   (i) $p \wedge q \equiv q \wedge p$
   (ii) $p \vee q \equiv q \vee p$
2. Associative laws:
   (i) $(p \wedge q) \wedge r = p \wedge (q \wedge r)$
   (ii) $(p \vee q) \vee r = p \vee (q \vee r)$
3. Distributive laws:
   (i) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   (ii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (q \vee r)$
4. Identity laws:
   (i) $p \wedge t \equiv p$
   (ii) $p \vee c \equiv p$
5. Negation laws:
   (i) $p \vee (\sim p) \equiv t$
   (ii) $p \wedge (\sim p) \equiv c$
6. Idempotent laws:
   (i) $p \wedge p \equiv p$
   (ii) $p \vee p \equiv p$
7. De Morgan's Laws:
   (i) $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$
   (ii) $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$
8. Universal bound laws:
   (i) $p \vee t \equiv t$
   (ii) $p \wedge c \equiv c$
9. Absorption laws:
   (i) $p \vee (p \wedge q) \equiv p$
   (ii) $p \wedge (p \vee q) \equiv p$
10. Double negative laws:
    $\sim (\sim p) \equiv p$
11. Negation of t and c
    (i) $\sim t \equiv c$
    (ii) $\sim c \equiv t$

The principle of Duality Let s and t be statements that contain no logical connectives other than $\wedge$ and $\vee$. If $s \leftrightarrow t$, then $s^d \leftrightarrow t^d$ where s^d denotes dual of s etc.

Quantifiers are the phrases like "There exists" and "for every".

SOLVED EXAMPLES

Concept-based Straight Objective Type Questions

Example 1: Which of the following is not a statement?

(a) 17 is a prime number
(b) 22 is an odd number
(c) What a beautiful flower !
(d) New Delhi is Capital of India

Ans. (c)

Solution: (*a*), (*b*) and (*d*) can be assigned values true or false:

Example 2: Let p and q be the statements:

p : Rakshit gets 100% marks in mathematics
q : Rakshit is a good dancer.
The verbal form of $\sim p \to q$ is

(a) If Rakshit does not get 100% marks in mathematics, then Rakshit is a good dancer
(b) If Rakshit gets 100% marks in mathematics then Rakshit is a good dancer
(c) Mathematics is good for dancing
(d) Rakshit is good in mathematics and dance

Ans. (a)

Solution: Verbal interpretation of $\sim p \to q$ is
If Rakshit does not get 100% marks in methematics then Rakshit is a good dancer.

Example 3: If truth value of p is T, q is F, then truth values of $(p \to q)$ and $(q \to p) \vee (\sim p)$ is are respectively

(a) F, F (b) F, T
(c) T, F (d) T, T

Ans. (b)

Solution: If $p = T$, $q = F$, then $p \to q$ is F. and $q \to p$ is T.
$\therefore$ truth value of $(q \to p) \vee (\sim p)$ is T

Example 4: If p, q, r are three statements and truth value of $p \wedge q \to r$ is F, then truth values of p, q, r are respectively

(a) T, F, T (b) T, T, F
(c) F, T, T (d) F, F, T

Ans. (b)

Solution: As $p \wedge q \to r$ has truth value F, $p \wedge q$ is T and r is F
$\therefore p = T, q = T, r = F$

Example 5: If p is any statement, then which of the following is a contradiction?

(a) $p \wedge p$ (b) $p \wedge \sim p$
(c) $p \vee (\sim p)$ (d) $(\sim p) \wedge (\sim p)$

Ans. (b)

Solution: $p \wedge (\sim p) \equiv F$ for both the truth values of p.

Example 6: Let p and q be two statements. If truth value of $p \to (\sim p \wedge q)$ is F, then truth values of p, q are respectively:

(a) F, F (b) T, F
(c) F, T (d) T, T

Ans. (b)

Solution: As $p \to (\sim p \vee q)$ has truth value F, p must be T and $\sim p \vee q$ must be F. As $\sim p$ is F, q must be F. Thus, truth values of p and q are respectively T and F.

Example 6: For integers m and n, both greater than 1, consider the following three statements

p : m divides n
q : m divides n^2
r : m is prime
Then

(a) $q \wedge r \to p$ (b) $p \wedge q \to r$
(c) $q \to r$ (d) $q \to p$

Ans. (a)

Solution: We know that if m is prime and $m|n^2$ the $nm|n$. Thus, $q \wedge r \to p$

Example 7: The statement $\sim (p \leftrightarrow \sim q)$ is

(a) equivalent to $\sim p \leftrightarrow q$ (b) a tautology
(c) *a* fallacy (d) equivalent to $p \leftrightarrow q$

Ans. (d)

Solution: $\sim (p \leftrightarrow \sim q)$

$$\equiv \sim((p \to \sim q) \wedge (\sim q \to p))$$
$$\equiv \sim((\sim p \vee \sim q) \wedge (q \vee p))$$

$\equiv \sim(\sim q \vee \sim p) \vee (\sim(q \vee p))$

$\equiv (\sim(q) \wedge \sim(\sim p)) \vee (\sim q \wedge \sim p)$

$\equiv (q \wedge p) \vee (\sim q \wedge \sim p)$

$\equiv [(q \wedge p) \vee (\sim q)] \wedge [(q \wedge p) \vee (\sim p)]$

$\equiv [(q \vee (\sim q)) \wedge (p \vee (\sim q))] \wedge [(q \vee (\sim p)) \wedge (p \vee (\sim p))]$

$\equiv [t \wedge (\sim q \vee p)] \wedge [((\sim p) \vee q)) \wedge t]$

$\equiv (\sim q \vee p) \wedge (\sim p \vee q)$

$\equiv (q \rightarrow p) \wedge (p \rightarrow q)$

$\equiv p \leftrightarrow q$

Alternative Solution. Use the following table.

p	**q**	**~q**	**p ↔ ~q**	**~(p ↔ ~q)**	**p ↔ q**
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

From the last two colums, we get $\sim(p \leftrightarrow (\sim q)) \equiv p \leftrightarrow q$

Example 8: The statement $\sim(p \wedge q) \vee q$:

(a) is a tautology
(b) is equivalent to $(p \wedge q) \vee (\sim q)$
(c) is equivalent to $p \vee q$
(d) is a contradiction

Ans. (a)

Solution: $\sim(p \wedge q) \vee q$

$\equiv ((\sim p) \vee (\sim q)) \vee q$

$\equiv (\sim p) \vee ((\sim q) \vee q)$

$\equiv (\sim p) \vee t \equiv t$

Thus, $\sim(p \wedge q) \vee q$ is a tautology.

Example 9: If p is any logical statement, then:

(a) $p \wedge (\sim p)$ is a tautology.
(b) $p \vee (\sim p)$ is a contradiction
(c) $p \wedge p \equiv p$
(d) $p \vee (\sim p) \equiv p$

Ans. (c)

Solution: Idempotent law $p \wedge p \equiv p$ is always true.

Example 10: If p, q are two statements, then $\sim(\sim p \wedge q) \wedge (p \wedge q)$ is logically equivalent to

(a) p (b) q
(c) $p \wedge q$ (d) $(\sim p) \vee q$

Ans. (a)

Solution: $\sim(\sim p \wedge q) \wedge (p \vee q)$

$\equiv ((\sim(\sim p)) \vee (\sim q)) \wedge (p \vee q)$

$\equiv (p \vee (\sim q)) \wedge (p \vee q)$

$\equiv p \vee (\sim q \wedge q) \equiv p \vee c \equiv p$

LEVEL 1

Straight Objective Type Questions

Example 11: Which of the following is not a negation of the statement p: $\sqrt{5}$ is rational?

(a) It is not the case that $\sqrt{5}$ is rational
(b) $\sqrt{5}$ is not rational
(c) $\sqrt{5}$ is an irrational number
(d) none of these

Ans. (d)

Solution: (a), (b) and (c) are negation of p.

Example 12: Negation of $\sqrt{5}$ is irrational or 3 is rational is

(a) $\sqrt{5}$ is rational or 3 is irrational
(b) $\sqrt{5}$ is rational and 3 is rational
(c) $\sqrt{5}$ is rational and 3 is irrational
(d) none of these

Ans. (c)

Solution: Use De Morgan's Law.

Example 13: Contrapositive of the statement.
If a number is divisible by 9, then it is divisible by 3, is

(a) If a number is not divisible by 3, it is not divisible by 9.
(b) If a number is not divisible by 3, it is divisible by 9.
(c) If a number is not divisible by 9, it is not divisible by 3.
(d) none of these

Ans. (a)

Solution: Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

Example 14: Converse of the statement:
If a number n is even, then n^2 is even, is

(a) If a number n^2 is even, then n is even
(b) If n is not even, then n^2 is not even

(c) Neither n nor n^2 is even
(d) none of these

Ans. (a)

Solution: Converse of $p \to q$ is $q \to p$.

Example 15: Which of the following statement is not equivalent to $p \leftrightarrow q$?

(a) p if and only if q
(b) q if and only if p
(c) p is necessary and sufficient for q
(d) none of these

Ans. (d)

Solution: Each of (a), (b), (c) is equivalent to $p \leftrightarrow q$.

Example 16: Negation of the statement

p: for every real number, either $x > 1$ or $x < 1$ is

(a) There exist a real number x such that neither $x > 1$ nor $x < 1$
(b) There exist a real number x such that $0 < x < 1$
(c) There exist a real number x such that neither $x \geq 1$ nor $x \leq 1$
(d) none of these

Ans. (a)

Solution: Negation of $p \vee q$ is $(\sim p) \wedge (\sim q)$.

Example 17: Contrapositive of

p: "If x and y are integers such that xy is odd then both x and y are odd" is

(a) If both x and y are odd, then xy is odd
(b) If both x and y are even, then xy is even
(c) If x or y is odd, then xy is odd
(d) If it is false that both x and y are odd then the product xy is odd

Ans. (d)

Solution: Contrapositive of $p \to q$ is $\sim q \to \sim p$.

Example 18: The statement $p \to (q \to p)$ is equivalent to

(a) $p \to (p \to q)$ (b) $p \to (p \vee q)$
(c) $p \to (p \wedge q)$ (d) $p \to (p \leftrightarrow q)$

Ans. (b)

Solution:
$$\begin{aligned} p \to (q \to p) &\equiv \sim p \vee (q \to p) \\ &\equiv (\sim p) \vee (\sim q \vee p) \\ &\equiv (\sim q) \vee (p \vee \sim p) \\ &\equiv (\sim q) \vee T = T \end{aligned}$$

$\therefore$ $p \to (q \to p)$ is a tautology.

Also
$$\begin{aligned} p \to (p \vee q) &\equiv \sim p \vee (p \vee q) \\ &\equiv (\sim p \vee p) \vee q \equiv T \vee q = T \end{aligned}$$

$\therefore$ $p \to (p \vee q)$ is also a tautology.

Thus, $p \to (q \to p)$ is equivalent to $p \to (p \vee q)$.

Example 19: The statement $p \to (q \vee r)$ is **not** equivalent to

(a) $(p \to q) \vee (p \to r)$
(b) $p \wedge (\sim q) \to r$
(c) $p \wedge (\sim r) \to q$
(d) $p \wedge q \to (p \wedge r) \vee (q \wedge r)$

Ans. (d)

Solution:
$$\begin{aligned} p \to (q \vee r) &\equiv (\sim p) \vee (q \vee r) \\ &\equiv (\sim p \vee q) \vee (\sim p \vee r) \\ &\equiv (p \to q) \vee (p \to r) \end{aligned}$$

Also,
$$\begin{aligned} p \to (q \vee r) &\equiv (\sim p) \vee (q \vee r) \equiv (\sim p \vee q) \vee r \\ &\equiv \sim (p \wedge (\sim q)) \vee r \\ &\equiv p \wedge (\sim q) \to r \end{aligned}$$

Interchanging the roles of q and r in the above paragraph, we find

$$p \to (q \vee r) \equiv p \wedge (\sim q) \to r \equiv p \wedge (\sim r) \to q$$

For $p = T$, $q = F$, $r = F$, $p \to (q \vee r)$ is F, but
$(p \wedge q) \to (p \vee r) \vee (q \wedge r)$ is T.

Therefore, $p \to (q \vee r)$ and

$$p \wedge q \to (p \wedge r) \vee (q \wedge r)$$

are not equivalent.

Example 20: Negation of $p \to q$ is

(a) $p \wedge (\sim q)$ (b) $\sim p \vee q$
(c) $\sim q \to \sim p$ (d) $p \vee (\sim q)$

Ans. (a)

Solution: $\sim (p \to q) \equiv \sim (\sim p \vee q) \equiv \sim (\sim p) \wedge (\sim q)$

[De Morgan's Laws]

$\therefore$ $\sim (p \to q) \equiv p \wedge (\sim q)$

Example 21: Contrapositive of $p \to (q \to r)$ is logically equivalent to

(a) $p \to (q \to r)$ (b) $(q \to r) \to \sim p$
(c) $p \vee q \to r$ (d) $(q \to r) \to p$

Ans. (a)

Solution: Contrapositive of $p \to (q \to r)$ is
$$\begin{aligned} &\sim (q \to r) \to \sim p \\ &\equiv \sim [\sim (q \to r)] \vee (\sim p) \\ &\equiv (q \to r) \vee (\sim p) \equiv (\sim p) \vee (q \to r) \\ &\equiv p \to (q \to r) \end{aligned}$$

Example 22: Let S be a non-empty subset of $\mathbf{R}$. Consider the following statement:

P : There is a rational number $x \in S$ such that $x > 0$.

Which of the following statements is the negation of the statement P ?

(a) Every rational number $x \in S$ satisfies $x \leq 0$.
(b) $x \in S$ and $x \leq 0 \Rightarrow x$ is not a rational number.

(c) There is a rational number $x \in S$ such that $x \leq 0$.

(d) There is no rational number $x \in S$ such that $x \leq 0$.

Ans. (a)

◎ **Solution:** The statement P can be written as follows :

$$P : \exists\, x \in Q \cap S \text{ such that } x > 0$$

Negation of P is $\sim P : \forall\, x \in Q \cap S$ satisfies $x \leq 0$.

◉ **Example 23:** The negation of the following statement :

P : Neha lives in Ludhiana or she lives in Gurudaspur.

(a) Neha does not live in Ludhiana or she does not live in Gurudaspur.

(b) Neha does not live in Ludhiana and she does not live in Gurudaspur

(c) Neha does not live in Punjab.

(d) None of these.

Ans. (b)

◎ **Solution:** Let p and q be the statements defined as follows :

p : Neha lives in Ludhiana

q : Neha lives in Gurudaspur.

The statement P is

$$P : p \vee q$$

Negation of p is

$$\sim P : \sim (p \vee q)$$

or $\quad \sim P : (\sim p) \wedge (\sim q) \quad [\because \sim (p \vee q) \equiv (\sim p) \wedge (\sim q)]$

Thus, negation of P is given by (b)

◉ **Example 24:** The converse of the statement "If $a < b$ then $x + a < x + b$", is

(a) If $a > b$ then $x + a > x + b$

(b) If $a \geq b$ then $x + a \geq x + b$

(c) If $x + a < x + b$ then $a < b$

(d) If $x + a \geq x + b$ then $a \geq b$

Ans. (c)

◎ **Solution:** See theory.

◉ **Example 25:** Which of the following is the conditional statement $p \rightarrow q$?

(a) p is necessary for q

(b) p is sufficient for q

(c) p only if q

(d) if q then p

Ans. (b)

◎ **Solution:** As $p \rightarrow q$, therefore truth of p is sufficient for truth of q.

◉ **Example 26:** Converse of the statement

"if x^2 is odd then x is odd" is

(a) if x^2 is even then x is even

(b) if x is odd then x^2 is odd

(c) if x is odd then x^2 is even

(d) if x is even then x^2 is odd

Ans. (b)

◎ **Solution:** See theory.

◉ **Example 27:** Consider the following statements :

P: Suman is brilliant

Q: Suman is rich

R: Suman is honest

The negation of the "Suman is brilliant and dishonest if and only if Suman is rich" can be expressed as

(a) $\sim (P \wedge \sim R) \leftrightarrow Q$ (b) $\sim p \wedge (Q \leftrightarrow \sim R)$

(c) $\sim (Q \leftrightarrow (P \wedge \sim R))$ (d) $\sim Q \leftrightarrow \sim p \wedge R$

Ans. (c)

◎ **Solution:** $P \wedge \sim R$ stands for Suman is brilliant and dishonest. Thus, $P \wedge \sim R \leftrightarrow Q$ stands for Suman is brilliant and dishonest if and only if Suman is rich.

Its negation is $\sim (P \wedge \sim R \leftrightarrow Q)$ or $\sim (Q \leftrightarrow P \wedge \sim R)$

◉ **Example 28:** The only statement among the following that is a tautology is:

(a) $A \wedge (A \vee B)$ (b) $A \vee (A \wedge B)$

(c) $[A \wedge (A \rightarrow B)] \rightarrow B$ (d) $B \rightarrow [A \wedge (A \rightarrow B)]$

Ans. (c)

◎ **Solution:** Note that

$$A \wedge (A \vee B) \text{ is } F \text{ when } A = F,$$
$$A \vee (A \wedge B) \text{ is } F \text{ when } A = F, B = F,$$
$$\text{and } B \rightarrow [A \wedge (A \rightarrow B)] \text{ is } F \text{ when } A = F, B = T$$

$\therefore$ We check only (c)

$$\begin{aligned}
& [A \wedge (A \rightarrow B)] \rightarrow B \\
\equiv\ & [A \wedge (\sim A \vee B)] \rightarrow B \\
\equiv\ & [(A \wedge (\sim A))] \vee (A \wedge B)] \rightarrow B \\
\equiv\ & A \wedge B \rightarrow B \\
\equiv\ & \sim (A \wedge B) \vee B \equiv \sim [(A \wedge B) \wedge (\sim B)] \\
\equiv\ & \sim [A \wedge (B \wedge \sim B)] \equiv \sim [A \wedge F] \equiv \sim F \equiv T
\end{aligned}$$

Thus,

$$[A \wedge (A \rightarrow B)] \rightarrow B \text{ is a tautology.}$$

◉ **Example 29:** The negation of the statement

"If I become teacher, then I will open a school."

(a) Either I will not become a teacher or I will not open a school.

(b) Neither I will become a teacher nor I will open a school

(c) I will not become a teacher or I will not open a school

(d) I will become a teacher and I will not open a school

Ans. (d)

◎ **Solution:** Let p : I become a teacher

q : I will open a school.

The given statement is $p \to q \equiv (\sim p) \vee q$

It negation is $\sim(\sim p) \vee q) \equiv p \wedge (\sim q)$

Thus, negation of the given statement is I will become a teacher and I will not open a school.

Assertion-Reason Type Questions

Example 30:

Statement-1 $\sim(p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$

Statement-2 $\sim(p \leftrightarrow \sim q)$ is a tautology

Ans. (c)

◎ **Solution:**

Table for basic logical connectives.

p	***q***	**~ *q***	***p* ↔ ~ *q***	**~ (*p* → ~*q*)**	**(*p* ↔ *q*)**
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

Note that $\sim(p \leftrightarrow \sim q)$ is not tautology.

∴ Statement-2 is false.

From table $\sim(p \leftrightarrow \sim q)$ is equivalent to $(p \leftrightarrow q)$

Thus, Statement-1 is true.

Example 31: Let p, q and r be the statements.

p : X is a rectangle

q : X is a square

$r : p \to q$

Statement-1: $p \to r$ is a tautology.

Statement-2: A tautology is equivalent to T.

Ans. (d)

◎ **Solution:** $p \to r \equiv p \to (p \to q)$

$$\equiv (\sim p) \vee (p \to q) \equiv (\sim p) \vee [(\sim p) \vee q]$$

$$\equiv [(\sim p) \vee (\sim p)] \vee q$$

$$\equiv (\sim p) \vee q \equiv p \to q \equiv r$$

Thus, Statement-1 is not a tautology.

Example 32: Let p, q, r be three statements.

Statement-1: If $p \to q$ and $q \to r$ then $p \to r$ is a tautology.

Statement-2: $(p \to q) \wedge (q \to r) \leftrightarrow (p \to r)$

Ans. (c)

◎ **Solution:** Statement-2 is not true.

It can be checked by taking

$$p = T, q = F, r = T$$

We can write Statement-1 as

$$(p \to q) \wedge (q \to r) \to (p \to r)$$

Assume it to be F then

$$p \to r \text{ is } F \text{ and } (p \to q) \wedge (q \wedge r) \text{ is } T.$$

But $p \to r$ is $F \Leftrightarrow p = T$ and $r = F$.

Now, $(T \to q) \wedge (q \to F)$ must be T

⇒ $T \to q$ is T and $q \to F$ is T

⇒ q is T and q is F.

A contradiction.

∴ $(p \to q) \wedge (q \to r) \to (p \to r)$

cannot be F for any assignment of T and F to p, q, r.

∴ $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.

We may abbreviate the above procedure as follows:

1. $$\begin{array}{ccccccccccc} (p & \to & q) & \wedge & (q & \to & r) & \to & (p & \to & r) \\ & & & & & & & \text{F} & & & \end{array}$$
2. $$\begin{array}{ccccccccccc} (p & \to & q) & \wedge & (q & \to & r) & \to & (p & \to & r) \\ & & & \text{T} & & & & \text{F} & & \text{F} & \end{array}$$
3. $$\begin{array}{ccccccccccc} (p & \to & q) & \wedge & (q & \to & r) & \to & (p & \to & r) \\ \text{T} & \text{T} & & \text{T} & & \text{T} & \text{F} & \text{F} & \text{T} & \text{F} & \text{F} \end{array}$$
4. $$\begin{array}{ccccccccccc} (p & \to & q) & \wedge & (q & \to & r) & \to & (p & \to & r) \\ \text{T} & \text{T} & \text{T} & \text{T} & \text{F} & \text{T} & \text{F} & \text{F} & \text{T} & \text{F} & \text{F} \\ & & & \uparrow & & \uparrow & & & & & \end{array}$$

This assignment is not possible.

Example 33: Let p, q, r and s be three statements.

Statement-1: $(p \to r) \wedge (\sim p \to q) \wedge (q \to s) \wedge (\sim r \to s)$ is a tautology.

Statement-2: $(p \to q) \wedge (\sim p \to q) \to q$ is a tautology.

Ans. (b)

◎ **Solution:** Assume Statement-1 is false. Then

1. $$\begin{array}{ccccccccccccccccc} (p & \to & r) & \wedge & (\sim & p & \to & q) & \wedge & (q & \to & s) & \to & (\sim & r & \to & s) \\ & & & \text{T} & & & & & \text{T} & & & & \text{F} & & & \text{F} & \end{array}$$
2. $$\begin{array}{ccccccccccccccccc} (p & \to & r) & \wedge & (\sim & p & \to & q) & \wedge & (q & \to & s) & \to & (\sim & r & \to & s) \\ & \text{T} & & \text{T} & & & \text{T} & & \text{T} & & \text{T} & & \text{F} & \text{T} & \text{F} & \text{F} & \text{F} \end{array}$$
3. $$\begin{array}{ccccccccccccccccc} (p & \to & r) & \wedge & (\sim & p & \to & q) & \wedge & (q & \to & s) & \to & (\sim & r & \to & s) \\ \text{F} & \text{T} & \text{F} & \text{T} & \text{T} & \text{F} & \text{T} & \text{F} & \text{T} & \text{F} & \text{T} & \text{F} & \text{F} & \text{T} & \text{F} & \text{F} & \text{F} \end{array}$$

Look at the encircled part. This means $\sim p \to q$ is both T and F.

A contradiction.

∴ Statement-1 is always true.

Next assume Statement-2 is false.

Then

$$\begin{array}{c}(p \to q) \wedge (\sim p \to q) \to q \\ \text{F T F T TF T T F F}\end{array}$$

Not possible

∴ Statement-2 is a tautology.

Example 34: Statement-1: $\sim (A \Leftrightarrow \sim B)$ is equivalent to $A \Leftrightarrow B$

Statement-2: $A \vee (\sim(A \wedge \sim B))$ is a tautology.

Ans. (b)

Solution: We have

$$\begin{aligned} & A \vee [\sim (A \wedge \sim B)] \\ &\equiv A \vee [\sim A \vee B] \\ &\equiv (A \vee \sim A) \vee B \equiv T \vee B \equiv T \end{aligned}$$

Thus, Statement-2 is true.

We have

$$\begin{aligned} &\equiv \sim [A \Leftrightarrow \sim B] \\ &\equiv \sim [(A \to \sim B) \wedge (\sim B \to A)] \\ &\equiv \sim [(\sim A \vee \sim B) \wedge (B \vee A)] \\ &\equiv \sim [\sim(A \wedge B) \wedge (A \vee B)] \\ &\equiv [(A \wedge B) \vee (\sim A \wedge \sim B)] \\ &\equiv [(A \wedge B) \vee (\sim A)] \wedge [(A \wedge B) \vee (\sim B)] \\ &\equiv [(A \vee \sim A) \vee (B \vee \sim A)] \wedge [(A \vee \sim B) \wedge (B \vee \sim B)] \\ &\equiv [T \wedge (\sim A \vee B)] \wedge [(A \vee \sim B) \wedge T] \\ &\equiv (A \to B) \wedge (B \to A) \equiv A \Leftrightarrow B \end{aligned}$$

Example 35:

Statement-1: Consider the following statements

p : Delhi is in India

q : Mumbai is not in Italy

Then negation of $p \vee q$ is Delhi is not in India and Mumbai is in Italy.

Statement-2: For two statements p and q

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

Ans. (a)

Solution: Statement-2 is true. [See Theory]

As $\sim (p \vee q) \equiv \sim p \wedge \sim q$

≡ ~ (Delhi is in India) and ~(Mumbai is not is Italy)

≡ Delhi is not in India and Mumbai is in Italy.

LEVEL 2

Straight Objective Type Questions

Example 36: Let p, q and r be the statements:

p: city X is in U.P.

q: city X is in India

$r : p \to q$

Contrapositive of r is

(a) if city X is not in India then X is not in U.P.
(b) city X is neither in U.P. nor in India
(c) city X is in India but not in U.P.
(d) none of these

Ans. (a)

Solution: Contrapositive of $p \to q$ is $\sim q \to \sim p$.

Example 37: If p, q are r are as in Example 31, then which one of the following represents converse of $p \to q$.

(a) If X is a rectangle then X is a square.
(b) If X is a rectangle then X is not a square
(c) X is a rectangle but X is not a square
(d) none of these

Ans. (a)

Solution: Converse of $p \to q$ is $q \to p$.

Example 38: Let p, q, r be three statements, then

$(p \to (q \to r)) \leftrightarrow ((p \wedge q) \to r)$, is a

(a) tautology (b) contradiction
(c) fallacy (d) none of these

Ans. (a)

Solution:

$$\begin{aligned} p \to (q \to r) &\equiv \sim p \vee (q \to r) \\ &\equiv \sim p \vee (\sim q \vee r) \\ &\equiv [(\sim p) \vee (\sim q)] \vee r \\ &\equiv \sim (p \wedge q) \vee r \\ &\equiv p \wedge q \to r \end{aligned}$$

∴ given statement is a tautology.

Example 39: Let p and q be two statements, then $\sim (\sim p \wedge q) \wedge (p \vee q)$ is logically equivalent to

(a) q (b) $p \wedge q$
(c) p (d) $p \vee \sim q$

Ans. (c)

Solution:

$$\begin{aligned} \sim (\sim p \wedge q) \wedge (p \vee q) &\equiv [\sim (\sim p) \vee (\sim q)] \wedge (p \vee q) \\ &\equiv [p \vee (\sim q) \wedge (p \vee q] \\ &\equiv p \vee [(\sim q) \wedge q] \\ &\equiv p \vee F \equiv p \end{aligned}$$

EXERCISES

Concept-based Straight Objective Type Questions

1. Which of the following pairs are **not** logically equivalent?
   (a) $p \vee (p \wedge q)$ and p
   (b) $p \vee t$ and p
   (c) $\sim(p \vee q)$ and $(\sim p) \wedge (\sim q)$
   (d) $p \vee c$ and p
2. The statement $(p \wedge q) \vee (\sim p \vee (p \wedge(\sim q)))$ is logically equivalent to
   (a) $p \wedge q$
   (b) p
   (c) q
   (d) t
3. If $(\sim p) \vee q \rightarrow \sim q$ has value F, then p, q are respectively
   (a) F, F
   (b) T, F
   (c) p is T or F, F
   (d) none of these
4. The statement $p \wedge \sim q \rightarrow r$ is logically equivalent to
   (a) $\sim p \vee q \vee r$
   (b) $(\sim p \wedge q) \vee r$
   (c) $(p \wedge \sim q) \vee r$
   (d) $(\sim p \vee q) \wedge r$
5. The statement $(p \vee \sim q) \wedge (\sim p \vee \sim q)$ is logically equivalent to
   (a) p
   (b) q
   (c) $\sim p$
   (d) $\sim q$
6. Let p, q, r be the following:
   p : Rohit is healthy
   q : Rohit is wealthy
   r : Rohit is wise.

   The statement $p \vee q \rightarrow \sim r$ means
   (a) If Rohit is healthy or wealthy then Rohit is not wise
   (b) If Rohit is healthy and wealthy then Rohit is wise
   (c) Rohit is neither healthy nor wealthy nor wise
   (d) none of these
7. Let p, q, r be the following three statements:
   p : n is prime
   q : n is odd
   r : n is 2
   Then $p \rightarrow q \vee r$ means
   (a) If n is prime then n is odd or 2
   (b) If n is prime and odd then n cannot be 2
   (c) If n is odd or 2 then n is prime
   (d) If n is odd then n is prime or 2
8. The contrapositive of the statement "I go to college if its not a holiday"
   (a) If I do not go to the college then its a holiday
   (b) If I go the college then its not a holiday
   (c) I go the college and its not a holiday
   (d) If I go to college then its a holiday
9. Suppose p and q are two statements such that $p \rightarrow q$ is false, then which one of the following **not** true?
   (a) Truth value of $\sim p \vee q$ is F
   (b) Truth values of $p \wedge (\sim q)$ is T
   (c) Truth values of $(\sim p) \wedge (\sim q)$ is T
   (d) Truth values of $p \vee q$ is T
10. Which of the following is equivalent to $p \rightarrow q \vee r$?
   (a) $p \wedge (\sim q) \rightarrow r$
   (b) $p \vee (\sim r) \rightarrow q$
   (c) $p \wedge (\sim r) \rightarrow \sim q$
   (d) $p \vee (\sim r) \rightarrow q$

LEVEL 1

Straight Objective Type Questions

11. Let p be the proposition: Mathematics is interesting and q be mathematics is difficult, then $p \wedge q$ means
   (a) Mathematics is interesting or difficult
   (b) Mathematics is interesting and difficult
   (c) Mathematics is interesting implies it is difficult
   (d) Mathematics is interesting is equivalent to saying that it is difficult
12. If $p \rightarrow (\sim p \vee q)$ is false, the truth value of p and q are respectively
   (a) T, T
   (b) T, F
   (c) F, T
   (d) F, F
13. The contrapositive of $p \rightarrow q$ is
   (a) $q \rightarrow p$
   (b) $\sim p \rightarrow \sim q$
   (c) $\sim q \rightarrow \sim p$
   (d) $p \rightarrow q$
14. The contrapositive of $(p \vee q) \rightarrow r$ is
   (a) $r \rightarrow (p \vee q)$
   (b) $\sim r \rightarrow (\sim p) \wedge (\sim q)$
   (c) $(\sim p) \vee (\sim q) \rightarrow \sim r$
   (d) none of these

15. $\sim p \wedge q$ is logically equivalent to

(a) $p \to q$ (b) $q \to p$
(c) $\sim (p \to q)$ (d) $\sim (q \to p)$

16. Negation of $q \vee \sim (p \wedge r)$ is

(a) $\sim q \vee \sim (p \vee r)$ (b) $\sim q \wedge (p \wedge r)$
(c) $\sim q \wedge (p \wedge r)$ (d) $\sim q \vee (p \wedge r)$

17. Negation of the statement "If a number is prime then it odd", is

(a) A number is not prime but odd
(b) A number is prime but it is not odd
(c) A number is neither prime nor odd
(d) none of these

18. If p, q are two propositions, then $\sim (p \vee q) \equiv \sim p \wedge \sim q$, is

(a) a tautology (b) a contradiction
(c) a simple statement (d) none of these

19. Which one of the following is a tautology?

(a) $(p \to q) \wedge p \to q$ (b) $(p \to q) \vee p \to q$
(c) $(p \to q) \vee p \to q$ (d) $(p \to q) \wedge (\sim q) \to p$

20. Which of the following is not equivalent to $\sim p \wedge q$?

(a) $\sim (q \to p)$ (b) $\sim (p \vee \sim q)$
(c) $\sim p \to \sim q$ (d) $\sim (p \vee q)$

21. Which of the following is equivalent to $p \leftrightarrow q$?

(a) $(\sim p \vee q) \vee (p \vee q)$ (b) $(p \wedge q) \vee (\sim p \wedge \sim q)$
(c) $(p \vee q) \wedge (p \vee \sim q)$ (d) $(p \wedge q) \vee (p \vee q)$

Assertion-Reason Type Questions

22. Let p, q, r be three statements.

Statement-1: $p \leftrightarrow q \equiv (p \to q) \wedge (\sim q \vee p)$ is a tautology.

Statement-2: $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$ is a tautology.

23. Let p and q be two statements.

Statement-1: $(p \vee q) \vee \sim (\sim p \wedge q)$ is logically equivalent to p.

Statement-2: $p \vee T \equiv p$

24. Let p, q and r be three statements.

Statement-1: $[p \vee (q \wedge r)] \vee [\sim (p \vee q)] \equiv p \vee q$ is a tautology.

Statement-2: $[p \vee q \to r] \leftrightarrow [\sim r \to (\sim p) \wedge (\sim q)]$ is a tautology.

25. Let p, q and r be three statements.

Statement-1: Negation of $p \wedge (q \vee r)$ is $\sim p \vee (\sim q \wedge \sim r)$

Statement-2: Negation of $p \vee q$ is $(\sim p) \wedge (\sim q)$; and that of $p \wedge q$ is $(\sim p) \vee (\sim q)$

LEVEL 2

Straight Objective Type Questions

26. Let p and q be two statements, then $q \leftrightarrow (\sim p \vee \sim q)$ is logically equivalent to

(a) p (b) q
(c) $p \to q$ (d) $\sim p \wedge q$

27. Dual of $(p \to q) \to r$ is

(a) $(q \to p) \wedge r$ (b) $p \to (q \to r)$
(c) $(p \vee \sim q) \vee r$ (d) none of these.

28. Statement $(p \vee q) \to (p \wedge q)$ is equivalent

(a) F (b) $p \leftrightarrow q$
(c) T (d) $q \to p \wedge q$

Previous Years' AIEEE/JEE Main Questions

1. The statement $p \to (q \to p)$ is equivalent to

(a) $p \to (q \leftrightarrow p)$ (b) $p \to (p \to q)$
(c) $p \to (p \vee q)$ (d) $p \to (p \wedge q)$ **[2008]**

2. Let p be the statement "x is an irrational number", q be the statement "y is a transcendental number", and r be the statement "x is a rational number iff y is a transcendental number".

Statement-1: r is equivalent to either q or p.

Statement-2: r is equivalent to $\sim (p \leftrightarrow \sim q)$.

[2008]

3. **Statement-1:** $\sim (p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$.

Statement -2: $\sim (p \leftrightarrow \sim q)$ is tautology.

[2009]

4. Let S be a non-empty subset of **R.** Consider the following statement :

P : There is a rational number $x \in S$ such that $x > 0$.

Which of the following statements is the negation of the statement P ?

(a) Every rational number $x \in S$ satisfies $x \le 0$.
(b) $x \in S$ and $x \le 0 \Rightarrow x$ is not rational.
(c) There is a rational number $x \in S$ such that $x \le 0$.
(d) There is no rational number x such that $x \le 0$.

[2010]

5. Consider the following statements

P : Suman is brilliant
Q : Suman is rich
R : Suman is honest

The negation of the statement "Suman is brilliant and dishonest if and only if Suman is rich" can be expressed as

(a) $\sim (P \wedge \sim R) \leftrightarrow Q$
(b) $\sim P \wedge (Q \wedge \sim R)$
(c) $\sim (Q \leftrightarrow (P \wedge \sim R))$
(d) $\sim Q \leftrightarrow \sim P \wedge R$ **[2011]**

6. The only statement among the followings that is a tautology is

(a) $A \wedge (A \vee \mathrm{B})$
(b) $A \vee (A \wedge B)$
(c) $[A \wedge (A \to B)] \to B$
(d) $B \to [A \wedge (A \to B)]$ **[2011]**

7. The negation of the statement

"If I become a teacher, then I will open a school", is

(a) Either I will not become a teacher or I will not open a school.
(b) Neither I will become a teacher nor I will open a school.
(c) I will not become a teacher or I will open a school.
(d) I will become a teacher and I will not open a school.

[2012]

8. Consider:

Statement-1: $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is a fallacy

Statement-2: $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology.

[2013]

9. The statement $p \to (q \to p)$ is equivalent to

(a) $p \to q$ (b) $p \to (p \vee q)$
(c) $p \to (p \to q)$ (d) $p \to (p \wedge q)$

[2013, online]

10. Let p and q be any two logical statements and r : $p \to (\sim p \vee q)$.

If r has a truth value F, the truth values of p and q are respectively

(a) F, F (b) T, T
(c) T, F (d) F, T **[2013, online]**

11. **Statement-1:** The statement $A \to (B \to A)$ is equivalent to $A \to (A \vee B)$.

Statement-2: The statement $\sim[(A \wedge B) \to (\sim A \vee B)]$ is a tautology. **[2013, online]**

12. For integers m and n, both greater than 1, consider the following three statements:

P : m divides n

Q : m divides n^2

R : m is prime

then

(a) $Q \wedge R \to P$ (b) $P \wedge Q \to R$
(c) $Q \to R$ (d) $Q \to P$ **[2013, online]**

13. The statement $\sim(p \leftrightarrow \sim q)$ is

(a) equivalent to $\sim p \leftrightarrow q$ (b) a tautology
(c) a fallacy (d) equivalent to $p \leftrightarrow q$

[2014]

14. The contrapositive of the statement " if I am not feeling well, then I will go to the doctor" is

(a) If I am feeling well, then I will not go to the doctor
(b) If I will go to the doctor, then I am feeling well
(c) If I will not go to the doctor, then I am feeling well
(d) If I will go to the doctor, then I am not feeling well **[2014, online]**

15. The proposition $\sim(p \vee \sim q) \vee \sim(p \vee q)$ is logically equivalent to

(a) p (b) q
(c) $\sim p$ (d) $\sim q$ **[2014, online]**

16. Let p, q, r denote three arbitrary statements. The logically equivalent of the statement $p \rightarrow (q \vee r)$ is

(a) $(p \rightarrow \sim q) \wedge (p \rightarrow r)$ (b) $(p \rightarrow q) \vee (p \rightarrow r)$
(c) $(p \rightarrow q) \wedge (p \rightarrow \sim r)$ (d) $p \vee q \rightarrow \rho$ **[2014, online]**

17. The contrapositive of the statement “I go to school if it does not rain” is

(a) if it rains, I do not go to school
(b) if I do not go to school, it rains
(c) if it rains, I go to school
(d) if I go to school, it rains **[2014, online]**

18. The negation of $\sim s \vee (\sim r \wedge s)$ is equivalent to:

(a) $s \wedge \sim r$ (b) $s \wedge (r \wedge \sim)$
(c) $s \vee (r \vee \sim s)$ (d) $s \wedge r$ **[2015]**

19. Contrapositive of the statement “If it is raining then I will not come”, is

(a) if I will come, then it is not raining
(b) if I will not come, then it is raining
(c) if I will not come, then it is not raining
(d) if I will come, then it is raining **[2015, online]**

20. Consider the following statements:

P : Suman is brilliant

O : Suman is rich

R : Suman is honest

The negation of the statement,

"Suman is brilliant and dishonest if and only if Suman is rich" can be equivalently expressed as:

(a) $\sim Q \leftrightarrow P \wedge R$ (b) $\sim Q \leftrightarrow \sim P \vee R$
(c) $\sim Q \leftrightarrow P \vee \sim R$ (d) $\sim Q \leftrightarrow P \wedge \sim R$ **[2015, online]**

21. The Boolean Expression $(p \wedge -q) \vee q \vee (\sim p \wedge q)$ is equivalent to

(a) $\sim p \wedge q$ (b) $p \wedge q$
(c) $p \vee q$ (d) $p \vee \sim q$ **[2016]**

22. Consider the following two statements:

P: If 7 is an odd number, then 7 is divisible by 2

Q: If 7 is a prime number, then 7 is an odd number

If V_1 is the truth value of the contrapositive of P and V_2 is the truth value of contrapositive of Q, then the ordered pair (V_1, V_2) equals:

(a) (F, F) (b) (F, T)
(c) (T, F) (d) (T, T) **[2016, online]**

23. The contrapositive of the following statement,

"If the side of a square doubles, then its area increases four times", is

(a) if the area of a square increases four times, then its side is not doubled
(b) if the area of a square increases four times, then its side is doubled
(c) if the area of a square does not increase four times, then its side is not doubled
(d) if the side of a square is not doubled, then its area does not increase four times **[2016, online]**

Previous Years' B-Architecture Entrance Examination Questions

1. The statement $\sim(p \wedge q) \vee q$:

(a) is a tautology
(b) is equivalent to $(p \wedge q) \vee \sim q$
(c) is equivalent to $p \vee q$
(d) is a contradiction **[2009]**

2. The contrapositive of the statement, "If x is a prime number and x divides ab then x divides a or x divides b", can be symbolically represented using logical connectives, on appropriately defined statements p, q, r, s, as

(a) $(\sim r \vee \sim s) \rightarrow (\sim p \wedge \sim q)$
(b) $(r \wedge s) \rightarrow (\sim p \wedge \sim q)$
(c) $(\sim r \wedge \sim s) \rightarrow (\sim p \vee \sim q)$
(d) $(r \vee s) \rightarrow (\sim p \vee \sim q)$ **[2010]**

3. **Statement-1:**

$\sim (A \Leftrightarrow \sim B)$ is equivalent to $A \Leftrightarrow B$.

Statement-2:

$A \vee (\sim(A \wedge \sim B))$ a tautology. **[2011]**

4. **Statement-1:** Consider the statements

p : Delhi is in India

q : Mumbai is not in Italy

Then the negation of the statement $p \vee q$, is Delhi is not in India and Mumbai is in Italy.

Statement-2: For any two statements p and q
$\sim(p \vee q) = \sim p \vee \sim q$ **[2012]**

5. If p is any logical statement, then
   (a) $p \wedge (\sim p)$ is a tautology
   (b) $p \vee (\sim p)$ is a contradiction
   (c) $p \wedge p = p$ (d) $p \vee (\sim p) = p$ **[2013]**
6. Let p and q be any two propositions.
   Statement-1: $(p \to q) \leftrightarrow q \vee \sim p$ is a tautology.
   Statement-2: $\sim(\sim p \wedge q) \wedge (p \vee q) \leftrightarrow p$ is a fallacy.
   (a) Both statements 1 and statements 2 are true
   (b) Both statements 1 and statement 2 are false
   (c) Statement 1 is true and statement 2 is false
   (d) Statement 1 is false and statement 2 is true **[2014]**
7. The statement
   $[p \wedge (p \to q)] \to q$, is
   (a) a fallacy
   (b) a tautology
   (c) neither a fallacy nor a tautology
   (d) not a compound statement **[2015]**
8. The negation of $A \to (A \vee \sim B)$ is
   (a) a tautology
   (b) equivalent to $(A \vee \sim B) \to A$
   (c) equivalent to $A \to (A \wedge \sim B)$
   (d) a fallacy **[2016]**

Answers

Concept-based

1. (b) **2.** (d) **3.** (c) **4.** (a)
5. (d) **6.** (a) **7.** (a) **8.** (a)
9. (c) **10.** (a)

Level 1

11. (b) **12.** (b) **13.** (c) **14.** (b)
15. (d) **16.** (b) **17.** (b) **18.** (a)
19. (a) **20.** (d) **21.** (b) **22.** (b)
23. (c) **24.** (d) **25.** (a)

Level 2

26. (d) **27.** (a) **28.** (b)

Previous Years' AIEEE/JEE Main Questions

1. (c) **2.** None of the answer matches **3.** (c)
4. (a) **5.** (c) **6.** (c) **7.** (d)
8. (b) **9.** (b) **10.** (c) **11.** (c)
12. (a) **13.** (d) **14.** (c) **15.** (c)
16. (b) **17.** (b) **18.** (d) **19.** (a)
20. (b) **21.** (c) **22.** (b) **23.** (a)

Previous Years' B-Architecture Entrance Examination Questions

1. (a) **2.** (c) **3.** (b) **4.** (c)
5. (c) **6.** (c) **7.** (b) **8.** (d)

Hints and Solutions

Concept-based

1. $p \vee t \equiv t$ not p. [see Results on Logical equivalences]
2. $(p \wedge q) \vee (\sim p \vee (p \wedge (\sim q)))$
   $$\equiv (p \wedge q) \vee ((\sim p \vee p) \wedge (\sim p \vee \sim q))$$
   $$\equiv (p \wedge q) \vee (t \wedge \sim(p \wedge q))$$
   $$\equiv (p \wedge q) \vee (\sim(p \wedge q)) \equiv t$$
3. $(\sim p \vee q) \to \sim q$ is F if $\sim p \vee q$ is T but $\sim q$ is F or q is T. But then $\sim p \vee q$ is T irrespective of value of p.
4. Use $p \to q$ is logically equivalent to $\sim p \vee q$.
5. $(p \vee \sim q) \wedge (\sim p \vee \sim q)$
   $$\equiv (p \wedge (\sim p)) \vee (\sim q) \equiv c \vee (\sim q) \equiv \sim q$$
6. Use $p \vee q$ mean Rohit is healthy or wealthy
7. If n is prime then n is odd or 2.
8. Let p : It is a holiday
   q : I go the college.
   The given statement is
   $\sim p \to q$
   Its contrapositive is $\sim q \to p$.
9. $p \to q$ is false mean p is T and q is F.
   $\sim p \vee q = (F) \vee (F) = F$, $p \wedge (\sim q) = T$
   $(\sim p) \wedge (\sim q) = F$, $p \vee q = F$
   $\therefore$ (c) is incorrect.
10. $p \to q \vee r \equiv \sim p \vee (q \vee r) \equiv (\sim p \vee q) \vee r$
    $$\equiv p \wedge (\sim q) \to r$$

Level 1

11. Mathematics is interesting and difficult
12. $p \to (\sim p \vee q)$ is F if $p = T$ and $\sim p \vee q$ $= F$ or $F \vee q = F$ or if $q = F$.

13. Contrapositive of $p \to q$ is $\sim q \to \sim p$.

14. Contrapositive of $p \vee q \to r$ is $\sim r \to \sim (p \vee q)$ i.e. $\sim r \to (\sim p) \wedge (\sim q)$

15. See Theory

16. $\sim [q \vee \sim (p \wedge r)] \equiv \sim q \wedge (p \wedge r)$

17. Negation of $p \to q = \sim p \vee q$ is $\sim (\sim p \vee q) = p \wedge \sim q$, that is, a number is prime and but it is not odd.

18. It is De Morgan's law.

19. $(p \to q) \wedge p \to q$.
F T F T T F F
↑—————↑
Contradiction.

Thus, $(p \to q) \wedge p \to q$ can never take value F.

20. $\sim p \wedge q \equiv \sim (p \vee \sim q)$ by De Morgan's Law.

21. Use $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$
$\equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

22. $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$\equiv (p \to q) \wedge (\sim q \vee p)$

and $p \vee q \to r \equiv \sim (p \vee q) \vee r$
$\equiv (\sim p \wedge \sim q) \vee r$
$\equiv (\sim p \vee r) \wedge (\sim q \vee r)$
$\equiv (p \to r) \wedge (q \to r)$

23. $(p \vee q) \wedge \sim (\sim p \wedge q)$
$\equiv (p \vee q) \wedge [\sim (\sim p) \vee (\sim q)]$
$\equiv (p \vee q) \wedge [p \vee (\sim q)]$
$\equiv p \vee [q \wedge (\sim q)] \equiv p \vee F \equiv p$

Note that $p \vee T = T$ not p.

24. $[p \vee (q \wedge r)] \vee [\sim (p \vee q)]$
$= [(p \vee q) \wedge (p \vee r)] \vee [\sim (p \vee q)]$
$= [\sim (p \vee q)] \vee [(p \vee q) \wedge (p \vee r)]$
$= T \wedge (p \vee r) = p \vee r$

Next, $\sim r \to (\sim p) \wedge (\sim q)$
$\equiv \sim r \to \sim (p \vee q)$
$\equiv p \vee q \to r$.

25. $\sim [p \wedge (q \vee r)]$
$\equiv \sim p \vee [\sim (q \vee r)]$
$\equiv \sim p \vee (\sim q \wedge \sim r)$

Level 2

26. $q \leftrightarrow (\sim p \vee \sim q)$
$\equiv q \leftrightarrow \sim (p \wedge q)$
$\equiv [q \to \sim (p \wedge q)] \wedge [\sim (p \wedge q) \to q]$
$\equiv [\sim q \vee \sim (p \wedge q)] \wedge [(p \wedge q) \vee q]$
$\equiv \sim [q \wedge (p \wedge q)] \wedge q$
$\equiv \sim (p \wedge q) \wedge q$
$\equiv (\sim p \vee \sim q) \wedge q$
$\equiv (\sim p \wedge q) \vee (\sim q \wedge q)$
$\equiv (\sim p \wedge q)$

27. $(p \to q) \to r$
$\equiv (\sim p \vee q) \to r$
$\equiv \sim (\sim p \vee q) \vee r$
$\equiv [p \wedge (\sim q)] \vee r$.

Dual of $(p \to q) \to r$ is

$[p \vee (\sim q)] \wedge r \equiv (q \to p) \wedge r$

28. $(p \vee q) \to (p \wedge q)$
$\equiv \sim (p \vee q) \vee (p \wedge q)$
$\equiv (\sim p \wedge \sim q) \vee (p \wedge q)$
$\equiv [\sim p \vee (p \wedge q)] \wedge [\sim q \vee (p \wedge q)]$
$\equiv [\sim p \vee q] \wedge [\sim q \vee p] \equiv (p \to q) \wedge (q \to p)$
$\equiv p \leftrightarrow q$

Previous Years' AIEEE/JEE Main Questions

1. $p \to (q \to p) \equiv \sim p \vee (q \to p)$
$\equiv (\sim p) \vee (\sim q \vee p)$
$\equiv (\sim q) \vee (p \vee \sim p)$
$\equiv (\sim q) \vee T = T$

$\therefore$ $p \to (q \to p)$ is a tautology.

Also $p \to (p \vee q) \equiv \sim p \vee (p \vee q)$
$\equiv (\sim p \vee q) \vee q \equiv T \vee q = T$

$\therefore$ $p \to (p \vee q)$ is also an tautology.

Thus, $p \to (q \to p)$ is equivalent to $p \to (p \vee q)$.

2. Note that the statement r is $\sim p \leftrightarrow q$.

Now, $\sim p \leftrightarrow q \equiv (\sim p \to q) \wedge (q \to \sim p)$
$\equiv [\sim(\sim p) \vee q \wedge (\sim q \vee \sim p)]$
$\equiv (p \vee q) \wedge [\sim(p \wedge q)]$
$\neq p \vee q$

Next, $p \leftrightarrow \sim q \equiv \sim q \leftrightarrow p$
$\equiv (p \vee q) \wedge [\sim(p \wedge q)]$

$\Rightarrow \sim(p \leftrightarrow \sim q) \equiv \sim(p \vee q) \vee (p \wedge q)$

Thus, neither Statement-1 nor Statement-2 is true.

3.

p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$	$p \leftrightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

Note that $\sim(p \leftrightarrow \sim q)$ is not a tautology.

$\therefore$ Statement-2 is false.

From table $\sim(p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$.

Thus, Statement-1 is true.

4. Negation of P is

"for each rational number $x \in S$, $x \leq 0$".

5. $P \wedge \sim R$ stands for

Suman is brilliant and dishonest.

Thus $P \wedge \sim R \leftrightarrow Q$ stands for Suman is brilliant and dishonest if and only if Suman is rich.

Its negation is $\sim(P \wedge \sim R \leftrightarrow Q)$ or $\sim(Q \leftrightarrow P \wedge \sim R)$

6. $A \wedge (A \vee B)$ is F when $A = F$

$A \vee (A \wedge B)$ is F when $A = F$, $B = F$

We have

$[A \wedge (A \to B)] \to B$

$\equiv [A \wedge (\sim A \vee B)] \to B$

$\equiv [(A \wedge (\sim A)) \vee (A \wedge B)] \to B$

$\equiv A \wedge B \to B$

$\equiv \sim (A \wedge B) \vee B$

$\equiv [(\sim A) \vee (\sim B)] \vee B$

$\equiv (\sim A) \vee [(\sim B) \vee B]$

$\equiv (\sim A) \vee T \equiv T$

$\therefore$ $[A \wedge (A \to B)] \to B$ is a tautology.

7. Let p : I become a teacher

q : I will open a school.

The given statement is $p \to q \equiv (\sim p) \vee q$

Its negation is $\sim ((\sim p) \vee q) \equiv p \wedge (\sim q)$

Thus, negation of the given Statement is 1 will become a teacher and I will not open a school.

8. As $\sim q \to \sim p \equiv p \to q$, Statement 2 can be written as $(p \to q) \leftrightarrow (p \to q)$

Thus, Statement 2 is a tautology.

Also, $(p \wedge \sim q) \wedge (\sim p \wedge q)$

$\equiv (p \wedge \sim p) \wedge (\sim q \wedge q) = F \wedge F \equiv F$, which is a fallacy. However Statement 2 is not a correct reason for Statement 1.

9. See Solution to Question 1.

10. Statement r is

$p \to (\sim p \vee q)$

If r is false, then

p must be T

and $\sim p \vee q$ must be F

$\Rightarrow p$ is T and $F \vee q$ is F

$\Rightarrow p$ is T and q is F.

11. $A \wedge B \to (\sim A \vee B)$

$\equiv \sim (A \wedge B) \vee (\sim A \vee B)$

$\equiv (\sim A \vee \sim B) \vee (\sim A \vee B)$

$\equiv (\sim A) \vee (\sim B \vee B) \equiv (\sim A) \vee T \equiv \sim A$

$\Rightarrow \sim[(A \wedge B) \to (\sim A \vee B)] \equiv A$

$\therefore$ Statement-2 is false.

Next, $A \to (B \to A)$

$\equiv A \to (\sim B \vee A)$

$\equiv \sim A \vee (\sim B \vee A)$

$\equiv (\sim B) \vee (\sim A \vee A)$

$\equiv (\sim B) \vee T \equiv T$

$\equiv (\sim \mathrm{A} \vee \mathrm{A}) \vee \mathrm{B}$

$\equiv \sim A \vee (A \vee B)$

$\equiv A \to A \vee B$

Thus, Statement-1 is true.

12. $Q \wedge R \to P$

In words it means

m is prime and $m|n^2$

$\Rightarrow m|n$.

13. Sec Solution to Questions 3.

14. Let p, q be statements

p: I am not feeling well

q: I will go to the doctor

Contrapositive of $p \to q$ is $\sim q \to \sim p$

i.e. if I do not go to the docor then I am feeling well.

15. $\sim(p \vee \sim q) \vee \sim(p \vee q)$

$= \sim[(p \vee \sim q) \wedge (p \vee q)]$

$= \sim[p \vee (\sim q \wedge q)]$

$= \sim[p \vee F] = \sim p$

16. $p \to q \vee r$

$\equiv \sim p \vee (q \vee r)$

$\equiv (\sim p \vee q) \vee (\sim p \vee r)$

$\equiv (p \to q) \vee (p \to r)$

17. Let p and q be the statements

p : It does not rain

q : I go to school

The given statement is $p \to q$.

Its contrapositive is $\sim q \to \sim p$ i.e., if I do not go to school it rains.

18. $\sim[\sim s \vee (\sim r \wedge s)]$

$\equiv \sim(\sim s) \wedge \sim(\sim r \wedge s)$

$\equiv s \wedge (r \vee \sim s)$

$\equiv (s \wedge r) \vee (s \wedge \sim s)$

$\equiv (s \wedge r) \vee F \equiv s \wedge r$

19. Let p and q be the statements:

p: It is raining

q: I will not come.

Contrapositive of $p \to q$ is

$\sim q \to \sim p$

that is, if I (will) come then it is not raining.

20. Suman is brilliant and dishonest if and only if Suman is rich, can be expressed as

    $P \wedge (\sim R) \leftrightarrow Q$

    Its negation is

    $\sim Q \leftrightarrow \sim P \vee R$

21. $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$

    $= (p \wedge \sim q) \vee [q \vee (\sim p \wedge q)]$

    $= (p \wedge \sim q) \vee q$ [absorption law]

    $= (p \vee q) \wedge (\sim q \vee q)$

    $= (p \vee q) \wedge T = p \vee q$

22. As a statement and its counter positive have the same truth values, truth values of (V_1, V_2)

    = truth values of (P, Q)

    $= (F, T)$

23. Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$. Thus, contrapositive of given statement is (c), that is, if the area of a square does not increase four times, then its side is not doubled.

Previous Years' B-Architecture Entrance Examination Questions

1. $\sim(p \wedge q) \vee q$

    $\equiv (\sim p \vee \sim q) \vee q$

    $\equiv \sim p \vee (\sim q \vee q)$

    $\equiv \sim p \vee T \equiv T$

2. Given statement is

    $p \wedge q \rightarrow r \vee s$

    Its contrapositive is

    $\sim(r \vee s) \rightarrow \sim(p \wedge q)$

    $\Leftrightarrow (\sim r \wedge \sim s) \rightarrow (\sim p \vee \sim q)$

3. $A \vee (\sim(A \wedge \sim B))$

    $\equiv A \vee (\sim A \vee B)$

    $\equiv (A \vee \sim A) \vee B$

    $\equiv T \vee B \equiv T$

    For Statement-1, see to Question 3 in the previous year AIEEE/JEE Questions.

    Thus, Statement-1 is true. However Statement-2 is not a correct reason for statement-1.

4. Statement-2 is true. [See theory]

    As $\sim(p \vee q) \equiv \sim p \wedge \sim q$

    $\equiv$ ~(Delhi is in India) and ~(Mumbai is not is Italy)

    $\equiv$ Delhi is not in India and Mumbai is in Italy.

5. $p \wedge (\sim p) \equiv F$;

    $p \vee (\sim p) \equiv T$

    and $p \wedge p \equiv p$ is true for each logical statement p.

6. $\sim(\sim p \wedge q) \wedge (p \vee q)$

    $\equiv (p \vee \sim q) \wedge (p \vee q)$

    $\equiv p \vee (\sim q \wedge q) \equiv p \vee F \equiv p.$

    As, $\sim(\sim p \wedge q) \wedge (p \vee q) \leftrightarrow p$

    $p \leftrightarrow p$

    is a tautology, Statement-2 is false.

    For truth Statement-1 see theory.

7. $p \wedge (p \rightarrow q) \rightarrow q$

    $\equiv [p \wedge (\sim p \vee q)] \rightarrow q$

    $\equiv [(p \wedge (\sim p)) \vee (p \wedge q)] \rightarrow q$

    $\equiv [F \vee (p \wedge q)] \rightarrow q$

    $\equiv (p \wedge q) \rightarrow q$

    $\equiv \sim(p \wedge q) \vee q$

    $\equiv (\sim p \vee \sim q) \vee q$

    $\equiv \sim p \vee (\sim q \vee q)$

    $\equiv (\sim p) \vee T \equiv T$

8. $A \rightarrow (A \vee \sim B)$

    $\equiv \sim A \vee (A \vee \sim B)$

    $\equiv (\sim A \vee A) \vee (\sim B)$

    $\equiv T \vee (\sim B) \equiv T$

    $\therefore$ Negative of $A \rightarrow A \vee \sim B$ is $\sim T = F$.

Architecture Entrance – 2017

1. If A and B be two finite sets that in total number of subsets of A is 960 more than the total number of subsets of B, then $n(a) - n(b)$ (where $n(X)$ denotes the number of elements in set X) is equal to:
   (a) 6 (b) 2
   (c) 3 (d) 4

2. The order and the degree of the differential equation of all ellipses with centre at the origin, major axis along x-axis and eccentricity $\frac{\sqrt{3}}{2}$ are, respectively:
   (a) 2, 2 (b) 1, 1
   (c) 2 , 1 (d) 1, 2

3. If A and B are two independent events such that P(A) $= \frac{3}{10}$ and P(A ∪ B) $= \frac{4}{5}$, then P(A ∩ B) is equal to:
   (a) $\frac{3}{35}$ (b) $\frac{1}{5}$
   (c) $\frac{1}{10}$ (d) $\frac{3}{14}$

4. Let S be the set of all real values of 'a' for which the following system of linear equations.
   $ax + 2y + 5z = 1$
   $2x + y + 3z = 1$
   $3y + 7z = 1$
   Is consistent. Then the set S is:
   (a) an empty set (b) equal to R
   (c) equal to R – {1} (d) equal to {1}

5. If an equilateral triangle, having centroid at the origin has a side along the line, $x + y = 2$, then the area (in sq. units) of this triangle is:
   (a) $3\sqrt{6}$ (b) 6
   (c) $6\sqrt{3}$ (d) $\frac{9}{2}\sqrt{3}$

6. If the shortest distance between the lines $x + 2\lambda = 2y = -12z$, $x = y + 4\lambda = 6z - 12\lambda$ is $4\sqrt{2}$ units, then a value of λ is:
   (a) $\frac{\sqrt{2}}{2}$ (b) 2
   (c) $\sqrt{2}$ (d) $2\sqrt{2}$

7. If the digits at ten's and hundred's places in $(11)^{2016}$ are x and y respectively, then the ordered pair (x, y) is equal to:-
   (a) (1, 8) (b) (1, 6)
   (c) (6, 1) (d) (8, 1)

8. Which one of the following points does not lie on the normal to the hyperbola, $\frac{x^2}{16} - \frac{y^2}{9} = 1$ drawn at the point $(8, 3\sqrt{3})$?
   (a) $\left(10, \frac{1}{\sqrt{3}}\right)$ (b) $\left(13, -\frac{1}{\sqrt{3}}\right)$
   (c) $\left(12, \frac{1}{\sqrt{3}}\right)$ (d) $\left(11, \sqrt{3}\right)$

9. An urn contains 5 red, 4 black and 3 white marbles, Then the number of ways in which 4 marbles can be drawn from it so that at most 3 of them are red, is:
   (a) 495 (b) 455
   (c) 460 (d) 490

10. If $(4 + 3i)^2 = 7 + 24i$, then a value of $\left(7+\sqrt{-576}\right)^{\frac{1}{2}} - \left(7-\sqrt{-576}\right)^{\frac{1}{2}}$ is:-
   (a) $-6i$ (b) $-3i$
   (c) $2i$ (d) 6

11. The equation of the circle, which is the mirror image of the circle, $x^2 + y^2 - 2x = 0$, in the line, $y = 3 - x$ is:
   (a) $x^2 + y^2 - 6x - 4y + 12 = 0$
   (b) $x^2 + y^2 - 6x - 8y + 24 = 0$
   (c) $x^2 + y^2 - 8x - 6y + 24 = 0$
   (d) $x^2 + y^2 - 4x - 6y + 12 = 0$

12. $\lim\limits_{x \to 0} \frac{\log(\sin 7x + \cos 7x)}{sin 3x}$ equals: -
   (a) $\frac{1}{3}\log 7$ (b) $\frac{7}{3}$
   (c) $\frac{14}{3}$ (d) $\frac{1}{3}$

13. If the line, $y = mx$, bisects the area of the region $\{(x, y) : 0 \le x \le \frac{3}{2}, 0 \le y \le 1 + 4x - x^2\}$, then m equals:
   (a) $\frac{39}{16}$ (b) $\frac{9}{8}$
   (c) $\frac{13}{3}$ (d) $\frac{13}{6}$

14. The product of the perpendiculars drawn from the foci of the ellipse, $\frac{x^2}{9} + \frac{y^2}{25} = 1$ upon the tangent to it at the point $\left(\frac{3}{2}, \frac{5\sqrt{3}}{2}\right)$, is:

(a) $3\sqrt{13}$ (b) 9

(c) $\frac{189}{13}$ (d) 18

15. An observer at a point P on the top of a hill near the sea-shore notices that the angle of depression of a ship moving towards the hill in a straight line at a constant speed is 30°. After 45 minutes, this angle becomes 45°. If T (in minutes) is the total time taken by the ship to move to a point in the sea where the angle of depression from P of the ship is 60°, then T is equal to:

(a) $45\left(1+\frac{1}{\sqrt{3}}\right)$ (b) $45\left(1+\sqrt{3}\right)$

(c) $45\left(1+\frac{2}{\sqrt{3}}\right)$ (d) $45\left(2+\frac{1}{\sqrt{3}}\right)$

16. Two numbers are selected at random (without replacement) from the first six positive integers. If X denotes the smaller of the two numbers, then the expectation of X, is:

(a) $\frac{5}{3}$ (b) $\frac{14}{3}$

(c) $\frac{13}{3}$ (d) $\frac{7}{3}$

17. The perpendicular distance from the point (3, 1, 1) on the plane passing through the point (1, 2, 3) and containing the line.

$\mathbf{r} = \mathbf{i} + \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$ is

(a) $\frac{3}{\sqrt{11}}$ (b) $\frac{1}{\sqrt{11}}$

(c) $\frac{4}{\sqrt{41}}$ (d) 0

18. The integral $I = \int \frac{x+2}{\left(x^2+3x+3\right)\sqrt{x+1}}\,dx$ is equal to

(a) $\frac{1}{\sqrt{3}}\cot^{-1}\left[\frac{x\sqrt{3}}{\sqrt{x+1}}\right]+c$

(b) $\frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{x}{\sqrt{3(x+1)}}\right]+c$

(c) $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x}{\sqrt{3(x+1)}}\right]+c$

(d) $\frac{2}{\sqrt{3}}\cot^{-1}\left[\frac{x}{\sqrt{3(x+1)}}\right]+c$

(Where C is a constant of integration)

19. The value of $E = \frac{1}{\cos 285^\circ}+\frac{1}{\sqrt{3}\sin 255^\circ}$ is

(a) $\sqrt{3}-\sqrt{2}$ (b) $2\sqrt{2}$

(c) $\frac{4\sqrt{2}}{\sqrt{3}}$ (d) $\frac{2\sqrt{2}}{3}$

20. Let a_1, a_2, a_3, a_4, a_5 be a G.P. of positive real numbers such that the A.M., of a_2 and a_4 is 117 and the G.M. of a_2 and a_4 is 108. Then the A.M. of a_1 and a_5 is

(a) 145.5 (b) 108

(c) 117 (d) 144.5

21. The integral $\int_{\pi/24}^{5\pi/24} \frac{dx}{1+\sqrt[3]{\tan 2x}}$ is equal to

(a) $\frac{\pi}{18}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{12}$ (d) $\frac{\pi}{6}$

22. Three vectors **a**, **b** and **c** are such that $|\mathbf{a}|=1$, $|\mathbf{b}|=2$, $|\mathbf{c}| = 4$ and $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. Then the value of $4\mathbf{a}\cdot\mathbf{b} + 3\mathbf{b}\cdot\mathbf{c} + 3\mathbf{c}\cdot\mathbf{a}$ is equal to:

(a) 27 (b) –68

(c) –26 (d) –34

23. For all real numbers x, y and z, the determinant

$$\Delta = \begin{vmatrix} 2x & xy-xz & y \\ 2x+z+1 & xy-xz+yz-z^2 & 1+y \\ 3x+1 & 2xy-2xz & 1+y \end{vmatrix}$$ is equal to:

(a) $(y-xz)(z-x)$ (b) zero

(c) $(x-y)(y-z)(z-x)$ (d) $(x-yz)(y-z)$

24. If λ_1 and λ_2 are the two values of λ such that the roots α and β of the quadratic equation, $\lambda(x^2 - x) + x + 5 = 0$ satisfy $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{4}{5} = 0$, then $\frac{\lambda_1}{\lambda^2_2}+\frac{\lambda_2}{\lambda^2_1}$ is equal to:

(a) 488 (b) 536

(c) 512 (d) 501

25. If the sum of the first 15 terms of the series $3 + 7 + 14 + 24 + 37 + \ldots$ is $15k$, then k is equal to:

(a) 126 (b) 122

(c) 81 (d) 119

26. Water is running into an underground right circular conical reservoir, which is 10 m deep and radius of its base is 5 m. If the rate of change in the volume of water in the reservoir is $\frac{3}{2}\pi \text{m}^3$/min., then the rate (in m/min) at which water rises in it, when the water level is 4m, is:

(a) $\frac{3}{2}$ (b) $\frac{3}{8}$

(c) $\frac{1}{8}$ (d) $\frac{1}{4}$

27. A bag contains three coins, one of which has head on both sides, another is a biased coin that shows up heads 90% of the time and the third one is an unbiased

coin. A coin is taken out from the bag at random and tossed. If it shows up a head, then the probability that it is the unbiased coin, is:

(a) $\frac{3}{8}$ (b) $\frac{5}{12}$

(c) $\frac{5}{24}$ (d) $\frac{1}{3}$

28. If the functiuon f : **R** → **R**, defined by $f(x)=\begin{cases} ax & x<2 \\ ax^2-bx+3, & x\geq 2\end{cases}$ is differentiable, then the value of $f'(-3)+f'(3)$ is equal to:

(a) 0 (b) 3

(c) 4 (d) $\frac{15}{2}$

29. Which one of the following statements is a tautology?

(a) $p \to (p \to q)$ (b) $(p \vee q) \to q$

(c) $p \vee (p \to q)$ (c) $p \vee (q \to p)$

30. The sum of the abscissae of the points where the curves, $y = kx^2 + (5k + 3)x + 6k + 5$, ($k \in$ **R**), touch the x – axis, is equal to:

(a) $-\frac{4}{3}$ (b) $-\frac{19}{3}$

(c) $-\frac{10}{3}$ (d) $\frac{5}{3}$

Answers

1. (d)	**2.** (b)	**3.** (d)	**4.** (b)
5. (c)	**6.** (c)	**7.** (c)	**8.** (a)
9. (d)	**10.** (a)	**11.** (a)	**12.** (b)
13. (d)	**14.** (b)	**15.** (a)	**16.** (d)
17. (d)	**18.** (c)	**19.** (c)	**20.** (a)
21. (c)	**22.** (c)	**23.** (c)	**24.** (a)
25. (b)	**26.** (b)	**27.** (c)	**28.** (b)
29. (c)	**30.** (c)		

Hints and Solutions

1. Let $n(\text{A}) = r$, $n(\text{B}) = s$, then

$2^r - 2^s = 260$

$\Rightarrow 2^s(2^{r-s}-1) = 2^6(2^4 - 1)$

$\therefore\ s = 6,\ r - s = 4$

$\Rightarrow n(\text{A}) - n(\text{B}) = 4.$

2. $b^2 = a^2(1 - e^2) = a^2(1 - 3/4) = a^2/4$

$\therefore$ equation of ellipse is

$\frac{x^2}{a^2}+\frac{4y^2}{a^2}=1$ or $x^2 + 4y^2 = a^2$

Differentiating we get

$2x+8y\frac{dy}{dx}=0$

or $4y\frac{dy}{dx} + x = 0$

It's a differential equation of order one and degree one

3. $\frac{4}{5}$ = P(A ∪ B) = P(A) + P(B) – P(A ∩ B)

$\Rightarrow \frac{4}{5}$ = P(A) + P(B) – P(A)P(B)

[∵A and B are independent]

$= \frac{3}{10}+\left(1-\frac{3}{10}\right)\text{P(B)}$

$\Rightarrow \frac{1}{2}= \frac{7}{10}\text{P(B)} \Rightarrow \text{P(B)} = \frac{5}{7}$

Also, P(A ∩ B) = P(A)P(B) = $\left(\frac{3}{10}\right)\left(\frac{5}{7}\right)= \frac{3}{14}$

4. Multiply the first equation by 2 and subtract second equation from it to get

$2(a - 1)x + 3y + 7z = 1$

Also, $3y + 7z = 1$ [3rd equation]

$\therefore\ 2(a - 1)x = 0$

For $a \neq 1$, $x = 0$ and we can solve

$2y + 5z = 1$

$y + 3z = 1$

$3y + 7z = 1$

To obtain $y = -2$, $z = 1$

$\therefore$ For all $a \neq 1$, the system of equations is consistent.

For $a = 1$, (1) is satisfied each $x \in$ **R**.

We choose a value of x and solve

$y + 3z = 1 - 2x$

$3y + 5z = 1$

To obtain values of y and z.

Thus, the system of equations is consistent for each a ∈ **R**.

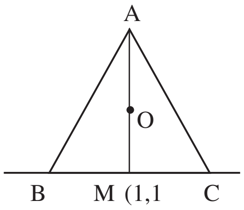
5. As triangle is equilateral, O is orthocentre of the triangle, and

$y = x$ is altitude of the triangle, then

$$\text{AM} = \frac{\sqrt{3}}{2}a$$

$$\Rightarrow \text{OM} = \frac{1}{3}(\text{AM})$$

$$\Rightarrow \sqrt{2} = \frac{1}{3}\left(\frac{\sqrt{3}}{2}a\right)$$



$$\Rightarrow a = 2\sqrt{6}$$

$$\text{Area of ABC} = \frac{\sqrt{3}}{4}(a^2) = 6\sqrt{3}$$

6. Two lines can be written as

$$\frac{x+2\lambda}{12} = \frac{y}{6} = \frac{z}{-1} \qquad (1)$$

$$\text{and} \quad \frac{x}{6} = \frac{y+4\lambda}{6} = \frac{z-2\lambda}{1} \qquad (2)$$

A vector perpendicular to (1) and (2) is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \hat{\mathbf{k}} \\ 12 & 6 & -1 \\ 6 & 6 & 1 \end{vmatrix}$$

$$= 12\mathbf{i} - 18\mathbf{j} + 36\hat{\mathbf{k}}$$

A unit vector perpendicular to (1) and (2) is

$$\hat{\boldsymbol{n}} = \frac{1}{7}\left(2\mathbf{i} - 3\mathbf{j} + 6\hat{\mathbf{k}}\right)$$

Thus, shortest distance between 1 and (2) is

$$\frac{1}{7}|(2)(2\lambda) + (-3)(-4\lambda) + 6(2\lambda)| = 4|\lambda|$$

Now, $4|\lambda| = 4\sqrt{2} \Rightarrow \lambda = \pm\sqrt{2}$

Thus, a possible value of λ is $\sqrt{2}$.

7. $(11)^{2016} = (10 + 1)^{2016}$

$$\sum_{k=0}^{2014}\left({}^{2016}C_k\right)(10)^{2016-k} + {}^{2016}C_{2015}(10) + 1$$

$= 1000m + 161$

$\therefore\ x = 6,\ y = 1$

8. An equation of tangent at $(8,\ 3\sqrt{3})$

$$\frac{8}{16}x - \frac{3\sqrt{3}}{9}y = 1 \text{ or } \frac{1}{2}x - \frac{1}{\sqrt{3}}y = 1$$

$\therefore$ an equation of normal at $(8,\ 3\sqrt{3})$ is

$$\frac{1}{\sqrt{3}}(x-8) + \frac{1}{2}\left(y - 3\sqrt{3}\right) = 0$$

$$\text{or } 2(x-8) + \sqrt{3}\left(y - 3\sqrt{3}\right) = 0$$

Note that $(10,\ \frac{1}{\sqrt{3}})$ does not lie on this normal.

9. Number of ways of selecting 4 marbles such that at most 3 are red

= number of ways of selectin 4 marbles – number of selecting 4 red marbles

$= {}^{12}C_4 - {}^{5}C_4 = 495 - 5 = 490.$

10. $7 + 24i = (4 + 3i)^2$

$\Rightarrow (7 + 24i)^{1/2} = \pm(4 + 3i)$

Also, $(7 - 24i)^{1/2} = \pm(4 - 3i)$

Now, $\left(7 + \sqrt{-576}\right)^{\frac{1}{2}} - \left(7 - \sqrt{-576}\right)^{\frac{1}{2}}$

$= \left(7 + 24i\right)^{\frac{1}{2}} - \left(7 - 24i\right)^{\frac{1}{2}}$

$= (-4 - 3i) - (-4 + 3i) = -6i$

11. Centre of circle $x^2 + y^2 - 2x = 0$ is A(1, 0) nd its radius is 1

We find image of (1, 0) in the line $y = 3 - x$

Let image of A(1, 0) in $y = 3 - x$ be (h, k),

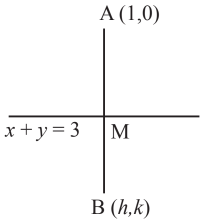
Then mid point $M\left(\frac{1+h}{2}, \frac{k}{2}\right)$ of AB lies on $x + y = 3$

$$\therefore\ \frac{1+h}{2} + \frac{k}{2} = 3 \Rightarrow h + k = 5$$

Also, slope of AB : $\frac{k-0}{h-1} = 1$

$\Rightarrow k = h - 1$

Thus, $h = 3$, $k = 2$



$\therefore$ Equation of required circle

$$(x - 3)^2 + (y - 2)^2 = 1$$

Or $x^2 + y^2 - 6x - 4y + 12 = 0$

12. $\lim\limits_{x \to 0} \frac{\log(\sin 7x + \cos 7x)}{\sin 3x}$

$$= \frac{1}{2}\lim_{x\to 0}\frac{\log(1 + 2\sin 7x \cos 7x)}{\sin 3x} \times$$

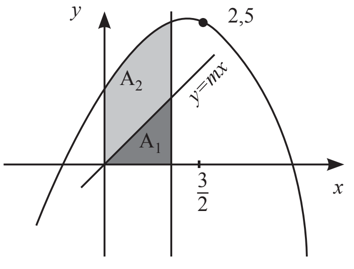
$$= \frac{14}{3 \times 2}\lim_{x\to 0}\left[\frac{\log(1+\sin 14x)}{\sin 14x} \times \frac{\sin 14x}{14x} \times \frac{3x}{\sin 3x}\right]$$

$$= \frac{7}{3}(1)\ (1)\ (1) = \frac{7}{3}$$

13. Refer Figure

$y = 1 + 4x - x^2$

$= 5 - (2 - x)^2$



We are given that

$A_1 = A_2$

$\Rightarrow 2A_1 = A_1 + A_2$

$$\Rightarrow 2\int_0^{3/2}(mx)\,dx = \int_0^{3/2}\left(1+4x-x^2\right)dx$$

$$\Rightarrow m\left(x^2\right)]_0^{3/2} = \left(x+2x^2-\frac{1}{3}x^3\right)_0^{3/2}$$

$$\frac{9}{4}m = \frac{3}{2}+\frac{9}{2}-\frac{1}{3}\left(\frac{3}{2}\right)^3 = \frac{39}{8}$$

$$\Rightarrow m = \frac{13}{6}$$

14. If e is eccentricity of the given ellipse, then

$9 = 25(1 - e^2) \Rightarrow e = 4/5$

Foci of the ellipse are, (0, 4) and (0, –4)

Also, equation of tongent to the ellipse at $\left(\frac{3}{2}, \frac{5\sqrt{3}}{2}\right)$ is

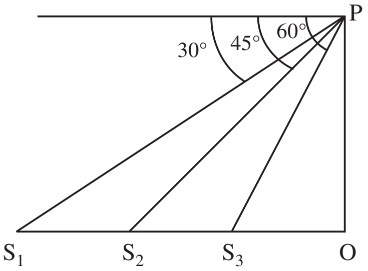
$$\frac{3}{2}\left(\frac{x}{9}\right)+\frac{5\sqrt{3}}{2}\left(\frac{y}{25}\right) = 1$$

$$\Rightarrow \frac{1}{6}x+\frac{\sqrt{3}}{10}y = 1$$

Now, $p_1p_2 = \dfrac{\left|0+\left(\frac{\sqrt{3}}{10}\right)(-4)-1\right|\left|0+\left(\frac{\sqrt{3}}{10}\right)(4)-1\right|}{\left(\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{\sqrt{3}}{10}\right)^2}\right)^2}$

$$= \frac{\left|\left(\frac{3}{100}\right)(16)-1\right|}{\frac{52}{900}} = 9$$

15. Let OP = a, then



$$\frac{OS_1}{OP} = \cot(30^0) = \sqrt{3}$$

$\Rightarrow OS_1 = \sqrt{3}a$

Similarly $OS_2 = a$, $OS_3 = \dfrac{a}{\sqrt{3}}$

Let speed of the ship be v km/min, then

$$\sqrt{3}a - a = 45v \Rightarrow a = \frac{45v}{\sqrt{3}-1}$$

Now,

$$S_1S_3 = \sqrt{3}a - \left(\frac{a}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}a$$

$$\Rightarrow T = \frac{S_1S_3}{v} = \frac{2}{\sqrt{3}}\frac{45}{\sqrt{3}-1} = \frac{45\left(\sqrt{3}+1\right)}{\sqrt{3}}$$

$$= 45\left(1+\frac{1}{\sqrt{3}}\right).$$

16. Total number of ways of selecting two numbers from 1, 2, 3, 4, 5, 6 is

${}^6C_2 = 15$

Now,

P(X = 1) = P(selecting 1 and a number from 2, 3, 4, 5, 6)

$= \dfrac{5}{15}$

$P(X = 2) = \dfrac{4}{15}$, $P(X = 3) = \dfrac{3}{15}$

$P(X = 4) = \dfrac{2}{15}$, $P(X = 5) = \dfrac{1}{15}$

$$E(X) = \frac{1}{15}\left[(1)(5)+(2)(4)+(3)(3)+(4)(2)+(5)(1)\right]$$

$$= \frac{1}{15}(35) = \frac{7}{3}$$

17. Equation of plane containing the line

$\mathbf{r} = \mathbf{i}+\mathbf{j}+\lambda(2\mathbf{i}+\mathbf{j}+4\mathbf{k})$

and the point (1, 2, 3) is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 1-1 & 1-2 & 0-3 \\ 2 & 1 & 4 \end{vmatrix} = 0$$

or, $\begin{vmatrix} x-1 & y-2 & z-3 \\ 0 & -1 & -3 \\ 2 & 1 & 4 \end{vmatrix} = 0$

$(-1)(x - 1) + (-6)(y - 2) + 2(z - 3) = 0$

Or

$x + 6y - 2z - 7 = 0$

Note that (3, 1, 1) lies on the plane

∴ distance of (3, 1, 1) from

$x + 6y - 2z - 7 = 0$ is 0

18. Put $x + 1 = t^2$, then

$$I = \int\frac{\left(t^2+1\right)(2t)\,dt}{\left[\left(t^2-1\right)^2+3\left(t^2-1\right)+3\right]t}$$

$$= 2\int\frac{t^2+1}{t^4+t^2+1}dt$$

$$= 2\int\frac{\left(1+\frac{1}{t^2}\right)dt}{t^2+1+\frac{1}{t^2}}$$

Put $t - 1/t = u$, then $\left(1+\frac{1}{t^2}\right)dt = du$

$$\therefore\ I = 2\int\frac{du}{u^2+3} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+c$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}t}\right)+c = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3(x+1)}}\right)+c$$

19. $\cos(285°) = \cos(360° - 75°) = \cos(75°)$

$= \cos(45° + 30°)$

$= \cos45°\cos30° - \sin45°\sin30°$

$= \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

And $\sin(255°) = \sin(180°+ 75°) = -\sin(75°)$

$= -\sin(45° + 30°)$

$= -\left(\dfrac{\sqrt{3}+1}{2\sqrt{2}}\right)$

$$\therefore\ E = 2\sqrt{2}\left(\frac{1}{\sqrt{3}-1}-\frac{1}{\sqrt{3}\left(\sqrt{3}+1\right)}\right)$$

$$= 2\sqrt{2}\left(\frac{3+\sqrt{3}-\sqrt{3}+1}{\sqrt{3}(3-1)}\right)$$

$$= \frac{4\sqrt{2}}{\sqrt{3}}.$$

20. Let a be the first term and r be the common ratio of the G.P.

We are given that

$a_2 + a_4 = 234,\ \sqrt{a_2a_4} = 108$

$\Rightarrow a_2 + a_4 = 234,\ a_3 = 108$

$$\frac{a_2+a_4}{a_3}=\frac{234}{108} \Rightarrow r+\frac{1}{r}=\frac{13}{6}$$

We have

$a_1 + a_5 = \dfrac{a_3}{r^2}+a_3r^2$

$$= 108\left[\left(r+\frac{1}{r}\right)^2-2\right]$$

$$= 108\left[\left(\frac{13}{6}\right)^2-2\right] = 291$$

$\therefore$ Arithmetic mean of a_1, a_5 is 145.5.

21. $$I = \int_{\frac{\pi}{24}}^{\frac{5\pi}{24}}\frac{dx}{1+\sqrt[3]{\tan 2x}} \tag{1}$$

$$= \int_{\frac{\pi}{24}}^{\frac{5\pi}{24}}\frac{dx}{1+\sqrt[3]{\tan\left(2\left(\frac{5\pi}{24}+\frac{\pi}{24}-x\right)\right)}}$$

$$= \int_{\frac{\pi}{24}}^{\frac{5\pi}{24}}\frac{dx}{1+\sqrt[3]{\cot 2x}}dx$$

$$= \int_{\frac{\pi}{24}}^{\frac{5\pi}{24}}\frac{\sqrt[3]{\tan 2x}}{\sqrt[3]{\tan 2x+1}}dx \tag{2}$$

Adding (1) and (2), we get

$$2I = \int_{\frac{\pi}{24}}^{\frac{5\pi}{24}}dx = \frac{5\pi}{24}-\frac{\pi}{24}=\frac{\pi}{6}$$

$$\Rightarrow I = \frac{\pi}{12}.$$

22. $\mathbf{c} = -\mathbf{a}-\mathbf{b}$

$\Rightarrow |\mathbf{c}|^2 = |\mathbf{b}|^2+|\mathbf{a}|^2+2\mathbf{a}\cdot\mathbf{b}$

$\Rightarrow 16 = 1+4+2\mathbf{a}\cdot\mathbf{b}$

$\Rightarrow 2\mathbf{a}\cdot\mathbf{b} = 11$

Now,

$4\mathbf{a}\cdot\mathbf{b}+3\mathbf{b}\cdot\mathbf{c}+3\mathbf{c}\cdot\mathbf{a}$

$= 4\mathbf{a}\cdot\mathbf{b}+3\mathbf{c}\cdot(\mathbf{b}+\mathbf{a})$

$= 2(11)+3\mathbf{c}\cdot(-\mathbf{c})$

$= 22 - 3|\mathbf{c}|^2$

$= 22 - 3(16) = -26.$

23. Using $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$, We get

$$\Delta = \begin{vmatrix} 2x & x(y-z) & y \\ z+1 & z(y-z) & 1 \\ x+1 & x(y-z) & 1 \end{vmatrix}$$

$$= (y-z)\begin{vmatrix} 2x & x & y \\ z+1 & z & 1 \\ x+1 & x & 1 \end{vmatrix}$$

Using $C_1 \to C_1 - C_2 - C_3$, we get

$$\Delta = (y-z)\begin{vmatrix} x-y & x & y \\ 0 & z & 1 \\ 0 & x & 1 \end{vmatrix}$$

$= (x - y)(y - z)(z - x).$

24. $\alpha + \beta = 1 - 1/\lambda$

$\alpha\beta = 5/\lambda$

Now,

$$-\frac{4}{5} = \frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$\Rightarrow -\frac{4}{5}\alpha\beta = (\alpha+\beta)^2 - 2\alpha\beta$$

$$\Rightarrow \left(-\frac{4}{5}\right)\left(\frac{5}{\lambda}\right)=\left(1-\frac{1}{\lambda}\right)^2-\frac{10}{\lambda}$$

$$\Rightarrow -\frac{4}{\lambda} = 1-\frac{2}{\lambda}+\frac{1}{\lambda^2}-\frac{10}{\lambda}$$

$$\Rightarrow \frac{1}{\lambda^2}-\frac{8}{\lambda}+1=0 \text{ or } \lambda^2 - 8\lambda + 1 = 0$$

$\therefore\ \lambda_1 + \lambda_2 = 8,\ \lambda_1\lambda_2 = 1$

Now,

$$\frac{\lambda_1}{\lambda_2{}^2}+\frac{\lambda_2}{\lambda_1{}^2}=\lambda_1{}^3+\lambda_2{}^3 \qquad \left[\because \lambda_2=\frac{1}{\lambda_1}\right]$$

$$= (\lambda_1 + \lambda_3)^3 - 3\lambda_1\lambda_2(\lambda_1 + \lambda_2)$$

$$= 8^3 - 3(8) = 488$$

25. Let

$S = 3 + 7 + 14 + 24 + 37 + \ldots. + t_{n-1} + t_n$

$S = 0 + 3 + 7 + 14 +\ldots. + + t_{n-1} + t_n$

Subtracting we get

$0 = 3 + (4 + 7 + 10 + 13 + \ldots + (n-1) \text{ terms}) - t_n$

$$\Rightarrow t_n = 3+\frac{n-1}{2}\left[2(4)+3(n-2)\right]$$

$$= 3+\frac{1}{2}(n-1)(3n+2) = \frac{1}{2}\left(3n^2-n+4\right)$$

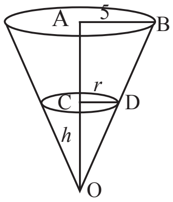
Now,

$$15k = \sum_{j=1}^{15} tj = \frac{1}{2}\sum_{j=1}^{15}\left(3j^2-j+4\right)$$

$$= \frac{3}{2}\times\frac{1}{6}(15)(15+1)(30+1)-\frac{1}{4}(15)(15+1)+2(15)$$

$\Rightarrow k = 4(31) - 4 + 2 = 122.$

26. Let h be the height of the water and r the radius of the cone formed by the water at time t.



$\Delta OAB \sim \Delta OCD$

$$\frac{OA}{AB} = \frac{OC}{CD}$$

$$\Rightarrow \frac{10}{5} = \frac{h}{r} \Rightarrow r = \frac{h}{2}$$

Let V = volume of water, the

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$$

$$\Rightarrow \frac{3}{2}\pi = \frac{1}{4}\pi h^2\frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{6}{h^2} \Rightarrow \left.\frac{dh}{dt}\right|h=4 = \frac{6}{16} = \frac{3}{8}.$$

27. Let E_1, E_2, E_3 be the following events.

E_1: coin with two heads is chosen

E_2: coin with 90% chance of getting head is chosen

E_3: unbiased coin is chosen

And let A be the event that a head occurs when the coin is tossed.

We have $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$P(A|E_1) = 1,\ P(A|E_2) = 0.9,\ P(A|E_3) = \frac{1}{2}$

By the Bayes rule

$P(E_3|A)$

$$= \frac{P(E_3)P(A\,|\,E_3)}{P(E_1)P(A\,|\,E_1)+P(E_2)P(A\,|\,E_2)+P(E_3)P(A\,|\,E_3)}$$

$$= \frac{\frac{1}{2}}{1+\frac{9}{10}+\frac{1}{2}}$$

$$= \frac{5}{10+9+5} = \frac{5}{24}.$$

28. As f is differentiable on **R**,

It is differentiable as well as continuous at $x = 2$

$$\therefore \lim_{x\to 2-} f(x) = \lim_{x\to 2+} f(x)$$

$\Rightarrow 2a = 4a - 4b + 3$

$\Rightarrow -2a + 2b = 3$ (1)

Also, $Lf'(2) = Rf'(2)$

$$\lim_{x\to 2-}\frac{ax-2a}{x-2} = \lim_{x\to 2+}\frac{\left(ax^2-bx+3\right)-\left(4a-2b+3\right)}{x-2}$$

$\Rightarrow a = 4a - b \Rightarrow b = 3a$ (2)

From (1) and (2), $a = \frac{3}{4},\ b = \frac{9}{4}$

Also,

$f'(-3) + f'(3)$

$= a + (2a)(3) - b = 7a - b = 3.$

29. Note: $pV(p \to q)$ is a tautology

Reason: If p is T, then

$pV(p \to q)$ is T,

And if p is F, then $p \to q$ is T,

And therefore, $pV(p \to q)$ is T

Thus, in all the cases, $pV(p \to q)$ is T

30. As $y = kx^2 + (5k + 3)x + 6k + 5$ touches

x–axis is,

$0 = (5k + 3)^2 - 4k(6k + 5)$

$= 25k^2 + 30k + 9 - 24k2 - 20k$

$= k^2 + 10k + 9 = (k + 1)(k + 9)$

$\Rightarrow k = -1, k = -9$

When $k = -1$

$y = -x^2 - 2x - 1 = -(x + 1)^2$

In this case $x = -1$

When $k = -9$

$y = -9x^2 - 42x - 49$

$= -(3x + 7)2$

In this case $x = -\frac{7}{3}$

Thus, the required sum of abscissae

$$= -1 - \frac{7}{3} = -\frac{10}{3}.$$

JEE (Main) 2017 Questions with Solutions Mathematics (8th April - online)

1. Let $f(x) = 2^{10}x + 1$ and $g(x) = 3^{10}x - 1$. If $(f\circ g)(x) = x$, then x is equal to:

   (a) $\dfrac{3^{10}-1}{3^{10}-2^{-10}}$
   (b) $\dfrac{2^{10}-1}{2^{10}-3^{-10}}$
   (c) $\dfrac{1-3^{-10}}{2^{10}-3^{-10}}$
   (d) $\dfrac{1-2^{-10}}{3^{10}-2^{-10}}$

2. Let $p(x)$ be a quadratic polynomial such that $p(0) = 1$. If $p(x)$ leaves remainder 4 when divided by $x - 1$ and it leaves remainder 6 when divided by $x + 1$; then:

   (a) $p(2) = 11$
   (b) $p(2) = 19$
   (c) $p(-2) = 19$
   (d) $p(-2) = 11$

3. Let z ∈ C, the set of complex numbers. Then the equation, $2|z + 3i| - |z - i| = 0$ Represents:

   (a) a circle with radius $\frac{8}{3}$.
   (b) a circle with diameter $\frac{10}{3}$.
   (c) an ellipse with length of major axis $\frac{16}{3}$.
   (d) an ellipse with length of minor axis $\frac{16}{9}$.

4. The number of real values of λ for which the system of linear equations

   $2x + 4y - \lambda z = 0$
   $4x + \lambda y + 2z = 0$
   $\lambda x + 2y + 2z = 0$

   has infinitely many solutions, is:

   (a) 0
   (b) 1
   (c) 2
   (d) 3

5. Let A be any 3 × 3 invertible matrix. Then which one of the following is not always true?

   (a) $\text{adj }(A) = |A|\cdot A^{-1}$
   (b) $\text{adj }(\text{adj}(A)) = |A|\cdot A$
   (c) $\text{adj }(\text{adj}(A)) = |A|^2\cdot(\text{adj}(A))^{-1}$
   (d) $\text{adj }(\text{adj}(A)) = |A|.(\text{adj}(A))^{-1}$

6. If all the words, with or without meaning, are written using the letters of the word QUEEN and are arranged as in English dictionary, then the position of the word QUEEN is:

   (a) 44th
   (b) 45th
   (c) 46th
   (d) 47th

7. If $(27)^{999}$ is divided by 7, then the remainder is:

   (a) 1
   (b) 2
   (c) 3
   (d) 6

8. If the arithmetic mean of two numbers a and b $a > b > 0$, is five times their goemetric mean, then $\dfrac{a+b}{a-b}$ is equal to:

   (a) $\dfrac{\sqrt{6}}{2}$
   (b) $\dfrac{3\sqrt{2}}{4}$
   (c) $\dfrac{7\sqrt{3}}{12}$
   (d) $\dfrac{5\sqrt{6}}{12}$

9. If the sum of the first n terms of the series $\sqrt{3}+\sqrt{75}+\sqrt{243}+\sqrt{507}+\ldots$ is $435\sqrt{3}$ then n equals:

   (a) 18
   (b) 15
   (c) 13
   (d) 29

10. $\displaystyle\lim_{x\to 3}\frac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}$ is equal to:

    (a) $\sqrt{3}$
    (b) $\dfrac{1}{\sqrt{2}}$
    (c) $\dfrac{\sqrt{3}}{2}$
    (d) $\dfrac{1}{2\sqrt{2}}$

11. The tangent at the point (2, –2) to the curve, $x^2y^2 - 2x = 4(1 - y)$ does not pass through the point:

    (a) $\left(4, \frac{1}{3}\right)$
    (b) (8, 5)
    (c) (–4, –9)
    (d) (–2, –7)

12. If $y = \left[x+\sqrt{x^2-1}\right]^{15}+\left[x-\sqrt{x^2-1}\right]^{15}$ then $\left(x^2-1\right)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}$ is equal to:

    (a) $125\,y$
    (b) $224\,y^2$
    (c) $225\,y^2$
    (d) $225\,y$

13. If a point P has co-ordinates (0, –2) and Q is any point on the circle, $x^2 + y^2 - 5x - y + 5 = 0$, then the maximum value of $(PQ)^2$ is:

    (a) $\dfrac{25+\sqrt{6}}{2}$
    (b) $14+5\sqrt{3}$
    (c) $\dfrac{47+10\sqrt{6}}{2}$
    (d) $8+5\sqrt{3}$

14. The integral I = $\int\sqrt{1+2\cot x(\operatorname{cosec} x+\cot x)}\,dx$ $\left(0<x<\frac{\pi}{2}\right)$ is equal to:
(where C is a constant of integration)

(a) $4\log\left(\sin\frac{x}{2}\right)+C$ (b) $2\log\left(\sin\frac{x}{2}\right)+C$

(c) $2\log\left(\cos\frac{x}{2}\right)+C$ (d) $4\log\left(\cos\frac{x}{2}\right)+C$

15. the integral I = $\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}\frac{8\cos 2x}{(\tan x+\cot x)^3}dx$ equals:

(a) $\frac{15}{128}$ (b) $\frac{15}{64}$

(c) $\frac{13}{32}$ (d) $\frac{13}{256}$

16. The area (in sq. units) of the smaller portion enclosed between the curves, $x^2 + y^2 = 4$ and $y^2 = 3x$, is:

(a) $\frac{1}{2\sqrt{3}}+\frac{\pi}{3}$ (b) $\frac{1}{\sqrt{3}}+\frac{2\pi}{3}$

(c) $\frac{1}{2\sqrt{3}}+\frac{2\pi}{3}$ (d) $\frac{1}{\sqrt{3}}+\frac{4\pi}{3}$

17. The curve satisfying the differential equation, $y\mathrm{d}x - (x + 3y^2)\mathrm{d}y = 0$ and passing through the point (1, 1), also passes through the point:

(a) $\left(\frac{1}{4},-\frac{1}{2}\right)$ (b) $\left(-\frac{1}{3},\frac{1}{3}\right)$

(c) $\left(\frac{1}{3},-\frac{1}{3}\right)$ (d) $\left(\frac{1}{4},\frac{1}{2}\right)$

18. The locus of the point of intersection of the straight lines,
$tx - 2y - 3t = 0$
$x - 2ty + 3 = 0$ ($r \in \mathbf{R}$), is :

(a) an ellipse with eccentricity $\frac{2}{\sqrt{5}}$
(b) an ellipse with the length of major axis 6
(c) a hyperbola with eccentricity $\sqrt{5}$
(d) a hyperbola with the length of conjugate axis 3

19. If two parallel chords of a circle, having diameter 4 units, lie on the opposite sides of the centre and subtend angles $\cos^{-1}\left(\frac{1}{7}\right)$ and $\sec^{-1}(7)$ at the centre respectively, then the distance between these chords, is:

(a) $\frac{4}{\sqrt{7}}$ (b) $\frac{8}{\sqrt{7}}$

(c) $\frac{8}{7}$ (d) $\frac{16}{7}$

20. If the common tangents to the parabola, $x^2 = 4y$ and the circle, $x^2 + y^2 = 4$ intersect at the point P, then the distance of P from the origin, is:

(a) $\sqrt{2}+1$ (b) $2\left(3+2\sqrt{2}\right)$

(c) $2\left(\sqrt{2}+1\right)$ (d) $3+2\sqrt{2}$

21. Consider an ellipse, whose centre is at the origin and its major axis is along the x-axis. If its eccentricity is $\frac{3}{5}$ and the distance between its foci is 6, then the area (in sq. units) of the quadrilateral inscribed in the ellipse, whith the vertices as the vertices of the ellipse, is:

(a) 8 (b) 32

(c) 80 (d) 40

22. The coordinates of the foot of the perpendicular from the point (1, –2, 1) on the plane containing the lines, $\frac{x+1}{6}=\frac{y-1}{7}=\frac{z-3}{8}$ and $\frac{x-1}{3}=\frac{y-2}{5}=\frac{z-3}{7}$, is:

(a) (2, –4, 2) (b) (–1, 2, –1)

(c) (0, 0, 0) (d) (1, 1, 1)

23. The line of intersection of the planes $\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+\mathbf{k})=1$ and $\mathbf{r}\cdot(\mathbf{i}+4\mathbf{j}-2\mathbf{k})=2$, is:

(a) $\frac{-}{\ }=\frac{y}{7}=\frac{z-\frac{5}{7}}{13}$

(b) $\frac{x-\frac{4}{7}}{2}=\frac{y}{-7}=\frac{z+\frac{5}{7}}{13}$

(c) $\frac{x-\frac{6}{13}}{2}=\frac{y-\frac{5}{13}}{-7}=\frac{z}{-13}$

(d) $\frac{x-\frac{6}{13}}{2}=\frac{y-\frac{5}{13}}{7}=\frac{z}{-13}$

24. The area (in sq. units) of the parallelogram whose diagonals are along the vectors $8\mathbf{i}-6\mathbf{j}$ and $3\mathbf{i}+4\mathbf{j}-12\mathbf{k}$, is:

(a) 26 (b) 65

(c) 20 (d) 52

25. The mean age of 25 teachers in a school is 40 years. A teacher retires at the age of 60 years and a new teacher is appointed in his place. If now the mean

age of the teachers in this school is 39 years, then the age (in years) of the newly appointed teacher is:

(a) 25 (b) 30

(c) 35 (d) 40

26. Three persons P,Q and R independentely try to hit a target. If the probabilities of their hitting the target are $\frac{3}{4}$, $\frac{1}{2}$ and $\frac{5}{8}$ respectively, then the probabiity that the target is hit by P or Q but not by R is:

(a) $\frac{21}{64}$ (b) $\frac{9}{64}$

(c) $\frac{15}{64}$ (d) $\frac{39}{64}$

27. An unbiased coin is tossed eight times. The probability of obtaining at least one head and at least one tail is:

(a) $\frac{255}{256}$ (b) $\frac{127}{128}$

(c) $\frac{63}{64}$ (d) $\frac{1}{2}$

28. If S = $\left\{x \in [0, 2\pi] : \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix} = 0\right\}$,

Then $\sum_{x \in s} \tan\left(\frac{\pi}{3} + x\right)$ is equal to:

(a) $4 + 2\sqrt{3}$ (b) $-2 + \sqrt{3}$

(c) $-2 - \sqrt{3}$ (d) $-4 - 2\sqrt{3}$

29. the value of $y = \tan^{-1}\left[\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right]$, $|x| < \frac{1}{2}$, $x \neq 0$, is equal to:

(a) $\frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$ (b) $\frac{\pi}{4} + \cos^{-1} x^2$

(c) $\frac{\pi}{4} - \frac{1}{2}\cos^{-1} x^2$ (d) $\frac{\pi}{4} - \cos^{-1} x^2$

30. The proposition (~p) ∨ (p∧~q) is equivalent to:

(a) p ∨ ~q (b) p → ~q

(c) p ∧ ~q (d) q → p

Answers

1. (d) **2.** (c) **3.** (a) **4.** (b)

5. (d) **6.** (c) **7.** (d) **8.** (d)

9. (b) **10.** (b) **11.** (d) **12.** (d)

13. (b) **14.** (b) **15.** (a) **16.** (d)

17. (b) **18.** (d) **19.** (b) **20.** (c)

21. (d) **22.** (c) **23.** (c) **24.** (b)

25. (c) **26.** (a) **27.** (b) **28.** (d)

29. (a) **30.** (b)

Hints and Solutions

1. $(fog)(x) = x$

$\Rightarrow f(g(x)) = x$

$\Rightarrow f(3^{10}x - 1) = x$

$\Rightarrow 2^{10}(3^{10}x - 1) + 1 = x$

$\Rightarrow 3^{10}x - 1 + 2^{-10} = 2^{-10}x$

$\Rightarrow (3^{10} - 2^{-10})x = 1 - 2^{-10}$

$\Rightarrow x = \frac{1 - 2^{-10}}{3^{10} - 2^{-10}}$

2. Let $p(x) = (x - 1)(ax + b) + 4$

$p(0) = (-1)(0 + b) + 4 = 1$

$\Rightarrow -b + 4 = 1 \Rightarrow b = 3$

Also, $6 = p(-1) = (-1 - 1)(-a + b) + 4$

$\Rightarrow 1 = a - b \Rightarrow a = 4$

$\therefore p(x) = (x - 1)(4x + 3) + 4$

$\Rightarrow p(2) = (2 - 1)(11) + 4 = 15$

$p(-2) = (-2 - 1)(-8 + 3) + 4 = 19$

3. Let $z = x + iy$

$2|x + iy + 3i| = |x + iy - i|$

$\Rightarrow 4[x^2 + (y + 3)^2] = x^2 + (y - 1)^2$

$\Rightarrow 3x^2 + 3y^2 + 26y + 35 = 0$

It's a circle with radius

$= \sqrt{\left(\frac{13}{3}\right)^2 - \frac{35}{3}} = \frac{8}{3}$

4. As the system has infinitely many solutions

$$\begin{vmatrix} 2 & 4 & -\lambda \\ 4 & \lambda & 2 \\ \lambda & 2 & 2 \end{vmatrix} = 0$$

$$\Rightarrow 2\begin{vmatrix} \lambda & 2 \\ 2 & 2 \end{vmatrix} - 4\begin{vmatrix} 4 & 2 \\ \lambda & 2 \end{vmatrix} - \lambda\begin{vmatrix} 4 & \lambda \\ \lambda & 2 \end{vmatrix} = 0$$

$\Rightarrow 2(2\lambda - 4) - 4(8 - 2\lambda) - \lambda(8 - \lambda^2) = 0$

$\Rightarrow \lambda^3 - 4\lambda - 40 = 0$

Let $g(x) = x^3 - 4x - 40$

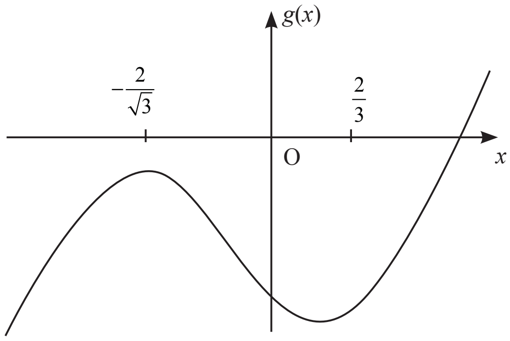
$g'(x) = 3x^2 - 4$

$g'(x) = 0 \Rightarrow x = \pm\ 2/\sqrt{3}$

$g''(x) = 6x$

As $g''\left(\dfrac{2}{\sqrt{3}}\right) > 0,\ g''\left(-\dfrac{2}{\sqrt{3}}\right) < 0$

Graph of $y(x)$ is



$\therefore$ $g(x) = 0$ has exactly one root.

$\Rightarrow$ there is exactly one value of λ

5. We know $A^{-1} = \dfrac{1}{|A|}\text{adj}A$

$\Rightarrow\quad |A|A^{-1} = \text{adj}A$

$\Rightarrow\quad \text{adj}\ (\text{adj}A) = \text{adj}\ (|A|A^{-1})$

$= |A|^2\ \text{adj}(A^{-1})$

$= |A|^2(|A^{-1}|(A^{-1})^{-1})$

$= |A|A$

Also, $(\text{adj}A)^{-1} = \dfrac{1}{|adjA|}\text{adj}(\text{adj}A)$

$= \dfrac{1}{|A|^2}\text{adj}(\text{adj}A)$

$\Rightarrow\quad \text{adj}(\text{adj}A) = |A|^2\ (\text{adj}(A))^{-1}$

Thus, $\text{adj}\ (\text{adj}\ A) = |A|(\text{adj}\ A)^{-1}$

is not necessarily true.

6. Letters of the word QUEEN are

E,E,N,Q,U

Words beginning with E (4!)	R 1 to 24
Words beginning with N$\left(\dfrac{4!}{2!}\right)$	25 to 36
Words beginning with QE (3!)	37 to 42
Words beginning with QN $\left(\dfrac{3!}{2!}\right)$	43 to 45

QUEEN is the next word and has rank 46th.

7. $(27)^{994} = (28 - 1)^{999} = (7 \times 4 - 1)^{999}$

$= 7m - 1 = 7(m - 1) + 6$

$\therefore$ remainder is 6

8. $\dfrac{a+b}{2} = 5\sqrt{ab}$

$\Rightarrow\ \dfrac{a+b}{\sqrt{ab}} = 10 \Rightarrow \sqrt{\dfrac{a}{b}} + \sqrt{\dfrac{b}{a}} = 10$

$\Rightarrow\ \dfrac{a}{b} + \dfrac{b}{a} + 2 = 100$

$\Rightarrow\ \dfrac{a}{b} + \dfrac{b}{a} = 98$

$\Rightarrow\ a^2 + b^2 + 2ab = 100ab$

And $a^2 + b^2 - 2ab = 96ab$

$\Rightarrow\ \dfrac{(a+b)^2}{(a-b)^2} = \dfrac{100}{96} = \dfrac{25}{24}$

$\Rightarrow\ \dfrac{a+b}{a-b} = \dfrac{5}{2\sqrt{6}} = \dfrac{5\sqrt{6}}{12}$

9. $\sqrt{3} + \sqrt{75} + \sqrt{243} + \sqrt{507} + \ldots$

Upto n terms $= 435\sqrt{3}$

$\Rightarrow\ \sqrt{3} + 5\sqrt{3} + 9\sqrt{3} + 13\sqrt{3} + \ldots$ upto n terms

$= 435\sqrt{3}$

$\Rightarrow$ 1 + 5 + 9 + 13 + …. Upto n terms = 435

$\Rightarrow\ \dfrac{n}{2}\{2(1) + 4(n-1)\} = 435$

$\Rightarrow\ 2n^2 - 30n + 29n - 435 = 0$

$\Rightarrow\ 2n(n - 15) + 29(n - 15) = 0$

$\Rightarrow\ (2n + 29)(n - 15) = 0$

$\Rightarrow\ n = 15$ as $n \in \mathbf{N}$.

10. $\displaystyle\lim_{x\to 3}\frac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}$

$$= \lim_{x\to 3}\frac{\left(\sqrt{3x}-3\right)\left(\sqrt{3x}+3\right)\left(\sqrt{2x-4}+\sqrt{2}\right)}{\left(\sqrt{2x-4}-\sqrt{2}\right)\left(\sqrt{2x-4}+\sqrt{2}\right)\left(\sqrt{3x}+3\right)}$$

$$= \lim_{x\to 3}\frac{(3x-9)\left(\sqrt{2x-4}+\sqrt{2}\right)}{(2x-4-2)\left(\sqrt{3x}+3\right)}$$

$$= \frac{3}{2}\lim_{x\to 3}\frac{\sqrt{2x-4}+\sqrt{2}}{\sqrt{3x}+3}$$

$$= \frac{3}{2}\ \frac{\sqrt{2}+\sqrt{2}}{\sqrt{3^2}+3} = \frac{1}{\sqrt{2}}.$$

11. $x^2y^2 - 2x = 4(1 - y)$

$\Rightarrow \quad x^2y^2 - 2x + 4y - 4 = 0$

Differentiating w.r.t. x, we get

$2xy^2 + 2x^2y\,\frac{dy}{dx} - 2 + 4\,\frac{dy}{dx} = 0$

When $x = 2$, $y = -2$

$16 + 2(2^2)(-2)\,\left.\frac{dy}{dx}\right]_{(2,-2)} - 2 + 4\,\left.\frac{dy}{dx}\right]_{(2,-2)} = 0$

$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{(2,-2)} = \frac{14}{12} = \frac{7}{6}$

Equation of tangent at (2, –2) is

$y + 2 = \frac{7}{6}(x - 2)$

It does not pass through (–2, –7).

12. $$y = \left[x+\sqrt{x^2-1}\right]^{15} + \left[x-\sqrt{x^2-1}\right]^{15}$$

$$\frac{dy}{dx} = 15\left[x+\sqrt{x^2-1}\right]^{14}\left\{1+\frac{x}{\sqrt{x^2-1}}\right\}$$

$$+\ 15\left[x-\sqrt{x^2-1}\right]^{14}\left\{1-\frac{x}{\sqrt{x^2-1}}\right\}$$

$$= \frac{15}{\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{15} - \left(x-\sqrt{x^2-1}\right)^{15}\right]$$

$$\Rightarrow \sqrt{x^2-1}\,\frac{dy}{dx} = 15\left[\left(x+\sqrt{x^2-1}\right)^{15} - \left(x-\sqrt{x^2-1}\right)^{15}\right]$$

Differentiating w.r.t. x, we get

$$\sqrt{x^2-1}\,\frac{d^2y}{dx^2} + \frac{x}{\sqrt{x^2-1}}\frac{dy}{dx}$$

$$= \frac{225}{\sqrt{x^2-1}}\left(1+\sqrt{x^2-1}\right)^{15} + \frac{225}{\sqrt{x^2-1}}\left(1+\sqrt{x^2-1}\right)^{15}$$

$$\Rightarrow (x^2 - 1)\,\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 225y$$

13. Rewrite the circle as

$$\left(x-\frac{5}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{3}{2} \qquad (1)$$

A point on (1) is

$x = \frac{5}{2}+\sqrt{\frac{3}{2}}\cos\theta$, $y = \frac{1}{2}+\sqrt{\frac{3}{2}}\sin\theta$, $0 \le \theta < 2\pi$.

$$\therefore\ (PQ)^2 = \left(\frac{5}{2}+\sqrt{\frac{3}{2}}\cos\theta\right)^2 + \left(\frac{5}{2}+\sqrt{\frac{3}{2}}\sin\theta\right)^2$$

$$= \frac{25}{4}+5\sqrt{\frac{3}{2}}\cos\theta + \frac{3}{2}\cos^2\theta$$

$$+\ \frac{25}{4}+5\sqrt{\frac{3}{2}}\sin\theta + \frac{3}{2}\sin^2\theta$$

$$= 14 + 5\sqrt{3}\cos\left(\theta-\frac{\pi}{4}\right)$$

$\therefore$ max $(PQ)^2 = 14 + 5\sqrt{3}$

14. $1 + 2\cot x\,(\operatorname{cosec} x + \cot x)$

$= 1 + 2\cot x\operatorname{cosec} x + 2\cot^2 x$

$= (1 + \cot^2 x) + 2\cot x\operatorname{cosec} x + \cot^2 x$

$= \operatorname{cosec}^2 x + 2\cot x\operatorname{cosec} x + \cot^2 x$

$= (\operatorname{cosec} x + \cot x)^2$

$$\therefore\ I = \int(\operatorname{cosec} x + \cot x)\,dx = \int\frac{1+\cos x}{\sin x}dx$$

$$= \int\frac{2\cos^2(x/2)}{2\cos(x/2)\sin(x/2)}dx = \int\cot\left(\frac{x}{2}\right)dx$$

$$= 2\log\left(\sin\frac{x}{2}\right) + c.$$

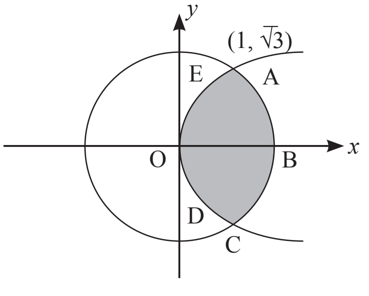
15. $\tan x + \cot x = \frac{1+\tan^2 x}{\tan x} = \frac{2}{\sin 2x}$

$$\therefore\ I = \int_{\pi/12}^{\pi/4} 8\cos 2x\frac{\sin^3 2x}{2^3}dx$$

$$= \frac{1}{4}\times\frac{1}{2}\left(\sin^4 2x\right)\Big]_{\pi/12}^{\pi/4}$$

$$= \frac{1}{8}\left(1-\left(\frac{1}{2}\right)^4\right) = \frac{15}{128}.$$

16.



Required area is shaded in the figure.

Area = area (ODCBAEO)

= 2 Area (O B A E O)

$$= 2\int_0^{\sqrt{3}}\sqrt{4-y^2}\,dy - 2\int_0^{\sqrt{3}}\frac{y^2}{3}dy$$

$$= \left[y\sqrt{4-y^2}+4\sin^{-1}\left(\frac{y}{2}\right)\right]_0^{\sqrt{3}} - \frac{2}{9}\left[y^3\right]^{\sqrt{3}}$$

$$= \sqrt{3}(1) + 4\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{2}{9}\left(3\sqrt{3}\right)$$

$$= \sqrt{3} + 4\left(\frac{\pi}{3}\right) - \frac{2}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} + \frac{4}{3}\pi.$$

17. Write the given differential equation as

$$\frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 3y \qquad (1)$$

$$\text{I.F.} = e^{\int\left(-\frac{1}{y}\right)dy} = e^{-\ln(y)} = \frac{1}{y}$$

Multiply (1) by $\frac{1}{y}$ to obtain

$$\frac{d}{dy}\left(\frac{1}{y}x\right) = 3$$

$$\Rightarrow \frac{x}{y} = 3y + C$$

As it passes through (1, 1), we get

$$1 = 3 + C \Rightarrow C = -2$$

$$\therefore \ x = (3y - 2)y$$

It passes through $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

18. $tx - 2y - 3t = 0$

$x - 2ty + 3 = 0$

$$\frac{x}{-6-6t^2} = \frac{-y}{3t+3t} = \frac{1}{-2t^2+2}$$

$$\Rightarrow x = \frac{-6\left(t^2+1\right)}{-2\left(t^2-1\right)} = \frac{3\left(t^2+1\right)}{t^2-1}$$

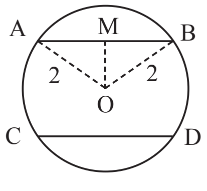
$$y = \frac{6t}{2\left(t^2-1\right)} = \frac{3t}{t^2-1}$$

$$\therefore \left(\frac{x}{3}\right)^2 - 4\left(\frac{y}{3}\right)^2 = \frac{\left(t^2+1\right)^2 - 4t^2}{\left(t^2-1\right)^2} = \frac{\left(t^2-1\right)^2}{\left(t^2-1\right)^2} = 1$$

$$\Rightarrow \frac{x^2}{9} - \frac{y^2}{(3/2)^2} = 1$$

It's a hyperbola with length of conjugate axis as 3

19. As $\sec^{-1}(7) = \cos^{-1}(1/7)$, both the chords are equidistant from the centre.



Let AB = $2a$.

From ΔOAB

Cos(∠AOB)

$$= \frac{2^2+2^2-(2a)^2}{(2)(2)(2)}$$

$$\Rightarrow \frac{1}{7} = \frac{8-4a^2}{8} = 1 - \frac{a^2}{2}$$

$$\Rightarrow \frac{a^2}{2} = 1 - \frac{1}{7} = \frac{6}{7} \Rightarrow a = 2\sqrt{\frac{3}{7}}$$

Now, $OM^2 = OA^2 - AM^2$

$$= 2^2 - a^2 = 4 - \frac{12}{7} = \frac{16}{7}$$

$$\Rightarrow OM = \frac{4}{\sqrt{7}}$$

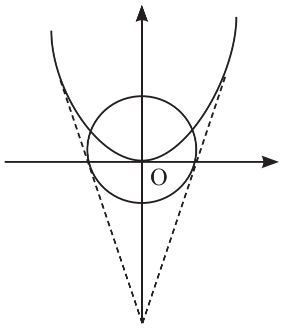
∴ distance between porallel chords

$$= 2(OM) = \frac{8}{\sqrt{7}}.$$

20. Equation of any tangent to $x^2 = 4y$ is

$$x = my + \frac{1}{m}$$

It will be tangent to $x^2 + y^2 = 4$ if



$$\frac{\left|m(0) - 0 + \frac{1}{m}\right|}{\sqrt{m^2+1}} = 2$$

$$\Rightarrow \left(\frac{1}{m}\right)^2 = 4\left(m^2+1\right)$$

$$\Rightarrow 4m^4 + 4m^2 + 1 = 2$$

$$\Rightarrow (2m^2 + 1)^2 = 2$$

$$\Rightarrow 2m^2 + 1 = \sqrt{2}$$

$$\Rightarrow m^2 = \frac{\sqrt{2}-1}{2}$$

Let $m_1 = \sqrt{\frac{\sqrt{2}-1}{2}}$, $m_2 = -\sqrt{\frac{\sqrt{2}-1}{2}}$

∴Two common tangents are

$$x = m_1y + \frac{1}{m_1},\ x = m_2y + \frac{1}{m_2}$$

These two intersect at

$$p\left(0, \frac{1}{m_1m_2}\right)$$

$$\therefore \quad OP = \frac{1}{m_1 m_2} = \frac{2}{\sqrt{2}-1} = 2\left(\sqrt{2}+1\right).$$

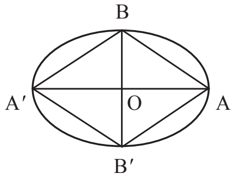
21. $$2ae = 6 \Rightarrow 2a\left(\frac{3}{5}\right) = 6$$

$\Rightarrow$ $2a = 10 \Rightarrow a = 5$

Now, $b^2 = a^2 - a^2e^2$

$= 25 - 9 = 16$

$\Rightarrow b = 4$



Area of quadrilateral

= 4[area ΔOAB]

$= (4)\left(\frac{1}{2}\right)(5)(4) = 40.$

22. Equation of plane containing

Both the lines is

$$\begin{vmatrix} x+1 & y-1 & z-3 \\ 6 & 7 & 8 \\ 3 & 5 & 7 \end{vmatrix} = 0$$

$\Rightarrow$ $(49 - 40)(x + 1) - (42 - 24)(y - 1)$

$+ (30 - 21)(z - 3) = 0$

$\Rightarrow$ $(x + 1) - 2(y - 1) + (z - 3) = 0$

Note that it passes through (1, 2, 3)

As points (2, –4, 2), (–1, –2, –1) do not lie on (1), ensures (a), (b) are not possible.

Also, (0, 0, 0) lies on (1) and line joining (1, –2, 1) and (0, 0, 0) is perpendicular to (1)

23. Let a, b, c, be direction ratios of the line of intersection of the planes

$$3x - y + z = 1 \qquad (1)$$

and

$$x + 4y - 2z = 2 \qquad (2)$$

Then

$3a - b + c = 0$

$a + 4b - 2c = 0$

Solving we get

$$\frac{a}{2-4} = \frac{-b}{-6-1} = \frac{c}{12+1}$$

$$\Rightarrow \quad \frac{a}{-2} = \frac{b}{7} = \frac{c}{13}$$

Note that (4/7, 0, 5/7) doesn't lie on (1)

Also (6/13, 5/13, 0) lies on both (1) and (2)

$\therefore$ the correct answer is (c)

24. Area of parallelogram is

$$= \frac{1}{2}|\mathbf{d}_1 \times \mathbf{d}_2|$$

$$= \frac{1}{2}\left|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -6 & 0 \\ 3 & 4 & -12 \end{vmatrix}\right|$$

$$= \frac{1}{2}\left|72\mathbf{i} + 96\mathbf{j} + 50\mathbf{k}\right|$$

$$= \sqrt{36^2 + 48^2 + 25^2} = \sqrt{4225} = 65.$$

25. Let age of new teacher be x years, then

$$\frac{40 \times 25 - 60 + x}{25} = 39$$

$$\Rightarrow \; 40 - \frac{60 - x}{25} = 39$$

$$\Rightarrow \quad \frac{60 - x}{25} = 1 \Rightarrow x = 35.$$

26. P(P or Q but not R hits the target)

= P((P ∪ Q) ∩ R′)

= P(P ∪ Q) P(R′) = [1 – P (P′ ∩ Q′)] P (R′)

= [1 – P(P′) P (Q′)]P (R′)

$$= \left[1 - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\right]\left(\frac{3}{8}\right) = \frac{21}{64}.$$

27. X = number of heads obtained,

Then X ~ B (n, p) where $n = 8$ $p = ½$

P(obtaining at leart one head and at leant one tail)

= 1 – P(no head) – P(no tail)

$$= 1 - \left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^8 = 1 - \frac{1}{2^7}$$

$$= \frac{127}{128}$$

28. Let $$\Delta(x) = \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}$$

$= \cos^3 x - \sin^3 x$

$= (\cos x - \sin x)(\cos^2 x + \sin^2 x + \cos x \sin x)$

$= (\cos x - \sin x)\left(1 + \frac{1}{2}\sin 2x\right)$

Now, $\Delta(x) = 0 \Rightarrow \tan x = 1$

$$\Rightarrow \ x = \frac{\pi}{4}, \frac{5\pi}{4}$$

$$\therefore \ S = \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$$

Now, $\tan\left(\frac{\pi}{4}+\frac{\pi}{3}\right) + \tan\left(\frac{5\pi}{4}+\frac{\pi}{3}\right)$

$$= \tan\left(\frac{\pi}{4}+\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$$

$$= 2\frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{3}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{3}\right)}$$

$$= 2\frac{1+\sqrt{3}}{1-\sqrt{3}} = -2\ \frac{\left(\sqrt{3}+1\right)^2}{3-1}$$

$$= -(3 + 1 + 2\sqrt{3}) = -4 - 2\sqrt{3}$$

29. Put $x^2 = \cos 2\theta$, then

$$1 + x^2 = 1 + \cos 2\theta = 2\cos^2\theta$$

And $1 - x^2 = 1 - \cos 2\theta = 2\sin^2\theta$

$$\therefore \ y = \tan^{-1}\left(\frac{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$$

$$= \frac{\pi}{4}+\theta = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

30. $(\sim p) \vee (p \wedge \sim q)$

$= (\sim p \vee p) \wedge (\sim p \vee \sim q)$

$= T \wedge (\sim p \vee \sim q)$

$= \sim p \vee \sim q$

$= p \rightarrow \sim q$

JEE (Main) 2017 Questions with Solution (9th April – online)

1. The function $f : \mathbf{N} \to \mathbf{N}$ defined by $f(x) = x - 5\left[\frac{x}{5}\right]$, where **N** is the set of natural numbers and $[x]$ denotes the greatest integer less than or equal to x, is:
   (a) one–one and onto.
   (b) one–one but not onto
   (c) onto but not one–one
   (d) neither one–one or onto
2. The sum of all the real values of x satisfying the equation $2^{(x-1)(x^2+5x-50)} = 1$ is:
   (a) 16 (b) 14
   (c) –4 (d) –5
3. The equation $I_m\left(\frac{iz-2}{z-i}\right)+1 = 0$, $n \in C, z \neq i$ represents a part of a circle having radius equal to:
   (a) 2 (b) 1
   (c) $\frac{3}{4}$ (d) $\frac{1}{2}$
4. For two 3 × 3 matrices A and B, let A + B = 2B′ and 3A + 2B = I_3, where B′ is the transpose of B and I_3 is 3 × 3 identity matrix. Then:
   (a) 5A + 10B = $2I_3$ (b) 10 A + 5B = $3I_3$
   (c) B + 2A = I_3 (d) 3A + 6B = $2I_3$
5. If $x = a$, $y = b$, $z = c$ is a solution of the system of linear equations
   $x + 8y + 7z = 0$
   $9x + 2y + 3z = 0$
   $x + y + z = 0$
   such that the point (a, b, c) lies on the plane $x + 2y + z = 6$, then $2a + b + c$ equals:
   (a) –1 (b) 0
   (c) 1 (d) 2
6. The number of ways in which 5 boys and 3 girls can be seated on a round table if a particular boy B_1 and a particular girl G_1 never sit adjacent to each other, is:
   (a) 5 × 6! (b) 6 × 6!
   (c) 7! (d) 5 × 7!
7. The coefficient of x^{-5} in the binomial expansion of $\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10}$, where $x \neq 0, 1$, is:
   (a) 1 (b) 4
   (c) –4 (d) –1
8. If three positive numbers a, b and c are in A.P. such that $abc = 8$, then the minimum possible value of b is:
   (a) 2 (b) $4^{\frac{1}{3}}$
   (c) $4^{\frac{2}{3}}$ (d) 4
9. Let $S_n = \frac{1}{1^3}+\frac{1+2}{1^3+2^3}+\frac{1+2+3}{1^3+2^3+3^3}+ \ldots\ldots\ldots + \frac{1+2+\ldots\ldots+n}{1^3+2^3+\ldots\ldots n^3}$. If 100 $S_n = n$ then n is equal to:
   (a) 199 (b) 99
   (c) 200 (d) 19
10. The value of k for which the function $f(x)=\begin{cases}\left(\frac{4}{5}\right)^{\frac{\tan 4x}{\tan 5x}}, 0<x<\frac{\pi}{2}\\ k+\frac{2}{5}, x=\frac{\pi}{2}\end{cases}$ is continuous at $x=\frac{\pi}{2}$, is
    (a) $\frac{17}{20}$ (b) $\frac{2}{5}$
    (c) $\frac{3}{5}$ (d) $-\frac{2}{5}$
11. If $2x = y^{\frac{1}{5}}+y^{-\frac{1}{5}}$ and $(x^2-1)\frac{d^2y}{dx^2}+\lambda x\frac{dy}{dx}+ky = 0$ then $\lambda + k$ is equal to
    (a) –23 (b) –24
    (c) 26 (d) –26
12. The function f defined by $f(x) = x^3 - 3x^2 + 5x + 7$, is
    (a) increasing in **R**
    (b) decreasing in **R**
    (c) decreasing in (0, ∞) and increasing in (–∞, 0)
    (d) increasing in (0, ∞) and decreasing in (–∞, 0)
13. Let f be polynomial function such that $f(3x) = f'(x) \cdot f''(x)$, for all $x \in \mathbf{R}$, then
    (a) $f(2) + f'(2) = 28$
    (b) $f''(2) - f'(2) = 0$
    (c) $f''(2) - f(2) = 4$
    (d) $f(2) - f'(2) + f''(2) = 10$
14. If $f\left(\frac{3x-4}{3x+4}\right) = x+2, x \neq -\frac{4}{3}$, and $\int f(x)dx$ = A log |1 – x| + Bx + C, then the ordered pair

(A, B) is equal to (where C is a constant of integration,

(a) $\left(\frac{8}{3},\frac{2}{3}\right)$ (b) $\left(-\frac{8}{3},\frac{2}{3}\right)$

(c) $\left(-\frac{8}{3},-\frac{2}{3}\right)$ (d) $\left(\frac{8}{3},-\frac{2}{3}\right)$

15. If $\int_1^2 \frac{dx}{\left(x^2-2x+4\right)^{\frac{3}{2}}}=\frac{k}{k+5}$, then k is equal to:

(a) 1 (b) 2

(c) 3 (d) 4

16. If $\lim\limits_{n\to\infty}\frac{1^a+2^a+......+n^a}{(n+1)^{a-1}\left[(na+1)+(na+2)+.....+(na+n\right]}=\frac{1}{60}$ for some positive real number a, then a is equal to

(a) 7 (b) 8

(c) $\frac{15}{2}$ (d0 $\frac{17}{2}$

17. A tangent to the curve, $y = f(x)$ at P(x, y) meets x–axis at A and y–axis at B. If AP : BP = 1 : 3 and $f(1) = 1$, then the curve also passes through the point:

(a) $\left(\frac{1}{3},24\right)$ (b) $\left(\frac{1}{2},4\right)$

(c) $\left(2,\frac{1}{8}\right)$ (d) $\left(3,\frac{1}{28}\right)$

18. A square, of each side 2, lies above the x–axis and has one vertex at the origin. If one of the sides passing through the origin makes an angle 30° with the positive direction of the x–axis, then the sum of the x–coordinates of the vertices of the square is:

(a) $2\sqrt{3}-1$ (b) $2\sqrt{3}-2$

(c) $\sqrt{3}-2$ (d) $\sqrt{3}-1$

19. A line drawn through the point P(4, 7) cuts the circle $x^2 + y^2 = 9$ at the points A and B. Then PA · PB is equal to:

(a) 53 (b) 56

(c) 74 (d) 65

20. The eccentricity of an ellipse having centre at the origin, axes along the co–ordinate axes and passing through the points (4, –1) and (–2, 2) is:

(a) $\frac{1}{2}$ (b) $\frac{2}{\sqrt{5}}$

(c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{4}$

21. If $y = mx + c$ is the normal at a point on the parabola $y^2 = 8x$ whose focal distance is 8 units, then $|c|$ is equal to:

(a) $2\sqrt{3}$ (b) $8\sqrt{3}$

(c) $10\sqrt{3}$ (d) $16\sqrt{3}$

22. If a variable plane, at a distance of 3 units from the origin, intersects the coordinate axes at A, B and C, then the locus of the centroid of ΔABC is

(a) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=1$ (b) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=3$

(c) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{9}$ (d) $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=9$

23. If the line, $\frac{x-3}{1}=\frac{y+2}{-1}=\frac{z+\lambda}{-2}$ lies in the plane, $2x - 4y + 3z = 2$, then the shortest distance between this line and the line $\frac{x-1}{12}=\frac{y}{9}=\frac{z}{4}$ is:

(a) 2 (b) 1

(c) 0 (d) 3

24. If the vector $\mathbf{b}=3\mathbf{j}+4\mathbf{k}$ is written as the sum of a vector $\mathbf{b}_1$ parallel to $\mathbf{a}=\mathbf{i}+\mathbf{j}$ and a vector $\mathbf{b}_2$, perpendicular to **a**, then $\mathbf{b}_1\times\mathbf{b}_2$ is equal to:

(a) $-3\mathbf{i}+3\mathbf{j}-9\mathbf{k}$ (b) $6\mathbf{i}-6\mathbf{j}+\frac{9}{2}\mathbf{k}$

(c) $-6\mathbf{i}+6\mathbf{j}-\frac{9}{2}\mathbf{k}$ (d) $3\mathbf{i}-3\mathbf{j}+9\mathbf{k}$

25. From a group of 10 men and 5 women, four member committees are to be formed each of which must contain at least one woman. Then the probability for these committees to have more women than men, is:

(a) $\frac{21}{220}$ (b) $\frac{3}{11}$

(c) $\frac{1}{11}$ (d) $\frac{2}{23}$

26. Let E and F be two independent events. The probability that both E and F happen is $\frac{1}{12}$ and the probability that neither E nor F happens is $\frac{1}{2}$, then a value of $\frac{P(E)}{P(F)}$ is

(a) $\frac{4}{3}$ (b) –

(c) $\frac{1}{3}$ (d) $\frac{5}{12}$

27. The sum of 100 observations and the sum of their squares are 400 and 2475, respectively. Later on, three observations, 3, 4 and 5, were found to be incorrect. If the incorrect observations are omitted, then the variance of the remaining observations is:

(a) 8.25 (b) 8.50
(c) 8.00 (d) 9.00

28. A value of x satisfying the equation $\sin[\cot^{-1}(1 + x)] = \cos[\tan^{-1}x]$ is
(a) $-\frac{1}{2}$ (b) -1
(c) 0 (d) $\frac{1}{2}$

29. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60°. If the area of the quadrilateral is $4\sqrt{3}$, then the perimeter of the quadrilateral is:
(a) 12.5 (b) 13.2
(c) 12 (d) 13

30. Contra positive of the statement
‘If two numbers are not equal, then their squares are not equal’, is:
(a) If the squares of two numbers are equal, then the numbers are equal.
(b) If the squares of two numbers are equal, then the numbers are not equal.
(c) If the squares of two numbers are not equal, then the numbers are not equal.
(d) If the squares of two numbers are not equal, then the numbers are equal.

Answers

1. (d)	**2.** (c)	**3.** (c)	**4.** (b)
5. (c)	**6.** (a)	**7.** (a)	**8.** (a)
9. (a)	**10.** (c)	**11.** (b)	**12.** (a)
13. (b)	**14.** (b)	**15.** (a)	**16.** (a)
17. (c)	**18.** (b)	**19.** (b)	**20.** (c)
21. (c)	**22.** (a)	**23.** (c)	**24.** (b)
25. (c)	**26.** (a)	**27.** (d)	**28.** (a)
29. (d)	**30.** (a)		

Hints and Solutions

1. Note that $f(5) = 5-5\left[\frac{5}{5}\right] = 0 = f(10)$.

Thus, function f is not defined as $0 \notin \mathbf{N}$, the codomain.

As, for $0 \le r \le 4$

$f(5\text{m} + r)$

$= 5\text{m} + r - 5\text{m}$

$= r$

Range $f = (0, 1, 2, 3, 4)$

In view of one may answer it as (d).

2. $$2^{(x-1)(x^2+5x-50)} = 1 \qquad \text{(i)}$$

$\Rightarrow$ $(x - 1)(x^2 + 5x - 50) = 0$

$\Rightarrow$ $(x - 1)(x + 10)(x - 5) = 0$

$\therefore$ $x = 1, 5, -10$

$\therefore$ sum of the real values satisfying (i) is $1 + 5 - 10 = -4$.

3. Let $z = x + iy$

Now, $\text{Im}\left(\frac{iz-2}{z-1}\right)+1 = 0$

$\Rightarrow$ $\text{Im}\frac{ix-y-2}{x+i(y-1)}+1 = 0$

$\Rightarrow$ $\text{Im}\frac{(ix-y-2)(x-i(y-1))}{x^2+(y-1)^2}+1 = 0$

$\Rightarrow$ $\frac{x^2+(y+2)(y-1)}{x^2+(y-1)^2}+1 = 0$

$\Rightarrow$ $2x^2+(y-1)(2y+1) = 0$

$\Rightarrow$ $2x^2 + 2y^2 - y - 1 = 0$

$x^2+y^2-\frac{1}{2}y-\frac{1}{2} = 0$

$\Rightarrow$ $r^2 = \left(\frac{1}{4}\right)^2+\frac{1}{2}=\frac{9}{16}$

$\Rightarrow r = \frac{3}{4}$

4. $A + B = 2B'$ (1)

$3A + 2B = I_3$ (2)

From (1), (2), we get

$A' + B' = 2B$ (3)

and $3A' + 2B' = I_3$ (4)

From (1) and (3)

$A - A' + B - B' = 2(B' - B)$

$\Rightarrow$ $A - A' = 3(B' - B)$ (5)

From (2) and (4)

$3(A - A') + 2(B - B') = 0$ (6)

From (5) and (6)

$9(B' - B) - 2(B' - B) = 0$

$\Rightarrow$ $7(B' - B) = 0 \Rightarrow B' = B$

$\Rightarrow$ From (1) we get A = B

and from (2) $A = B = \frac{1}{5}I_3$

Now, $5A + 10B = 15A = 3I_3 \neq 2I_2$

$10A + 5B = 15A = 3I_3$

It is easy to check (c), (d) are not true.

5. $x + 8y + 7z = 0$ (1)

$9x + 2y + 3z = 0$ (2)

$x + y + z = 0$ (3)

From (2), (3)

$$\frac{x}{2-3} = \frac{-y}{9-3} = \frac{z}{9-2}$$

$\Rightarrow \frac{x}{-1} = \frac{y}{-6} = \frac{z}{7} = k$ (say)

$\Rightarrow x = -k, y = -6k, z = 7k$

Note $(-k, -6k, 7k)$ satisfies (1)

Let $a = -k$, $b = -6k$, $c = 7k$

As (a, b, c) lies on $x + 2y + z = 6$

we get,

$-k - 12k + 7k = 6 \Rightarrow k = -1$

$\therefore a = 1, b = 6, c = -7$

Now, $2a + b + c = 2 + 6 - 7 = 1$

6. Required number of ways

= Total number of ways – Number of ways in which B_1 and G_1 are together

$= 7! - (2!)6! = (5)(6!)$

7. $$\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} = \frac{\left(x^{\frac{1}{3}}\right)^3 + 1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1}$$

$$= \frac{\left(x^{\frac{1}{3}} + 1\right)\left(x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1\right)}{\left(x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1\right)}$$

Also, $$\frac{x-1}{x - x^{\frac{1}{2}}} = \frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\sqrt{x}\left(\sqrt{x}-1\right)} = 1 + x^{-\frac{1}{2}}$$

$$\therefore \left(\frac{x+1}{x^{\frac{2}{3}} - x^{\frac{1}{3}} + 1} - \frac{x-1}{x - x^{\frac{1}{2}}}\right)^{10}$$

$$= \left(x^{\frac{1}{3}} - x^{-\frac{1}{2}}\right)^{10}$$

$$T_{r+1} = {}^{10}C_r \left(x^{\frac{1}{3}}\right)^{10-r} \left(-x^{\frac{-1}{2}}\right)^r$$

$$= {}^{10}C_r x^{\frac{10-r}{3} - \frac{r}{2}} (-1)^r$$

For coefficient of x^{-5}, set

$\frac{10-r}{3} - \frac{r}{2} = -5$

$\Rightarrow 20 - 2r - 3r = -30$

$\Rightarrow r = 10$

$\therefore$ coefficient of x^{-5} is ${}^{10}C_{10}(-1)^{10} = 1$

8. $3b = a + b + c \geq 3(abc)^{\frac{1}{3}} = 3(8)^{\frac{1}{3}}$

$\Rightarrow b \geq 2$

$\therefore$ minimum value of b is 2.

9. $$t_r = \frac{1+2+.....+r}{1^3+2^3+.....+r^3}$$

$$= \frac{\frac{r(r+1)}{2}}{\frac{r^2(r+1)^2}{4}} = \frac{2}{r(r+1)}$$

$$= 2\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$\Rightarrow S_n = 2\sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right) = 2\left(1 - \frac{1}{n+1}\right)$

$\Rightarrow S_n = \frac{2n}{n+1}$

Now, $S_n = \frac{n}{100}$ $\therefore \frac{2n}{n+1} = \frac{n}{100}$

$\Rightarrow n = 199$

10. As f is continuous at $x = \frac{\pi}{2}$,

$$f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}}\left(\frac{4}{5}\right)^{\frac{\tan 4x}{\tan 5x}} = \left(\frac{4}{5}\right)^{\lim_{x\to\frac{\pi}{2}}\frac{\tan 4x}{\tan 5x}} = \left(\frac{4}{5}\right)^{\circ} = 1$$

$\therefore k + \frac{2}{5} = 1 \quad \Rightarrow k = \frac{3}{5}$.

11. $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$

$\Rightarrow \left(y^{\frac{1}{5}}\right)^2 - 2x\left(y^{\frac{1}{5}}\right) + 1 = 0$

$\Rightarrow y^{\frac{1}{5}} = x \pm \sqrt{x^2-1}$

$\Rightarrow y = \left(x \pm \sqrt{x^2-1}\right)^5$

$$\Rightarrow \frac{dy}{dx} = 5\left(x \pm \sqrt{x^2-1}\right)^4\left(1 \pm \frac{x}{\sqrt{x^2-1}}\right)$$

$$= \pm\frac{5y}{\sqrt{x^2-1}}$$

$\Rightarrow \quad \sqrt{x^2-1}\frac{dy}{dx}=\pm 5y$

$\Rightarrow \quad \sqrt{x^2-1}\frac{d^2y}{dx^2}+\frac{x}{\sqrt{x^2-1}}\frac{dy}{dx} = \pm 5\frac{dy}{dx}$

$\Rightarrow \quad \left(x^2-1\right)\frac{d^2y}{dx^2}+x\frac{dy}{dx} = \pm 5\left(\sqrt{x^2-1}\frac{dy}{dx}\right)$

$= \pm 5\left(\pm 5y\right)=25y$

$\therefore \quad \lambda = 1,\ k = -25$

$\Rightarrow \quad \lambda + k = -24$

12. $f(x) = x^3 - 3x^2 + 5x + 7$

$f'(x) = 3x^2 - 6x + 5 = 3(x-1)^2 + 2 > 0 \quad \vee x\in R$

Thus, f is increasing in **R**.

13. Let $\deg(f(x)) = n$, then

$f(3x) = f'(x)f''(x)$

$\Rightarrow \quad \deg(f(3x)) = \deg(f'(x)) + \deg(f''(x))$

$\Rightarrow \quad n = (n-1) + (n-2)$

$\Rightarrow \quad n = 3$

Let $f(x) = ax^3 + bx^2 + cx + d$, then

$f(3x) = f'(x)f''(x)$

$\Rightarrow \quad 27ax^3 + 9bx^2 + 3cx + d$

$= (3ax^2 + 2bx + c)(6ax + 2b)$

$\Rightarrow \quad 27a = 18a^2$

$\Rightarrow \quad a=\frac{3}{2} \qquad (\because a\neq 0)$

Next, $9b = 6ab + 12ab = 18ab = 27b$

$\Rightarrow \quad b = 0$

and $3c = 6ac + 4b^2 = 9c + 0$

$\Rightarrow \quad c = 0$

and $d = 2bc = 0$

Thus, $f(x) = \frac{3}{2}x^3$

$\Rightarrow f'(x) = \frac{9}{2}x^2$, $f''(x) = 9x$

$f''(2) - f'(2) = 0.$

14. $f\left(\frac{3x-4}{3x+4}\right) = x + 2$

Put $\frac{3x-4}{3x+4} = t \quad \Rightarrow 1-\frac{8}{3x+4} = t$

$1 - t = \frac{8}{3x+4}$

$\Rightarrow \quad 3x + 4 = \frac{8}{1-t}$

$\Rightarrow \quad x = \frac{1}{3}\left(\frac{8}{1-t}-4\right) = \frac{1}{3}\left(\frac{8-4+4t}{1-t}\right)$

$= \frac{4}{3}\left(\frac{1+t}{1-t}\right)$

$\therefore\ f(x) = \frac{4}{3}\left(\frac{1+x}{1-x}\right)+2 = \frac{4}{3}\left(\frac{2}{1-x}-1\right)+2$

$= \frac{8}{3}\frac{1}{1-x}+\frac{2}{3}$

$\Rightarrow \quad \int f(x)dx=-\frac{8}{3}\log|1-x|+\frac{2}{3}x+C$

$\therefore \quad A = -\frac{8}{3}$

$B = \frac{2}{3}.$

15. $I = \int_1^2 \frac{dx}{\left(x^2-2x+4\right)^{\frac{3}{2}}}$

$= \int_1^2 \frac{dx}{\left[\left(x-1\right)^2+3\right]^{\frac{3}{2}}}$

Put $x - 1 = \sqrt{3}\tan\theta$

$I = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2\theta}{3\sqrt{3}\sec^3\theta}d\theta = \frac{1}{3}\int_0^{\frac{\pi}{6}}\cos\theta d\theta$

$= \frac{1}{3}\left[\sin\theta\right]_0^{\frac{\pi}{6}} = \frac{1}{6}$

Now, $\quad \frac{k}{k+5}=\frac{1}{6}$

$\Rightarrow \quad k = 1$

16. $L = \lim_{n\to\infty}\frac{1^a+2^a+.......+n^a}{(n+1)^{a-1}\left[(na+1)+(na+2)+....(na+n)\right]}$

$= \lim_{n\to\infty}\frac{n^a\sum_{k=1}^{n}\left(\frac{k}{n}\right)^a}{n^a\left(1+\frac{1}{n}\right)^{a-1}\sum_{k=1}^{n}\left(a+\frac{k}{n}\right)}$

$= \lim_{n\to\infty}\frac{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^a}{\left(1+\frac{1}{n}\right)^{a-1}\frac{1}{n}\sum_{k=1}^{n}\left(a+\frac{k}{n}\right)}$

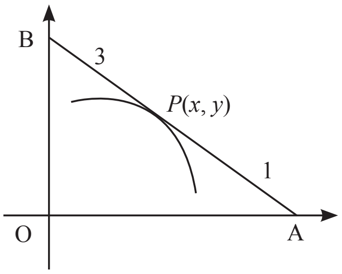
$= \frac{\int_0^1 x^a dx}{\int_0^1 (a+x)dx} = \frac{\frac{1}{a+1}}{\frac{1}{2}\left((a+1)^2-a^2\right)}$

$= \frac{2}{(a+1)(2a+1)}=\frac{1}{60}$

$\Rightarrow \quad (a + 1)(2a + 1) = 120$

$\Rightarrow \quad a = 7 \qquad (\because a>0)$

17. An equation of tangent at $P(x, y)$ is



$$Y - y = \frac{dy}{dx}(X - x)$$

It meets x – axis in

$$A\left(x - \frac{dx}{dy}y, 0\right)$$

and y–axis in $B\left(0, y - x\frac{dy}{dx}\right)$

Coordinates of the point which divide it in the ratio 1 : 3 is

$$\left(\frac{3}{4}\left(x - \frac{dx}{dy}y\right), \frac{1}{4}\left(y - x\frac{dy}{dx}\right)\right) = (x, y)$$

$$\Rightarrow \frac{3}{4}\left(x - y\frac{dx}{dy}\right) = x, \quad \frac{1}{4}\left(y - x\frac{dy}{dx}\right) = y$$

$$\Rightarrow 3\frac{dx}{x} + \frac{dy}{y} = 0$$

$\Rightarrow \log(x^3y) = \text{const.}$

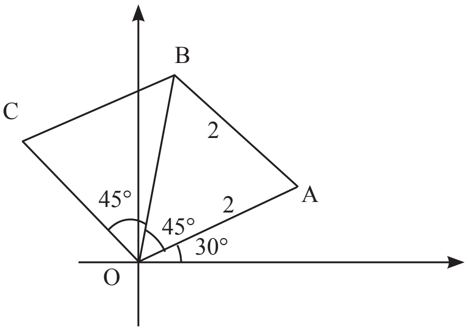
$\Rightarrow x^3y = c$

When $x = 1$, $y = 1$, therefore $c = 1$

Thus, required curve $x^3y = 1$ (i)

Note that $\left(2, \frac{1}{8}\right)$ satisfies (i)

18. x–coordinate of A is $2\cos 30° = \sqrt{3}$, x–coordinate of B is $2\sqrt{2}\cos 75°$ and x–coordinate of C is $2\cos(120°) = 2\cos(90° + 30°) = -2\sin 30° = -1$



Thus, sum of x–coordinates

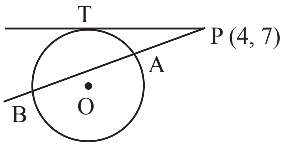
$$= 0 + \sqrt{3} + 2\sqrt{2}\cos 75° - 1$$

$$= \sqrt{3} + 2\sqrt{2}\cos(45° + 30°) - 1$$

$$= \sqrt{3} + 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\right) - 1$$

$$= \sqrt{3} + \sqrt{3} - 1 - 1 = 2\sqrt{3} - 2$$

19.



$(PA)(PB) = PT^2$ and $PT^2 = 4^2 + 7^2 - 9 = 56$

20. Let equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

As it passes through (4, –1) and (–2, 2) we get

$$\frac{16}{a^2} + \frac{1}{b^2} = 1 \text{ and } \frac{4}{a^2} + \frac{4}{b^2} = 1$$

Solving we get $b^2 = 5$, $a^2 = 20$

Now $a^2(1 - e^2) = b^2$

$\Rightarrow 20(1 - e^2) = 5$

$$\Rightarrow e = \frac{\sqrt{3}}{2}$$

21. $c = -2am - am^3 = -4m - 2m^3$

$\therefore y = mx - 4m - 2m^3$

is a normal at $P(2m^2, 4m)$

Distance of P from focus

= distance of P from directrix ($x = -2$)

$= 2(m^2 + 1)$

As $2(m^2 + 1) = 8$

$\Rightarrow m = \pm\sqrt{3}$

For $m = -\sqrt{3}$, $c = 4\sqrt{3} + 2(3\sqrt{3}) = 10\sqrt{3}$

22. Let the centroid of ΔABC be $G(\alpha, \beta, \gamma)$ then coordinates of A, B, C are $(3\alpha, 0, 0)$, $(0, 3\beta, 0)$, $(0, 0, 3\gamma)$ respectively

Equation of the plane through A, B and C is

$$\frac{x}{3\alpha} + \frac{y}{\beta} + \frac{z}{3\gamma} = 1$$

Its distance from (0, 0, 0) is

$$\frac{|0 - 3|}{\sqrt{\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}}} = 3$$

$$\Rightarrow \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = 1$$

Thus, locus of centroid of ΔABC is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=1$$

23. As $\frac{x-3}{1}=\frac{y+2}{-1}=\frac{z+\lambda}{-2}$ lie in the plane

$2x-4y+3z=2$

$(3,-2,-\lambda)$ lies on the plane.

$\therefore$ 2(3) – 4(–2) + 3(–λ) = 2

$\Rightarrow$ λ = 4

Now $\quad \mathbf{d}_1=\mathbf{i}-\mathbf{j}-2\mathbf{k}$ and $\mathbf{d}_2+12\mathbf{i}+9\mathbf{j}-4\mathbf{k}$

$$\mathbf{d}_1\times\mathbf{d}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & -2\\ 12 & 9 & 4\end{vmatrix}$$

$$= 14\mathbf{i}-28\mathbf{j}+21\mathbf{k}$$

$$= 7(2\mathbf{i}-4\mathbf{j}+3\mathbf{k})$$

Also, $\begin{vmatrix}3-1 & -2-0 & -4-0\\ 1 & -1 & -2\\ 12 & 9 & 4\end{vmatrix}$

= 14(2) – 28(–2) + 21(–4)

= 7(4 + 8 – 12) = 0

$\therefore$ Shortest distance between two lines is 0.

24. We have $\mathbf{b}=\mathbf{b}_1+\mathbf{b}_2$ where

$\mathbf{b}_1=\alpha\mathbf{a}_1, \mathbf{b}_2\cdot\mathbf{a} = 0$

$\therefore \quad \mathbf{a}\cdot\mathbf{b}=\alpha\mathbf{a}\cdot\mathbf{a}$

$\Rightarrow \quad 3=2\alpha \Rightarrow \alpha=\frac{3}{2}$

Thus, $\mathbf{b}_1=\frac{3}{2}\mathbf{a}\Rightarrow\vec{\mathbf{b}}_2=\vec{\mathbf{b}}=\frac{3}{2}\vec{\mathbf{a}}$

$\Rightarrow \quad \mathbf{b}_1\times\mathbf{b}_2=\frac{3}{2}\mathbf{a}\times\left(\mathbf{b}-\frac{3}{2}\mathbf{a}\right) = \frac{3}{2}\mathbf{a}\times\mathbf{b}$

$$= \frac{3}{2}(\mathbf{i}+\mathbf{j})\times(3\mathbf{j}+4\mathbf{k})$$

$$= \frac{3}{2}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & 0\\ 0 & 3 & 4\end{vmatrix}$$

$$= \frac{3}{2}(4\mathbf{i}-4\mathbf{j}+3\mathbf{k}) = 6\mathbf{i}-6\mathbf{j}+\frac{9}{2}\mathbf{k}.$$

25. Number of ways of choosing committee so that there is at least one woman is ${}^{15}C_4-{}^{10}C_4$ = 1365 – 210 = 1155

Number of ways in which there are more women than men in the committee

$= \left({}^{10}C_0\right)\left({}^{5}C_4\right)+\left({}^{10}C_1\right)\left({}^{5}C_3\right)$

= (1)(5) + (10)(10)

= 105

$\therefore$ Probability of the required event

$= \frac{105}{1155}=\frac{1}{11}.$

26. $\quad P(E\cap F)=\frac{1}{12} \Rightarrow P(E)P(F)=\frac{1}{12}$

and $P(E'\cap F')=\frac{1}{2} \Rightarrow P(E')P(F')=\frac{1}{2}$

$\Rightarrow \quad (1-P(E))(1-P(F))=\frac{1}{2}$

$\Rightarrow \quad 1-(P(E)+P(F))+\frac{1}{12}=\frac{1}{2}$

$\Rightarrow \quad P(E)+P(F)=\frac{7}{12}$

$\therefore$ Equation whose roots are P(E), P(F) is

$x^2-\left(\frac{7}{12}\right)x+\frac{1}{12} = 0$

or $\quad 12x^2-7x+1 = 0$

$\Rightarrow \quad (3x-1)(4x-1) = 0$

$\Rightarrow \quad x=\frac{1}{3},\frac{1}{4}$

$\therefore \quad \frac{P(E)}{P(F)}=\frac{\frac{1}{3}}{\frac{1}{4}}$ or $\frac{\frac{1}{4}}{\frac{1}{3}}$

$\Rightarrow \quad \frac{P(E)}{P(F)}=\frac{4}{3}$ or $\frac{3}{4}$.

27. Correct $\sum x=400-3-4-5=388$

Correct $\sum x^2=2475-3^2-4^2-5^2 = 2425$

$\therefore$ Correct $\sigma^2=\frac{1}{97}(2425)-\left(\frac{1}{97}(388)\right)^2$

$= 25 - 4^2 = 9.$

28. Let $\alpha=\cot^{-1}(1+x)$

$\Rightarrow \quad \cot\alpha=1+x$

$\Rightarrow \quad \sin\alpha=\frac{1}{\sqrt{1+\cot^2\alpha}}$

$$= \frac{1}{\sqrt{2+2x+x^2}}$$

Next, let $\beta = \tan^{-1} x$

$\Rightarrow \quad \tan\beta = x$

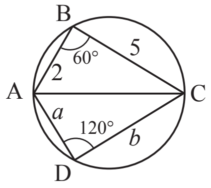
$\Rightarrow \quad \cos\beta = \frac{1}{\sqrt{1+\tan^2\beta}} = \frac{1}{\sqrt{1+x^2}}$

$\therefore \quad \sin\alpha = \cos\beta$

$\Rightarrow \quad 2 + 2x + x^2 = 1 + x^2$

$\Rightarrow \quad x = -\frac{1}{2}$.

29. Area of $\Delta ABC = \frac{1}{2}(2)(5)\sin 60° = \frac{5\sqrt{3}}{2}$



$\therefore \quad$ area of $\Delta ACD = 4\sqrt{3} - \frac{5\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

Thus, $\frac{3\sqrt{3}}{2} = \frac{1}{2}(ab)\sin 120° = \frac{1}{2}(ab)\left(\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad ab = 6$

Also, $AC^2 = 2^2 + 5^2 - (2)(2)(5)\cos 60°$

$= a^2 + b^2 - 2ab\cos 120°$

$\Rightarrow \quad 19 = a^2 + b^2 - 2ab\left(-\frac{1}{2}\right) = a^2 + b^2 + 6$

$\Rightarrow \quad a^2 + b^2 = 13$

$\Rightarrow \quad a^2 + b^2 = 13,\ ab = 6$

$\Rightarrow \quad a = 2,\ b = 3$

Thus, perimeter = 2 + 5 + 2 + 3 = 12.

30. Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$

$\therefore$ required contrapositive statement is

If the squares of two numbers are equal, then the numbers are equal.

JEE Main 2017 (Offline)

1. Let k be an integer such that the triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq. units. Then the orthocentre of this triangle is at the point:

(a) $\left(2, \frac{1}{2}\right)$

(b) $\left(2, -\frac{1}{2}\right)$

(c) $\left(1, \frac{3}{4}\right)$

(d) $\left(1, -\frac{3}{4}\right)$

2. If, for a positive integer n, the quadratic equation,

$$x(x+1)+(x+1)(x+2)+....$$
$$+ (x+\overline{n-1})\,(x+n)=10n$$

has two consecutive intergal solutions, then n is equal to:

(a) 11

(b) 12

(c) 9

(d) 10

3. The function $f:\mathbf{R}\rightarrow\left[-\frac{1}{2},\frac{1}{2}\right]$ defined as $f(x)=\frac{x}{1+x^2}$, is

(a) neither injective nor surjective.

(b) invertible.

(c) injective but not surjective.

(d) surjective but not injective.

4. The following statement $(p\rightarrow q)\rightarrow[(\sim p\rightarrow q)]$ is:

(a) a fallacy

(b) a tautology

(c) equivalent to $\sim p\rightarrow q$

(d) equivalent to $p\rightarrow \sim q$

5. If S is the set of distinct values of 'b' for which the following, system of linear equations

$$x+y+z=1$$
$$x+ay+z=1$$
$$ax+by+z=0$$

has no solution, then S is:

(a) A singleton

(b) An empty set

(c) An infinite set

(d) A finite set contaning two or more elements

6. The area (in sq. units) of the region $\{(x,y):\ x\geq 0, x+y\leq 3, x^2\leq 4y$ and $y\leq 1+\sqrt{x}\}$ is:

(a) $\frac{5}{2}$

(b) $\frac{59}{12}$

(c) $\frac{3}{2}$

(d) $\frac{7}{3}$

7. For any three positive real numbers a, b and c, $9(25a^2+b^2)+25(c^2-3ac)=15b(3a+c)$.

Then:

(a) a, b and c are in G.P.

(b) b, c and a are in G.P.

(c) b, c and a are in A.P.

(d) a, b and c are in A.P.

8. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is:

(a) 484

(b) 485

(c) 468

(d) 469

9. The normal to the curve $y(x-2)\,(x-3)=x+6$ at the point where the curve intersects the y-axis passes through the point:

(a) $\left(\frac{1}{2},\frac{1}{3}\right)$

(b) $\left(-\frac{1}{2},-\frac{1}{2}\right)$

(c) $\left(\frac{1}{2},\frac{1}{2}\right)$

(d) $\left(\frac{1}{2},-\frac{1}{3}\right)$

10. A hyperbola passes through the point $\mathrm{P}(\sqrt{2},\sqrt{3})$ and has foci at $(\pm 2, 0)$. Then the tangent to this hyperbola at P also passes through the point:

(a) $\left(-\sqrt{2},-\sqrt{3}\right)$

(b) $\left(3\sqrt{2},2\sqrt{3}\right)$

(c) $\left(2\sqrt{2},3\sqrt{3}\right)$

(d) $\left(\sqrt{3},\sqrt{2}\right)$

11. Let a, b, $c\in$ **R**. If $f(x)=ax^2+bx+c$ is such that $a+b+c=3$ and $f(x+y)=f(x)+f(y)+xy, \forall\ x, y\in$ **R**, then $\sum_{n=1}^{10} f(n)$ is equal to:

(a) 255

(b) 330

(c) 165

(d) 190

12. Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + \mathbf{j}$ Let **c** be a vector such that $|\mathbf{c} - \mathbf{a}| = 3, |(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$ and the angle between **c** and **a** × **b** be 30°. Then **a** · **c** is equal to:

(a) $\frac{1}{8}$ (b) $\frac{25}{8}$

(c) 2 (d) 5

13. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that AP = 2AB. If $\angle BPC = \beta$ then tanβ is equal to:

(a) $\frac{4}{9}$ (b) $\frac{6}{7}$

(c) $\frac{1}{4}$ (d) $\frac{2}{9}$

14. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower-bed, is:

(a) 30 (b) 12.5

(c) 10 (d) 25

15. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1 + \cos x}$ is equal to:

(a) –1 (b) –2

(c) 2 (d) 4

16. If (2+sin x) $\frac{dy}{dx}$ + (y + 1) cos x = 0 and $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ is equal to:

(a) $\frac{4}{3}$ (b) $\frac{1}{3}$

(c) $-\frac{2}{3}$ (d) $-\frac{1}{3}$

17. Let $I_n = \int \tan^n x \, dx$, ($n$ > 1). If $I_4 + I_6 = a\tan^5 x + bx^5 + c$, where c is a constant of integration, then the ordered pair (a, b) is equal to:

(a) $\left(-\frac{1}{5}, 0\right)$ (b) $\left(-\frac{1}{5}, 1\right)$

(c) $\left(\frac{1}{5}, 0\right)$ (d) $\left(\frac{1}{5}, -1\right)$

18. Let ω be a complex number such that 2ω + 1 = z where $z = \sqrt{-3}$. If

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$$

then k is equal to :

(a) 1 (b) $-z$

(c) z (d) –1

19. The value of

$$\left({}^{21}C_1 - {}^{10}C_1\right) + \left({}^{21}C_2 - {}^{10}C_2\right) + \left({}^{21}C_3 - {}^{10}C_3\right) + \left({}^{21}C_4 - {}^{10}C_4\right) + \ldots\ldots \left({}^{21}C_{10} - {}^{10}C_{10}\right)$$ is :

(a) $2^{20} - 2^{10}$ (b) $2^{21} - 2^{11}$

(c) $2^{21} - 2^{10}$ (d) $2^{20} - 2^{9}$

20. $\lim\limits_{x \to \frac{\pi}{2}} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals:

(a) $\frac{1}{4}$ (b) $\frac{1}{24}$

(c) $\frac{1}{16}$ (d) $\frac{1}{8}$

21. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of cos 4x is:

(a) $-\frac{7}{9}$ (b) $-\frac{3}{5}$

(c) $\frac{1}{3}$ (d) $\frac{2}{9}$

22. If the image of the point P(1, –2, 3) in the plane, $2x + 3y - 4z + 22 = 0$ measured parallel to the line, $\frac{x}{1} = \frac{y}{4} = \frac{7}{5}$ is Q, then PQ is equal to:

(a) $6\sqrt{5}$ (b) $3\sqrt{5}$

(c) $2\sqrt{42}$ (d) $\sqrt{42}$

23. The distance of the point (1,3, –7) from the plane passing through the point (1, –1, –1), having normal perpendicular to both the lines $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{-2}$ and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$, is :

(a) $\frac{10}{\sqrt{74}}$ (b) $\frac{20}{\sqrt{74}}$

(c) $\frac{10}{\sqrt{83}}$ (d) $\frac{5}{\sqrt{83}}$

24. If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}$. $g(x)$, then $g(x)$ equal:

    (a) $\frac{3}{1+9x^3}$ (b) $\frac{9}{1+9x^3}$

    (c) $\frac{3x\sqrt{x}}{1-9x^3}$ (d) $\frac{3x}{1-9x^3}$

25. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, $y = |x|$ is:

    (a) $4\left(\sqrt{2}+1\right)$ (b) $2\left(\sqrt{2}+1\right)$

    (c) $2\left(\sqrt{2}-1\right)$ (d) $4\left(\sqrt{2}-1\right)$

26. A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn, one-by-one, with replacement, then the variance of the number of green balls drawn is:

    (a) $\frac{6}{25}$ (b) $\frac{12}{5}$

    (c) 6 (d) 4

27. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directrices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is:

    (a) $x + 2y = 4$ (b) $2y - x = 2$

    (c) $4x - 2y = 1$ (d) $4x + 2y = 7$

28. If two different numbers are taken from the set $\{0, 1, 3, \ldots.., 10\}$; then the probability that their sum as well as absolute difference are both multiple of 4, is :

    (a) $\frac{7}{55}$ (b) $\frac{6}{55}$

    (c) $\frac{12}{55}$ (d) $\frac{14}{45}$

29. For three events A, B and C, P (Exactly one of A or B occurs) = P(Exactly one of B or C occurs) =P(Exactly one of C or A occurs)$=\frac{1}{4}$ and P(All the three events occur simultaneously) $=\frac{1}{16}$.

    Then the probability that at least one of the events occurs, is:

    (a) $\frac{3}{16}$ (b) $\frac{7}{32}$

    (c) $\frac{7}{16}$ (d) $\frac{7}{46}$

30. If A = $\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj $(3A^2+12A)$ is equal to:

    (a) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$ (b) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$

    (c) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (d) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$

Answers

1. (a)	**2.** (a)	**3.** (d)	**4.** (b)
5. (a)	**6.** (a)	**7.** (c)	**8.** (b)
9. (c)	**10.** (c)	**11.** (b)	**12.** (c)
13. (d)	**14.** (d)	**15.** (c)	**16.** (b)
17. (c)	**18.** (b)	**19.** (a)	**20.** (c)
21. (a)	**22.** (d)	**23.** (c)	**24.** (b)
25. (d)	**26.** (b)	**27.** (c)	**28.** (b)
29. (c)	**30.** (c)		

Hints and Solutions

1. $$\frac{1}{2}\left|\begin{vmatrix} k & -3k & 1 \\ 5 & k & 1 \\ -k & 2 & 1 \end{vmatrix}\right| = 28$$

   $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$

   To get

   $$\begin{vmatrix} k & -3k & 1 \\ 5-k & 4k & 0 \\ -2k & 2+3k & 0 \end{vmatrix} = \pm\ 56$$

   $\Rightarrow (5 - k)(2 + 3k) + 8k^2 = \pm 56$

   $\Rightarrow 10 + 13k - 3k^2 + 8k^2 = \pm 56$

   $\Rightarrow 5k^2 + 13k^2 + 66 = 0$ or $5k^2 + 13k - 46 = 0$

   But $5k^2 + 13k + 66 = 0$ does not have real roots.

   Next, $5k^2 + 13k - 46 = 0$ has two roots, viz. 2 and $-\frac{23}{5}$

   As k is an integer, $k = 2$.

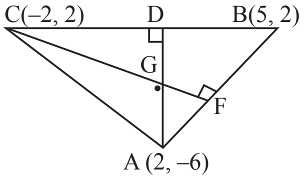
Thus, vertices of triangle are

(2, –6), (5, 2) and (–2, 2)

Equation of altitude AD is $x = 2$

Slope of AB is $\frac{-6-2}{2-5} = \frac{8}{3}$

∴ Slope os CF is $-\frac{3}{8}$



Equation of CF is

$y - 2 = -\frac{3}{8}(x + 2)$

Intersection of $x = 2$, is $y = \frac{1}{2}$.

Thus, orthocenter H of ΔABC is $(2, \frac{1}{2})$

2. Equation

$x(x + 1) + (x + 1)(x + 2) + \ldots\ldots + (x + n - 1)(x + n) = 10n$

Can be written as

$nx^2 + (1 + 3 + \ldots.. + 2n - 1)x + \sum_{k=1}^{n-1} k(k+1) = 10n$

$\Rightarrow nx^2 + n^2x + \frac{1}{6}(n - 1)n(2n - 1) + \frac{1}{2}n(n - 1)$

$= 10n$

$\Rightarrow x^2 + nx + \frac{1}{3}(n^2 - 31) = 0 \qquad \text{(i)}$

If α, β are roots of (i), then

$|\alpha - \beta| = 1$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$

$\Rightarrow n^2 - \frac{4}{3}(n^2 - 31) = 1$

$\Rightarrow -n^2 + 121 = 0$

$\Rightarrow n = 11.$

3. Note that

$\frac{x}{1+x^2} = \frac{1}{3}$

$\Rightarrow x^2 - 3x + 1 = 0$

$\Rightarrow x = \frac{3 \pm \sqrt{5}}{2}$

$\therefore\ f\left(\frac{3-\sqrt{5}}{2}\right) = \frac{1}{3} = f\left(\frac{3+\sqrt{5}}{2}\right)$

Thus, f is not injective.

For $\alpha \neq 0$,

$\frac{x}{1+x^2} = \alpha \Leftrightarrow \alpha x^2 - x + \alpha = 0$

$x = \frac{1 \pm \sqrt{1-4\alpha^2}}{2\alpha}$

$\therefore f\left(\frac{1 \pm \sqrt{1-4\alpha^2}}{2\alpha}\right) = \alpha$

Also, $f(0) = 0$.

Thus, f is surjective.

4. Let P be the statement $p \to q$;

Q be the statement $\sim p \to q$ and

R be the statement $Q \to q$.

p	q	~p	P	Q	Q→P	P→(Q→q)
T	T	F	T	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	F	T	T	F	T	T

Thus, $(p \to q) \to [(\sim p \to q) \to q]$

is a tautology.

5. Let

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix}$$

Applying, $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$, to obtain

$$D = \begin{vmatrix} 0 & 0 & 1 \\ 0 & a-1 & 1 \\ a-1 & b-1 & 1 \end{vmatrix} = -(a - 1)^2$$

If $D \neq 0$, the system has unique solution, therefore,

$D = 0 \Rightarrow a = 1$

System of Equations become

$x + y + z = 1$

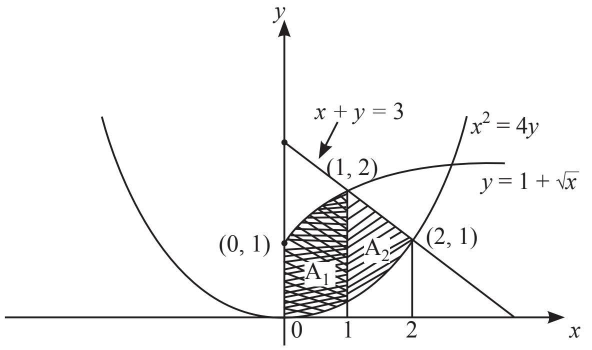
$x + by + z = 0$

$\Rightarrow (1 - b)y = 1$

For system to have no solution, $b = 1$

6. $y = 1 + \sqrt{x}$ and

$y = 3 - x$ meet at (1, 2)



Also, $x^2 = 4y$ and $x + y = 3$ meet at (2, 1)

Required area

$= A_1 + A_2$

$$= \int_0^1 \left(1+\sqrt{x}-\frac{1}{4}x^2\right)dx \ + \ \int_1^2 \left(3-x-\frac{1}{4}x^2\right)dx$$

$$= \left(x+\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{12}x^3\right)\Bigg]_0^1 + \left(3x-\frac{1}{2}x^2-\frac{1}{12}x^3\right)\Bigg]_1^2$$

$$= \left(1+\frac{2}{3}-\frac{1}{12}\right)+\left(6-2-\frac{8}{12}-3+\frac{1}{2}+\frac{1}{12}\right)$$

$= \frac{5}{2}$ sq.units

7. $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$

$\Rightarrow (15a)^2 + (3b)^2 + (5c)^2 - 45ab - 15bc - 75ac = 0$

$\Rightarrow ((15a)^2 + (3b)^2 - 90ab) + ((3b)^2 + (5c)^2 - 30bc)$
$+ ((15a)^2 + (5c)^2 - 150ac) = 0$

$\Rightarrow (15a - 3b)^2 + (3b - 5c)^2 + (15a - 5c)^2 = 0$

$\Rightarrow 15a - 3b = 0,\ 3b - 5c = 0$ and $15a - 5c = 0$

$\Rightarrow \frac{a}{1} = \frac{b}{5} = \frac{c}{3} = k$(say)

$\Rightarrow a = k,\ b = 5k,\ c = 3k$

$\therefore$ b, c, a are in A.P.

8. Different possible arrangements are given in the following table

	H W	H W	H W	H W
M	3 0	2 1	1 2	0 3
F	0 3	1 2	2 1	3 0

$\therefore$ The number of ways

$= ({}^4C_3)\ ({}^4C_3) + ({}^4C_2)({}^3C_1)({}^3C_1)({}^4C_2)$
$+ ({}^4C_1)({}^3C_2)({}^3C_2)({}^4C_1) + ({}^3C_3)({}^3C_3)$

$= 485$

9. The curve meets the y–axis at (0, 1). We have

$$y = \frac{x+6}{(x-2)(x-3)}$$

$$= \frac{9}{x-3} - \frac{8}{x-2}$$

$$\frac{dy}{dx} = -\frac{9}{(x-3)^2} + \frac{8}{(x-2)^2}$$

$$\therefore \frac{dy}{dx}\Bigg]_{(0,1)} = -\frac{9}{9} + \frac{8}{4} = 1$$

Thus, slope of normal at (0, 1) is –1.

Equation of normal at (0, 1) is

$y - 1 = -1(x - 0)$ or $x + y = 1$.

It passes through $\left(\frac{1}{2}, \frac{1}{2}\right)$

10. Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

We know $b^2 = a^2(e^2 - 1)$

$\Rightarrow a^2 + b^2 = (ae)^2 = 2^2 = 4$

Also,

$$\frac{2}{a^2} - \frac{3}{b^2} = 1$$

$$\Rightarrow \frac{2}{a^2} - \frac{3}{4-a^2} = 1$$

$\Rightarrow 8 - 2a^2 - 3a^2 = a^2(4 - a^2)$

$\Rightarrow a^4 - 9a^2 + 8 = 0 \Rightarrow a^2 = 1, 8$

As $a^2 + b^2 = 4$, $a^2 \neq 8$

Thus, $a^2 = 1$ and $b^2 = 3$

$\therefore$ equation of hyperbola is

$$x^2 - \frac{1}{3}y^2 = 1$$

An equation of tangent at $\left(\sqrt{2}, \sqrt{3}\right)$ is

$$\sqrt{2}x - \frac{\sqrt{3}}{3}y = 1$$

It passes through $\left(2\sqrt{2}, 3\sqrt{3}\right)$.

11. $f(x + y) = f(x) + f(y) + xy$

Also, $f(1) = 3$

$f(k + 1) = f(k) + f(1) + (k)(1)$

$= f(k) + (k + 3)$

Let

$$S = f(1) + f(2) + \ldots\ldots + f(k-1) + f(k)$$

$$S = \qquad f(1) + \ldots\ldots + f(k-2) + f(k-1) + f(k)$$

Subtract to obtain

$$0 = f(1) + (f(2) - f(1)) + (f(3) - f(2)) + \ldots + (f(k) - f(k-1)) - f(k)$$

$$\Rightarrow f(k) = 3 + 4 + 5 + \ldots\ldots + (k+2)$$

$$= \frac{1}{2}k(3+k+2) = \frac{1}{2}k(k+5)$$

Now, $\displaystyle\sum_{n=1}^{10} f(n) = \frac{1}{2}\sum_{n=1}^{10}\left(n^2+5n\right)$

$$= \frac{1}{2}\left[\frac{(10)(11)(21)}{6} + \frac{5}{2}(10)(11)\right]$$

$$= \frac{1}{4}(10)(11)(7+5) = 330$$

12. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix}$

$$= 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\Rightarrow |\mathbf{a}\times\mathbf{b}| = 3$$

As $|(\mathbf{a}\times\mathbf{b})\times\mathbf{c}| = 3$

$$\Rightarrow |\mathbf{a}\times\mathbf{b}||\mathbf{c}|\sin 30^\circ = 3$$

$$\Rightarrow 3|\mathbf{c}|\left(\frac{1}{2}\right) = 3 \Rightarrow |\mathbf{c}| = 2$$

Also,

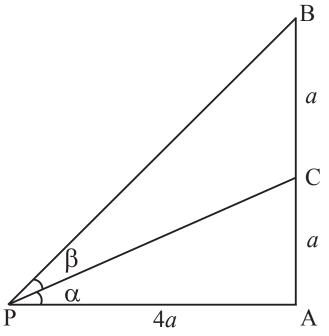
$$3 = |\mathbf{c} - \mathbf{a}|$$

$$\Rightarrow 9 = |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{c}$$

$$\Rightarrow \mathbf{a}\cdot\mathbf{c} = \frac{1}{2}(9 + 4 - 9) = 2$$

13. Let AC = a, then AB = $2a$

Also, AP = 2(AB) = $4a$



We have

$$PB^2 = 16a^2 + 4a^2 = 20a^2$$

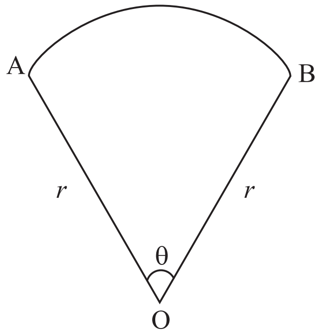
$$PC^2 = 16a^2 + a^2 = 17a^2$$

$$\therefore \cos\beta = \frac{20a^2 + 17a^2 - a^2}{2\left(\sqrt{20}a\right)\left(\sqrt{17}a\right)} = \frac{36}{4\sqrt{85}} = \frac{9}{\sqrt{85}}$$

$$\tan\beta = \frac{\sqrt{1-\cos^2\beta}}{\cos\beta} = \frac{\sqrt{85-81}}{9} = \frac{2}{9}$$

14. Let radius of circle be r, and ∠AOB = θ,

Then arc(AB) = θr



We are given

$$r + r + \theta r = 20$$

$$\Rightarrow \theta = \frac{20-2r}{r}$$

Also, area A of sector is given by

$$A = \frac{\theta}{2\pi}\left(\pi r^2\right)$$

$$= \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}r^2\left(\frac{20-2r}{r}\right) = r(10-r)$$

$$= 25 - (5-r)^2$$

Thus, A is maximum when $r = 5$ and maximum A is 25m^2

15. Let $I = \displaystyle\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ (1)

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos\left(\frac{3\pi}{4}+\frac{\pi}{4}-x\right)}$$

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x} \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x}\right)dx$$

$$= \int_{\pi/4}^{3\pi/4} \frac{2}{1-\cos^2 x}dx = 2\int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2 x\,dx$$

$$\Rightarrow \mathrm{I} = \left(-\cot x\right)\Big]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = -\cot\left(\frac{3\pi}{4}\right)+\cot\left(\frac{\pi}{4}\right)$$

$\Rightarrow \mathrm{I} = 2.$

16. Rewrite the equation as

$$\frac{dy}{y+1} = -\frac{\cos x}{2+\sin x}dx$$

Integrating, we get

$\ln|y + 1| = -\ln|2 + \sin x| + \ln c$

$\Rightarrow \ln|(y + 1)(2 + \sin x) = \ln c$

$\Rightarrow (y + 1)\ (2 + \sin x) = \pm\ \mathrm{c} = \mathrm{A}$ (say)

We are given $y(0) = 1$

$\therefore\ (1 + 1)(2 + \sin 0) = \mathrm{A} \Rightarrow \mathrm{A} = 4$

Thus, $y = -1 + \dfrac{4}{2+\sin x}$

$$\Rightarrow\ y\left(\frac{\pi}{2}\right) = -1+\frac{4}{2+1} = \frac{1}{3}$$

17. $\mathrm{I}_4 + \mathrm{I}_6 = \int\left(\tan^4 x+\tan^6 x\right)dx$

$= \int\tan^4 x\sec^2 x dx$

$= \dfrac{1}{5}\tan^5 x+c$

$\therefore\ a = \dfrac{1}{5},\ b = 0$

18. $w = \dfrac{1}{2}\left(-1+\sqrt{3}i\right)\ \Rightarrow 2w + 1 = \sqrt{3}i$

Also, $w^7 = w$ and $w^2 + 1 = -w$

Using $\mathrm{R}_1\rightarrow \mathrm{R}_1 + \mathrm{R}_2 + \mathrm{R}_3$ in the given determinant, We get

$$\Delta = \begin{vmatrix}3 & 0 & 0\\ 1 & w & w^2\\ 1 & w^2 & w\end{vmatrix}$$

$= 3(w^2 - w^4) = 3(w^2 - w)$

$\therefore\ k = w^2 - w$

$= w^2 + w - 2w$

$= -1 - 2w = -z$

19. Let

$\mathrm{S} = {}^{21}\mathrm{C}_1 + {}^{21}\mathrm{C}_2 + \ldots\ldots + {}^{21}\mathrm{C}_{10}$ (1)

Using ${}^{\mathrm{n}}\mathrm{C}_{\mathrm{r}} = {}^{\mathrm{n}}\mathrm{C}_{\mathrm{n-r}}$, We get

$\mathrm{S} = {}^{21}\mathrm{C}_{20} + {}^{21}\mathrm{C}_{19} + \ldots. + {}^{21}\mathrm{C}_{11}$ (2)

Adding (1), (2) we get

$$2\mathrm{s} = \sum_{k=1}^{20}\left({}^{21}C_k\right) = 2^{21} - 1 - 1 = 2^{21} - 2$$

$\Rightarrow \mathrm{S} = 2^{20}\text{-}1$

Next, let

$\mathrm{T} = {}^{10}\mathrm{C}_1 + {}^{10}\mathrm{C}_2 + \ldots. + {}^{10}\mathrm{C}_{10}$

$= 2^{10} - 1$

$\therefore\ ({}^{21}\mathrm{C}_1 - {}^{10}\mathrm{C}_1) + ({}^{21}\mathrm{C}_2 - {}^{10}\mathrm{C}_2) + \ldots. + ({}^{21}\mathrm{C}_{10} - {}^{10}\mathrm{C}_{10})$

$= (2^{20} - 1) - (2^{10} - 1) = 2^{20} - 2^{10}$

20. $\displaystyle\lim_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{\left(\pi-2x\right)^3}$

$$= \lim_{x\to\frac{\pi}{2}}\frac{\left(\cos x\right)\left(1-\sin x\right)}{8\sin x\left(\frac{\pi}{2}-x\right)^3}$$

$$= \lim_{\theta\to 0}\frac{\cos\left(\frac{\pi}{2}+\theta\right)\left[1-\sin\left(\frac{\pi}{2}+\theta\right)\right]}{8\left(\sin\left(\frac{\pi}{2}+\theta\right)\right)\left(-\theta\right)^3} \qquad \left[x = \frac{\pi}{2}+\theta\right]$$

$$= \lim_{\theta\to 0}\frac{\left(\sin\theta\right)\left(1-\cos\theta\right)}{8\left(\cos\theta\right)\theta^3}$$

$$= \frac{1}{8}\lim_{\theta\to 0}\frac{\left(\sin\theta\right)\left(1-\cos\theta\right)\left(1+\cos\theta\right)}{\left(\cos\theta\right)\left(1+\cos\theta\right)\theta^3}$$

$$= \frac{1}{8}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^3\frac{1}{\left(\cos\theta\right)\left(1+\cos\theta\right)}$$

$$= \frac{1}{8}(1)^3\ \frac{1}{(1)(1+1)} = \frac{1}{16}.$$

21. $5(\tan^2 x - \cos^2 x) = 2\cos(2x) + 9$

$\Rightarrow 5\tan^2 x = 5\cos^2 x + 2(2\cos^2 x - 1) + 9$

$\Rightarrow 5\tan^2 x = 9\cos^2 x + 7$

$\Rightarrow 5\sec^2 x = 9\cos^2 x + 12$

$\Rightarrow 9\cos^4 x + 12\cos^2 x - 5 = 0$

$\Rightarrow (3\cos^2 x - 1)(3\cos^2 x + 5) = 0$

$\Rightarrow \cos^2 x = \dfrac{1}{3}$ $\qquad [\because 3\cos^2 x + 5 > 0]$

$\Rightarrow \cos 2x = 2\cos^2 x - 1 = -\dfrac{1}{3}$

$\therefore\ \cos(4\mathrm{x}) = 2\cos^2(2x) - 1 = -\dfrac{7}{9}.$

22. Equation of line through P(1, –2, 3)

And parallel to $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$ is

$$\frac{x-1}{1}=\frac{y+2}{4}=\frac{z-3}{5}=r(\text{say})$$

$\Rightarrow x = r + 1,\ y = 4r - 2,\ z = 5r + 3.$

Let coordinates of point Q be $(r + 1, 4r - 2, 5r + 3)$

As Q lies on $2x + 3y - 4z + 22 = 0$,

$2(r + 1) + 3(4r - 2) - 4(5r + 3) + 22 = 0$

$\Rightarrow -6r + 6 = 0$

$\Rightarrow r = 1$

Now,

$$PQ^2 = (r + 1 - 1)^2 + (4r - 2 + 2)^2 + (5r + 3 - 3)^2$$

$= r^2(1 + 16 + 25) = 42r^2 = 42$

$\Rightarrow PQ = \sqrt{42}$

23. Note that normal to plane is

$\mathbf{N} = (\mathbf{i}-2\mathbf{j}+3\mathbf{k})\times(2\mathbf{i}-\mathbf{j}-\mathbf{k})$

$= 5\mathbf{i}+7\mathbf{j}+3\mathbf{k}$

Therefore equation of plane is

$$5(x - 1) + 7(y + 1) + 3(z + 1) = 0 \qquad (1)$$

Distance of (1, 3, -7) from (1) is

$$\frac{|5(1-1)+7(3+1)+3(-7+1)|}{\sqrt{25+49+9}} = \frac{10}{\sqrt{83}}$$

24. $y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$

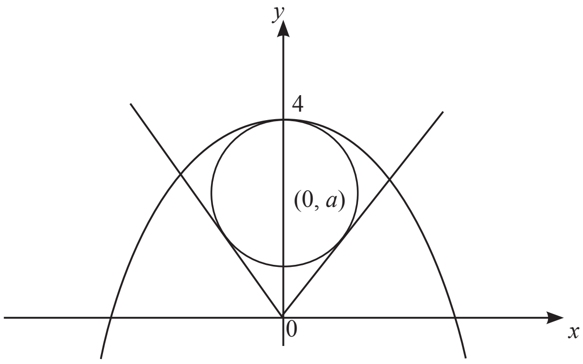
$$= \tan^{-1}\left(\frac{2\left(3x^{\frac{3}{2}}\right)}{1-\left(3x^{\frac{3}{2}}\right)^2}\right)$$

$$= 2\tan^{-1}\left(3x^{\frac{3}{2}}\right)$$

$$\frac{dy}{dx}=\frac{2.\left(3x^{\frac{1}{2}}\right)\left(\frac{3}{2}\right)}{1+9x^3} = \frac{9}{1+9x^3}\sqrt{x}$$

$$\therefore g(x) = \frac{9}{1+9x^3}$$

25. As the circle touches $y = |x|$, that is, $y = x$, $y = -x$, $y > 0$, the centre of circle lies on the positive of y–axis. Let centre of the circle be $(0, a)$ where $a > 0$.



Then the radius $r = 4 - a$

Also, length of perpendicular from $(0, a)$ to $y = x$ is r

$\therefore\ r = \frac{a}{\sqrt{2}}$

Thus, $4 - a = \frac{a}{\sqrt{2}} \Rightarrow 4 = \left(1+\frac{1}{\sqrt{2}}\right)a$

$\Rightarrow a = \frac{4\sqrt{2}}{\sqrt{2}+1}$

$\Rightarrow r = \frac{4}{\sqrt{2}+1} = 4\left(\sqrt{2}-1\right).$

26. Let p = probability of drawing a green ball in a single draw.

$= \frac{15}{25} = \frac{3}{5}$

n = number of balls drawn = 10, and

x = number of green balls drawn,

Then $x \sim B(n, p)$

$\text{Var}(x) = np(1 - p)$

$= 10\left(\frac{3}{5}\right)\left(1-\frac{3}{5}\right) = \frac{12}{5}$

27. We are given $e = \frac{1}{2}$, and

$-\frac{a}{e} = -4 \Rightarrow 2a = 4$ or $a = 2$

$\therefore\ b^2 = a^2(1 - e^2) = 4\left(1-\frac{1}{4}\right) = 3$

$\therefore$ Equation of ellipse is

$$\frac{x^2}{4}+\frac{y^2}{3} = 1$$

And equation of tangent at $\left(1, \frac{3}{2}\right)$ is

$$\frac{1}{4}x+\frac{3}{2}\frac{y}{3}=1$$

$x + 2y = 4$

$\therefore$ An equation of normal at $\left(1,\frac{3}{2}\right)$ is

$2(x - 1) - \left(y-\frac{3}{2}\right) = 0$

Or $2x - y - \frac{1}{2} = 0$ or $4x - 2y = 1$

28. Two numbers can be chosen in ${}^{11}C_2 = 55$ ways.

Suppose selected number are a, b with $a > b$, then

$a + b = 4r$, $a - b = 4s$

$\Rightarrow 2a = 4(r + s)$, $2b = 4(r - s)$

$\Rightarrow a = 2(r + s)$, $b = 2(r - s)$

$\therefore$ both a, b are even, and possible pairs of (a, b) are

(4, 0), (8, 0), (6, 2), (10, 2), (8, 4), (10, 6),

Thus, probability of required event is $\frac{6}{55}$.

29. P(Exactly one of A, B) $= \frac{1}{4}$

$\Rightarrow P(A \cap B') + P(A' \cap B) = \frac{1}{4}$

$\Rightarrow P(A) + P(B) - 2P(A \cap B) = \frac{1}{4}$ (1)

Similarly,

$P(B) + P(C) - 2P(B \cap C) = \frac{1}{4}$ (2)

And $P(C) + P(A) - 2P(C \cap A) = \frac{1}{4}$ (3)

Adding (1), (2), (3), we get

$2[P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)] = \frac{3}{4}$

$\Rightarrow P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = \frac{3}{8}$ (4)

Also, $P(A \cap B \cap C) = \frac{1}{16}$ (5)

From (4), (5), we get

$P(A \cup B \cup C) = \frac{3}{8}+\frac{1}{16}= \frac{7}{16}$.

30. $A^2 = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$

$\therefore 3A^2 + 12A$

$= \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$

$\Rightarrow \text{adj}(3A^2 + 12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$.